



HAL
open science

Research findings associated with the concept of function and their implementation in the design of mathematics textbooks tasks

Mario Sánchez Aguilar, Apolo Castañeda, Rosa Isela González-Polo

► **To cite this version:**

Mario Sánchez Aguilar, Apolo Castañeda, Rosa Isela González-Polo. Research findings associated with the concept of function and their implementation in the design of mathematics textbooks tasks. CERME 10, Feb 2017, Dublin, Ireland. hal-01950531

HAL Id: hal-01950531

<https://hal.science/hal-01950531>

Submitted on 10 Dec 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Research findings associated with the concept of function and their implementation in the design of mathematics textbooks tasks

Mario Sánchez Aguilar¹, Apolo Castañeda² and Rosa Isela González-Polo³

¹Instituto Politécnico Nacional, CICATA Legaria, Mexico; mosanchez@ipn.mx

²Instituto Politécnico Nacional, CICATA Legaria, Mexico; acastane@ipn.mx

³Instituto Cumbres Toluca, Mexico; rosaiselag@gmail.com

The aim of this paper is to illustrate how the research results generated in the field of mathematics education could be implemented in the design of mathematics textbooks tasks. First we present research findings related to the concept of function, particularly findings related to representation registers and to the conceptualization of function as a process. Next we illustrate with examples obtained from a high school textbook, how these research findings can be implemented in the design of mathematical tasks. We close the manuscript with a reflection on the implications for research that this kind of implementation may have.

Keywords: Textbook development, functions, task design, implementation of research findings.

Introduction

As noted in the call for papers for the Thematic Working Group 23, *implementation research* can encompass a wide range of different kinds of didactical designs on a broad range of formats. In this work we focus on the use of research findings in the design of mathematics textbooks. More particularly, we will address research findings related to the learning of the mathematical concept of function and its implementation in the design of tasks included in a mathematics textbook for upper secondary level.

Textbooks play an important role in the teaching and learning of mathematics. For example, textbooks can affect teaching strategies by conveying pedagogical messages to mathematics teachers (Fan & Kaeley, 2000); also, mathematics textbooks can contribute to the creation and strengthening of students' misconceptions (Kajander & Lovric, 2009), and even the content selection and presentation of materials in a textbook appear to influence learners' participation and success in mathematics (Macintyre & Hamilton, 2010). Due to the huge influence that textbooks can exert in the dynamics of the mathematics classroom, over time there has been a growing interest in developing new and high-quality textbooks, however as pointed out by Li, Zhang & Ma (2014), "there are a very limited number of studies available that examine and discuss textbook design and the process of textbook development" (p. 306). Even more rare are the studies considering the implementation of research results in the design and development of mathematics textbooks, which it is the main focus of this manuscript.

The aim of this paper is to illustrate how research results produced in the field of mathematics education can be implemented in the design and development of tasks for mathematics textbooks. To achieve this we identify some research findings related to the concept of function and illustrate how they were used in the development of tasks for a high school mathematics textbook that is currently in use in the Mexican education system.

On the concept of function and research findings associated with it

We have focused our attention on the concept of function because it is a fundamental concept in the mathematical knowledge and as such plays a major role in mathematics textbooks (Mesa, 2000), particularly in those of middle and upper level. Because of its status and importance in the corpus of mathematical knowledge, educational research around this concept has been developed for several years and its results could be useful in the process of designing textbooks. In the next section we mention some research findings associated with this concept that we ourselves have implemented in the design of mathematical tasks included in upper secondary level Mexican textbooks.

What do we mean by «research findings»?

The products or results generated by research in mathematics education can be varied and fluctuate in a range that goes from tangible products (such as textbooks, educational activities, software) to more abstract products (such as constructs and theories). When we use the term «research findings» in this work we refer to information that has been obtained or discovered through empirical research and that can be expressed through observations, identification of obstacles and students' modes of thinking, didactical suggestions, etc.

We argue that this type of information may be applicable in designing mathematical tasks included in textbooks. Next we present some examples of research findings connected to the learning of the concept of function, whose implementation will be illustrated below.

What research says about the learning of the concept of function?

Different representations of the concept should be encouraged. Some students' difficulties connected with the concept of function can be attributed to the procedural emphasis with which this concept is taught, and also to the lack of variety in the representation contexts in which this concept is illustrated and manipulated.

