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FOURIER MULTIPLIERS IN  $SL_n(\mathbf{R})$ JAVIER PARCET, ÉRIC RICARD  
AND MIKAEL DE LA SALLE

ABSTRACT. We establish precise regularity conditions for  $L_p$ -boundedness of Fourier multipliers in the group algebra of  $SL_n(\mathbf{R})$ . Our main result is inspired by Hörmander-Mikhlin criterion from classical harmonic analysis, although it is substantially and necessarily different. Locally, we get sharp growth rates of Lie derivatives around the singularity and nearly optimal regularity order. The asymptotics also match Mikhlin formula for a exponentially growing metric with respect to the word length. Additional decay comes imposed by this growth and Mikhlin condition for high order terms. Lafforgue/de la Salle's rigidity theorem fits here. The proof includes a new relation between Fourier and Schur  $L_p$ -multipliers for nonamenable groups. In  $SL_n(\mathbf{R})$ , this holds in terms of Harish-Chandra's almost  $L_2$  matrix coefficients. By transference, matters are reduced to a rather nontrivial  $RC_p$ -inequality for  $SL_n(\mathbf{R})$ -twisted forms of Riesz transforms associated to fractional laplacians.

Our second result gives a new and much stronger rigidity theorem for radial multipliers in  $SL_n(\mathbf{R})$ . More precisely, additional regularity and Mikhlin type conditions are proved to be necessary up to an order  $\sim |\frac{1}{2} - \frac{1}{p}|(n-1)$  for large enough  $n$  in terms of  $p$ . Locally, necessary and sufficient growth rates match up to that order. Asymptotically, extra decay for the symbol and its derivatives imposes more accurate and additional rigidity in a wider range of  $L_p$ -spaces. This rigidity increases with the rank, so we can construct radial generating functions satisfying our Hörmander-Mikhlin sufficient conditions in a given rank  $n$  and failing the rigidity conditions for ranks  $m \gg n$ . We also prove automatic regularity and rigidity estimates for first and higher order derivatives of K-biinvariant multipliers in the rank 1 groups  $SO(n, 1)$ .

## Introduction

We study the relation between regularity and  $L_p$ -boundedness for multipliers in the group algebra of  $SL_n(\mathbf{R})$ . In Euclidean harmonic analysis, this central topic orbits around the Hörmander-Mikhlin fundamental condition [26, 44]. It defines a large class of Fourier multipliers —including Riesz transforms and Littlewood-Paley partitions of unity— which are crucial in Fourier summability or pseudodifferential operator theory. Given a measurable function  $m : \mathbf{R}^n \rightarrow \mathbf{C}$ , its Fourier multiplier is the linear map determined by

$$\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi).$$

Then,  $T_m$  is  $L_p$ -bounded on  $\mathbf{R}^n$  for  $1 < p < \infty$  whenever

$$(HM) \quad |\partial_\xi^\gamma m(\xi)| \lesssim |\xi|^{-|\gamma|} \quad \text{for all } 0 \leq |\gamma| \leq \left[\frac{n}{2}\right] + 1.$$

This condition imposes  $m$  to be a bounded function, smooth over  $\mathbf{R}^n \setminus \{0\}$  satisfying certain local and asymptotic behavior. Locally,  $m$  admits a singularity at 0 with a mild control of derivatives around it up to order  $[\frac{n}{2}] + 1$ . This singularity links to deep concepts in harmonic analysis and justifies the key role of Hörmander-Mikhlin theorem in Fourier multiplier  $L_p$ -theory. Asymptotically, the same derivatives decay faster and faster to 0, at a polynomial rate given by the differentiation order. It is optimal in the sense that we may not consider less classical derivatives —a Sobolev type formulation (recalled below in this paper) admits differentiability orders up to  $\frac{n}{2} + \varepsilon$ — or larger upper bounds for them.

The Hörmander-Mikhlin theorem has been investigated during the last decades for nilpotent groups by Christ, Cowling, Müller, Ricci or Stein among others. In the context of semisimple Lie groups, these questions have only been considered under additional symmetry assumptions. Let  $G$  be a real semisimple (noncompact and connected) Lie group with finite center and  $K$  be maximal compact in  $G$ . Consider the Riemannian symmetric space  $G/K$  equipped with its  $G$ -invariant measure under left multiplication. If  $\mathfrak{a}$  is a Cartan subalgebra for  $(G, K)$ , the  $G$ -invariant maps on  $L_2(G/K)$  can be identified via the spherical transform with Weyl-group-invariant elements in  $L_\infty(\mathfrak{a}^*)$ . Hörmander-Mikhlin criteria for this class of multipliers were first considered by Clerc and Stein [8], which established a necessary analyticity condition and a weak form of Mikhlin sufficient condition. Stanton and Tomas [54] obtained nearly optimal results in rank one based on precise local/asymptotic expansion formulae for spherical functions on  $G/K$ . Anker [1] finally discovered satisfactory Mikhlin conditions in high ranks. We refer to [52] for an interesting generalization in  $SL_2(\mathbf{R})$  and to [2, 27, 28, 38] for related results.

In this paper we work with the full semisimple Lie group  $SL_n(\mathbf{R})$  and place it in the frequency side. Its dual is no longer a group and it is described as a group von Neumann algebra, a key model of quantum (nonclassical) group. The interest of Fourier multipliers over group algebras was early recognized in the pioneering work of Haagerup [19], as well as in the research carried out thereafter in the context of approximation properties [5, 9, 10, 20]. The corresponding theory of Fourier  $L_p$ -multipliers is basic in noncommutative harmonic analysis, with potential applications in geometric group theory and operator algebra. It has recently gained a considerable momentum [16, 29, 30, 31, 34, 36, 43, 48].

Given a locally compact unimodular group  $G$  with left regular representation  $\lambda$ , its group von Neumann algebra  $\mathcal{L}(G)$  is the weak-\* closure in  $\mathcal{B}(L_2(G))$  of  $\text{span}(\lambda(G))$ . If  $\mu$  denotes the Haar measure of  $G$ , we may approximate every element affiliated to  $\mathcal{L}(G)$  by operators of the form

$$f = \int_G \widehat{f}(g)\lambda(g) d\mu(g)$$

for smooth enough  $\widehat{f}$ . If  $e$  is the unit in  $G$ , the Haar trace  $\tau$  is then determined by  $\tau(f) = \widehat{f}(e)$ . Given a symbol  $m : G \rightarrow \mathbf{C}$ , its associated Fourier multiplier is the map  $T_m : \lambda(g) \mapsto m(g)\lambda(g)$  which satisfies

$$\widehat{T_m f}(g) = \tau(T_m f \lambda(g)^*) = m(g)\tau(f \lambda(g)^*) = m(g)\widehat{f}(g).$$

In other words, it intertwines pointwise multiplication with the Fourier transform.

**I. Main results.** The rigidity theorems in [34, 36] establish the failure of the completely bounded approximation property (CBAP) in the noncommutative  $L_p$  space over the group von Neumann algebra of any lattice in  $SL_n(\mathbf{R})$ , for certain values of  $p$ . Roughly speaking, what is behind in harmonic analysis words is that Fourier  $L_p$  summability fails in the group algebra of  $SL_n(\mathbf{R})$ , when  $|1/p - 1/2|$  is large enough in terms of the rank. This constitutes an  $L_p$  refinement of Haagerup's theorem on the failure of weak amenability for high rank semisimple Lie groups [9, 20]. As far as these group algebras are concerned, Fourier multiplier theory has been limited so far to rigidity theorems and the search of *necessary* conditions to this end. In this paper, we provide *sufficient* conditions for  $L_p$  boundedness of Fourier multipliers in the group algebra of  $SL_n(\mathbf{R})$ .

The aforementioned rigidity gets in conflict with the classical Hörmander-Mikhlin criterion. Indeed, condition (HM) certainly includes  $\mathcal{C}_0$ -functions with arbitrarily slow decay. On the contrary, any K-biinvariant symbol  $m \in \mathcal{C}_0(SL_3(\mathbf{R}))$  satisfies the asymptotic rigidity estimate

$$(AR) \quad \left| m \begin{pmatrix} e^s & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-s} \end{pmatrix} \right| \leq C e^{-\delta|s|} \|T_m\|_{cb(L_p(\mathcal{L}(SL_3(\mathbf{R})))}$$

for all  $p > 4$  and any  $\delta < 1/2 - 2/p$ . This is the key inequality in [36]. Considerably stronger rigidity estimates for radial Fourier multipliers are proved in Theorem B below. Asymptotic rigidity is not witnessed by Hörmander-Mikhlin conditions and sufficient conditions for  $L_p$ -boundedness must incorporate it. In this respect, it is important to clarify the metric we shall be working with. Let us write  $\|\cdot\|$  for the operator norm in  $M_n(\mathbf{R})$  and  $|\cdot|$  for the normalized Hilbert-Schmidt norm in  $S_2(\ell_2(n))$ . Consider any distance function  $|g| = \text{dist}(g, e)$  satisfying the following equivalences

$$|g| \approx \begin{cases} |g - e| & \text{locally at } e, \\ \max\{\|g\|, \|g^{-1}\|\} & \text{asymptotically.} \end{cases}$$

It is clarifying to compare  $|g|$  with the Lie algebra metric. Set  $|g| = e^{|s|} - 1$  for  $g = \exp(sZ)$  diagonal, with  $Z$  of norm 1 in the Cartan algebra  $\mathfrak{a}$ . This formula still applies for nondiagonal elements close to  $e$ , for which  $e^{|s|} - 1 \approx |s|$ . Asymptotically this exponential growth admits a K-biinvariant extension to  $SL_n(\mathbf{R}) = \mathbf{K} \exp(\mathfrak{a}_+) \mathbf{K}$

$$L(g) := \max\{\|g\|, \|g^{-1}\|\} \approx e^{|s|} \quad \text{for } g \in \mathbf{K} \exp(s\mathbf{X}) \mathbf{K}.$$

The distance function  $|g|$  is therefore comparable to the Euclidean metric around the identity. On the contrary, the Euclidean breakdown for large distances may be explained by the asymptotic rigidity in (AR) and Theorem B below. Note in passing that  $(g, h) \mapsto \log L(gh^{-1})$  is a standard pseudometric, comparable to the word length from a compact symmetric generating set in  $SL_n(\mathbf{R})$ . Once the metric is fixed, we work with the natural differential operators. Consider the left-invariant vector fields generated by an orthonormal basis  $X_1, X_2, \dots, X_{n^2-1}$  of the Lie algebra  $\mathfrak{sl}_n(\mathbf{R})$ . The corresponding Lie derivatives

$$\partial_{X_j} m(g) = \left. \frac{d}{ds} \right|_{s=0} m(g \exp(sX_j))$$

do not commute for  $j \neq k$ . This justifies to define the set of multi-indices  $\gamma$  as ordered tuples  $\gamma = (j_1, j_2, \dots, j_k)$  with  $1 \leq j_i \leq n^2 - 1$  and  $|\gamma| = k \geq 0$ , which correspond to the Lie differential operators

$$d_g^\gamma m(g) = \partial_{X_{j_1}} \partial_{X_{j_2}} \cdots \partial_{X_{j_{|\gamma|}}} m(g) = \left( \prod_{1 \leq k \leq |\gamma|}^{\rightarrow} \partial_{X_{j_k}} \right) m(g).$$

**Theorem A.** *Assume that  $m \in \mathcal{C}^{\lfloor \frac{n^2}{2} \rfloor + 1}(SL_n(\mathbf{R}) \setminus \{e\})$  satisfies*

$$(\star) \quad |g|^{|\gamma|} |d_g^\gamma m(g)| \leq C_{\text{hm}} \quad \text{for all } |\gamma| \leq \left\lfloor \frac{n^2}{2} \right\rfloor + 1.$$

*Then,  $T_m$  is completely  $L_p$ -bounded for all  $1 < p < \infty$  by a constant  $C_p C_{\text{hm}}$ .*

Theorem A gives the first sufficient condition for  $L_p$ -boundedness in the group algebra of  $\mathrm{SL}_n(\mathbf{R})$ . Compared to (HM), we have replaced Euclidean derivatives by Lie derivatives and the Euclidean norm  $|\xi|$  by  $|g|$ :

a) *Local analysis.* Both  $|g|$  and  $d_g^\gamma$  are comparable to their Euclidean models at small distances to  $e$  and Theorem A gives a satisfactory form of (HM). The growth of derivatives around the singularity match the sharp Euclidean estimates. As  $\dim \mathrm{SL}_n(\mathbf{R}) = n^2 - 1$ , the differentiability order also nearly matches the optimal one in the Euclidean (HM). More precisely, we match optimal Euclidean order for  $n$  odd and we loose up to one derivative for  $n$  even. In fact, the local form of  $(\star)$  can be replaced by (weaker) *Sobolev conditions* of order  $n^2/2 + \varepsilon$ . Moreover, less regularity suffices for small  $|1/p - 1/2|$  in the spirit of Calderón-Torchinsky [4]. In conclusion, local singularities (at the unit  $e$  or anywhere else) are admissible and the regularity around them is apparently close to optimal. As far as we know, there is no result in the literature—including rigidity conditions—which gives any information on the local behavior of Fourier  $L_p$ -multipliers. Our local conditions are much more flexible than in [1, 8, 54], due to the necessary analyticity there.

b) *Asymptotic analysis.* Condition  $(\star)$  coincides with (HM) for the given metric up to order  $[n^2/2] + 1$ . As stated, it apparently poses a contradiction with the asymptotic rigidity in (AR). However, this is not the case in the metric we work with, since high order Hörmander-Mikhlin conditions impose the same decay rates for lower order terms. More precisely, the exponential growth of  $L$  with respect to the word length implies the following inequality for any  $\phi \in \mathcal{C}^1(\mathrm{SL}_n(\mathbf{R}) \setminus \{e\})$  and  $\beta > 2$

$$\sup_{X \in \mathfrak{sl}_n(\mathbf{R})} L(g)^\beta |\partial_X \phi(g)| \leq 1 \Rightarrow L(g)^\beta |\phi(g) - \alpha| \leq C_\beta \quad \text{for some } \alpha \in \mathbf{C},$$

where the supremum runs over all unit vectors in the Lie algebra of  $\mathrm{SL}_n(\mathbf{R})$ . This forces the symbol  $m - \alpha: \mathrm{SL}_n(\mathbf{R}) \rightarrow \mathbf{C}$  to decay at the same rate  $\beta_0 = [n^2/2] + 1$  as the highest order derivative in  $(\star)$ , see Remark 3.8. The known rigidity theorems in this context [34, 36] are no longer in conflict with it. In fact, Weyl's integration formula implies that the critical integrability index for  $L$  is  $[n^2/2]$ , so that  $L^{-\beta_0}$  is in  $L_1(\mathrm{SL}_n(\mathbf{R}))$  and the asymptotic part of Theorem A reduces to the local part by a simple patching argument. In particular, the asymptotic bound  $L^{1-\beta_0} / \log^{1+\delta} L$  for  $m$  and its Lie derivatives suffices. Conceivable, the logarithmic factor could be removed for  $n$  even, under the smaller regularity order  $[(n^2 - 1)/2] + 1 = [n^2/2]$ . Our asymptotic conditions are (necessarily) more rigid than in [1, 8, 54].

After Theorem A, we have no reason to believe that our asymptotic estimates are even close to optimal, our analysis just gives a satisfactory comparison with the Euclidean setting in a natural metric for  $\mathrm{SL}_n(\mathbf{R})$ . However, Theorem B below proves that Mikhlin type conditions are also necessary up to certain regularity order for radial multipliers. Given an open interval  $J \subset \mathbf{R}$  and  $\alpha > 0$ , let  $\mathcal{C}^\alpha(J)$  be the space of functions which admit  $[\alpha]$  continuous derivatives in  $J$  and such that the  $[\alpha]$ -th derivative of  $\varphi$  is Hölder continuous of order  $\alpha - [\alpha]$  on every compact subset of  $J$ . We shall also write  $\mathcal{C}^{\alpha-}(J)$  for the space

$$\mathcal{C}^{\alpha-}(J) = \bigcap_{\beta < \alpha} \mathcal{C}^\beta(J).$$

Given  $g \in \mathrm{SL}_n(\mathbf{R})$ , we use normalized Hilbert-Schmidt norms  $|g|^2 = \frac{1}{n} \mathrm{tr}(g^* g)$ .

**Theorem B.** Consider a radial  $SL_n(\mathbf{R})$ -symbol  $m(g) = \varphi(|g|)$  for  $n \geq 3$ . Assume that the Schur multiplier  $S_m(g, h) = m(gh^{-1})$  is  $S_p$ -bounded for some  $p > 2 + \frac{2}{n-2}$  so that  $\alpha_0 := (n-2)/2 - (n-1)/p > 0$ . Then  $\varphi$  is of class  $C^{\alpha_0}(1, \infty)$  when  $\alpha_0 \notin \mathbf{Z}$  and of class  $C^{\alpha_0^-}(1, \infty)$  otherwise. Moreover, if

$$\alpha = \alpha_0 \delta_{\alpha_0 \notin \mathbf{Z}} + (\alpha_0 - \varepsilon) \delta_{\alpha_0 \in \mathbf{Z}},$$

the following local/asymptotic estimates hold for the function  $\varphi$ :

i)  $\varphi$  has a limit  $\varphi_\infty$  at  $\infty$  and

$$|\varphi(x) - \varphi_\infty| \leq C_{p,n}^\varepsilon \frac{\|S_m\|_{\mathcal{B}(S_p(L_2(G)))}}{x^{c_0}},$$

where  $c_0 = n / \left[ \frac{3}{1 - \frac{2}{p}} \right]$  for  $\alpha > 1$  and  $c_0 = \alpha \frac{n}{n-2}$  for  $\alpha < 1$ .

ii) Given  $x > 1$  and an integer  $1 \leq k \leq [\alpha]$

$$|\partial^k \varphi(x)| \leq C_{p,n}^\varepsilon \frac{\|S_m\|_{\mathcal{B}(S_p(L_2(G)))}}{(x-1)^k x^{c_k}} \quad \text{where} \quad c_k = \frac{n}{\left[ \frac{2k+1}{1 - \frac{2}{p}} \right]}.$$

iii) The Hölder constants in a neighborhood of  $x$

$$\limsup_{y \rightarrow x} \frac{|\partial^{[\alpha]} \varphi(x) - \partial^{[\alpha]} \varphi(y)|}{|x-y|^{\alpha-[\alpha]}} \leq C_{p,n}^\varepsilon \frac{\|S_m\|_{\mathcal{B}(S_p(L_2(G)))}}{\left( (x-1)x^{\frac{n}{n-2}} \right)^\alpha}.$$

Theorem B gives a major strengthening of the rigidity theorems in this context [20, 34, 36]. It is also valid for radial Fourier multipliers—using the cb-norm of  $T_m$  instead—improving (AR) and higher dimensional forms in various ways, notably by the conditions in Theorem B ii). The range of  $p$ 's for which it applies also improves the best known results [34]. Its Euclidean form for radial multipliers  $m(\xi) = \varphi(|\xi|)$  in dimension  $d \geq 2$  is the necessary condition

$$(TT) \quad p > 2 + \frac{2}{d-1} \Rightarrow |\xi|^k |\partial^k \varphi(\xi)| \leq C_{p,d} \|T_m: L_p(\mathbf{R}^d) \rightarrow L_p(\mathbf{R}^d)\|$$

for  $k < (d-1)/2 - d/p$ . Comparing (TT) with (HM) enlightens the structure of radial multipliers. Indeed, taking  $p$  arbitrarily large, Mihlin conditions are necessary up to order  $[d/2] - 1$  and sufficient from  $[d/2] + 1$  on. Condition (TT) has its roots in the regularity of the Hankel transform of radial  $L_p$  functions, which goes back to Schoenberg [53] and culminated in [56, 57]. It is the most satisfactory result for radial multipliers before the celebrated characterization [14, 25].

Theorem B exactly reproduces (TT) around the singularity ( $x = 1$  and  $\xi = 0$  respectively) when Euclidean dimension  $d$  is replaced by  $n - 1$ . Thus, Theorem B confirms that the growth rate around the singularity in Theorem A is optimal for low order derivatives. (TT) also suggests that the rank of  $p$ 's in Theorem B could be best possible. Asymptotic rigidity arises from the extra decay provided by  $c_0, c_1, c_2, \dots, c_{[\alpha]}$  and it increases with the rank. Remarkably, there exist radial multipliers satisfying  $(\star)$  in rank  $n$  and failing rigidity for ranks  $m \gg n$ .

As application, we also prove in Theorem 4.6 related rigidity estimates in  $SO(n, 1)$  for first and higher order derivatives. This result is especially satisfactory, since this group is rank 1 and weakly amenable. It is certainly a surprise that Lafforgue's methods around strong property (T) shed some light here.

**II. Structure of the proof.** Hörmander-Mikhlin criteria for group algebras have been recently investigated in [16, 29, 30]. The lack of finite-dimensional orthogonal representations for  $\mathrm{SL}_2(\mathbf{R})$  and Kazhdan’s property (T) for higher ranks implies that Euclidean geometry only mirrors the geometry of  $\mathrm{SL}_n(\mathbf{R})$  via nonorthogonal actions, which are beyond the scope of the above mentioned papers. Here are some benchmarks which can help the reader to follow our argument:

a) *A local measurement of nonamenability.* Almost every form of transference since the pioneer contributions of Cotlar or Calderón involve some kind of amenability assumption. This was dodged in [29, 30] for those nonamenable groups which act orthogonally in some finite-dimensional real Hilbert space. Among others, this excludes  $\mathrm{SL}_n(\mathbf{R})$ . Given  $\Omega, \Sigma$  relatively compact open neighborhoods of the identity in a nonamenable group  $G$ , we shall introduce a constant  $0 < \delta_\Sigma(\Omega) < 1$  which quantifies the ‘ $\Sigma$ -nonamenability’ relative to  $\Omega$ . When  $G = \mathrm{SL}_n(\mathbf{R})$ , this is closely related to the Harish-Chandra’s function [11, 46]. Our proof starts with a complete contraction  $j_p : L_p(\mathcal{L}(\mathrm{SL}_n(\mathbf{R}))) \rightarrow S_p(L_2(\mathrm{SL}_n(\mathbf{R})))$ , which admits a key partial converse for  $p \in 2\mathbf{Z}_+$ :

$$\|f\|_p \leq_{\mathrm{cb}} \frac{1}{1 - \delta_\Sigma(\Omega_p)} \|j_p(f)\|_p \quad \text{when } \mathrm{supp} \widehat{f} \subset \Omega.$$

Here  $\Omega_p = \Omega\Omega^{-1} \dots \Omega^{\pm 1}$  with  $p/2$  terms. This is valid for other nonamenable  $G$ .

b) *Local transference Fourier  $\rightarrow$  Schur  $\rightarrow$  Twisted multipliers.* Generalizing previous work of Bożejko/Fendler, the isometry between Fourier and Schur  $L_p$ -multipliers was proved in [7, 45] for amenable groups. In fact, Schur multiplier norms are dominated by Fourier multiplier ones even for nonamenable groups. The reverse inequality remains open. At the moment, we find no reason to believe that it holds true for nonamenable groups. However,  $\Omega$ -supported Fourier multipliers are dominated by  $\Sigma' \times \Sigma'$ -truncated Schur multipliers ( $\Sigma' = \Omega^{-1}\Sigma$ ), up to a constant which equals 1 for  $G$  amenable and blows up as  $\Omega \rightarrow G$  otherwise. More precisely, if  $\mathrm{supp} m \subset \Omega$  and  $p \in 2\mathbf{Z}_+$ , we shall prove that

$$(LT) \quad \|T_m\|_{\mathrm{cb}(L_p(\mathcal{L}(G)))} \leq \frac{1}{1 - \delta_\Sigma(\Omega_p)} \|S_m\|_{\mathrm{cb}(S_p(L_2(\Sigma')))}.$$

Next, if  $\beta : G \rightarrow \mathbf{R}^k$  is a cocycle map associated to a volume-preserving action  $\alpha : G \curvearrowright \mathbf{R}^k$  and  $\mathcal{R}_\Sigma$  is the algebra of bounded functions  $f : \mathbf{R}^k \rightarrow \mathcal{B}(L_2(\Sigma))$ , we shall lift our multiplier  $m = \dot{m} \circ \beta$  to control the right hand side of (LT) by the  $\mathrm{cb}$ - $L_p$ -norm of the ‘‘twisted Fourier multiplier’’ given by

$$\widetilde{T}_{\dot{m}} : (f_{gh})_{g,h \in \Sigma} \mapsto (T_{\dot{m}_g}(f_{gh}))_{g,h \in \Sigma} \quad \text{with } T_{\dot{m}_g}(\widehat{f_{gh}})(\xi) = \dot{m}(\alpha_g(\xi)) \widehat{f_{gh}}(\xi).$$

c) *Twisted Riesz transforms for fractional laplacians.* Calderón-Zygmund and other classical methods are inefficient to bound twisted Fourier multipliers. Instead, we shall adapt a key result from [30], which identifies Hörmander-Mikhlin multipliers as Littlewood-Paley averages of fractional Riesz transforms. It does not apply in the group algebra of  $\mathrm{SL}_n(\mathbf{R})$  for lack of orthogonal cocycles, but local transference opens the door to a twisted form in  $\mathcal{R}_\Sigma$  via the infinite-dimensional, but orthogonal cocycles naturally linked to fractional powers of Euclidean laplacians. A duality argument then shows that it suffices to prove a square function inequality for twisted Riesz transforms. The behavior of these maps is highly asymmetric: the  $\alpha$ -twist affects the row index  $g$ , but not the column index  $h$ . The column case follows

by a combination of harmonic analysis and operator space techniques, whereas the row case seems to be false in the full algebra  $\mathcal{R}_\Sigma$ . At this point, the proof turns more technical. Roughly, we restrict to the image of our local embedding into  $\mathcal{R}_\Sigma$  and invert transference to rewrite the row square function in the group algebra. In this context, group inversion—which is locally smooth around the unit, so that Hörmander-Mikhlin conditions are stable up to  $\Omega$ -constants—allows to switch rows to columns. Then we crucially use a new form of Littlewood-Paley theorem for Schur multipliers over  $SL_n(\mathbf{R})$ , which yields the local form of Theorem A. Our local argument holds for any compact  $\Omega$ , but only yields optimal estimates around the singularity. The constants grow too fast in terms of  $\text{diam}(\Omega)$  and the alluded patching argument gives better asymptotic estimates.

d) *On the rigidity theorem. The proof of Theorem B starts with the same idea which already worked in previous rigidity theorems [34, 36]. Given  $p$  as in the statement of Theorem B, we first prove  $C^\alpha$ -regularity for  $SO(n-1)$ -biinvariant  $S_p$ -multipliers on  $SO(n)$ . This follows in turn from related estimates in the  $n-1$ -dimensional sphere for the averaging operator which was introduced in [36]. The main novelty in our argument is to infer global regularity and decay estimates for radial multipliers in  $SL_n(\mathbf{R})$ , as stated in Theorem B. The key point is to factorize the radial generating function  $\varphi$  as a composition  $\varphi(x) = \psi_r \circ H_r(x)$  (with  $r$  depending on  $x$ ), where  $\psi_r$  is smooth as a consequence of our auxiliary estimates in  $SO(n)$ , and the derivatives of  $H_r$  decay fast enough. Then, the assertion in Theorem B follows from an application of Faà di Bruno's formula. Quite surprisingly, the same technique gives "higher order" rigidity estimates in the rank 1 groups  $SO(n, 1)$  as explained above.*

### 1. Local measurement of nonamenability

Let  $G$  be a locally compact unimodular group with Haar measure  $\mu$  and left regular representation  $\lambda$ . Let  $\Omega$  be a relatively compact neighborhood of the identity in  $G$  and let  $\Sigma \subset G$  be open. If  $\Sigma$  is large enough, we can always find  $\phi : G \rightarrow \mathbf{R}_+$  in  $L_2(G)$  and a constant  $0 < \delta_\phi(\Omega) < 1$  such that

- $\text{supp} \phi \subset \Sigma$ ,
- $\int_G |\phi(h)|^2 d\mu(h) = 1$ ,
- $\int_G |\phi(gh) - \phi(h)|^2 d\mu(h) \leq 2\delta_\phi(\Omega)$  for all  $g \in \Omega$ .

