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A block moment method to handle spectral condensation phenomenon in parabolic control problems

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41 null-controllability of the one dimensional heat equation using a nonhomogeneous
42 boundary condition as a control. For this purpose, they give a direct strategy re-
43 ducing the null-controllability property to a moment problem that the control should
44 satisfy (see Sec. 1.4 for a presentation of the moment problem). The moment method
45 they propose consists in solving this problem using a biorthogonal family in $L^2(0, T)$
46 to the family of exponentials associated to the eigenvalues of \mathcal{A}^* . Let us mention that
47 this idea of reducing a (optimal) control problem to a moment problem is already
48 present in the works [21] by J.V. Egorov and [26] by L.I. Gal'chuk.

49 Later on, A.V. Fursikov, O.Yu. Imanuvilov [25] and G. Lebeau, L. Robbiano [34]
50 used Carleman estimates to solve the boundary and internal null-controllability prob-
51 lem of the heat equation in any space dimension. These two papers have gener-
52 ated plenty of null-controllability results for various parabolic problems. The com-
53 mon qualitative behavior of these results is that for scalar parabolic equations null-
54 controllability holds in arbitrary time (i.e. $T_0 = 0$) and without any restriction on the
55 control domain.

56 Among all of these results let us mention the peculiar work [17] by S. Dolecki.
57 He considered a one dimensional heat equation, with a scalar control located at one
58 point inside the space interval, and proved that choosing suitably the location of this
59 control point one can achieve any value in $[0, +\infty]$ for the minimal null-control time
60 T_0 . Until the years 2000's this work seemed to be considered too peculiar and the
61 possible presence of a positive minimal null-control time (a very natural property
62 in the hyperbolic case) was expected to be generically not possible in a parabolic
63 setting. However, this point of view has been reconsidered recently in various works
64 as for instance: [3] for abstract control problems, [5] for the abstract problem (1) with
65 applications to systems of one dimensional coupled parabolic equations, and [10] for
66 a degenerate parabolic two dimensional equation of Grushin type.

67 Since then, the minimal null-control time property was investigated on various
68 examples, still in the setting of coupled one dimensional parabolic systems [6, 18,
69 36, 42] or in the setting of degenerate parabolic scalar equations [9, 11, 12, 13, 19].
70 For coupled parabolic systems a geometric control condition may also be needed for
71 approximate controllability to hold [15, 38], proving once again that hyperbolic-like
72 behavior can be observed in the parabolic setting. We will however not consider this
73 last feature in this work.

74 Concerning the study of the abstract control problem (1), some characterization
75 are provided in the series of works [27, 28, 29, 30, 31] using the formalism of Carleson
76 measures. However the precise question of an abstract characterization of the minimal
77 null-control time has not been much considered. A formula has been given for the
78 minimal null-control time of system (1), in [5], in a setting encompassing coupled
79 one dimensional parabolic equations with a scalar control. Its value depends on the
80 condensation index of the eigenvalues of \mathcal{A}^* (see Sec. 7.5 for a precise definition)
81 and the observation of the associated eigenvectors. In this work the authors assume
82 that the eigenvectors form a Riesz basis of the state space. Let us also mention the
83 work [7] where the null-control time is studied through a resolvent-like inequality
84 (introduced in [20]) that is a quantification of the well-known Fattorini-Hautus test
85 for approximate controllability (see [22, 38]). It is an abstract characterization that
86 might not be easily computable on actual examples but provides a common setting
87 for all the previous examples exhibiting a positive null-control time. The last two
88 mentioned results also rely on the moment method. Note that, even if the Carleman
89 approach is very powerful, it does not seem to be applicable to all the systems of
90 interest: in many situations (including the ones discussed in Section 5.2) the moment

91 method is still the only successful technique up to now.

92 To highlight the limitations of the existing literature on such problems and the
 93 improvements we propose, let us consider the following control problem

$$94 \quad \begin{cases} \partial_t y + \begin{pmatrix} -\partial_{xx} & 1 \\ 0 & -\partial_{xx} + c(x) \end{pmatrix} y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & (t, x) \in (0, T) \times (0, 1), \\ y(t, 0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad y(t, 1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & t \in (0, T), \\ y(0, x) = y_0(x), & x \in (0, 1), \end{cases}$$

95 where $c \in L^2(0, 1; \mathbb{R})$ is a given potential. We insist on the fact that our goal is
 96 not to study this particular example but to develop a general characterization. The
 97 application to this particular example is detailed in Sec. 5.2. The study of the minimal
 98 null-control time for this system for an arbitrary potential c is not covered by the
 99 literature for several reasons.

- 100 • First, depending on c , the underlying operator can have geometrically double
 101 eigenvalues. This induces (a finite number of) non-observable modes and thus
 102 prevents even approximate controllability. We thus propose to extend the
 103 study of the minimal null-control time to a given subspace of initial conditions.
 104 This allows to still analyze the controllability properties in this case.
- 105 • Even if the potential c is such that the eigenvalues are geometrically simple
 106 it can happen that some of them are algebraically double. In this case, to
 107 the best of our knowledge, the only existing result is [4] which ensures null-
 108 controllability in arbitrary time if the eigenvalues are well separated (i.e.
 109 satisfy the classical gap condition recalled in (27)).
- 110 • Finally, if the potential c is such that the eigenvalues are geometrically and
 111 algebraically simple, to the best of our knowledge, the only existing result
 112 can be found in [5]. Under an extra assumption (on the observability of
 113 eigenfunctions), it provides null-controllability at any time T satisfying

$$114 \quad T > T^* = \limsup_{\lambda \in \sigma(\mathcal{A}^*)} \frac{-\ln \text{dist}(\lambda, \sigma(\mathcal{A}^*) \setminus \{\lambda\})}{\lambda}.$$

115 However, their arguments to disprove null-controllability at time $T < T^*$
 116 should not apply in this example for every such potential c as the family of
 117 eigenvectors forms a complete family but might not be a Riesz basis. There-
 118 fore, the above formula for T^* may dramatically overestimate the actual null-
 119 control time for the system. We will see in Section 5.2.1 that it may happen
 120 that $T^* = +\infty$ whereas the system is actually null-controllable at any time
 121 $T > 0$.

122 We will use quite general assumptions in our analysis answering all these concerns.
 123 Doing so, we will prove that the difference between the Riesz basis assumption and
 124 the complete family assumption for the eigenvectors is not only technical and a new
 125 phenomenon can appear: the condensation of eigenvalues can be compensated by the
 126 condensation of eigenvectors.

127 We continue this introduction by stating more precisely the problem under con-
 128 sideration and the obtained results.

129 1.2. Functional setting.

130 Let X be a separable Hilbert space, whose inner product and norm are denoted
131 by $\langle \bullet, \bullet \rangle$ and $\|\bullet\|$ respectively. We shall systematically identify X to its dual through
132 the Riesz theorem. Let $(\mathcal{A}, D(\mathcal{A}))$ be an unbounded operator in X such that $-\mathcal{A}$
133 generates a C^0 -semigroup in X and $(\mathcal{A}^*, D(\mathcal{A}^*))$ its adjoint in X . Up to a suitable
134 translation, we can assume that 0 is in the resolvent set of \mathcal{A} . We denote by X_1 (resp.
135 X_1^*) the Hilbert space $D(\mathcal{A})$ (resp. $D(\mathcal{A}^*)$) equipped with the norm $\|x\|_1 := \|\mathcal{A}x\|$
136 (resp. $\|x\|_{1^*} := \|\mathcal{A}^*x\|$). We define X_{-1} as the completion of X with respect to the
137 norm

$$138 \quad \|y\|_{-1} := \sup_{z \in X_1^*} \frac{\langle y, z \rangle}{\|z\|_{1^*}}.$$

139 Notice that X_{-1} is isometrical to the topological dual of X_1^* using X as a pivot space
140 (see for instance [45, Proposition 2.10.2]); the corresponding duality bracket will be
141 denoted by $\langle \bullet, \bullet \rangle_{-1, 1^*}$.

142 Let U be an Hilbert space (that we will identify to its dual) and $\mathcal{B} : U \rightarrow X_{-1}$ be
143 a linear continuous control operator. We denote by $\mathcal{B}^* : X_1^* \rightarrow U$ its adjoint in the
144 duality described above.

145 **PROPOSITION 1.1.** *Under the above assumptions, for any $T > 0$, any $y_0 \in X_{-1}$,
146 and any $u \in L^2(0, T; U)$, there exists a unique $y \in C^0([0, T]; X_{-1})$ solution to (1) in
147 the sense that it satisfies for any $t \in [0, T]$ and any $z_t \in X_1^*$,*

$$148 \quad (2) \quad \langle y(t), z_t \rangle_{-1, 1^*} - \langle y_0, e^{-t\mathcal{A}^*} z_t \rangle_{-1, 1^*} = \int_0^t \left(u(s), \mathcal{B}^* e^{-(t-s)\mathcal{A}^*} z_t \right)_U ds.$$

149 Moreover there exists $C_T > 0$ such that

$$150 \quad \sup_{t \in [0, T]} \|y(t)\|_{-1} \leq C_T (\|y_0\|_{-1} + \|u\|_{L^2(0, T; U)}).$$

151 The proof of this result is recalled in Appendix 7.1. Let us mention that this notion of
152 solution is very weak. In most works concerning controllability properties for abstract
153 systems like (1), an extra admissibility assumption is made on the control operator \mathcal{B}
154 to ensure more regularity for the solutions. Note however that this is not mandatory
155 to prove wellposedness of the system in the weak sense above. We will discuss below
156 the regularity properties of the control problem.

157 Let $(X_\diamond^*, \|\cdot\|_{\diamond^*})$ be an Hilbert space such that $X_1^* \subset X_\diamond^* \subset X$ with dense and
158 continuous embeddings. We assume that X_\diamond^* is stable by the semigroup generated by
159 $-\mathcal{A}^*$ (see Remark 1.1). We also define $X_{-\diamond}$ as the subspace of X_{-1} defined by

$$160 \quad X_{-\diamond} := \left\{ y \in X_{-1}; \|y\|_{-\diamond} := \sup_{z \in X_1^*} \frac{\langle y, z \rangle_{-1, 1^*}}{\|z\|_{\diamond^*}} < +\infty \right\},$$

161 which is also isometrical to the dual of X_\diamond^* with X as a pivot space. The corresponding
162 duality bracket will be denoted by $\langle \bullet, \bullet \rangle_{-\diamond, \diamond^*}$. Thus, we end up with the following five
163 functional spaces

$$164 \quad X_1^* \subset X_\diamond^* \subset X \subset X_{-\diamond} \subset X_{-1}.$$

165 We say that the control operator \mathcal{B} is an admissible control operator for (1) with
166 respect to the space $X_{-\diamond}$ if

$$167 \quad (3) \quad \int_0^T \left\| \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} z \right\|_U^2 dt \leq C \|z\|_{\diamond^*}^2, \quad \forall z \in X_1^*.$$

168 This implies that, by density, we can give a meaning to the map

$$169 \quad (t \mapsto \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} z) \in L^2(0, T; U),$$

170 for any $z \in X_\diamond^*$.

171 In this setting, following the lines of [16, Theorem 2.37] we obtain the following
172 regularity result for the solutions.

173 **PROPOSITION 1.2.** *Assume that (3) holds. Then, for any $T > 0$, any $y_0 \in X_{-\diamond}$,
174 and any $u \in L^2(0, T; U)$, there exists a unique $y \in C^0([0, T]; X_{-\diamond})$ solution to (1) in
175 the sense that it satisfies for any $t \in [0, T]$ and any $z_t \in X_\diamond^*$,*

$$176 \quad \langle y(t), z_t \rangle_{-\diamond, \diamond} - \langle y_0, e^{-t\mathcal{A}^*} z_t \rangle_{-\diamond, \diamond} = \int_0^t \left(u(s), \mathcal{B}^* e^{-(t-s)\mathcal{A}^*} z_t \right)_U ds.$$

177 Moreover there exists $C_T > 0$ such that

$$178 \quad \sup_{t \in [0, T]} \|y(t)\|_{-\diamond} \leq C_T (\|y_0\|_{-\diamond} + \|u\|_{L^2(0, T; U)}).$$

REMARK 1.1. *Note that a similar regularity result holds if we don't assume that
 X_\diamond^* is stable by the semigroup generated by $-\mathcal{A}^*$ except that we need to restrict our-
selves to initial data $y_0 \in X$. In that case the solution satisfies for any $t \in [0, T]$ and
any $z_t \in X_\diamond^*$,*

$$\langle y(t), z_t \rangle_{-\diamond, \diamond} - \langle y_0, e^{-t\mathcal{A}^*} z_t \rangle_{-\diamond, \diamond} = \int_0^t \left(u(s), \mathcal{B}^* e^{-(t-s)\mathcal{A}^*} z_t \right)_U ds,$$

$$179 \quad \sup_{t \in [0, T]} \|y(t)\|_{-\diamond} \leq C_T (\|y_0\| + \|u\|_{L^2(0, T; U)}).$$

181 **REMARK 1.2.** *The case where $X_\diamond^* = X_1^*$ means that we do not have any additional
182 regularity property for \mathcal{B} . Conversely, the case $X_\diamond^* = X$ means that we have the best
183 regularity we can hope for system (1) (this is the usual definition of an admissible
184 control operator as in [16, 45]).*

185 To give more accurate results, we aim at analyzing the minimal null-control time
186 problem for each specified set of initial data. This is the object of the following
187 definition.

188 **DEFINITION 1.1.** *Let Y_0 be a closed subspace of $X_{-\diamond}$.*

189 *We say that system (1) is null-controllable at time T from Y_0 if for any $y_0 \in Y_0$
190 there exists $u \in L^2(0, T; U)$ such that the associated solution of (1) satisfies $y(T) = 0$.*

191 As a specific choice of Y_0 one can think of $Y_0 = X_{-\diamond}$, in which case we recover the
192 classical notion of null-controllability. On the opposite side, if Y_0 is a one dimensional
193 subspace $Y_0 = \text{Span}\{y_0\}$, then the notion above amounts to consider only the null-
194 controllability of the system for that particular initial condition y_0 .

195 From now on, we will assume that the space Y_0 is given, and we denote by P_{Y_0} the
196 orthogonal projection onto Y_0 with respect to $\|\bullet\|_{-\diamond}$ and by $P_{Y_0}^* \in L(X_\diamond^*)$ its adjoint
197 in the duality $X_{-\diamond}, X_\diamond^*$. Notice that these definitions yield

$$198 \quad (4) \quad \|P_{Y_0}^* z\|_{\diamond^*} \leq \|z\|_{\diamond^*}, \quad \forall z \in X_\diamond^*.$$

199

Notations. We give here some notations that will be used throughout this article.

- For any integers $a, b, c \in \mathbb{N}$, we shall define the following subsets of \mathbb{N} :

$$\llbracket a, b \rrbracket := [a, b] \cap \mathbb{N},$$

$$\llbracket a, b \rrbracket_{\neq c} := \llbracket a, b \rrbracket \setminus \{c\}.$$

200

- For any complex number $\mu \in \mathbb{C}$ we define $e_\mu : (0, +\infty) \rightarrow \mathbb{C}$ to be the exponential function

201

$$(5) \quad e_\mu : s \mapsto e^{-\mu s}.$$

202

203

- We shall denote by $C_{\gamma_1, \dots, \gamma_l} > 0$ a constant possibly varying from one line to another but depending only on the parameters $\gamma_1, \dots, \gamma_l$.

204

205

- For any multi-index $\alpha \in \mathbb{N}^n$, we denote its length by $|\alpha| = \sum_{j=1}^n \alpha_j$ and its maximum by $|\alpha|_\infty = \max_{j \in \llbracket 1, n \rrbracket} \alpha_j$.

206

207

For $\alpha, \mu \in \mathbb{N}^n$, we say that $\mu \leq \alpha$ if and only if $\mu_j \leq \alpha_j$ for any $j \in \llbracket 1, n \rrbracket$.

208

- For any finite subset $A \subset \mathbb{C}$, we will make use of the polynomial P_A defined by

209

$$(6) \quad P_A(x) := \prod_{\mu \in A} (x - \mu).$$

210

It satisfies in particular, for any $\lambda \in A$,

211

$$P'_A(\lambda) = \prod_{\substack{\mu \in A \\ \mu \neq \lambda}} (\lambda - \mu).$$

212

213

1.3. Presentation of the main results.

214

1.3.1. Spectral assumptions. In addition to the hypothesis described in the introduction and that are necessary for the well-posedness and regularity of our controlled problem, we shall make now the following structural assumptions.

215

216

- First of all, we shall only consider scalar controls in this paper, that is $U = \mathbb{R}$.

217

We will study in a forthcoming paper the extension of our analysis to the case of non scalar controls.

218

219

- We assume that the spectrum of \mathcal{A}^* is only made of a countable number of geometrically simple eigenvalues (this is mandatory since we only consider scalar controls) denoted by Λ . We shall also assume for simplicity that the eigenvalues are all real (see however the discussion in Section in Sec. 6.1).

220

221

222

223

Replacing \mathcal{A} by $\mathcal{A} + \gamma$ for $\gamma > 0$ large enough if necessary, we can always assume for instance that $\Lambda \subset [1, +\infty)$.

224

225

- For any eigenvalue $\lambda \in \Lambda$, we denote by $\alpha_\lambda \geq 1$ its algebraic multiplicity and we assume that there exists an integer $\eta \geq 1$ such that $\alpha_\lambda \leq \eta$ for any $\lambda \in \Lambda$.

226

227

- The main structural assumptions on the eigenvalues Λ we shall make in this paper are the following:

228

229

- Asymptotic behavior:

230

$$(7) \quad \sum_{\lambda \in \Lambda} \frac{1}{\lambda} < +\infty.$$

231

- **Weak gap condition** with parameters $p \in \mathbb{N}$ and $\rho > 0$:

232

$$(8) \quad \# \left(\Lambda \cap [\mu, \mu + \rho] \right) \leq p, \quad \forall \mu \in [0, +\infty).$$

233

234 In the case $p = 1$, the weak gap condition above simply reduces to

$$235 \quad |\lambda - \lambda'| > \rho, \quad \forall \lambda, \lambda' \in \Lambda, \lambda \neq \lambda',$$

236 which is the usual gap condition used for instance in [23]. If the spectrum Λ
237 is increasingly indexed as $\Lambda = (\lambda_m)_{m \geq 1}$ the weak gap condition (8) reads

$$238 \quad \lambda_{m+p} - \lambda_m > \rho, \quad \forall m \geq 1.$$

239 As we will use a different labelling of the spectrum in this paper we shall not
240 use these notations anymore in what follows.

241 • We denote by $(\phi_\lambda^0)_{\lambda \in \Lambda}$ an associated family of eigenvectors of \mathcal{A}^* . These
242 eigenvectors are chosen to be normalized in X_\diamond^* .

243 As we are interested in null-controllability properties of system (1), we will
244 first assume that

$$245 \quad (9) \quad \mathcal{B}^* \phi_\lambda^0 \neq 0, \quad \text{for any } \lambda \in \Lambda.$$

246 This is a necessary condition for the approximate controllability of system (1),
247 and is therefore mandatory if we expect null-controllability to hold. In our
248 setting, the assumption (9) is also a sufficient condition for approximate con-
249 trollability (see [22, 38]).

250 When the considered set of initial data Y_0 is not the whole space $X_{-\diamond}$, the
251 approximate controllability condition (9) can be too strong and we can relax
252 it. We will discuss this point in section 6.2.

253 • For each $\lambda \in \Lambda$, we denote by $(\phi_\lambda^l)_{l \in \llbracket 1, \alpha_\lambda - 1 \rrbracket}$ a Jordan chain associated with
254 ϕ_λ^0 , that is a family satisfying

$$255 \quad \mathcal{A}^* \phi_\lambda^l = \lambda \phi_\lambda^l + \phi_\lambda^{l-1}, \quad \forall l \in \llbracket 1, \alpha_\lambda - 1 \rrbracket.$$

256 By (9), we may uniquely determine such Jordan chain if we impose in addition
257 that the generalized eigenvectors satisfy

$$258 \quad (10) \quad \mathcal{B}^* \phi_\lambda^l = 0, \quad \forall l \in \llbracket 1, \alpha_\lambda - 1 \rrbracket.$$

259 This particular choice of the Jordan chain is not mandatory but will sim-
260 plify the forthcoming computations. In the case were the eigenvalues are
261 algebraically simple ($\eta = 1$) we drop the superscript 0 for the eigenvectors.

• We introduce the notation

$$\Phi := \{\phi_\lambda^l, \lambda \in \Lambda, l \in \llbracket 0, \alpha_\lambda - 1 \rrbracket\},$$

262 for the family of all the (generalized) eigenvectors of \mathcal{A}^* . We assume that Φ
263 is complete in X_\diamond^* i.e. for any $y \in X_{-\diamond}$, we have

$$264 \quad (11) \quad \left(\langle y, \phi \rangle_{-\diamond, \diamond} = 0, \quad \forall \phi \in \Phi \right) \implies y = 0.$$

265 We emphasize the fact that we will not make any additional assumptions on the
266 family Φ . This is a very important difference with related results in the literature
267 in which, most of the time, it is assumed that Φ forms a Riesz basis of X_\diamond^* . This is
268 discussed in Sections 1.3.4 and 3.

269 **1.3.2. Groupings of eigenvalues.** To introduce our formula for the minimal
 270 null-control time it is convenient to define adapted groupings for the spectrum Λ . We
 271 highlight that this notion does not exactly coincide with the condensation groupings
 272 introduced by Bernstein [14], even though it is closely related.

273 **DEFINITION 1.2.** Let $p \in \mathbb{N}^*$ and $r, \rho > 0$. A sequence of sets $(G_k)_{k \geq 1} \subset \mathcal{P}(\Lambda)$
 274 is said to be a grouping for Λ with parameters p, r, ρ , and we will write $(G_k)_{k \geq 1} \in$
 275 $\mathcal{G}(\Lambda, p, r, \rho)$, if it is a covering of Λ

$$276 \quad \Lambda = \bigcup_{k \geq 1} G_k,$$

277 with the additional properties that for every $k \geq 1$,

$$278 \quad g_k := \#G_k \leq p,$$

$$279 \quad \sup(G_k) < \inf(G_{k+1}),$$

$$280 \quad \text{dist}(G_k, G_{k+1}) \geq r,$$

$$281$$

$$282 \quad (12) \quad \text{diam } G_k < \rho,$$

283 and

$$284 \quad (13) \quad \text{diam } G_k < \rho.$$

285 We prove in Appendix (Proposition 7.1) that such a grouping always exists for
 286 any Λ satisfying the weak gap condition (8).

287 Once we are given such a grouping, we shall always adopt the following labelling
 288 of the elements of Λ

$$289 \quad G_k = \{\lambda_{k,1}, \dots, \lambda_{k,g_k}\}$$

290 with $\lambda_{k,1} < \dots < \lambda_{k,g_k}$, and the (generalized) eigenvectors will be relabelled accord-
 291 ingly

$$292 \quad \phi_{k,j}^l := \phi_{\lambda_{k,j}}^l, \quad \forall k \geq 1, \forall j \in \llbracket 1, g_k \rrbracket, \forall l \in \llbracket 0, \alpha_{k,j} - 1 \rrbracket,$$

293 where in the same fashion $\alpha_{k,j} := \alpha_{\lambda_{k,j}}$. For any $k \geq 1$, we gather the multiplicities
 294 associated with the elements of G_k in the multi-index $\alpha_k = (\alpha_{\lambda_{k,1}}, \dots, \alpha_{\lambda_{k,g_k}}) \in \mathbb{N}^{g_k}$

295 **1.3.3. Minimal control time definition.** From now on, we assume given a
 296 grouping $(G_k)_k$ in $\mathcal{G}(\Lambda, p, r, \rho)$. Thanks to the assumption (9), we can define the
 297 following family of elements in X_\diamond^*

$$298 \quad (14) \quad \psi_{k,j}^l := \frac{P_{Y_0}^*(\phi_{k,j}^l)}{\mathcal{B}^* \phi_{k,j}^0}, \quad \forall k \geq 1, \forall j \in \llbracket 1, g_k \rrbracket, \forall l \in \llbracket 0, \alpha_{k,j} - 1 \rrbracket.$$

299 Let

$$300 \quad (15) \quad T_0(Y_0) := \limsup_{k \rightarrow \infty} \frac{\ln \left(\max_{\mu \leq \alpha_k} \left\| \psi[\lambda_{k,1}^{(\mu_1)}, \dots, \lambda_{k,g_k}^{(\mu_{g_k})}] \right\|_{\diamond^*} \right)}{\lambda_{k,1}}$$

301 where the notation $\psi[\dots]$ stands for the generalized divided differences (see Section
 302 7.3.2, in particular Proposition 7.7). From Proposition 7.11, notice that the quantity
 303 $\psi[\lambda_{k,1}^{(\mu_1)}, \dots, \lambda_{k,g_k}^{(\mu_{g_k})}]$ appearing in the previous definition is a linear combination of the
 304 elements

$$305 \quad \{\psi_{k,j}^l; j \in \llbracket 1, g_k \rrbracket, l \in \llbracket 0, \alpha_{k,j} - 1 \rrbracket\}$$

306 whose coefficients can be explicitly computed on actual control problems (see Sec. 5)
 307 and that only depends on the group G_k and on the multiplicity multi-index μ .

