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An arithmetical function related to Báez-Duarte's criterion for the Riemann hypothesis

Michel Balazard

To the memory of my friend, Luis Báez-Duarte.

Abstract

In this mainly expository article, we revisit some formal aspects of Báez-Duarte's criterion for the Riemann hypothesis. In particular, starting from Weingartner's formulation of the criterion, we define an arithmetical function v, which is equal to the Möbius function if, and only if the Riemann hypothesis is true. We record the basic properties of the Dirichlet series of v, and state a few questions.

KEYWORDS

Riemann hypothesis, arithmetical functions, Dirichlet series, Hilbert space MSC classification : 11M26

1 The spaces D and D_0

We will denote by \mathbb{N} (resp. \mathbb{N}^*) the set of non-negative (resp. positive) integers, by *H* the Hilbert space $L^2(0,\infty;t^{-2}dt)$, with inner product

$$\langle f,g\rangle = \int_0^\infty f(t)\overline{g(t)}\,\frac{dt}{t^2},$$

and by $Vect(\mathscr{F})$ the set of finite linear combinations of elements of a family \mathscr{F} of elements of *H*.

For $k \in \mathbb{N}^*$, we define

$$e_k(t) = \{t/k\} \quad (t > 0),$$

where $\{u\} = u - \lfloor u \rfloor$ denotes the fractional part of the real number u, and $\lfloor u \rfloor$ its integer part. The functions e_k belong to H, as do the functions χ and κ defined by

$$\chi(t) = [t \ge 1]$$
; $\kappa(t) = t[0 < t < 1]$

(here, and in the following, we use Iverson's notation : [P] = 1 if the assertion P is true, [P] = 0 if it is false).

Let *D* be the closed subspace of functions $f \in H$ of the type

$$f(t) = \lambda t + \varphi(t), \tag{1}$$

where φ is constant on each interval $[j, j+1], j \in \mathbb{N}$ (for j = 0, the constant must be 0). The functions e_k belong to D.

Let D_0 be the subspace of D defined by taking $\lambda = 0$ in (1), that is, the subspace of functions $\varphi \in H$ which are constant on each interval $[j, j+1], j \in \mathbb{N}$. The functions χ and $e_k - e_1/k$ belong to D_0 .

A hilbertian basis for D_0 is given by the family of step functions ε_k defined by

$$\varepsilon_k(t) = \sqrt{k(k+1)} \cdot [k \le t < k+1] \quad (k \in \mathbb{N}^*, t > 0).$$

The mapping $h \mapsto (h(j))_{j \ge 1}$ is a Hilbert space isomorphism of D_0 onto the sequence space \mathfrak{h} of complex sequences $(x_j)_{j \ge 1}$ such that

$$\sum_{j\geq 1} \frac{|x_j|^2}{j(j+1)} < \infty \cdot$$

Observe that, for $f \in D$, written as (1), one has

$$\lambda = \langle f, \kappa \rangle \tag{2}$$

$$f = \lambda e_1 + h$$
, where $h \in D_0$. (3)

Thus, the subspace D is the (non orthogonal) direct sum of $Vect(e_1)$ and D_0 .

In formula (2), the function κ could be replaced by its orthogonal projection κ' on *D*. The definition of the families (ψ_n) of Proposition 2 and (g_n) of Proposition 4 below could be modified accordingly. We compute κ' in the appendix.

To every function in *D*, one can associate certain arithmetical functions. Let $f \in D$, with λ and *h* as in (2), (3). We first define the arithmetical function

$$u(n) = u(n; f) = -\lambda + h(n) - h(n-1) \quad (n \in \mathbb{N}^*).$$
(4)

With this definition, we see that the function φ of (1) is given by

$$\varphi(t) = -\lambda t + f(t) = -\lambda t + \lambda \{t\} + h(t) = \sum_{n \le t} u(n)$$

Thus, f(t) is the remainder term in the approximation of the sum function $\varphi(t)$ of the arithmetical function u by the linear function $-\lambda t$. The fact that f belongs to H implies, and is stronger than, the asymptotic relation f(t) = o(t).