Consider the best possible constant

$$\begin{aligned} \delta_\Sigma(\Omega) &= 1 - \sup_{\substack{\|\phi\|_2=1 \\ \text{supp} \phi \subset \Sigma}} \inf_{g \in \Omega} \langle \lambda(g)\phi, \phi \rangle_{L_2(G)} \\ &= \inf_{\substack{\|\phi\|_2=1 \\ \text{supp} \phi \subset \Sigma}} \sup_{g \in \Omega} \frac{1}{2} \int_G |\phi(gh) - \phi(h)|^2 d\mu(h). \end{aligned}$$

Then, the following dichotomy holds

- i) If  $G$  is amenable, then  $\delta_G(\Omega) = 0$  for all  $\Omega$ .
- ii) If  $G$  is nonamenable, then  $\lim_{\Omega \rightarrow G} \delta_G(\Omega) = 1$ .



It easily follows from Dixmier's characterization of amenability and an ultraproduct argument. This suggests that  $\delta_\Sigma(\Omega)$  measures the “ $\Sigma$ -nonamenability relative to  $\Omega$ ” with constants  $\delta_\Sigma(\Omega) \approx 0$  for  $\Omega$  small and  $\delta_\Sigma(\Omega) \approx 1$  for  $\Omega$  large. Along this paper, we shall not use precise estimates of these constants in terms of  $(\Omega, \Sigma)$ , but they can be relevant for future applications.

**Remark 1.1.** The Iwasawa decomposition  $\mathrm{SL}_n(\mathbf{R}) = \mathrm{KP}$  takes place with maximal compact group  $\mathrm{K} = \mathrm{SO}(n)$  and parabolic part  $\mathrm{P}$  formed by the subgroup of upper triangular matrices. Let  $\Delta$  be the left  $\mathrm{K}$ -invariant extension  $\Delta(kp) = \Delta(p)$  of the left-modular function in  $\mathrm{P}$ . The Harish-Chandra function  $\Xi : \mathrm{SL}_n(\mathbf{R}) \rightarrow \mathbf{R}_+$  is then defined as

$$\Xi(g) = \int_{\mathrm{K}} \Delta(gk)^{-\frac{1}{2}} dk.$$

Let us write  $B_R$  for the ball of radius  $R$  around the identity in the pseudometric  $\log L$ , as defined in the Introduction. Then, the following inequalities hold for the pair  $(\Omega, \Sigma) = (B_R, B_{2nR})$  and every  $q > 2$

$$\exp\left(-\frac{1}{2}\left[\frac{n^2}{2}\right]R\right) \lesssim 1 - \delta_\Sigma(\Omega) \approx \inf_{g \in \Omega} \Xi(g) \leq C_q \exp\left(-\frac{1}{q}\left[\frac{n^2}{2}\right]R\right).$$

The lower/upper bounds arise from well-known estimates [23] of the Harish-Chandra function  $\Xi$  in terms of the modular function  $\Delta$ . The upper bound in the equivalence above reduces to an inequality for left  $\mathrm{K}$ -invariant matrix coefficients  $\langle \lambda(g)\phi, \phi \rangle$  [11, 46]. These are the most elementary Fourier multipliers which are bounded in the Fourier and group algebras of  $\mathrm{SL}_n(\mathbf{R})$ . The lower bound requires fairly precise estimates for Følner sequences in the parabolic part  $\mathrm{P}$ . We omit the argument since these bounds will not play a significant role in this paper.

**1.1. Matrix algebras.** Let

$$j : \mathcal{L}(\mathrm{G}) \rightarrow \mathcal{B}(L_2(\mathrm{G}))$$

be the canonical embedding

$$\begin{aligned} j(f) &= j\left(\int_{\mathrm{G}} \widehat{f}(g)\lambda(g) d\mu(g)\right) \\ &= \int_{\mathrm{G} \times \mathrm{G}} \widehat{f}(gh^{-1})e_{gh} d\mu(g)d\mu(h) \quad \text{for } \widehat{f} \in \mathcal{C}_c(\mathrm{G}). \end{aligned}$$

The  $e_{gh}$  stand for infinitesimal matrix units and the last integral must be understood in the weak-\* sense. The map  $j$  is a \*-homomorphism, but it is not trace preserving and it fails to be  $L_p$ -bounded. It has been further studied for amenable actions in [15]. Given  $1 \leq p \leq \infty$  and  $0 \leq \theta \leq 1$ , define

$$(1.1) \quad j_{p\theta}^\phi : f \mapsto \Phi_{p,1-\theta} j(f) \Phi_{p,\theta}$$

where  $\Phi_{p,\theta}$  is the pointwise multiplication map by  $\phi(g)^{2\theta/p}$  for some  $\phi$  satisfying the above conditions with constant  $\delta_\phi(\Omega)$ , with the convention  $\phi(g)^0 = 1$  (even if  $\phi(g) = 0$ ). As an operator affiliated to  $\mathcal{B}(L_2(\mathrm{G}))$  we may think of  $\Phi_{p,\theta}$  as the diagonal matrix with entries  $\phi(g)^{2\theta/p}$ . Consider the constants

$$\varepsilon_\phi(\Omega) = \frac{\delta_\phi(\Omega) - \delta_\Sigma(\Omega)}{1 - \delta_\phi(\Omega)} \quad \text{so that} \quad \lim_{\delta_\phi \rightarrow \delta_\Sigma} \varepsilon_\phi(\Omega) = 0.$$

Finally, when  $p \in 2\mathbf{Z}_+$ , we also define  $\Omega_p = \Omega\Omega^{-1}\Omega\Omega^{-1} \dots \Omega^{\pm 1}$  with  $p/2$  terms.

**Lemma 1.2.** *The following properties hold:*

- i)  $\|j_{p\theta}^\phi : L_p(\mathcal{L}(G)) \rightarrow S_p(L_2(G))\|_{\text{cb}} \leq 1$  for  $p \geq 2$ .  
ii) *If in addition  $\widehat{f}(g) = 0$  for all  $g \notin \Omega$ , then we get*

$$\|f\|_p \leq_{\text{cb}} \frac{1 + \varepsilon_\phi(\Omega_p)}{1 - \delta_\Sigma(\Omega_p)} \|j_{p0}^\phi(f)\|_p \quad \text{for } p \in 2\mathbf{Z}_+.$$

*Of course, the same stament above holds for  $j_{p1}^\phi$  instead of  $j_{p0}^\phi$ .*

**Proof.** The first property trivially holds for  $p = \infty$ . Thus, by interpolation it suffices to prove it for  $p = 2$ . Given  $0 \leq \theta \leq 1$ , define  $(2/q_0, 2/q_1) = (1 - \theta, \theta)$  so that every  $f \in L_2(\mathcal{L}(G))$  factorizes as  $f = f_0 f_1$  with  $\|f_0\|_{q_0} \|f_1\|_{q_1} = \|f\|_2$ . This yields

$$\|j_{20}^\phi(f)\|_2 = \|j_{q_0 0}^\phi(f_0) j_{q_1 1}^\phi(f_1)\|_2 \leq \|j_{q_0 0}^\phi(f_0)\|_{q_0} \|j_{q_1 1}^\phi(f_1)\|_{q_1},$$

which reduces the problem to show that  $j_{q_0 0}^\phi$  and  $j_{q_1 1}^\phi$  are complete contractions. By symmetry in the argument and again by interpolation, we are reduced to study  $j_{20}^\phi$ , which is a complete isometry since

$$\begin{aligned} \|j_{20}^\phi(f)\|_2^2 &= \text{tr} \left[ \left| \left( \phi(g) \widehat{f}(gh^{-1}) \right)_{g,h} \right|^2 \right] \\ &= \int_{G \times G} \phi(g)^2 |\widehat{f}(gh^{-1})|^2 d\mu(g) d\mu(h) \\ &= \left( \int_G \phi(g)^2 d\mu(g) \right) \left( \int_G |\widehat{f}(h)|^2 d\mu(h) \right) = \|f\|_2^2. \end{aligned}$$

To prove the second property, we note that

$$\begin{aligned} &\langle j_{2,1-\theta}^\phi(f), j_{2,\theta}^\phi(f') \rangle \\ &= \text{tr} \left[ \left( \phi(g)^\theta \widehat{f}(gh^{-1}) \phi(h)^{1-\theta} \right)_{g,h}^* \left( \phi(g)^{1-\theta} \widehat{f}'(gh^{-1}) \phi(h)^\theta \right)_{g,h} \right] \\ &= \int_{G \times G} \phi(g) \overline{\widehat{f}(gh^{-1})} \widehat{f}'(gh^{-1}) \phi(h) d\mu(g) d\mu(h) = \int_G \overline{\widehat{f}(g)} \widehat{f}'(g) (1 + a_g) d\mu(g) \end{aligned}$$

with  $a_g = \int_G \phi(gh) \phi(h) d\mu(h) - 1$  for  $\overline{\widehat{f}(g)} \widehat{f}'(g) \neq 0$  and  $a_g = 0$  otherwise. If  $g \in \Omega$

$$|a_g| = \frac{1}{2} \left| \int_G (\phi(gh)^2 + \phi(h)^2 - 2\phi(gh)\phi(h)) d\mu(h) \right| \leq \delta_\phi(\Omega).$$

Therefore, given  $f \in L_2(\mathcal{L}(G))$  with  $\widehat{f}(g) = 0$  for  $g \notin \Omega$ , we may prove the second property as follows. Let  $f'$  be in the unit ball of  $L_2(\mathcal{L}(G))$  so that  $\|f\|_2 = \langle f, f' \rangle$ . Then we get

$$\begin{aligned} \|f\|_2 &\leq \left| \int_G \overline{\widehat{f}(g)} \widehat{f}'(g) (1 + a_g) d\mu(g) \right| + \left| \int_\Omega \overline{\widehat{f}(g)} \widehat{f}'(g) a_g d\mu(g) \right| \\ &\leq |\langle j_{2,1-\theta}^\phi(f), j_{2,\theta}^\phi(f') \rangle| + \left( \int_\Omega |\widehat{f}'(g)|^2 |a_g|^2 d\mu(g) \right)^{\frac{1}{2}} \|f\|_2 \\ &\leq \|j_{2,1-\theta}^\phi(f)\|_2 + \delta_\phi(\Omega) \|f\|_2. \end{aligned}$$

This proves that

$$(1.2) \quad \|f\|_2 \leq \frac{1}{1 - \delta_\phi(\Omega)} \|j_{2\theta}^\phi(f)\|_2 = \frac{1 + \varepsilon_\phi(\Omega)}{1 - \delta_\Sigma(\Omega)} \|j_{2\theta}^\phi(f)\|_2 \quad \text{for } 0 \leq \theta \leq 1.$$

Now let  $f \in L_p(\mathcal{L}(G))$  for some  $p \in 2\mathbf{Z}_+$  greater than 2. Consider its polar decomposition  $f = u_f|f| = |f^*|u_f$  and let  $q > 2$  satisfy  $2/p + 2/q = 1$ . Then we pick

$$f' = \begin{cases} \frac{|f^*| |f^*|^{\frac{p}{2}-2}}{\| |f^*| |f^*|^{\frac{p}{2}-2} \|_q} & \text{if } p \in 4\mathbf{Z}_+, \\ \frac{|f^*| |f^*|^{\frac{p}{2}-2} u_f}{\| |f^*| |f^*|^{\frac{p}{2}-2} u_f \|_q} & \text{if } p \notin 4\mathbf{Z}_+. \end{cases}$$

This gives an element of the unit ball of  $L_q(\mathcal{L}(G))$  satisfying  $\|f\|_p = \|ff'\|_2$ . On the other hand, note that the support of the Fourier spectrum of  $ff'$  is the same as that of  $|f^*|^{p/2} = (ff^*)^{p/4}$  for  $p \in 4\mathbf{Z}_+$  and  $|f^*|^{p/2-1}f = (ff^*)^{(p-2)/4}f$  when  $p \in 2\mathbf{Z}_+ \setminus 4\mathbf{Z}_+$ . In both cases, the Fourier spectrum is supported in the set  $\Omega_p$  defined before the statement. Moreover, taking  $\theta = 1 - 2/p$  and applying the  $L_2$ -inequality above we find

$$\begin{aligned} \|f\|_p = \|ff'\|_2 &\leq \frac{1 + \varepsilon_\phi(\Omega_p)}{1 - \delta_\Sigma(\Omega_p)} \|j_{2\theta}^\phi(ff')\|_2 \\ &= \frac{1 + \varepsilon_\phi(\Omega_p)}{1 - \delta_\Sigma(\Omega_p)} \|j_{p0}^\phi(f)j_{q1}^\phi(f')\|_2 \leq \frac{1 + \varepsilon_\phi(\Omega_p)}{1 - \delta_\Sigma(\Omega_p)} \|j_{p0}^\phi(f)\|_p \end{aligned}$$

by Hölder's inequality and the contractivity of  $j_{q1}^\phi$ . The cb-analogue is similar.  $\square$

**1.2. Local transference.** Given  $m : G \rightarrow \mathbf{C}$ , let  $S_m(g, h) = m(gh^{-1})$  be the Schur multiplier associated to  $m$ , as defined in the Introduction. The main results in [7, 45] give that Fourier and Schur multipliers share the same cb- $L_p$ -norm for amenable groups. In the nonamenable setting, Schur multiplier cb-norms are always dominated by Fourier multiplier cb-norms. The reverse inequality remains open. If we consider  $\Sigma' = \Omega^{-1}\Sigma$ , the distortion constants  $\delta_\Sigma(\Omega_p)$  provide a local form of such an inequality between  $\Omega$ -supported Fourier multipliers and the corresponding  $\Sigma' \times \Sigma'$ -truncated Schur multipliers

$$S_m^{\Sigma'}(a) = \left( m(gh^{-1})a_{g,h} \right)_{g,h \in \Sigma'} \quad \text{for } a = \left( a_{g,h} \right)_{g,h \in \Sigma'}.$$

We shall write  $S_m$  instead of  $S_m^{\Sigma'}$  when the truncation is clear from the context.

**Theorem 1.3.** *Let  $G$  be a locally compact unimodular group. Consider a relatively compact neighborhood of the identity  $\Omega$  and an arbitrary open subset  $\Sigma$  in  $G$ . Let  $m : G \rightarrow \mathbf{C}$  be a bounded symbol supported in  $\Omega$ . Then, the following inequality holds for  $p \in 2\mathbf{Z}_+$*

$$\|T_m : L_p(\mathcal{L}(G)) \rightarrow L_p(\mathcal{L}(G))\|_{\text{cb}} \leq \frac{1}{1 - \delta_\Sigma(\Omega_p)} \|S_m : S_p(L_2(\Sigma')) \rightarrow S_p(L_2(\Sigma'))\|_{\text{cb}}.$$

**Proof.** As  $\text{supp } m \subset \Omega$ , Lemma 1.2 ii) gives

$$(1.3) \quad \|T_m f\|_p \leq_{\text{cb}} \frac{1 + \varepsilon_\phi(\Omega_p)}{1 - \delta_\Sigma(\Omega_p)} \|S_m(j_{p0}^\phi f)\|_{S_p(L_2(\Sigma'))}$$

since  $j_{p0}^\phi(T_m f) = S_m(j_{p0}^\phi f)$ . Note that  $\phi(g)^{2/p}m(gh^{-1})$  appears as a factor of the  $(g, h)$ -entry. In particular, since  $\mathrm{supp}(\phi \otimes m) \subset \Sigma \times \Omega$ , we easily deduce that each nonvanishing entry  $(g, h) \in \Sigma' \times \Sigma'$ . This implies that  $S_m \circ j_{p0}^\phi = S_m^{\Sigma'} \circ j_{p0}^\phi$  and Lemma 1.2 i) yields the assertion for  $p \geq 2$  by taking  $\varepsilon_\phi$  arbitrarily small.  $\square$

**Remark 1.4.** The proof really gives

$$\|T_m\|_{\mathrm{cb}(L_p(\mathcal{L}(\mathrm{G})))} \leq \frac{1}{1 - \delta_\Sigma(\Omega_p)} \|S_m\|_{\mathrm{cb}(S_p(L_2(\Sigma'), L_2(\Sigma)))}.$$

In other words, the same inequality holds with the  $\Sigma \times \Sigma'$ -truncation instead.

**Problem 1.5.** The above result yields the expected inequality with constant 1 for amenable groups [7, 45], since  $\delta_G(\Omega) = 0$  for all  $\Omega$ . Our inequalities are new for nonamenable groups, but unfortunately leave several questions unsolved. Can we generalize Lemma 1.2 and Theorem 1.3 to noninteger values  $p \geq 2$ ? What is the behavior for  $p < 2$ ? On the other hand, we do not recover the  $L_2$ -isometry and Bożejko-Fendler's  $L_\infty$ -isometry [3]. In other words, we should expect our constants in Theorem 1.3 to be close to 1 when  $p$  approaches 2 or  $\infty$ . A quick review of our argument shows that this would be the case if the constants in (1.2) converge fast enough to 1 when  $\theta$  approaches 0 or 1. Is that true? Last but not least, is there a nonlocal upper bound for nonamenable groups? Proving such an inequality or providing a counterexample would enlighten very much this relation. We have not investigated similar inequalities for nonamenable nonunimodular groups.

**1.3. Nonorthogonal cocycles.** Let  $\alpha : \mathrm{G} \rightarrow \mathrm{SL}_d(\mathbf{R})$  be any volume-preserving continuous representation and let  $\beta : \mathrm{G} \rightarrow \mathbf{R}^d$  be a continuous map satisfying the cocycle law  $\alpha_g(\beta(h)) = \beta(gh) - \beta(g)$  for  $g, h \in \mathrm{G}$ . We call  $\beta$  orthogonal when the action  $\alpha$  is orthogonal. In contrast with [29, 30], nonorthogonal cocycles will be admissible in what follows. A  $\beta$ -lifted multiplier for the symbol  $m : \mathrm{G} \rightarrow \mathbf{C}$  is any function  $\dot{m} : \mathbf{R}^d \rightarrow \mathbf{C}$  satisfying the identity  $m = \dot{m} \circ \beta$ . We need some additional notation:

- $\mathcal{R}_\Sigma = L_\infty(\mathbf{R}^d) \bar{\otimes} \mathcal{B}(L_2(\Sigma))$ .
- $\dot{m}_g(\xi) = \dot{m}(\alpha_g(\xi))$  for  $g \in \mathrm{G}$  and  $\xi \in \mathbf{R}^d$ .
- $\tilde{T}_{\dot{m}}(f)(x) = \left( T_{\dot{m}_g}(f_{gh})(x) \right)_{g, h \in \Sigma}$  when  $f = \left( f_{gh} \right)_{g, h \in \Sigma}$ .
- Characters in  $\mathbf{R}^d$ :  $\exp_\xi(x) = \exp(2\pi i \langle x, \xi \rangle)$  with  $\xi \in \mathbf{R}^d$ .

The map  $\tilde{T}_{\dot{m}}$  is a “twisted Fourier multiplier” which acts over  $\Sigma \times \Sigma$  matrix-valued functions in  $\mathbf{R}^d$ , see Section 2 for further details. Twisted Fourier multipliers are very relevant for this paper and will be studied later on. In the following result, we adapt the arguments from [7, 45] in conjunction with de Leeuw’s transference [37] to relate Schur and twisted multipliers.

**Proposition 1.6.** *If  $p \geq 1$ , we obtain*

$$\|S_m : S_p(L_2(\Sigma)) \rightarrow S_p(L_2(\Sigma))\|_{\mathrm{cb}} \leq \|\tilde{T}_{\dot{m}} : L_p(\mathcal{R}_\Sigma) \rightarrow L_p(\mathcal{R}_\Sigma)\|_{\mathrm{cb}}$$

for any continuous Fourier symbol  $\dot{m} : \mathbf{R}^d \rightarrow \mathbf{C}$  satisfying the identity  $m = \dot{m} \circ \beta$ .

**Proof.** Consider the unitary  $u$  on  $L_2(\mathbf{R}^d \times \Sigma)$

$$u f(g, \xi) = \exp_{\beta(g^{-1})}(\xi) f(\xi, g) = e^{2\pi i \langle \beta(g^{-1}), \xi \rangle} f(\xi, g).$$

Define  $\pi_u: S_p(L_2(\Sigma)) \rightarrow L_\infty(\mathbf{R}^d; S_p((L_2(\Sigma)))$  by

$$\begin{aligned}\pi_u(a) &= u^*(1 \otimes a)u \\ &= \left( \exp_{-\beta(g^{-1})} a_{gh} \exp_{\beta(h^{-1})} \right)_{g,h \in \Sigma} = \left( \exp_{\alpha_{g^{-1}}(\beta(gh^{-1}))} a_{gh} \right)_{g,h \in \Sigma}.\end{aligned}$$

The last identity follows from the cocycle law for  $\beta$ . Note the intertwining identity

$$\begin{aligned}\tilde{T}_{\dot{m}}(\pi_u(a)) &= \left( T_{\dot{m}_g}(\exp_{\alpha_{g^{-1}}(\beta(gh^{-1}))} a_{gh}) \right)_{g,h \in \Sigma} \\ &= \left( \dot{m}(\beta(gh^{-1})) \exp_{\alpha_{g^{-1}}(\beta(gh^{-1}))} a_{gh} \right)_{g,h \in \Sigma} = \pi_u(S_m(a)).\end{aligned}$$

Since  $\pi_u$  is a normal representation of  $\mathcal{B}(L_2(\Sigma))$  when  $p = \infty$ , one concludes easily by weak- $*$  density. The case  $p < \infty$  requires some normalization in the spirit of de Leeuw. If  $\gamma_s(x) = \exp(-s|x|^2)$  for  $s > 0$ , we get

$$\|\gamma_{s/p}\|_{L_p(\mathbf{R}^d)} = \left(\frac{\pi}{s}\right)^{d/2p}.$$

If  $\frac{1}{p} + \frac{1}{q} = 1$  and  $a, b \in S_2(L_2(\Sigma))$ , we claim that

$$(1.4) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\|\gamma_{\varepsilon/p}\|_p \|\gamma_{\varepsilon/q}\|_q} \left\langle \tilde{T}_{\dot{m}}(\gamma_{\varepsilon/p} \pi_u(a)), \gamma_{\varepsilon/q} \pi_u(b) \right\rangle = \langle S_m(a), b \rangle.$$

Before justifying the claim, note that it implies the statement since

$$(1.5) \quad \|\gamma_s \pi_u(a)\|_{L_p(\mathcal{R}_\Sigma)}^p = \int_{\mathbf{R}^d} |\gamma_s(x)|^p \|\pi_u(a)(x)\|_{S_p(L_2(\Sigma))}^p dx = \|\gamma_s\|_p^p \|a\|_{S_p(L_2(\Sigma))}^p.$$

Therefore, as the same identity holds for  $\gamma_s \pi_u(b)$  in  $L_q(\mathcal{R}_\Sigma)$ , we obtain

$$|\langle S_m(a), b \rangle| \leq \|\tilde{T}_{\dot{m}}: L_p(\mathcal{R}_\Sigma) \rightarrow L_p(\mathcal{R}_\Sigma)\|_{\text{cb}} \|a\|_p \|b\|_q.$$

This is enough to conclude by density of  $S_2$  in  $S_p$  and  $S_q$ . To prove (1.4) we use the Plancherel formula. More precisely, by approximation we may also assume that  $a$  and  $b$  have continuous kernels and

$$\begin{aligned}\left\langle \tilde{T}_{\dot{m}}(\gamma_{\varepsilon/p} \pi_u(a)), \gamma_{\varepsilon/q} \pi_u(b) \right\rangle \\ = \int_{\Sigma \times \Sigma} \langle \dot{m}_g[\widehat{\gamma_{\varepsilon/p} \pi_u(a)}_{gh}], [\widehat{\gamma_{\varepsilon/q} \pi_u(b)}_{gh}] \rangle d\mu(g) d\mu(h).\end{aligned}$$

If we set  $\eta_{gh} = \alpha_{g^{-1}}(\beta(gh^{-1}))$ , we note that

$$[\widehat{\gamma_{\varepsilon/p} \pi_u(a)}_{gh}](\xi) = a_{gh} [\widehat{\gamma_{\varepsilon/p} \exp_{\eta_{gh}}}] (\xi) = a_{gh} \underbrace{\left(\frac{p\pi}{\varepsilon}\right)^{\frac{d}{2}} \exp\left(-\frac{p\pi^2|\eta_{gh} - \xi|^2}{\varepsilon}\right)}_{\psi_{p,\varepsilon}(\eta_{gh} - \xi)}.$$

Thus, the left hand term in (1.4) can be rewritten as follows

$$\lim_{\varepsilon \rightarrow 0} \left(\frac{\varepsilon}{\pi}\right)^{\frac{d}{2}} \int_{\Sigma \times \Sigma} a_{gh} \overline{b_{gh}} \left( \int_{\mathbf{R}^d} \dot{m}_g(\xi) \psi_{p,\varepsilon}(\eta_{gh} - \xi) \psi_{q,\varepsilon}(\eta_{gh} - \xi) d\xi \right) d\mu(g) d\mu(h).$$

Next, using the change of variables  $\xi = \varepsilon^{\frac{1}{2}} \rho + \eta_{gh}$ , the inner integral equals

$$\left(\frac{pq\pi^2}{\varepsilon}\right)^{\frac{d}{2}} \int_{\mathbf{R}^d} \dot{m}_g(\varepsilon^{\frac{1}{2}} \rho + \eta_{gh}) \exp(-\pi^2(p+q)|\rho|^2) d\rho.$$

Altogether, the dominated convergence theorem gives the desired identity since

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{1}{\|\gamma_{\varepsilon/p}\|_p \|\gamma_{\varepsilon/q}\|_q} \left\langle \widetilde{T}_{\dot{m}}(\gamma_{\varepsilon/p} \pi_u(a)), \gamma_{\varepsilon/q} \pi_u(b) \right\rangle \\ &= (\pi pq)^{\frac{d}{2}} \left( \int_{\mathbf{R}^d} e^{-\pi^2(p+q)|\rho|^2} d\rho \right) \left( \int_{\Sigma \times \Sigma} \dot{m}_g(\eta_{gh}) a_{gh} \overline{b_{gh}} d\mu(g) d\mu(h) \right) \\ &= \left( \frac{pq}{p+q} \right)^{\frac{d}{2}} \int_{\Sigma \times \Sigma} \dot{m}(\alpha_g(\eta_{gh})) a_{gh} \overline{b_{gh}} d\mu(g) d\mu(h) = \langle S_m(a), b \rangle. \end{aligned}$$

We have used  $p+q=pq$ . The proof for the complete boundedness is identical.  $\square$

**Remark 1.7.** There are many variants of Proposition 1.6. The following will be of particular interest below. Namely, using the same argument one can prove for  $\dot{m} : \mathbf{R}^d \rightarrow \mathbf{C}$  continuous that the (non-Toeplitz) Schur multiplier with symbol  $\widetilde{m}(g, h) = \dot{m}(\alpha_{g^{-1}}(\beta(gh^{-1})))$  satisfies the inequality

$$\|S_{\widetilde{m}} : S_p(L_2(\mathbf{G})) \rightarrow S_p(L_2(\mathbf{G}))\|_{\text{cb}} \leq \|T_{\dot{m}} : L_p(\mathbf{R}^d) \rightarrow L_p(\mathbf{R}^d)\|_{\text{cb}}.$$

This is the analogous result moving the twist from the Fourier to the Schur side. Just note that  $T_{\dot{m}} \circ \pi_u = \pi_u \circ S_{\widetilde{m}}$  and (1.4) still holds with the same replacements.