308 In the simpler case where the eigenvalues are assumed to be algebraically simple
 309 (i.e. $\eta = 1$) we can immediately give a more explicit formula for $T_0(Y_0)$. Indeed, in this
 310 case one recovers the standard divided differences (whose definition and properties are
 311 recalled in Sec. 7.3.1) and thus

$$312 \quad (16) \quad T_0(Y_0) = \limsup_{k \rightarrow \infty} \frac{\ln \left(\max_{\substack{m, l \in \llbracket 1, g_k \rrbracket \\ m \leq l}} \|\psi[\lambda_{k,m}, \dots, \lambda_{k,l}]\|_{\diamond^*} \right)}{\lambda_{k,1}}.$$

313 Then, using Corollary 7.1 and (13) it comes that the computation of all those divided
 314 differences is not needed and the formula reduces to

$$315 \quad (17) \quad T_0(Y_0) = \limsup_{k \rightarrow \infty} \frac{\ln \left(\max_{l \in \llbracket 1, g_k \rrbracket} \|\psi[\lambda_{k,1}, \dots, \lambda_{k,l}]\|_{\diamond^*} \right)}{\lambda_{k,1}}$$

$$316 \quad (18) \quad = \limsup_{k \rightarrow \infty} \frac{\ln \left(\max_{l \in \llbracket 1, g_k \rrbracket} \left\| \sum_{j=1}^l \frac{\psi_{k,j}}{\prod_{i \in \llbracket 1, l \rrbracket, i \neq j} (\lambda_{k,j} - \lambda_{k,i})} \right\|_{\diamond^*} \right)}{\lambda_{k,1}}$$

317 where the last equality comes from the use of Newton formula (see Proposition 7.3).

318 REMARK 1.3. *The definition above corresponds to a given grouping of the spec-*
 319 *trum, however the minimal null-control result stated in Theorem 1.1 will show that its*
 320 *value does not depend on this particular choice of a grouping. As a consequence, for*
 321 *specific examples, one can compute the minimal null control time $T_0(Y_0)$ using any*
 322 *convenient such grouping in a class $\mathcal{G}(\Lambda, p, r, \rho)$.*

323 For the sake of simplicity, for any $y_0 \in X_{-\diamond}$ we denote by $T_0(y_0)$ the quantity
 324 $T_0(\text{Span}(y_0))$. Of course, we have the following proposition relating $T_0(Y_0)$ and $T_0(y_0)$
 325 for $y_0 \in Y_0$.

326 PROPOSITION 1.3. *For any closed subspace $Y_0 \subset X_{-\diamond}$,*

$$327 \quad \sup_{y_0 \in Y_0} T_0(y_0) = T_0(Y_0).$$

328 This assertion is proved in Subsec. 7.4.

329 REMARK 1.4. *Let us discuss the sign of $T_0(Y_0)$.*

- 330 • *In the case $Y_0 = X_{-\diamond}$ (the operator $P_{Y_0}^*$ thus reduces to the identity), the*
 331 *minimal time $T_0(Y_0)$ is always non-negative. Indeed, from the case $\mu =$*
 332 *$(1, 0, \dots, 0)$ in the definition (15) of T_0 we have that*

$$333 \quad T_0(X_{-\diamond}) \geq \limsup_{k \rightarrow \infty} \frac{\ln \frac{\|\phi_{k,1}^0\|_{\diamond^*}}{|\mathcal{B}^* \phi_{k,1}^0|}}{\lambda_{k,1}}.$$

334 From the admissibility condition (3) applied to $z = \phi_{k,1}^0$, we deduce the fol-
 335 lowing upper bound $|\mathcal{B}^* \phi_{k,1}^0| \leq C_T \sqrt{\lambda_{k,1}} \|\phi_{k,1}^0\|_{\diamond^*}$. Thus,

$$336 \quad \limsup_{k \rightarrow \infty} \frac{\ln \frac{\|\phi_{k,1}^0\|_{\diamond^*}}{|\mathcal{B}^* \phi_{k,1}^0|}}{\lambda_{k,1}} \geq 0,$$

338 which proves that $T_0(X_{-\diamond}) \in [0, +\infty]$.

339 • In the general case where Y_0 is a strict closed subspace of $X_{-\diamond}$, it may happen
340 that $T_0(Y_0) < 0$.

For instance, if we choose $y_0 \in X_1$ to be an eigenvector of \mathcal{A} for an eigenvalue
 $\lambda \in \Lambda$, then we have $T_0(y_0) = -\infty$. Indeed, we first observe that

$$\langle y_0, \phi_{\lambda'}^0 \rangle_{-\diamond, \diamond} = (y_0, \phi_{\lambda'}^0) = 0, \quad \forall \lambda' \in \Lambda, \lambda' \neq \lambda,$$

341 which implies, with $Y_0 = \text{Span}(y_0)$ that $P_{Y_0}^* \phi_{\lambda'}^0 = 0$ for any $\lambda' \neq \lambda$. We
342 deduce that the logarithms in the definition of $T_0(y_0)$ are all equal to $-\infty$ for
343 k large enough.

344 **1.3.4. Null-controllability result.** The main result of this paper reads as fol-
345 lows (see also the extension discussed in Sec. 6.1).

346 **THEOREM 1.1.** *Assume that the operators \mathcal{A} and \mathcal{B} satisfy the assumptions given*
347 *in Sec. 1.3.1. Let $T > 0$ and $T_0(Y_0)$ be defined by (15). Then,*

- 348 i. *If $T_0(Y_0) < +\infty$ and $T > T_0(Y_0)$, the system (1) is null-controllable from Y_0*
349 *at time T .*
350 ii. *If $T_0(Y_0) > 0$ and $T < T_0(Y_0)$, the system (1) is not null-controllable from Y_0*
351 *at time T .*

352 In the case where $T > T_0(Y_0)$ we actually prove in Corollary 2.2 a more accurate result
353 giving a uniform (with respect to Λ in a certain class) estimate of the controllability
354 cost. Those uniform estimates are important in various contexts when one wants to
355 achieve bounds on the control for parameter-dependent problems (see for instance [1,
356 2] for an application in numerical analysis of null-controllability problems, or [35] for
357 an application in oscillating coefficient problems). Moreover, this uniformity property
358 will be crucial in Section 4 to infer the results on multiple eigenvalues from the ones
359 on simple eigenvalues.

360 Let us briefly mention that our strategy of proof relies on an adapted block reso-
361 lution of the associated moment problem (see Theorems 2.1 and 4.1). In the case of
362 spectral condensation this new method of resolution ensures sharper results than the
363 one given by standard biorthogonal families. However, as a by-product, in the case
364 of algebraically simple eigenvalues we recover the known optimal estimates for such
365 biorthogonal families (see Corollary 2.1). In the case of algebraically multiple eigenval-
366 ues we provide new estimates for such biorthogonal families (see Corollary 4.1). Before
367 describing with more details this strategy of proof let us make some comments.

- 368 • There are settings in which formulas for the minimal null-control time are
369 already known in the literature for instance when the eigenvalues are alge-
370 braically simple and:
371 – when the condensation index of Λ (see Appendix 7.5 for a precise defi-
372 nition) is equal to 0 (see [7, Remark 1.15]);
373 – or when the family $(\phi_\lambda)_{\lambda \in \Lambda}$ of eigenvectors forms a Riesz basis of X_\diamond^*
374 (see [5]).

375 Obviously, in those settings we recover the known expressions. This is dis-
376 cussed in Sec. 3.1 and Sec. 3.2.

- 377 • However, we also prove that the Riesz basis assumption considered in [5] is not
378 only technical. More precisely, we show in Proposition 3.2, that if the Riesz
379 basis assumption does not hold, then the actual minimal control time is less
380 or equal than the value T^* given by the formula in this reference (see (62)).

381 Moreover, we present in Sec. 5.1, a few examples that are built such that the
 382 value of T^* is any chosen element of $[0, +\infty]$ whereas the minimal null-control
 383 time $T_0(X_{-\diamond})$ is in fact 0.

384 This highlights a new phenomenon: when $(\phi_\lambda)_{\lambda \in \Lambda}$ does not form a Riesz
 385 basis, it may happen that the eigenvectors condensate (or more precisely the
 386 eigenvectors normalized with respect to the observation i.e. $\frac{\phi_\lambda}{\mathcal{B}^* \phi_\lambda}$) and this
 387 condensation can compensate for the condensation of eigenvalues.

388 • The weak gap condition (8) is particularly well adapted to the applications we
 389 have in mind, namely coupled one dimensional parabolic equations in which
 390 case the spectrum is given by a finite union of sequences satisfying a classical
 391 gap condition (see for instance Lemma 2.1).

392 The restriction to the one dimensional case in those applications comes from
 393 the assumption (7). Although this assumption can be seen as a restriction
 394 due to the use of moment method, as we are considering scalar controls ($U =$
 395 \mathbb{R}) it is also a necessary null-controllability condition (see for instance [37,
 396 Appendix A]).

397 • As we precised the space of initial conditions in this study of minimal null-
 398 control time, it directly comes that finite linear combination of eigenvectors
 399 are null-controllable in arbitrary small time: the existence of positive mini-
 400 mal null-control time is definitely a high-frequency phenomenon as already
 401 observed in Remark 1.4.

402 1.3.5. Structure of the article.

403 We end this introduction by describing the global strategy used to prove Theo-
 404 rem 1.1 and giving some further bibliographical comments. Section 2 is dedicated to
 405 the proof of Theorem 1.1 in the case of algebraically simple eigenvalues. We provide
 406 in Sec. 3 a comparison of our results with available results of the literature. In Sec. 4
 407 we prove that the uniform estimates obtained in Sec. 2 allow to prove Theorem 1.1 in
 408 the general case of algebraically multiple eigenvalues. To highlight the new cases and
 409 phenomenon covered by our analysis we present different examples in Sec. 5. Then
 410 we propose some extensions in Sec. 6. To ease the reading we gather various technical
 411 results in Sec. 7.

412 1.4. Strategy of proof.

413 The proof of the positive controllability result (that is point i. of Theorem 1.1)
 414 relies on a block resolution of the moment problem. Let us give more details about
 415 this strategy.

416 Let $y_0 \in Y_0$ and $u \in L^2(0, T; \mathbb{R})$ given. Using Proposition 1.2, it comes that

$$417 \quad \langle y(T), \phi \rangle_{-\diamond, \diamond} - \langle y_0, e^{-T\mathcal{A}^*} \phi \rangle_{-\diamond, \diamond} = \int_0^T u(t) \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} \phi dt, \quad \forall \phi \in \Phi.$$

418 As the family Φ of (generalized) eigenvectors is assumed to form a complete family in
 419 X_\diamond^* (see (11)) it comes that $y(T) = 0$ if and only if the control u solves the following
 420 countable set of equations

$$421 \quad (19) \quad - \langle y_0, e^{-T\mathcal{A}^*} \phi_\lambda^l \rangle_{-\diamond, \diamond} = \int_0^T u(t) \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} \phi_\lambda^l dt, \quad \forall \lambda \in \Lambda, \forall l \in \llbracket 0, \alpha_\lambda - 1 \rrbracket.$$

422 Using the formalism of generalized divided differences, we can give a convenient ex-

423 pression of the action of the semi-group on the generalized eigenvectors as follows

$$\begin{aligned}
e^{-tA^*} \phi_\lambda^l &= e^{-\lambda t} \sum_{p=0}^l \frac{(-t)^p}{p!} \phi_\lambda^{l-p} \\
&= \sum_{p=0}^l \frac{e_t^{(p)}(\lambda)}{p!} \phi_\lambda^{l-p} \\
&= \sum_{p=0}^l e_t[\lambda^{(p+1)}] \phi[\lambda^{(l-p+1)}] \\
&= (e_t \phi)[\lambda^{(l+1)}],
\end{aligned}
\tag{20}$$

425 this last equality coming from Definition 7.3. Then, $y(T) = 0$ if and only if for any
426 $\lambda \in \Lambda$ and any $l \in \llbracket 0, \alpha_\lambda - 1 \rrbracket$,

$$\int_0^T u(T-t) \mathcal{B}^* ((e_t \phi)[\lambda^{(l+1)}]) dt = - \left\langle y_0, (e_T \phi)[\lambda^{(l+1)}] \right\rangle_{-\diamond, \diamond}.$$

428 By (10), and since $y_0 \in Y_0$, this reduces to find $u \in L^2(0, T; U)$ such that for any
429 $\lambda \in \Lambda$ and any $l \in \llbracket 0, \alpha_\lambda - 1 \rrbracket$,

$$(\mathcal{B}^* \phi_\lambda^0) \int_0^T u(T-t) \frac{(-t)^l}{l!} e^{-\lambda t} dt = - \left\langle y_0, P_{Y_0}^* (e_T \phi)[\lambda^{(l+1)}] \right\rangle_{-\diamond, \diamond},$$

431 that is, using (9) and (14),

$$(21) \quad \int_0^T u(T-t) \frac{(-t)^l}{l!} e^{-\lambda t} dt = - \left\langle y_0, (e_T \psi)[\lambda^{(l+1)}] \right\rangle_{-\diamond, \diamond}, \quad \forall \lambda \in \Lambda, \forall l \in \llbracket 0, \alpha_\lambda - 1 \rrbracket,$$

433 To solve this so-called moment problem the classical strategy introduced in [23]
434 consists in designing a biorthogonal family in $L^2(0, T)$ to

$$\{t \mapsto t^l e^{-\lambda t}; \lambda \in \Lambda, l \in \llbracket 0, \alpha_\lambda - 1 \rrbracket\}$$

436 with associated estimates. Then, thanks to these estimates, a suitable control is de-
437 fined. Usually in this procedure each biorthogonal element is estimated separately.
438 Thus, this method is somehow inopert to analyse the possible condensation of eigen-
439 vectors (which is related to possible cancellations in linear combinations of right-hand
440 sides of (21)). We will thus propose to solve this moment problem using the grouping
441 introduced in Section 1.3.2, in order to cope with such possible compensations. We
442 then look for a solution u in the form

$$(22) \quad u(t) = - \sum_{k \geq 1} q_k(T-t)$$

444 where each q_k will solve the moment problem corresponding to the group G_k . More
445 precisely, such a control will formally solve (21) if

$$(23) \quad \begin{cases} \int_0^T q_k(t) \frac{(-t)^{l'}}{l'!} e^{-\lambda_{k',j'} t} dt = 0, \quad \forall k' \neq k, \forall j' \in \llbracket 1, g_{k'} \rrbracket, \forall l' \in \llbracket 0, \alpha_{k',j'} - 1 \rrbracket, \\ \int_0^T q_k(t) \frac{(-t)^l}{l!} e^{-\lambda_{k,j} t} dt = \left\langle y_0, (e_T \psi)[\lambda_{k,j}^{(l+1)}] \right\rangle_{-\diamond, \diamond}, \\ \forall k \geq 1, \forall j \in \llbracket 1, g_k \rrbracket, \forall l \in \llbracket 0, \alpha_{k,j} - 1 \rrbracket. \end{cases}$$

447 Then the proof of point **i.** of Theorem 1.1 reduces to the resolution of such a
 448 block moment problem with suitable estimates (see Theorem 4.1). First, we solve
 449 in Theorem 2.1 the block moment problem in the case where the eigenvalues are
 450 algebraically simple i.e.

$$451 \quad (24) \quad \begin{cases} \int_0^T q_k(t) e^{-\lambda_{k',j'} t} dt = 0, & \forall k' \neq k, \forall j' \in \llbracket 1, g_{k'} \rrbracket, \\ \int_0^T q_k(t) e^{-\lambda_{k,j} t} dt = e^{-\lambda_{k,j} T} \langle y_0, \psi_{k,j} \rangle_{-\diamond, \diamond}, & \forall k \geq 1, \forall j \in \llbracket 1, g_k \rrbracket. \end{cases}$$

452 This construction uses a Laplace transform isomorphism together with a suitable
 453 restriction argument (Proposition 2.4). The obtained estimates on q_k will allow to
 454 prove convergence of the series (22) when $T > T_0(Y_0)$. Those estimates are uniform
 455 with respect to Λ in a certain class (see Definition 2.1) which will allow in Sec. 4 to
 456 infer the resolution of (23) in the general case.

457 **REMARK 1.5.** *Contrarily to the classical strategy, notice that the sequence $(q_k)_k$*
 458 *is not a biorthogonal family to*

$$459 \quad \{t \mapsto t^l e^{-\lambda t}; \lambda \in \Lambda, l \in \llbracket 0, \alpha_\lambda - 1 \rrbracket\}.$$

460 *The function q_k is only orthogonal to those functions corresponding to groups other*
 461 *than G_k . Inside the group G_k its definition is adapted to solve the moment prob-*
 462 *lem (21). Through the right-hand side (adapted to each initial condition) we will*
 463 *possibly take into account the insufficient observation of eigenvectors, the conden-*
 464 *sation of eigenvalues but also the condensation of eigenvectors. This construction*
 465 *can thus be seen as a block moment method. As we consider at the same time the*
 466 *eigenvalues associated to a same group this will lead to sharper estimates than the one*
 467 *coming from the design of a biorthogonal family (i.e. when considering each eigenvalue*
 468 *individually).*

469 *However, as already mentioned, our strategy still allows to prove the existence and*
 470 *sharp estimates on biorthogonal families (see Corollary 2.1 and Corollary 4.1). Let*
 471 *us mention that, to the best of our knowledge, the estimate we obtain in Corollary 4.1*
 472 *for a biorthogonal family in presence of algebraic multiplicity of eigenvalues without*
 473 *the standard gap condition was not known. Even though these biorthogonal families*
 474 *are not always suitable to deal with controllability properties in presence of spectral*
 475 *condensation (this is why we designed this block resolution of the moment problem)*
 476 *they can be useful for other problems.*

477 Let us mention that, in the context of control problems with a spectrum satisfying the
 478 weak-gap condition, divided differences were already used for instance in [8]. Among
 479 other things, in this work, the authors give a necessary and sufficient condition for the
 480 family of (generalized) divided differences

$$481 \quad \{t \mapsto e_t[i\lambda_{k,1}], \dots, t \mapsto e_t[i\lambda_{k,1}, \dots, i\lambda_{k,g_k}], k \geq 1\}$$

482 to form a Riesz basis of $L^2(0, T; \mathbb{C})$. Although it can seem to be related, our analysis
 483 is quite different. In our work, the divided differences are only used as a technical
 484 tool and the spectral assumptions in the two works are unrelated.

485 Classically, the proof of point **ii.** of Theorem 1.1, relies on the design of a sequence
 486 of counterexamples disproving the associated observability inequality. As dealing
 487 with null-controllability from a proper subspace of initial conditions is not classical

488 let us recall the following lemma that characterizes this controllability through an
 489 observability inequality.

490 LEMMA 1.1 (see for instance [1, Lemma 2.1]). *Let $M > 0$.*

491 *The following two propositions are equivalent.*

492 1. *For any $y_0 \in Y_0$ there exists a $u \in L^2(0, T; U)$ such that $y(T) = 0$ and*

493
$$\|u\|_{L^2(0, T; U)} \leq M \|y_0\|_{-O}.$$

494 2. *For any $z_T \in X_\diamond^*$, the following partial observability holds:*

495 (25)
$$\left\| P_{Y_0}^* \left(e^{-T\mathcal{A}^*} z_T \right) \right\|_{\diamond^*}^2 \leq M^2 \int_0^T \left\| \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} z_T \right\|_U^2 dt.$$

496 *In this case, the best constant M satisfying those properties is called the cost of con-*
 497 *trollability from Y_0 at time T and is denoted $M(Y_0, T)$.*

498 **2. The case of simple eigenvalues.**

499 **2.1. Null-controllability in large time.**

500 The goal of this section is to prove point **i.** of Theorem 1.1 in the case of alge-
 501 braically simple eigenvalues. Thus, in all this section we assume that $\eta = 1$.

502 As explained in Subsec. 1.4, we will now focus on the construction of a solution
 503 to (24). Of course as we want to design a control $u \in L^2(0, T; U)$ the estimate of
 504 $\|q_k\|_{L^2(0, T; \mathbb{R})}$ will play a crucial role to prove that the series (22) makes sense. Actually
 505 we will prove sharp estimates that are uniformly valid for Λ in a certain class. These
 506 uniform estimates can be used for various applications and will be crucial to deal with
 507 algebraic multiplicity of eigenvalues in Sec. 4. We start by precisizing the class of Λ we
 508 will consider. .

509 DEFINITION 2.1. *Let $p \in \mathbb{N}^*$, $\rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$. We say that a count-*
 510 *able family Λ belongs to the class $\mathcal{L}_w(p, \rho, \mathcal{N})$ if Λ satisfies the weak-gap condition (8)*
 511 *with parameters p and ρ and if for any $\varepsilon > 0$ we have*

512 (26)
$$\sum_{\substack{\lambda \in \Lambda \\ \lambda \geq \mathcal{N}(\varepsilon)}} \frac{1}{\lambda} < \varepsilon.$$

513 This definition is directly inspired by the pioneering work [24]. More precisely,
 514 the class of sequences used in [24] is similar to $\mathcal{L}_w(1, \rho, \mathcal{N})$, but it is however slightly
 515 different since in (26) the summation condition is given on the value of λ itself whereas
 516 in the above reference the condition is on the index of the eigenvalue in Λ (which is
 517 supposed to be sorted increasingly). Despite this small difference (whose aim is to
 518 simplify some computations) the results we shall take from [24] that use this definition
 519 are also valid with this alternative definition and thus we set $\mathcal{L}(\rho, \mathcal{N}) := \mathcal{L}_w(1, \rho, \mathcal{N})$.

520 REMARK 2.1 (The usual gap condition). *With our definition, a sequence Λ be-*
 521 *longs to $\mathcal{L}(\rho, \mathcal{N})$ if it satisfies the classical gap condition*

522 (27)
$$|\lambda' - \lambda| > \rho, \quad \forall \lambda, \lambda' \in \Lambda, \quad \lambda \neq \lambda',$$

523 *and the asymptotic behavior estimate (26).*

524 As we will see in the examples (Section 5), the typical situation where sequences
 525 satisfying the weak gap condition appear is when one glues a finite number of se-
 526 quences, each of them satisfying a standard gap condition as in Remark 2.1. This is
 527 formalized in the following lemma.

LEMMA 2.1. *Let $p, \tilde{p} \in \mathbb{N}^*$, $\rho, \tilde{\rho} > 0$ and $\mathcal{N}, \tilde{\mathcal{N}} : (0, +\infty) \rightarrow \mathbb{R}$ given. Then, for
 any $\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$ and $\tilde{\Lambda} \in \mathcal{L}_w(\tilde{p}, \tilde{\rho}, \tilde{\mathcal{N}})$, we have*

$$\Lambda \cup \tilde{\Lambda} \in \mathcal{L}_w(\bar{p}, \bar{\rho}, \bar{\mathcal{N}}),$$

528 with $\bar{p} = p + \tilde{p}$, $\bar{\rho} = \min(\rho, \tilde{\rho})$ and $\bar{\mathcal{N}}(\varepsilon) = \max(\mathcal{N}(\varepsilon/2), \tilde{\mathcal{N}}(\varepsilon/2))$.