For $f \in D$, we will also consider the arithmetical function $w = \mu * u$, where μ denotes the Möbius function,

$$w(n) = w(n; f) = \sum_{d|n} \mu(n/d)u(d; f) \quad (n \in \mathbb{N}^*).$$

For instance,

$$u(n; \boldsymbol{\chi}) = [n = 1]$$
; $w(n; \boldsymbol{\chi}) = \mu(n)$ $(n \in \mathbb{N}^*)$

The arithmetical functions *u* and *w* depend linearly on *f* and the correspondences are one-to-one.

Proposition 1 For $f \in D$,

$$f = 0 \Leftrightarrow u = 0 \Leftrightarrow w = 0.$$

Proof

The second equivalence follows from $w = u * \mu$ and u = w * 1 (Möbius inversion). It remains to prove that $u = 0 \Rightarrow f = 0$. By (4), u = 0 implies $h(n) = \lambda n$ for all n, hence $\lambda = 0$ since $h \in D_0$, and h = 0.

Since u = w * 1, one has

$$f(t) = \lambda t + \sum_{n \le t} u(n) = \lambda t + \sum_{n \ge 1} w(n) \lfloor t/n \rfloor.$$

In Proposition 7 below, we will prove the identity

$$\sum_{n\geq 1} \frac{w(n)}{n} = -\lambda,\tag{5}$$

so that, for every f in D and every t > 0, one has

$$f(t) = -\sum_{n \ge 1} w(n)e_n(t).$$
(6)

Of course, it does not mean that the series $\sum_{n\geq 1} w(n; f)e_n$ converges in H (in fact, it diverges if $f = \chi$, cf. [1], Theorem 2.2, p. 6), but, if it does, its sum is -f.

2 Vasyunin's biorthogonal system

In Theorem 7 of his paper [7], Vasyunin defined a family $(f_k)_{k\geq 2}$, which, together with the family $(e_k - e_1/k)_{k\geq 2}$, yields a biorthogonal system in D_0 , which means that

$$\langle e_j - e_1/j, f_k \rangle = [j = k] \quad (j \ge 2, k \ge 2).$$
 (7)

We will recall Vasyunin's construction, which can be thought of as a Hilbert space formulation of Möbius inversion, and add several comments.

2.1 The sequence (φ_k)

First one defines, for $k \in \mathbb{N}^*$, a step function $\varphi_k \in D_0$ by

$$\varphi_k(t) = k(k-1)[k-1 \le t < k] - k(k+1)[k \le t < k+1]$$

(Vasyunin's φ_k have the opposite sign, according to his definition for e_k). Thus

$$\varphi_k = \sqrt{k(k-1)} \cdot \varepsilon_{k-1} - \sqrt{k(k+1)} \cdot \varepsilon_k \quad (k \in \mathbb{N}^*),$$

with $\varepsilon_0 = 0$ by convention. One sees that the family $(\varphi_k)_{k \ge 1}$ is total in D_0 .

One checks that

$$\langle h, \boldsymbol{\varphi}_k \rangle = h(k-1) - h(k) \quad (k \in \mathbb{N}^*), \tag{8}$$

for $h \in D_0$ with constant value h(k) on [k, k+1] (h(0) = 0). In particular,

$$\langle e_j - e_1/j, \varphi_k \rangle = [j \mid k] - 1/j \quad (j \ge 1, k \ge 1).$$

Using the family (φ_k) , one can write the values u(n; f), for $f \in D$, as scalar products.

Proposition 2 For $f \in D$, with λ and h as in (2), (3), one has

$$u(n;f) = \langle f, \Psi_n \rangle,$$

where

$$\psi_n = (\langle e_1, \varphi_n \rangle - 1)\kappa - \varphi_n \quad (n \in \mathbb{N}^*).$$

In particular, $f \mapsto u(n; f)$ is a continuous linear form on D, for every $n \in \mathbb{N}^*$.

