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An arithmetical function related to Báez-Duarte's criterion for the Riemann hypothesis

Michel Balazard

To the memory of my friend, Luis Báez-Duarte.

Abstract

In this mainly expository article, we revisit some formal aspects of Báez-Duarte's criterion for the Riemann hypothesis. In particular, starting from Weingartner's formulation of the criterion, we define an arithmetical function ν , which is equal to the Möbius function if, and only if the Riemann hypothesis is true. We record the basic properties of the Dirichlet series of ν , and state a few questions.

KEYWORDS

Riemann hypothesis, arithmetical functions, Dirichlet series, Hilbert space
MSC classification : 11M26

1 The spaces D and D_0

We will denote by \mathbb{N} (resp. \mathbb{N}^*) the set of non-negative (resp. positive) integers, by H the Hilbert space $L^2(0, \infty; t^{-2} dt)$, with inner product

$$\langle f, g \rangle = \int_0^\infty f(t) \overline{g(t)} \frac{dt}{t^2},$$

and by $\text{Vect}(\mathcal{F})$ the set of finite linear combinations of elements of a family \mathcal{F} of elements of H .

For $k \in \mathbb{N}^*$, we define

$$e_k(t) = \{t/k\} \quad (t > 0),$$

where $\{u\} = u - [u]$ denotes the fractional part of the real number u , and $[u]$ its integer part. The functions e_k belong to H , as do the functions χ and κ defined by

$$\chi(t) = [t \geq 1] \quad ; \quad \kappa(t) = t[0 < t < 1]$$

(here, and in the following, we use Iverson's notation : $[P] = 1$ if the assertion P is true, $[P] = 0$ if it is false).

Let D be the closed subspace of functions $f \in H$ of the type

$$f(t) = \lambda t + \varphi(t), \tag{1}$$

where φ is constant on each interval $[j, j+1[$, $j \in \mathbb{N}$ (for $j = 0$, the constant must be 0). The functions e_k belong to D .

Let D_0 be the subspace of D defined by taking $\lambda = 0$ in (1), that is, the subspace of functions $\varphi \in H$ which are constant on each interval $[j, j+1[$, $j \in \mathbb{N}$. The functions χ and $e_k - e_1/k$ belong to D_0 .

A hilbertian basis for D_0 is given by the family of step functions ε_k defined by

$$\varepsilon_k(t) = \sqrt{k(k+1)} \cdot [k \leq t < k+1] \quad (k \in \mathbb{N}^*, t > 0).$$

The mapping $h \mapsto (h(j))_{j \geq 1}$ is a Hilbert space isomorphism of D_0 onto the sequence space \mathfrak{h} of complex sequences $(x_j)_{j \geq 1}$ such that

$$\sum_{j \geq 1} \frac{|x_j|^2}{j(j+1)} < \infty.$$

Observe that, for $f \in D$, written as (1), one has

$$\lambda = \langle f, \kappa \rangle \tag{2}$$

$$f = \lambda e_1 + h, \text{ where } h \in D_0. \tag{3}$$

Thus, the subspace D is the (non orthogonal) direct sum of $\text{Vect}(e_1)$ and D_0 .

In formula (2), the function κ could be replaced by its orthogonal projection κ' on D . The definition of the families (ψ_n) of Proposition 2 and (g_n) of Proposition 4 below could be modified accordingly. We compute κ' in the appendix.

To every function in D , one can associate certain arithmetical functions. Let $f \in D$, with λ and h as in (2), (3). We first define the arithmetical function

$$u(n) = u(n; f) = -\lambda + h(n) - h(n-1) \quad (n \in \mathbb{N}^*). \tag{4}$$

With this definition, we see that the function φ of (1) is given by

$$\varphi(t) = -\lambda t + f(t) = -\lambda t + \lambda \{t\} + h(t) = \sum_{n \leq t} u(n).$$

Thus, $f(t)$ is the remainder term in the approximation of the sum function $\varphi(t)$ of the arithmetical function u by the linear function $-\lambda t$. The fact that f belongs to H implies, and is stronger than, the asymptotic relation $f(t) = o(t)$.