529 *Proof.* Let us first prove the weak gap condition. For any $\mu \geq 0$, we have

$$\begin{aligned} 530 \quad [\mu, \mu + \bar{\rho}] \cap (\Lambda \cup \tilde{\Lambda}) &= ([\mu, \mu + \bar{\rho}] \cap \Lambda) \cup ([\mu, \mu + \bar{\rho}] \cap \tilde{\Lambda}) \\ 531 \quad &\subset ([\mu, \mu + \rho] \cap \Lambda) \cup ([\mu, \mu + \tilde{\rho}] \cap \tilde{\Lambda}), \end{aligned}$$

and taking the cardinal, we get

$$\#[\mu, \mu + \bar{\rho}] \cap (\Lambda \cup \tilde{\Lambda}) \leq p + \tilde{p} = \bar{p}.$$

533 For the asymptotic behavior of the sequences, we have

$$534 \quad \sum_{\substack{\lambda \in \Lambda \cup \tilde{\Lambda} \\ \lambda > \bar{\mathcal{N}}(\varepsilon)}} \frac{1}{\lambda} \leq \sum_{\substack{\lambda \in \Lambda \\ \lambda > \mathcal{N}(\varepsilon)}} \frac{1}{\lambda} + \sum_{\substack{\lambda \in \tilde{\Lambda} \\ \lambda > \tilde{\mathcal{N}}(\varepsilon)}} \frac{1}{\lambda} \leq \sum_{\substack{\lambda \in \Lambda \\ \lambda > \mathcal{N}(\varepsilon/2)}} \frac{1}{\lambda} + \sum_{\substack{\lambda \in \tilde{\Lambda} \\ \lambda > \tilde{\mathcal{N}}(\varepsilon/2)}} \frac{1}{\lambda} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

535 The claim is proved. □

536 The following straightforward facts will also be useful.

537 REMARK 2.2. *Let $\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$.*

- 538 • *Any $\tilde{\Lambda} \subset \Lambda$ also satisfy $\tilde{\Lambda} \in \mathcal{L}_w(p, \rho, \mathcal{N})$.*
- 539 • *For any $h > 0$, $\Lambda + h \in \mathcal{L}_w(p, \rho, \mathcal{N})$.*

540 Using this class we prove the following theorem.

541 THEOREM 2.1. *Let $T \in (0, +\infty]$. Let $p \in \mathbb{N}^*$, $r, \rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$.*

542 *Assume that $\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$ and let $(G_k)_k \in \mathcal{G}(\Lambda, p, r, \rho)$ be an associated grouping.*

543 *For any $\varepsilon > 0$, there exists a constant $C_{\varepsilon, T, p, r, \rho, \mathcal{N}} > 0$ such that for any $k \geq 1$,*
 544 *for any $\omega_{k,1}, \dots, \omega_{k,g_k} \in \mathbb{C}$, there exists $q_k \in L^2(0, T; \mathbb{C})$ satisfying*

$$545 \quad (28) \quad \begin{cases} \int_0^T q_k(t) e^{-\lambda_{k',j'} t} dt = 0, & \forall k' \neq k, \forall j' \in \llbracket 1, g_{k'} \rrbracket, \\ \int_0^T q_k(t) e^{-\lambda_{k,j} t} dt = \omega_{k,j}, & \forall j \in \llbracket 1, g_k \rrbracket, \end{cases}$$

546 and

$$547 \quad (29) \quad \|q_k\|_{L^2(0, T; \mathbb{C})} \leq C_{\varepsilon, T, p, r, \rho, \mathcal{N}} e^{\varepsilon \lambda_{k,1}} \max_{i \in \llbracket 1, g_k \rrbracket} \left| \omega[\lambda_{k,1}, \dots, \lambda_{k,i}] \right|.$$

548 *Moreover, up to the factor $e^{\varepsilon \lambda_{k,1}}$, this last estimate is sharp: any solution $q_k \in$
 549 $L^2(0, T; \mathbb{C})$ of (28) satisfies*

$$550 \quad (30) \quad \|q_k\|_{L^2(0, T; \mathbb{C})} \geq \tilde{C}_p \max_{i \in \llbracket 1, g_k \rrbracket} \left| \omega[\lambda_{k,1}, \dots, \lambda_{k,i}] \right|,$$

551 for some $\tilde{C}_p > 0$.

552 The proof of Theorem 2.1 is conducted all along Sections 2.1.1 and 2.1.2.

553 Before going on with the proof, let us notice that the resolution of the block
 554 moment problem (28) for a specific choice of $\omega_{k,j}$ allows to prove, as a by-product,
 555 the existence and uniform estimates of a biorthogonal family to the exponentials
 556 $(e_\lambda)_{\lambda \in \Lambda}$ where e_λ is defined by (5).

557 COROLLARY 2.1. *Let $T \in (0, +\infty]$. Let $p \in \mathbb{N}^*$, $r, \rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$.
 558 Assume that $\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$ and let $(G_k)_k \in \mathcal{G}(\Lambda, p, r, \rho)$ be an associated grouping.
 559 For any $k \geq 1$, for any $j \in \llbracket 1, g_k \rrbracket$, there exists $q_{k,j} \in L^2(0, T; \mathbb{R})$ satisfying*

$$560 \quad (31) \quad \int_0^T q_{k,j}(t) e^{-\lambda_{k',j'} t} dt = \delta_{k,k'} \delta_{j,j'}, \quad \forall k, k' \geq 1, \forall j \in \llbracket 1, g_k \rrbracket, \forall j' \in \llbracket 1, g_{k'} \rrbracket,$$

561 where δ denotes the Kronecker symbol. Moreover, for any $\varepsilon > 0$, there exists a
 562 constant $C_{\varepsilon, T, p, r, \rho, \mathcal{N}} > 0$ such that for any $k \geq 1$ and for any $j \in \llbracket 1, g_k \rrbracket$,

$$563 \quad \|q_{k,j}\|_{L^2(0, T; \mathbb{R})} \leq C_{\varepsilon, T, p, r, \rho, \mathcal{N}} \frac{e^{\varepsilon \lambda_{k,1}}}{|P'_{G_k}(\lambda_{k,j})|},$$

564 where P_{G_k} is defined in (6).

Moreover, up to the factor $e^{\varepsilon \lambda_{k,1}}$, this estimate is optimal since any function $q_{k,j}$
 satisfying (31), satisfies the lower bound

$$\|q_{k,j}\|_{L^2(0, T; \mathbb{R})} \geq \tilde{C}_p \frac{1}{|P'_{G_k}(\lambda_{k,j})|}$$

565 for some $\tilde{C}_p > 0$.

566 *Proof.* Let $k \geq 1$ and $j \in \llbracket 1, g_k \rrbracket$. Let $q_{k,j} \in L^2(0, T; \mathbb{C})$ be the solution of the
 567 block moment problem (28) given by Theorem 2.1 associated with the right-hand side
 568 $\omega_{k,j'} = \delta_{j,j'}$ for any $j' \in \llbracket 1, g_k \rrbracket$. Since those values of ω are real we can change $q_{k,j}$ in
 569 its real part without changing its properties. Then, the equalities (31) follow directly.
 570 Moreover we have

$$571 \quad \|q_{k,j}\|_{L^2(0, T; \mathbb{R})} \leq C_{\varepsilon, T, p, r, \rho, \mathcal{N}} e^{\varepsilon \lambda_{k,1}} \max_{i \in \llbracket 1, g_k \rrbracket} |\omega[\lambda_{k,1}, \dots, \lambda_{k,i}]|.$$

572 From the Newton formula (see Proposition 7.3) it comes that for any $i \in \llbracket 1, g_k \rrbracket$,

$$573 \quad \omega[\lambda_{k,1}, \dots, \lambda_{k,i}] = \begin{cases} 0, & \text{if } i < j, \\ \frac{1}{\prod_{m \in \llbracket 1, i \rrbracket, m \neq j} (\lambda_{k,j} - \lambda_{k,m})}, & \text{if } i \geq j. \end{cases}$$

574 To conclude the proof of Corollary 2.1 we prove that there exists $C_{p,\rho} > 0$ such that
 575 for any $k \geq 1$, $j \in \llbracket 1, g_k \rrbracket$ and any $i \in \llbracket j, g_k \rrbracket$,

$$576 \quad (32) \quad \prod_{m \in \llbracket 1, i \rrbracket, m \neq j} |\lambda_{k,j} - \lambda_{k,m}| \geq C_{p,\rho} |P'_{G_k}(\lambda_{k,j})|.$$

577 Indeed, we have

$$578 \quad \frac{\prod_{m \in \llbracket 1, i \rrbracket, m \neq j} |\lambda_{k,j} - \lambda_{k,m}|}{|P'_{G_k}(\lambda_{k,j})|} = \frac{\prod_{m \in \llbracket 1, i \rrbracket, m \neq j} |\lambda_{k,j} - \lambda_{k,m}|}{\prod_{m \in \llbracket 1, g_k \rrbracket, m \neq j} |\lambda_{k,j} - \lambda_{k,m}|} = \frac{1}{\prod_{m \in \llbracket i+1, g_k \rrbracket} |\lambda_{k,j} - \lambda_{k,m}|}.$$

579 By (13), we get

$$580 \quad |\lambda_{k,j} - \lambda_{k,m}| \leq \rho, \quad \forall m \in \llbracket i+1, g_k \rrbracket.$$

581 Thus,

$$582 \quad (33) \quad \frac{\prod_{m \in \llbracket 1, i \rrbracket \neq j} |\lambda_{k,j} - \lambda_{k,m}|}{|P'_{G_k}(\lambda_{k,j})|} \geq \left(\frac{1}{\rho}\right)^{g_k - i}.$$

583 As the right-hand side only takes a finite number of values, inequality (33) proves (32)
584 and ends the proof of Corollary 2.1.

585 The lower bound directly follows from (30) and the inequality

$$586 \quad \max_{i \in \llbracket 1, g_k \rrbracket} \left| \omega[\lambda_{k,1}, \dots, \lambda_{k,i}] \right| \geq \left| \omega[\lambda_{k,1}, \dots, \lambda_{k,g_k}] \right| = \frac{1}{|P'_{G_k}(\lambda_{k,j})|}. \quad \square$$

587 2.1.1. Resolution of block moment problems in infinite time.

588 In this section, we start by proving Theorem 2.1 in the case of simple eigenvalues
589 and with $T = +\infty$. More precisely, we prove the following proposition.

590 PROPOSITION 2.1. *Let $p \in \mathbb{N}^*$, $r, \rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$. Assume that*
591 *$\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$ and let $(G_k)_k \in \mathcal{G}(\Lambda, p, r, \rho)$ be an associated grouping.*

592 *For any $\varepsilon > 0$, there exists a constant $C_{\varepsilon, p, r, \rho, \mathcal{N}} > 0$ such that for any $k \geq 1$, for*
593 *any $\omega_{k,1}, \dots, \omega_{k,g_k} \in \mathbb{C}$, there exists $\tilde{q}_k \in L^2(0, +\infty; \mathbb{C})$ satisfying*

$$594 \quad (34) \quad \begin{cases} \int_0^{+\infty} \tilde{q}_k(t) e^{-\lambda_{k',j'} t} dt = 0, & \forall k' \neq k, \forall j' \in \llbracket 1, g_{k'} \rrbracket, \\ \int_0^{+\infty} \tilde{q}_k(t) e^{-\lambda_{k,j} t} dt = \omega_{k,j}, & \forall j \in \llbracket 1, g_k \rrbracket, \end{cases}$$

595 and

$$596 \quad \|\tilde{q}_k\|_{L^2(0, +\infty; \mathbb{C})} \leq C_{\varepsilon, p, r, \rho, \mathcal{N}} e^{\varepsilon \lambda_{k,1}} \max_{i \in \llbracket 1, g_k \rrbracket} \left| \omega[\lambda_{k,1}, \dots, \lambda_{k,i}] \right|.$$

597 The proof relies on the construction of an holomorphic function satisfying suitable
598 properties and estimates. The resolution of the block moment problem (34) then
599 comes from the isomorphism induced by the Laplace transform.

600 *Proof.* Let us start by recalling classical properties of the Laplace transform (see
601 for instance [43, pp. 19-20] and the references therein). Let $H^2(\mathbb{C}^+)$ the space of
602 holomorphic functions F on $\mathbb{C}^+ = \{z \in \mathbb{C}; \Re(z) > 0\}$ such that

$$603 \quad \sup_{\sigma > 0} \|F(\sigma + i\bullet)\|_{L^2(\mathbb{R}; \mathbb{C})} < +\infty,$$

604 endowed with the norm

$$605 \quad \|F\|_{H^2(\mathbb{C}^+)}^2 := \sup_{\sigma > 0} \|F(\sigma + i\bullet)\|_{L^2(\mathbb{R}; \mathbb{C})}^2 = \int_{\mathbb{R}} |F(i\tau)|^2 d\tau.$$

606 Then the Laplace transform

$$607 \quad \mathbb{L} : f \in L^2(0, +\infty; \mathbb{C}) \mapsto \left(F : \lambda \in \mathbb{C}^+ \mapsto \int_{\mathbb{R}} e^{-\lambda t} f(t) dt \right) \in H^2(\mathbb{C}^+)$$

608 is an isomorphism.

609 We shall construct for each k , a function $J_k \in H^2(\mathbb{C}^+)$ satisfying

$$610 \quad (35) \quad J_k(\lambda) = 0, \quad \forall \lambda \in \Lambda \setminus G_k,$$

$$611 \quad (36) \quad J_k(\lambda_{k,j}) = \omega_{k,j}, \quad \forall j \in \llbracket 1, g_k \rrbracket,$$

613 and such that for any $\varepsilon > 0$, there exists $C_{\varepsilon,p,r,\rho,\mathcal{N}} > 0$ such that

$$614 \quad \int_{\mathbb{R}} |J_k(i\tau)|^2 d\tau \leq C_{\varepsilon,p,r,\rho,\mathcal{N}} e^{\varepsilon\lambda_{k,1}} \max_{i \in \llbracket 1, g_k \rrbracket} |\omega[\lambda_{k,1}, \dots, \lambda_{k,i}]|, \quad \forall k \geq 1.$$

615 Taking advantage of the isomorphism property of the Laplace transform we will then
616 set $\tilde{q}_k := \mathbb{L}^{-1}(J_k)$, to conclude the proof.

617 *Construction of J_k .*

618 We define J_k as

$$619 \quad J_k : z \in \mathbb{C}^+ \mapsto \frac{P_k(z)}{(1+z)^p} W_k(z)$$

620 where P_k is a polynomial of degree less than p which is precised below and W_k is the
621 following Blaschke-type product

$$622 \quad W_k(z) = \prod_{j=1}^p \left(\prod_{\lambda \in \Lambda_j \setminus G_k} \frac{\lambda - z}{\lambda + z} \right),$$

623 where

$$624 \quad (37) \quad \Lambda_j := \{\lambda_{l, \min(j, g_l)}, \quad l \geq 1\}.$$

625 The sequence Λ_j contains the j -th element of each group G_l , except if this group
626 contains less than j elements, in which case, we replace it by the largest element of
627 G_l that is λ_{l, g_l} . In particular, we observe that Λ_j is a subsequence of Λ .

628 From (7), we deduce that $\sum_{\lambda \in \Lambda_j} \frac{1}{\lambda} < +\infty$, so that for any j , the associated
629 infinite product uniformly converges on any compact of \mathbb{C}^+ . As a consequence, W_k
630 is well-defined and holomorphic in \mathbb{C}^+ (see for instance [41, Chapter 15]). It follows
631 that J_k is also holomorphic on \mathbb{C}^+ .

632 We shall need the following property, whose proof is technical and postponed to
633 Section 2.1.4.

634 **PROPOSITION 2.2.** *There exists a constant $C_{\varepsilon,p,r,\rho,\mathcal{N}} > 0$ such that for any $k \geq 1$,
635 any $l \in \llbracket 0, p \rrbracket$, and any $\theta \in \text{Conv}(G_k)$,*

$$636 \quad (38) \quad \left| \left(\frac{1}{W_k} \right)^{(l)}(\theta) \right| \leq C_{\varepsilon,p,r,\rho,\mathcal{N}} e^{\varepsilon\lambda_{k,1}}.$$

637 From the definition of W_k it comes that (35) is satisfied. Next, it comes that (36)
638 is equivalent to

$$639 \quad P_k(\lambda_{k,j}) = \frac{(1 + \lambda_{k,j})^p}{W_k(\lambda_{k,j})} \omega_{k,j}, \quad \forall j \in \llbracket 1, g_k \rrbracket.$$

640 Let

$$641 \quad f : s \in \mathbb{R} \mapsto (1+s)^p, \quad \text{and} \quad f_k : s \in \mathbb{R} \mapsto \frac{f(s)}{W_k(s)}.$$

642 To satisfy (36), we define P_k as the Lagrange interpolating polynomial at points $\lambda_{k,j}$
 643 with values $(f_k\omega)[\lambda_{k,j}] := f_k(\lambda_{k,j})\omega_{k,j}$ that is, in Newton form,

$$644 \quad P_k(z) := \sum_{j=1}^{g_k} (f_k\omega)[\lambda_{k,1}, \dots, \lambda_{k,j}] \prod_{i=1}^{j-1} (z - \lambda_{k,i}).$$

645 Thus, to conclude it remains to estimate $\int_{\mathbb{R}} |J_k(i\tau)|^2 d\tau$.

646 *Estimate of J_k .*

647 Notice that since the eigenvalues in Λ are real, for any $k \geq 1$ and any $\tau \in \mathbb{R}$, we
 648 have $|W_k(i\tau)| = 1$. This implies

$$649 \quad (39) \quad |J_k(i\tau)| \leq g_k \frac{(|\tau| + \lambda_{k,g_k})^{p-1}}{(1 + \tau^2)^{p/2}} \max_{j \in \llbracket 1, g_k \rrbracket} |(f_k\omega)[\lambda_{k,1}, \dots, \lambda_{k,j}]|$$

650 and thus $J_k \in H^2(\mathbb{C}^+)$.

651 Using Leibniz formula (see Proposition 7.6),

$$652 \quad (40) \quad |(f_k\omega)[\lambda_{k,1}, \dots, \lambda_{k,j}]| \leq \sum_{i=1}^j |f_k[\lambda_{k,i}, \dots, \lambda_{k,j}]| |\omega[\lambda_{k,1}, \dots, \lambda_{k,i}]|.$$

653 Using again Leibniz formula (see Proposition 7.6),

$$654 \quad (41) \quad f_k[\lambda_{k,i}, \dots, \lambda_{k,j}] = \sum_{m=i}^j f[\lambda_{k,i}, \dots, \lambda_{k,m}] \left(\frac{1}{W_k} \right) [\lambda_{k,m}, \dots, \lambda_{k,j}].$$

655 The two factors in each term of this sum are estimated using Lagrange theorem (see
 656 Proposition 7.4):

657 • First, we have

$$658 \quad f[\lambda_{k,i}, \dots, \lambda_{k,m}] = \frac{f^{(m-i)}(\theta_k)}{(m-i)!}$$

659 with $\theta_k \in [\lambda_{k,i}, \lambda_{k,m}]$. It comes that there exists $C_p > 0$ such that

$$660 \quad (42) \quad |f[\lambda_{k,1}, \dots, \lambda_{k,m}]| \leq C_p (1 + \lambda_{k,m})^p \leq C_p (1 + \lambda_{k,g_k})^p.$$

661 • Second, we have

$$662 \quad \left(\frac{1}{W_k} \right) [\lambda_{k,m}, \dots, \lambda_{k,j}] = \frac{1}{(j-m+1)!} \left(\frac{1}{W_k} \right)^{(j-m+1)} (\tilde{\theta}_k)$$

663 with $\tilde{\theta}_k \in [\lambda_{k,m}, \lambda_{k,j}]$.

664 By using (38), it follows that

$$665 \quad (43) \quad \left| \left(\frac{1}{W_k} \right) [\lambda_{k,m}, \dots, \lambda_{k,j}] \right| \leq C_{\varepsilon,p,\rho,\mathcal{N}} e^{\varepsilon\lambda_{k,g_k}}.$$

666 Recall that (13) implies $\lambda_{k,g_k} - \lambda_{k,1} < \rho$. Then, using (42) and (43) into the identity
 667 (41) proves that there exists $C_{\varepsilon,p,r,\rho,\mathcal{N}} > 0$ such that for any $i, j \in \llbracket 1, g_k \rrbracket, i \leq j$, we
 668 have

$$669 \quad |f_k[\lambda_{k,i}, \dots, \lambda_{k,j}]| \leq C_{\varepsilon,p,r,\rho,\mathcal{N}} e^{\varepsilon\lambda_{k,1}}.$$

670 Plugging it in (40) we obtain

$$671 \quad \max_{j \in \llbracket 1, g_k \rrbracket} |(f_k \omega)[\lambda_{k,1}, \dots, \lambda_{k,j}]| \leq C_{\varepsilon, p, r, \rho, \mathcal{N}} e^{\varepsilon \lambda_{k,1}} \max_{i \in \llbracket 1, g_k \rrbracket} \left| \omega[\lambda_{k,1}, \dots, \lambda_{k,i}] \right|.$$

672 Finally, getting back to estimate (39), and using the isomorphism property of \mathbf{L}
673 ends the proof of Proposition 2.1. \square

674 2.1.2. From infinite time horizon to finite time horizon.

675 In this section we first prove that the estimates on the solution on $(0, +\infty)$ of the
676 block moment problem (34) for simple eigenvalues given in Proposition 2.1 implies
677 the resolution on $(0, T)$ of the similar block moment problem (28). More precisely we
678 prove the following.

679 PROPOSITION 2.3. *Let $p \in \mathbb{N}^*$, $\rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$. Assume that
680 $\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$. For any $T > 0$, there exists a constant $C_{T, p, \rho, \mathcal{N}} > 0$ such that for
681 any $\tilde{q} \in L^2(0, +\infty; \mathbb{C})$ there exists $q \in L^2(0, T; \mathbb{C})$ satisfying*

$$682 \quad \int_0^T q(t) e^{-\lambda t} dt = \int_0^{+\infty} \tilde{q}(t) e^{-\lambda t} dt, \quad \forall \lambda \in \Lambda,$$

683 and

$$684 \quad \|q\|_{L^2(0, T; \mathbb{C})} \leq C_{T, p, \rho, \mathcal{N}} \|\tilde{q}\|_{L^2(0, +\infty; \mathbb{C})}.$$

685 For any $T \in (0, +\infty]$, we set

$$686 \quad A(\Lambda, T) := \overline{\text{Span}\{e_\lambda; \lambda \in \Lambda\}}^{L^2(0, T; \mathbb{C})},$$

687 where e_λ is defined in (5). The proof of Proposition 2.3 mainly relies on the following
688 proposition that gives an estimate on the inverse of the restriction operator.

689 PROPOSITION 2.4. *Let $p \in \mathbb{N}^*$, $\rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$. Assume that
690 $\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$. Let $T > 0$ be fixed. Then, the restriction operator*

$$691 \quad (44) \quad R_{\Lambda, T} : q \in A(\Lambda, +\infty) \mapsto q|_{(0, T)} \in A(\Lambda, T)$$

692 is an isomorphism. Moreover there exists a constant $C_{T, p, \rho, \mathcal{N}} > 0$ such that

$$693 \quad (45) \quad \|R_{\Lambda, T}^{-1}\| \leq C_{T, p, \rho, \mathcal{N}}.$$

694 In the case $p = 1$, this result is due to Fattorini and Russell [24, Theorem 1.3]. Our
695 proof follows closely the strategy developed in this reference and takes advantage of
696 the uniform estimates we established in the previous sections.

697 *Proof.* The fact that $R_{\Lambda, T}$ is an isomorphism is proved in [43] under the sole
698 assumption (7). The only thing to prove is thus the bound (45).

699 The proof is done by contradiction. Assume that the estimate does not hold for
700 given T, p, ρ , and \mathcal{N} , then there exists a sequence $(\Lambda^m)_{m \geq 1}$ belonging to the same
701 class $\mathcal{L}_w(p, \rho, \mathcal{N})$, such that

$$702 \quad (46) \quad \|R_{\Lambda^m, T}^{-1}\| \xrightarrow{m \rightarrow \infty} +\infty.$$

703 For each m , by Proposition 7.1, we consider a grouping $(G_k^m)_k \in \mathcal{G}(\Lambda^m, p, \rho/p, \mathcal{N})$,
 704 and from (46) we know that there exists coefficients $a_{k,j}^m$ such that the finite linear
 705 combination

$$P^m : t \mapsto \sum_{k=1}^{K^m} \sum_{j=1}^{g_k^m} a_{k,j}^m e^{-\lambda_{k,j}^m t},$$

707 satisfies

$$708 \quad \|P^m\|_{L^2(0,\infty;\mathbb{C})} = 1, \quad \text{and} \quad \|P^m\|_{L^2(0,T;\mathbb{C})} \xrightarrow{m \rightarrow \infty} 0.$$

709 Let $0 < \varepsilon < \frac{T}{2}$ be fixed and let $\mathbb{C}_{2\varepsilon}^+ = \{z \in \mathbb{C}; \Re(z) > 2\varepsilon\}$. We prove that the
 710 sequence $z \mapsto P^m(z)$ is uniformly bounded on any compact of $\mathbb{C}_{2\varepsilon}^+$.

711 Let $m \geq 1$ and $z \in \mathbb{C}_{2\varepsilon}^+$. Then for any $k \in \{1, \dots, K^m\}$ the application of Propo-
 712 sition 2.1 to the sequence Λ^m yields the existence of $\tilde{q}_k^{m,z} \in L^2(0, +\infty; \mathbb{C})$ satisfying

$$713 \quad (47) \quad \begin{cases} \int_0^{+\infty} \tilde{q}_k^{m,z}(t) e^{-\lambda_{k',j'}^m t} dt = 0, & \forall k' \neq k, \quad \forall j' \in \llbracket 1, g_{k'}^m \rrbracket, \\ \int_0^{+\infty} \tilde{q}_k^{m,z}(t) e^{-\lambda_{k,j}^m t} dt = e^{-\lambda_{k,j}^m z}, & \forall j \in \llbracket 1, g_k^m \rrbracket, \end{cases}$$

714 and

$$715 \quad \|\tilde{q}_k^{m,z}\|_{L^2(0,+\infty;\mathbb{C})} \leq C_{\varepsilon,p,\rho,\mathcal{N}} e^{\varepsilon \lambda_{k,1}^m} \left(\max_{j \in \llbracket 1, g_k^m \rrbracket} |e_z[\lambda_{k,1}^m, \dots, \lambda_{k,j}^m]| \right),$$

716 where e_z is defined in (5).