Proof

By (2), (4) and (8), one has

$$\begin{split} u(n;f) &= -\langle f, \kappa \rangle - \langle h, \varphi_n \rangle \\ &= -\langle f, \kappa \rangle - \langle f - \langle f, \kappa \rangle e_1, \varphi_n \rangle \\ &= -\langle f, \kappa \rangle - \langle f, \varphi_n \rangle + \langle e_1, \varphi_n \rangle \langle f, \kappa \rangle \\ &= \langle f, \psi_n \rangle \quad (n \in \mathbb{N}^*). \end{split}$$

We compute the scalar product $\langle e_1, \varphi_n \rangle$ in the appendix.

The next proposition describes the behavior of the series $\sum_k \varphi_k/k$.

Proposition 3 The series

$$\sum_{k\geq 1}\frac{\varphi_k}{k}$$

is weakly convergent in D_0 , with weak sum $-\chi$.

Proof

The partial sum

$$\sum_{k\leq K}\frac{\varphi_k}{k}$$

is the step function with values

0 on (0,1) and
$$(K+1,\infty)$$

-1 on (1,*K*)
-(*K*+1) on (*K*,*K*+1)

This partial sum is thus equal to $-\chi$ on every fixed bounded segment of $(0,\infty)$, if K is large enough, and the norm of this partial sum in H is the constant $\sqrt{2}$. The result follows.

2.2 The sequence (f_k)

Vasyunin defined

$$f_k = \sum_{d|k} \mu(k/d) \varphi_d \quad (k \in \mathbb{N}^*).$$

Equivalently,

$$\varphi_k = \sum_{d|k} f_d \quad (k \in \mathbb{N}^*),$$

by Möbius inversion ; this implies that the family $(f_k)_{k\geq 1}$ is also total in D_0 .

A slightly more general form of (7), namely

$$\langle e_j - e_1/j, f_k \rangle = [j = k] - [k = 1]/j \quad (j, k \in \mathbb{N}^*),$$
(9)

is proved by means of the identity

$$\sum_{j|d|k} \mu(k/d) = [j=k].$$

Using the family (f_k) , one can write the values w(n; f), for $f \in D$, as scalar products.

Proposition 4 For $f \in D$, with λ and h as in (2), (3), one has

$$w(n;f) = \langle f, g_n \rangle,$$

where

$$g_n = (\langle e_1, f_n \rangle - [n=1])\kappa - f_n \quad (n \in \mathbb{N}^*).$$

In particular, $f \mapsto w(n; f)$ is a continuous linear form on D, for every $n \in \mathbb{N}^*$.

Proof

By Proposition 2, one has

$$w(n;f) = \sum_{d|n} \mu(n/d)u(d;f)$$
$$= \langle f, \sum_{d|n} \mu(n/d)\psi_d \rangle \quad (n \in \mathbb{N}^*).$$

Now,

$$\sum_{d|n} \mu(n/d) \psi_d = \sum_{d|n} \mu(n/d) \big((\langle e_1, \varphi_d \rangle - 1) \kappa - \varphi_d \big)$$
$$= (\langle e_1, f_n \rangle - [n = 1]) \kappa - f_n.$$

We compute the scalar product $\langle e_1, f_n \rangle$ in the appendix.

In order to study the series $\sum_k f_k/k$, we will need the following auxiliary proposition.

Proposition 5 Let

$$f(x) = \sum_{k \le x} \eta(k) \quad (x > 0),$$

where η is a complex arithmetical function such that $\eta(k) = O(1/k)$, for $k \ge 1$. Then, for every fixed $\alpha > 1$,

$$\sum_{k\geq 1} |f(x/k) - f(x/(k+1))|^{\alpha} = O(1) \quad (x>0).$$

Proof

The series is in fact a finite sum, as

$$f(x/k) = f(x/(k+1)) = 0$$
 $(k > x).$

We will use the estimate

.

$$f(y) - f(x) \ll \sum_{x < k \le y} \frac{1}{k} \ll \frac{1}{x} + \ln(y/x) \quad (y > x \ge 1).$$