For $f \in D$, we will also consider the arithmetical function $w = \mu * u$, where μ denotes the Möbius function,

$$w(n) = w(n; f) = \sum_{d|n} \mu(n/d) u(d; f) \quad (n \in \mathbb{N}^*).$$

For instance,

$$u(n; \chi) = [n = 1] \quad ; \quad w(n; \chi) = \mu(n) \quad (n \in \mathbb{N}^*).$$

The arithmetical functions u and w depend linearly on f and the correspondences are one-to-one.

Proposition 1 For $f \in D$,

$$f = 0 \Leftrightarrow u = 0 \Leftrightarrow w = 0.$$

Proof

The second equivalence follows from $w = u * \mu$ and $u = w * 1$ (Möbius inversion). It remains to prove that $u = 0 \Rightarrow f = 0$. By (4), $u = 0$ implies $h(n) = \lambda n$ for all n , hence $\lambda = 0$ since $h \in D_0$, and $h = 0$. \square

Since $u = w * 1$, one has

$$f(t) = \lambda t + \sum_{n \leq t} u(n) = \lambda t + \sum_{n \geq 1} w(n) \lfloor t/n \rfloor.$$

In Proposition 7 below, we will prove the identity

$$\sum_{n \geq 1} \frac{w(n)}{n} = -\lambda, \quad (5)$$

so that, for every f in D and every $t > 0$, one has

$$f(t) = - \sum_{n \geq 1} w(n) e_n(t). \quad (6)$$

Of course, it does not mean that the series $\sum_{n \geq 1} w(n; f) e_n$ converges in H (in fact, it diverges if $f = \chi$, cf. [1], Theorem 2.2, p. 6), but, if it does, its sum is $-f$.

2 Vasyunin's biorthogonal system

In Theorem 7 of his paper [7], Vasyunin defined a family $(f_k)_{k \geq 2}$, which, together with the family $(e_k - e_1/k)_{k \geq 2}$, yields a biorthogonal system in D_0 , which means that

$$\langle e_j - e_1/j, f_k \rangle = [j = k] \quad (j \geq 2, k \geq 2). \quad (7)$$

We will recall Vasyunin's construction, which can be thought of as a Hilbert space formulation of Möbius inversion, and add several comments.

2.1 The sequence (φ_k)

First one defines, for $k \in \mathbb{N}^*$, a step function $\varphi_k \in D_0$ by

$$\varphi_k(t) = k(k-1)[k-1 \leq t < k] - k(k+1)[k \leq t < k+1]$$

(Vasyunin's φ_k have the opposite sign, according to his definition for e_k). Thus

$$\varphi_k = \sqrt{k(k-1)} \cdot \varepsilon_{k-1} - \sqrt{k(k+1)} \cdot \varepsilon_k \quad (k \in \mathbb{N}^*),$$

with $\varepsilon_0 = 0$ by convention. One sees that the family $(\varphi_k)_{k \geq 1}$ is total in D_0 .

One checks that

$$\langle h, \varphi_k \rangle = h(k-1) - h(k) \quad (k \in \mathbb{N}^*), \quad (8)$$

for $h \in D_0$ with constant value $h(k)$ on $[k, k+1[$ ($h(0) = 0$). In particular,

$$\langle e_j - e_1/j, \varphi_k \rangle = [j | k] - 1/j \quad (j \geq 1, k \geq 1).$$

Using the family (φ_k) , one can write the values $u(n; f)$, for $f \in D$, as scalar products.

Proposition 2 For $f \in D$, with λ and h as in (2), (3), one has

$$u(n; f) = \langle f, \psi_n \rangle,$$

where

$$\psi_n = (\langle e_1, \varphi_n \rangle - 1)\kappa - \varphi_n \quad (n \in \mathbb{N}^*).$$

In particular, $f \mapsto u(n; f)$ is a continuous linear form on D , for every $n \in \mathbb{N}^*$.