717 The previous right-hand side is estimated using Lagrange theorem (see Propo-
 718 sition 7.4). As the function e_z is complex-valued we apply it on both its real and
 719 imaginary parts. This yields

$$720 \quad \left(\max_{j \in \llbracket 1, g_k^m \rrbracket} |e_z[\lambda_{k,1}^m, \dots, \lambda_{k,j}^m]| \right) \leq C_{p,\rho} |z|^p e^{-\Re(z) \lambda_{k,1}^m}.$$

721 Thus,

$$722 \quad \|\tilde{q}_k^{m,z}\|_{L^2(0,+\infty;\mathbb{C})} \leq C_{\varepsilon,p,\rho,\mathcal{N}} |z|^p e^{-(\Re(z)-\varepsilon) \lambda_{k,1}^m}.$$

723 Then, using (47) it comes that, for m sufficiently large,

$$724 \quad \langle P^m, \overline{\tilde{q}_k^{m,z}} \rangle_{L^2(0,\infty;\mathbb{C})} = \sum_{k'=1}^{K^m} \sum_{j=1}^{g_{k'}^m} a_{k',j}^m \int_0^{+\infty} e^{-\lambda_{k',j}^m t} \tilde{q}_k^{m,z}(t) dt = \sum_{j=1}^{g_k^m} a_{k,j}^m e^{-\lambda_{k,j}^m z}.$$

725 From Cauchy-Schwarz inequality we deduce that

$$726 \quad \left| \sum_{j=1}^{g_k^m} a_{k,j}^m e^{-\lambda_{k,j}^m z} \right| \leq \|P^m\|_{L^2(0,+\infty;\mathbb{C})} \|\tilde{q}_k^{m,z}\|_{L^2(0,+\infty;\mathbb{C})} \leq C_{\varepsilon,p,\rho,\mathcal{N}} |z|^p e^{-(\Re(z)-\varepsilon) \lambda_{k,1}^m}.$$

727 Summing these inequalities we obtain that for any $z \in \mathbb{C}_{2\varepsilon}^+$,

$$728 \quad |P^m(z)| \leq \sum_{k=1}^{K^m} \left| \sum_{j=1}^{g_k^m} a_{k,j}^m e^{-\lambda_{k,j}^m z} \right| \leq C_{\varepsilon,p,\rho,\mathcal{N}} |z|^p \sum_{k \geq 1} e^{-(\Re(z)-\varepsilon) \lambda_{k,1}^m},$$

729 From the properties of the groupings (see Definition 1.2), it comes that $\lambda_{k,1}^m \geq \frac{\rho}{p}(k-1)$.

730 Thus, for any $z \in \mathbb{C}_{2\varepsilon}^+$,

$$731 \quad (48) \quad |P^m(z)| \leq C_{\varepsilon,p,\rho,\mathcal{N}} |z|^p e^{-\frac{\rho}{p}(\Re(z)-\varepsilon)} \sum_{k \geq 1} e^{-\varepsilon \frac{\rho}{p}(k-2)} \leq C_{\varepsilon,p,\rho,\mathcal{N}} |z|^p e^{-\frac{\rho}{p}\Re(z)}.$$

732 This gives that $(P^m)_m$ is a sequence of holomorphic functions uniformly bounded
733 on any compact of $\mathbb{C}_{2\varepsilon}^+$. From Montel's theorem it comes that we can extract a
734 subsequence converging uniformly on any compact of $\mathbb{C}_{2\varepsilon}^+$ to an holomorphic function
735 P .

736 Now recall that $\|P^m\|_{L^2(0,T;\mathbb{C})}$ goes to 0 as m goes to infinity. This implies that
737 $P(t) = 0$ for any $t \in (2\varepsilon, T)$. The function P being holomorphic it comes that it
738 vanishes on $\mathbb{C}_{2\varepsilon}^+$. Using (48) and the Lebesgue dominated-convergence theorem yields

$$739 \quad \|P^m\|_{L^2(0,+\infty;\mathbb{C})} \xrightarrow{m \rightarrow \infty} 0.$$

740 This is in contradiction with $\|P^m\|_{L^2(0,+\infty;\mathbb{C})} = 1$ and ends the proof of Proposi-
741 tion 2.4. \square

742 We now have all the ingredients to prove Proposition 2.3.

743 *Proof (of Proposition 2.3).* This proof follows closely the one of [4, Section 4]
744 and [5, Lemma 4.2]. From [4, Corollary 4.3], as Λ satisfies (7), it comes that $A(\Lambda, +\infty)$
745 is a proper subspace of $L^2(0, +\infty; \mathbb{C})$. Let Π_Λ the associated orthogonal projection.
746 Let $\tilde{q} \in L^2(0, +\infty, \mathbb{C})$. Then, by construction, we have

$$747 \quad (49) \quad \int_0^{+\infty} \Pi_\Lambda \tilde{q}(t) e^{-\lambda t} dt = \int_0^{+\infty} \tilde{q}(t) e^{-\lambda t} dt, \quad \forall \lambda \in \Lambda.$$

748 From Proposition 2.4, the restriction operator $R_{\Lambda,T}$ defined by (44) is an isomorphism.
749 Thus, setting $q := (R_{\Lambda,T}^{-1})^* \Pi_\Lambda \tilde{q}$ ends the proof of Proposition 2.3. Indeed, there exists
750 $C_{T,p,\rho,\mathcal{N}} > 0$ such that

$$751 \quad \|q\|_{L^2(0,T;\mathbb{C})} \leq C_{T,p,\rho,\mathcal{N}} \|\tilde{q}\|_{L^2(0,+\infty;\mathbb{C})},$$

752 and, using (49), for every $\lambda \in \Lambda$,

$$753 \quad \int_0^T q(t) e^{-\lambda t} dt = \langle (R_{\Lambda,T}^{-1})^* \Pi_\Lambda \tilde{q}, e_\lambda \rangle_{L^2(0,T)} = \langle \Pi_\Lambda \tilde{q}, R_{\Lambda,T}^{-1} R_{\Lambda,T} e_\lambda \rangle_{L^2(0,+\infty)}$$

$$754 \quad = \int_0^{+\infty} \tilde{q}(t) e^{-\lambda t} dt. \quad \square$$

755

756 We can now conclude the proof of Theorem 2.1 for simple eigenvalues.

757 *Proof (of Theorem 2.1).* The resolution of the block moment problem (28) as well
758 as the estimate (29) follow directly from Propositions 2.1 and 2.3.

759 The only thing left to prove is the lower bound (30). Let $q_k \in L^2(0, T; \mathbb{C})$ be any
760 solution of (28). Using the linearity of divided differences, equalities (28) imply that
761 for any $i \in \llbracket 1, g_k \rrbracket$

$$762 \quad (50) \quad \omega[\lambda_{k,1}, \dots, \lambda_{k,i}] = \int_0^T q_k(t) e_t[\lambda_{k,1}, \dots, \lambda_{k,i}] dt$$

763 where e_t is defined by (5). From the Lagrange theorem (see Proposition 7.4) and the
 764 fact that $e_t^{(i-1)}$ is decreasing on $(0, +\infty)$, it comes that for any $t \in (0, T)$, we have

$$765 \quad |e_t[\lambda_{k,1}, \dots, \lambda_{k,i}]| \leq t^p e^{-\lambda_{k,1}t} \leq t^p e^{-t},$$

766 since we assumed that $\Lambda \subset [1, +\infty)$. Thus applying the Cauchy-Schwarz inequality
 767 to (50) gives

$$768 \quad |\omega[\lambda_{k,1}, \dots, \lambda_{k,i}]| \leq \|q_k\|_{L^2(0,T;\mathbb{C})} \left(\int_0^T t^{2p} e^{-t} dt \right)^{\frac{1}{2}} \leq (2p)! \|q_k\|_{L^2(0,T;\mathbb{C})}$$

769 which proves (30) and ends the proof of Theorem 2.1. \square

770 2.1.3. Construction of the control.

771 We now have all the ingredients to prove the positive controllability result.

772 *Proof (of point i. of Theorem 1.1 for simple eigenvalues).*

773 Assume that $T_0(Y_0) < +\infty$ and let us consider an initial data $y_0 \in Y_0$. Without
 774 loss of generality we assume that $\|y_0\|_{-\diamond} = 1$.

775 Let $T \in (T_0(Y_0), +\infty)$ and $\varepsilon > 0$ be such that $T > T_0(Y_0) + 2\varepsilon$. From Proposi-
 776 tion 1.3, it comes that $T > T_0(y_0) + 2\varepsilon$.

777 For any $k \geq 1$ and $j \in \llbracket 1, g_k \rrbracket$ we set

$$778 \quad \omega_{k,j} := e^{-\lambda_{k,j}T} \langle y_0, \psi_{k,j} \rangle_{-\diamond, \diamond}.$$

779 Let $(q_k)_{k \geq 1}$ be the solution of the block moment problem (28) given in Theorem 2.1.

780 There exists a constant $C_{\varepsilon, T, p, r, \rho, \mathcal{N}} > 0$ such that

$$781 \quad \|q_k\|_{L^2(0,T;\mathbb{R})} \leq C_{\varepsilon, T, p, r, \rho, \mathcal{N}} e^{\varepsilon \lambda_{k,1}} \max_{i \in \llbracket 1, g_k \rrbracket} |\omega[\lambda_{k,1}, \dots, \lambda_{k,i}]|, \quad \forall k \geq 1.$$

782 Let $\xi_{k,j} := \langle y_0, \psi_{k,j} \rangle_{-\diamond, \diamond}$. Notice that $\omega_{k,j} = e_T(\lambda_{k,j}) \xi_{k,j}$, where e_T is defined in (5).

783 From Leibniz formula (see Proposition 7.6),

$$784 \quad (51) \quad \omega[\lambda_{k,1}, \dots, \lambda_{k,i}] = \sum_{m=1}^i e_T[\lambda_{k,m}, \dots, \lambda_{k,i}] \xi[\lambda_{k,1}, \dots, \lambda_{k,m}].$$

785 In this expression, $\xi[\dots]$ stands for the divided differences associated with the values
 786 $(\lambda_{k,1}, \xi_{k,1}), \dots, (\lambda_{k,g_k}, \xi_{k,g_k})$. From Lagrange theorem (see Proposition 7.4) it comes
 787 that

$$788 \quad e_T[\lambda_{k,m}, \dots, \lambda_{k,i}] = \frac{e_T^{(i-m)}(\theta_k)}{(i-m)!} = \frac{(-T)^{i-m}}{(i-m)!} e^{-\theta_k T}$$

789 with $\theta_k \in [\lambda_{k,m}, \lambda_{k,i}]$. Using the definition (17) of $T_0(y_0)$, it comes that,

$$790 \quad (52) \quad \left| \xi[\lambda_{k,1}, \dots, \lambda_{k,m}] \right| = \left| \langle y_0, \psi[\lambda_{k,1}, \dots, \lambda_{k,m}] \rangle_{-\diamond, \diamond} \right| \leq C e^{\lambda_{k,1}(T_0(y_0) + \varepsilon)}.$$

791 Thus, there exists $C_{T,p} > 0$ such that

$$792 \quad \left| \omega[\lambda_{k,1}, \dots, \lambda_{k,i}] \right| \leq C_{T,p} e^{\lambda_{k,1}(T_0(y_0) + \varepsilon - T)}.$$

793 Then, as $T > T_0(y_0) + 2\varepsilon$, the series (22) is convergent in $L^2(0, T; \mathbb{R})$ and defines
 794 a control u that solves the moment problem (24), which implies that the associated
 795 solution of (1) satisfies $y(T) = 0$. \square

796 With the same strategy we can prove a more accurate result. Namely we get the
797 following uniform bound for the control cost.

798 **COROLLARY 2.2.** *Let $p \in \mathbb{N}^*$, $r, \rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$. Assume that*
799 *$\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$ and let $(G_k)_k \in \mathcal{G}(\Lambda, p, r, \rho)$ be an associated grouping. Assume that*
800 *$\eta = 1$ and $T_0(Y_0) < +\infty$. For any $T > T_0(Y_0)$, let $\mathcal{C}^* > 0$ be such that*

$$801 \quad (53) \quad \max_{l \in [1, g_k]} \|\psi[\lambda_{k,1}, \dots, \lambda_{k,l}]\|_{\diamond^*} \leq \mathcal{C}^* e^{\lambda_{k,1} \frac{T_0(Y_0)+T}{2}}, \quad \forall k \geq 1.$$

802 *Then, there exists a constant $C_{T_0(Y_0), T, p, r, \rho, \mathcal{N}} > 0$ such that for any $y_0 \in Y_0$, there*
803 *exists a control $u \in L^2(0, T; \mathbb{R})$ such that the associated solution y of (1) satisfies*
804 *$y(T) = 0$ and*

$$805 \quad \|u\|_{L^2(0, T; \mathbb{R})} \leq C_{T_0(Y_0), T, p, r, \rho, \mathcal{N}} \mathcal{C}^* \|y_0\|_{-\diamond}.$$

806 *Proof.* We follow the same strategy as in the proof of point i. of Theorem 1.1
807 with $\varepsilon = \frac{T - T_0(Y_0)}{4}$ but we do not use (52). Instead notice that using (53) we have

$$808 \quad \left| \xi[\lambda_{k,1}, \dots, \lambda_{k,m}] \right| = \left| \langle y_0, \psi[\lambda_{k,1}, \dots, \lambda_{k,m}] \rangle_{-\diamond, \diamond} \right| \leq \mathcal{C}^* e^{\lambda_{k,1} \frac{T_0(Y_0)+T}{2}} \|y_0\|_{-\diamond}.$$

809 From (51) it comes that

$$810 \quad \left| \omega[\lambda_{k,1}, \dots, \lambda_{k,m}] \right| \leq C_{T, p, r, \rho, \mathcal{N}} e^{-\lambda_{k,1} T} \mathcal{C}^* e^{\lambda_{k,1} \frac{T_0(Y_0)+T}{2}} \|y_0\|_{-\diamond}.$$

811 Thus, writing that $\|u\| \leq \sum_{k \geq 1} \|q_k\|$, we get

$$812 \quad \|u\|_{L^2(0, T; \mathbb{R})} \leq C_{T, p, r, \rho, \mathcal{N}} \mathcal{C}^* \|y_0\|_{-\diamond} \sum_{k \geq 1} e^{\lambda_{k,1} \frac{T_0(Y_0)-T}{2}}.$$

813 From Definition 1.2 it comes that $\lambda_{k,1} \geq r(k-1)$ which ends the proof of Corol-
814 lary 2.2. \square

815 **2.1.4. Estimates on Blaschke products.**

816 This aim of this section is to prove the technical estimate stated in Proposition 2.2.
817 This relies on an extension of the following result by Fattorini and Russell.

818 **LEMMA 2.2** (see [24, Theorem 1.1]). *Let $\gamma > 0$ and $\mathcal{J} : \mathbb{R}^+ \rightarrow \mathbb{R}$. Let $\mathcal{L}(\gamma, \mathcal{J})$ be*
819 *the class introduced in Remark 2.1. For any $\Sigma \in \mathcal{L}(\gamma, \mathcal{J})$ and $\sigma \in \Sigma$ we define*

$$820 \quad \mathcal{W}_\sigma^\Sigma : z \in \mathbb{C}^+ \mapsto \prod_{\substack{\sigma' \in \Sigma \\ \sigma' \neq \sigma}} \frac{\sigma' - z}{\sigma' + z}.$$

821 *Then, for any $\varepsilon > 0$, there exists $C_{\varepsilon, \gamma, \mathcal{J}} > 0$ such that*

$$822 \quad |\mathcal{W}_\sigma^\Sigma(\sigma)| \geq C_{\varepsilon, \gamma, \mathcal{J}} e^{-\varepsilon \sigma}.$$

823 **REMARK 2.3.** *To be complelety accurate let us precise that [24, Theorem 1.1] does*
824 *not exactly state such estimate since this theorem only deals with the estimate of a*
825 *biorthogonal family. However, the estimate given in this theorem together with the*
826 *[24, equality (2.1)] given during its proof directly yield Lemma 2.2.*

827 The generalisation we propose is the following.

828 LEMMA 2.3. Let $\gamma > 0$ and $\mathcal{J} : \mathbb{R}^+ \rightarrow \mathbb{R}$. For any $\varepsilon > 0$, there exists $C_{\varepsilon, \gamma, \mathcal{J}} > 0$
 829 such that, for any sequence $\Sigma \in \mathcal{L}(\gamma, \mathcal{J})$, for any $\sigma \in \Sigma$, we have

$$830 \quad |\mathcal{W}_\sigma^\Sigma(z)| \geq C_{\varepsilon, \gamma, \mathcal{J}} e^{-\varepsilon\sigma}, \quad \forall z \in \mathbb{C}^+, \text{ s.t. } |z - \sigma| \leq \frac{\gamma}{2}$$

831 *Proof.* For any $\sigma' > 0$, since $(\sigma' - \Re(z))^2 \leq (\sigma' + \Re(z))^2$, it comes that

$$832 \quad \left| \frac{\sigma' - z}{\sigma' + z} \right|^2 = \frac{(\sigma' - \Re(z))^2 + \Im(z)^2}{(\sigma' + \Re(z))^2 + \Im(z)^2} \geq \frac{(\sigma' - \Re(z))^2}{(\sigma' + \Re(z))^2},$$

833 and thus,

$$834 \quad (54) \quad |\mathcal{W}_\sigma^\Sigma(z)| \geq |\mathcal{W}_{\sigma'}^\Sigma(\Re(z))|.$$

We introduce the family $\tilde{\Sigma}$ obtained from Σ by replacing σ by $\Re(z)$, that is

$$\tilde{\Sigma} := (\Sigma \setminus \{\sigma\}) \cup \{\Re(z)\}.$$

Since only one value has been modified, $\tilde{\Sigma}$ also satisfies

$$\sum_{\tilde{\sigma} \in \tilde{\Sigma}} \frac{1}{\tilde{\sigma}} < +\infty.$$

835 As,

$$836 \quad (55) \quad |\Re(z) - \sigma| \leq |z - \sigma| \leq \frac{\gamma}{2},$$

837 it comes that $\tilde{\Sigma}$ satisfies the gap condition (27) with ρ replaced by $\frac{\gamma}{2}$. Notice that
 838 $\{\Re(z)\} \in \mathcal{L}(1, \varepsilon \mapsto \frac{1}{\varepsilon})$. Thus using Remark 2.2 and the arguments of the proof of
 839 Lemma 2.1 it comes that $\tilde{\Sigma} \in \mathcal{L}(\frac{\gamma}{2}, \tilde{\mathcal{J}})$ with $\tilde{\mathcal{J}}$ depending only on \mathcal{J} .

840 Obviously, as the terms $\sigma' \in \Sigma$ that are different from σ have not been modified
 841 it comes that

$$842 \quad \mathcal{W}_\sigma^\Sigma = \mathcal{W}_{\Re(z)}^{\tilde{\Sigma}}.$$

843 Applying Lemma 2.2 it comes that for any $\varepsilon > 0$, there is $C_{\varepsilon, \gamma, \mathcal{J}} > 0$ such that

$$844 \quad |\mathcal{W}_{\Re(z)}^{\tilde{\Sigma}}(\Re(z))| \geq C_{\varepsilon, \gamma, \mathcal{J}} e^{-\varepsilon \Re(z)}.$$

845 Finally, recalling (54) and (55), we obtain

$$846 \quad |\mathcal{W}_\sigma^\Sigma(z)| = |\mathcal{W}_{\Re(z)}^{\tilde{\Sigma}}(z)| \geq |\mathcal{W}_{\Re(z)}^{\tilde{\Sigma}}(\Re(z))| \geq C_{\varepsilon, \gamma, \mathcal{J}} e^{-\varepsilon \Re(z)} \geq C_{\varepsilon, \gamma, \mathcal{J}} e^{-\varepsilon \frac{\gamma}{2}} e^{-\varepsilon\sigma}$$

847 which ends the proof of Lemma 2.3. \square

848 We now turn to the estimates we need for the derivatives of $\frac{1}{\mathcal{W}_\sigma^\Sigma}$.

849 PROPOSITION 2.5. Let $\gamma > 0$ and $\mathcal{J} : \mathbb{R}^+ \rightarrow \mathbb{R}$.

850 Then, for any $l \geq 0$, for any $\varepsilon > 0$, there exists $C_{l, \varepsilon, \gamma, \mathcal{J}} > 0$ such that for any
 851 $\Sigma \in \mathcal{L}(\gamma, \mathcal{J})$,

$$852 \quad \left| \left(\frac{1}{\mathcal{W}_\sigma^\Sigma} \right)^{(l)}(\sigma) \right| \leq C_{l, \varepsilon, \gamma, \mathcal{J}} e^{\varepsilon\sigma}, \quad \forall \sigma \in \Sigma.$$

853 *Proof.* The case $l = 0$ is nothing but the estimate given in Lemma 2.3.
 854 Let

$$855 \quad D_{\sigma,\gamma} := \left\{ z \in \mathbb{C}^+ ; |z - \sigma| \leq \frac{\gamma}{2} \right\}, \quad \mathcal{C}_{\sigma,\gamma} := \left\{ z \in \mathbb{C}^+ ; |z - \sigma| = \frac{\gamma}{2} \right\}.$$

857 As $\mathcal{W}_\sigma^\Sigma$ does not vanish in an open neighbourhood of $D_{\sigma,\gamma}$ it comes that $\frac{1}{\mathcal{W}_\sigma^\Sigma}$ is
 858 holomorphic on this domain. Thus applying Cauchy formula yields

$$859 \quad \left(\frac{1}{\mathcal{W}_\sigma^\Sigma} \right)^{(l)}(\sigma) = \frac{l!}{2i\pi} \int_{\mathcal{C}_{\sigma,\gamma}} \frac{\frac{1}{\mathcal{W}_\sigma^\Sigma}(z)}{(z - \sigma)^{l+1}} dz.$$

860 From Lemma 2.3 it comes that for any $\varepsilon > 0$ there exists $C_{\varepsilon,\gamma,\mathcal{J}} > 0$ such that

$$861 \quad \left| \left(\frac{1}{\mathcal{W}_\sigma^\Sigma} \right)(z) \right| \leq C_{\varepsilon,\gamma,\mathcal{J}} e^{\varepsilon\sigma}, \quad \forall z \in \mathcal{C}_{\sigma,\gamma}.$$

862 This directly implies

$$863 \quad \left| \left(\frac{1}{\mathcal{W}_\sigma^\Sigma} \right)^{(l)}(\sigma) \right| \leq C_{\varepsilon,\gamma,\mathcal{J}} \frac{l!}{\gamma^l} e^{\varepsilon\sigma}$$

864 and ends the proof of Proposition 2.5. \square

865 We shall now move to the proof of Proposition 2.2 which is the main objective of
 866 this section.

867 *Proof (of Proposition 2.2).* Recall that the function $\mathcal{N} : \mathbb{R}^+ \rightarrow \mathbb{R}$ is the one
 868 appearing in (26) and that the subsequences Λ_j are defined in (37).

869 We recall that the index k is fixed, as well as the value $\theta \in \text{Conv}(G_k)$. We intro-
 870 duce the new sequence $\tilde{\Lambda}_j$ obtained from Λ_j by replacing the k -th value $\lambda_{k,\min(j,g_k)}$
 871 by θ i.e.

$$872 \quad \tilde{\Lambda}_j := (\Lambda_j \setminus \{\lambda_{k,\min(j,g_k)}\}) \cup \{\theta\}.$$

873 Notice that, using Proposition 7.1, the fact that Λ_j is a subsequence of Λ such that
 874 each term belong to a different group, and by the assumption on θ , we obtain that
 875 $\tilde{\Lambda}_j$ satisfies the gap condition (27) with ρ replaced by $\gamma = \frac{\rho}{p}$. Notice that $\{\theta\} \in$
 876 $\mathcal{L}(1, \varepsilon \mapsto \frac{1}{\varepsilon})$. Thus using Remark 2.2 and the arguments of the proof of Lemma 2.1
 877 it comes that $\tilde{\Lambda}_j \in \mathcal{L}(\gamma, \tilde{\mathcal{J}})$ with $\tilde{\mathcal{J}}$ depending only on \mathcal{N} .

878 With these notations and Proposition 7.1 it comes that

$$879 \quad \frac{1}{W_k(z)} = \prod_{j=1}^p \frac{1}{W_\theta^{\tilde{\Lambda}_j}(z)}.$$

880 Finally, using Leibniz rule (for derivatives), evaluating the result at $z = \theta$ and using
 881 Proposition 2.5 yield the claim. \square

882 2.2. Lack of null-controllability in small time.

883 The goal of this section is to prove the point ii. of Theorem 1.1 in the case of
 884 algebraically simple eigenvalues. Thus, in all this section we assume that $\eta = 1$.
 885 To do so we will disprove the observability inequality given by Lemma 1.1 at time
 886 $T < T_0(Y_0)$.