Thus,

$$f(x/k) - f(x/(k+1)) \ll \frac{k}{x} + \frac{1}{k} \ll \frac{1}{k} \quad (k \le \sqrt{x}),$$

and

$$\sum_{k \le \sqrt{x}} \left| f(x/k) - f\left(x/(k+1)\right) \right|^{\alpha} \ll \sum_{k \ge 1} \frac{1}{k^{\alpha}} \ll 1 \quad (x > 0).$$

If $k > \sqrt{x}$, then

$$\frac{x}{k} - \frac{x}{k+1} < 1,$$

so that the interval]x/(k+1), x/k] contains at most one integer, say *n*, and, if *n* exists, one has $k = \lfloor x/n \rfloor$ and

$$f(x/k) - f\left(x/(k+1)\right) = \eta(n) \ll \frac{1}{n}$$

Hence

$$\sum_{k > \sqrt{x}} \left| f(x/k) - f(x/(k+1)) \right|^{\alpha} \ll \sum_{n \ge 1} \frac{1}{n^{\alpha}} \ll 1 \quad (x > 0).$$

The result follows.

Proposition 6 The series

 $\sum_{k\geq 1}\frac{f_k}{k}$

is weakly convergent in D_0 (hence in H), with weak sum 0.

Proof

Let $K \in \mathbb{N}^*$. One has

$$S_K = \sum_{k \le K} \frac{f_k}{k} = \sum_{d \le K} \frac{m(K/d)}{d} \varphi_d,$$

where

$$m(x) = \sum_{n \le x} \frac{\mu(n)}{n} \quad (x > 0).$$

Hence,

$$S_{K} = \sum_{d \le K} \frac{m(K/d)}{d} \left(\sqrt{d(d-1)} \cdot \varepsilon_{d-1} - \sqrt{d(d+1)} \cdot \varepsilon_{d} \right)$$
$$= \sum_{d \le K-1} \left(\frac{m(K/(d+1))}{d+1} - \frac{m(K/d)}{d} \right) \sqrt{d(d+1)} \cdot \varepsilon_{d} - \sqrt{1+1/K} \cdot \varepsilon_{K}$$

For every fixed $d \in \mathbb{N}^*$, the fact that $\langle S_K, \varepsilon_d \rangle$ tends to 0 when *K* tends to infinity follows from this formula and the classical result of von Mangoldt, that m(x) tends to 0 when *x* tends to infinity. It remains to show that $||S_K||$ is bounded. One has

$$\begin{split} \|S_K\|^2 &= \sum_{d \le K-1} d(d+1) \Big(\frac{m(K/d)}{d} - \frac{m(K/(d+1))}{d+1} \Big)^2 + 1 + 1/K \\ &\le 2 \sum_{d \le K-1} d(d+1) \Big(\frac{m(K/d) - m(K/(d+1))}{d} \Big)^2 \\ &+ 2 \sum_{d \le K-1} d(d+1) \Big(\frac{m(K/(d+1))}{d(d+1)} \Big)^2 + 1 + 1/K \\ &\ll 1 + \sum_{d \le K-1} \Big(m(K/d) - m(K/(d+1)) \Big)^2 \end{split}$$

The boundedness of $||S_K||$ then follows from Proposition 5. We are now able to prove (5).

Proposition 7 Let $f \in D$, with λ and h as in (2), (3). The series

$$\sum_{n\geq 1}\frac{w(n;f)}{n}$$

is convergent and has sum $-\lambda$.

Proof

Putting $\beta_N = \sum_{n \le N} f_n / n$ for $N \in \mathbb{N}^*$, one has

$$\sum_{n \le N} \frac{g_n}{n} = \sum_{n \le N} \frac{(\langle e_1, f_n \rangle - [n=1])\kappa - f_n}{n}$$
$$= (\langle e_1, \beta_N \rangle - 1)\kappa - \beta_N,$$

which tends weakly to $-\kappa$, as N tends to infinity, by Proposition 6.

Hence,

$$\sum_{n \le N} \frac{w(n; f)}{n} = \sum_{n \le N} \frac{\langle f, g_n \rangle}{n} = \langle f, \sum_{n \le N} g_n / n \rangle \to -\langle f, \kappa \rangle = -\lambda \quad (N \to \infty).$$

3 Dirichlet series

For $f \in D$ we define

$$F(s) = \sum_{n \ge 1} \frac{u(n;f)}{n^s},$$

and we will say that F is the Dirichlet series of f.