**Corollary 1.8.** *Let  $\mathbf{G}$  be a locally compact unimodular group. Consider a relatively compact neighborhood of the identity  $\Omega$  and an arbitrary open subset  $\Sigma$  in  $\mathbf{G}$ . Let  $m : \mathbf{G} \rightarrow \mathbf{C}$  be a continuous symbol supported by  $\Omega$  and satisfying  $m = \dot{m} \circ \beta$ . Then the following inequality holds for  $p \in 2\mathbf{Z}_+$*

$$\|T_m : L_p(\mathcal{L}(\mathbf{G})) \rightarrow L_p(\mathcal{L}(\mathbf{G}))\|_{\text{cb}} \leq \frac{1}{1 - \delta_{\Sigma}(\Omega_p)} \|\widetilde{T}_{\dot{m}} : L_p(\mathcal{R}_{\Sigma}) \rightarrow L_p(\mathcal{R}_{\Sigma})\|_{\text{cb}}.$$

**Proof.** The inequality in the statement with  $\mathcal{R}_{\Sigma'}$  in place of  $\mathcal{R}_{\Sigma}$  immediately follows from Theorem 1.3 and Proposition 1.6. According to Remark 1.4, we may replace  $S_p(L_2(\Sigma'))$  by  $S_p(L_2(\Sigma'), L_2(\Sigma))$ . Then, the argument in Proposition 1.6 still applies for rectangular matrices  $a, b \in S_2(L_2(\Sigma'), L_2(\Sigma))$ . Indeed, the left term in (1.4) is then written in terms of an integral over  $\Sigma \times \Sigma'$  instead. This yields an upper bound which equals the  $\text{cb-}L_p$ -norm of the twisted Fourier multiplier over the subspace of  $\Sigma \times \Sigma'$  matrix-valued functions. However, the twist of our multiplier only affects the  $g$ -variable and acts trivially in the  $h$ -variable. Thus, the  $\text{cb-}$ norm is unaffected after replacing  $\Sigma'$  by  $\Sigma$ . This completes the proof.  $\square$

**Remark 1.9.** The lifted symbol  $\dot{m}$  is called regulated when it satisfies the Lebesgue differentiation theorem everywhere. Both Proposition 1.6 and Corollary 1.8 hold for regulated symbols as well, for the same reasons as in [37].

## 2. Twisted multipliers

Along this paper, a twisted Fourier multiplier will be any linear map sending a matrix-valued function  $f$  to its Schur product  $M \bullet f$  with a matrix  $M$  of Fourier multipliers. Unless specified otherwise, we will work with matrices defined over a continuous parameter  $(g, h)$  in  $\Sigma \times \Sigma$ . More precisely, if  $f = (f_{gh}) : \mathbf{R}^d \rightarrow S_2(L_2(\Sigma))$  we consider the map which sends  $f$  to

$$\widetilde{T}_M f = M \bullet f = \left( T_{M_{gh}}(f_{gh}) \right)_{g, h \in \Sigma} = \int_{\mathbf{R}^d} \left( M_{gh}(\xi) \widehat{f_{gh}}(\xi) \right)_{g, h \in \Sigma} \exp_{\xi} d\xi.$$

This determines the action of  $\tilde{T}_M$  on  $L_p(\mathcal{R}_\Sigma)$  for  $p < \infty$ . Typically,  $M_{gh}$  will be a continuous deformation —independent of  $h$ , so that  $(g, \xi) \mapsto M_g(\xi)$  is a bounded measurable function— of a bounded symbol  $M: \mathbf{R}^d \rightarrow \mathbf{C}$ . In that case, we may also define the twist action on  $\mathcal{R}_\Sigma$  by  $(\tilde{T}_M f)\zeta = \tilde{V}_M(f\zeta)$  for the  $L_2(\Sigma)$ -valued map

$$(\tilde{V}_M h)_g = T_{M_g}(h_g).$$

Let  $m = \dot{m} \circ \beta$  for some cocycle  $\beta: G \rightarrow \mathbf{R}^d$  associated to the volume-preserving action  $\alpha$ . Using distortion of amenability when  $\text{supp } m \subset \Omega$ , we have related the  $L_p$ -cb-boundedness of the Fourier multiplier in  $L_p(\mathcal{L}(G))$  associated to  $m$  with the  $L_p$ -cb-boundedness of the  $\Sigma \times \Sigma$  twisted multiplier

$$M_{gh}(\xi) = \dot{m}_g(\xi) = \dot{m}(\alpha_g(\xi)).$$

We shall work with relatively compact  $\Sigma$ 's in what follows. Euclidean harmonic analysis appears to be insufficient to establish  $L_p$ -bounds for this operator —even for small  $\Sigma$  and smooth  $\alpha$ — and noncommutative harmonic analysis tools become necessary. A key result in [30] establishes that Hörmander-Mikhlin multipliers are Littlewood-Paley averages of families of Riesz transforms associated to a fractional laplacian. The lack of finite-dimensional orthogonal cocycles for  $\text{SL}_n(\mathbf{R})$  forces us to express our twisted multiplier in terms of “twisted Riesz transforms” for fractional laplacians. We start with some partial estimates based on these methods.

**2.1. Fractional Riesz transforms.** Given  $0 < \varepsilon < 1$ , let

$$(2.1) \quad \mathcal{H}_\varepsilon = L_2(\mathbf{R}^d, \mu_\varepsilon) \quad \text{with} \quad d\mu_\varepsilon(x) = \frac{dx}{|x|^{d+2\varepsilon}}.$$

The map  $b_\varepsilon: \mathbf{R}^d \rightarrow \mathcal{H}_\varepsilon$  given by  $b_\varepsilon(\xi) = \exp(2\pi i \langle \xi, \cdot \rangle) - 1$  is an orthogonal cocycle with respect to the action  $\alpha_{\varepsilon, \xi}(f) = \exp(2\pi i \langle \xi, \cdot \rangle) f$ . The length function associated to it  $\psi_\varepsilon(\xi) = \langle b_\varepsilon(\xi), b_\varepsilon(\xi) \rangle_{\mathcal{H}_\varepsilon}$  is the fractional laplacian length

$$\psi_\varepsilon(\xi) = 2 \int_{\mathbf{R}^d} (1 - \cos(2\pi \langle \xi, x \rangle)) \frac{dx}{|x|^{d+2\varepsilon}} = c_{d, \varepsilon} |\xi|^{2\varepsilon} \quad \text{with} \quad c_{d, \varepsilon} \approx \frac{\pi^{d/2}}{\Gamma(d/2)} \frac{1}{\varepsilon(1-\varepsilon)},$$

see [30, Example 1.4.A]. As usual, define

$$\mathbf{H}_\alpha^2(\mathbf{R}^d) = \left\{ f: \mathbf{R}^d \rightarrow \mathbf{C} : \|(1 + |\cdot|^2)^{\frac{\alpha}{2}} \widehat{f}\|_{L_2(\mathbf{R}^d)} < \infty \right\}.$$

Define also  $\mathbf{W}_{d, \varepsilon}^2(\mathbf{R}^d)$  as the completion of  $\mathcal{C}_c^\infty(\mathbf{R}^d \setminus \{0\})$  for the norm

$$\|f\|_{\mathbf{W}_{d, \varepsilon}^2(\mathbf{R}^d)} := \left\| | \cdot |^{\frac{d}{2} + \varepsilon} \widehat{\sqrt{\psi_\varepsilon} f} \right\|_{L_2(\mathbf{R}^d)}.$$

**Remark 2.1.** The norm in  $\mathbf{W}_{d, \varepsilon}^2(\mathbf{R}^d)$  is dilation invariant

$$\|f\|_{\mathbf{W}_{d, \varepsilon}^2(\mathbf{R}^d)} = \|f(\lambda \cdot)\|_{\mathbf{W}_{d, \varepsilon}^2(\mathbf{R}^d)} \quad \text{for all } \lambda > 0.$$

We shall use several times along the paper Littlewood-Paley partitions of unity. Given a smooth function  $\eta: \mathbf{R}^d \rightarrow \mathbf{R}_+$  with  $\chi_{B_1(0)} \leq \eta \leq \chi_{B_2(0)}$ , the model of such a partition of unity is

$$(2.2) \quad \varphi_j(\xi) = \left( \eta(2^{-j}\xi) - \eta(2^{1-j}\xi) \right)^{\frac{1}{2}} \quad \text{with } j \in \mathbf{Z}.$$

**Lemma 2.2.** *If  $0 < \varepsilon \leq 1 - \frac{d}{2} + [\frac{d}{2}]$*

$$\begin{aligned} \sup_{j \in \mathbf{Z}} \|\varphi_j^2 M\|_{W_{d,\varepsilon}^2(\mathbf{R}^d)} &\lesssim \sup_{j \in \mathbf{Z}} \|\varphi_0^2 M(2^j \cdot)\|_{H_{\frac{d}{2}+\varepsilon}^2(\mathbf{R}^d)} \\ &\lesssim \max_{0 \leq |\gamma| \leq [\frac{d}{2}]+1} \operatorname{ess\,sup}_{\xi \in \mathbf{R}^d} |\xi^{|\gamma|} |\partial_\xi^\gamma M(\xi)|. \end{aligned}$$

The second inequality in the statement is standard [17, Theorem 5.2.7]. The first inequality follows since  $\varphi_j(\xi) = \varphi_0(2^{-j}\xi)$  and  $W_{d,\varepsilon}^2(\mathbf{R}^d)$  has a dilation invariant norm. The inequality then reduces to show that

$$f \in H_{\frac{d}{2}+\varepsilon}^2(\mathbf{R}^d) \mapsto |\cdot|^\varepsilon \psi f \in H_{\frac{d}{2}+\varepsilon}^2(\mathbf{R}^d)$$

is bounded for a smoothing  $\psi$  of the characteristic function of  $\operatorname{supp} \varphi_0$  vanishing around 0. This easily follows by complex interpolation using endpoint spaces  $H_\alpha^2$  with  $\alpha \in 2\mathbf{Z}_+$ . As noted in [30], the main advantages of working with this new Sobolev norm (instead of the classical space  $H_\alpha^2(\mathbf{R}^d)$ ) come from dilation invariance in Remark 2.1 and, especially, from the crucial result below.

**Lemma 2.3.** *The following map is a unitary*

$$\Psi_\varepsilon : \mathcal{H}_\varepsilon = L_2(\mathbf{R}^d, \mu_\varepsilon) \ni h \mapsto \frac{1}{\sqrt{\psi_\varepsilon(\xi)}} \int_{\mathbf{R}^d} b_\varepsilon(\xi) h d\mu_\varepsilon \in W_{d,\varepsilon}^2(\mathbf{R}^d).$$

Moreover, the space  $W_{d,\varepsilon}^2(\mathbf{R}^d)$  consists of bounded continuous functions on  $\mathbf{R}^d \setminus \{0\}$ .

**Proof.** A first useful observation is that  $\xi \in \mathbf{R}^d \mapsto b_\varepsilon(\xi) \in \mathcal{H}_\varepsilon$  is continuous, and therefore  $\xi \in \mathbf{R}^d \setminus \{0\} \mapsto b_\varepsilon(\xi)/\sqrt{\psi_\varepsilon(\xi)} \in \mathcal{H}_\varepsilon$  is bounded by 1 and continuous. This implies that for every  $h \in \mathcal{H}_\varepsilon$ , the function  $\Psi_\varepsilon(h)$  is a bounded continuous function on  $\mathbf{R}^d \setminus \{0\}$ , and  $\|\Psi_\varepsilon(h)\|_\infty \leq \|h\|_{\mathcal{H}_\varepsilon}$ .

We proceed by constructing first the inverse map. Set  $u(m) := |\cdot|^{d+2\varepsilon} \widehat{\sqrt{\psi_\varepsilon} m}$  for any function  $m$  in  $\mathcal{C}_c^\infty(\mathbf{R}^d \setminus \{0\})$ . Since  $m$  vanishes around 0, it turns out that  $\sqrt{\psi_\varepsilon} m$  belongs to  $\mathcal{C}_c^\infty(\mathbf{R}^d)$  and its Fourier transform is in the Schwartz class. This proves that  $u(m)$  belongs to  $\mathcal{H}_\varepsilon$  and we get

$$\|u(m)\|_{\mathcal{H}_\varepsilon} = \left\| \frac{1}{|\cdot|^{\frac{d}{2}+\varepsilon}} u(m) \right\|_{L_2(\mathbf{R}^d)} = \left\| |\cdot|^{\frac{d}{2}+\varepsilon} \widehat{\sqrt{\psi_\varepsilon} m} \right\|_{L_2(\mathbf{R}^d)} = \|m\|_{W_{d,\varepsilon}^2(\mathbf{R}^d)}.$$

By density,  $u$  extends to an isometry from  $W_{d,\varepsilon}^2(\mathbf{R}^d)$  into  $\mathcal{H}_\varepsilon$ . Moreover, we have

$$\sqrt{\psi_\varepsilon(\xi)} \Psi_\varepsilon(u(m))(\xi) = \int b_\varepsilon(\xi)(x) \widehat{\sqrt{\psi_\varepsilon} m}(x) dx = \langle \widehat{b_\varepsilon(\xi)}, \sqrt{\psi_\varepsilon} m \rangle$$

where  $\widehat{b_\varepsilon(\xi)} = \delta_\xi - \delta_0$  is the Fourier transform of the tempered distribution  $b_\varepsilon(\xi)$ . This gives  $\sqrt{\psi_\varepsilon(\xi)} \Psi_\varepsilon(u(m))(\xi) = \sqrt{\psi_\varepsilon(\xi)} m(\xi)$  as the test function  $\sqrt{\psi_\varepsilon} m$  vanishes at 0. In particular, by the observation made at the beginning of the proof,  $W_{d,\varepsilon}^2(\mathbf{R}^d)$  consists of bounded continuous functions on  $\mathbf{R}^d \setminus \{0\}$ . By density,  $\Psi_\varepsilon \circ u$  is the identity map on  $W_{d,\varepsilon}^2(\mathbf{R}^d)$ .

All we are left to prove is that  $u$  is surjective, or equivalently that the only element  $h$  of  $\mathcal{H}_\varepsilon$  orthogonal to  $u(m)$  for every  $m \in \mathcal{C}_c^\infty(\mathbf{R}^d \setminus \{0\})$  is  $h = 0$ . Given  $h \in \mathcal{H}_\varepsilon$  and  $m \in \mathcal{C}_c^\infty(\mathbf{R}^d \setminus \{0\})$ , we have

$$\int_{\mathbf{R}^d} hu(m) d\mu_\varepsilon = \langle \widehat{h}, \sqrt{\psi_\varepsilon} m \rangle$$



for  $\widehat{h}$  the Fourier transform of  $h$  seen as a tempered distribution. If the preceding is 0 for every  $m \in \mathcal{C}_c^\infty(\mathbf{R}^d \setminus \{0\})$ , we obtain that the support of  $\widehat{h}$  is contained in 0 or equivalently that  $h$  is a polynomial function. But the only polynomial function in  $\mathcal{H}_\varepsilon$  is the zero polynomial. This proves that  $u$  is surjective, as desired.  $\square$

Twisted forms of fractional laplacian Riesz transforms will play a crucial role later in this paper. Let us introduce these maps for future reference. The Riesz transform for the  $\varepsilon$ -fractional laplacian pointing towards  $u \in L_2(\mathbf{R}^d; \mu_\varepsilon)$  is

$$R_{\psi_\varepsilon, u} f(x) = \int_{\mathbf{R}^d} \underbrace{\frac{\langle b_\varepsilon(\xi), u \rangle_{\mathcal{H}_\varepsilon}}{\sqrt{\psi_\varepsilon(\xi)}}}_{\rho_{\varepsilon, u}(\xi)} \widehat{f}(\xi) \exp_\xi(x) d\xi.$$

The associated G-twisted Riesz transforms are

$$(2.3) \quad \widetilde{R}_{\psi_\varepsilon, u}(f) = \left( R_{\psi_\varepsilon, u}^g(fgh) \right)_{g, h \in \Sigma},$$

$$(2.4) \quad \widetilde{R}_{\psi_\varepsilon, \mathbf{u}} \left( \sum_{j \in \mathbf{Z}} f_j \otimes \delta_j \right) = \sum_{j \in \mathbf{Z}} \widetilde{R}_{\psi_\varepsilon, u_j}(f_j) \otimes \delta_j,$$

for the multipliers  $R_{\psi_\varepsilon, u}^g$  with symbol  $\rho_{\varepsilon, u}(\alpha_g(\xi))$  and any family  $\mathbf{u} = (u_j) \subset \mathcal{H}_\varepsilon$ .

**2.2. Calderón-Zygmund methods.** The theory of singular integral operators acting over matrix-valued functions has been recently developed to include endpoint estimates [29, 47]. If  $f$  is a function affiliated to  $\mathcal{R}_\Sigma = L_\infty(\mathbf{R}^d) \otimes \mathcal{B}(L_2(\Sigma))$  and  $k(x, y)$  is a linear operator acting on  $\mathcal{B}(L_2(\Sigma))$  for every  $(x, y) \in \mathbf{R}^{2d} \setminus \Delta$ , the  $k$ -singular integral acting on  $f$  is formally given by

$$Tf(x) = \int_{\mathbf{R}^d} k(x, y) (f(y)) dy \quad \text{for } x \notin \text{supp}_{\mathbf{R}^d} f.$$

The next result follows from [55, Sect VI.4.4 Prop 2 (b)], see also [59, Lemma 4.3].

**Lemma 2.4.** *Given  $A: \mathbf{R}^d \rightarrow \mathcal{H}$  for some Hilbert space  $\mathcal{H}$ , we get*

$$\|A\|_{L_\infty(\mathbf{R}^d; \mathcal{H})} + \int_{|x| \geq 2|w|} \|\widehat{A}(x-w) - \widehat{A}(x)\|_{\mathcal{H}} dx \leq C_{d, \varepsilon} \sup_{j \in \mathbf{Z}} \|\varphi_0^2 A(2^j \cdot)\|_{\mathcal{H}_{\frac{d}{2} + \varepsilon}(\mathbf{R}^k; \mathcal{H})}.$$

In the case of twisted Fourier multipliers, Calderón-Zygmund conditions can be streamlined using Lemma 2.4 and noncommutative techniques. Let us now present these conditions for arbitrary column-twisted multipliers

$$\widetilde{T}_M(f)(x) = \left( \int_{\mathbf{R}^d} M_g(\xi) \widehat{f}_{gh}(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi \right)_{g, h \in \Sigma}.$$

Our analysis requires Hardy and BMO spaces over  $\mathcal{R}_\Sigma$ . Let  $\mathcal{Q}$  be the set of balls in  $\mathbf{R}^d$  and let  $f_Q$  be the  $Q$ -average of a matrix-valued function  $f$ . We also use the standard notation of  $\lambda_Q$  for the ball concentric with  $Q$  and radius multiplied by  $\lambda$ . The Hardy and BMO column-norms of  $f$  are defined in [41] as

$$(2.5) \quad \|f\|_{H_p^c(\mathcal{R}_\Sigma)} = \left\| \left( \int_{\mathbf{R}_+} \left| \frac{\partial}{\partial t} P_t f \right|^2 t dt \right)^{\frac{1}{2}} \right\|_{L_p(\mathcal{R}_\Sigma)},$$

$$(2.6) \quad \|f\|_{\text{BMO}_c^c(\mathcal{R}_\Sigma)} = \sup_{Q \in \mathcal{Q}} \left\| \left( \frac{1}{|Q|} \int_Q |f(x) - f_Q|^2 dx \right)^{\frac{1}{2}} \right\|_{\mathcal{B}(L_2(\Sigma))}.$$

Here  $(P_t)_{t>0}$  denotes the Poisson semigroup

$$P_t(f)(x) = e^{-t(-\Delta)^{\frac{1}{2}}} f(x) = \left( \int_{\mathbf{R}^d} e^{-t|\xi|} \widehat{f}_{gh}(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi \right)_{g,h \in \Sigma}.$$

Let us now consider the following regularity conditions

A) *Uniform Sobolev smoothness*

$$\sup_{g \in \Sigma} \sup_{j \in \mathbf{Z}} \left\| \varphi_0^2 M_g(2^j \cdot) \right\|_{H^{\frac{d}{2} + \varepsilon}(\mathbf{R}^d)} \leq C_{\text{sob}} < \infty.$$

B) *Schur factorization of the twist*

$$M_g(\xi) = \langle A_g, B_\xi \rangle_{\mathcal{K}}$$

for some Hilbert  $\mathcal{K}$  satisfying  $\sup_{(g,\xi) \in \Sigma \times \mathbf{R}^d} \|A_g\|_{\mathcal{K}} \|B_\xi\|_{\mathcal{K}} \leq C_{\text{sch}} < \infty$ .

**Proposition 2.5.** *The following results hold:*

- i) *Condition A implies  $\widetilde{T}_M : L_p(\mathcal{R}_\Sigma) \xrightarrow{\text{cb}} H_p^c(\mathcal{R}_\Sigma)$  for  $2 \leq p < \infty$ .*
- ii) *Conditions A + B imply  $\widetilde{T}_M : L_p(\mathcal{R}_\Sigma) \xrightarrow{\text{cb}} L_p(\mathcal{R}_\Sigma)$  for  $1 < p < \infty$ .*

**Proof.** Lemma 2.4 readily implies that the twisted multiplier  $\widetilde{T}_M$  is completely bounded in  $L_2(\mathcal{R}_\Sigma)$ . On the other hand, Theorem 6.2 in Mei's paper [41] establishes

$$\begin{aligned} H_p^c(\mathcal{R}_\Sigma) &\simeq [\text{BMO}_{\mathcal{R}_\Sigma}^c, L_2(\mathcal{R}_\Sigma)]_{2/p}, \\ L_p(\mathcal{R}_\Sigma) &\simeq [\text{BMO}_{\mathcal{R}_\Sigma}, L_2(\mathcal{R}_\Sigma)]_{2/p}, \end{aligned}$$

where  $\text{BMO}_{\mathcal{R}_\Sigma} = \text{BMO}_{\mathcal{R}_\Sigma}^r \cap \text{BMO}_{\mathcal{R}_\Sigma}^c$  and  $\|f\|_{\text{BMO}_{\mathcal{R}_\Sigma}^r} = \|f^*\|_{\text{BMO}_{\mathcal{R}_\Sigma}^c}$ . It therefore suffices to give estimates in  $\text{BMO}_{\mathcal{R}_\Sigma}^c$  or  $\text{BMO}_{\mathcal{R}_\Sigma}$  assuming A or A+B accordingly.

