



HAL
open science

Using manipulatives in upper secondary math education-a semiotic analysis

Marit Hvalsøe Schou

► **To cite this version:**

Marit Hvalsøe Schou. Using manipulatives in upper secondary math education-a semiotic analysis. CERME 10, Feb 2017, Dublin, Ireland. hal-01950537

HAL Id: hal-01950537

<https://hal.science/hal-01950537v1>

Submitted on 10 Dec 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Using manipulatives in upper secondary math education – a semiotic analysis

Marit Hvalsøe Schou

Laboratory for Coherent Education and Learning, University of Southern Denmark;
maritschou@imada.sdu.dk

The use of manipulatives in upper secondary education is not common, although they hold learning potential in contributing to students' conceptualizing and establishing of relations. In this case study it is investigated how the concept of recursive sequences as a description of growth can be seen in the written reflections from students having engaged in activities with manipulatives like matches and LEGO® and with only minor teacher interference. The analysis of students' answers is carried out by a combination of Radford's Theory of Objectification and Peirce's Theory of Signs. By interpreting the type of signs, the kind of arguing and the nature of the words used by students in various activities involving manipulatives, it is possible to acquire knowledge of the significance of the manipulatives in students' conceptualizing.

Keywords: Upper secondary mathematics education, manipulatives, semiotics, conceptualizing, objectification.

Introduction

For centuries manipulatives have been used in mathematics education as a pedagogical means for learning at all levels. A visit to the Institute of Mathematics at the University of Göttingen, Germany reveals an entire floor display of manipulatives used in university teaching (Stewart, Mühlhausen, & Miyazaki, 1993). Today the use of manipulatives is still common in primary and lower secondary school as a way of making the teaching concrete and meaningful. On higher levels where deduction and generalisation are important issues, manipulatives are rarely seen and research on their use is scarce (Bartolini & Martignone, 2014). Previous studies show that manipulatives can help students perform algebraic reflections and actions in pattern-generalizing tasks in primary and lower secondary school by joint student-teacher interaction (Radford, 2008). Here it will be investigated how the use of manipulatives can be seen in student reflections on the concept of growth in this upper secondary education.

Theoretical framework

Generalisation depends on symbolization (Otte, 2006) and many teachers use symbolic representations extensively without connecting them to other representations thereby preventing their students the chance to explore or reflect on these relations (Steinbring, 2005). The use of manipulatives is an alternative way of bringing in abstraction and generalisation in teaching, enhancing the development of mathematical knowledge (Mitchelmore, 2002). In his Theory of Objectification, Radford (2013, p. 26) states: "Learning [...] is the noticing of something that is revealed in the emerging intention projected onto the signs [...] in the course of practical concrete activity [...] and is hence transformed into knowledge."

A semiotic analysis of the representations and the communication about these representations, i.e. the signs the students produce, is used to investigate the mathematical knowledge developed. Each

observation of student work is treated as *a semiotic bundle*, a system of signs produced by a student or a group of students while they solve a problem or discuss a mathematical question. (Arzarello, Paola, Robutti, & Sabena, 2009).

Presmeg et al. (2016) demonstrate the usefulness of Peirce's semiotics in analyzing students' work with representations. Peirce defines:

A sign, or *representamen*, is something, which stands to somebody or something in some respect or capacity. It addresses somebody, that is, creates in the mind of that person an equivalent sign, or perhaps a more developed sign. That sign which it creates I call the *interpretant* of the first sign. The sign stands for something, its *object*. [Peirce 1965, 2.228, italics in original]

A sign to Peirce thus consists of a triad (representamen – object – sign). Each of these can be interpreted in terms of his categories (of which there are three: “possibility, existence and law”). Here I use in particular two trichotomies: *icon*, *index* and *symbol* referring to how the sign stands for its object, and the *token-type* distinction (omitting *quali-sign*).

An *icon* is a sign that shares a likeness with the object it represents either as an *image*, a *diagram* or a *metaphor*. An *image* has a resemblance or a simple quality in common with the object – a drawing of a real thing; a *diagram* is a sign representing relations, e.g., $x^2 + y^2 = r^2$ represents the points of a circle. A *metaphor* represents a parallelism where the character of one sign is expressed in a law-like manner in the other, like when a symptom, fever, of illness is seen as an increased temperature at a thermometer. An *index* is related to the object it represents by either a causal connection or by ‘a purposeful act of connecting the signs’ like using ‘*f*’ as the name of a certain mathematical function. Finally a *symbol* is connected to its object by habit or law, e.g. designating π to refer to the ratio between the circumference and the diameter of a circle.