Research has shown that many students possess prototypical visions of the concept of function. For instance, they tend to assume that functions are linear or quadratic in cases where this assumption is unwarranted, so for example they tend to think that “u-shaped” graphs are parabolas (Schwarz & Hershkowitz, 1999); Carlson & Oehrtman (2005) suggest that this difficulty may be related to the fact that many teachers introduce the concept of function through prototypical examples, which often are linear or quadratic. Thus, they suggest that the concept of function instruction should include more opportunities to experience different types of functions emphasizing different contexts of representation.

This is in line with the observations of Duval (2000). He claims that the conceptual understanding of a mathematical object becomes more robust when there is coordination between representation registers. Each register highlights certain characteristics and properties of the mathematical object, and the interaction between these registers allows for a broader conceptual understanding, so it is important to promote tasks that favor the transit between such representation registers, particularly the transit in directions that are not usually addressed in school, such as transit from the graphical register towards the algebraic one.

A dynamic vision of function as a process should be promoted. A common practice in the teaching of mathematics is to represent functions as static objects, however it has been suggested that

students must possess dynamic interpretations of this concept in order to favor a conceptualization of function as a process (Carlson, Oehrtman & Engelke, 2010). For instance, Figure 1 represents the area under a curve defined by a function $f(x)$, and it does not promote a dynamic conceptualization of the area function; to achieve such dynamic conceptualization the student should imagine that point b moves and in doing so the shaded area S increase or decrease its size.

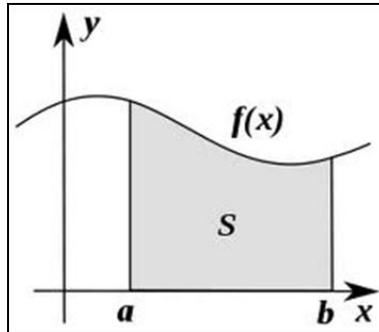


Figure 1: «Static» representation of an area under a curve

This type of static conceptualizations is closely related to an *action view of functions* (Dubinsky & Harel, 1992):

An *action* conception of function would involve the ability to plug numbers into an algebraic expression and calculate. It is a static conception in that the subject will tend to think about it one step at a time (e.g., one evaluation of an expression). (p. 85)

However, an action view of functions may result in an impoverished conceptualization of the concept; for instance, students with an action view often think of a function graph as being only a curve, a fixed object in the plane, they do not think the graph as defining a general rule where a set of input values are mapped to a set of output values (Carlson & Oehrtman, 2005). It is desirable to move students from an *action view of functions* to a *process view of functions*:

A process conception of function involves a dynamic transformation of quantities according to some repeatable means that, given the same original quantity, will always produce the same transformed quantity. The subject is able to think about the transformation as a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result of what was done. (Dubinsky & Harel, 1992, p. 85)

Is difficult to achieve this transition from static to a dynamic view of functions, however it has been suggested that technological tools can help in this transition. For example Borba & Confrey (1996) have suggested an approach to the study of functions based on visualization and the use of software; the approach focuses on the relationship between graphs and tabular values, and on the relationship between graphs and algebraic representations. For instance students could be asked to use the software to graph and explore how the coefficients of a quadratic function relate to translations, stretches and reflections of its graph.

Examples of implementation of research findings in the design of mathematics textbooks tasks

As we have claimed before, we believe that research findings as those previously presented can be implemented in the design of tasks for mathematics textbooks. We are aware that there may be different types of «implementation of research findings»; although it is not our intention to discuss such distinction here, we do want to clarify that in this work the «implementation of research findings» is interpreted as taking results or suggestions produced through research, and to use them as a source of inspiration for the design of mathematical tasks. To illustrate this point, next we present examples of tasks that were designed taking into consideration the research findings previously discussed. These tasks are included in the text González-Polo & Castañeda (2014), which was developed by the second and the third authors of this article. This is a textbook for high school level that is currently in use in the Mexican educational system; high school in Mexico traditionally consists of three years of education divided into six semesters, and this book is used in the fourth semester. Its print run for 2015 was 10,000 copies. The tasks proposed in the textbook are unpublished, but some of them have been inspired by tasks used as tools in the development of research in mathematics education.

Tasks to study functions in different representation registers

As mentioned before, research suggests that the concept of function should be studied and manipulated in different representation registers, but also should be promoted the transit between such representation registers, especially in directions that are not habitually addressed in school.