887 *Proof.* Let $(k_n)_{n \geq 1}$ be an extraction such that

$$888 \quad \lim_{n \rightarrow \infty} \frac{\ln \left(\max_{l \in \llbracket 1, g_{k_n} \rrbracket} \left\| \sum_{j=1}^l \frac{\psi_{k_n, j}}{\prod_{i \in \llbracket 1, l \rrbracket, i \neq j} (\lambda_{k_n, j} - \lambda_{k_n, i})} \right\|_{\diamond^*} \right)}{\lambda_{k_n, 1}} = T_0(Y_0).$$

889 For any $n \geq 1$, let l_n be an integer achieving the maximum appearing in the previous
 890 formula. Since $1 \leq l_n \leq g_{k_n} \leq p$ for any n , we can find a $l^* \in \llbracket 1, p \rrbracket$ such that $l_n = l^*$
 891 for an infinite number of values of n . Therefore, there exists another extraction
 892 $(\kappa_n)_{n \geq 1}$ such that

$$893 \quad (56) \quad \lim_{n \rightarrow \infty} \frac{\ln \|x_n\|_{\diamond^*}}{\lambda_{\kappa_n, 1}} = T_0(Y_0),$$

894 where

$$895 \quad x_n := \sum_{j=1}^{l^*} \frac{\psi_{\kappa_n, j}}{\prod_{i \in \llbracket 1, l^* \rrbracket, i \neq j} (\lambda_{\kappa_n, j} - \lambda_{\kappa_n, i})} = \psi[\lambda_{\kappa_n, 1}, \dots, \lambda_{\kappa_n, l^*}].$$

896 Moreover, we can assume that for any $m \in \llbracket 1, l^* - 1 \rrbracket$

$$897 \quad \limsup_{n \rightarrow +\infty} \frac{\ln \|\psi[\lambda_{\kappa_n, 1}, \dots, \lambda_{\kappa_n, m}]\|_{\diamond^*}}{\lambda_{\kappa_n, 1}} < T_0(Y_0),$$

898 since if it is not the case, we can reduce the value of l^* accordingly.

899 Let T be any number such that, for some $\varepsilon > 0$,

$$900 \quad (57) \quad \max_{m \in \llbracket 1, l^* - 1 \rrbracket} \left(\limsup_{n \rightarrow +\infty} \frac{\ln \|\psi[\lambda_{\kappa_n, 1}, \dots, \lambda_{\kappa_n, m}]\|_{\diamond^*}}{\lambda_{\kappa_n, 1}} \right) < T < T_0(Y_0) - \varepsilon,$$

901 where the left-hand side term is replaced by 0 when $l^* = 1$. We will prove that system
 902 (1) is not null-controllable at time T from Y_0 , which will obviously imply the same
 903 result for all the values of T in $(0, T_0(Y_0))$.

904 By (56), for n sufficiently large we have

$$905 \quad (58) \quad \|x_n\|_{\diamond^*} \geq e^{\lambda_{\kappa_n, 1}(T_0(Y_0) - \varepsilon)}.$$

906 Note that, by (14), we have $x_n = P_{Y_0}^*(z_n)$ with

$$907 \quad z_n := \sum_{j=1}^{l^*} \frac{\frac{\phi_{\kappa_n, j}}{\mathcal{B}^* \phi_{\kappa_n, j}}}{\prod_{i \in \llbracket 1, l^* \rrbracket, i \neq j} (\lambda_{\kappa_n, j} - \lambda_{\kappa_n, i})}.$$

908 If system (1) is null-controllable at time T from Y_0 , then using Lemma 1.1 with
 909 $z_T = z_n$, it comes that

$$910 \quad (59) \quad \left\| P_{Y_0}^*(e^{-TA^*} z_n) \right\|_{\diamond^*}^2 \leq C \int_0^T \left| \sum_{j=1}^{l^*} \frac{e^{-\lambda_{\kappa_n, j} t}}{\prod_{i \in \llbracket 1, l^* \rrbracket, i \neq j} (\lambda_{\kappa_n, j} - \lambda_{\kappa_n, i})} \right|^2 dt.$$

911 By definition of z_n we have

$$912 \quad e^{-TA^*} z_n = \sum_{j=1}^{l^*} e^{-\lambda_{\kappa_n, j} T} \frac{\psi_{\kappa_n, j}}{\prod_{i \in \llbracket 1, l^* \rrbracket, i \neq j} (\lambda_{\kappa_n, j} - \lambda_{\kappa_n, i})}.$$

913 From Newton formula (see Proposition 7.3) and Lagrange theorem (see Proposi-
 914 tion 7.4), for any $t \in [0, T]$, there exists $\xi_{n,t}$ in $[\lambda_{\kappa_n,1}, \lambda_{\kappa_n,l^*}]$ such that

$$915 \quad \left| \sum_{j=1}^{l^*} \frac{e^{-\lambda_{\kappa_n,j}t}}{\prod_{i \in [1, l^*] \neq j} (\lambda_{\kappa_n,j} - \lambda_{\kappa_n,i})} \right| \leq \frac{t^{l^*}}{l^*!} e^{-\xi_{n,t}t} \leq \frac{t^{l^*}}{l^*!} e^{-\lambda_{\kappa_n,1}t}.$$

916 Thus, the right-hand side of (59) goes to 0 as n goes to $+\infty$. Notice that if $l^* = 1$ this
 917 last inequality is direct and one does not need to use Newton formula nor Lagrange
 918 theorem.

919 From (58) and (13) it comes that

$$920 \quad \|e^{-\lambda_{\kappa_n,l^*}T} P_{Y_0}^*(z_n)\|_{\diamond^*} = e^{-\lambda_{\kappa_n,l^*}T} \|x_n\|_{\diamond^*} \geq e^{-\rho T} e^{\lambda_{\kappa_n,l^*}(T_0(Y_0) - \varepsilon - T)} \xrightarrow{n \rightarrow +\infty} +\infty.$$

921 We will use this inequality to prove that the left side of (59) goes to $+\infty$. For
 922 this purpose we show that

$$923 \quad (60) \quad \left\| P_{Y_0}^*(e^{-T\mathcal{A}^*} z_n) - e^{-\lambda_{\kappa_n,l^*}T} P_{Y_0}^*(z_n) \right\|_{\diamond^*} \xrightarrow{n \rightarrow \infty} 0.$$

924 Together with (12) it will disprove the observability inequality (59) and thus null-
 925 controllability of the system (1) at time T from Y_0 .

926 Notice that, if $l^* = 1$, then $e^{-T\mathcal{A}^*} z_n = e^{-\lambda_{\kappa_n,l^*}T} z_n$ and thus (60) is obvious. We
 927 assume in the rest of the proof that $l^* > 1$. Notice that from Propositions 7.3 and 7.6
 928 it comes that

$$929 \quad P_{Y_0}^*(e^{-T\mathcal{A}^*} z_n) = \sum_{j=1}^{l^*} \frac{e_T(\lambda_{\kappa_n,j}) \psi_{\kappa_n,j}}{\prod_{i \in [1, l^*] \neq j} (\lambda_{\kappa_n,j} - \lambda_{\kappa_n,i})} = (e_T \psi)[\lambda_{\kappa_n,1}, \dots, \lambda_{\kappa_n,l^*}]$$

$$930 \quad = \sum_{m=1}^{l^*} e_T[\lambda_{\kappa_n,l^*}, \dots, \lambda_{\kappa_n,m}] \psi[\lambda_{\kappa_n,m}, \dots, \lambda_{\kappa_n,1}],$$

$$931$$

932 with e_T defined in (5), and also

$$933 \quad P_{Y_0}^*(z_n) = \psi[\lambda_{\kappa_n,1}, \dots, \lambda_{\kappa_n,p^*}].$$

934 Thus,

$$935 \quad P_{Y_0}^*(e^{-T\mathcal{A}^*} z_n) - e^{-\lambda_{\kappa_n,l^*}T} P_{Y_0}^*(z_n) = \sum_{m=1}^{l^*} e_T[\lambda_{\kappa_n,l^*}, \dots, \lambda_{\kappa_n,m}] \psi[\lambda_{\kappa_n,m}, \dots, \lambda_{\kappa_n,1}]$$

$$936 \quad - e_T[\lambda_{\kappa_n,l^*}] \psi[\lambda_{\kappa_n,1}, \dots, \lambda_{\kappa_n,l^*}]$$

$$937 \quad = \sum_{m=1}^{l^*-1} e_T[\lambda_{\kappa_n,l^*}, \dots, \lambda_{\kappa_n,m}] \psi[\lambda_{\kappa_n,m}, \dots, \lambda_{\kappa_n,1}]$$

$$938$$

939 From Lagrange theorem (see Proposition 7.4), there is $\xi_m \in [\lambda_{\kappa_n,m}, \lambda_{\kappa_n,l^*}]$ such that

$$940 \quad |e_T[\lambda_{\kappa_n,m}, \dots, \lambda_{\kappa_n,l^*}]| = \frac{|e_T^{(l^*-m)}(\xi_m)|}{(l^*-m)!} \leq \frac{T^{l^*-m}}{(l^*-m)!} e^{-\lambda_{\kappa_n,1}T}$$

941 and thus

$$942 \quad \left\| P_{Y_0}^*(e^{-T\mathcal{A}^*} z_n) - e^{-\lambda_{\kappa_n, l^*} T} P_{Y_0}^*(z_n) \right\|_{\mathcal{O}^*} \leq C \sum_{m=1}^{l^*-1} e^{-\lambda_{\kappa_n, 1} T} \|\psi[\lambda_{\kappa_n, 1}, \dots, \lambda_{\kappa_n, m}]\|_{\mathcal{O}^*}.$$

943 Using the conditions (57) on T finally implies

$$944 \quad \left\| P_{Y_0}^*(e^{-T\mathcal{A}^*} z_n) - e^{-\lambda_{\kappa_n, l^*} T} P_{Y_0}^*(z_n) \right\|_{\mathcal{O}^*} \xrightarrow{n \rightarrow \infty} 0$$

945 and ends the proof on the lack of null-controllability at time $T < T_0(Y_0)$. \square

946 The computations done in the previous proof provide the following lower bound
947 for the cost of controllability from Y_0 at time T as defined by Lemma 1.1.

948 **COROLLARY 2.3.** *Let $p \in \mathbb{N}^*$, $r, \rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$. Assume that
949 $\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$ and let $(G_k)_k \in \mathcal{G}(\Lambda, p, r, \rho)$ be an associated grouping. Let $T > 0$
950 and $Y_0 \subset X_{-\diamond}$. Assume that $\eta = 1$. If system (1) is null-controllable from Y_0 at time
951 T then,*

$$952 \quad M(Y_0, T) \geq \sup_{k \geq 1} \max_{l \in \llbracket 1, g_k \rrbracket} \frac{l! \sqrt{\lambda_{k, 1}}}{T^l} \|(e_T \psi)[\lambda_{k, 1}, \dots, \lambda_{k, l}]\|_{\mathcal{O}^*}$$

953 where e_T is defined by (5), ψ is defined by (14) and $M(Y_0, T)$ the cost of controllability
954 from Y_0 at time T is defined in Lemma 1.1.

955 *Proof.* Let $k \geq 1$ and $l \in \llbracket 1, g_k \rrbracket$. If system (1) is null-controllable at time T from
956 Y_0 , we apply Lemma 1.1 with

$$957 \quad z_T := \sum_{j=1}^l \frac{\frac{\phi_{k, j}}{\mathbf{B}^* \phi_{k, j}}}{\prod_{i \in \llbracket 1, l \rrbracket \neq j} (\lambda_{k, j} - \lambda_{k, i})}.$$

958 By definition of z_T we have

$$959 \quad e^{-T\mathcal{A}^*} z_T = \sum_{j=1}^l e^{-\lambda_{k, j} T} \frac{\frac{\phi_{k, j}}{\mathbf{B}^* \phi_{k, j}}}{\prod_{i \in \llbracket 1, l \rrbracket \neq j} (\lambda_{k, j} - \lambda_{k, i})},$$

960 and thus

$$961 \quad M(Y_0, T)^2 \geq \frac{\|P_{Y_0}^*(e^{-T\mathcal{A}^*} z_T)\|_{\mathcal{O}^*}^2}{\int_0^T \left| \sum_{j=1}^l \frac{e^{-\lambda_{k, j} t}}{\prod_{i \in \llbracket 1, l \rrbracket \neq j} (\lambda_{k, j} - \lambda_{k, i})} \right|^2 dt}$$

962 From Newton formula (see Proposition 7.3) and Lagrange theorem (see Proposi-
963 tion 7.4), for any $t \in [0, T]$, there exists ξ_t in $[\lambda_{k, 1}, \lambda_{k, l}]$ such that

$$964 \quad \left| \sum_{j=1}^l \frac{e^{-\lambda_{k, j} t}}{\prod_{i \in \llbracket 1, l \rrbracket \neq j} (\lambda_{k, j} - \lambda_{k, i})} \right| \leq \frac{t^l}{l!} e^{-\xi_t t} \leq \frac{T^l}{l!} e^{-\lambda_{k, 1} t}.$$

965 Using Newton formula (see Proposition 7.3), we have

$$966 \quad P_{Y_0}^*(e^{-T\mathcal{A}^*} z_T) = (e_T \psi)[\lambda_{k, 1}, \dots, \lambda_{k, l}],$$

967 which ends the proof of Corollary 2.3. \square

968 **3. Comparison with some already known results.**

969 In this section, we prove that we actually recover the known formulas for the
 970 minimal null-control time when there is no condensation of eigenvalues or when the
 971 eigenvectors are assumed to form a Riesz basis of X_\diamond^* . Doing so we will highlight in
 972 Proposition 3.2 that the actual minimal null-control time is always smaller than the
 973 value predicted by the formula that would be valid under the Riesz basis assumption.
 974 As all these results were proved for algebraically simple eigenvalues we assume in all
 975 this section that $\eta = 1$.

976 Notice that the proofs in all this section only rely on the definition of the minimal
 977 null-control time (18) and thus do not depend on Theorem 1.1.

978 **3.1. When there is no condensation of eigenvalues.**

979 In this section we prove that, if the condensation index of the sequence Λ vanishes
 980 (the definition of $c(\Lambda)$ is recalled in Appendix 7.5) then the expression (18) coincides
 981 with the known expression relating the minimal time for null-controllability to the
 982 observation of the eigenvectors ϕ_λ through the operator \mathcal{B}^* .

983 PROPOSITION 3.1. *Assume that \mathcal{A} and \mathcal{B} satisfy the assumptions of Theorem 1.1*
 984 *with $\eta = 1$. If $c(\Lambda) = 0$, then, we have*

985
$$T_0(X_{-\diamond}) = \limsup_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Lambda}} \frac{-\ln |\mathcal{B}^* \phi_\lambda|}{\lambda}.$$

986 This result was already proved in [5] with the additional assumption that the
 987 family of eigenvectors $\Phi = (\phi_\lambda)_{\lambda \in \Lambda}$ forms a Riesz basis of X_\diamond^* or in [7, Remark 1.15]
 988 in a more general framework encompassing the one studied here.

989 *Proof.* Notice that when $Y_0 = X_{-\diamond}$, the operator $P_{Y_0}^*$ reduces to the identity.
 990 Thus, considering $l = 1$ in (18) always lead to

991
$$T_0(X_{-\diamond}) \geq \limsup_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Lambda}} \frac{-\ln |\mathcal{B}^* \phi_\lambda|}{\lambda}.$$

992 We assume that

993 (61)
$$T_0(X_{-\diamond}) > \limsup_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Lambda}} \frac{-\ln |\mathcal{B}^* \phi_\lambda|}{\lambda},$$

994 and we will prove that $c(\Lambda) > 0$.

995 We shall reason as in the proof of point ii. of Theorem 1.1 (see Sec. 2.2) but
 996 starting with the formula (16) instead of (17). We can find an integer $l^* \geq 1$, an
 997 extraction $(\kappa_n)_{n \geq 1}$ and integers m_n such that $1 \leq m_n \leq m_n + l^* - 1 \leq g_{\kappa_n}$ and such
 998 that if

999
$$x_n := \psi[\lambda_{\kappa_n, m_n}, \dots, \lambda_{\kappa_n, m_n + l^* - 1}],$$

1000 we have

1001
$$\lim_{n \rightarrow \infty} \frac{\ln \|x_n\|_{\diamond^*}}{\lambda_{\kappa_n, 1}} = T_0(X_{-\diamond}).$$

1002 Moreover, we can assume that for any $l \in \llbracket 1, l^* - 1 \rrbracket$, we have for some $\varepsilon > 0$

$$1003 \quad \tilde{T}(l) := \limsup_{n \rightarrow +\infty} \frac{\ln \left(\max_{\substack{m, r \in \llbracket 1, g_{\kappa_n} \rrbracket \\ m \leq r \\ r-m < l}} \|\psi[\lambda_{\kappa_n, m}, \dots, \lambda_{\kappa_n, r}]\|_{\diamond^*} \right)}{\lambda_{\kappa_n, 1}} < T_0(X_{-\diamond}) - \varepsilon,$$

1004 since, if it is not the case, we can reduce the value of l^* accordingly. Note that, as
 1005 $\|P_{Y_0}^* \phi_\lambda\|_{\diamond^*} \leq 1$, by (61), we know that $l^* > 1$.

1006 From the definition of divided differences (see Definition 7.1), it comes that

$$1007 \quad x_n = \frac{\psi[\lambda_{\kappa_n, m_n+1}, \dots, \lambda_{\kappa_n, m_n+l^*-1}] - \psi[\lambda_{\kappa_n, m_n}, \dots, \lambda_{\kappa_n, m_n+l^*-2}]}{\lambda_{\kappa_n, m_n+l^*-1} - \lambda_{\kappa_n, m_n}}.$$

1008 For n sufficiently large, we have

$$1009 \quad \|x_n\|_{\diamond^*} \geq e^{\lambda_{\kappa_n, 1}(T_0(X_{-\diamond}) - \varepsilon/2)}.$$

1010 Using the definition of $\tilde{T}(l^* - 1)$ it comes that, for n large enough,

$$1011 \quad \|\psi[\lambda_{\kappa_n, m_n+1}, \dots, \lambda_{\kappa_n, m_n+l^*-1}]\|_{\diamond^*} + \|\psi[\lambda_{\kappa_n, m_n}, \dots, \lambda_{\kappa_n, m_n+l^*-2}]\|_{\diamond^*} \\ 1012 \quad \leq e^{\lambda_{\kappa_n, 1}(\tilde{T}(l^*-1) + \varepsilon/2)}.$$

1013 Thus, since $l^* \geq 2$, we can combine the last two estimates to obtain

$$1014 \quad |\lambda_{\kappa_n, m_n+1} - \lambda_{\kappa_n, m_n}| \leq |\lambda_{\kappa_n, m_n+l^*-1} - \lambda_{\kappa_n, m_n}| \\ 1015 \quad \leq e^{-\lambda_{\kappa_n, 1}(T_0(X_{-\diamond}) - \varepsilon - \tilde{T}(l^*-1))} \\ 1016 \quad \leq e^{\rho(T_0(X_{-\diamond}) - \varepsilon - \tilde{T}(l^*-1))} e^{-\lambda_{\kappa_n, m_n}(T_0(X_{-\diamond}) - \varepsilon - \tilde{T}(l^*-1))}.$$

1017 In particular, we have

$$1018 \quad \limsup_{n \rightarrow \infty} \frac{-\ln |P'_{G_{\kappa_n}}(\lambda_{\kappa_n, m})|}{\lambda_{\kappa_n, m_n}} \geq \limsup_{n \rightarrow \infty} \frac{-\ln |\lambda_{\kappa_n, m_n+1} - \lambda_{\kappa_n, m_n}|}{\lambda_{\kappa_n, m_n}} \\ 1019 \quad \geq T_0(X_{-\diamond}) - \varepsilon - \tilde{T}(l^* - 1) > 0.$$

1020 Using Proposition 7.12, we conclude that $c(\Lambda) > 0$, and the claim is proved. \square

1022 3.2. When there is a Riesz basis of eigenvectors.

1023 As already mentioned the null-control problem for (1) has been considered in [5]
 1024 with the additional assumption that the family $(\phi_\lambda)_{\lambda \in \Lambda}$ forms a Riesz basis of X_{\diamond^*} .
 1025 Observe that it is equivalent to ask that $(\phi_\lambda / \|\phi_\lambda\|)_{\lambda \in \Lambda}$ is a Riesz basis of X .

1026 With this additional assumption, the minimal null-control time from $Y_0 = X_{-\diamond}$
 1027 was proved to be equal to

$$1028 \quad (62) \quad T^* := \limsup_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Lambda}} \frac{\ln \frac{1}{|\mathcal{B}^* \phi_\lambda| |E'_\Lambda(\lambda)|}}{\lambda},$$

1029 where the interpolating function E_Λ is defined in (125).

1030 REMARK 3.1. Notice that, since ϕ_λ is normalized in X_\diamond^* , there exists $C > 0$ such
 1031 that

$$1032 \quad \frac{1}{C\lambda} \leq \|\phi_\lambda\| \leq C, \quad \forall \lambda,$$

1033 so that the value of T^* in (62) does not change if one considers the normalization of
 1034 eigenvectors in X instead of in X_\diamond^* .

1035 In our setting, we prove that the formula above for T^* is always an upper bound
 1036 of the actual minimal null-control time, without assuming the Riesz basis condition.

1037 PROPOSITION 3.2. Assume that \mathcal{A} and \mathcal{B} satisfy the assumptions of Theorem 1.1
 1038 with $\eta = 1$. Then, $T_0(X_{-\diamond}) \leq T^*$ where T^* is defined by (62).

1039 *Proof. First step:* we begin by proving that the grouping designed in Proposi-
 1040 tion 7.1 ensures a simpler expression for T^* . Let $(G_k)_{k \geq 1} \in \mathcal{G}(\Lambda, p, r, \rho)$ be a grouping
 1041 as introduced in Section 1.3.2. For each $\lambda \in \Lambda$, we denote by $G^{[\lambda]}$ the unique group
 1042 in $(G_k)_{k \geq 1}$ that contains λ . Then, we have

$$1043 \quad (63) \quad T^* = \limsup_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Lambda}} \frac{\ln \frac{1}{|\mathcal{B}^* \phi_\lambda| |P'_{G^{[\lambda]}(\lambda)}|}}{\lambda},$$

1044 where, for each group G , the polynomial P_G is defined by (6).

1045 • Let G be a group of eigenvalues and $\lambda \in G$. We prove that, for any finite subset M
 1046 of $\Lambda \setminus G$, whose cardinal is denoted by $n := \#M$, we have

$$1047 \quad (64) \quad \prod_{\mu \in M} |\lambda - \mu| \geq r^n \left\lfloor \frac{n}{2p} \right\rfloor!$$

1048 where $r := \frac{\rho}{p}$. To this end, for any $j \in \llbracket 1, p \rrbracket$, we define

$$1049 \quad M_j := \{\mu \in M ; \exists k \geq 1 \text{ such that } \mu = \lambda_{k,j}\}.$$

1050 Since the groups are covering Λ , we have a disjoint union $M = \bigcup_{j=1}^p M_j$. It follows that

1051 there exists $j_0 \in \llbracket 1, p \rrbracket$ such that $\#M_{j_0} \geq \left\lfloor \frac{n}{p} \right\rfloor$. From (12) it comes that

$$1052 \quad |\lambda - \mu| \geq r, \quad \forall \mu \in M,$$

1053 and

$$1054 \quad |\mu - \mu'| \geq r, \quad \forall j \in \llbracket 1, p \rrbracket, \forall \mu, \mu' \in M_j, \mu \neq \mu'.$$

1055 Then,

$$1056 \quad \prod_{\mu \in M} |\lambda - \mu| = \left(\prod_{\substack{j=1 \\ j \neq j_0}}^p \prod_{\mu \in M_j} |\lambda - \mu| \right) \left(\prod_{\mu \in M_{j_0}} |\lambda - \mu| \right)$$

$$1057 \quad \geq \left(r^{\#(M \setminus M_{j_0})} \right) \left(r^{\#(M_{j_0})} \right) \left\lfloor \frac{\#M_{j_0}}{2} \right\rfloor!$$

1058 This proves (64).
 1059

1060 • From (64) we apply [5, Theorem 3.8] to obtain that for any subsequence $(\lambda_n)_{n \geq 1} \subset$
 1061 Λ ,

$$1062 \quad (65) \quad \lim_{n \rightarrow \infty} \left(\frac{\ln \frac{1}{|E'_\Lambda(\lambda_n)|}}{\lambda_n} - \frac{\ln \frac{1}{|P'_{G[\lambda_n]}(\lambda_n)|}}{\lambda_n} \right) = 0.$$

1063 This directly implies (63).

1064 **REMARK 3.2.** Notice that (64) is not the exact assumption required in [5, Theo-
 1065 rem 3.8]. For this result the authors assumed

$$1066 \quad (66) \quad \prod_{\mu \in M} |\lambda - \mu| \geq r^n n!,$$

1067 with the same notation as in the proof above. We claim that with the exact same proof
 1068 it is sufficient to assume (64). Indeed, in the proof of [5, Theorem 3.8], the only point
 1069 where assumption (66) is used is the Second step in the middle of page 2097. Then
 1070 the term $n!$ is estimated asymptotically using Stirling formula to prove that the term
 1071 $\Gamma_{k,1}$ goes to 0 as k goes to ∞ . As the rest of the proof is long, technical and remain
 1072 unchanged when replacing (66) by (64) we do not reproduce it here for the sake of
 1073 brevity.