We will denote by σ the real part of the complex variable *s*. The following proposition summarizes the basic facts about the correspondance between elements *f* of *D* and their Dirichlet series *F*. We keep the notations of (2) and (3).

Proposition 8 For $f \in D$, the Dirichlet series F(s) is absolutely convergent in the half-plane $\sigma > 3/2$, and convergent in the half-plane $\sigma > 1$. It has a meromorphic continuation to the half-plane $\sigma > 1/2$ (we will denote it also by F(s)), with a unique pole in s = 1, simple and with residue $-\lambda$. In the strip $1/2 < \sigma < 1$, one has

$$F(s)/s = \int_0^\infty f(t)t^{-s-1}dt.$$
 (10)

If $f \in D_0$, that is $\lambda = 0$, there is no pole at s = 1, and the Mellin transform (10) represents the analytic continuation of F(s)/s to the half-plane $\sigma > 1/2$. Moreover, the Dirichlet series F(s)converges on the line $\sigma = 1$.

Proof

If h = 0 in (3), the arithmetical function u is the constant $-\lambda$, and $F = -\lambda \zeta$. In this case, the assertion about (10) follows from (2.1.5), p. 14 of [6].

If $\lambda = 0$, then $f = h \in D_0$ and u(n) = h(n) - h(n-1) by (4). Therefore,

$$\begin{split} \sum_{n \ge 1} \frac{|u(n)|}{n^{\sigma}} &\leq 2 \sum_{n \ge 1} \frac{|h(n)|}{n^{\sigma}} \\ &\leq 2 \big(\sum_{n \ge 1} \frac{|h(n)|^2}{n^2} \big)^{1/2} \big(\sum_{n \ge 1} \frac{1}{n^{2\sigma-2}} \big)^{1/2} \\ &\leq 2 \zeta (2\sigma-2)^{1/2} \|h\| < \infty, \end{split}$$

if $\sigma > 3/2$, where we used Cauchy's inequality for sums.

The convergence of the series F(1) follows from the formula $u(n) = -\langle h, \varphi_n \rangle$ and Proposition 3. It implies the convergence of F(s) in the half-plane $\sigma > 1$.

Using the Bunyakovsky-Schwarz inequality for integrals, and the fact that h = 0 on (0, 1), one sees that the integral (10) now converges absolutely and uniformly in every half-plane $\sigma \ge 1/2 + \varepsilon$ (with $\varepsilon > 0$), thus defining a holomorphic function in the half-plane $\sigma > 1/2$. It is the analytic continuation of F(s)/s since one has, for $\sigma > 3/2$,

$$\int_0^\infty h(t)t^{-s-1}dt = \frac{1}{s} \sum_{n \ge 1} h(n) \left(n^{-s} - (n+1)^{-s} \right)$$
$$= \frac{1}{s} \sum_{n \ge 1} \frac{h(n) - h(n-1)}{n^s} = \frac{F(s)}{s}.$$

Finally, the convergence of the Dirichlet series F(s) on the line $\sigma = 1$ follows from the convergence at s = 1 and the holomorphy of F on the line, by a theorem of Marcel Riesz (cf. [5], Satz I, p. 350).

One combines the two cases, h = 0 and $\lambda = 0$, to obtain the statement of the proposition.

The Dirichlet series F(s) of functions in D_0 are precisely those which converge in some halfplane and have an analytic continuation to $\sigma > 1/2$ such that F(s)/s belongs to the Hardy space H^2 of this last half-plane. As we will not use this fact in the present paper, we omit its proof.

We now investigate the Dirichlet series

$$\frac{F(s)}{\zeta(s)} = \sum_{n \ge 1} \frac{w(n; f)}{n^s}.$$

Proposition 9 Let $f \in D$, and let F(s) be the Dirichlet series of f. The Dirichlet series $F(s)/\zeta(s)$ is absolutely convergent if $\sigma > 3/2$, and convergent if $\sigma \ge 1$.