To implement this research-based suggestion, we have designed activities that require the student to transit from a graphical representation register to an algebraic register, when the usual is to ask students to start from an algebraic expression to generate a table of values, and from this table trace the graph of the function. Figure 2 shows an example of this kind of task.

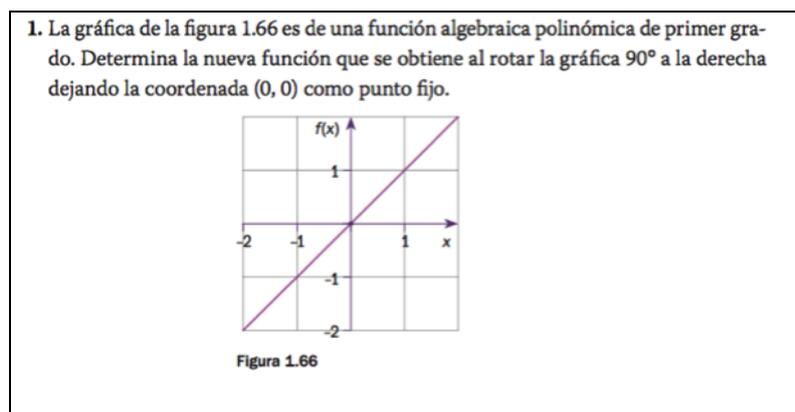


Figure 2: Task that requires the student to transit from a graphical register to an algebraic register

The English translation of the task instruction is as follows: «The graph in figure 1.66 belongs to a first degree polynomial function. Determine the new function obtained by rotating the graph 90° to the right leaving the coordinate $(0, 0)$ as a fixed point». In this task the student must start working on a graphical register—rotating the graph 90° clockwise—and then determine the algebraic expression that defines the new function, which in this case would be $f(x) = -x$.

A2 1. A partir de la forma general $f(x) = a^x$ se puede:

	Multiplicar la función por una constante	$f_1(x) = k \cdot a^x$
	Multiplicar el exponente por una constante	$f_2(x) = a^{kx}$
	Sumar al exponente una constante	$f_3(x) = a^{k+x}$
	Sumar una constante a la función	$f_4(x) = a^x + k$

a) Utiliza un graficador para obtener las gráficas de las siguientes funciones.

• $k = 2$	$f_1(x) = 2 \cdot 2^x$	$f_2(x) = 2^{2x}$	$f_3(x) = 2^{2+x}$	$f_4(x) = 2^x + 2$
• $k = 3$	$g_1(x) = 3 \cdot 2^x$	$g_2(x) = 2^{3x}$	$g_3(x) = 2^{3+x}$	$g_4(x) = 2^x + 3$
• $k = 5$	$h_1(x) = 5 \cdot 2^x$	$h_2(x) = 2^{5x}$	$h_3(x) = 2^{5+x}$	$h_4(x) = 2^x + 5$

b) Analiza el comportamiento de cada grupo de gráficas y explica cuál es el efecto en la gráfica al variar la constante k .

Figure 4: Task requesting the student to use graphing software to explore the effect of parameters on the graph of a function

The task takes as its starting point the function $f(x) = a^x$ and the constant k . Then the student is asked to use software to explore the effects that different integer values of the parameter k produces in the graph of the following functions:

$$f_1(x) = k \cdot a^x$$

$$f_2(x) = a^{k \cdot x}$$

$$f_3(x) = a^{k+x}$$

$$f_4(x) = a^x + k$$

Discussion

In this article we have tried to illustrate how some research findings related to the learning of the concept of function can be implemented in the design of tasks for mathematics textbooks. If one of our aims as mathematics educators is to bring products that are generated in our discipline closer to the school society (teachers, students, administrators, parents, etc.), then the textbooks are a privileged outlet for this purpose since it allows to bring research findings into the heart of formal mathematics instruction: the mathematics classroom.

Our enthusiasm as authors of textbooks and as researchers in mathematics education incline us to think that these tasks with a research-based design can be productive and beneficial for students' mathematical learning, but what evidence is there to support these enthusiast assumptions? It would be necessary to develop studies from different perspectives that could allow us to understand how the textbook mathematical tasks are enacted in the classroom, and the type of conceptions and perspectives that they produce on students.