Proof

By (2), (4) and (8), one has

$$\begin{aligned} u(n; f) &= -\langle f, \kappa \rangle - \langle h, \varphi_n \rangle \\ &= -\langle f, \kappa \rangle - \langle f - \langle f, \kappa \rangle e_1, \varphi_n \rangle \\ &= -\langle f, \kappa \rangle - \langle f, \varphi_n \rangle + \langle e_1, \varphi_n \rangle \langle f, \kappa \rangle \\ &= \langle f, \psi_n \rangle \quad (n \in \mathbb{N}^*). \end{aligned} \quad \square$$

We compute the scalar product $\langle e_1, \varphi_n \rangle$ in the appendix.

The next proposition describes the behavior of the series $\sum_k \varphi_k/k$.

Proposition 3 The series

$$\sum_{k \geq 1} \frac{\varphi_k}{k}$$

is weakly convergent in D_0 , with weak sum $-\chi$.

Proof

The partial sum

$$\sum_{k \leq K} \frac{\varphi_k}{k}$$

is the step function with values

$$\begin{aligned} &0 \text{ on } (0, 1) \text{ and } (K+1, \infty) \\ &-1 \text{ on } (1, K) \\ &-(K+1) \text{ on } (K, K+1) \end{aligned}$$

This partial sum is thus equal to $-\chi$ on every fixed bounded segment of $(0, \infty)$, if K is large enough, and the norm of this partial sum in H is the constant $\sqrt{2}$. The result follows. \square

2.2 The sequence (f_k)

Vasyunin defined

$$f_k = \sum_{d|k} \mu(k/d) \varphi_d \quad (k \in \mathbb{N}^*).$$

Equivalently,

$$\varphi_k = \sum_{d|k} f_d \quad (k \in \mathbb{N}^*),$$

by Möbius inversion ; this implies that the family $(f_k)_{k \geq 1}$ is also total in D_0 .

A slightly more general form of (7), namely

$$\langle e_j - e_1/j, f_k \rangle = [j = k] - [k = 1]/j \quad (j, k \in \mathbb{N}^*), \quad (9)$$

is proved by means of the identity

$$\sum_{j|d|k} \mu(k/d) = [j = k].$$

Using the family (f_k) , one can write the values $w(n; f)$, for $f \in D$, as scalar products.

Proposition 4 For $f \in D$, with λ and h as in (2), (3), one has

$$w(n; f) = \langle f, g_n \rangle,$$

where

$$g_n = (\langle e_1, f_n \rangle - [n = 1])\kappa - f_n \quad (n \in \mathbb{N}^*).$$

In particular, $f \mapsto w(n; f)$ is a continuous linear form on D , for every $n \in \mathbb{N}^*$.

Proof

By Proposition 2, one has

$$\begin{aligned} w(n; f) &= \sum_{d|n} \mu(n/d) u(d; f) \\ &= \langle f, \sum_{d|n} \mu(n/d) \psi_d \rangle \quad (n \in \mathbb{N}^*). \end{aligned}$$

Now,

$$\begin{aligned} \sum_{d|n} \mu(n/d) \psi_d &= \sum_{d|n} \mu(n/d) ((\langle e_1, \varphi_d \rangle - 1)\kappa - \varphi_d) \\ &= (\langle e_1, f_n \rangle - [n = 1])\kappa - f_n. \end{aligned} \quad \square$$

We compute the scalar product $\langle e_1, f_n \rangle$ in the appendix.

In order to study the series $\sum_k f_k/k$, we will need the following auxiliary proposition.

Proposition 5 Let

$$f(x) = \sum_{k \leq x} \eta(k) \quad (x > 0),$$

where η is a complex arithmetical function such that $\eta(k) = O(1/k)$, for $k \geq 1$.

Then, for every fixed $\alpha > 1$,

$$\sum_{k \geq 1} |f(x/k) - f(x/(k+1))|^\alpha = O(1) \quad (x > 0).$$

Proof

The series is in fact a finite sum, as

$$f(x/k) = f(x/(k+1)) = 0 \quad (k > x).$$

We will use the estimate

$$f(y) - f(x) \ll \sum_{x < k \leq y} \frac{1}{k} \ll \frac{1}{x} + \ln(y/x) \quad (y > x \geq 1).$$