1074 *Second step:* we end the proof of Proposition 3.2. Recall that from (32) we have
 1075 that there exists $C_{p,\rho} > 0$ such that for any $k \geq 1$, $l \in \llbracket 1, g_k \rrbracket$ and any $j \in \llbracket 1, l \rrbracket$,

$$1076 \quad \prod_{i \in \llbracket 1, l \rrbracket \neq j} |\lambda_{k,j} - \lambda_{k,i}| \geq C_{p,\rho} |P'_{G_k}(\lambda_{k,j})|.$$

1077 As we have considered normalized eigenvectors, and by (4), for any $k \geq 1$ and
 1078 any $l \in \llbracket 1, g_k \rrbracket$, we have

$$\begin{aligned} 1079 \quad \left\| \sum_{j=1}^l \frac{\psi_{k,j}}{\prod_{i \in \llbracket 1, l \rrbracket \neq j} (\lambda_{k,j} - \lambda_{k,i})} \right\|_{\diamond^*} &\leq \left\| \sum_{j=1}^l \frac{\frac{\phi_{k,j}}{\mathcal{B}^* \phi_{k,j}}}{\prod_{i \in \llbracket 1, l \rrbracket \neq j} (\lambda_{k,j} - \lambda_{k,i})} \right\|_{\diamond^*} \\ 1080 &\leq l \max_{j \in \llbracket 1, l \rrbracket} \left\| \frac{\frac{\phi_{k,j}}{\mathcal{B}^* \phi_{k,j}}}{\prod_{i \in \llbracket 1, l \rrbracket \neq j} (\lambda_{k,j} - \lambda_{k,i})} \right\|_{\diamond^*} \\ 1081 &\leq l \max_{j \in \llbracket 1, l \rrbracket} \frac{1}{|\mathcal{B}^* \phi_{k,j}| \prod_{i \in \llbracket 1, l \rrbracket \neq j} |\lambda_{k,j} - \lambda_{k,i}|}. \end{aligned}$$

1083 Using (32) this leads to

$$1084 \quad \left\| \sum_{j=1}^l \frac{\psi_{k,j}}{\prod_{i \in \llbracket 1, l \rrbracket \neq j} (\lambda_{k,j} - \lambda_{k,i})} \right\|_{\diamond^*} \leq Cl \max_{j \in \llbracket 1, l \rrbracket} \frac{1}{|\mathcal{B}^* \phi_{k,j}| |P'_{G_k}(\lambda_{k,j})|}.$$

1085 Thus,

$$1086 \frac{\ln \max_{l \in \llbracket 1, g_k \rrbracket} \left(\left\| \sum_{j=1}^l \frac{\psi_{k,j}}{\prod_{i \in \llbracket 1, l \rrbracket, i \neq j} (\lambda_{k,j} - \lambda_{k,i})} \right\|_{\diamond^*} \right)}{1087 \lambda_{k,1}} \leq \max_{j \in \llbracket 1, g_k \rrbracket} \frac{\ln \frac{1}{|\mathcal{B}^* \phi_{k,j}| |P_{G_k}^l(\lambda_{k,j})|} \lambda_{k,j}}{\lambda_{k,j}} + \frac{\ln(Cl)}{\lambda_{k,1}}.$$

1088 Then, using (63), we obtain

$$1089 T_0(X_{-\circ}) \leq T^*. \quad \square$$

1090 We now prove that we indeed recover exactly the expression of the minimal
1091 time (62) (or (63)) when we assume that the eigenvectors form a Riesz basis.

1092 **PROPOSITION 3.3.** *Assume that \mathcal{A} and \mathcal{B} satisfy the assumptions of Theorem 1.1*
1093 *with $\eta = 1$ and that $(\phi_\lambda)_{\lambda \in \Lambda}$ forms a Riesz basis of X_\diamond^* . Then, $T_0(X_{-\circ}) = T^*$ where*
1094 *T^* is defined by (63).*

REMARK 3.3. *It will appear clearly in the proof that the Riesz basis assumption*
is much stronger than what we really need. The only thing that we actually use at
the very beginning of the proof, is that the spectral radius of the inverse of the Gram
matrix $M_k := \text{Gram}_{X_\diamond^}(\phi_{k,1}, \dots, \phi_{k,g_k})$ satisfies*

$$\sup_{k \geq 1} \rho(M_k^{-1}) < +\infty.$$

A careful inspection of the proof shows that it is in fact sufficient to assume that

$$\limsup_{k \rightarrow \infty} \frac{\ln \rho(M_k^{-1})}{\lambda_{k,1}} = 0.$$

1095 *Note in particular that, in practice, estimating such a spectral radius in each group is*
1096 *much simpler than proving that the whole family is a Riesz basis.*

1097 *Proof.* As we assumed that $(\phi_\lambda)_{\lambda \in \Lambda}$ is a Riesz basis of X_\diamond^* it comes that there
1098 exists $C > 0$ such that for any $k \geq 1$, for any $\alpha_{k,1}, \dots, \alpha_{k,g_k} \in \mathbb{R}$,

$$1099 \max_{j \in \llbracket 1, g_k \rrbracket} |\alpha_{k,j}| \leq \left(\sum_{j=1}^{g_k} \alpha_{k,j}^2 \right)^{\frac{1}{2}} \leq C \left\| \sum_{j=1}^{g_k} \alpha_{k,j} \phi_{k,j} \right\|_{\diamond^*},$$

1100 and thus

$$1101 \max_{j \in \llbracket 1, g_k \rrbracket} |\alpha_{k,j}| \leq C \max_{l \in \llbracket 1, g_k \rrbracket} \left\| \sum_{j=1}^l \alpha_{k,j} \phi_{k,j} \right\|_{\diamond^*}.$$

1102 Setting

$$1103 \alpha_{k,j} := \frac{1}{|\mathcal{B}^* \phi_{k,j}| \prod_{i \in \llbracket 1, g_k \rrbracket, i \neq j} (\lambda_{k,j} - \lambda_{k,i})},$$

1104 yield

$$1105 \max_{j \in \llbracket 1, g_k \rrbracket} \frac{1}{|\mathcal{B}^* \phi_{k,j}| \prod_{i \in \llbracket 1, g_k \rrbracket, i \neq j} |\lambda_{k,j} - \lambda_{k,i}|} \leq C \left\| \sum_{j=1}^{g_k} \frac{\frac{\phi_{k,j}}{\mathcal{B}^* \phi_{k,j}}}{\prod_{i \in \llbracket 1, g_k \rrbracket, i \neq j} (\lambda_{k,j} - \lambda_{k,i})} \right\|_{\diamond^*}$$

$$1106 \leq C \max_{1 \leq l \leq g_k} \left(\left\| \sum_{j=1}^l \frac{\frac{\phi_{k,j}}{\mathcal{B}^* \phi_{k,j}}}{\prod_{i \in \llbracket 1, l \rrbracket, i \neq j} (\lambda_{k,j} - \lambda_{k,i})} \right\|_{\diamond^*} \right).$$

1107

1108 It follows that for any $j \in \llbracket 1, g_k \rrbracket$,

$$1109 \quad \frac{1}{|\mathcal{B}^* \phi_{k,j}| |P'_{G_k}(\lambda_{k,j})|} = \frac{1}{|\mathcal{B}^* \phi_{k,j}| \prod_{i \in \llbracket 1, g_k \rrbracket \neq j} |\lambda_{k,j} - \lambda_{k,i}|}$$

$$1110 \quad \leq C \max_{1 \leq l \leq g_k} \left(\left\| \sum_{j=1}^l \frac{\frac{\phi_{k,j}}{\mathcal{B}^* \phi_{k,j}}}{\prod_{i \in \llbracket 1, l \rrbracket \neq j} (\lambda_{k,j} - \lambda_{k,i})} \right\|_{\diamond^*} \right).$$

1112 Thus, taking the logarithm,

$$1113 \quad \frac{-\ln |\mathcal{B}^* \phi_{k,j}| |P'_{G_k}(\lambda_{k,j})|}{\lambda_{k,j}} \leq \frac{\ln \max_{l \in \llbracket 1, g_k \rrbracket} \left(\left\| \sum_{j=1}^l \frac{\frac{\phi_{k,j}}{\mathcal{B}^* \phi_{k,j}}}{\prod_{i \in \llbracket 1, l \rrbracket \neq j} (\lambda_{k,j} - \lambda_{k,i})} \right\|_{\diamond^*} \right)}{\lambda_{k,1}} + \frac{\ln C}{\lambda_{k,1}}.$$

1115 Since by definition we have $G_k = G^{[\lambda_{k,j}]}$, this ends the proof of Proposition 3.3. \square

1116 4. The case of multiple eigenvalues.

1117 In this section we prove Theorem 1.1 in the case where we allow algebraic multi-
1118 plicity for the eigenvalues i.e. $\eta \geq 2$. As previously, the main issue is the resolution
1119 of the block moment problem given in (23). This is detailed in the next subsection.

1120 4.1. Resolution of block moment problems.

1121 We prove in this subsection the following theorem which is the generalization of
1122 Theorem 2.1.

1123 **THEOREM 4.1.** *Let $T \in (0, +\infty]$. Let $p \in \mathbb{N}^*$, $r, \rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$.
1124 Assume that $\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$ and let $(G_k)_k \in \mathcal{G}(\Lambda, p, r, \rho)$ be an associated grouping.
1125 We also consider an integer $\eta \geq 1$.*

1126 *For any $\varepsilon > 0$, there exists a constant $C_{\varepsilon, T, p, r, \rho, \eta, \mathcal{N}} > 0$ such that for any $k \geq 1$,
1127 for any multi-index $\alpha_k \in \mathbb{N}^{g_k}$ with $|\alpha_k|_\infty \leq \eta$, any set of values $\omega_{\alpha_k} \in \mathbb{C}^{|\alpha_k|}$, there
1128 exists $q_k \in L^2(0, T; \mathbb{C})$ satisfying*

$$1129 \quad (67) \quad \begin{cases} \int_0^T q_k(t) \frac{(-t)^{l'}}{l'!} e^{-\lambda_{k',j'} t} dt = 0, & \forall k' \neq k, \forall j' \in \llbracket 1, g_{k'} \rrbracket, \forall l' \in \llbracket 0, \eta \rrbracket, \\ \int_0^T q_k(t) \frac{(-t)^l}{l!} e^{-\lambda_{k,j} t} dt = \omega_{k,j}^l, & \forall j \in \llbracket 1, g_k \rrbracket, \forall l \in \llbracket 0, \alpha_{k,j} - 1 \rrbracket, \end{cases}$$

1130 and the bound

$$1131 \quad (68) \quad \|q_k\|_{L^2(0, T; \mathbb{C})} \leq C_{\varepsilon, T, p, r, \rho, \eta, \mathcal{N}} e^{\varepsilon \lambda_{k,1}} \max_{\substack{\mu \in \mathbb{N}^{g_k} \\ \mu \leq \alpha_k}} \left| \omega[\lambda_{k,1}^{(\mu_1)}, \dots, \lambda_{k, g_k}^{(\mu_k)}] \right|.$$

1132 Moreover, up to the factor $e^{\varepsilon \lambda_{k,1}}$, this last estimate is sharp: any solution q_k of (67),
1133 satisfy

$$1134 \quad (69) \quad \|q_k\|_{L^2(0, T; \mathbb{C})} \geq \tilde{C}_{p, \eta} \max_{\substack{\mu \in \mathbb{N}^{g_k} \\ \mu \leq \alpha}} \left| \omega[\lambda_{k,1}^{(\mu_1)}, \dots, \lambda_{k, g_k}^{(\mu_k)}] \right|,$$

1135 for some $\tilde{C}_{p, \eta} > 0$.

1136 In the case $p = 1$ (usual gap condition), a solution to (67) is given by the biorthogonal
 1137 family built in [4]. Here, we extend this resolution using a weak gap condition (8) and
 1138 we prove that the obtained estimates are uniform with respect to Λ in a given class
 1139 $\mathcal{L}_w(\bullet, \bullet, \bullet)$.

1140 COROLLARY 4.1. *Let $T \in (0, +\infty]$. Let $p \in \mathbb{N}^*$, $r, \rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$.
 1141 Assume that $\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$ and let $(G_k)_k \in \mathcal{G}(\Lambda, p, r, \rho)$ be an associated grouping.*

1142 *We consider an integer $\eta \geq 1$ and for any k we suppose given a multi-index
 1143 $\alpha_k \in \mathbb{N}^{g_k}$ such that $|\alpha_k|_\infty \leq \eta$.*

1144 *Then, for any $k \geq 1$, for any $j \in \llbracket 1, g_k \rrbracket$ and any $l \in \llbracket 0, \alpha_{k,j} - 1 \rrbracket$ there exists
 1145 $q_{k,j,l} \in L^2(0, T; \mathbb{C})$ satisfying*

$$1146 \quad \int_0^T q_{k,j,l}(t) \frac{(-t)^{l'}}{l'!} e^{-\lambda_{k',j'} t} dt = \delta_{k,k'} \delta_{j,j'} \delta_{l,l'},$$

1147 *for any $k, k' \geq 1$, any $j \in \llbracket 1, g_k \rrbracket$, $j' \in \llbracket 1, g_{k'} \rrbracket$ and any $l \in \llbracket 0, \alpha_{k,j} - 1 \rrbracket$, $l' \in$
 1148 $\llbracket 0, \alpha_{k',j'} - 1 \rrbracket$. Moreover, for any $\varepsilon > 0$, there exists a constant $C_{\varepsilon, T, p, r, \rho, \eta, \mathcal{N}} > 0$ such
 1149 that for any $k \geq 1$, any $j \in \llbracket 1, g_k \rrbracket$ and any $l \in \llbracket 0, \alpha_{k,j} - 1 \rrbracket$, we have*

$$1150 \quad \|q_{k,j,l}\|_{L^2(0, T; \mathbb{C})}$$

$$1152 \quad \leq C_{\varepsilon, T, p, r, \rho, \eta, \mathcal{N}} \frac{e^{\varepsilon \lambda_{k,1}}}{\prod_{i \in \llbracket 1, g_k \rrbracket \neq j} |\lambda_{k,j} - \lambda_{k,i}|^{\alpha_{k,i}}} \frac{1}{\left(\min_{i \in \llbracket 1, g_k \rrbracket \neq j} |\lambda_{k,j} - \lambda_{k,i}| \right)^{\alpha_{k,j} - l - 1}}.$$

1154 The proof of Corollary 4.1 is left to the reader: it follows closely the one of Corollary 2.1
 1155 and makes use of the estimate given in Proposition 7.11 instead of the Newton formula
 1156 for standard divided differences.

1157 REMARK 4.1. *Contrary to the estimate in Corollary 2.1, the above estimate is
 1158 not optimal in general, even if we do not consider the exponential factor. Indeed,
 1159 some cancellations can occur depending on the relative positions and multiplicities of
 1160 the eigenvalues that are not taken into account in the above general bound. In actual
 1161 examples, one needs to compute carefully the coefficients of the generalized divided
 1162 differences introduced in Proposition 7.11 to see whether or not a sharper estimate
 1163 can be obtained.*

1164 Here also, the proof of Theorem 4.1 relies on the resolution of the block moment
 1165 problem (67) with $T = +\infty$ and then on a restriction argument. For pedagogical
 1166 reasons (the proof being less technical) let us present first this restriction argument
 1167 (which is the generalization of Proposition 2.3).

1168 PROPOSITION 4.1. *Let $p \in \mathbb{N}^*$, $r, \rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$. Assume that
 1169 $\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$ and let $(G_k)_k \in \mathcal{G}(\Lambda, p, r, \rho)$ be an associated grouping. We also
 1170 consider an integer $\eta \geq 1$.*

*For any $T > 0$, there exists a constant $C_{T, p, r, \rho, \eta, \mathcal{N}} > 0$ such that for any $\tilde{q} \in$
 $L^2(0, +\infty; \mathbb{C})$, there exists $q \in L^2(0, T; \mathbb{C})$ satisfying*

$$\int_0^T q(t) \frac{(-t)^l}{l!} e^{-\lambda t} dt = \int_0^{+\infty} \tilde{q}(t) \frac{(-t)^l}{l!} e^{-\lambda t} dt, \quad \forall \lambda \in \Lambda, \forall l \in \llbracket 0, \eta \rrbracket,$$

and the estimate

$$\|q\|_{L^2(0, T; \mathbb{C})} \leq C_{T, p, r, \rho, \eta, \mathcal{N}} \|\tilde{q}\|_{L^2(0, +\infty; \mathbb{C})}.$$

Proof. For any $h > 0$, we define

$$\Lambda_h := \bigcup_{l=0}^{\eta} (\Lambda + lh).$$

1171 Using Remark 2.2 and Lemma 2.1 we have that $\Lambda_h \in \mathcal{L}_w(p\eta, \rho, \tilde{\mathcal{N}})$ for some $\tilde{\mathcal{N}}$ which
 1172 does not depend on h . We suppose given a fixed \tilde{q} and, for any $h > 0$, we can
 1173 apply Proposition 2.3 with the sequence Λ_h and obtain the existence of a function
 1174 $q_h \in L^2(0, T; \mathbb{C})$ such that

$$1175 \quad (70) \quad \int_0^T q_h(t) e^{-(\lambda+lh)t} dt = \int_0^{+\infty} \tilde{q}(t) e^{-(\lambda+lh)t} dt, \quad \forall \lambda \in \Lambda, \forall l \in \llbracket 0, \eta \rrbracket,$$

and satisfying moreover the uniform estimate

$$\|q_h\|_{L^2(0, T; \mathbb{C})} \leq C_{T, p\eta, r, \rho, \tilde{\mathcal{N}}} \|\tilde{q}\|_{L^2(0, +\infty; \mathbb{C})}, \quad \forall h > 0.$$

1176 We can then find a subsequence $(q_{h_n})_n$ that weakly converges towards some $q \in$
 1177 $L^2(0, T; \mathbb{C})$ such that $\|q\|_{L^2(0, T; \mathbb{C})} \leq C_{T, p\eta, r, \rho, \tilde{\mathcal{N}}} \|\tilde{q}\|_{L^2(0, +\infty; \mathbb{C})}$. We will show that q
 1178 solves the required equations.

1179 Let $\lambda \in \Lambda$ and $l \in \llbracket 0, \eta - 1 \rrbracket$ be fixed. Combining the equations (70) to make
 1180 appear divided differences, we have the equality

$$1181 \quad (71) \quad \int_0^T q_{h_n}(t) e_t[\lambda, \dots, \lambda + lh_n] dt = \int_0^{+\infty} \tilde{q}(t) e_t[\lambda, \dots, \lambda + lh_n] dt,$$

where e_t is defined in (5). The Lagrange theorem (see Proposition 7.4) implies that,
 for any t and any n , there is a $\xi_{t,n} \in [\lambda, \lambda + lh_n]$ such that

$$e_t[\lambda, \dots, \lambda + lh_n] = \frac{(-t)^l}{l!} e^{-\xi_{t,n} t},$$

which implies that $|e_t[\lambda, \dots, \lambda + lh_n]| \leq \frac{t^l}{l!} e^{-\lambda t}$ and

$$e_t[\lambda, \dots, \lambda + lh_n] \xrightarrow{n \rightarrow \infty} \frac{(-t)^l}{l!} e^{-\lambda t}.$$

1182 By the Lebesgue dominated-convergence theorem we deduce the strong convergence
 1183 in $L^2(0, +\infty; \mathbb{C})$ of $t \mapsto e_t[\lambda, \dots, \lambda + lh_n]$ towards $t \mapsto \frac{(-t)^l}{l!} e^{-\lambda t}$ and the claim follows
 1184 by weak-strong convergence in (71). \square

1185 Let us now turn to the resolution of the block moment problem (67) for $T = +\infty$.
 1186 The next proposition is the generalization of Proposition 2.1.

1187 **PROPOSITION 4.2.** *Let $p \in \mathbb{N}^*$, $r, \rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$. Assume that*
 1188 *$\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$ and let $(G_k)_k \in \mathcal{G}(\Lambda, p, r, \rho)$ be an associated grouping. We also*
 1189 *consider an integer $\eta \geq 1$.*

1190 *For any $\varepsilon > 0$, there exists a constant $C_{\varepsilon, p, r, \rho, \eta, \mathcal{N}} > 0$ such that for any $k \geq 1$, for*
 1191 *any multi-index $\alpha_k \in \mathbb{N}^{g_k}$ with $|\alpha_k|_{\infty} \leq \eta$, and any set of values $\omega_{\alpha_k} \in \mathbb{C}^{|\alpha_k|}$, there*
 1192 *exists $q_k \in L^2(0, +\infty; \mathbb{C})$ satisfying*

$$1193 \quad \begin{cases} \int_0^{+\infty} q_k(t) \frac{(-t)^{l'}}{l'!} e^{-\lambda_{k', j'} t} dt = 0, & \forall k' \neq k, \forall j' \in \llbracket 1, g_{k'} \rrbracket, \forall l' \in \llbracket 0, \eta \rrbracket, \\ \int_0^{+\infty} q_k(t) \frac{(-t)^l}{l!} e^{-\lambda_{k, j} t} dt = \omega_{k, j}^l, & \forall j \in \llbracket 1, g_k \rrbracket, \forall l \in \llbracket 0, \alpha_{k, j} - 1 \rrbracket, \end{cases}$$

1194 and the bound

$$1195 \quad (72) \quad \|q_k\|_{L^2(0,+\infty;\mathbb{C})} \leq C_{\varepsilon,p,r,\rho,\eta,\mathcal{N}} e^{\varepsilon\lambda_{k,1}} \max_{\substack{\mu \in \mathbb{N}^{g_k} \\ \mu \leq \alpha_k}} \left| \omega[\lambda_{k,1}^{(\mu_1)}, \dots, \lambda_{k,g_k}^{(\mu_k)}] \right|.$$

1196 Before getting to the proof let us mention that Propositions 4.1 and 4.2 imply
 1197 Theorem 4.1. The lower bound (69) is proved in the exact same way as (30) and is
 1198 thus left to the reader.

Proof. As in the previous proof, for $h > 0$, we define

$$\Lambda_h := \bigcup_{l=0}^{\eta} (\Lambda + lh),$$

that belongs to the class $\mathcal{L}_w(p\eta, \rho, \tilde{\mathcal{N}})$. For any $k \geq 1$, we set

$$G_{k,h} := \bigcup_{l=0}^{\eta} (G_k + lh).$$

1199 For any $h < r/(2\eta)$, the family $(G_{k,h})_k$ is a grouping in $\mathcal{G}(\Lambda_h, p\eta, r/2, \rho + r/2)$.

1200 Now, we are given a fixed index k . We observe that, there exists a $h_0 \in (0, r/(2\eta))$
 1201 (possibly depending on k) such that, for any $h < h_0$, the sets $G_k, G_k + h, \dots, G_k + \eta h$
 1202 are pairwise disjoint.

Since we need to take into account precisely the multiplicities we are interested in, encoded in the multi-index α_k , we introduce the modified k -th group

$$\tilde{G}_{k,h} = \bigcup_{j=1}^{g_k} \{\lambda_{k,j}, \lambda_{k,j} + h, \dots, \lambda_{k,j} + (\alpha_{k,j} - 1)h\} \subset G_{k,h},$$

and the new family

$$\tilde{\Lambda}_h = \left(\bigcup_{\substack{l \geq 1 \\ l \neq k}} G_{l,h} \right) \cup \tilde{G}_{k,h},$$

1203 which satisfies $\tilde{\Lambda}_h \subset \Lambda_h$ and therefore also belongs to the class $\mathcal{L}_w(p\eta, \rho, \tilde{\mathcal{N}})$.

1204 By construction, the family of points in $\tilde{G}_{k,h}$, that we denote by $\mu_{k,h,1} < \dots <$
 1205 $\mu_{k,h,|\alpha_k|}$ is an approximation of the weighted family $((\lambda_{k,1}, \dots, \lambda_{k,g_k}), \alpha_k)$ in the sense
 1206 of Definition 7.2. Let $F : \mathbb{R} \rightarrow \mathbb{C}$ be a smooth function satisfying the conditions

$$1207 \quad (73) \quad \frac{1}{l!} F^{(l)}(\lambda_{k,j}) = \omega_{k,j}^l, \quad \forall j \in \llbracket 1, g_k \rrbracket, \forall l \in \llbracket 0, \alpha_{k,j} - 1 \rrbracket.$$

1208 For each $h > 0$, we apply Proposition 2.1 to the family $\tilde{\Lambda}_h$ to find a solution $q_{k,h} \in$
 1209 $L^2(0, +\infty; \mathbb{C})$ to the following moment problem

$$1210 \quad (74) \quad \begin{cases} \int_0^{+\infty} q_{k,h}(t) e^{-(\lambda_{k',j'} + hl)t} dt = 0, & \forall k' \neq k, \forall j' \in \llbracket 1, g_{k'} \rrbracket, \forall l \in \llbracket 0, \eta \rrbracket, \\ \int_0^{+\infty} q_{k,h}(t) e^{-(\lambda_{k,j} + hl)t} dt = F(\lambda_{k,j} + hl), & \forall j \in \llbracket 1, g_k \rrbracket, \forall l \in \llbracket 0, \alpha_{k,j} - 1 \rrbracket, \end{cases}$$

and satisfying the following bound, with a constant uniform with respect to h ,

$$\|q_{k,h}\|_{L^2(0,+\infty;\mathbb{C})} \leq C_{\varepsilon,\eta p,r,\rho,\tilde{N}} e^{\varepsilon\lambda_{k,1}} \max_{i \in \llbracket 1, \alpha_k \rrbracket} \left| F[\mu_{k,h,1}, \dots, \mu_{k,h,i}] \right|.$$

1211 By Proposition 7.7, we know that the right-hand side in the above estimate converges
 1212 when $h \rightarrow 0$ towards a similar quantity with generalized divided differences instead
 1213 of the usual divided differences. It follows that we can extract a subsequence $(q_{k,h_n})_n$
 1214 that weakly converges in $L^2(0, +\infty; \mathbb{C})$ towards a function q_k that satisfies the bound
 1215 (72).