Proof

The Dirichlet series F(s) converges for $\sigma > 1$, and converges absolutely for $\sigma > 3/2$ (Proposition 8). The Dirichlet series $1/\zeta(s)$ converges absolutely for $\sigma > 1$. The Dirichlet product $F(s)/\zeta(s)$ thus converges absolutely for $\sigma > 3/2$, and converges for $\sigma > 1$.

If s = 1, the series is convergent by Proposition 7. Since the function $F(s)/\zeta(s)$ is holomorphic in the closed half-plane $\sigma \geq 1$, Riesz' convergence theorem applies again to ensure convergence on the line $\sigma = 1$. \square

Báez-Duarte's criterion for the Riemann hypothesis 4

We now define

$$\mathscr{B} = \operatorname{Vect}(e_n, n \in \mathbb{N}^*)$$
; $\mathscr{B}_0 = \operatorname{Vect}(e_n - e_1/n, n \in \mathbb{N}^*, n \ge 2).$

Since $e_n \in D$ and $e_n - e_1/n \in D_0$ for all $n \in \mathbb{N}^*$, one sees that

$$\overline{\mathscr{B}} \subset D \quad ; \quad \overline{\mathscr{B}_0} \subset D_0 \quad ; \quad \overline{\mathscr{B}_0} = \overline{\mathscr{B}} \cap D_0.$$

The subspace $\overline{\mathscr{B}}$ is the (non orthogonal) direct sum of Vect(e_1) and $\overline{\mathscr{B}}_0$.

We will consider the orthogonal projection $\tilde{\chi}$ (resp. $\tilde{\chi}_0$) of χ on $\overline{\mathscr{B}}$ (resp. $\overline{\mathscr{B}}_0$). In 2003, Báez-Duarte gave the following criterion for the Riemann hypothesis.

Proposition 10 The following seven assertions are equivalent.

(i) $\overline{\mathscr{B}} = D$; (i)₀ $\overline{\mathscr{B}}_0 = D_0$ (*ii*) $\boldsymbol{\chi} \in \overline{\mathscr{B}}$; (*ii*)₀ $\boldsymbol{\chi} \in \overline{\mathscr{B}}_0$ (*iii*) $\widetilde{\chi} = \chi$; (*iii*)₀ $\widetilde{\chi}_0 = \chi$ (iv) the Riemann hypothesis is true.

In fact, Báez-Duarte's paper [2] contains the proof of the equivalence of (ii) and (iv); the other equivalences are mere variations. The statements $(i)_0$, $(ii)_0$ and $(iii)_0$ allow one to work in the sequence space \mathfrak{h} instead of the function space H; see [3] for an exposition in this setting.

The main property of Dirichlet series of elements of \mathcal{B} is given in the following proposition.

Proposition 11 If $f \in \overline{\mathcal{B}}$, the Dirichlet series $F(s)/\zeta(s)$ has a holomorphic continuation to the halfplane $\sigma > 1/2$.

Proof

Write $f = \lambda e_1 + h$, with $\lambda \in \mathbb{R}$ and $h \in D_0$. If h = 0, one has $F = -\lambda \zeta$ and the result is true. Now suppose $\lambda = 0$. The function h is the limit in H of finite linear combinations, say h_i $(j \ge 1)$, of the $e_k - e_1/k$ ($k \ge 2$), when $j \to \infty$. The Dirichlet series of $e_k - e_1/k$ is

$$(k^{-1}-k^{-s})\zeta(s)$$

so that the result is true for each h_i . It remains to see what happens when one passes to the limit.

By the relation between the Dirichlet series of h_j and the Mellin transform of h_j , one sees that the Mellin transform of h_j must vanish at each zero ρ of ζ in the half-plane $\sigma > 1/2$, with a multiplicity no less than the corresponding multiplicity of ρ as a zero of ζ . Thus

$$\int_{1}^{\infty} h_j(t) t^{-\rho - 1} \ln^k t \, dt = 0 \tag{11}$$

for every zero ρ of the Riemann zeta function, such that $\Re \rho > 1/2$, and for every non-negative integer k smaller than the multiplicity of ρ as a zero of ζ . When $j \to \infty$, one gets (11) with h_j replaced by h, which proves the result for h.