Thus,

$$f(x/k) - f(x/(k+1)) \ll \frac{k}{x} + \frac{1}{k} \ll \frac{1}{k} \quad (k \leq \sqrt{x}),$$

and

$$\sum_{k \leq \sqrt{x}} |f(x/k) - f(x/(k+1))|^\alpha \ll \sum_{k \geq 1} \frac{1}{k^\alpha} \ll 1 \quad (x > 0).$$

If $k > \sqrt{x}$, then

$$\frac{x}{k} - \frac{x}{k+1} < 1,$$

so that the interval $]x/(k+1), x/k]$ contains at most one integer, say n , and, if n exists, one has $k = \lfloor x/n \rfloor$ and

$$f(x/k) - f(x/(k+1)) = \eta(n) \ll \frac{1}{n}.$$

Hence

$$\sum_{k > \sqrt{x}} |f(x/k) - f(x/(k+1))|^\alpha \ll \sum_{n \geq 1} \frac{1}{n^\alpha} \ll 1 \quad (x > 0).$$

The result follows. □

Proposition 6 *The series*

$$\sum_{k \geq 1} \frac{f_k}{k}$$

is weakly convergent in D_0 (hence in H), with weak sum 0.

Proof

Let $K \in \mathbb{N}^*$. One has

$$S_K = \sum_{k \leq K} \frac{f_k}{k} = \sum_{d \leq K} \frac{m(K/d)}{d} \varphi_d,$$

where

$$m(x) = \sum_{n \leq x} \frac{\mu(n)}{n} \quad (x > 0).$$

Hence,

$$\begin{aligned} S_K &= \sum_{d \leq K} \frac{m(K/d)}{d} (\sqrt{d(d-1)} \cdot \varepsilon_{d-1} - \sqrt{d(d+1)} \cdot \varepsilon_d) \\ &= \sum_{d \leq K-1} \left(\frac{m(K/(d+1))}{d+1} - \frac{m(K/d)}{d} \right) \sqrt{d(d+1)} \cdot \varepsilon_d - \sqrt{1+1/K} \cdot \varepsilon_K \end{aligned}$$

For every fixed $d \in \mathbb{N}^*$, the fact that $\langle S_K, \varepsilon_d \rangle$ tends to 0 when K tends to infinity follows from this formula and the classical result of von Mangoldt, that $m(x)$ tends to 0 when x tends to infinity.

It remains to show that $\|S_K\|$ is bounded. One has

$$\begin{aligned} \|S_K\|^2 &= \sum_{d \leq K-1} d(d+1) \left(\frac{m(K/d)}{d} - \frac{m(K/(d+1))}{d+1} \right)^2 + 1 + 1/K \\ &\leq 2 \sum_{d \leq K-1} d(d+1) \left(\frac{m(K/d) - m(K/(d+1))}{d} \right)^2 \\ &\quad + 2 \sum_{d \leq K-1} d(d+1) \left(\frac{m(K/(d+1))}{d(d+1)} \right)^2 + 1 + 1/K \\ &\ll 1 + \sum_{d \leq K-1} \left(m(K/d) - m(K/(d+1)) \right)^2 \end{aligned}$$

The boundedness of $\|S_K\|$ then follows from Proposition 5. □

We are now able to prove (5).

Proposition 7 *Let $f \in D$, with λ and h as in (2), (3). The series*

$$\sum_{n \geq 1} \frac{w(n; f)}{n}$$

is convergent and has sum $-\lambda$.

Proof

Putting $\beta_N = \sum_{n \leq N} f_n/n$ for $N \in \mathbb{N}^*$, one has

$$\begin{aligned} \sum_{n \leq N} \frac{g_n}{n} &= \sum_{n \leq N} \frac{(\langle e_1, f_n \rangle - [n=1])\kappa - f_n}{n} \\ &= (\langle e_1, \beta_N \rangle - 1)\kappa - \beta_N, \end{aligned}$$

which tends weakly to $-\kappa$, as N tends to infinity, by Proposition 6.

Hence,

$$\sum_{n \leq N} \frac{w(n; f)}{n} = \sum_{n \leq N} \frac{\langle f, g_n \rangle}{n} = \langle f, \sum_{n \leq N} g_n/n \rangle \rightarrow -\langle f, \kappa \rangle = -\lambda \quad (N \rightarrow \infty). \quad \square$$

3 Dirichlet series

For $f \in D$ we define

$$F(s) = \sum_{n \geq 1} \frac{u(n; f)}{n^s},$$

and we will say that F is the Dirichlet series of f .