1216 Finally, by the same argument as in the proof of Proposition 4.1 above, we can
 1217 combine the equations (74) to make appear divided differences on both side and pass
 1218 to the weak-strong limit in the integral to finally get

$$1219 \quad \begin{cases} \int_0^{+\infty} q_k(t) \frac{(-t)^l}{l!} e^{-\lambda_{k',j'} t} dt = 0, & \forall k' \neq k, \forall j' \in \llbracket 1, g_{k'} \rrbracket, \forall l \in \llbracket 0, \eta \rrbracket, \\ \int_0^{+\infty} q_k(t) \frac{(-t)^l}{l!} e^{-\lambda_{k,j} t} dt = F[\lambda_{k,j}^{(l)}], & \forall j \in \llbracket 1, g_k \rrbracket, \forall l \in \llbracket 0, \alpha_{k,j} - 1 \rrbracket, \end{cases}$$

1220 which is exactly our claim since, by the computation rule (121) and by (73), we have
 1221 $F[\lambda_{k,j}^{(l)}] = \omega_{k,j}^l$. \square

1222 4.2. Proof of the minimal null-control time property.

1223 In this section we end the proof of Theorem 1.1. The extension of Corollaries 2.2
 1224 and 2.3 as well as their proofs to the case $\eta \geq 2$ are straightforward and left to the
 1225 reader.

1226 *Controllability in large time: proof of point i. of Theorem 1.1.*

1227 Let $T > T_0(Y_0)$ and $y_0 \in Y_0$. For any $k \geq 1$, let $q_k \in L^2(0, T; \mathbb{C})$ be given by
 1228 Theorem 4.1 with

$$1229 \quad \omega_{k,j}^l := \left\langle y_0, (e_T \psi)[\lambda_{k,j}^{(l+1)}] \right\rangle_{-\diamond, \diamond}.$$

1230 As in Sec. 2.1.3, since $T > T_0(Y_0)$, the estimates (68) imply that

$$1231 \quad u := - \sum_{k \geq 1} q_k(T - \bullet) \in L^2(0, T; \mathbb{C}).$$

1232 Moreover, as q_k solves the block moment problem (67) it comes that u solves the
 1233 moment problem (21) and thus $y(T) = 0$.

1234 *Lack of null-controllability in small time: proof of point ii. of Theorem 1.1.*

1235 Let

$$1236 \quad (75) \quad \tilde{\phi}_{k,j}^l := \frac{\phi_{k,j}^l}{\mathcal{B}^* \phi_{k,j}^0}, \quad \forall k \geq 1, \forall j \in \llbracket 1, g_k \rrbracket, \forall l \in \llbracket 0, \alpha_{k,j} - 1 \rrbracket.$$

1237 Then, $\psi_{k,j}^l = P_{Y_0}^* \tilde{\phi}_{k,j}^l$.

1238 Let $(k_n)_{n \geq 1}$ be an extraction such that

$$1239 \quad \lim_{n \rightarrow \infty} \frac{\ln \left(\max_{\mu \leq \alpha_k} \left\| \psi \left[\lambda_{k_n,1}^{(\mu_1)}, \dots, \lambda_{k_n, g_{k_n}}^{(\mu_{g_{k_n}})} \right] \right\|_{\diamond^*} \right)}{\lambda_{k_n,1}} = T_0(Y_0).$$

1240 For any $n \geq 1$, let $\mu^n \leq \alpha_k$ be a multi-index in \mathbb{N}^{g_k} achieving the maximum appearing
 1241 in the previous formula. Since $\alpha_k \leq (\eta, \dots, \eta)$, we can find an integer l^* and a multi-
 1242 index $\mu^* \in \mathbb{N}^{l^*}$ such that $\mu^n = \mu^*$ for an infinite number of values of n . Therefore,
 1243 there exists another extraction $(\kappa_n)_{n \geq 1}$ such that

$$1244 \quad \lim_{n \rightarrow \infty} \frac{\ln \left\| \psi \left[\lambda_{\kappa_n, 1}^{(\mu_1^*)}, \dots, \lambda_{\kappa_n, l^*}^{(\mu_{l^*}^*)} \right] \right\|_{\diamond^*}}{\lambda_{\kappa_n, 1}} = T_0(Y_0),$$

1245 Moreover, we can assume that for any $\mu \in \mathbb{N}^{l^*}$ satisfying $\mu \leq \mu^*$ and $\mu \neq \mu^*$, then

$$1246 \quad \limsup_{n \rightarrow +\infty} \frac{\ln \left\| \psi \left[\lambda_{\kappa_n, 1}^{(\mu_1)}, \dots, \lambda_{\kappa_n, l^*}^{(\mu_{l^*})} \right] \right\|_{\diamond^*}}{\lambda_{\kappa_n, 1}} < T_0(Y_0),$$

1247 since if it is not the case, we can reduce the value of μ^* accordingly.

1248 For any $n \geq 1$, let

$$1249 \quad z_n := \tilde{\phi} \left[\lambda_{\kappa_n, 1}^{(\mu_1^*)}, \dots, \lambda_{\kappa_n, l^*}^{(\mu_{l^*}^*)} \right],$$

1250 where $\tilde{\phi}_{k,j}^l$ is defined by (75). From (20) we obtain that

$$1251 \quad e^{-tA^*} z_n = (e_t \tilde{\phi}) \left[\lambda_{\kappa_n, 1}^{(\mu_1^*)}, \dots, \lambda_{\kappa_n, l^*}^{(\mu_{l^*}^*)} \right].$$

1252 Recall that $\mathcal{B}^* \phi_{k,j}^l = 0$ for any $l \geq 1$. Then plugging z_n in the observability inequal-
 1253 ity (25) and following the lines of Sec. 2.2, proves the lack of null-controllability of (1)
 1254 from Y_0 at time $T < T_0(Y_0)$ and ends the proof of Theorem 1.1.

1255 5. Examples.

1256 In this section we study various examples. In Sec. 5.1, we design ‘abstract ex-
 1257 amples’ to highlight the phenomenon described in Section 1.3.4: the condensation
 1258 of eigenvectors can compensate the condensation of eigenvalues. More precisely we
 1259 design an example which is null-controllable in arbitrary time but with an arbitrary
 1260 condensation of the eigenvalues. We also give examples to illustrate the new settings
 1261 covered by our analysis when the eigenvalues are algebraically multiple in the absence
 1262 of a gap condition. The interest of these abstract examples is to highlight the different
 1263 phenomena as the computations are straightforward.

1264 Finally, we provide in Sec. 5.2, actual examples of one dimensional coupled
 1265 parabolic controlled systems that have motivated the present study. The precise anal-
 1266 ysis of null-controllability for those systems was not possible using existing results in
 1267 the literature.

1268 5.1. Abstract examples: a possible compensation of condensation of 1269 eigenvalues.

1270 The design of these abstract examples is inspired from the work [3]. Our goal is
 1271 to illustrate, in particular, the fact that, even if the control operator has no influence
 1272 on the minimal null-control time, the knowledge of the condensation index of the
 1273 eigenvalues of the operator \mathcal{A} is not sufficient to understand the null-controllability
 1274 properties of system (1).

1275 Let A be a positive definite self-adjoint operator with compact resolvent in a
 1276 Hilbert space H whose eigenvalues $(\mu_k)_{k \geq 1}$ are assumed to be sorted in increasing

1277 order. One can think of A , for instance, as the Laplace operator $-\partial_{xx}$ or any Sturm-
 1278 Liouville operator with homogeneous Dirichlet boundary conditions.

1279 If we denote by $(\varphi_k)_{k \geq 1}$ a corresponding Hilbert basis of eigenvectors, A may be
 1280 written

$$1281 \quad A = \sum_{k \geq 1} \mu_k (\bullet, \varphi_k)_H \varphi_k, \quad D(A) = \left\{ x \in H; \sum_{k \geq 1} \mu_k^2 (x, \varphi_k)_H^2 < +\infty \right\},$$

1282 where $(\bullet, \bullet)_H$ denotes the scalar product in H . We assume that $(\mu_k)_{k \geq 1}$ satisfies (7)
 1283 and (8) with $p = 1$, i.e., satisfies the so-called gap property. Let $\rho > 0$ be such that

$$1284 \quad (76) \quad 0 < \rho < \inf_{k \geq 1} (\mu_{k+1} - \mu_k)$$

1285 and $f : \sigma(A) \rightarrow \mathbb{R}$ a positive function defined on $\sigma(A)$ the spectrum of A satisfying

$$1286 \quad (77) \quad 0 < f(\mu_k) < \rho, \quad \forall k \geq 1.$$

1287 Let $f(A)$ be the operator defined on $D(A)$ by

$$1288 \quad f(A) := \sum_{k \geq 1} f(\mu_k) (\bullet, \varphi_k)_H \varphi_k.$$

1289 Let $x_0 \in H$ fixed satisfying

$$1290 \quad (78) \quad |(x_0, \varphi_k)_H| \geq e^{-\sqrt{\mu_k}}, \quad \forall k \geq 1.$$

1291

1292 **REMARK 5.1.** *This vector x_0 will be used to design the control operator \mathcal{B} . This*
 1293 *assumption will ensure that the terms $\mathcal{B}^* \phi_\lambda$ appearing in the definition (18) have*
 1294 *no influence. This will allow us to really emphasize the role of the condensation of*
 1295 *eigenvectors.*

1296 **5.1.1. Perturbation of a 2×2 Jordan block.**

1297 Let $X = H \times H$,

$$1298 \quad (79) \quad \mathcal{A} = \begin{pmatrix} A & I \\ 0 & A + f(A) \end{pmatrix}, \quad D(\mathcal{A}) = D(A) \times D(A),$$

1299 and

$$1300 \quad (80) \quad \mathcal{B} : u \in \mathbb{R} \mapsto u \begin{pmatrix} 0 \\ x_0 \end{pmatrix} \in X.$$

1301 It is easy to see that $(-\mathcal{A}, D(\mathcal{A}))$ generates a \mathcal{C}_0 -semigroup on X and that $\mathcal{B} : \mathbb{R} \rightarrow X$
 1302 is bounded. Thus we consider for this example that $X_\diamond^* = X = X_{-\diamond}$ and $Y_0 = X$.

1303 The spectrum of $(\mathcal{A}^*, D(\mathcal{A}))$ is given by

$$1304 \quad \Lambda = \{\mu_k, \mu_k + f(\mu_k); k \geq 1\}.$$

1305 **PROPOSITION 5.1.** *Let us consider the controlled system (1) with \mathcal{A} and \mathcal{B} given*
 1306 *by (79)-(80).*

1307 *i. For any function f satisfying (77), null-controllability from X holds in any*
 1308 *time i.e. $T_0(X) = 0$.*

1309 *ii. For any $\tau \in [0, +\infty]$, there exists a function f satisfying (77) such that $c(\Lambda) =$*
 1310 *τ .*

1311 This gives a first example in this setting where the minimal time is not related to
 1312 the condensation index. As it will appear from the proof, see (81), this is due to a
 1313 condensation of eigenvectors compensating the condensation of eigenvalues.

1314 *Proof.* The proof of point *ii.* directly follows from straightforward computations
 1315 using Proposition 7.12 with the explicit choices $f : s \mapsto \rho e^{-\sqrt{s}}$, $f : s \mapsto \rho e^{-cs}$ with
 1316 $c > 0$ or $f : s \mapsto \rho e^{-s^2}$.

1317 We now turn to the computation of the minimal null-control time. Using (76)
 1318 and (77), it comes that (7) and (8) are satisfied with $p = 2$. We define our grouping
 1319 by setting $\lambda_{k,1} := \mu_k$ and $\lambda_{k,2} := \mu_k + f(\mu_k)$. The associated normalized eigenvectors
 1320 are

$$1321 \quad \phi_{k,1} := \frac{1}{\sqrt{1 + f(\mu_k)^2}} \begin{pmatrix} -f(\mu_k) \\ 1 \end{pmatrix} \varphi_k, \quad \phi_{k,2} := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k,$$

1322 which do form a complete family in X . Moreover, for all $k \geq 1$,

$$1323 \quad \mathcal{B}^* \phi_{k,1} = \frac{1}{\sqrt{1 + f(\mu_k)^2}} (x_0, \varphi_k)_H, \quad \text{and} \quad \mathcal{B}^* \phi_{k,2} = (x_0, \varphi_k)_H,$$

1324 so that, with (14), we have

$$1325 \quad \psi_{k,1} = \frac{1}{(x_0, \varphi_k)_H} \begin{pmatrix} -f(\mu_k) \\ 1 \end{pmatrix} \varphi_k, \quad \psi_{k,2} = \frac{1}{(x_0, \varphi_k)_H} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k.$$

1326 From Definition (17), we have

$$1327 \quad T_0(X) = \limsup_{k \rightarrow \infty} \frac{\ln \left(\max \left\{ \|\psi_{k,1}\|, \frac{\|\psi_{k,2} - \psi_{k,1}\|}{f(\mu_k)} \right\} \right)}{\mu_k}.$$

1328 One has

$$1329 \quad (81) \quad \frac{\|\psi_{k,2} - \psi_{k,1}\|}{f(\mu_k)} = \frac{1}{|(x_0, \varphi_k)_H|} \left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k \right\| = \frac{1}{|(x_0, \varphi_k)_H|}.$$

1330 Using (78) and (77) we easily deduce that

$$1331 \quad T_0(X) = \limsup_{k \rightarrow \infty} \frac{\ln \frac{1}{|(x_0, \varphi_k)_H|}}{\mu_k} = 0. \quad \square$$

1332 **REMARK 5.2.** *Notice that,*

$$1333 \quad \|\phi_{k,2} - \phi_{k,1}\|^2 = \frac{2 \left(1 + f(\mu_k)^2 - \sqrt{1 + f(\mu_k)^2} \right)}{1 + f(\mu_k)^2} \xrightarrow[k \rightarrow \infty]{} 0,$$

1334 *thus the eigenvectors of \mathcal{A}^* do not form a Riesz basis of X . If this family were a*
 1335 *Riesz basis, then we would deduce from [5] that the minimal null-control time would*
 1336 *be equal to the condensation index $c(\Lambda)$.*

1337 REMARK 5.3. Let us consider in the same setting the evolution problem (1) given
 1338 by

$$1339 \quad \mathcal{A} = \begin{pmatrix} A & I \\ 0 & A \end{pmatrix}.$$

1340 In this case, the operator \mathcal{A}^* has spectrum $\sigma(\mathcal{A}^*) = \{\mu_k; k \geq 1\}$ with algebraically
 1341 double eigenvalues satisfying the gap property and an associated Hilbert basis of (gen-
 1342 eralized) eigenvectors given by

$$1343 \quad \phi_k^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k, \quad \text{and} \quad \phi_k^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k.$$

1344 Notice that from (81) one has

$$1345 \quad \frac{\psi_{k,2} - \psi_{k,1}}{f(\mu_k)} = \frac{1}{(x_0, \varphi_k)_H} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k = \frac{\phi_k^1}{\mathcal{B}^* \phi_k^0}.$$

1346 Thus, the analysis of (79)-(80), is unchanged if ones sets $f = 0$.

1347 5.1.2. Algebraically multiple eigenvalues.

1348 Let $X = H \times H \times H$. Let $\beta > 0$ and $g : \sigma(A) \rightarrow \mathbb{R}$ be such that

$$1349 \quad (82) \quad g(\mu_k) = \rho e^{-\beta \mu_k},$$

1350 with ρ satisfying (76). Let

$$1351 \quad (83) \quad \mathcal{A} = \begin{pmatrix} A & I & 0 \\ 0 & A & 0 \\ 0 & 0 & A + g(A) \end{pmatrix}, \quad D(\mathcal{A}) = D(A) \times D(A) \times D(A),$$

1352 and

$$1353 \quad (84) \quad \mathcal{B} : u \in \mathbb{R} \mapsto u \begin{pmatrix} 0 \\ x_0 \\ x_0 \end{pmatrix}.$$

1354 Again \mathcal{B} is a bounded control operator and we also set for this example $X_\diamond^* = X = X_{-\diamond}$
 1355 and $Y_0 = X$.

1356 The spectrum of $(\mathcal{A}^*, D(\mathcal{A}))$ is given by

$$1357 \quad \Lambda = \{\mu_k, \mu_k + g(\mu_k); k \geq 1\}.$$

1358 PROPOSITION 5.2. Let us consider the controlled system (1) with \mathcal{A} and \mathcal{B} given
 1359 by (83)-(84). Then,

$$1360 \quad (85) \quad T_0(X) = 2\beta = 2c(\Lambda).$$

1361 REMARK 5.4. In this case, the family of (generalized) eigenvectors do form a
 1362 Hilbert basis in X . However due to the presence of algebraically multiple eigenvalues
 1363 one cannot compute the value of the minimal null-control time using [5]. Its value
 1364 is still related to the condensation index of Λ but also depends on the multiplicity of
 1365 each eigenvalue in the system.

1366 *Proof.* From (82), we see that the eigenvalues are geometrically simple. Then, it
 1367 comes that (7) and (8) are satisfied with $p = 2$. We define our grouping by setting
 1368 $\lambda_{k,1} := \mu_k$ and $\lambda_{k,2} := \mu_k + g(\mu_k)$.

1369 In this setting, the eigenvalue $\lambda_{k,1}$ is algebraically double and $\lambda_{k,2}$ is algebraically
 1370 simple. The associated eigenvectors and generalized eigenvectors of \mathcal{A}^* are

$$1371 \quad \phi_{k,1}^0 := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k, \quad \phi_{k,1}^1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \varphi_k, \quad \phi_{k,2}^0 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \varphi_k,$$

1372 which obviously form a complete family in X . Moreover, for all $k \geq 1$,

$$1373 \quad \mathcal{B}^* \phi_{k,1}^0 = \mathcal{B}^* \phi_{k,2}^0 = (x_0, \varphi_k)_H,$$

1374 leading to

$$1375 \quad \psi_{k,1}^0 = \frac{1}{(x_0, \varphi_k)_H} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \varphi_k, \quad \psi_{k,2}^0 = \frac{1}{(x_0, \varphi_k)_H} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \varphi_k,$$

1376 and

$$1377 \quad \psi_{k,1}^1 = \frac{1}{(x_0, \varphi_k)_H} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \varphi_k.$$

1378 To compute the minimal time $T_0(X)$, let us estimate the different terms appearing
 1379 in (17). We have $\psi[\lambda_{k,1}] = \psi_{k,1}^0$ and $\psi[\lambda_{k,1}, \lambda_{k,1}] = \psi_{k,1}^1$ implying

$$1380 \quad \|\psi[\lambda_{k,1}]\| = \|\psi[\lambda_{k,1}, \lambda_{k,1}]\| = \frac{1}{|(x_0, \varphi_k)_H|}.$$

1381 Using Proposition 7.10, it only remains to compute and estimate the generalized
 1382 divided difference $\psi[\lambda_{k,1}, \lambda_{k,1}, \lambda_{k,2}]$. This comes from (121) and (122) as follows

$$1383 \quad \psi[\lambda_{k,1}, \lambda_{k,2}] = \frac{1}{(x_0, \varphi_k)_H} \frac{1}{\lambda_{k,2} - \lambda_{k,1}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \varphi_k = \frac{1}{(x_0, \varphi_k)_H} \frac{1}{g(\mu_k)} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \varphi_k,$$

1384 and

$$1385 \quad \begin{aligned} \psi[\lambda_{k,1}, \lambda_{k,1}, \lambda_{k,2}] &= \frac{\psi[\lambda_{k,1}, \lambda_{k,2}] - \psi[\lambda_{k,1}, \lambda_{k,1}]}{\lambda_{k,2} - \lambda_{k,1}} \\ 1386 &= \frac{1}{(x_0, \varphi_k)_H} \frac{1}{g(\mu_k)^2} \begin{pmatrix} -g(\mu_k) \\ -1 \\ 1 \end{pmatrix} \varphi_k. \end{aligned}$$

1387 Thus, using (82) we obtain

$$1389 \quad \begin{aligned} \|\psi[\lambda_{k,1}, \lambda_{k,1}, \lambda_{k,2}]\| &= \frac{1}{(x_0, \varphi_k)_H} \frac{1}{g(\mu_k)^2} \sqrt{g(\mu_k)^2 + 2} \\ 1390 &= \frac{1}{(x_0, \varphi_k)_H} \rho^{-2} e^{2\beta\mu_k} \sqrt{\rho^2 e^{-2\beta\mu_k} + 2}. \end{aligned}$$

1391 Then, for k large enough, we have

$$1393 \quad \max \left\{ 1, \rho^{-2} e^{2\beta\mu_k} \sqrt{\rho^2 e^{-2\beta\mu_k} + 2} \right\} = \rho^{-2} e^{2\beta\mu_k} \sqrt{\rho^2 e^{-2\beta\mu_k} + 2},$$

1394 and, using (78), this leads to (85). □

1395 **5.1.3. Competition between different perturbations.**

1396 Let $X = H \times H \times H$. Let $\alpha, \beta > 0$ with $\alpha \neq \beta$ and $f, g : \sigma(A) \rightarrow \mathbb{R}$ be such that

1397
$$f(\mu_k) = \rho e^{-\alpha\mu_k}, \quad g(\mu_k) = \rho e^{-\beta\mu_k},$$

1398 with ρ satisfying (76). Let

1399 (86)
$$\mathcal{A} = \begin{pmatrix} A & I & 0 \\ 0 & A + f(A) & 0 \\ 0 & 0 & A + g(A) \end{pmatrix}, \quad D(\mathcal{A}) = D(A) \times D(A) \times D(A),$$

1400 and

1401 (87)
$$\mathcal{B} : u \in \mathbb{R} \mapsto u \begin{pmatrix} 0 \\ x_0 \\ x_0 \end{pmatrix}.$$

1402 Again \mathcal{B} is a bounded control operator and we still set for this example $X_\diamond^* = X = X_{-\diamond}$
 1403 and $Y_0 = X$.

1404 PROPOSITION 5.3. *Let us consider the controlled system (1) with \mathcal{A} and \mathcal{B} given*
 1405 *by (86)-(87). Then,*

1406
$$T_0(X) = \beta + \min\{\alpha, \beta\}.$$

1407 *Proof.* The spectrum of $(\mathcal{A}^*, D(\mathcal{A}))$ is given by

1408
$$\Lambda = \{\mu_k, \mu_k + f(\mu_k), \mu_k + g(\mu_k); k \geq 1\}.$$

1409 By construction, these eigenvalues are geometrically simple. Then, it comes that (7)
 1410 and (8) are satisfied with $p = 3$. We define our grouping by setting

1411
$$\lambda_{k,1} := \mu_k, \quad \lambda_{k,2} := \mu_k + f(\mu_k), \quad \text{and} \quad \lambda_{k,3} := \mu_k + g(\mu_k).$$

1412 Notice that the eigenvalues are not necessarily increasingly sorted inside the k^{th} group
 1413 depending on the relative positions of α and β but, due to the invariance of divided
 1414 differences with respect to permutations, this does not change our analysis.