The *token-type* distinction applies to the sign alone. A *token* refers to the sign itself; consider the letter ‘e’, this letter appears many times – as a *token* – on this page. It is also possible to refer to ‘e’ as a general – as a *type*, stating that ‘e’ appears a particular number of times on this page.

A thorough explanation of Peirce's trichotomies of signs is offered in (Short, 2007) whereas mathematical examples can be found in (Otte, 2006) and (Sáenz-Ludlow & Kadunz, 2015).

The teaching design rests on the Theory of Didactical Situations, TDS (Brousseau, 2006). The epistemological foundation of TDS is that new knowledge is obtained in two steps: first as personal knowledge *connaissance* in working with a given problem and second as formal, scientific knowledge *savoir*, which can be share with and understood by others.

TDS distinguishes between *didactical situations* where the teacher participates, and *a-didactical situations* where students engage in activities on their own. *Devolution*, where the teacher introduces and hands over the problem, *validation* and *institutionalisation*, where the ideas and results are tested and then related to the scientific knowledge of the solution, are all didactical situations. The main focus of this case study is, however, the a-didactical situation where students interact with the *milieu*, which contains the tools that the students have at their disposal (symbolic or material as well as the other students in the group). An important issue in designing the milieu is to ensure that it provides feedback on how the students progress in order to solve the problem.

Within this framework the study will try to answer the question: *How does the use of manipulatives in teaching reflect in students' answers about growth, exemplified by recursive sequences?*

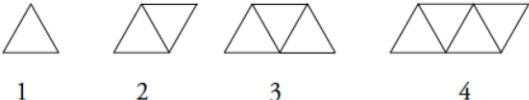
Method

The case study considers one lesson (100 min.) of a ten-week teaching sequence about differential equations in an upper secondary mathematics class with 25 students. The aim of the lesson was to enhance the students' understanding of the concept "growth" using recursive sequences. After a short devolution where the purpose of the lesson, describing the growth of certain patterns, and the milieu containing worksheets and manipulatives like matches and LEGO® were introduced, the students worked in groups of 4 for 60 min. During this period teacher intervention was very sparse. Figure 1 shows an example of an activity/worksheet. As preparation for the validation and institutionalisation the students were asked to share results, ideas and reflections with their fellow group members in a logbook prompted by questions given by the teacher. Part of the logbook writing was done in class and the rest at home using a Google Docs document shared by the group members. The empirical data consists of the logbooks and field notes from classroom observations.

The logbook entries were analysed through a qualitative content analysis (Hsieh & Shannon, 2005) for words indicating the concept of growth, like 'grow' and 'rise' (coded with underscore in the statements). In addition, words referring to the manipulatives (coded in **bold**) were noted.

To describe the patterns and how they evolved, the students introduced a 'sequence of signs'. They explained each sign and how they came on to it, and they used the sign as a starting point for the next sign they introduced. In order to examine the knowledge obtained by the students during the teaching sequence, data was analysed with respect to (1) the type of sign, applying Peirce's categories, (2) how the use of manipulatives show, according to the nature of the marked words, and (3) mathematical reasoning, according to Radford's definition of generalizing patterns (Radford, 2008).

The sequence of figures shown is made of matches. The first is made of 3 matches, the second of 5 matches etc. The sequence can be continued with figure no. 5, 6, 7, 8 ...



Build and describe the next figures and tell how many matches you need to build them.

Figure no. 2 has been made by adding matches to figure no. 1 in a specific way. Figure no. 2 has been made from no. 3 etc. Explain how you move from one figure to the next in the sequence. Describe the pattern for adding new matches.

Explain how this pattern affects the *number* of matches, you need, to go from one figure to the next in the sequence. That is how can the number of matches in one figure be found from the number of matches in the previous figure?

If you want to know how many matches you need to build figure no. 37 it is rather time consuming to build and count the number. Use the pattern you have described above to predict how many matches you need for figure no. 37.






Figure 1: An example of an activity

Results

In this section, we will look at two logbook entries and one episode observed in class that show how the use of manipulatives contributes to obtaining knowledge of growth.

Figure 2 shows the first logbook entry on the question: *Choose the pattern from one of the activities and explain how the pattern developed. Use drawings/pictures/symbols describing the growth.*

For a start we observed how the number of **matches** grew. We found that there was a rise on two **matches** in every step. You can almost say that in each step a new triangle is mirrored on to the right side of the **figure**.

Figure no.					
	1	2	3	4	5
No. of matches	3	5	7	9	11
<u>Rise</u> in number	2	2	2	2	2

We decided to make a formula to calculate the numbers of **matches** in figure no. 37. As the number rises with 2 for each figure, 2 is a constant and has to be multiplied by the figure no. But then the number is one too little. Therefore we add another constant in the formula namely +1.