We will denote by σ the real part of the complex variable s . The following proposition summarizes the basic facts about the correspondance between elements f of D and their Dirichlet series F . We keep the notations of (2) and (3).

Proposition 8 For $f \in D$, the Dirichlet series $F(s)$ is absolutely convergent in the half-plane $\sigma > 3/2$, and convergent in the half-plane $\sigma > 1$. It has a meromorphic continuation to the half-plane $\sigma > 1/2$ (we will denote it also by $F(s)$), with a unique pole in $s = 1$, simple and with residue $-\lambda$. In the strip $1/2 < \sigma < 1$, one has

$$F(s)/s = \int_0^\infty f(t)t^{-s-1}dt. \quad (10)$$

If $f \in D_0$, that is $\lambda = 0$, there is no pole at $s = 1$, and the Mellin transform (10) represents the analytic continuation of $F(s)/s$ to the half-plane $\sigma > 1/2$. Moreover, the Dirichlet series $F(s)$ converges on the line $\sigma = 1$.

Proof

If $h = 0$ in (3), the arithmetical function u is the constant $-\lambda$, and $F = -\lambda\zeta$. In this case, the assertion about (10) follows from (2.1.5), p. 14 of [6].

If $\lambda = 0$, then $f = h \in D_0$ and $u(n) = h(n) - h(n-1)$ by (4). Therefore,

$$\begin{aligned} \sum_{n \geq 1} \frac{|u(n)|}{n^\sigma} &\leq 2 \sum_{n \geq 1} \frac{|h(n)|}{n^\sigma} \\ &\leq 2 \left(\sum_{n \geq 1} \frac{|h(n)|^2}{n^2} \right)^{1/2} \left(\sum_{n \geq 1} \frac{1}{n^{2\sigma-2}} \right)^{1/2} \\ &\leq 2\zeta(2\sigma-2)^{1/2} \|h\| < \infty, \end{aligned}$$

if $\sigma > 3/2$, where we used Cauchy's inequality for sums.

The convergence of the series $F(1)$ follows from the formula $u(n) = -\langle h, \varphi_n \rangle$ and Proposition 3. It implies the convergence of $F(s)$ in the half-plane $\sigma > 1$.

Using the Bunyakovsky-Schwarz inequality for integrals, and the fact that $h = 0$ on $(0, 1)$, one sees that the integral (10) now converges absolutely and uniformly in every half-plane $\sigma \geq 1/2 + \varepsilon$ (with $\varepsilon > 0$), thus defining a holomorphic function in the half-plane $\sigma > 1/2$. It is the analytic continuation of $F(s)/s$ since one has, for $\sigma > 3/2$,

$$\begin{aligned} \int_0^\infty h(t)t^{-s-1}dt &= \frac{1}{s} \sum_{n \geq 1} h(n)(n^{-s} - (n+1)^{-s}) \\ &= \frac{1}{s} \sum_{n \geq 1} \frac{h(n) - h(n-1)}{n^s} = \frac{F(s)}{s}. \end{aligned}$$

Finally, the convergence of the Dirichlet series $F(s)$ on the line $\sigma = 1$ follows from the convergence at $s = 1$ and the holomorphy of F on the line, by a theorem of Marcel Riesz (cf. [5], Satz I, p. 350).

One combines the two cases, $h = 0$ and $\lambda = 0$, to obtain the statement of the proposition. \square

The Dirichlet series $F(s)$ of functions in D_0 are precisely those which converge in some half-plane and have an analytic continuation to $\sigma > 1/2$ such that $F(s)/s$ belongs to the Hardy space H^2 of this last half-plane. As we will not use this fact in the present paper, we omit its proof.