**i) The  $\text{BMO}_{\mathcal{R}_\Sigma}^c$ -estimate.** We shall only prove boundedness, since cb-boundedness follows from the same argument. Given  $f \in \mathcal{R}_\Sigma$  and  $Q \in \mathcal{Q}$ , set  $f_{1Q} = f\chi_{5Q}$  and  $f_{2Q} = f\chi_{\mathbf{R}^d \setminus 5Q}$ . By the triangle and Kadison-Schwarz inequalities

$$\begin{aligned} &\left\| \left( \frac{1}{|Q|} \int_Q |\widetilde{T}_M f(z) - (\widetilde{T}_M f)_Q|^2 dz \right)^{\frac{1}{2}} \right\|_{\mathcal{B}(L_2(\Sigma))} \\ &\leq 2 \left\| \left( \frac{1}{|Q|} \int_Q |\widetilde{T}_M f_{1Q}(z)|^2 dz \right)^{\frac{1}{2}} \right\|_{\mathcal{B}(L_2(\Sigma))} \\ &\quad + \left\| \left( \frac{1}{|Q|} \int_Q |\widetilde{T}_M f_{2Q}(z) - (\widetilde{T}_M f_{2Q})_Q|^2 dz \right)^{\frac{1}{2}} \right\|_{\mathcal{B}(L_2(\Sigma))} = 2A_{1Q} + A_{2Q}. \end{aligned}$$

Let  $\widetilde{V}_M$  be the  $L_2(\mathbf{R}^d; L_2(\Sigma))$ -bounded map

$$(\widetilde{V}_M f)_g(x) = \int_{\mathbf{R}^d} M_g(\xi) \widehat{f}_g(\xi) e^{-2\pi i \langle x, \xi \rangle} d\xi.$$

$\widetilde{T}_M$  is cb-bounded in  $L_2(\mathcal{R}_\Sigma)$  and a ‘‘right-module’’ since  $(\widetilde{T}_M f)\zeta = \widetilde{V}_M(f\zeta)$  for  $\zeta \in L_2(\Sigma)$ —warning: it is not a left-module— so we may estimate  $A_{1Q}$  as follows

$$\begin{aligned} A_{1Q} &= \sup_{\|\zeta\|_{L_2(\Sigma)}=1} \left( \frac{1}{|Q|} \int_Q \|\widetilde{V}_M(f_{1Q}\zeta)(z)\|_{L_2(\Sigma)}^2 dz \right)^{\frac{1}{2}} \\ &\lesssim \sup_{\|\zeta\|_{L_2(\Sigma)}=1} \left( \frac{1}{|Q|} \int_{\mathbf{R}^d} \|f_{1Q}(z)\zeta\|_{L_2(\Sigma)}^2 dz \right)^{\frac{1}{2}} \leq 5^{\frac{d}{2}} \|f\|_{\mathcal{R}_\Sigma}. \end{aligned}$$

On the other hand, Jensen's inequality gives

$$A_{2Q} \leq \sup_{y,z \in Q} \left\| \widetilde{T}_M f_{2Q}(z) - \widetilde{T}_M f_{2Q}(y) \right\|_{\mathcal{B}(L_2(\Sigma))} =: \sup_{y,z \in Q} B_Q(y, z).$$

Recall that the kernel  $K$  of  $\widetilde{T}_M$  is the matrix  $K(x-y) = \text{diag}(\widehat{M}_g(x-y))_{g \in \Sigma}$  acting by left matrix multiplication. Since we have  $|x-z| \geq 2|y-z|$  for all  $(x, y, z) \in (\mathbf{R}^d \setminus 5Q) \times Q \times Q$ , we may write for  $f_{2Q}^\zeta := f_{2Q}\zeta$

$$\begin{aligned} B_Q(y, z) &\leq \sup_{\|\zeta\|_{L_2(\Sigma)}=1} \left\| \int_{|x-z| \geq 2|y-z|} (K(x-y) - K(x-z)) f_{2Q}^\zeta(x) dx \right\|_{L_2(\Sigma)} \\ &= \sup_{\substack{\|\zeta\|_{L_2(\Sigma)}=1 \\ \|D\|_{L_2(\Sigma)}=1}} \left[ \int_{|x-z| \geq 2|y-z|} \left\langle \underbrace{D, (K(x-y) - K(x-z)) f_{2Q}^\zeta(x)}_{\|D_K(x, y, z)\|} \right\rangle_{L_2(\Sigma)} dx \right], \end{aligned}$$

where  $D_K(x, y, z)$  is the vector in  $L_2(\Sigma)$  given by

$$\overline{D_K^g(x, y, z)} = \overline{D}_g(\widehat{M}_g(x-y) - \widehat{M}_g(x-z)).$$

This readily yields

$$\sup_{Q \in \mathcal{Q}} B_Q(y, z) \leq \sup_{\|D\|_{L_2(\Sigma)}=1} \underbrace{\left[ \int_{|x-z| \geq 2|y-z|} \|D_K(x, y, z)\|_{L_2(\Sigma)} dx \right]}_{D_K(y, z)} \|f\|_{\mathcal{R}_\Sigma},$$

so that it remains to estimate the integral above. Let us consider the change of variables  $(x-z, y-z) \mapsto (x, w)$  and consider  $A(\xi) = (\overline{D}_g M_g(\xi))_{g \in \Sigma} \in L_2(\Sigma)$ . Then it turns out that  $D_K(y, z) = D_K(y-z, 0) = D_K(w)$  coincides with the integral term in the statement of Lemma 2.4. Taking  $B_{gj} = \overline{D}_g \varphi_0^2 M_g(2^j \cdot)$  and  $\alpha = \frac{d}{2} + \varepsilon$ , this yields the desired inequality

$$\begin{aligned} D_K(w) &\leq \sup_{j \in \mathbf{Z}} \|\varphi_0^2 A(2^j \cdot)\|_{\mathbb{H}_{\frac{d}{2}+\varepsilon}^2(\mathbf{R}^d; \mathcal{H})} \\ &= \sup_{j \in \mathbf{Z}} \left( \int_{\mathbf{R}^d} \int_{\Sigma} \left| \int_{\mathbf{R}^d} (1 + |\xi|^2)^{\frac{\alpha}{2}} \widehat{B}_{gj}(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi \right|^2 d\mu(g) dx \right)^{\frac{1}{2}} \\ &= \sup_{j \in \mathbf{Z}} \left( \int_{\Sigma} |D_g|^2 \|\varphi_0^2 M_g(2^j \cdot)\|_{\mathbb{H}_{\frac{d}{2}+\varepsilon}^2(\mathbf{R}^d)}^2 d\mu(g) \right)^{\frac{1}{2}} \lesssim C_{\text{sob}}. \end{aligned}$$

**ii) The  $\text{BMO}_{\mathcal{R}_\Sigma}$ -estimate.** The identity

$$\widetilde{T}_M^* = \widetilde{T}_M$$

holds for the duality pairing  $L_p(\mathbf{R}^d; S_p(L_2(\Sigma)))^* = L_q(\mathbf{R}^d; S_q^{\text{op}}(L_2(\Sigma)))$ . Since conditions A/B are stable under complex conjugation, we may assume  $p > 2$ . By complex interpolation, we are reduced to prove the  $\text{BMO}_{\mathcal{R}_\Sigma}$ -estimate. As we have already justified the column estimate, it remains to prove the row one. In other words, the same inequalities above for the adjoint  $\widetilde{T}_M(f)^*$ . A brief look at our argument for the column case shows that the given estimate for  $A_{2Q}$  is adjoint invariant. Therefore, we just need to prove

$$\sup_{Q \in \mathcal{Q}} \left\| \left( \frac{1}{|Q|} \int_Q |\widetilde{T}_M(f_{1Q})^*(z)|^2 dz \right)^{\frac{1}{2}} \right\|_{\mathcal{B}(L_2(\Sigma))} \lesssim_{\text{cb}} \|f\|_{\mathcal{R}_\Sigma}.$$

Letting  $\tilde{T}_M^\dagger(f) = \tilde{T}_M(f^*)^*$ , this follows from

$$\left\| \left( \int_{\mathbf{R}^d} \tilde{T}_M^\dagger f(x)^* \tilde{T}_M^\dagger f(x) dx \right)^{\frac{1}{2}} \right\|_{\mathcal{B}(L_2(\Sigma))} \lesssim_{\text{cb}} \left\| \left( \int_{\mathbf{R}^d} f(x)^* f(x) dx \right)^{\frac{1}{2}} \right\|_{\mathcal{B}(L_2(\Sigma))}.$$

It is also straightforward to show that we have

$$\tilde{T}_M^\dagger f = \left( T_{M_h}^\dagger (f_{gh}) \right) = \left( T_{\overline{M}_h(-\cdot)} (f_{gh}) \right).$$

Indeed, the  $\dagger$ -operation on  $T_{M_h}$  transforms the Fourier symbol  $M_h(\xi)$  into  $\overline{M}_h(-\xi)$  but Schur factorization is stable under this kind of transformation. In other words, we need to justify the  $L_2$ -column inequality

$$(2.7) \quad \left\| \int_{\mathbf{R}^d} \left| \left( T_{M_h} (f_{gh}) \right) \right|^2(x) dx \right\|_{\mathcal{B}(L_2(\Sigma))}^{\frac{1}{2}} \lesssim_{\text{cb}} \left\| \int_{\mathbf{R}^d} \left| (f_{gh}) \right|^2(x) dx \right\|_{\mathcal{B}(L_2(\Sigma))}^{\frac{1}{2}}$$

According to Plancherel theorem, the left hand side can be written as

$$\text{LHS}_{(2.7)}^2 = \sup_{\|\zeta\|_{L_2(\Sigma)}=1} \int_{\Sigma} \int_{\mathbf{R}^d} \left| \int_{\Sigma} M_h(\xi) \widehat{f}_{gh}(\xi) \zeta(h) d\mu(h) \right|^2 d\xi d\mu(g).$$

Assume that  $M_h(\xi) = \langle A_h, B_\xi \rangle_{\mathcal{K}}$  as in the statement. According to Grothendieck's characterization [49, Proposition 1.1], this is equivalent to being a Schur multiplier in  $\mathcal{B}(S_\infty(L_2(\Sigma), L_2(\mathbf{R}^d)))$ . This yields

$$\text{LHS}_{(2.7)}^2 \leq \sup_{\|\zeta\|_{L_2(\Sigma)}=1} \int_{\Sigma} \int_{\mathbf{R}^d} \left\| \int_{\Sigma} A_h \widehat{f}_{gh}(\xi) \zeta(h) d\mu(h) \right\|_{\mathcal{K}}^2 d\xi d\mu(g) \times \sup_{\xi \in \mathbf{R}^d} \|B_\xi\|_{\mathcal{K}}^2.$$

Now, given an ONB  $(w_j)_{j \geq 1}$  of  $\mathcal{K}$ , we set

$$A\zeta_h^j = \langle A_h \zeta(h), w_j \rangle_{\mathcal{K}} \quad \text{and} \quad A\zeta_j = (A\zeta_h^j)_{h \in \Sigma}$$

and obtain by Plancherel

$$\begin{aligned} \text{LHS}_{(2.7)}^2 &\lesssim \int_{\Sigma} \int_{\mathbf{R}^d} \left\| \int_{\Sigma} A_h f_{gh}(x) \zeta(h) d\mu(h) \right\|_{\mathcal{K}}^2 dx d\mu(g) \\ &= \sum_{j \geq 1} \int_{\Sigma} \int_{\mathbf{R}^d} \left| \int_{\Sigma} A\zeta_h^j f_{gh}(x) d\mu(h) \right|^2 dx d\mu(g) \\ &= \sum_{j \geq 1} \left\langle A\zeta_j, \int_{\mathbf{R}^d} \left| (f_{gh}(x)) \right|^2 dx A\zeta_j \right\rangle_{L_2(\Sigma)} \leq \sum_{j \geq 1} \|A\zeta_j\|_{L_2(\Sigma)}^2 \times \text{RHS}_{(2.7)}^2. \end{aligned}$$

The assertion then follows, since it is clear that  $\sum_{j \geq 1} \|A\zeta_j\|_{L_2(\Sigma)}^2 \leq \sup_{h \in \Sigma} \|A_h\|_{\mathcal{K}}^2$ .  $\square$

**Remark 2.6.** Compared to [29], Proposition 2.5 is also valid for twists not coming from orthogonal actions and Mihlin regularity has been optimized to order  $[\frac{d}{2}] + 1$ .

**Remark 2.7.** The  $L_p$ -norm of noncommutative square functions for  $p > 2$  is defined through the so called  $RC_p$  spaces [51]. Namely, given a family  $f_j \in L_p(\mathcal{M})$

$$(2.8) \quad \left\| \sum_j f_j \otimes \delta_j \right\|_{L_p(\mathcal{M}; RC_p)} = \max \left\{ \left\| \left( \sum_j f_j^* f_j \right)^{\frac{1}{2}} \right\|_p, \left\| \left( \sum_j f_j f_j^* \right)^{\frac{1}{2}} \right\|_p \right\}.$$

Proposition 2.5 adapts to families  $(T_{M_j}) : L_p(\mathcal{R}_\Sigma) \rightarrow L_p(\mathcal{R}_\Sigma; RC_p)$  of twisted multipliers. Condition A becomes an  $\ell_2$ -valued Hörmander-Mihlin condition for the symbols  $(M_{jg})$  uniformly in  $g \in \Sigma$ . Condition B is more interesting. Consider  $(g, j, \xi) \in \Sigma \times \mathbf{Z} \times \mathbf{R}^d$ . The variable  $g \in \Sigma$  is always a column variable. The variable

$j \in \mathbf{Z}$  is a row/column according to whether we take values in  $R_p$  or  $C_p$ . Finally the variable  $\xi \in \mathbf{R}^d$  is a row/column according to whether we want a  $\text{BMO}_{\mathcal{R}_\Sigma}^r$  or a  $\text{BMO}_{\mathcal{R}_\Sigma}^c$  estimate respectively. Thus, we get:

i) For  $L_p(\mathcal{R}_\Sigma) \rightarrow L_p(\mathcal{R}_\Sigma; R_p)$ , we need

$$M_{jg}(\xi) = \langle A_g, B_{\xi j} \rangle_{\mathcal{K}_1} = \langle A'_{g\xi}, B'_j \rangle_{\mathcal{K}_2}$$

where  $\sup_{g,j,\xi} \|A_g\|_{\mathcal{K}_1} \|B_{\xi j}\|_{\mathcal{K}_1} + \sup_{g,j,\xi} \|A'_{g\xi}\|_{\mathcal{K}_2} \|B'_j\|_{\mathcal{K}_2}$  is finite.

ii) For  $L_p(\mathcal{R}_\Sigma) \rightarrow L_p(\mathcal{R}_\Sigma; C_p)$ , we need

$$M_{jg}(\xi) = \langle A_{gj}, B_\xi \rangle_{\mathcal{K}_1} = \langle A'_{gj\xi}, \mathbf{1} \rangle_{\mathcal{K}_2}$$

where  $\sup_{g,j,\xi} \|A_{gj}\|_{\mathcal{K}_1} \|B_\xi\|_{\mathcal{K}_1} + \sup_{g,j,\xi} \|A'_{gj\xi}\|_{\mathcal{K}_2}$  is again finite.

**2.3. The twisted column estimate.** Twisted Riesz transforms  $\tilde{R}_{\psi_\varepsilon, u}$  and  $\tilde{R}_{\psi_\varepsilon, \mathbf{u}}$  were introduced in (2.3) and (2.4). Given  $p > 2$ , in this section we shall establish the column inequality  $L_p(\mathcal{R}_\Sigma; C_p) \rightarrow L_p(\mathcal{R}_\Sigma; C_p)$  for  $\tilde{R}_{\psi_\varepsilon, \mathbf{u}}$  and certain families of directions  $\mathbf{u} = (u_j)$  in  $\mathcal{H}_\varepsilon$  which are determined by a symbol  $M$  satisfying the Hörmander-Mikhlin conditions. Recall that the symbol of  $R_{\psi_\varepsilon, u_j}^g$  equals

$$(2.9) \quad \rho_{\varepsilon, u_j}(\alpha_g(\xi)) = \frac{\langle b_\varepsilon(\alpha_g(\xi)), u_j \rangle_{\mathcal{H}_\varepsilon} |\xi|^\varepsilon}{\sqrt{\psi_\varepsilon(\xi)} |\alpha_g(\xi)|^\varepsilon} = \frac{\langle b_\varepsilon(\xi), u_{gj} \rangle_{\mathcal{H}_\varepsilon} |\xi|^\varepsilon}{\sqrt{\psi_\varepsilon(\xi)} |\alpha_g(\xi)|^\varepsilon}.$$

Indeed, the last identity follows from

$$\begin{aligned} \langle b_\varepsilon(\alpha_g(\xi)), u_j \rangle_{\mathcal{H}_\varepsilon} &= \int_{\mathbf{R}^d} (\exp_{\alpha_g(\xi)}(x) - 1) u_j(x) d\mu_\varepsilon(x) \\ &= \int_{\mathbf{R}^d} (\exp_\xi(s) - 1) u_j(\alpha'_g(s)) \underbrace{\frac{|s|^{d+2\varepsilon}}{|\alpha'_g(s)|^{d+2\varepsilon}}}_{u_{gj}(s)} d\mu_\varepsilon(s) \end{aligned}$$

with  $\alpha'_g = (\alpha_g^*)^{-1}$ . In particular,  $\tilde{R}_{\psi_\varepsilon, u_j} = R_{\psi_\varepsilon, \tilde{u}_j} \circ \tilde{H}$  with:

- A “homogeneous twisted” multiplier

$$\tilde{H}(f)(x) = \left( \int_{\mathbf{R}^n} \frac{|\xi|^\varepsilon}{|\alpha_g(\xi)|^\varepsilon} \hat{f}_{gh}(\xi) \exp_\xi(x) d\xi \right)_{g,h \in \Sigma}.$$

- A “multidirectional” Riesz transform

$$\begin{aligned} R_{\psi_\varepsilon, \tilde{u}_j}(f)(x) &= \left( R_{\psi_\varepsilon, u_{gj}}(f_{gh}) \right)_{g,h \in \Sigma} \\ &= \left( \int_{\mathbf{R}^n} \frac{\langle b_\varepsilon(\xi), u_{gj} \rangle_{\mathcal{H}_\varepsilon}}{\sqrt{\psi_\varepsilon(\xi)}} \hat{f}_{gh}(\xi) \exp_\xi(x) d\xi \right)_{g,h \in \Sigma}. \end{aligned}$$

Therefore, it suffices to prove that

$$\begin{aligned} \|\tilde{H}(f)\|_{L_p(\mathcal{R}_\Sigma)} &\leq_{\text{cb}} C_{p,d}(\Sigma) \|f\|_{L_p(\mathcal{R}_\Sigma)}, \\ \left\| \sum_{j \in \mathbf{Z}} R_{\psi_\varepsilon, \tilde{u}_j}(f_j) \otimes e_{j1} \right\|_{L_p(\mathcal{R}_\Sigma; C_p)} &\leq_{\text{cb}} C_{p,d}(\Sigma) \left\| \sum_{j \in \mathbf{Z}} f_j \otimes e_{j1} \right\|_{L_p(\mathcal{R}_\Sigma; C_p)}. \end{aligned}$$

### 2.3.1. Multidirectional Riesz transforms.

**Proposition 2.8.** *Let  $p \geq 2$  and  $\Sigma$  relatively compact, then*

$$\left\| \sum_{j \in \mathbf{Z}} R_{\psi_\varepsilon, \tilde{u}_j}(f_j) \otimes e_{j1} \right\|_{L_p(\mathcal{R}_\Sigma; C_p)} \leq_{\text{cb}} C_{p,d}(\Sigma) \left\| \sum_{j \in \mathbf{Z}} f_j \otimes e_{j1} \right\|_{L_p(\mathcal{R}_\Sigma; C_p)}$$

provided  $\frac{\langle b_\varepsilon(\xi), u_j \rangle}{\sqrt{\psi_\varepsilon(\xi)}} = \varphi_j M$  for some  $M$  satisfying that  $\sup_{j \in \mathbf{Z}} \|\varphi_0^2 M(2^j \cdot)\|_{H^{\frac{d}{2} + \varepsilon}} < \infty$ .

**Proof.** Given an orthonormal basis  $e_1, e_2, \dots$  of  $\mathcal{H}_\varepsilon$ ,  $u_{gj} = \sum_k \langle u_{gj}, e_k \rangle e_k$ . This gives  $R_{\psi_\varepsilon, u_{gj}} = \sum_k \langle u_{gj}, e_k \rangle R_{\psi_\varepsilon, e_k}$ . By [30, Theorem A1 and Remark 1.8] we obtain

$$\left\| \sum_{j \in \mathbf{Z}} \sum_{k \geq 1} R_{\psi_\varepsilon, e_k}(f_j) \otimes e_{j1} \otimes e_{k1} \right\|_{L_p(\mathcal{R}_\Sigma; C_p(\mathbf{N} \times \mathbf{Z}))} \leq_{\text{cb}} C_p \left\| \sum_{j \in \mathbf{Z}} f_j \otimes e_{j1} \right\|_{L_p(\mathcal{R}_\Sigma; C_p(\mathbf{Z}))}.$$

It therefore suffices to prove that

$$(2.10) \quad \Lambda_u : \sum_{j \in \mathbf{Z}} \sum_{k \geq 1} \left( a_{gh}^{jk} \right) \otimes e_{j1} \otimes e_{k1} \mapsto \sum_{j \in \mathbf{Z}} \left( \sum_{k \geq 1} \langle u_{gj}, e_k \rangle a_{gh}^{jk} \right) \otimes e_{j1}$$

is a cb-bounded map  $\Lambda_u : S_p(L_2(\Sigma); C_p(\mathbf{N} \times \mathbf{Z})) \rightarrow S_p(L_2(\Sigma); C_p(\mathbf{Z}))$  for  $p \geq 2$ . Factorization of  $S_p = C_p \otimes_h R_p$  in terms of the Haagerup tensor product [50] yields

$$\begin{aligned} S_p(L_2(\Sigma); C_p(\mathbf{Z})) &= C_p(L_2(\Sigma) \otimes_2 \ell_2(\mathbf{Z})) \otimes_h R_p(L_2(\Sigma)), \\ S_p(L_2(\Sigma); C_p(\mathbf{N} \times \mathbf{Z})) &= C_p(L_2(\Sigma) \otimes_2 \ell_2(\mathbf{N} \times \mathbf{Z})) \otimes_h R_p(L_2(\Sigma)). \end{aligned}$$

The map  $\Lambda_u$  acts trivially on  $R_p(L_2(\Sigma))$  —the  $h$ -variable— so that cb-boundedness is equivalent to that of the map  $C_p(L_2(\Sigma) \otimes_2 \ell_2(\mathbf{N} \times \mathbf{Z})) \rightarrow C_p(L_2(\Sigma) \otimes_2 \ell_2(\mathbf{Z}))$ . On the other hand,  $C_p$ -spaces are homogeneous Hilbertian operator spaces [51], which reduces the problem to prove boundedness instead of complete boundedness. Now the Hilbertian nature of the spaces make them isomorphic for all values of  $p$  and it suffices to prove the boundedness of  $\Lambda_u$  in  $S_2[C_2]$ . We have

$$\begin{aligned} \|\Lambda_u(a)\|_{S_2[C_2]}^2 &= \int_{\Sigma \times \Sigma} \sum_{j \in \mathbf{Z}} \left| \sum_{k \geq 1} \langle u_{gj}, e_k \rangle a_{gh}^{jk} \right|^2 d\mu(g) d\mu(h) \\ &\leq \sup_{\substack{j \in \mathbf{Z} \\ g \in \Sigma}} \sum_{k \geq 1} \left| \langle u_{gj}, e_k \rangle \right|^2 \|a\|_{S_2[C_2]}^2 = \sup_{\substack{j \in \mathbf{Z} \\ g \in \Sigma}} \|u_{gj}\|_{\mathcal{H}_\varepsilon}^2 \|a\|_{S_2[C_2]}^2. \end{aligned}$$

By compactness of  $\Sigma \subset \mathbf{G}$

$$(2.11) \quad \begin{aligned} \sup_{g \in \Sigma} \|u_{gj}\|_{\mathcal{H}_\varepsilon}^2 &= \sup_{g \in \Sigma} \int_{\mathbf{R}^d} |u_j(\alpha'_g(s))|^2 \frac{|s|^{2d+4\varepsilon}}{|\alpha'_g(s)|^{2d+4\varepsilon}} \frac{ds}{|s|^{d+2\varepsilon}} \\ &= \sup_{g \in \Sigma} \int_{\mathbf{R}^d} |u_j(s)|^2 \frac{|\alpha_g^*(s)|^{d+2\varepsilon}}{|s|^{d+2\varepsilon}} d\mu_\varepsilon(s) \leq C_d(\Sigma) \|u_j\|_{\mathcal{H}_\varepsilon}^2. \end{aligned}$$

The uniform bound for  $u_j = \Psi_\varepsilon^{-1}(\varphi_j M)$  follows from Lemmas 2.2 and 2.3.  $\square$

**Remark 2.9.** If  $p > 2$ , it turns out that

$$\left\| \tilde{T}_{\tilde{m}} : L_p(\mathcal{R}_\Sigma) \rightarrow L_p(\mathcal{R}_\Sigma) \right\|_{\text{cb}} \leq C_{p,d}(\Sigma) \left\| \tilde{R}_{\psi_\varepsilon, \mathbf{u}} : L_p(\mathcal{R}_\Sigma; RC_p) \rightarrow L_p(\mathcal{R}_\Sigma; RC_p) \right\|_{\text{cb}}$$

for some Riesz directions  $\mathbf{u} = (u_j)$  dictated by  $\tilde{m}$ . It follows from Propositions 2.8 and 2.11 below that the column estimate holds. However, the first factorization in Remark 2.7 i) fails for twisted Riesz transforms, and everything indicates that the row estimate does not hold in general. This is partly explained from the asymmetric

nature of our twist. This will difficulty will be sorted out in the next section using a new Littlewood-Paley type inequality for Schur multipliers, together with a local inversion trick which allows to write the row estimates in terms of column ones.

2.3.2. *The homogeneous twisted multiplier.* We now study the map  $\tilde{H}$  using the above Calderón-Zygmund methods. In addition, we need a standard Sobolev bound for Schur multipliers which we now recall. Given a pair of cubes  $Q_j \subset \mathbf{R}^{d_j}$ , let  $S : Q_1 \times Q_2 \rightarrow \mathbf{C}$ . The following result establishes a sufficient condition on  $S$  to be a Schur multiplier in  $\mathcal{B}(S_\infty(L_2(Q_1), L_2(Q_2)))$ .