One combines the two cases, h = 0 and $\lambda = 0$, to obtain the statement of the proposition.

5 The *v* function

5.1 Weingartner's form of Báez-Duarte's criterion

For $N \in \mathbb{N}^*$, we will consider the orthogonal projections of χ on the subspaces $\operatorname{Vect}(e_1, \ldots, e_N)$ and $\operatorname{Vect}(e_2 - e_1/2, \ldots, e_N - e_1/N)$:

$$\chi_N = \sum_{k=1}^N c(k,N)e_k \tag{12}$$

$$\chi_{0,N} = \sum_{k=2}^{N} c_0(k,N)(e_k - e_1/k), \qquad (13)$$

thus defining the coefficients c(k,N) and $c_0(k,N)$. In [8], Weingartner gave a formulation of Báez-Duarte's criterion in terms of the coefficients $c_0(k,N)$ of (13). The same can be done with the c(k,N) of (12). First, we state a basic property of these coefficients.

Proposition 12 For every $k \in \mathbb{N}^*$, the coefficients c(k,N) in (12) and $c_0(k,N)$ in (13) (here, with $k \ge 2$) converge when N tends to infinity.

Proof

With the notations of $\S4$,

$$\widetilde{\chi} = \lim_{N \to \infty} \chi_N$$

 $\widetilde{\chi}_0 = \lim_{N \to \infty} \chi_{0,N},$

where the limits are taken in H.

Using the identity (6), we observe that, for every $N \in \mathbb{N}^*$,

$$c(k,N) = -w(k; \chi_N)$$
 $(k \ge 1)$
 $c_0(k,N) = -w(k; \chi_{0,N})$ $(k \ge 2),$

Therefore, Proposition 4 yields, for every k,

$$\begin{aligned} c(k,N) &\to -w(k;\widetilde{\boldsymbol{\chi}}) & (N \to \infty) \\ c_0(k,N) &\to -w(k;\widetilde{\boldsymbol{\chi}_0}) & (N \to \infty). \end{aligned}$$

Definition 1 The arithmetical functions v and v_0 are defined by

$$\mathbf{v}(n) = w(n; \widetilde{\boldsymbol{\chi}})$$
$$\mathbf{v}_0(n) = w(n; \widetilde{\boldsymbol{\chi}}_0).$$

Note that

$$v_0(1) = \lim_{N \to \infty} \sum_{2 \le k \le N} \frac{c_0(k,N)}{k} = -\sum_{k \ge 2} \frac{v_0(k)}{k},$$

by Proposition 7.

We can now state Báez-Duarte's criterion in Weingartner's formulation.

Proposition 13 The following assertions are equivalent.

(i)
$$\mathbf{v} = \boldsymbol{\mu}$$

(ii) $\mathbf{v}_0 = \boldsymbol{\mu} \text{ on } \mathbb{N}^* \setminus \{1\}$
(iii) the Riemann hypothesis is true.

Proof

By Báez-Duarte's criterion, (*iii*) is equivalent to $\chi = \tilde{\chi}$. By Proposition 1, this is equivalent to $w(n; \chi) = w(n; \tilde{\chi})$ for all $n \ge 1$, that is, $\mu = \nu$.

Similarly, (*iii*) implies $\mu = v_0$. Conversely, if $\mu(n) = v_0(n)$ for all $n \ge 2$, then $w(n; \chi - \tilde{\chi_0}) = 0$ for $n \ge 2$, which means that $\chi - \tilde{\chi_0}$ is a scalar multiple of e_1 . This implies $\chi = \tilde{\chi_0}$ since χ and $\tilde{\chi_0}$ belong to D_0 .

5.2 The Dirichlet series $\sum_{n} v(n) n^{-s}$

Since $v(n) = w(n; \tilde{\chi})$, the following proposition is a corollary of Propositions 9 and 11.