We now investigate the Dirichlet series

$$\frac{F(s)}{\zeta(s)} = \sum_{n \geq 1} \frac{w(n; f)}{n^s}.$$

Proposition 9 *Let $f \in D$, and let $F(s)$ be the Dirichlet series of f . The Dirichlet series $F(s)/\zeta(s)$ is absolutely convergent if $\sigma > 3/2$, and convergent if $\sigma \geq 1$.*

Proof

The Dirichlet series $F(s)$ converges for $\sigma > 1$, and converges absolutely for $\sigma > 3/2$ (Proposition 8). The Dirichlet series $1/\zeta(s)$ converges absolutely for $\sigma > 1$. The Dirichlet product $F(s)/\zeta(s)$ thus converges absolutely for $\sigma > 3/2$, and converges for $\sigma > 1$.

If $s = 1$, the series is convergent by Proposition 7. Since the function $F(s)/\zeta(s)$ is holomorphic in the closed half-plane $\sigma \geq 1$, Riesz' convergence theorem applies again to ensure convergence on the line $\sigma = 1$. \square

4 Báez-Duarte's criterion for the Riemann hypothesis

We now define

$$\mathcal{B} = \text{Vect}(e_n, n \in \mathbb{N}^*) \quad ; \quad \mathcal{B}_0 = \text{Vect}(e_n - e_1/n, n \in \mathbb{N}^*, n \geq 2).$$

Since $e_n \in D$ and $e_n - e_1/n \in D_0$ for all $n \in \mathbb{N}^*$, one sees that

$$\overline{\mathcal{B}} \subset D \quad ; \quad \overline{\mathcal{B}_0} \subset D_0 \quad ; \quad \overline{\mathcal{B}_0} = \overline{\mathcal{B}} \cap D_0.$$

The subspace $\overline{\mathcal{B}}$ is the (non orthogonal) direct sum of $\text{Vect}(e_1)$ and $\overline{\mathcal{B}_0}$.

We will consider the orthogonal projection $\tilde{\chi}$ (resp. $\tilde{\chi}_0$) of χ on $\overline{\mathcal{B}}$ (resp. $\overline{\mathcal{B}_0}$). In 2003, Báez-Duarte gave the following criterion for the Riemann hypothesis.

Proposition 10 *The following seven assertions are equivalent.*

- (i) $\overline{\mathcal{B}} = D \quad ; \quad (i)_0 \quad \overline{\mathcal{B}_0} = D_0$
- (ii) $\chi \in \overline{\mathcal{B}} \quad ; \quad (ii)_0 \quad \chi \in \overline{\mathcal{B}_0}$
- (iii) $\tilde{\chi} = \chi \quad ; \quad (iii)_0 \quad \tilde{\chi}_0 = \chi$
- (iv) *the Riemann hypothesis is true.*

In fact, Báez-Duarte's paper [2] contains the proof of the equivalence of (ii) and (iv) ; the other equivalences are mere variations. The statements (i)₀, (ii)₀ and (iii)₀ allow one to work in the sequence space \mathfrak{h} instead of the function space H ; see [3] for an exposition in this setting.

The main property of Dirichlet series of elements of $\overline{\mathcal{B}}$ is given in the following proposition.

Proposition 11 *If $f \in \overline{\mathcal{B}}$, the Dirichlet series $F(s)/\zeta(s)$ has a holomorphic continuation to the half-plane $\sigma > 1/2$.*

Proof

Write $f = \lambda e_1 + h$, with $\lambda \in \mathbb{R}$ and $h \in D_0$. If $h = 0$, one has $F = -\lambda \zeta$ and the result is true.

Now suppose $\lambda = 0$. The function h is the limit in H of finite linear combinations, say h_j ($j \geq 1$), of the $e_k - e_1/k$ ($k \geq 2$), when $j \rightarrow \infty$. The Dirichlet series of $e_k - e_1/k$ is

$$(k^{-1} - k^{-s})\zeta(s),$$

so that the result is true for each h_j . It remains to see what happens when one passes to the limit.