**Lemma 2.10.** *If  $\gamma = (1, 1, \dots, 1) \in \mathbf{R}^{d_1+d_2}$*

$$\begin{aligned} & \left\| \left( S(x, y) A(x, y) \right) \right\|_{S_\infty(L_2(Q_1), L_2(Q_2))} \\ & \leq_{\text{cb}} \sum_{\rho \leq \gamma} \left\| \partial^\rho S \right\|_{L_2(Q_1 \times Q_2)} \left\| \left( A(x, y) \right) \right\|_{S_\infty(L_2(Q_1), L_2(Q_2))}. \end{aligned}$$

**Proof.** If  $\ell_j = \text{length}(Q_j)$  and  $\mathbf{Z}_j = \ell_j^{-1} \mathbf{Z}^{d_j}$

$$S(x, y) = \sum_{(p, q) \in \mathbf{Z}_1 \times \mathbf{Z}_2} \widehat{S}(p, q) \underbrace{\exp(2\pi i \langle x, p \rangle)}_{u_x(p)} \underbrace{\exp(2\pi i \langle y, q \rangle)}_{u_y(q)}.$$

By Grothendieck's characterization [49, Proposition 1.1], it suffices to factorize the symbol  $S(x, y) = \langle A_x, B_y \rangle_{\mathcal{K}}$  for some Hilbert space  $\mathcal{K}$  with uniformly bounded vectors  $A_x, B_y \in \mathcal{K}$ . Using the Fourier expansion above, this will be the case when the Fourier coefficients are summable, since we may pick

$$\begin{aligned} A_x &= \sum_{(p, q) \in \mathbf{Z}_1 \times \mathbf{Z}_2} |\widehat{S}(p, q)|^{\frac{1}{2}} u_x(p) \otimes \delta_{p, q}, \\ B_y &= \sum_{(p, q) \in \mathbf{Z}_1 \times \mathbf{Z}_2} |\widehat{S}(p, q)|^{\frac{1}{2}} \text{sgn}(\widehat{S}(p, q)) u_y(q) \otimes \delta_{p, q}. \end{aligned}$$

Elementary integration by parts and Plancherel theorem give

$$\sum_{(p, q) \in \mathbf{Z}_1 \times \mathbf{Z}_2} |\widehat{S}(p, q)| \lesssim \sum_{\rho \leq \gamma} \left\| \partial^\rho S \right\|_{L_2(Q_1 \times Q_2)}.$$

Multi-indices of order  $j$  are used for  $(p, q)$  with  $d_1 + d_2 - j$  vanishing entries.  $\square$

**Proposition 2.11.** *If  $p \geq 2$*

$$\left\| \tilde{H}(f) \right\|_{L_p(\mathcal{R}_\Sigma)} \leq_{\text{cb}} C_{p, d}(\Sigma) \|f\|_{L_p(\mathcal{R}_\Sigma)}.$$

**Proof.** According to Proposition 2.5, it suffices to show that the twisted symbol is uniformly Sobolev-smooth and admits Schur factorization. The stronger Mihklin smoothness condition  $\sup_g |\partial_\xi^\gamma M_g(\xi)| \leq C_\Sigma |\xi|^{-|\gamma|}$  for  $M_g(\xi) = |\xi|^\varepsilon |\alpha_g(\xi)|^{-\varepsilon}$  holds for all multi-index  $\gamma$ . This is a simple exercise which follows from the compactness of  $\Sigma$ . It remains to prove Schur's factorization. Equivalently

$$\frac{|\xi|^\varepsilon}{|\alpha_g(\xi)|^\varepsilon} = \frac{1}{|\alpha_g(\frac{\xi}{|\xi|})|^\varepsilon} \in \mathcal{B}(S_\infty(L_2(\Sigma), L_2(\mathbf{R}^d)))$$

as a Schur multiplier in  $(g, \xi) \in \Sigma \times \mathbf{R}^d$ . Of course, Schur factorization is stable by composition, so we may replace  $(g, \xi)$  by  $(\alpha_g, \xi) \in \Pi \times \mathbf{R}^d$  for some compact set  $\Pi \subset \text{SL}_d(\mathbf{R})$ . Moreover, since boundedness and cb-boundedness are equivalent for

this class of Schur multipliers, we may use the homogeneity of our symbol to reduce it to  $(\alpha_g, \xi) \in \Pi \times \mathbf{S}^{d-1}$ . The set  $\Pi \times \mathbf{S}^{d-1}$  is a compact manifold in  $\mathbf{R}^{d^2} \times \mathbf{R}^d$  and our symbol  $H(g, \xi) = H_g(\xi)$  admits a smooth extension —still denoted by  $H$ — to an open neighborhood of it. In particular, since row/column restriction is a continuous operation for Schur multipliers, we may cover that open (relatively compact) set by a finite number  $Q_{1j} \times Q_{2j} \subset \mathbf{R}^{d^2} \times \mathbf{R}^d$  of pairs of cubes. This gives

$$\|H\|_{\mathcal{B}(S_\infty(L_2(\Sigma), L_2(\mathbf{S}^{d-1})))} \leq \sum_{j=1}^N \|H\|_{\mathcal{B}(S_\infty(L_2(Q_{1j}), L_2(Q_{2j})))}.$$

The assertion follows from Lemma 2.10 and the smoothness of our symbol.  $\square$

### 3. Proof of Theorem A

As explained in the Introduction, the main challenge in the proof of Theorem A comes from the local behavior of the multiplier around the singularity, whereas the asymptotic behavior will follow at the end from an elementary patching argument due to the exponential nature of the metric. Thus, we assume momentarily that  $m: \mathbf{G} \rightarrow \mathbf{C}$  is supported in a compact neighborhood around the identity.

#### 3.1. Local inversion. Set

$$\mathcal{I}(A) = (A + e)^{-1} - e \quad \text{for } A \in GL_n(\mathbf{R}) - e.$$

Let  $K$  be a compact set in  $GL_n(\mathbf{R}) - e$  containing 0. In the following we shall use local stability properties of  $\mathcal{I}$  relative to  $K$ , which we now collect. Let us fix  $d = n^2$  for the rest of this section and let  $(\varphi_j)_{j \in \mathbf{Z}}$  be the Littlewood-Paley partition of unity (2.2) in  $\mathbf{R}^d$ . Construct the partition of unity

$$(3.1) \quad \sigma_j = \frac{1}{2N_K + 1} \sum_{k=j-N_K}^{j+N_K} \varphi_k^2 \quad \text{for some } N_K \in \mathbf{N}.$$

Let us recall that  $(2N_K + 1)\sigma_j \equiv 1$  in the set  $\{\xi \in \mathbf{R}^d : 2^{j-N_K} \leq |\xi| \leq 2^{j+N_K}\}$ .

**Lemma 3.1.** *If  $\text{supp } m \subset \mathcal{I}(K)$  and  $\mu_j = (\sigma_j m) \circ \mathcal{I}$*

$$\sup_{j \in \mathbf{Z}} \|\mu_j\|_{W_{d,\varepsilon}^2(\mathbf{R}^d)} \leq C_{K,d} \sup_{j \in \mathbf{Z}} \|\varphi_0^2 m(2^j \cdot)\|_{H_{\frac{d}{2}+\varepsilon}^2(\mathbf{R}^d)}.$$

**Proof.** If  $A \in K$  and  $|\cdot|$  denotes the Hilbert-Schmidt norm, we first observe that  $|\mathcal{I}(A)| \approx |A|$  up to a constant  $C_K$  depending only of  $K$ . Indeed, matrix inversion is operator Lipschitz on any compact  $K + e$  in  $GL_n(\mathbf{R})$ , so we get the upper estimate over  $K$  with some constant  $c_K$ . The lower estimate follows since  $A = \mathcal{I}((A+e)^{-1} - e)$  and  $\mathcal{I}(K)$  is another compact set in  $GL_n(\mathbf{R}) - e$ .

We shall use as well a standard fact on Sobolev norms of composition by smooth functions [58]. Assume that  $\text{supp } f \subset K \subset \Delta \subset \overline{\Delta} \subset \Lambda$  for some open domains  $\Delta$  and  $\Lambda$ . Given  $\Phi \in C^\infty(\Lambda)$  and  $\Psi: \Lambda \rightarrow \Lambda$  a diffeomorphism satisfying  $\Psi^2 = id$ , let us set

$$a = \max_{|\gamma| \leq [\alpha]+1} \|\partial^\gamma \Psi\|_{L^\infty(\Psi(\Delta))}.$$

Then, the following inequality holds

$$(3.2) \quad \|f \circ \Psi\|_{H_a^2(\mathbf{R}^d)} \leq C_{K,a,\alpha} \|f\|_{H_a^2(\mathbf{R}^d)}.$$



Fix a relatively compact open set  $\Delta$  in  $\mathrm{GL}_n(\mathbf{R}) - e$  containing  $\mathbf{K}$  as above, on which the derivatives up to some order of  $\mathcal{I}$  over  $\mathcal{I}(\Delta)$  can be controlled by a constant determined by  $\mathbf{K}$ . Applying the triangular inequality, the dilation invariance in  $W_{d,\varepsilon}^2(\mathbf{R}^d)$  and Lemma 2.2 (for a fixed  $j$ ), we get

$$\|\mu_j\|_{W_{d,\varepsilon}^2(\mathbf{R}^d)} \leq \sup_{\ell \in \mathbf{Z}} \|[(\varphi_\ell^2 m) \circ \mathcal{I}](2^\ell \cdot)\|_{H^2_{\frac{d}{2}+\varepsilon}(\mathbf{R}^d)}.$$

As in the proof of Lemma 2.2, this requires to check that  $[(\varphi_\ell^2 m) \circ \mathcal{I}](2^\ell \cdot)$  is supported in certain corona around the unit sphere of  $\mathbf{R}^d$  which is independent of the value of  $\ell \in \mathbf{Z}$ . Indeed, we know that  $\mathrm{supp} m \circ \mathcal{I}(2^\ell \cdot) \subset 2^{-\ell} \mathbf{K}$ . Given  $\xi \in 2^{-\ell} \mathbf{K}$  and according to our first observation, we get  $|\mathcal{I}(2^\ell \xi)| \approx |2^\ell \xi|$  up to a constant determined by  $\mathbf{K}$  but independent of  $\ell \in \mathbf{Z}$ . This proves the required condition. Let us now define  $\mathcal{I}_\ell(\xi) = 2^{-\ell} \mathcal{I}(2^\ell \xi)$ , so that  $[(\varphi_\ell^2 m) \circ \mathcal{I}](2^\ell \cdot) = (\varphi_0^2 m(2^\ell \cdot)) \circ \mathcal{I}_\ell = f_\ell \circ \Psi_\ell$  where  $f_\ell = \varphi_0^2 m(2^\ell \cdot)$  is supported by  $2^{-\ell} \mathcal{I}(\mathbf{K}) \cap \{\frac{1}{2} \leq |\xi| \leq 2\}$  and  $\Psi_\ell = \mathcal{I}_\ell$  is defined in  $\Lambda_\ell = 2^{-\ell}(\mathrm{GL}_n(\mathbf{R}) - e)$ . It is clear that  $f_\ell = 0$  when  $2^\ell > 2 \mathrm{diam}(\mathcal{I}(\mathbf{K}))$ . Using our observation one more time, this implies that  $2^\ell \leq 2C_{\mathbf{K}} \mathrm{diam}(\mathbf{K})$ . Moreover, given  $\Delta_\ell = 2^{-\ell} \Delta$  around  $2^{-\ell} \mathbf{K}$ , we get by construction

$$\max_{0 \leq |\gamma| \leq [\frac{d}{2} + \varepsilon] + 1} \|\partial^\gamma \Psi_\ell\|_{L^\infty(\Psi_\ell(\Delta_\ell))} \leq 2^{\ell|\gamma| - 1} C'_K \leq C''_K \mathrm{diam}(\mathbf{K})^{|\gamma|}.$$

Therefore, the assertion in the statement follows from inequality (3.2) above.  $\square$

**3.2. The local theorem.** We have now all the ingredients to prove the local form of Theorem A. We begin with an  $S_p$ -column inequality for Schur multipliers in  $\mathrm{SL}_n(\mathbf{R})$  which satisfy a uniform Mikhlin bound.

**Proposition 3.2.** *Let  $S = (m_j)_{j \in \mathbf{Z}}$  be a sequence of functions  $\mathrm{SL}_n(\mathbf{R}) \rightarrow \mathbf{C}$  and let  $\Sigma \subset \mathrm{SL}_n(\mathbf{R})$  be a relatively compact subset. Assume that  $m_j(g) = \hat{m}_j(g - e)$  for certain  $\hat{m}_j: \mathbf{R}^d \rightarrow \mathbf{C}$  satisfying*

$$C_{\mathrm{hm}}(S) := \sup_{j \in \mathbf{Z}} \|\hat{m}_j\|_{W_{d,\varepsilon}^2(\mathbf{R}^d)} < \infty \quad \text{for some } 0 < \varepsilon < 1.$$

Then, for any  $1 < p < \infty$  and any sequence  $A_j \in S_p(L_2(\Sigma))$ , we have

$$\left\| \sum_{j \in \mathbf{Z}} S_{m_j}(A_j) \otimes \delta_j \right\|_{S_p[C_p]} \leq C_{p,d}(\Sigma) C_{\mathrm{hm}}(S) \left\| \sum_{j \in \mathbf{Z}} A_j \otimes \delta_j \right\|_{S_p[C_p]}.$$

**Proof.** By a straightforward adaptation of the proof of Proposition 1.6, the norm on  $S_p[C_p]$  of the map in the statement is bounded above by norm on  $L_p(\mathcal{R}_\Sigma; C_p)$  of the map

$$\tilde{T}_S : \sum_{j \in \mathbf{Z}} a_j \otimes \delta_j \mapsto \sum_{j \in \mathbf{Z}} \tilde{T}_{\hat{m}_j}(a_j) \otimes \delta_j.$$

Moreover, Lemma 2.3 yields

$$\tilde{T}_{\hat{m}_j} = \tilde{R}_{\psi_\varepsilon, u_j}$$

for a sequence  $(u_j)_j$  in  $\mathcal{H}_\varepsilon = L_2(\mathbf{R}^d, \mu_\varepsilon)$  which is uniformly bounded in norm by the constant  $C_{\mathrm{hm}}((m_j)_j)$ . Next, we follow the exact same argument as in Section 2.3 and factor

$$\tilde{R}_{\psi_\varepsilon, u_j} = R_{\psi_\varepsilon, \tilde{u}_j} \circ \tilde{H}.$$

Therefore,  $\tilde{T}_S$  is bounded above by the product of the completely bounded norm of  $\tilde{H}$  with the norm on  $L_p(\mathcal{R}_\Sigma; C_p)$  of  $\sum_j a_j \otimes \delta_j \mapsto \sum_j R_{\psi_\varepsilon, \tilde{u}_j}(a_j) \otimes \delta_j$ . The assertion then follows by applying Proposition 2.8 and Proposition 2.11.  $\square$

**Theorem 3.3.** *Let  $\Sigma$  be a relatively compact subset of  $G = SL_n(\mathbf{R})$  and  $m: G \rightarrow \mathbf{C}$ . Assume  $m(g) = \dot{m}(g-e)$  for some  $\dot{m}: \mathbf{R}^d \rightarrow \mathbf{C}$  satisfying the Mihlin type condition*

$$|\partial_\xi^\gamma \dot{m}(\xi)| \leq |\xi|^{-|\gamma|} \quad \text{for all } 0 \leq |\gamma| \leq \left\lceil \frac{n^2}{2} \right\rceil + 1.$$

*Then, the following inequality holds for each  $1 < p < \infty$*

$$\|S_m: S_p(L_2(\Sigma)) \rightarrow S_p(L_2(\Sigma))\|_{\text{cb}} \leq C_p(\Sigma).$$

**Proof.** We shall prove that  $\|S_m(A)\|_p \leq C_p(\Sigma)\|A\|_p$  for  $A \in S_p(L_2(\Sigma))$ . The completely bounded norm is also dominated by the same constant with the same argument. Alternatively, we can use [36, Theorem 1.19], which implies that, if  $\Sigma$  is open (which we can always assume by enlarging it), the norm and completely bounded norm of the Schur multiplier  $S_m$  coincide. Next, by duality we may restrict to the case  $p \geq 2$ . Moreover, by replacing  $\dot{m}$  by  $\psi \dot{m}$  for a suitable compactly supported smooth function  $\psi$  equal to 1 on  $\Sigma\Sigma^{-1}$ , the Schur multiplier  $S_m$  is not affected while the Mihlin condition for  $\dot{m}$  holds up to a constant  $C_\Sigma$  determined by  $\Sigma$ . Therefore, we may and will assume that there is a compact subset  $K$  in  $GL_n(\mathbf{R}) - e$  such that  $\dot{m} = 0$  outside of  $K$ .

**A. Reduction to an  $RC_p$  inequality.** By Remark 1.7, the classical Littlewood theorem can be transferred to twisted Schur multipliers. More precisely, consider  $\varphi_j$  as in (2.2) and the Schur multiplier  $S_j$  on  $L_2(G)$  with symbol

$$(g, h) \mapsto \varphi_j^2(\alpha_{g^{-1}}(\beta(gh^{-1}))) = \varphi_j^2(\beta(h^{-1}) - \beta(g^{-1})) \quad \text{for } \beta(g) = g - e.$$

This gives an unconditional decomposition of the identity

$$(3.3) \quad \left\| \sum_{j \in \mathbf{Z}} S_j(A) \otimes \delta_j \right\|_{S_p[RC_p]} \approx \|A\|_{S_p(L_2(G))},$$

see (2.8) for the definition of  $S_p[RC_p]$ . Next, we construct a partition of unity of the form (3.1). More precisely, pick a positive integer  $N_\Sigma$  (to be fixed) and consider

$$\sigma_j = \frac{1}{2N_\Sigma + 1} \sum_{k=j-N_\Sigma}^{j+N_\Sigma} \varphi_k^2.$$

It is clear that  $\sum_j \sigma_j = 1$  and

- $\text{supp } \varphi_j \subset \{\xi \in \mathbf{R}^d : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$ ,
- $(2N_\Sigma + 1)\sigma_j \equiv 1$  in  $\{\xi \in \mathbf{R}^d : 2^{j-N_\Sigma} \leq |\xi| \leq 2^{j+N_\Sigma}\}$ .

Thus, since  $\Sigma$  is relatively compact in  $SL_n(\mathbf{R})$  and the action  $\alpha_g(h) = gh$  is continuous, there must exist  $N_\Sigma \in \mathbf{Z}_+$  determined by  $\Sigma$  such that  $(2N_\Sigma + 1)\sigma_j(\alpha_g(\xi)) \equiv 1$  for all  $(j, g, \xi) \in \mathbf{Z} \times \Sigma \times \text{supp } \varphi_j$ . This yields for  $(g, h) \in \Sigma \times \Sigma$

$$\varphi_j^2(\alpha_{g^{-1}}(\beta(gh^{-1})))m(gh^{-1}) = (2N_\Sigma + 1)\varphi_j^2(\alpha_{g^{-1}}(\beta(gh^{-1})))\sigma_j(\beta(gh^{-1}))m(gh^{-1}).$$

This means that

$$S_j(S_m(A)) = (2N_\Sigma + 1)S_{m_j}(S_j(A))$$

for  $m_j = (\sigma_j \circ \beta)m$ . Therefore, by (3.3) the assertion is equivalent to

$$(3.4) \quad \left\| \sum_{j \in \mathbf{Z}} S_{m_j}(A_j) \otimes \delta_j \right\|_{S_p[RC_p]} \leq C_p(\Sigma) \left\| \sum_{j \in \mathbf{Z}} A_j \otimes \delta_j \right\|_{S_p[RC_p]}$$

for  $A_j = S_j(A)$ . We shall prove (3.4) for every family  $(A_j)_{j \in \mathbf{Z}} \in S_p[RC_p]$ .

**B. Rows to columns by local inversion.** We have

$$S_{m_j}(A^*)^* = S_{\mu_j}(A) \quad \text{for} \quad \mu_j(g) = \overline{m_j(g^{-1})}.$$

This allows then to write the row term in (3.4) as follows

$$\left\| \sum_{j \in \mathbf{Z}} S_{m_j}(A_j) \otimes \delta_j \right\|_{S_p[R_p]} = \left\| \sum_{j \in \mathbf{Z}} S_{\mu_j}(A_j^*) \otimes \delta_j \right\|_{S_p[C_p]}.$$

Therefore, the  $RC_p$ -inequality (3.4) follows from the following inequalities

$$(3.5) \quad \left\| \sum_{j \in \mathbf{Z}} S_{\bullet_j}(A_j) \otimes \delta_j \right\|_{S_p[C_p]} \leq C_p(\Sigma) \left\| \sum_{j \in \mathbf{Z}} A_j \otimes \delta_j \right\|_{S_p[C_p]}$$

for  $\bullet = m$  and  $\bullet = \mu$ . The case  $\bullet = m$  is the content of Proposition 3.2, since  $\dot{m}_j = \sigma_j \dot{m}$  is uniformly bounded in  $W_{d,\varepsilon}^2(\mathbf{R}^d)$  by virtue of the triangular inequality and Lemma 2.2. On the other hand, group inversion  $g \mapsto g^{-1}$  is smooth in any relatively compact set of  $G$ . Thus  $\mu_j$  satisfies (up to constants) the same Mikhlin conditions as  $m_j$ . To be more precise, using the map  $\mathcal{I}$  from Section 3.1

$$\mu_j(g) = \dot{\mu}_j(g - e) \quad \text{with} \quad \dot{\mu}_j(\xi) = \overline{\dot{m}_j((\xi + e)^{-1} - e)} = \overline{(\sigma_j \dot{m}) \circ \mathcal{I}(\xi)}.$$

By Lemmas 2.2 and 3.1 (the norm in  $W_{d,\varepsilon}^2(\mathbf{R}^d)$  is conjugate invariant) we get

$$(3.6) \quad \sup_{j \in \mathbf{Z}} \|\dot{\mu}_j\|_{W_{d,\varepsilon}^2(\mathbf{R}^d)} \leq C_\Sigma \max_{|\gamma| \leq \lfloor \frac{d}{2} \rfloor + 1} \|\xi^{|\gamma|} \partial_\xi^\gamma \dot{m}(\xi)\|_\infty < \infty.$$

So (3.5) for  $\bullet = \mu$  also follows from Proposition 3.2. This proves the assertion.  $\square$

**Local form of Theorem A.** *Assume that  $m: \mathrm{SL}_n(\mathbf{R}) \rightarrow \mathbf{C}$  is supported by a relatively compact neighborhood  $\Omega$  of the identity and satisfies  $(\star)$ . Then,  $T_m$  is completely  $L_p$ -bounded for  $1 < p < \infty$  by  $C_p(\Omega)$ , with  $C_p(\Omega) \approx C_p C_{\mathrm{hm}}$  for  $\Omega$  small.*

**Proof.** By duality and interpolation we may assume that  $p \in 2\mathbf{Z}_+$ . Fix a smooth function  $\varphi: \mathbf{R} \rightarrow \mathbf{R}_+$  which is 0 outside of  $[1/2, 2]$  and equal to 1 at 1. Extend the multiplier  $m: \mathrm{SL}_n(\mathbf{R}) \rightarrow \mathbf{C}$  to a function  $M: \mathbf{R}^d = M_n(\mathbf{R}) \rightarrow \mathbf{C}$  by the formula  $M(A) = \varphi(\det A) m(A/(\det A)^{\frac{1}{n}})$  if  $\det(A) > 0$  and  $M(A) = 0$  otherwise. Let  $\dot{m}(\xi) = M(\xi + e)$ , so that  $m(g) = \dot{m}(g - e)$ . It follows from  $(\star)$  that

$$(3.7) \quad |\partial_\xi^\gamma \dot{m}(\xi)| \leq C(\Omega) C_{\mathrm{hm}} |\xi|^{-|\gamma|} \quad \text{for all} \quad 0 \leq |\gamma| \leq \left\lfloor \frac{n^2}{2} \right\rfloor + 1.$$

The local theorem then follows by combining Theorem 1.3 and Theorem 3.3.  $\square$

**Remark 3.4.** The proof above gives in fact a stronger result. Namely, by the left invariant nature of our Lie differential operators, it suffices to prove (3.7) in a small neighborhood of the identity. Then, the Mikhlin constants of  $\dot{m}$  are dominated by a subfamily of Lie derivatives of  $m$ . More precisely, it suffices to assume  $(\star)$  in Theorem A for a family  $\Gamma_0$  of multi-indices satisfying that every other multi-index  $\gamma$  with  $|\gamma| \leq \lfloor n^2/2 \rfloor + 1$  is the permutation of an element in  $\Gamma_0$ .

**Remark 3.5.** Given any  $\varepsilon > 0$ , the Sobolev condition

$$\sup_{j \in \mathbf{Z}} \|\varphi_0^2 \dot{m}(2^j \cdot)\|_{H_{\frac{d}{2}+\varepsilon}^2(\mathbf{R}^d)} < \infty$$

also suffices for the local form of Theorem A. Indeed, we just need to observe that both Lemmas 2.2 and 3.1 do provide upper bounds of the  $W_{d,\varepsilon}^2(\mathbf{R}^d)$ -norms in terms of the smaller Sobolev norms above. Our Calderón-Zygmund type estimates

in Proposition 2.5 —needed to bound the homogeneous twisted multiplier— are also given in terms of Sobolev norms. This Sobolev condition is standard in Euclidean harmonic analysis and less demanding than the Mihklin one.