The formula can then be written: The number of **matches** in the figure = the number of **matches** the figure grows with * figure number + 1.

We also considered other ways of describing the growth after having discussed our results in class. Here we noticed the constant rise with 2. This was the constant change between the earlier and the next figure. Therefore we could write down the relations:

$$T(1) = 3$$

$$T(2) = T(1) + 2$$

$$T(3) = T(2) + 2$$

$$T(4) = T(3) + 2$$

Figure 2: A logbook entry with marked words

The semiotic bundle in figure 2 shows how the students, based on the pattern they have built of matches, introduce three signs: a table inclusive a drawing of the pattern, a ‘formula’ written in words and a list of the first four terms of the recursive sequence describing the pattern. On figure 3 one can see how any one sign evolves from a previous sign. The photo on the left shows the pattern build of matches.

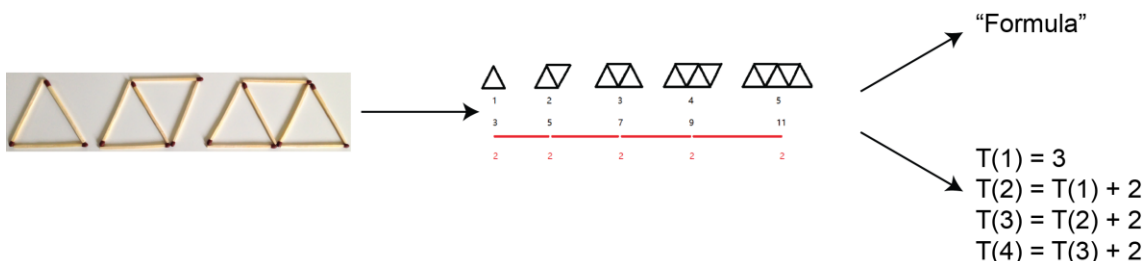


Figure 3: Signs introduced by the first group

The students begin by describing what they notice in the pattern they have built from matches, and they draw the pattern as the first line in a table. There is no explanation on the rest of the table but

my interpretation is that this is ‘a conditioned reflex’ as the students have made many tables like this with an independent variable (figure no.) and a dependent variable (no. of matches) when working with growth as a continuous function. The table is *iconic*: the previous text indicates that the students partly take it *diagrammatic* as it describes the relation between two subsequent figures. It also works in part as an *image* as the drawn pattern looks like the pattern made of matches.

In the table, the students also state the difference between two subsequent figures and this number is used for producing the next sign, a ‘formula’, from which they can calculate the number of matches in figure no. 37. The students notice that the increase is constant 2, which leads them to the first term of the expression. Using Radford’s definition of pattern generalization (Radford, 2008) we see that they use naïve inductions for producing their formula. Instead of arguing for the y -intercept, they guess and test the value. The formula itself is a partial translation of the previous arguments to a semi-mathematical language. This sign is partly *diagrammatic* stating a relation between the figure number and the number of matches and containing the *symbols* ‘+’ and ‘*’. It is interesting that the students do not finish the translation to a conventional formula (e.g. $f(x) = 2 \times x + 1$) introducing *indices* f and x . Most groups did at this stage. Instead they keep the original *iconical* terms ‘figure no.’ and ‘no. of matches’ closely related to the manipulatives.

Finally, after a validation and institutionalisation where the idea of explaining growth as a recursive sequence was discussed, the students decide to include this view as, not mentioning the important issue, though, that every new term is built on the previous term. They introduce an *index* T to stand for the number of matches but do not find a general expression using the symbol n for the figure number like $T(n) = T(n - 1) + 2$ seen in other logbook entries.

We see that the signs used for representing the object Growth include various kinds of icons as well as indices and symbols. In many of the icons there is a close link to the use of manipulatives i.e. the way the students obtain their ‘data’ and how they explain relations in the pattern. The manipulatives are not clearly visible in their reasoning, though. As a result of the analysis we may conclude that the understanding of growth for this group of students now includes some very concrete conceptions based on activities with manipulatives (matches), and these are pointing towards a more abstract understanding that has not yet been reached. The relations produced indicate that they take the pattern as a *token*. They have not yet reached the stage of considering the pattern as a *type* in order to produce general relations.

In some of the worksheets the use of manipulatives were not explicitly mentioned but replaced by drawings of squares, dots etc. Some groups, though, were using the manipulatives mentally, which in figure 4 is shown as a mixture of the terms ‘squares’/‘LEGO® blocks’ in their explanations.