By the relation between the Dirichlet series of h_j and the Mellin transform of h_j , one sees that the Mellin transform of h_j must vanish at each zero ρ of ζ in the half-plane $\sigma > 1/2$, with a multiplicity no less than the corresponding multiplicity of ρ as a zero of ζ . Thus

$$\int_1^\infty h_j(t) t^{-\rho-1} \ln^k t dt = 0 \quad (11)$$

for every zero ρ of the Riemann zeta function, such that $\Re \rho > 1/2$, and for every non-negative integer k smaller than the multiplicity of ρ as a zero of ζ . When $j \rightarrow \infty$, one gets (11) with h_j replaced by h , which proves the result for h .

One combines the two cases, $h = 0$ and $\lambda = 0$, to obtain the statement of the proposition. \square

5 The ν function

5.1 Weingartner's form of Báez-Duarte's criterion

For $N \in \mathbb{N}^*$, we will consider the orthogonal projections of χ on the subspaces $\text{Vect}(e_1, \dots, e_N)$ and $\text{Vect}(e_2 - e_1/2, \dots, e_N - e_1/N)$:

$$\chi_N = \sum_{k=1}^N c(k, N) e_k \quad (12)$$

$$\chi_{0,N} = \sum_{k=2}^N c_0(k, N) (e_k - e_1/k), \quad (13)$$

thus defining the coefficients $c(k, N)$ and $c_0(k, N)$. In [8], Weingartner gave a formulation of Báez-Duarte's criterion in terms of the coefficients $c_0(k, N)$ of (13). The same can be done with the $c(k, N)$ of (12). First, we state a basic property of these coefficients.

Proposition 12 *For every $k \in \mathbb{N}^*$, the coefficients $c(k, N)$ in (12) and $c_0(k, N)$ in (13) (here, with $k \geq 2$) converge when N tends to infinity.*

Proof

With the notations of §4,

$$\begin{aligned} \tilde{\chi} &= \lim_{N \rightarrow \infty} \chi_N \\ \tilde{\chi}_0 &= \lim_{N \rightarrow \infty} \chi_{0,N}, \end{aligned}$$

where the limits are taken in H .

Using the identity (6), we observe that, for every $N \in \mathbb{N}^*$,

$$\begin{aligned} c(k, N) &= -w(k; \chi_N) \quad (k \geq 1) \\ c_0(k, N) &= -w(k; \chi_{0,N}) \quad (k \geq 2), \end{aligned}$$

Therefore, Proposition 4 yields, for every k ,

$$\begin{aligned} c(k, N) &\rightarrow -w(k; \tilde{\chi}) & (N \rightarrow \infty) \\ c_0(k, N) &\rightarrow -w(k; \tilde{\chi}_0) & (N \rightarrow \infty). \end{aligned} \quad \square$$

Definition 1 *The arithmetical functions v and v_0 are defined by*

$$\begin{aligned} v(n) &= w(n; \tilde{\chi}) \\ v_0(n) &= w(n; \tilde{\chi}_0). \end{aligned}$$

Note that

$$v_0(1) = \lim_{N \rightarrow \infty} \sum_{2 \leq k \leq N} \frac{c_0(k, N)}{k} = - \sum_{k \geq 2} \frac{v_0(k)}{k},$$

by Proposition 7.

We can now state Báez-Duarte's criterion in Weingartner's formulation.

Proposition 13 *The following assertions are equivalent.*

- (i) $v = \mu$
- (ii) $v_0 = \mu$ on $\mathbb{N}^* \setminus \{1\}$
- (iii) *the Riemann hypothesis is true.*

Proof

By Báez-Duarte's criterion, (iii) is equivalent to $\chi = \tilde{\chi}$. By Proposition 1, this is equivalent to $w(n; \chi) = w(n; \tilde{\chi})$ for all $n \geq 1$, that is, $\mu = v$.

Similarly, (iii) implies $\mu = v_0$. Conversely, if $\mu(n) = v_0(n)$ for all $n \geq 2$, then $w(n; \chi - \tilde{\chi}_0) = 0$ for $n \geq 2$, which means that $\chi - \tilde{\chi}_0$ is a scalar multiple of e_1 . This implies $\chi = \tilde{\chi}_0$ since χ and $\tilde{\chi}_0$ belong to D_0 . □ □

5.2 The Dirichlet series $\sum_n v(n)n^{-s}$

Since $v(n) = w(n; \tilde{\chi})$, the following proposition is a corollary of Propositions 9 and 11.