**Remark 3.6.** We also deduce the following result of independent interest. Let  $(\varphi_j)_{j \in \mathbf{Z}}$  be a Littlewood-Paley partition of unity in  $\mathbf{R}^d$  of the form (2.2). Set  $\psi_j(g) = \varphi_j(g - e)$  for  $g \in G = \mathrm{SL}_n(\mathbf{R})$ . Let us write  $\Psi_j$  for the Fourier multiplier associated to the symbol  $\psi_j$ . Let  $M \in \mathbf{Z}$ . Then, the following holds for  $1 < p < \infty$

$$\left\| \sum_{j \leq M} \Psi_j(f) \otimes \delta_j \right\|_{L_p(\mathcal{L}(G); RC_p)} \leq_{\mathrm{cb}} C_p(M) \|f\|_{L_p(\mathcal{L}(G))}.$$

This becomes a cb-norm equivalence when  $\mathrm{supp} \hat{f} \subset \Omega = \bigcup_{j \leq M-1} \mathrm{supp} \psi_j$ . In other words, a *local form of the Littlewood-Paley theorem* in the group algebra of  $\mathrm{SL}_n(\mathbf{R})$ .

**Proof.** According to the noncommutative Khintchine inequality [39, 40], the  $RC_p$  norm/square functions in the statement can be linearized and rewritten as follows

$$\begin{aligned} \left\| \sum_{j \leq M_\Omega} \Psi_j(f) \otimes \delta_j \right\|_{L_p[RC_p]} &\simeq_{\mathrm{cb}} \mathbf{E}_\varepsilon \left\| \sum_{j \leq M_\Omega} \varepsilon_j \Psi_j(f) \right\|_p \\ &\leq_{\mathrm{cb}} \sup_{\varepsilon_j = \pm 1} \left\| \sum_{j \leq M_\Omega} \varepsilon_j \Psi_j(f) \right\|_p \leq_{\mathrm{cb}} 2 \sup_{A \subset \mathbf{Z}_\Omega} \|\Psi_A(f)\|_p, \end{aligned}$$

where  $\mathbf{Z}_\Omega = \{j \in \mathbf{Z} : j \leq M_\Omega\}$  and  $\Psi_A$  has symbol  $\psi_A = \sum_{j \in A} \psi_j$ . By the local form of Theorem A, the upper estimate will follow if the Euclidean symbols  $\psi_A(\xi + e)$  satisfy the Mihklin regularity imposed there with HM-constants uniformly bounded in  $A$ . This is however standard for Littlewood-Paley radial decompositions and follows by construction. Therefore, it remains to justify the lower estimate for  $f$  with frequency support well inside  $\Omega$ . Assume first that  $f \in L_2(\mathcal{L}(G))$  and consider any other  $f' \in L_2(\mathcal{L}(G))$ . We obtain for  $\Psi = \sum_j \Psi_j \otimes \delta_j$

$$\langle \Psi(f), \Psi(f') \rangle = \sum_{j \leq M} \langle \Psi_j(f), \Psi_j(f') \rangle = \sum_{j \leq M} \int_G \psi_j^2(g) \hat{f}(g) \overline{\hat{f}'(g)} d\mu(g) = \langle f, f' \rangle$$

since  $\sum_{j \leq M} \psi_j^2$  is identically 1 in the support of  $\hat{f}$  by hypothesis. Now, by density we may assume that  $f \in L_p \cap L_2$  and that its norm is nearly attained by duality against a norm 1 element  $f' \in L_q \cap L_2$ . Altogether, we get the lower estimate since  $\|f\|_p \approx_{\mathrm{cb}} \langle f, f' \rangle = \langle \Psi(f), \Psi(f') \rangle \lesssim_{\mathrm{cb}} \|\Psi(f)\|_p$ . This completes the proof.  $\square$

**3.3. The asymptotic condition.** The proof of Theorem A will be completed with a simple patching argument from its local form. The key point is to observe that condition  $(\star)$  in Theorem A implies that the symbol  $m: \mathrm{SL}_n(\mathbf{R}) \rightarrow \mathbf{C}$  differs from  $L_1(\mathrm{SL}_n(\mathbf{R}))$  by a constant function. This follows in turn from the exponential nature of the metric and Weyl's integration formula.

**Lemma 3.7.** *Given  $\phi \in C^1(\mathrm{SL}_n(\mathbf{R}) \setminus \{e\})$  and  $\beta > 2$*

$$\sup_{X \in \mathfrak{sl}_n(\mathbf{R})} L(g)^\beta |\partial_X \phi(g)| \leq 1 \Rightarrow L(g)^\beta |\phi(g) - \alpha| \leq C_\beta$$

for some  $\alpha \in \mathbf{C}$ . The supremum runs over all unit vectors in the Lie algebra.

**Proof.** We claim that  $\phi - \alpha \in \mathcal{C}_0(\mathrm{SL}_n(\mathbf{R}))$  for some  $\alpha \in \mathbf{C}$ . This claim gives the statement. Indeed, every  $g \in \mathrm{SL}_n(\mathbf{R})$  factorizes as  $g = u \exp(sX)$  for some unit vector  $X \in \mathfrak{sl}_n(\mathbf{R})$  and  $u \in \mathrm{SO}(n)$ . By assumption and K-biinvariance of  $L$ , we obtain

$$\begin{aligned} |\phi(g) - \alpha| &= \left| \sum_{k \geq 1} \phi(g \exp((k-1)X)) - \phi(g \exp(kX)) \right| = \left| \sum_{k \geq 1} \partial_X \phi(g \exp(s_k X)) \right| \\ &\leq \sum_{k \geq 1} L(u \exp((s + s_k)X))^{-\beta} = \sum_{k \geq 1} e^{-\beta(s+s_k)} \lesssim L(\exp(sX))^{-\beta} = L(g)^{-\beta} \end{aligned}$$

for some  $s_k \in (k-1, k)$ . Let us now justify the claim. Every  $g \in \mathrm{SL}_n(\mathbf{R})$  factorizes as  $g = u_1 \exp(Z) u_2$  with  $Z$  a diagonal matrix in  $\mathfrak{sl}_n(\mathbf{R})$  and  $u_1, u_2 \in \mathrm{SO}(n)$ . By the surjectivity of the exponential map onto  $\mathrm{SO}(n)$ , we get that  $u_j = \exp(A_j)$  for some skew-symmetric  $A_j \in \mathfrak{so}_n$  with  $\|A_j\| \leq 2\pi$ . Under this factorization, we have  $L(g) = \exp(\|Z\|)$  and we conclude that

$$(3.8) \quad \left| \phi(\exp(A_1) \exp(Z) \exp(A_2)) - \phi(\exp(A_1) \exp(Z)) \right| \\ = \|A_2\| \left| \frac{\partial_{A_2}}{\|A_2\|} \phi(\exp(A_1) \exp(Z) \exp(rA_2)) \right| \leq 2\pi \exp(-\beta\|Z\|)$$

for some  $0 < r < 1$ . Similarly, let us note that  $\exp(A_1) \exp(Z) = \exp(Z)w$  for  $w = \exp(-Z) \exp(A_1) \exp(Z) = \exp(Y)$  where  $Y = \exp(-Z)A_1 \exp(Z)$  belongs to  $\mathfrak{sl}_n(\mathbf{R})$ . Therefore, the following identity holds for some  $r \in (0, 1)$

$$(3.9) \quad \left| \phi(\exp(A_1) \exp(Z)) - \phi(\exp(Z)) \right| = \|Y\| \left| \frac{\partial_Y}{\|Y\|} \phi(\exp(Z) \exp(rY)) \right|.$$

Since  $\|Y\| \leq 2\pi \exp(2\|A\|)$  and  $L(\exp(Z) \exp(rY)) = L(\exp(rA_1) \exp(Z)) = e^{\|Z\|}$ , the above quantity is bounded by  $2\pi \exp(-(\beta-2)\|Z\|)$ , which decreases to 0 for any  $\beta > 2$  as  $Z \rightarrow \infty$ . According to (3.8) and (3.9), it suffices to prove that  $\phi - \alpha \in \mathcal{C}_0$  when restricted to diagonal matrices  $\exp Z$ . To prove it, we identify diagonal matrices in  $\mathfrak{sl}_n(\mathbf{R})$  with  $\mathbf{R}^{n-1}$  as follows

$$Z = \mathrm{diag}(z_1, z_2, \dots, z_n) \xrightarrow{\Lambda} (z_1, z_2, \dots, z_{n-1}) = z.$$

Consider the function  $\rho(z) = \phi(\exp Z)$ . If  $\Lambda(U) = u$ , we get

$$\begin{aligned} \partial_u \rho(z) &= \lim_{s \rightarrow 0} \frac{\rho(z + su) - \rho(z)}{s} \\ &= \lim_{s \rightarrow 0} \frac{\phi(\exp(Z \exp(sU))) - \phi(\exp(Z))}{s} = \partial_U \phi(\exp(Z)). \end{aligned}$$

Since  $\Lambda$  is a contraction, we deduce the following inequality for  $\rho$

$$\sup_{\|u\|=1} |\partial_u \rho(z)| \leq \sup_{\|X\|=1} |\partial_X \phi(\exp(Z))| \leq L(\exp(Z))^{-\beta} \leq \exp(-\beta\|z\|).$$

This readily implies that  $\rho$  has a limit  $\alpha$  at infinite and the same holds for  $\phi$ .  $\square$

**Remark 3.8.** By linearity of Lie differentiation  $\partial_{X_1+X_2} = \partial_{X_1} + \partial_{X_2}$ , the supremum in Lemma 3.7 may be replaced by a maximum over norm 1 matrices  $X \in \mathfrak{sl}_n(\mathbf{R})$  in the directions of a fixed orthogonal basis of the Lie algebra. In particular, if condition  $(\star)$  holds, we may apply Lemma 3.7 to  $\phi = d^\gamma m$  with  $|\gamma| = [n^2/2]$ . By assumption,  $\partial_X \phi(g)$  decays as  $L(g)^{-\beta}$  for  $\beta = [n^2/2] + 1 > 2$ . In addition,  $\phi \in \mathcal{C}_0$  so that  $\alpha = 0$  and  $L(g)^{|\gamma|+1} |d^\gamma m(g)| \leq C_\gamma$ . In other words, the asymptotic decay of derivatives of order  $[n^2/2]$  is the same as those of order  $[n^2/2] + 1$ . Iterating this

argument, Lemma 3.7 finally applies to  $(\phi, \beta) = (m, [n^2/2] + 1)$  up to a constant depending on  $n$ . Condition  $(\star)$  does not assume  $m \in \mathcal{C}_0$  and the lemma gives a constant  $\alpha \in \mathbf{C}$  with

$$L(g)^{\sigma_n+1}|m(g) - \alpha| \leq C_n \quad \text{for } \sigma_n = \left\lfloor \frac{n^2}{2} \right\rfloor.$$

It explains how  $(\star)$  incorporates Lafforgue/de la Salle rigidity [36] in Theorem A.

**Remark 3.9.** Let  $\sigma_n = [n^2/2]$ . By Weyl's integration formula

$$\int_{\mathrm{SL}_n(\mathbf{R})} f(g) d\mu(g) = \int_{\mathrm{SO}(n) \times \mathfrak{a}_+ \times \mathrm{SO}(n)} f(k_1 \exp Z k_2) \prod_{j>k} \sinh(Z_j - Z_k) dk_1 dZ dk_2,$$

we get  $\mu(B_R) \approx \exp(\sigma_n R)$  for

$$B_R = \{g \in \mathrm{SL}_n(\mathbf{R}) : \log L(g) \leq R\}.$$

Thus,  $\sigma_n$  is the critical integrability index for the metric  $L: L^{-\sigma_n - \varepsilon} \in L_1(\mathrm{SL}_n(\mathbf{R}))$ .

**Proof of Theorem A.** By Remark 3.8, we may assume  $m \in \mathcal{C}_0(\mathrm{SL}_n(\mathbf{R}) \setminus \{e\})$ , equivalently  $\alpha = 0$ . Let  $\Gamma$  be a cocompact lattice in  $\mathrm{SL}_n(\mathbf{R})$  with fundamental domain  $\Delta$ . Consider a relatively compact neighborhood of the identity  $\Omega$  containing the closure of  $\Delta$ . Given  $\phi \in \mathcal{C}_c^{\infty,+}(\mathrm{SL}_n(\mathbf{R}))$  supported in  $\Omega$  and identically 1 over  $\Delta$ , define

$$\Phi_\gamma(g) = \frac{\phi(\gamma g)}{\sum_{\rho \in \Gamma} \phi(\rho g)} \quad \text{for each } \gamma \in \Gamma.$$

It is clear by construction that the denominator above is greater or equal than 1 and the  $\Phi_\gamma$ 's form a smooth partition of unity in  $\mathrm{SL}_n(\mathbf{R})$  indexed by  $\Gamma$ . Let us decompose the symbol  $m = \sum_\gamma m_\gamma$  accordingly. By the triangle inequality, it suffices to prove that

$$A_p(m) = \sum_{\gamma \in \Gamma} \|T_{m_\gamma} : L_p(\mathcal{L}(\mathrm{SL}_n(\mathbf{R}))) \rightarrow L_p(\mathcal{L}(\mathrm{SL}_n(\mathbf{R})))\|_{\mathrm{cb}} \leq C_p C_{\mathrm{hm}}.$$

By translation invariance of Fourier multipliers, we may clearly replace  $m_\gamma$  by its left translate  $M_\gamma(g) = m(\gamma^{-1}g)\Phi_e(g) = m_\gamma(\gamma^{-1}g)$ . Then, the local form of Theorem A yields for  $\sigma_n = [n^2/2]$

$$\begin{aligned} A_p(m) &\leq C_p \max_{|\beta| \leq \sigma_n + 1} \sum_{\gamma \in \Gamma} \sup_{g \in \Omega} |g|^{|\beta|} |d_g^\beta M_\gamma(g)| \\ &\lesssim C_p \max_{|\beta| \leq \sigma_n + 1} \left( \sup_{g \in \Omega} |g|^{|\beta|} |d_g^\beta M_e(g)| + \sum_{\gamma \neq e} \sup_{g \in \Omega} |d_g^\beta M_\gamma(g)| \right). \end{aligned}$$

Next, Leibnitz rule and left invariance of Lie differentiation give

$$|d_g^\beta M_\gamma(g)| \leq \sum_{\rho \leq \beta} |d_g^\rho m(\gamma^{-1}g) d_g^{\beta-\rho} \Phi_e(g)| \lesssim \sum_{\rho \leq \beta} |d_g^\rho m(\gamma^{-1}g)|.$$

In particular, combining the above estimates we get the expected inequality

$$A_p(m) \leq C_p \max_{|\beta| \leq \sigma_n + 1} \left( \sup_{g \in \Omega} |g|^{|\beta|} |d_g^\beta m(g)| + \sum_{\gamma \neq e} \sup_{g \in \Omega} |d_g^\beta m(\gamma g)| \right) \leq C_p C_{\mathrm{hm}}.$$

The last inequality follows from Remark 3.9. Namely, the proof there gives that the Lie derivatives  $|d_g^\beta m(\gamma g)|$  are all dominated by  $C_{\mathrm{hm}} L(\gamma g)^{-(\sigma_n+1)}$ . Using relative compactness of  $\Omega$  and Weyl's integration formula as we did before, the above sum is dominated (up to absolute constants) by  $C_{\mathrm{hm}}$ . This completes the proof.  $\square$

#### 4. Proof of Theorem B

The first rigidity theorems for Fourier multipliers in  $\mathrm{SL}_n(\mathbf{R})$  which are relevant for this paper can be traced back to [11, 20, 46]. Haagerup's paper [20] is particularly relevant since it proves that  $\mathrm{SL}_n(\mathbf{R})$  and  $\mathrm{SL}_n(\mathbf{Z})$  fail to be weakly amenable for high ranks  $n \geq 3$ . This was strengthened in [34, 36] by disproving the CBAP for the noncommutative  $L_p$  spaces over the group algebra of any lattice in  $\mathrm{SL}_n(\mathbf{R})$  when  $|1/p - 1/2|$  is large enough in terms of the rank. In this section we prove Theorem B as stated in the Introduction, which strengthens in turn [34, 36] with more demanding rigidity conditions for  $L_p$ -multipliers.

The proof relies, as in [34, 36] (or [21, 33] for other higher rank Lie groups), on the idea developed in Lafforgue's work [35] on strong property (T): to prove first local Hölder rigidity on the level of compact Lie groups, and then to combine these local estimates to obtain estimates on the whole noncompact Lie group. The local Hölder rigidity results for the compact group  $\mathrm{SO}(n)$  are the content of Proposition 4.2, and their combination to explore the whole group are derived in Theorem 4.6 for  $\mathrm{SO}(n, 1)$  and at the end of the section for  $\mathrm{SL}_n(\mathbf{R})$ .

**4.1. Composition of Hölder continuous functions.** If  $J$  is an interval of  $\mathbf{R}$  and  $\alpha > 0$ , we denote by  $\mathcal{C}^\alpha(J)$  the space of all functions  $f: J \rightarrow \mathbf{R}$  which are  $[\alpha]$  times differentiable, and whose  $[\alpha]$ -th derivative is  $(\alpha - [\alpha])$ -Hölder-continuous on every compact subset of  $J$ . If  $X$  is a Banach space we denote  $\mathcal{C}^\alpha(J; X)$  the space of such functions with values in  $X$ . When  $J$  is compact,  $\mathcal{C}^\alpha(J; X)$  is a Banach space for the norm

$$\|f\|_{\mathcal{C}^\alpha(J; X)} = \max \left\{ \max_{k \leq [\alpha]} \sup_{x \in J} \|\partial^k f(x)\|, \sup_{\substack{x \neq y \\ x, y \in J}} \frac{\|\partial^{[\alpha]} f(x) - \partial^{[\alpha]} f(y)\|}{|x - y|^{\alpha - [\alpha]}} \right\}.$$

When no confusion is possible, we simply write  $\|f\|_{\mathcal{C}^\alpha}$  for this norm. In general the family of seminorms  $\|\cdot\|_{\mathcal{C}^\alpha(K; X)}$  for  $K$  a compact subinterval of  $J$  turn  $\mathcal{C}^\alpha(J; X)$  into a Fréchet space. We also denote

$$\mathcal{C}^{\alpha-}(J; X) = \bigcap_{\beta < \alpha} \mathcal{C}^\beta(J; X).$$

It is clear that the spaces  $\mathcal{C}^\alpha(J; X)$  are invariant under precomposition by a sufficiently smooth function. When  $\alpha < 1$ ,  $f \in \mathcal{C}^\alpha(J; X)$  and  $\varphi: I \rightarrow J$  is a  $\mathcal{C}^1$  function, the following inequality holds for every  $x, y \in I$

$$(4.1) \quad \|f \circ \varphi(x) - f \circ \varphi(y)\| \leq \|f\|_{\mathcal{C}^\alpha} \|\varphi'\|_\infty^\alpha |x - y|^\alpha.$$

We shall need quantitative estimates for higher derivatives, that we now collect.

**Proposition 4.1.** *Given  $\alpha > 0$  and two compact intervals  $I, J$  of  $\mathbf{R}$ , assume that  $\varphi: I \rightarrow J$  is a function of class  $\mathcal{C}^{[\alpha]}$ . Then, the map  $f \mapsto f \circ \varphi$  maps  $\mathcal{C}^\alpha(J; X)$  into  $\mathcal{C}^\alpha(I; X)$ . More precisely, if  $\alpha \geq 1$ , there is a constant  $C_\alpha > 0$  such that, for every such  $I, J, f, \varphi$ , every  $1 \leq k \leq [\alpha]$  and every  $x \in I$ , we have*

$$(4.2) \quad \|\partial^k(f \circ \varphi)(x)\| \leq C_\alpha \|f\|_{\mathcal{C}^k} \max_{1 \leq j \leq k} |\partial^j \varphi(x)|^{\frac{k}{j}}.$$

Moreover, given  $x, y \in I$  we also get the following inequality

$$(4.3) \quad \|\partial^{[\alpha]}(f \circ \varphi)(x) - \partial^{[\alpha]}(f \circ \varphi)(y)\| \\ \leq C_\alpha \|f\|_{C^\alpha} \left( \|\varphi'\|_\infty^\alpha |x - y|^{\alpha - [\alpha]} + \max_{1 \leq j \leq [\alpha]} \left( \|\partial^j \varphi\|_\infty^{\frac{[\alpha]}{j}} \right) |x - y| \right).$$

In particular, if  $\alpha \notin \mathbf{Z}$ , the  $(\alpha - [\alpha])$ -Hölder constant of  $\partial^{[\alpha]}(f \circ \varphi)$  at  $x$

$$(4.4) \quad \limsup_{y \rightarrow x} \frac{\|\partial^{[\alpha]}(f \circ \varphi)(x) - \partial^{[\alpha]}(f \circ \varphi)(y)\|}{|x - y|^{\alpha - [\alpha]}} \leq C_\alpha \|f\|_{C^\alpha} |\varphi'(x)|^\alpha.$$

**Proof.** Faà di Bruno's formula asserts for  $k \leq [\alpha]$  that

$$\partial^k(f \circ \varphi)(x) = \sum_{j=1}^k B_{k,j}(\varphi'(x), \dots, \partial^{k-j+1}\varphi(x))(\partial^j f \circ \varphi)(x).$$

The coefficients in the above sum are given by the Bell polynomials

$$B_{k,j}(z_1, z_2, \dots, z_{k-j+1}) = \sum \frac{k!}{i_1! i_2! \dots i_{k-j+1}!} \prod_{s=1}^{k-j+1} \left( \frac{z_s}{s!} \right)^{i_s},$$

where the sum is over all sequences  $i_1, \dots, i_{k-j+1}$  of non-negative integers such that  $i_1 + i_2 + \dots + i_{k-j+1} = j$  and  $i_1 + 2i_2 + \dots + (k-j+1)i_{k-j+1} = k$ . Elementary computations provide a constant  $C_k$  such that

$$|B_{k,j}(z_1, \dots, z_{k-j+1})| \leq C_k \max \left\{ |z_1|^k, |z_2|^{\frac{k}{2}}, \dots, |z_{k-j+1}|^{\frac{k}{k-j+1}} \right\}.$$

This immediately implies the inequality (4.2) in the statement.

Inequality (4.4) follows from (4.1) or (4.3), according to the value of  $\alpha$ . To prove (4.3), we consider Faà di Bruno's formula for  $k = [\alpha]$ . The term  $j = [\alpha]$  is  $\varphi'(x)^{[\alpha]} \partial^{[\alpha]} f(\varphi(x))$ . In particular, the difference  $\partial^{[\alpha]}(f \circ \varphi)(x) - \partial^{[\alpha]}(f \circ \varphi)(y)$  yields a term  $j = [\alpha]$  given by

$$\varphi'(x)^{[\alpha]} \left( \partial^{[\alpha]} f(\varphi(x)) - \partial^{[\alpha]} f(\varphi(y)) \right) + \left( \varphi'(x)^{[\alpha]} - \varphi'(y)^{[\alpha]} \right) \partial^{[\alpha]} f(\varphi(y)) = A + B.$$

These two terms are respectively bounded above as follows

$$A \leq |\varphi'(x)|^{[\alpha]} \|f\|_{C^\alpha} |\varphi(x) - \varphi(y)|^{\alpha - [\alpha]} \leq \|f\|_{C^\alpha} \|\varphi'\|_\infty^\alpha |x - y|^{\alpha - [\alpha]}, \\ B \leq \|((\varphi')^{[\alpha]})'\|_\infty \|f\|_{C^\alpha} |x - y| \leq C_\alpha \|f\|_{C^\alpha} \|\varphi'\|_\infty^{[\alpha]-1} \|\varphi''\|_\infty |x - y|.$$

The other terms in Faà di Bruno's formula for  $1 \leq j < [\alpha]$  can be split into two terms  $A_j + B_j$  as above. The terms  $A_j$  can be uniformly bounded, as for (4.2), by

$$C_\alpha \|f\|_{C^\alpha} \max_{1 \leq j \leq [\alpha]} \|\partial^j \varphi\|_\infty^{\frac{[\alpha]}{j}} |x - y|.$$

Letting  $\beta_{k,j}^\varphi(x) = B_{k,j}(\varphi'(x), \dots, \partial^{k-j+1}\varphi(x))$ , we have

$$|B_j| = |(\beta_{[\alpha],j}^\varphi(x) - \beta_{[\alpha],j}^\varphi(y))(\partial^j f \circ \varphi)(x)| \leq \|f\|_{C^\alpha} \|(\beta_{[\alpha],j}^\varphi)'\|_\infty |x - y|.$$

It is an straightforward exercise to show by direct calculation that

$$|(\beta_{k,j}^\varphi)'(x)| \leq C_\alpha B_{k+1,j}(|\varphi'(x)|, \dots, |\partial^{k-j+2}\varphi(x)|).$$

Thus we get  $|B_j| \leq C_\alpha \|f\|_{C^\alpha} \max_{1 \leq j \leq [\alpha]} \|\partial^j \varphi\|_\infty^{\frac{[\alpha]}{j}} |x - y|$ . This proves (4.3).  $\square$



**4.2.  $S_p$ -multipliers: Estimates on  $\mathrm{SO}(n)$ .** Given a locally compact group  $G$  and  $1 \leq p \leq \infty$ , we say that a bounded measurable function  $m: G \rightarrow \mathbf{C}$  is an  $S_p$ -multiplier (resp.  $S_p$ - $S_q$ -multiplier) if the map

$$S_m: \left( a_{g,h} \right)_{g,h \in G} \mapsto \left( m(gh^{-1})a_{g,h} \right)_{g,h \in G}$$

is bounded on  $S_p(L_2(G))$  (resp. from  $S_p(L_2(G))$  to  $S_q(L_2(G))$ ).

Let us identify  $\mathrm{SO}(n-1)$  with the subgroup of  $\mathrm{SO}(n)$  fixing the first coordinate vector  $e_1$  of  $\mathbf{R}^n$ . Then, the double quotient  $\mathrm{SO}(n-1) \backslash \mathrm{SO}(n) / \mathrm{SO}(n-1)$  identifies with  $[-1, 1]$  through

$$\mathrm{SO}(n-1)k\mathrm{SO}(n-1) \mapsto k_{1,1}.$$

Therefore, to an  $\mathrm{SO}(n-1)$ -biinvariant function  $\varphi: \mathrm{SO}(n) \rightarrow \mathbf{C}$  corresponds a unique function  $\tilde{\varphi}: [-1, 1] \rightarrow \mathbf{C}$  satisfying  $\varphi(k) = \tilde{\varphi}(k_{1,1})$ . In what follows, we shall fix  $n \geq 3$ ,  $p > 2 + \frac{2}{n-2}$  and set

$$(4.5) \quad \alpha_0 = \frac{n-2}{2} - \frac{n-1}{p} > 0 \quad \text{and} \quad \alpha = \alpha_0 \delta_{\alpha_0 \notin \mathbf{Z}} + (\alpha_0 - \varepsilon) \delta_{\alpha_0 \in \mathbf{Z}}$$

for an arbitrarily small  $\varepsilon$ . The following is a strengthening of [34, Proposition 3.1].