[...] In other words, the figure consists of a typical **tower of LEGO® blocks** where each level is 2 broader than the previous so that one **unit** is sticking out on each side. Therefore the number of extra squares in the figure rises for every figure because the bottom is 2 **squares bigger** every time. You start with $T(1) = 1$. That is the first **LEGO® block**. [...]

Figure 4: Logbook entry showing the use of mentally evoked manipulatives

The group has not built this pattern physically, but in the context of other activities with manipulatives they use a mental picture made of LEGO®. Figure 5 shows the squares from the worksheet and the mentally evoked manipulatives. Like in the entry in figure 2, the text indicates that the *iconic* sign is partly *diagrammatic* as it describes the relation between two subsequent figures and at the same time works as an *image* as the drawn pattern looks like the pattern made of LEGO® blocks, which the students have evoked in their mind.

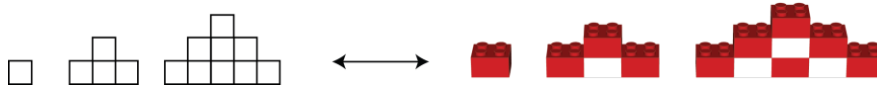


Figure 5: Alternating between to signs

As in figure 3, this group bring in various words describing growth; several terms refer to the manipulatives thereby showing the importance of the artefacts in the learning process.

The last example does not directly refer to the manipulatives from the activities but shows how a student seizes the idea of applying a concrete artefact in explaining a mathematical concept. The episode happened during the validation of the results from the activities described above. During the discussion it became obvious that many students had great difficulties with this new and different way of looking at the concept of growth. Their prior knowledge, connected to continuous functions, seemed very persistent. During the discussion one of the students reacted:

“No, no. We talk about the growth, the change – not the actual number. Look... [she jumps up from her chair, looks around and grabs a pile of **books** from the neighbouring table] ...If we start with, say 1 **book**, [she puts one book on the table] and then in the first step we put 2 **books** on top of it, and then in the next step 2 more **books** and 2 more **books** and so on. [For every step she places 2 books on top of the stack on the table, see figure 6]. It doesn't matter if there are 1 or 3 or 5 **books** in the **pile**. The important thing is that it grows with two **books** every time. *The difference* is two.“

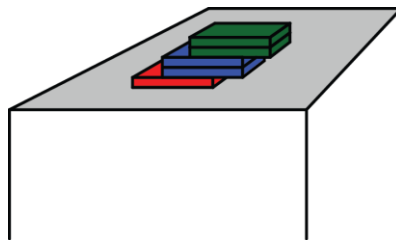


Figure 6: The pile of books holds many interpretations of a sign on Growth.

The student makes use of concrete artefacts in order to explain to her peers how to understand growth. She applies what is easily accessible in a classroom: books! As a representation of growth, the sign contain qualities of all three kinds of *icons*: the growing stable of books is an *image*, the relation of adding two more books in each step is a *diagram*, and by insisting on seeing the difference in each step (two more books every time) instead of the height of the stable, it becomes a *metaphor*. The student clearly states that this is just one way of visualizing how something grows. From the many different words used to indicate growth she demonstrates a wide understanding. It is an example of how manipulatives can be used as visualising a challenging concept rather than a starting point for general, symbolic expressions, which we saw in the logbook entries.

Discussion and conclusion

The case study demonstrates how the use of manipulatives shows in students' answers and reflections when describing a mathematical activity, e.g. building of patterns. By using concrete artefacts they are able to describe how patterns develop and translate this description to an expression of how the number of parts in each figure of the pattern grows. In the logbook entry (figure 2) the students use an intermediate stage mixing words and symbols, not reaching the final stage expressing the formula in formal mathematical language. Other groups, not considered here, argue for an algebraic expression.

Students use various words referring to different aspects of growth. Some are everyday language e.g. sticking out, broader, bigger, grow, rise and extra; whereas others are mathematical terms like add, multiply, change, growth etc. Often the everyday language is linked with the manipulatives, and the words might not have been evoked in a purely abstract context without the artefacts present. The semiotic analysis reveals that the signs introduced by students are predominantly iconic, many of these images. (Otte, 2006) argues “[...] iconic representations and perceptions are essential to introduce anything new into mathematical discourse”, and he notices, “Mathematics teachers very often dislike iconic representations and perceptions, believing them to be confusing and not controllable with respect to their impact” (Otte, 1983). This case study shows how students benefit from the use of artefacts by bringing in numerous iconic representations.