Proposition 14 *The Dirichlet series*

$$\sum_{n \geq 1} \frac{v(n)}{n^s}$$

is absolutely convergent for $\sigma > 3/2$, convergent for $\sigma \geq 1$, and has a holomorphic continuation to the half-plane $\sigma > 1/2$.

6 Questions

Here are three questions related to the preceding exposition.

Question 1 *Is it true that $\tilde{\chi} = \tilde{\chi}_0$?*

Question 2 *Let $f \in D$ such that the Dirichlet series $F(s)/\zeta(s)$ has a holomorphic continuation to the half-plane $\sigma > 1/2$. Is it true that $f \in \overline{\mathcal{B}}$?*

A positive answer would be a discrete analogue of Bercovici's and Foias' Corollary 2.2, p. 63 of [4].

Question 3 *Is the Dirichlet series*

$$\sum_{n \geq 1} \frac{v(n)}{n^s}$$

convergent in the half-plane $\sigma > 1/2$?

Another open problem is to obtain any quantitative estimate beyond the tautologies $\|\tilde{\chi} - \tilde{\chi}_N\| = o(1)$ and $\|\tilde{\chi}_0 - \tilde{\chi}_{0,N}\| = o(1)$ ($N \rightarrow \infty$).

Appendix : some computations

Scalar products

1. One has

$$\langle e_1, \varepsilon_k \rangle = \sqrt{k(k+1)} \int_k^{k+1} (t-k) \frac{dt}{t^2} = \sqrt{k(k+1)} (\ln(1+1/k) - 1/(k+1)). \quad (14)$$

2. For $k \in \mathbb{N}^*$, one has

$$\begin{aligned} \langle e_1, \varphi_k \rangle &= \int_{k-1}^k k(k-1)(t-k+1) \frac{dt}{t^2} - \int_k^{k+1} k(k+1)(t-k) \frac{dt}{t^2} \\ &= 2k^2 \ln k - k(k-1) \ln(k-1) - k(k+1) \ln(k+1) + 1 \\ &= -\omega(1/k), \end{aligned}$$

where

$$\begin{aligned} \omega(z) &= z^{-2} ((1-z) \ln(1-z) + (1+z) \ln(1+z)) - 1 \\ &= \sum_{j \geq 1} \frac{z^{2j}}{(j+1)(2j+1)} \quad (|z| \leq 1). \end{aligned}$$

3. For $n \in \mathbb{N}^*$, one has

$$\begin{aligned} \langle e_1, f_n \rangle &= \sum_{k|n} \mu(n/k) \langle e_1, \varphi_k \rangle = - \sum_{k|n} \mu(n/k) \omega(1/k) \\ &= - \sum_{j \geq 1} \frac{\sum_{k|n} \mu(n/k) k^{-2j}}{(j+1)(2j+1)} = - \sum_{j \geq 1} \frac{n^{-2j} \prod_{p|n} (1-p^{2j})}{(j+1)(2j+1)}. \end{aligned}$$

In particular,

$$\sup_{n \in \mathbb{N}^*} |\langle e_1, f_n \rangle| = \sum_{j \geq 1} \frac{1}{(j+1)(2j+1)} = \ln 4 - 1.$$

Projections

By (14), the orthogonal projection e'_1 of e_1 on D_0 is

$$e'_1 = \sum_{k \geq 1} \langle e_1, \varepsilon_k \rangle \varepsilon_k = \sum_{k \geq 1} \sqrt{k(k+1)} (\ln(1+1/k) - 1/(k+1)) \varepsilon_k.$$

Since $e'_1(k)$ has limit $1/2$ when k tends to infinity, one sees that $e_1 - e'_1$ "interpolates" between the fractional part (on $[0, 1[$) and the first Bernoulli function (at infinity). One has the hilbertian decomposition

$$D = D_0 \oplus \text{Vect}(e_1 - e'_1).$$

Since $\kappa \perp D_0$ and $\langle \kappa, e_1 \rangle = 1$, the orthogonal projection of κ on D is

$$\kappa' = \frac{e_1 - e'_1}{\|e_1 - e'_1\|^2}.$$

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