**Proposition 4.2.** *Let  $p$  and  $\alpha$  be as above for any  $\varepsilon$ . Assume that  $\varphi: \mathrm{SO}(n) \rightarrow \mathbf{C}$  is an  $\mathrm{SO}(n-1)$ -biinvariant  $S_p$ -multiplier in  $\mathrm{SO}(n)$ . Then, we get  $\tilde{\varphi} \in \mathcal{C}^\alpha((-1, 1))$ .*

**Remark 4.3.** The proof will actually show more:

- If  $\alpha_0 \in \mathbf{Z}$ ,  $\tilde{\varphi}$  is  $\alpha_0 - 1$  times differentiable and

$$|\partial^{\alpha_0-1} \tilde{\varphi}(x) - \partial^{\alpha_0-1} \tilde{\varphi}(y)| \lesssim |x-y| |\log|x-y||^{\frac{1}{p}}$$

holds uniformly on every compact subset of the interval  $(-1, 1)$ .

- The conclusion holds if one merely assumes that  $\varphi$  is an  $S_p$ - $S_\infty$ -multiplier.

We shall prove a dual statement. Let  $\mathbf{S}^{n-1}$  denote the unit sphere in  $\mathbf{R}^n$  equipped with the Lebesgue probability measure. For  $\delta \in [-1, 1]$ , let  $T_\delta$  be the (densely defined) operator on  $L_2(\mathbf{S}^{n-1})$  given by

$$T_\delta f(x) = \text{the average of } f \text{ on } \{y \in \mathbf{S}^{n-1} \mid \langle x, y \rangle = \delta\}.$$

Equivalently, using the identification  $\mathbf{S}^{n-1} \cong \mathrm{SO}(n-1) \backslash \mathrm{SO}(n)$  through the map  $\mathrm{SO}(n-1)g \mapsto g^{-1}e_1$ , we can consider  $L_2(\mathbf{S}^{n-1})$  as a subspace of  $L_2(\mathrm{SO}(n))$ . Then  $T_\delta$  is the operator on  $L_2(\mathrm{SO}(n))$  equal to

$$(4.6) \quad \int_{\mathrm{SO}(n-1) \times \mathrm{SO}(n-1)} \lambda(ugu') dudu' \in B(L_2(\mathrm{SO}(n)))$$

for  $g \in \mathrm{SO}(n)$  satisfying  $g_{11} = \delta$ . Here,  $\lambda$  denotes the left-regular representation.

**Proposition 4.4.** *The map  $\delta \in (-1, 1) \mapsto T_\delta$  belongs to  $\mathcal{C}^\alpha((-1, 1); S_p(L_2(\mathrm{SO}(n))))$ .*

**Remark 4.5.** This implies Proposition 4.2 because for an  $S_p$ -multiplier  $\varphi$  (or more generally an  $S_p$ - $S_\infty$  multiplier), we have  $S_\varphi(T_\delta) = \tilde{\varphi}(\delta)T_\delta$  and in particular  $\tilde{\varphi}(\delta) = \langle S_\varphi(T_\delta)\xi, \xi \rangle$  where  $\xi \in L_2(\mathrm{SO}(n))$  is the constant function equal to 1. So the function  $\tilde{\varphi}$ , which is the composition of  $(\delta \mapsto T_\delta)$  with the continuous linear map  $T \in S_p \mapsto \langle S_\varphi(T)\xi, \xi \rangle$ , is at least as regular as  $(\delta \mapsto T_\delta)$ .

**Proof.** As explained in [34, Lemma 3.2], there is an orthonormal basis in which the operators  $T_\delta$  are all diagonal, and in which the eigenvalue sequence is  $(\varphi_k(\delta))_{k \geq 0}$  with multiplicity

$$m_k = \frac{(n+k-3)!(n+2k-2)}{(n-2)!k!},$$

where

$$\varphi_k(x) = c_n \int_0^\pi (x + i\sqrt{1-x^2} \cos \theta)^k (\sin \theta)^{n-3} d\theta \quad \text{and} \quad c_n = \frac{\Gamma(\frac{n-1}{2})}{\sqrt{\pi} \Gamma(\frac{n-2}{2})}.$$

By derivating in the integral, we obtain

$$|\partial^r \varphi_k(x)| \leq C(n, r) \frac{(1+k)^r}{(\sqrt{1-x^2})^{2r-1}} \int_0^\pi |x + i\sqrt{1-x^2} \cos \theta|^{k-r} (\sin \theta)^{n-3} d\theta$$

for any nonnegative integer  $r \leq k$ . When  $r > k$ , we must replace the exponent  $k-r$  inside the integral by 0. As (3.2) in [34], there is a constant  $C(n, r)$  (depending on  $n$  and  $r$ ) such that this is less than

$$\frac{C(n, r)}{((1+k)(1-x^2))^{\frac{n-2}{2}}} \frac{(1+k)^r}{(\sqrt{1-x^2})^{2r-1}}.$$

In particular, if  $0 < c < 1$ , there exists  $C'(n, r)$  such that

$$|\partial^r \varphi_k(x)| \leq C'(n, r)(1+k)^{r+1-\frac{n}{2}} \quad \text{for every } x \in [-c, c].$$

And so, bounding  $m_k \leq A(n)(1+k)^{n-2}$ , we obtain that for every such  $x$

$$\sum_{k \geq 0} m_k |\partial^r \varphi_k(x)|^p \leq AC' \sum_{k \geq 0} (1+k)^{n-2+p(r+1-\frac{n}{2})} = AC' \sum_{k \geq 0} (1+k)^{p(r-\alpha_0)-1},$$

which converges if  $r < \alpha_0$ . Since  $r \in \mathbf{Z}$ , this holds iff  $r < \alpha$ . Taking  $r = [\alpha]$ , we deduce that  $\delta \mapsto T_\delta$  belongs to  $C^{[\alpha]}([-c, c]; S_p(L_2(\text{SO}(n))))$  for all  $c < 1$ . Its  $[\alpha]$ -th derivative  $\partial^{[\alpha]} T_x$  is the operator which, in the basis as above, is diagonal with eigenvalues  $(\partial^{[\alpha]} \varphi_k(x))_k$  and multiplicities  $(m_k)_k$ . We get

$$\|\partial^{[\alpha]} T_x - \partial^{[\alpha]} T_y\|_p^p = \sum_{k \geq 0} m_k |\partial^{[\alpha]} \varphi_k(x) - \partial^{[\alpha]} \varphi_k(y)|^p.$$

As above, we bound  $m_k \leq A(n)(1+k)^{n-2}$ . In addition, when  $x, y \in [-c, c]$ , we estimate the difference  $|\partial^{[\alpha]} \varphi_k(x) - \partial^{[\alpha]} \varphi_k(y)|$  with two bounds. If  $k|x-y| \leq 1$ , we use

$$|\partial^{[\alpha]} \varphi_k(x) - \partial^{[\alpha]} \varphi_k(y)| \leq \max_{|z| \leq c} |\partial^{[\alpha]} \varphi_k(z)| |x-y| \leq C'(1+k)^{[\alpha]+1-\frac{n}{2}} |x-y|.$$

If  $k|x-y| > 1$ , we use

$$|\partial^{[\alpha]} \varphi_k(x) - \partial^{[\alpha]} \varphi_k(y)| \leq 2 \max_{|z| \leq c} |\partial^{[\alpha]} \varphi_k(z)| \leq 2C'(1+k)^{[\alpha]+1-\frac{n}{2}}.$$

When  $\alpha_0 \notin \mathbf{Z}$ , we get  $\alpha = \alpha_0$  and obtain

$$\begin{aligned} \|\partial^{[\alpha]} T_x - \partial^{[\alpha]} T_y\|_p^p &\lesssim \sum_{k \leq \frac{1}{|x-y|}} (1+k)^{p([\alpha]-\alpha_0)-1} |x-y|^p \\ &+ \sum_{k > \frac{1}{|x-y|}} (1+k)^{p([\alpha]-\alpha_0)-1} \lesssim |x-y|^{p(\alpha-[\alpha])}. \end{aligned}$$

Additionally, when  $\alpha_0 \in \mathbf{Z}$ , we get  $[\alpha] = \alpha_0 - 1$  and the same estimate gives

$$\|\partial^{[\alpha]}T_x - \partial^{[\alpha]}T_y\|_p \lesssim |x - y| |\log |x - y||^{\frac{1}{p}}.$$

Therefore,  $\partial^{[\alpha]}T_x$  is  $(\alpha - [\alpha])$ -Hölder continuous on every compact subinterval.  $\square$

**4.3. Rigidity for K-biinvariant  $S_p$ -multipliers on  $\mathrm{SO}(n, 1)$ .** Theorem B will be deduced from Propositions 4.1 and 4.2 in the next subsection. Before that we explain, in a simpler situation, how the same idea as for Theorem B allows to prove some rigidity for K-biinvariant  $S_p$ -multipliers on the rank 1 simple Lie group  $\mathrm{SO}(n, 1)$  which contain  $\mathrm{SO}(n)$  as a subgroup. Of course, the results cannot be as strong as for  $\mathrm{SL}_n(\mathbf{R})$  in the sense that they cannot prescribe any rate of convergence at infinity as  $\mathrm{SO}(n, 1)$  is weakly amenable. But it turns out that there are some “higher order” rigidity, which only appears at the level of the derivatives. To our knowledge, Theorem 4.6 and its particular case Remark 4.7 for K-biinvariant matrix coefficients of  $\mathrm{SO}(n, 1)$  is the first application to rank 1 groups of the ideas around (the proof of) strong property (T) originating in [35].

Recall that  $\mathrm{SO}(n, 1)$  the group of  $(n + 1) \times (n + 1)$  matrices of determinant one and preserving the symmetric bilinear form

$$[(x_1, \dots, x_{n+1}), (y_1, \dots, y_{n+1})] = \sum_{i=1}^n x_i y_i - x_{n+1} y_{n+1}.$$

Denote by  $\mathbf{K} \simeq \mathrm{O}(n)$  the maximal compact subgroup of  $\mathrm{SO}(n, 1)$  given by

$$\mathbf{K} = \left\{ \begin{pmatrix} U & 0 \\ 0 & \det(U) \end{pmatrix} : U \in \mathrm{O}(n) \right\},$$

and by  $\mathbf{A} \simeq \mathbf{R}$  the group

$$\mathbf{A} = \left\{ D(s) = \begin{pmatrix} \cosh(s) & 0 & \sinh(s) \\ 0 & 1_{n-1} & 0 \\ \sinh(s) & 0 & \cosh(s) \end{pmatrix} : s \in \mathbf{R} \right\}.$$

Denote also by  $\mathbf{A}_+$  the subset corresponding to  $s \geq 0$ . The polar decomposition in  $\mathrm{SO}(n, 1)$  reads as  $\mathrm{SO}(n, 1) = \mathbf{K}\mathbf{A}_+\mathbf{K}$ . That is, every element of  $\mathrm{SO}(n, 1)$  can be written as  $g = kak'$  for  $k, k' \in \mathbf{K}$  and  $a \in \mathbf{A}_+$ . Moreover,  $a = D(s)$  is uniquely determined by  $\|g\| = \|a\|$ , that is  $\|g\| = e^s$ . Alternatively,  $a = D(s)$  is uniquely determined by  $\mathrm{tr}(g^*g) = \mathrm{tr}(a^*a) = n - 1 + 2 \cosh(2s)$ .

In particular, every K-biinvariant function  $m: \mathrm{SO}(n, 1) \rightarrow \mathbf{C}$  can be written as  $m(g) = \varphi(\mathrm{tr}(g^*g))$  for a function  $\varphi: [n + 1, \infty) \rightarrow \mathbf{C}$ . So the next result gives regularity properties for K-biinvariant multipliers of  $\mathrm{SO}(n, 1)$ .

**Theorem 4.6.** *Let  $\alpha = \alpha(\varepsilon)$  be defined as in (4.5) for some  $n \geq 3$  and  $p > 2 + \frac{2}{n-2}$ . Then, every K-biinvariant  $S_p$ - $S_\infty$  Schur multiplier of  $\mathbf{G} = \mathrm{SO}(n, 1)$  is of class  $\mathcal{C}^\alpha$ .*

*More precisely, let  $m(g) = \varphi(\mathrm{tr}(g^*g))$  for a function  $\varphi: [n + 1, \infty) \rightarrow \mathbf{C}$ . Assume that the Schur multiplier  $S_m(g, h) = m(gh^{-1})$  is  $S_p$ - $S_\infty$  bounded. Then  $\varphi$  is of class  $\mathcal{C}^\alpha(n + 1, \infty)$ , and the following local/asymptotic estimates hold:*

i) *Given  $x > n + 1$  and an integer  $1 \leq k \leq [\alpha]$*

$$|\partial^k \varphi(x)| \leq C_{p,n}^\varepsilon \frac{\|S_m\|_{\mathcal{B}(S_p(L_2(\mathbf{G})), S_\infty(L_2(\mathbf{G})))}}{(x - n - 1)^k}.$$

ii) *The Hölder constants in a neighborhood of  $x$*

$$\limsup_{y \rightarrow x} \frac{|\partial^{[\alpha]} \varphi(x) - \partial^{[\alpha]} \varphi(y)|}{|x - y|^{\alpha - [\alpha]}} \leq C_{p,n}^\varepsilon \frac{\|S_m\|_{\mathcal{B}(S_p(L_2(\mathbb{G})), S_\infty(L_2(\mathbb{G})))}}{(x - n - 1)^\alpha},$$

**Remark 4.7.** Coefficients of uniformly bounded representations on Hilbert spaces are particular cases of  $S_\infty$ -multipliers. So for  $p = \infty$ , the previous theorem has as a consequence that K-biinvariant coefficients of uniformly bounded representations of  $SO(n, 1)$  are of class  $\mathcal{C}^{\frac{n}{2}-1}$  if  $n$  is odd (and of class  $\mathcal{C}^{\frac{n}{2}-1-\varepsilon}$  for every  $\varepsilon > 0$  if  $n$  is even), and the derivatives and Hölder constants of their restrictions to  $\mathbb{A}$  are explicitly controlled by the bounds of Theorem 4.6. We are not aware of any such result in the literature, even for unitary representations. We recall however that K-biinvariant coefficients of irreducible unitary representations (and more generally K-finite coefficients of admissible representations) are  $\mathcal{C}^\infty$  [22], and that unitary representations are direct integrals of irreducible representations. But Harish-Chandra's estimates depend on the representation and therefore do not provide higher order regularity estimates for arbitrary unitary representations.

**Proof of Theorem 4.6.** For  $r > 0$ , the function

$$k \in SO(n) \mapsto m\left(D(r) \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} D(r)\right)$$

is an  $S_p$ - $S_\infty$ -multiplier of norm  $\leq 1$  by restriction, and is  $SO(n-1)$ -biinvariant because  $D = D(r)$  commutes with the image of  $SO(n-1)$  in  $\mathbb{K}$ .

In particular, if we consider the rotation matrix  $k_\delta$  of angle  $\arccos \delta$  in the space spanned by the first two coordinate vectors

$$k_\delta = \begin{pmatrix} \delta & -\sqrt{1-\delta^2} & 0 \\ \sqrt{1-\delta^2} & \delta & 0 \\ 0 & 0 & 1_{n-2} \end{pmatrix} \quad \text{for } \delta \in [0, 1],$$

Proposition 4.2 gives that the function  $\psi_r: \delta \mapsto m(Dk_\delta D)$  is of class  $\mathcal{C}^\alpha$  (uniformly in  $r$  and  $\varphi$ ) and we obtain

$$(4.7) \quad \sup_{r>0} \|\psi_r\|_{\mathcal{C}^\alpha([0,1-\eta])} < \infty \text{ for every } \eta > 0.$$

We can compute

$$\text{tr}((Dk_\delta D)^*(Dk_\delta D)) = 4(\sinh r)^4 \delta^2 + 2(\sinh 2r)^2 \delta + n - 3 + 4(\cosh r)^4 = a_r \delta^2 + b_r \delta + c_r.$$

Denote by  $g_r$  the inverse of  $\delta \in [0, 1] \mapsto \text{tr}((D(r)k_\delta D(r))^*(D(r)k_\delta D(r)))$ . That is

$$g_r(x) = \frac{-b_r}{2a_r} + \sqrt{\frac{b_r^2}{4a_r^2} + \frac{x - c_r}{a_r}}$$

for every  $x \in [c_r, a_r + b_r + c_r]$ . Observe that

$$\frac{b_r}{a_r} = \frac{(\sinh 2r)^2}{2(\sinh r)^4} \geq 2$$

Therefore, for every integer  $k \geq 1$ , we have

$$\max_{1 \leq j \leq k} \|\partial^j g_r\|_\infty^{\frac{k}{j}} \leq C_k b_r^{-k}.$$

Let  $r > 0$  be determined by  $x = c_r$ . Then Proposition 4.1 gives,

$$|\partial^k \varphi(x)| \leq C b_r^{-k} \quad \text{for every } x \in [c_r, a_r + b_r + c_r] \text{ and every integer } k \leq [\alpha]$$

and the  $(\alpha - [\alpha])$ -Hölder constant of  $\varphi$  at  $x$  is  $\leq C b_r^{-\alpha}$ . This proves the theorem as there is a constant  $C'$  such that  $1/C' b_r \leq c_r - n - 1 \leq C' b_r$ .  $\square$

**Remark 4.8.** Theorem 4.6 implies a cheap and weaker form of Theorem B. Indeed, when  $n \geq 4$  and  $m$  and  $\varphi$  are given as in Theorem B, then by restriction  $m$  defines a K-biinvariant multiplier on  $\text{SO}(n-1, 1)$ . In particular, Theorem 4.6 implies that  $\varphi$  is  $\mathcal{C}^{\beta-}$  for  $\beta = \frac{n-3}{2} - \frac{n-2}{p}$  and gives explicit estimates on the derivatives of the function  $\varphi$ . These estimates are weaker than the conclusion of Theorem B, as the regularity is lower and there are no additional factors  $c_k$ .

**4.4. Rigidity for radial  $S_p$ -multipliers on  $\text{SL}_n(\mathbf{R})$ .** Now we use our results so far to prove Theorem B. The idea is the same of for Theorem 4.6, but the details are more technical. In fact, we shall prove a form of Theorem B, which is slightly stronger in two senses:

- i) We shall allow Schur multipliers which are radial either in the normalized Hilbert-Schmidt norm  $|g|^2 = \frac{1}{n} \text{tr}(g^* g)$  or the operator norm  $\|\cdot\|$  on  $\text{SL}_n(\mathbf{R})$

$$\|g\| = \sup \left\{ \sum_{i,j=1}^n g_{i,j} x_i y_j \mid x, y \in \mathbf{R}^n, \sum_i x_i^2 = \sum_j y_j^2 = 1 \right\}.$$

- ii) The assumption that  $S_m$  is  $S_p$ -bounded by the weaker assumption that  $S_m$  is  $S_p - S_\infty$  bounded, and with  $\|S_m\|_{\mathcal{B}(S_p(L_2(G)))}$  replaced by the smaller quantity  $\|S_m\|_{\mathcal{B}(S_p(L_2(G)), B(L_2(G)))}$ .

Thus, let  $\varphi: (0, \infty) \rightarrow \mathbf{C}$  be a function satisfying that  $S_{\varphi(\|\cdot\|)}$  or  $S_{\varphi(\|\cdot\|)}$  maps  $S_p(L_2(G))$  to  $B(L_2(G))$  with norm 1. We start with the following crucial lemma where the letter  $C$  stands for a constant depending on  $p, n$  only.

**Lemma 4.9.** *The symbol  $\varphi$  is of class  $\mathcal{C}^\alpha$  on  $(1, \infty)$ . Moreover:*

- i) *If  $\alpha \leq 1$ , then*

$$|\varphi(x) - \varphi(y)| \leq \sup_{x \leq z \leq y} \frac{C}{((z-1)z^{\frac{n}{n-2}})^\alpha} |x - y|^\alpha$$

*for every pair  $x, y \in (1, \infty)$  satisfying that  $x \leq y \leq x^{1 + \frac{n}{n-2}}$ .*

- ii) *If  $\alpha > 1$ , then for every  $1 \leq k \leq [\alpha]$ ,*

$$|\partial^k \varphi(x)| \leq \frac{C}{(x-1)^k x^{\frac{n}{n-2}}}.$$

- iii) *The  $(\alpha - [\alpha])$ -Hölder constant of  $\partial^{[\alpha]} \varphi$  at  $x$  satisfies*

$$\limsup_{y \rightarrow x} \frac{|\partial^{[\alpha]} \varphi(x) - \partial^{[\alpha]} \varphi(y)|}{|x - y|^{\alpha - [\alpha]}} \leq \frac{C}{((x-1)x^{\frac{n}{n-2}})^\alpha}.$$

**Proof.** For  $r > 0$ , define  $s = -\frac{r}{n-1}$  and  $D = \text{diag}(e^r, e^s, \dots, e^s) \in \text{SL}_n(\mathbf{R})$ . The function  $k \in \text{SO}(n) \mapsto \varphi(\|DkD\|)$  is an  $S_p - S_\infty$ -multiplier of norm  $\leq 1$  by restriction to submatrices with entries  $(g, h)$  in  $D\text{SO}(n) \times D^{-1}\text{SO}(n)$ . Let us also recall that

it is  $SO(n-1)$ -biinvariant, because  $D$  commutes with  $SO(n-1)$ . Note additionally that the same properties hold for the normalized Hilbert-Schmidt norm  $|\cdot|$ .

In particular, if we consider the rotation matrix  $k_\delta$  of angle  $\arccos \delta$  in the space spanned by the first two coordinate vectors

$$k_\delta = \begin{pmatrix} \delta & -\sqrt{1-\delta^2} & 0 \\ \sqrt{1-\delta^2} & \delta & 0 \\ 0 & 0 & 1_{n-2} \end{pmatrix} \quad \text{for } \delta \in [0, 1],$$

Proposition 4.2 gives that the function  $\psi_r: \delta \mapsto \varphi(\|Dk_\delta D\|)$  is of class  $\mathcal{C}^\alpha$  uniformly in  $r$  and  $\varphi$  (the same holds one more time replacing the operator norm by the Hilbert-Schmidt norm) and we obtain

$$(4.8) \quad \sup_{r>0} \|\psi_r\|_{\mathcal{C}^\alpha([0,1-\eta])} < \infty \text{ for every } \eta > 0.$$

We can compute

$$Dk_\delta D = e^{r+s} \begin{pmatrix} e^{r-s}\delta & -\sqrt{1-\delta^2} & 0 \\ \sqrt{1-\delta^2} & e^{s-r}\delta & 0 \\ 0 & 0 & e^{s-r}1_{n-2} \end{pmatrix}.$$

We first consider radial multipliers in the operator norm  $\|\cdot\|$ . The matrix

$$\begin{pmatrix} e^{r-s}\delta & -\sqrt{1-\delta^2} \\ \sqrt{1-\delta^2} & e^{s-r}\delta \end{pmatrix}$$

has determinant 1 and Hilbert-Schmidt norm  $(2 + 4\delta^2 \sinh^2(r-s))^{1/2}$ , so its norm is equal to  $g(\delta \sinh(r-s))$  where  $g(x) = (1 + 2x^2 + 2\sqrt{x^2 + x^4})^{1/2}$ . Therefore,  $Dk_\delta D$  has norm  $e^{r+s}g(\delta \sinh(r-s))$ . We conclude that

$$\psi_r(\delta) = \varphi(e^{r+s}g(\delta \sinh(r-s))).$$

Taking  $\delta = 1$  in the computation of the norm of  $Dk_\delta D$ , it follows that  $g(\sinh u) = e^u$  for every  $u \geq 0$ . In other words, the inverse is  $g^{-1}(x) = \sinh(\log x) = \frac{1}{2}(x - \frac{1}{x})$ . So if we define  $H_r: [e^{r+s}, e^{2r}] \rightarrow [0, 1]$  by

$$(4.9) \quad H_r(x) = \frac{1}{2 \sinh(r-s)} \left( \frac{x}{e^{r+s}} - \frac{e^{r+s}}{x} \right) = \frac{\frac{x}{e^{r+s}} - \frac{e^{r+s}}{x}}{e^{r-s} - e^{s-r}},$$

we obtain that  $e^{r+s}g(H_r(x) \sinh(r-s)) = x$  and

$$(4.10) \quad \varphi = \psi_r \circ H_r \text{ on } [e^{r+s}, e^{2r}].$$

We are now in position to apply Proposition 4.1, which gives us bounds on the derivatives of  $\varphi$  and their Hölder constants in terms of the derivatives of  $H_r$ . So we compute, for  $x \in [e^{r+s}, e^{2r}]$ ,

$$H'_r(x) = \frac{1}{e^{2r} - e^{2s}} \left( 1 + \frac{e^{2r+2s}}{x^2} \right) \in \left[ \frac{1}{e^{2r} - e^{2s}}, \frac{2}{e^{2r} - e^{2s}} \right]$$

and

$$\partial^j H_r(x) = \frac{(-1)^{j-1} j!}{(e^{-2s} - e^{-2r}) x^{j+1}} \quad \text{for } j \geq 2.$$

For a fixed  $x$ ,  $|\partial^j H_r(x)|$  is a decreasing function of  $r$ , so the bounds we get will be optimal when  $r$  is maximal. In other words, when  $x = e^{r+s}$ . This determines the value of  $r$  and  $s$  as a function of  $x$ , so that

$$e^{2r} = x^{1+\frac{n}{n-2}} \quad \text{and} \quad e^{2s} = x^{1-\frac{n}{n-2}}.$$

This yields that  $|\partial^j H_r(x)| \approx \frac{1}{x-1}$  when  $x$  is close to one and  $|\partial^j H_r(x)| \approx x^{-(j+\frac{n}{n-2})}$  when  $x$  is large. Written concisely, we get the following estimates for the derivatives of  $H_r$

$$(4.11) \quad |\partial^j H_r(x)| \leq \frac{C}{(x-1)x^{j-1+\frac{n}{n-2}}} \Rightarrow \max_{1 \leq j \leq k} |\partial^j H_r(x)|^{\frac{k}{j}} \leq \frac{C}{(x-1)^k x^{\frac{n}{n-2}}}.$$

Proposition 4.1 together with (4.8), (4.10) and (4.11) implies that  $\varphi$  is of class  $\mathcal{C}^\alpha$  at  $x$  and estimates ii) and iii) in the statement follow. When  $\alpha < 1$ , we use (4.1) instead of Proposition 4.1 to deduce i). The condition  $x \leq y \leq x^{1+\frac{n}{n-2}}$  comes imposed by the domain of  $H_r$ . This proves the lemma for the operator norm.