While figure 3 shows that very simple patterns can be visualized as drawings without actual physical materials, classroom observations, not treated here, reveal that patterns in 3D demand the use of actual building blocks. In both cases students use the names of the manipulative i.e. “matches” or “LEGO®”. Even when the physical artefact is not needed, students can apply a mental image of the artefact in their mathematical reasoning.

In former research results about the use of manipulatives in lower secondary education, Radford (2008) emphasizes the importance of joined-labour between students and teacher. In this study, the students were expected to be much more self-reliant as they were soon to end their upper secondary education, hence the use of a-didactical situations. The results, though, show that even students in the last year of upper secondary school need guidance in order to make their procedures and ideas converging with the mathematics curriculum as stated in the Theory of Objectification (Radford, 2008).

There are reasons to be somewhat cautious of the results. Using Peirce's categories of the relationship between object and sign is complicated as Presmeg, Radford, Roth, and Kadunz (2016) state: “the distinctions are subtle because they depend on the interpretation of the learner”. They continue: “the distinctions may be useful to researchers or teachers for the purpose of identifying the subtlety of a learner's mathematical conceptions if differences in interpretation are taken into account.” Thus, there are reasons to be careful in the interpretation. Earlier work of, among others, Radford (2000) shows comparable results indicating that the result of this case study is veridical. The semiotic analysis is based on students' statements, and it cannot always be taken for granted that students are conscious about the precise meaning of the words they use. The fact, though, that the analysis of most logbook entries share the features exemplified here indicates that the approach used in this study is rather robust with respect to the lack of carefulness in student language.

The results points towards several areas worth further investigation. In his study Radford (2008) shows how students can be led to algebraic generalisation, introducing mathematical symbols for unknown quantities. This can be taken further by studying how students make sense in already evolved algebraic expression, and whether manipulatives can be supportive. In upper secondary education students are presented with a large number of such algebraic expressions, which they are expected to manipulate and use but often fail to understand. Another interesting aspect is the significance of *prior knowledge*. In figure 2 we saw how the students were led to look at the relation between a figure number in the pattern and the total amount of matches in that figure resulting in a functional expression. This behaviour was even more apparent in other entries, not shown here, and could very well originate from their knowledge of continuous functions. Although prior knowledge often helps students in new contexts, it can also prevent them from going in the desired direction.

References

- Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational studies in mathematics*, 70(2), 97–109.
- Bartolini, M. G., & Martignone, F. (2014). Manipulatives in Mathematics Education *Encyclopedia of Mathematics Education* (pp. 365–372): Springer.
- Brousseau, G. (2006). *Theory of didactical situations in mathematics: Didactique des mathématiques, 1970–1990* (Vol. 19): Springer Science & Business Media.
- Hsieh, H.-F., & Shannon, S. E. (2005). Three approaches to qualitative content analysis. *Qualitative health research*, 15(9), 1277–1288.
- Mitchelmore, M. C. (2002). The Role of Abstraction and Generalisation in the Development of Mathematical Knowledge.
- Otte, M. (1983). Textual strategies. *For the learning of Mathematics*, 3(3), 15–28.
- Otte, M. (2006). Mathematical epistemology from a Peircean semiotic point of view. *Educational studies in mathematics*, 61(1), 11–38.
- Peirce, C. S. (1965). *Collected Papers of Charles Sanders Peirce: Edited by Charles Hartshorne and Paul Weiss*: Harvard University Press and the Belknap Press.
- Presmeg, N., Radford, L., Roth, W.-M., & Kadunz, G. (2016). Semiotics in Theory and Practice in Mathematics Education *Semiotics in Mathematics Education* (pp. 5–29): Springer.
- Radford, L. (2000). *Students' processes of symbolizing in algebra*. Paper presented at the Proceedings of the 24th PME Conference.
- Radford, L. (2008). Iconicity and contraction: A semiotic investigation of forms of algebraic generalizations of patterns in different contexts. *ZDM*, 40(1), 83–96.
- Radford, L. (2013). Three key concepts of the theory of objectification: Knowledge, knowing, and learning. *Journal of Research in Mathematics Education*, 2(1), 7–44.
- Sáenz-Ludlow, A., & Kadunz, G. (2015). *Semiotics as a Tool for Learning Mathematics*: Springer.
- Short, T. L. (2007). *Peirce's theory of signs*: Cambridge University Press.
- Steinbring, (2005). *The construction of new mathematical knowledge in classroom interaction: An epistemological perspective* (Vol. 38): Springer Science & Business Media.
- Stewart, I., Mühlhausen, E., & Miyazaki, K. (1993). Riemann surface – crocheted in four colors. *The Mathematical Intelligencer*, 15(3), 49–55.