Proposition 14 The Dirichlet series

$$\sum_{n\geq 1}\frac{\nu(n)}{n^s}$$

is absolutely convergent for $\sigma > 3/2$, convergent for $\sigma \ge 1$, and has a holomorphic continuation to the half-plane $\sigma > 1/2$.

6 Questions

Here are three questions related to the preceding exposition.

Question 1 *Is it true that* $\tilde{\chi} = \tilde{\chi}_0$ *?*

Question 2 Let $f \in D$ such that the Dirichlet series $F(s)/\zeta(s)$ has a holomorphic continuation to the half-plane $\sigma > 1/2$. Is it true that $f \in \overline{\mathscr{B}}$?

A positive answer would be a discrete analogue of Bercovici's and Foias' Corollary 2.2, p. 63 of [4].

Question 3 Is the Dirichlet series

$$\sum_{n\geq 1}\frac{\nu(n)}{n^s}$$

convergent in the half-plane $\sigma > 1/2$?

Another open problem is to obtain any quantitative estimate beyond the tautologies $\|\tilde{\chi} - \tilde{\chi}_N\| = o(1)$ and $\|\tilde{\chi}_0 - \tilde{\chi}_{0,N}\| = o(1)$ $(N \to \infty)$.

Appendix : some computations

Scalar products

1. One has

$$\langle e_1, \varepsilon_k \rangle = \sqrt{k(k+1)} \int_k^{k+1} (t-k) \frac{dt}{t^2} = \sqrt{k(k+1)} \left(\ln(1+1/k) - 1/(k+1) \right).$$
 (14)

2. For $k \in \mathbb{N}^*$, one has

$$\begin{split} \langle e_1, \varphi_k \rangle &= \int_{k-1}^k k(k-1)(t-k+1) \frac{dt}{t^2} - \int_k^{k+1} k(k+1)(t-k) \frac{dt}{t^2} \\ &= 2k^2 \ln k - k(k-1) \ln(k-1) - k(k+1) \ln(k+1) + 1 \\ &= -\omega(1/k), \end{split}$$

where

$$\begin{split} \omega(z) &= z^{-2} \left((1-z) \ln(1-z) + (1+z) \ln(1+z) \right) - 1 \\ &= \sum_{j \ge 1} \frac{z^{2j}}{(j+1)(2j+1)} \quad (|z| \le 1). \end{split}$$

3. For $n \in \mathbb{N}^*$, one has

$$\begin{split} \langle e_1, f_n \rangle &= \sum_{k|n} \mu(n/k) \langle e_1, \varphi_k \rangle = -\sum_{k|n} \mu(n/k) \omega(1/k) \\ &= -\sum_{j \ge 1} \frac{\sum_{k|n} \mu(n/k) k^{-2j}}{(j+1)(2j+1)} = -\sum_{j \ge 1} \frac{n^{-2j} \prod_{p|n} (1-p^{2j})}{(j+1)(2j+1)} \cdot \end{split}$$

In particular,

$$\sup_{n \in \mathbb{N}^*} |\langle e_1, f_n \rangle| = \sum_{j \ge 1} \frac{1}{(j+1)(2j+1)} = \ln 4 - 1.$$

Projections

By (14), the orthogonal projection e'_1 of e_1 on D_0 is

$$e_1' = \sum_{k\geq 1} \langle e_1, \varepsilon_k \rangle \varepsilon_k = \sum_{k\geq 1} \sqrt{k(k+1)} \big(\ln(1+1/k) - 1/(k+1) \big) \varepsilon_k.$$

Since $e'_1(k)$ has limit 1/2 when k tends to infinity, one sees that $e_1 - e'_1$ "interpolates" between the fractional part (on [0,1[) and the first Bernoulli function (at infinity). One has the hilbertian decomposition

$$D = D_0 \oplus \operatorname{Vect}(e_1 - e_1').$$

Since $\kappa \perp D_0$ and $\langle \kappa, e_1 \rangle = 1$, the orthogonal projection of κ on *D* is

$$\kappa' = rac{e_1 - e_1'}{\|e_1 - e_1'\|^2} \cdot$$

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