Let us now consider radial Schur multipliers in the normalized Hilbert-Schmidt norm  $|\cdot|$ . In that case, the norm computations are straightforward and lead to the decomposition

$$(4.12) \quad \varphi = \psi_r \circ \tilde{H}_r \text{ on } [A_r, B_r]$$

with  $A_r^2 = \frac{1}{n}((n-2)e^{4s} + 2e^{2r+2s})$ ,  $B_r^2 = \frac{1}{n}((n-1)e^{4s} + e^{4r})$  and

$$\tilde{H}_r(x) = \sqrt{\frac{x^2/A_r^2 - 1}{B_r^2/A_r^2 - 1}} = \frac{A_r}{\sqrt{B_r^2 - A_r^2}} H(x/A_r).$$

The function  $H: x \mapsto \sqrt{x^2 - 1}$  has first derivative  $x(x^2 - 1)^{-\frac{1}{2}}$  and (by induction)  $j$ -th derivative of the form  $P_j(x)(x^2 - 1)^{\frac{1}{2}-j}$  for certain polynomial  $P_j$ . This gives

$$\partial^j \tilde{H}_r(x) = \frac{A_r^j}{\sqrt{(x+A_r)^{2j-1}}} P_j\left(\frac{x}{A_r}\right) \sqrt{\frac{1}{(B_r^2 - A_r^2)(x - A_r)^{2j-1}}}.$$

To bound it when  $x$  is close to 1, we choose  $r > 0$  so that  $B_r = x$ . We have that  $x - 1 \approx cr^2$  and  $B_r^2 - A_r^2 = dr^2$  for some  $c, d > 0$ . We clearly obtain for  $x$  close to 1

$$\max_{1 \leq j \leq k} |\partial^j H_r(x)|^{\frac{k}{j}} \leq \frac{C}{(x-1)^k}.$$

For  $x$  large, we choose  $r > 0$  so that  $x = e^{r+s}$ . It follows that  $x \sim cA_r$  and  $B_r \sim dx^{2\frac{n-1}{n-2}}$  for some  $c > 1$  and  $d > 0$ . Thus we get

$$\max_{1 \leq j \leq k} |H_r^{(j)}(x)|^{\frac{k}{j}} \leq \frac{C}{x^{k+\frac{n}{n-2}}},$$

for  $x$  big enough. Assertions i), ii) and iii) for the Hilbert-Schmidt norm then follow as above from (4.1) and Proposition 4.1. This completes the proof.  $\square$

The previous lemma looks a lot like the conclusion of Theorem B, except that the precise exponents are not correct. The correct exponents are obtained by applying Proposition 4.2 for  $\text{SO}(m)$  for various  $m \leq n$ .

**Lemma 4.10.** *Let  $p > 2 + \frac{2}{n-2}$  and let  $m \leq n$  be an integer such that  $\frac{m-2}{2} - \frac{m-1}{p}$  is strictly positive. Consider  $\beta \notin \mathbf{N}$  such that  $\beta \leq \frac{m-2}{2} - \frac{m-1}{p}$ . Then, the following estimates hold:*

- If  $\beta > 1$ , then for every integer  $1 \leq k \leq \beta$ ,

$$|\partial^k \varphi(x)| \leq \frac{C}{(x-1)^k x^{\frac{n}{m-2}}}.$$

- If  $\beta < 1$ , then

$$|\varphi(x) - \varphi(y)| \leq \sup_{x \leq z \leq y} \frac{C}{((z-1)z^{\frac{n}{m-2}})^\beta} |x-y|^\beta$$

for every pair  $x, y \in (1, \infty)$  satisfying that  $x \leq y \leq x^{1+\frac{n}{m-2}}$ .

**Proof.** The argument is the same as in Lemma 4.9, except that  $D$  is replaced by the diagonal matrix in  $\mathrm{SL}_n(\mathbf{R})$  with eigenvalues  $e^r$  with multiplicity 1,  $e^s$  with multiplicity  $m-1$  and  $e^t$  with multiplicity  $n-m$ , where  $r > 0$  and  $s, t$  are determined by  $r$  as follows

$$(s, t) = \left( -\frac{n-m+2}{n+m-2}r, \frac{m-2}{n+m-2}r \right).$$

As above, it turns out that  $k \in \mathrm{SO}(m) \mapsto \varphi(\|DkD\|)$  is an  $S_p$ - $S_\infty$ -multiplier by restriction. Moreover, the fact that  $D$  has  $m-1$  equal eigenvalues ensures that it is  $\mathrm{SO}(m-1)$ -biinvariant. In particular, we can use Proposition 4.2 for  $\mathrm{SO}(m)$  combined with Proposition 4.1.

Any choice of  $r, s, t$  with  $r + (m-1)s + (n-m)t = 0$  ensures that  $\det D = 1$  and that  $\|Dk_\delta D\| = \max\{e^{2s}, e^{2t}, e^{r+s}g(\delta \sinh(r-s))\}$ . Our particular choice gives in addition that the operator norm of  $Dk_\delta D$  is equal to  $e^{r+s}g(\delta \sinh(r-s))$ . The only difference is that if  $x, r, s$  are related by  $x = e^{r+s}$  and  $s = -\frac{n-m+2}{n+m-2}r$ , then one gets

$$e^{2r} = x^{1+\frac{n}{m-2}} \quad \text{and} \quad e^{2s} = x^{1-\frac{n}{m-2}}.$$

So if  $H_r$  is still defined by (4.9) and  $x, r, s$  are related as above, then the estimates on the derivatives of  $H_r$  become

$$|\partial^j H_r(x)| \leq \frac{C}{(x-1)x^{j-1+\frac{n}{m-2}}} \Rightarrow \max_{1 \leq j \leq k} |\partial^j H_r(x)|^{\frac{k}{j}} \leq \frac{C}{(x-1)^k x^{\frac{n}{m-2}}}.$$

Thus, the conclusion for the operator norm  $\|\cdot\|$  is the same as for Lemma 4.9.

On the other hand, the normalized Hilbert-Schmidt norm of  $Dk_\delta D$  has the form  $\sqrt{A_r^2 + \delta^2(B_r^2 - A_r^2)}$  with

$$\begin{aligned} B_r^2 &= \frac{1}{n} (e^{4r} + (m-1)e^{4s} + (n-m)e^{4t}), \\ A_r^2 &= \frac{1}{n} (2e^{2r+2s} + (m-2)e^{4s} + (n-m)e^{4t}). \end{aligned}$$

Therefore, when using radial Schur multipliers in the normalized Hilbert-Schmidt norm  $|\cdot|$ , if we define  $\tilde{H}_r$  by (4.12) with  $x, r, s$  related as above, the same analysis for the derivatives of  $\tilde{H}_r$  can be applied and proves the lemma.  $\square$

**Proof of Theorem B.** The conclusion of Theorem B follows from the preceding two lemmas. The fact that  $\varphi$  is of class  $\mathcal{C}^\alpha$  and the estimate on the local  $\alpha$ - $[\alpha]$  Hölder constant is contained in Lemma 4.9. It remains to justify the pointwise estimates on  $\varphi$  and its derivatives.



We start with the derivatives, for which the argument is direct. Fix an integer  $1 \leq k < \alpha$ . Consider  $m$ , the smallest integer such that  $\beta := \frac{m-2}{2} - \frac{m-1}{p}$  is strictly greater than  $k$ . A small computation shows that it satisfies

$$m - 2 = \left\lceil \frac{2k + 1}{1 - \frac{2}{p}} \right\rceil \geq 3.$$

Note that  $k < \beta \leq k + \frac{1}{2} - \frac{1}{p}$ , so  $\beta \notin \mathbf{Z}$  and Lemma 4.10 gives the expected estimate

$$(4.13) \quad |\partial^k \varphi(x)| \leq \frac{C}{(x-1)^k x^{\frac{n}{m-2}}} = \frac{C}{(x-1)^k x^{c_k}} \quad \text{for every } x \in (1, \infty).$$

To obtain pointwise estimates on  $\varphi$ , one more argument is needed. First observe that the case  $x \leq 2$  is trivial because the norm of an  $S_p$ -multiplier is always bounded below by the  $L_\infty$  norm of its symbol. So we can consider the case  $x \geq 2$ . When  $\alpha < 1$ , Lemma 4.9 implies in particular that

$$|\varphi(x) - \varphi(y)| \leq \frac{C}{x^{\alpha + \frac{\alpha n}{n-2}}} |x - y|^\alpha \leq \frac{C}{x^{\frac{\alpha n}{n-2}}} \quad \text{when } 2 < x \leq y \leq 2x < \infty.$$

Thus  $\varphi$  satisfies the Cauchy criterion, has a limit  $\varphi_\infty$  and

$$|\varphi(x) - \varphi_\infty| \leq \sum_{i \geq 0} |\varphi(2^i x) - \varphi(2^{i+1} x)| \leq \frac{C}{x^{\frac{\alpha n}{n-2}}}.$$

This gives the assertion for  $\alpha < 1$ . When  $\alpha > 1$ , the inequality

$$|\varphi(x) - \varphi_\infty| \leq \int_x^\infty |\varphi'(y)| dy \leq \frac{C}{x^{c_1}}$$

is immediate from (4.13). This concludes the proof of Theorem B.  $\square$

**Remark 4.11.** Compared to the best known result [34], Theorem B gives:

- i) Automatic regularity of class  $C^\alpha$ .
- ii) More accurate decay rates for asymptotic rigidity of  $\varphi$ .
- iii) Local Mihlin type conditions and asymptotic “higher order” rigidity.
- iv) A larger range of  $p$ 's (in terms of  $n$ ) for which the rigidity results hold.

## 5. Final comments

**A. Rank one.** According to [10], we know that  $\mathrm{SL}_2(\mathbf{R})$  is weakly amenable. In particular, the rigidity theorems that shaped our statement in Theorem A do not apply to it. More precisely, there are some particularly well-behaved multipliers in  $\mathrm{SL}_2(\mathbf{R})$  —completely  $L_p$ -bounded by 1 with Fourier symbols converging to 1 uniformly on compact sets— which strongly break the decay implicit in Theorem A (as described in Remark 3.8). Is there a substantial improvement of Theorem A for  $\mathrm{SL}_2(\mathbf{R})$ ? The group  $\mathrm{SL}_2(\mathbf{R})$  does not admit finite-dimensional orthogonal cocycles and, consequently, the Mihlin type conditions in [29, 30] do not apply. On the contrary,  $\mathrm{SL}_2(\mathbf{R})$  enjoys Haagerup property since it admits infinite-dimensional proper cocycles which lead to K-biinvariant associated length functions [12, Chapter IV]. In this respect, we may construct noncommutative Riesz transforms for any such cocycle  $\beta: \mathrm{SL}_2(\mathbf{R}) \rightarrow \mathcal{H}$  and any Riesz direction  $u$  in the cocycle Hilbert space

$$\widehat{R}_u f(g) = \frac{\langle \beta(g), u \rangle_{\mathcal{H}}}{\|\beta(g)\|_{\mathcal{H}}} \widehat{f}(g).$$

According to [30], the maps  $R_u$  are completely  $L_p$ -bounded and also satisfy more involved dimension free estimates. An optimal formulation of the Mihlin condition in  $SL_2(\mathbf{R})$  should include this natural class of multipliers. A quick inspection of the cocycle  $\beta$  from [12] gives an asymptotic decay of order  $(\log L)^{1/2}$ , which is much less rigid than the behavior imposed by Theorem A. This indicates that there might be room for improvement in the rank 1 case. This construction is not possible in higher ranks for the lack of such cocycles, due to Kazhdan property (T).

**B. Twisted multipliers.** We have claimed in the Introduction that classical harmonic analysis methods are not efficient to give  $L_p$ -bounds of twisted Fourier multipliers. The first illustration of that was given in [48]. There it was proved that twisted forms of  $u$ -directional Hilbert transforms are  $L_p$ -unbounded for all  $p \neq 2$  unless the  $G$ -orbit of  $u$  is a finite set. The hidden tool here is Fefferman's Kakeya type construction for his ball multiplier theorem [13]. On the contrary, the twisted one-dimensional Hilbert transforms are  $L_p$ -bounded for orthogonal actions [29]. This evidences that the twist is not stable under tensor product extensions! In addition, asymptotic Calderón-Zygmund methods worked in [29] for orthogonal actions, but become much less efficient for nonorthogonal ones due to the distortion effect of volume-preserving transformations. Namely, combining Proposition 2.5 i) with Junge's  $H_p^c \rightarrow L_p$  inequality as in the proof of [29, Theorem A], we get a sufficient condition for  $L_p$ -boundedness ( $1 < p < \infty$ ) of twisted multipliers which vanish around 0

$$\sup_{g \in \Sigma} |\xi|^{|\gamma|} |\partial_\xi^\gamma M_{\pm g}(\xi)| \lesssim 1 \Rightarrow \tilde{T}_m: L_p(\mathcal{R}_\Sigma) \xrightarrow{\text{cb}} L_p(\mathcal{R}_\Sigma)$$

for  $M_{\pm g}(\xi) = |\xi|^{\pm\delta} \dot{m}(\alpha_g(\xi))$ . This is effective for orthogonal actions. In the nonorthogonal case, multipliers with lower decay than in Theorem A satisfy the above condition for first-order derivatives, but fail it for higher orders (even for  $K$ -biinvariant multipliers in  $SL_2(\mathbf{R})$ ) due to the distortion produced by  $\alpha_g$ .

**C. Calderón-Torchinsky theorem.** Is it true that

$$(CT) \quad |g|^{|\gamma|} |d_g^\gamma m(g)| \leq C_{\text{hm}} \quad \text{for all } |\gamma| \leq [s] + 1$$

suffices for the complete  $L_p$ -boundedness of the Fourier multiplier  $T_m$  in the group algebra of  $SL_n(\mathbf{R})$  whenever  $|1/p - 1/2| < s/n^2$ ? This would be a natural analogue of Calderón-Torchinsky refinement of HM-condition [4, 18], originally formulated using Sobolev spaces for fractional derivatives in the spirit of Remark 3.5. It is not hard to show that (CT) suffices locally. Indeed, the proof of the local form of Theorem A can not be directly modified since we make crucial use of Riesz transforms, for which we need full regularity. However, the proof gives an upper bound in terms of the Mihlin condition for the lift  $\dot{m}$ . This bound still holds for the Sobolev condition in  $\dot{m}$ . In particular, the interpolation argument in [18] still applies. Unfortunately it seems much harder to interpolate the asymptotic behavior of the multiplier and we have no results in this direction.

The validity of (CT) beyond compactly supported symbols would be especially relevant in our context. Namely, Remark 3.8 shows that the Fourier symbol decays as the largest Lie derivative in (CT). In particular, (CT) imposes (as expected) less and less decay when  $p$  approaches 2. Moreover, working with fractional derivatives we might replace  $[s] + 1$  by  $s + \varepsilon$ , which can be arbitrarily close to 0. Consequently, it is especially interesting to find a Sobolev formulation of Theorem A and the

corresponding interpolated ‘CT-condition’ to give room for  $L_p$  multipliers with arbitrarily mild decay as  $p \rightarrow 2$ . Using left invariant Lie derivatives

$$\lambda(\partial_X m) = \int_G \frac{d}{ds} \Big|_{s=0} m(g \exp(sX)) \lambda(g) d\mu(g) = \lambda(m) \frac{d}{ds} \Big|_{s=0} \lambda(\exp(-sX)) = \lambda(m) a_X$$

for  $m$  regular enough. In particular, letting  $A = -\sum_j a_{X_j}^2$  for certain ONB  $X_1, X_2, \dots, X_{\dim \mathfrak{g}}$  of the Lie algebra  $\mathfrak{g}$ , we could define the Sobolev space  $H_{q,s}(\mathbb{G})$  with ‘ $s$ -derivatives in  $L_q$ ’ as follows

$$\|m\|_{H_{q,s}(\mathbb{G})} = \|\widehat{\lambda(m)(1+A)^{\frac{s}{2}}}\|_{L_q(\mathbb{G})}.$$

Compared to Remark 3.5 this definition is intrinsic to  $\mathbb{G}$ . Nevertheless, a Sobolev form of the Hörmander-Mikhlin condition requires to find a ‘dilation map’ in the group, so that the resulting condition recovers the Mikhlin one for Lie derivatives as  $q \rightarrow \infty$  for  $s \in \mathbf{Z}_+$ . We have no results in this direction.

**D. Other Lie groups.** The new techniques in this paper are beyond the scope of [16, 29, 30], notably since we include nonorthogonal cocycles. This opens a door to investigate regularity conditions for  $L_p$ -multipliers in many other unimodular Lie groups. This is work in progress which will appear elsewhere.

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## References

1. J.P. Anker,  $L_p$  Fourier multipliers on Riemannian symmetric spaces of the noncompact type. *Ann. of Math.* **132** (1990), 597-628.
2. J.P. Anker and N. Lohoué, Multiplicateurs sur certain espaces symétriques. *Amer. J. Math.* **108** (1986), 1303-1354.
3. M. Bożejko and G. Fendler, Herz-Schur multipliers and completely bounded multipliers of the Fourier algebra of a locally compact group. *Boll. Un. Mat. Ital.* **3** (1984), 297-302.
4. A.P. Calderón and A. Torchinsky, Parabolic maximal functions associated with a distribution II. *Adv. Math.* **24** (1977), 101-171.
5. J. de Cannière and U. Haagerup, Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups. *Amer. J. Math.* **107** (1985), 455-500.
6. M. Caspers, J. Parcet, M. Perrin and É. Ricard, Noncommutative de Leeuw theorems. *Forum of Mathematics  $\Sigma$*  **3** (2015), e21.
7. M. Caspers and M. de la Salle, Schur and Fourier multipliers of an amenable group acting on non-commutative  $L_p$ -spaces. *Trans. Amer. Math. Soc.* **367** (2015), 6997-7013.
8. J.L. Clerc and E.M. Stein,  $L^p$ -multipliers for noncompact symmetric spaces. *Proc. Nat. Acad. USA* **71** (1974), 3911-3912.
9. M. Cowling, B. Dorofaeff, A. Seeger and J. Wright, A family of singular oscillatory integral operators and failure of weak amenability. *Duke Math. J.* **127** (2005), 429-486.
10. M. Cowling and U. Haagerup, Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. *Invent. Math.* **96** (1989), 507-549.
11. M. Cowling, U. Haagerup and R. Howe, Almost  $L^2$  matrix coefficients. *J. reine angew. Math.* **387** (1988), 97-110

12. J. Erven and B.J. Falkowski, Low Order Cohomology and Applications. Lecture Notes Math. **877**. Springer-Verlag, 1981.
13. C. Fefferman, The multiplier problem for the ball. Ann. of Math. **94** (1971), 330-336.
14. G. Garrigós and A. Seeger, Characterizations of Hankel multipliers. Math. Ann. **342** (2008), 31-68.
15. A. González-Pérez, Crossed-product extensions of  $L_p$ -bounds for amenable actions. Preprint 2016.
16. A.M. González-Pérez, M. Junge and J. Parcet, Smooth Fourier multipliers in group algebras via Sobolev dimension. Ann. Sci. École Norm. Sup. **50** (2017), 879-925.
17. L. Grafakos, Classical Fourier Analysis. 2nd edition. Graduate Texts in Math. Springer, 2009.
18. L. Grafakos, D. He, P. Honzik, H.V. Nguyen, The Hörmander multiplier theorem I: The linear case revisited. Illinois J. Math. **61** (2017), 25-35.
19. U. Haagerup, An example of a non nuclear  $C^*$ -algebra, which has the metric approximation property. Invent. Math. **50** (1979), 279-293.
20. U. Haagerup, Group  $C^*$ -algebras without the completely bounded approximation property. J. Lie Theory **26** (2016), no. 3, 861-887.
21. U. Haagerup, T. de Laat, Simple Lie groups without the approximation property, Duke Math. J. **162** (2013) 925-964.
22. Harish-Chandra, Representations of a semisimple Lie group on a Banach space. I. Trans. Amer. Math. Soc. **75**, (1953) 185-243.
23. Harish-Chandra, Spherical functions on a semisimple Lie group I. Amer. J. Math. **80** (1958), 553-613.
24. Harish-Chandra, Discrete series for semisimple Lie groups II. Acta Math. **116** (1966), 1-111.
25. Y. Heo, F. Nazarov and A. Seeger, Radial Fourier multipliers in high dimensions. Acta Math. **206** (2011), 55-92.
26. L. Hörmander, Estimates for translation invariant operators in  $L^p$  spaces. Acta Math. **104** (1960), 93-140.
27. A.D. Ionescu, An endpoint estimate for the Kunze-Stein phenomenon and related maximal operator. Ann. of Math. **152** (2000), 259-275.
28. A.D. Ionescu, Singular integrals on symmetric spaces of real rank one. Duke Math. J. **114** (2002), 101-122.
29. M. Junge, T. Mei and J. Parcet, Smooth Fourier multipliers on group von Neumann algebras. Geom. Funct. Anal. **24** (2014), 1913-1980.
30. M. Junge, T. Mei and J. Parcet, Noncommutative Riesz transforms – Dimension free bounds and Fourier multipliers. J. Eur. Math. Soc. **20** (2018), 529-595.
31. M. Junge and Z.J. Ruan, Approximation properties for noncommutative  $L_p$ -spaces associated with discrete groups. Duke Math. J. **117** (2003), 313-341.
32. A.W. Knapp, Lie groups beyond an introduction. Progress in Math. **140**. Birkhäuser, 2002.
33. T. de Laat, Approximation properties for noncommutative  $L^p$ -spaces associated with lattices in Lie groups. J. Funct. Anal. **264** (2013) 2300-2322.
34. T. de Laat and M. de la Salle, Approximation properties for noncommutative  $L_p$  of high rank lattices and nonembeddability of expanders. J. Reine Angew. Math. **737** (2018), 46-69
35. V. Lafforgue, Un renforcement de la propriété (T). Duke. Math. J. **143** (2008), 559-602.
36. V. Lafforgue and M. de la Salle, Noncommutative  $L_p$ -spaces without the completely bounded approximation property. Duke. Math. J. **160** (2011), 71-116.
37. K. de Leeuw, On  $L_p$  multipliers. Ann. of Math. **81** (1965), 364-379.
38. N. Lohoué and M. Marias, Invariants géométriques des espaces localement symétriques et théorèmes de multiplicateurs. Math. Ann. **343** (2009), 639-667.
39. F. Lust-Piquard, Inégalités de Khintchine dans  $C_p$  ( $1 < p < \infty$ ). C.R. Acad. Sci. Paris **303** (1986), 289-292.
40. F. Lust-Piquard and G. Pisier, Non-commutative Khintchine and Paley inequalities. Ark. Mat. **29** (1991), 241-260.
41. T. Mei, Operator-valued Hardy spaces. Mem. Amer. Math. Soc. **188**, 2007.
42. T. Mei and J. Parcet, Pseudo-localization and noncommutative Littlewood-Paley inequalities. Int. Math. Res. Not. **9** (2009), 1433-1487.
43. T. Mei and É. Ricard, Free Hilbert Transforms. Duke Math. J. **166** (2017), 2153-2182.
44. S.G. Mikhlin, On the multipliers of Fourier integrals. Dokl. Akad. Nauk SSSR **109** (1956), 701-703.

45. S. Neuwirth and É. Ricard, Transfer of Fourier multipliers into Schur multipliers and sumsets in a discrete group. *Canad. J. Math.* **63** (2011), 1161-1187.
46. H. Oh, Uniform bounds for matrix coefficients of unitary representations and applications to Kazhdan constants. *Duke Math. J.* **113** (2002), 133-192.
47. J. Parcet, Pseudo-localization of singular integrals and noncommutative Calderón-Zygmund theory. *J. Funct. Anal.* **256** (2009), 509-593.
48. J. Parcet and K. Rogers, Twisted Hilbert transforms vs Kakeya sets of directions. *J. Reine Angew. Math.* **710** (2016), 137-172.
49. G. Pisier, Multipliers and lacunary sets in non-amenable groups. *Amer. J. Math.* **117** (1995), 337-376.
50. G. Pisier, Non-Commutative Vector Valued  $L_p$ -Spaces and Completely  $p$ -Summing Maps. *Astérisque* **247** (1998), 1-111.
51. G. Pisier, Introduction to Operator Space Theory. Cambridge University Press, 2003.
52. F. Ricci and B. Wróbel, Spectral multipliers for functions of fixed K-type on  $L^p(SL(2, \mathbf{R}))$ . arXiv:1809.09385.
53. I.J. Schoenberg, Metric spaces and completely monotone functions. *Ann. of Math.* **39** (1938), 811-841.
54. R.J. Stanton and P.A. Tomas, Expansions for spherical functions on noncompact symmetric spaces. *Acta Math.* **140** (1978), 251-276.
55. E.M. Stein, Harmonic Analysis. Princeton Math. Ser. **43**. Princeton Univ. Press., NJ, 1993.
56. P. A. Tomas, On radial Fourier multipliers. Ph.D. Thesis. Cornell University, 1974.
57. W. Trebels, Some necessary conditions for radial Fourier multipliers. *Proc. Amer. Math. Soc.* **58** (1976), 97-103.
58. F. Trèves, Basic Linear Partial Differential Equations. Dover Publications, 1975.
59. X. Xiong, Espaces de fonctions sur les tores quantiques. Ph.D. Thesis, 2015.

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