Regarding the actual design of the tasks, in her reflections on textbook design, Yerushalmy (2015) has suggested that the tasks appearing in textbooks—or more precisely the mathematical concepts involved in such tasks—can be organized around objects and operations that can mathematically and pedagogically support a variety of progressions and sequences. For example, in the case of functions, it can be considered an organizational map that clarifies the type of mathematical object involved in the task (like linear or quadratic functions), but also the type of operations required in the task such as represent, modify, transform, analyze, operate or compare), where each operation can take place in symbolic, graphic, or numeric representations (see figure 5).

Symbolic/ Graphic/ Numeric						
Operation	Represent	Modify	Transform	Analyze (function change)	Operate (with 2)	Compare
Objects						
Generic						
Linear						
Quadratic						

Figure 5: Example of an organizational map for the tasks included in a textbook. Taken from Yerushalmy (2015, p. 243)

This type of organizational maps can make the design of tasks more transparent, that is, to render explicit the mathematical objects involved in the tasks as well as the operations that are performed on them. These maps can work as a framework that helps both textbook designers and users to identify gaps in the presentation of concepts, and produce a sequencing of tasks that addresses the largest possible number of operations and contexts of representation with the intention to provide students with a richer picture of the mathematical objects studied.

Finally, it is important to note that in addition to the textbook González-Polo & Castañeda (2014), there is a teacher's guide explaining in more detail the theoretical background on which the design of the tasks is based, as well as their purpose. This kind of guide represents a fundamental support to achieve a classroom implementation of mathematical tasks that is faithful to the intentions of the task designers.

References

- Borba, M. C. & Confrey, J. (1996). A student's construction of transformations of functions in a multiple representational environment. *Educational Studies in Mathematics*, 31(3), 319–337. doi: 10.1007/BF00376325
- Carlson, M. & Oehrtman, M. (2005). Key aspects of knowing and learning the concept of function. *Research Sampler Series, 9, The Mathematical Association of America Notes Online*. Retrieved September 14, 2016, from <http://goo.gl/8mfGFt>
- Carlson, M., Oehrtman, M. & Engelke, N. (2010). The Precalculus Concept Assessment: a tool for assessing students' reasoning abilities and understandings. *Cognition & Instruction*, 28(2), 113–145. doi: 10.1080/07370001003676587

- Dubinsky, E., & Harel, G. (1992). The process conception of function. In G. Harel & E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy*. MAA Notes, 28, 85–106.
- Duval, R. (2000). Basic issues for research in mathematics education. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education* (Volume 1, pp. 55–69). Hiroshima: Nishiki Print Co. Ltd.
- Fan, L. & Kaeley, G. S. (2000). The influence of textbooks on teaching strategies: an empirical study. *Mid-Western Educational Researcher*, 13(4), 2–9.
- González-Polo, R. I. & Castañeda, A. (2014). *Matemáticas 4* [Mathematics 4]. Mexico: Mcmillan Education.
- Kajander, A. & Lovric, M. (2009). Mathematics textbooks and their potential role in supporting misconceptions. *International Journal of Mathematical Education in Science and Technology*, 40(2), 173–181. doi: 10.1080/00207390701691558
- Li, Y., Zhang, J. & Ma, T. (2014). School mathematics textbook design and development practices in China. In Y. Li & G. Lappan (Eds.), *Mathematics Curriculum in School Education* (pp. 305–326). Dordrecht: Springer. doi: 10.1007/978-94-007-7560-2_15
- Macintyre, T. & Hamilton, S. (2010). Mathematics learners and mathematics textbooks: a question of identity? Whose curriculum? Whose mathematics? *The Curriculum Journal*, 21(1), 3–23. doi: 10.1080/09585170903558224
- Mesa, V. M. (2000). *Conceptions of function promoted by seventh- and eighth-grade textbooks from eighteen countries* (Unpublished doctoral dissertation). University of Georgia. Athens, GA.
- Schwarz, B. B. & Hershkowitz, R. (1999). Prototypes: Brakes or levers in learning the function concept? The role of computer tools. *Journal for Research in Mathematics Education*, 30(4), 362–389.
- Yerushalmy, M. (2015). E-textbooks for mathematical guided inquiry: Design of tasks and task sequences. In A. Watson & M. Ohtani (Eds.), *Task Design In Mathematics Education. An ICMI study* (pp. 229–247). Switzerland: Springer. doi: 10.1007/978-3-319-09629-2_7