1415 These eigenvalues are algebraically and geometrically simple and the associated
 1416 eigenvectors are

1417
$$\phi_{k,1} := \frac{1}{\sqrt{1 + f(\mu_k)^2}} \begin{pmatrix} -f(\mu_k) \\ 1 \\ 0 \end{pmatrix} \varphi_k, \quad \phi_{k,2} := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \varphi_k, \quad \phi_{k,3} := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \varphi_k,$$

1418 which do form a complete family in X . Moreover, for all $k \geq 1$,

1419
$$\mathcal{B}^* \phi_{k,1} = \frac{1}{\sqrt{1 + f(\mu_k)^2}} (x_0, \varphi_k)_H, \quad \text{and} \quad \mathcal{B}^* \phi_{k,2} = \mathcal{B}^* \phi_{k,3} = (x_0, \varphi_k)_H,$$

1420 leading to

1421
$$\psi_{k,1} = \frac{1}{(x_0, \varphi_k)_H} \begin{pmatrix} -f(\mu_k) \\ 1 \\ 0 \end{pmatrix} \varphi_k, \quad \psi_{k,2} = \frac{1}{(x_0, \varphi_k)_H} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \varphi_k,$$

1422 and

$$1423 \quad \psi_{k,3} = \frac{1}{(x_0, \varphi_k)_H} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \varphi_k.$$

1424 To compute the minimal time $T_0(X)$, let us determine the different terms appearing
1425 in (17). We have $\psi[\lambda_{k,1}] = \psi_{k,1}$,

$$1426 \quad \psi[\lambda_{k,1}, \lambda_{k,2}] = \frac{1}{(x_0, \varphi_k)_H} \frac{1}{\lambda_{k,2} - \lambda_{k,1}} \begin{pmatrix} f(\mu_k) \\ 0 \\ 0 \end{pmatrix} \varphi_k = \frac{1}{(x_0, \varphi_k)_H} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \varphi_k,$$

1427

$$1428 \quad \psi[\lambda_{k,2}, \lambda_{k,3}] = \frac{1}{(x_0, \varphi_k)_H} \frac{1}{\lambda_{k,3} - \lambda_{k,2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \varphi_k = \frac{1}{(x_0, \varphi_k)_H} \frac{1}{g(\mu_k) - f(\mu_k)} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \varphi_k,$$

1429 and finally

$$1430 \quad \psi[\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}] = \frac{\psi[\lambda_{k,1}, \lambda_{k,2}] - \psi[\lambda_{k,2}, \lambda_{k,3}]}{\lambda_{k,1} - \lambda_{k,3}}$$

$$1431 \quad = \frac{1}{(x_0, \varphi_k)_H} \frac{1}{g(\mu_k)(g(\mu_k) - f(\mu_k))} \begin{pmatrix} g(\mu_k) - f(\mu_k) \\ -1 \\ 1 \end{pmatrix} \varphi_k.$$

1432

Since $\lim_{k \rightarrow +\infty} g(\mu_k) = 0$, we immediately see that, for k large enough, we have

$$\max \{ \|\psi[\lambda_{k,1}]\|, \|\psi[\lambda_{k,1}, \lambda_{k,2}]\|, \|\psi[\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}]\| \} = \|\psi[\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}]\|,$$

1433 so that using (78) and (17) we get

$$1434 \quad (88) \quad T_0(X) = \limsup_{k \rightarrow \infty} \frac{\ln \|\psi[\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}]\|}{\mu_k} = \limsup_{k \rightarrow \infty} \frac{-\ln |g(\mu_k)(g(\mu_k) - f(\mu_k))|}{\mu_k}.$$

1435 The analysis is now split into two cases:

1436 1. Assume first that

$$1437 \quad (89) \quad \beta < \alpha.$$

1438 We deduce from (88) that

$$1439 \quad T_0(X) = \limsup_{k \rightarrow \infty} \frac{-\ln e^{-2\beta\mu_k} (1 - e^{-(\alpha-\beta)\mu_k})}{\mu_k} = 2\beta.$$

1440 2. Assume now that

$$1441 \quad (90) \quad \beta > \alpha.$$

1442 We deduce from (88) that

$$1443 \quad T_0(X) = \limsup_{k \rightarrow \infty} \frac{-\ln e^{-(\alpha+\beta)\mu_k} (1 - e^{-(\beta-\alpha)\mu_k})}{\mu_k} = \alpha + \beta. \quad \square$$

1444 **REMARK 5.5.** *As previously the family of eigenvectors does not form a Riesz basis*
 1445 *since for instance we have $\|\phi_{k,1} - \phi_{k,2}\| \xrightarrow[k \rightarrow \infty]{} 0$. Thus, one cannot apply the results*
 1446 *of [5] that would give that the minimal null-control time is (see Appendix 7.5)*

$$1447 \quad c(\Lambda) = \limsup_{k \rightarrow \infty} \frac{-\ln(f(\mu_k)g(\mu_k))}{\mu_k} = \alpha + \beta.$$

1448 *Yet, in the case (90) we still have $T_0(X) = c(\Lambda)$. However, in the case (89) we have*
 1449 *$0 < T_0(X) = 2\beta < c(\Lambda)$. Notice that, in this case, setting $f = 0$ one recovers the*
 1450 *system studied in subsection 5.1.2 for which the minimal time is exactly 2β .*

1451 **5.2. Condensation in partial differential equations.**

1452 We provide in this section actual PDE examples covered by our analysis. First
 1453 of all, let us emphasize that our setting naturally covers a wide range of coupled one
 1454 dimensional parabolic equations. Indeed if there exists $p \in \mathbb{N}^*$ such that the spectrum
 1455 of \mathcal{A} is given by the union of p family

$$1456 \quad \Lambda^j = \left\{ \lambda_k^j ; k \geq 1 \right\}$$

1457 such that each family satisfies (8) and (7), then the structural assumptions on Λ are
 1458 automatically satisfied (see Lemma 2.1).

1459 **5.2.1. A system with two different potentials.**

1460 Let us consider the following boundary control system

$$1461 \quad (91) \quad \begin{cases} \partial_t y + \begin{pmatrix} -\partial_{xx} + c_1(x) & 1 \\ 0 & -\partial_{xx} + c_2(x) \end{pmatrix} y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & (t, x) \in (0, T) \times (0, 1), \\ y(t, 0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad y(t, 1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & t \in (0, T), \\ y(0, x) = y_0(x), \end{cases}$$

1462 where $c_1, c_2 \in L^2(0, 1; \mathbb{R})$. Without loss of generality we assume that c_1 and c_2 are non-
 1463 negative. The operator \mathcal{A} appearing in this system is defined in $X = (L^2(0, 1; \mathbb{R}))^2$
 1464 with domain $X_1^* = D(\mathcal{A}) = (H^2(0, 1; \mathbb{R}) \cap H_0^1(0, 1; \mathbb{R}))^2$. The control operator \mathcal{B}
 1465 is defined in a weak sense as in [45]. The expression of its adjoint is easier to rule out
 1466 and is given by

$$1467 \quad \mathcal{B}^* : \begin{pmatrix} f \\ g \end{pmatrix} \in X_1^* \mapsto \begin{pmatrix} 0 \\ B^* g \end{pmatrix} = \begin{pmatrix} 0 \\ -g'(0) \end{pmatrix}.$$

1468 Here we denoted by B^* the (scalar) normal derivative operator at $x = 0$ defined
 1469 on $H^2(0, 1; \mathbb{R})$. Standard parabolic regularity properties show that, if we define
 1470 $X_\diamond^* = (H_0^1(0, 1; \mathbb{R}))^2$, then the operator \mathcal{B} is admissible with respect to $X_{-\diamond} =$
 1471 $(H^{-1}(0, 1; \mathbb{R}))^2$, in the sense of (3). Therefore, for any $u \in L^2(0, T; \mathbb{R})$, (91) is well-
 1472 posed in $\mathcal{C}^0([0, T], (H^{-1}(0, 1; \mathbb{R}))^2)$.

1473 For any non-negative potential $c \in L^2(0, 1; \mathbb{R})$, we denote by A^c the definite posi-
 1474 tive self-adjoint operator in $L^2(0, 1; \mathbb{R})$ with domain $H^2(0, 1; \mathbb{R}) \cap H_0^1(0, 1; \mathbb{R})$ defined
 1475 by $A^c y = -\partial_{xx} y + c(x)y$. Its spectrum is denoted by $\Lambda^c \subset (\pi^2, +\infty)$ and satisfies

$$1476 \quad (92) \quad \inf_{\substack{\lambda, \mu \in \Lambda^c \\ \lambda \neq \mu}} |\sqrt{\lambda} - \sqrt{\mu}| > 0.$$

1477 We choose associated eigenfunctions denoted by φ_λ^c that are normalized in $L^2(0, 1; \mathbb{R})$
 1478 and that satisfy (see for instance [33, Theorem 4.11])

$$1479 \quad (93) \quad \varphi_\lambda^c(x) = \sqrt{2} \sin(\sqrt{\lambda}x) + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right), \quad \text{uniformly in } x,$$

1480

$$1481 \quad (94) \quad \partial_x \varphi_\lambda^c(x) = \sqrt{2}\sqrt{\lambda} \cos(\sqrt{\lambda}x) + \mathcal{O}(1), \quad \text{uniformly in } x.$$

1482 In particular, there exist $\bar{C}, \tilde{C} > 0$ such that,

$$1483 \quad (95) \quad \bar{C}\sqrt{\lambda} \leq |B^* \varphi_\lambda^{c_i}| = |\partial_x \varphi_\lambda^{c_i}(0)| \leq \tilde{C}\sqrt{\lambda}, \quad \forall \lambda \in \Lambda^{c_i}, \forall i = 1, 2.$$

The analysis will be based on the careful inspection of spectral properties of the adjoint operator

$$\mathcal{A}^* = \begin{pmatrix} A^{c_1} & 0 \\ 1 & A^{c_2} \end{pmatrix}.$$

1484 It is easily seen that the spectrum of \mathcal{A}^* is given by $\Lambda = \Lambda^{c_1} \cup \Lambda^{c_2}$. We will often use
 1485 the following straightforward property

$$1486 \quad (96) \quad (A^{c_i} \varphi_\lambda^{c_j}, \varphi_\lambda^{c_j}) \leq C\lambda, \quad \forall \lambda \in \Lambda^{c_j}, \forall i, j \in \{1, 2\},$$

1487 where C depends only on $\|c_1\|$ and $\|c_2\|$.

1488 Our controllability result concerning system (91) is the following.

1489 **THEOREM 5.1.** *For any non-negative potentials c_1, c_2 , there exists a closed sub-*
 1490 *space Y_0 of $(H^{-1}(0, 1; \mathbb{R}))^2$ of finite codimension such that:*

- 1491 • *For any $y_0 \notin Y_0$, system (91) is not approximately controllable.*
- 1492 • *For any $y_0 \in Y_0$, system (91) is null-controllable at any time $T > 0$.*

1493 **REMARK 5.6.** *The set Y_0 can be equal to the whole space $(H^{-1}(0, 1; \mathbb{R}))^2$, for*
 1494 *instance if c_1 and c_2 are close enough.*

1495 Before proving this theorem, we would like to emphasize the fact that for a system
 1496 like (91), the condensation index of its spectrum can be arbitrary. Therefore, Theorem
 1497 5.1 gives another example of a system which is null-controllable at any time $T > 0$
 1498 (for well-prepared initial data) despite the fact that the condensation index of the
 1499 spectrum is non zero.

1500 **PROPOSITION 5.4.** *For any $\tau \in [0, +\infty]$ there exist $c_1, c_2 \in L^2(0, 1; \mathbb{R})$ such that*
 1501 *the condensation index of the spectrum Λ of the operator \mathcal{A}^* satisfies $c(\Lambda) = \tau$.*

1502 *Proof.* This follows from inverse spectral theory. Indeed, it is proven in [40,
 1503 Chapter 3] for instance, that for any $\alpha \in \mathbb{R}$ and any sequence $(\nu_k)_{k \geq 1} \in l^2$, one can find
 1504 a potential $c \in L^2(0, 1; \mathbb{R})$ such that the spectrum of A^c is given by $(k^2\pi^2 + \alpha + \nu_k)_k$.
 1505 It is then clear that we can choose c_1 and c_2 such that the spectrums of A^{c_1} and A^{c_2}
 1506 are asymptotically as close as we want and then generate an arbitrary condensation
 1507 index for the spectrum of \mathcal{A}^* . Note that such potentials are not necessarily non-
 1508 negative, but this is actually not really needed in our analysis (we simply need that
 1509 the spectrum of A^c is made of positive eigenvalues).

1510 In the context of controlled parabolic problems, this was already noticed and used
 1511 in [36]. □

1512 We can now move to the proof of the main result of this section.

1513 *Proof (of Theorem 5.1).* The first part of the proof consists in a precise descrip-
 1514 tion of the spectral properties of \mathcal{A}^* .

1515 • For any $\lambda \in \Lambda^{c_2}$, we have a first eigenfunction of \mathcal{A}^* given by

$$1516 \quad (97) \quad \phi_\lambda^0 := \begin{pmatrix} 0 \\ \varphi_\lambda^{c_2} \end{pmatrix}.$$

Moreover, by (95), we have

$$\mathcal{B}^* \phi_\lambda^0 = B^* \varphi_\lambda^{c_2} \neq 0,$$

1517 so that all those eigenfunctions are observable.

- 1518 – If $\lambda \notin \Lambda^{c_1}$, this eigenfunction is algebraically and geometrically simple.
- 1519 – However, if $\lambda \in \Lambda^{c_2} \cap \Lambda^{c_1}$, this eigenvalue is (algebraically or geometri-
 1520 cally) double. Let us define

$$1521 \quad (98) \quad \beta_\lambda := (\varphi_\lambda^{c_1}, \varphi_\lambda^{c_2}).$$

1522 By (93), we see that there exists λ_0 such that

$$1523 \quad (99) \quad \frac{1}{2} \leq \beta_\lambda \leq 1, \quad \forall \lambda \in \Lambda^{c_2} \cap \Lambda^{c_1}, \text{ s.t. } \lambda > \lambda_0.$$

* If $\beta_\lambda = 0$ then there exists a solution of

$$(A^{c_2} - \lambda)\vartheta_\lambda = -\varphi_\lambda^{c_1},$$

that we can choose to satisfy $B^*\vartheta_\lambda = 0$ in such a way that

$$\tilde{\phi}_\lambda^0 := \begin{pmatrix} \varphi_\lambda^{c_1} \\ \vartheta_\lambda \end{pmatrix},$$

1524 is another independent eigenfunction of \mathcal{A}^* associated with λ that
 1525 satisfy $\mathcal{B}^* \tilde{\phi}_\lambda^0 = 0$. Note that, by (99), we know that β_λ can vanish
 1526 only for a finite number of values of λ .

* Assume now that $\beta_\lambda \neq 0$. In that case, λ is geometrically simple
 but there exists a generalized eigenfunction ϕ_λ^1 associated with ϕ_λ^0
 of the following form

$$\phi_\lambda^1 := \frac{1}{\beta_\lambda} \begin{pmatrix} \varphi_\lambda^{c_1} \\ \chi_\lambda \end{pmatrix},$$

where χ_λ is the unique solution of

$$(A^{c_2} - \lambda)\chi_\lambda = \beta_\lambda \varphi_\lambda^{c_2} - \varphi_\lambda^{c_1},$$

1527 that satisfy $B^*\chi_\lambda = 0$.

1528 We can express χ_λ in the basis $\varphi_\bullet^{c_2}$ as follows

$$1529 \quad \chi_\lambda = a_\lambda \varphi_\lambda^{c_2} - \sum_{\substack{\mu \in \Lambda^{c_2} \\ \mu \neq \lambda}} \frac{(\varphi_\lambda^{c_1}, \varphi_\mu^{c_2})}{\lambda - \mu} \varphi_\mu^{c_2},$$

1530 with

$$1531 \quad a_\lambda = \frac{1}{B^* \varphi_\lambda^{c_2}} \sum_{\substack{\mu \in \Lambda^{c_2} \\ \mu \neq \lambda}} \frac{(\varphi_\lambda^{c_1}, \varphi_\mu^{c_2})}{\lambda - \mu} B^* \varphi_\mu^{c_2}.$$

1532 • Consider now $\lambda \in \Lambda^{c_1} \setminus \Lambda^{c_2}$. We obtain another family of eigenfunctions given
 1533 by

$$1534 \quad (100) \quad \phi_\lambda^0 := \begin{pmatrix} \varphi_\lambda^{c_1} \\ \xi_\lambda \end{pmatrix},$$

where ξ_λ satisfies

$$(A^{c_2} - \lambda)\xi_\lambda = -\varphi_\lambda^{c_1}.$$

1535 This last equation has a unique solution since $\lambda \notin \Lambda^{c_2}$ and it can be expressed
 1536 as follows

$$1537 \quad (101) \quad \xi_\lambda = \sum_{\mu \in \Lambda^{c_2}} \frac{(\varphi_\lambda^{c_1}, \varphi_\mu^{c_2})}{\lambda - \mu} \varphi_\mu^{c_2}.$$

1538 We now state the following lemma, whose proof is postponed at the end of
 1539 this subsection.

1540 **LEMMA 5.1.** *There exists $C_1, C_2 > 0$ depending only on c_1, c_2 such that*

$$1541 \quad (102) \quad |B^* \xi_\lambda|^2 \geq C_1 \lambda - C_2, \quad \forall \lambda \in \Lambda^{c_1} \setminus \Lambda^{c_2}.$$

1542 This lemma shows in particular that $B^* \xi_\lambda$ can only vanish for a finite number
 1543 of values of λ .

1544 It is straightforward to prove that the family of (generalized) eigenfunctions we
 1545 just computed is complete in X . We can now define Y_0 to be the set of initial data
 1546 $y_0 \in X_{-\diamond}$ such that

$$1547 \quad \begin{cases} \langle y_0, \tilde{\phi}_\lambda^0 \rangle_{-\diamond, \diamond} = 0, & \forall \lambda \in \Lambda^{c_1} \cap \Lambda^{c_2}, \text{ s.t. } \beta_\lambda = 0, \text{ see (98),} \\ \langle y_0, \phi_\lambda^0 \rangle_{-\diamond, \diamond} = 0, & \forall \lambda \in \Lambda^{c_1} \setminus \Lambda^{c_2}, \text{ s.t. } B^* \xi_\lambda = 0, \text{ see (101).} \end{cases}$$

1548 By construction, this set is closed and of finite codimension, moreover it is clear that
 1549 initial data not belonging to this set are not approximately controllable. Note that this
 1550 definition actually excludes the influence of the possible presence of a geometrically
 1551 double eigenvalue in the system.

We will now endow the space X_\diamond^* with the following norm

$$\left\| \begin{pmatrix} f \\ g \end{pmatrix} \right\|_{\diamond^*}^2 := \langle A^{c_1} f, f \rangle_{H^{-1}, H_0^1} + \langle A^{c_2} g, g \rangle_{H^{-1}, H_0^1},$$

1552 which is equivalent to the usual H^1 -norm and more comfortable for the following
 1553 computations. Note that, if $f, g \in H^2(0, 1; \mathbb{R})$, this quantity is simply equal to
 1554 $(A^{c_1} f, f) + (A^{c_2} g, g)$.

From (92), we can find a $\rho > 0$ such that

$$|\sqrt{\lambda} - \sqrt{\mu}| > \rho, \quad \forall \lambda \neq \mu \in \Lambda^{c_i}, \forall i = 1, 2.$$

1555 This implies, in particular, that

$$1556 \quad (103) \quad |\lambda - \mu| > \rho(\sqrt{\lambda} + \sqrt{\mu}) > \rho, \quad \forall \lambda \neq \mu \in \Lambda^{c_i}, \forall i = 1, 2.$$

1557 Without loss of generality, we can assume that $\rho < \frac{\bar{C}}{2C_1}$ where \bar{C} and C_1 are respec-
 1558 tively defined in (95) and (102).

1559 It follows that Λ satisfies the summability condition (7), as well as the weak gap
 1560 condition (8) with $p = 2$. We can thus consider a grouping $(G_k)_k \in \mathcal{G}(\Lambda, 2, r, \rho)$ for
 1561 a suitable $r > 0$. We will now use the formula (16) we obtained for $T_0(Y_0)$ to prove
 1562 that the system is null-controllable from Y_0 at any time $T > 0$. For that we will
 1563 consider one of the groups G (we drop the index k which is not important here) and
 1564 give estimates of the corresponding divided differences.

- 1565 • Case 1 : $G = \{\lambda\}$ is of cardinal 1.
 – If $\lambda \in \Lambda^{c_2}$ we need to estimate the quantity

$$\|\psi[\lambda]\|_{\diamond^*}^2 := \left\| \frac{P_{Y_0}^* \phi_\lambda^0}{\mathcal{B}^* \phi_\lambda^0} \right\|_{\diamond^*}^2 \leq \left\| \frac{\phi_\lambda^0}{\mathcal{B}^* \phi_\lambda^0} \right\|_{\diamond^*}^2$$

except if $\lambda \in \Lambda^{c_1} \cap \Lambda^{c_2}$ and $\beta_\lambda = 0$. The computations above, and (95),
 show that

$$\|\psi[\lambda]\|_{\diamond^*}^2 \leq \frac{1}{|B^* \varphi_\lambda^{c_2}|^2} (A^{c_2} \varphi_\lambda^{c_2}, \varphi_\lambda^{c_2}) = \frac{\lambda}{|B^* \varphi_\lambda^{c_2}|^2} \leq \frac{1}{\bar{C}^2}.$$

- 1566 – If $\lambda \in \Lambda^{c_1} \setminus \Lambda^{c_2}$, recall that ϕ_λ^0 is given by (100) and that we need to
 1567 estimate the same quantity $\|\psi[\lambda]\|_{\diamond^*}$, in the case where $B^* \xi_\lambda \neq 0$. Since
 1568 λ is the only element in the group G , we know that $|\lambda - \mu| \geq r$ for any
 1569 other eigenvalue μ . With this remark, we can deduce that

$$\begin{aligned} 1570 \quad \|\psi[\lambda]\|_{\diamond^*}^2 &\leq \frac{1}{|B^* \xi_\lambda|^2} ((A^{c_1} \varphi_\lambda^{c_1}, \varphi_\lambda^{c_1}) + (A^{c_2} \xi_\lambda, \xi_\lambda)) \\ 1571 &= \frac{1}{|B^* \xi_\lambda|^2} \left(\lambda + \sum_{\mu \in \Lambda^{c_2}} \frac{\mu (\varphi_\lambda^{c_1}, \varphi_\mu^{c_2})^2}{(\lambda - \mu)^2} \right) \\ 1572 &\leq \frac{1}{|B^* \xi_\lambda|^2} \left(\lambda + \frac{1}{r^2} \sum_{\mu \in \Lambda^{c_2}} (\varphi_\lambda^{c_1}, \varphi_\mu^{c_2}) (\varphi_\lambda^{c_1}, A^{c_2} \varphi_\mu^{c_2}) \right). \\ 1573 \end{aligned}$$

1574 Using Parseval's identity, (96) and then (102), we finally obtain

$$\begin{aligned} 1575 \quad \|\psi[\lambda]\|_{\diamond^*}^2 &\leq \frac{1}{|B^* \xi_\lambda|^2} \left(\lambda + \frac{1}{r^2} (\varphi_\lambda^{c_1}, A^{c_2} \varphi_\lambda^{c_1}) \right) \\ 1576 &\leq \frac{\lambda}{|B^* \xi_\lambda|^2} \left(1 + \frac{C}{r^2} \right) \\ 1577 &\leq C. \end{aligned}$$

- 1579 – Finally, if $\lambda \in \Lambda^{c_1} \cap \Lambda^{c_2}$, then we need to estimate the contribution of
 1580 the generalized eigenvector $\|\psi[\lambda, \lambda]\|_{\diamond^*}^2 := \|P_{Y_0}^* \phi_\lambda^1 / (\mathcal{B}^* \phi_\lambda^0)\|_{\diamond^*}^2$. A com-

1581

putation similar to the one above, for $\lambda > \lambda_0$, leads to

1582

$$\|\psi[\lambda, \lambda]\|_{\diamond^*}^2 \leq \frac{1}{|B^* \phi_\lambda^0|^2 \beta_\lambda^2} ((A^{c_1} \varphi_\lambda^{c_1}, \varphi_\lambda^{c_1}) + (A^{c_2} \chi_\lambda, \chi_\lambda))$$

1583

$$= \frac{1}{|B^* \varphi_\lambda^{c_2}|^2 \beta_\lambda^2} \left(\lambda(1 + a_\lambda^2) + \sum_{\substack{\mu \in \Lambda^{c_2} \\ \mu \neq \lambda}} \frac{(\varphi_\lambda^{c_1}, \varphi_\mu^{c_2})^2}{(\lambda - \mu)^2} \mu \right)$$

1584

$$\leq \frac{1}{|B^* \varphi_\lambda^{c_2}|^2 \beta_\lambda^2} \left(\lambda(1 + a_\lambda^2) + \frac{1}{r^2} (A^{c_2} \varphi_\lambda^{c_1}, \varphi_\lambda^{c_1}) \right)$$

1585

$$\leq \frac{\lambda}{|B^* \varphi_\lambda^{c_2}|^2 \beta_\lambda^2} \left(1 + a_\lambda^2 + \frac{C}{r^2} \right)$$

1586

$$\leq C(1 + a_\lambda^2).$$

1588

Here, we have used (99) to bound from below the term β_λ . It remains to bound a_λ . We proceed as follows, by using (94), (103), and (96)

1589

1590

$$|a_\lambda| \leq \frac{1}{|B^* \varphi_\lambda^{c_2}|} \sum_{\substack{\mu \in \Lambda^{c_2} \\ \mu \neq \lambda}} \left| \frac{(\varphi_\lambda^{c_1}, \varphi_\mu^{c_2})}{\lambda - \mu} \right| |B^* \varphi_\mu^{c_2}|$$

1591

$$\leq \frac{C}{\sqrt{\lambda}} \sum_{\substack{\mu \in \Lambda^{c_2} \\ \mu \neq \lambda}} |(\varphi_\lambda^{c_1}, \varphi_\mu^{c_2})| \frac{\sqrt{\mu}}{\sqrt{\lambda} + \sqrt{\mu}}$$

1592

$$\leq \frac{1}{\sqrt{\lambda}} \left(\sum_{\substack{\mu \in \Lambda^{c_2} \\ \mu \neq \lambda}} |(\varphi_\lambda^{c_1}, \varphi_\mu^{c_2})|^2 \mu \right)^{\frac{1}{2}} \left(\sum_{\mu \in \Lambda^{c_2}} \frac{1}{\mu} \right)^{\frac{1}{2}}$$

1593

$$\leq \frac{C}{\sqrt{\lambda}} (A^{c_2} \varphi_\lambda^{c_1}, \varphi_\lambda^{c_1})^{\frac{1}{2}}$$

1594

$$\leq C.$$

1596

This concludes the proof of the uniform bound of $\|\psi[\lambda, \lambda]\|_{\diamond^*}$.

- Case 2 : $G = \{\lambda_1, \lambda_2\}$ is of cardinal 2. Since the diameter of G is smaller than ρ , we can choose the numbering such that $\lambda_1 \in \Lambda^{c_1} \setminus \Lambda^{c_2}$ and $\lambda_2 \in \Lambda^{c_2} \setminus \Lambda^{c_1}$. In particular, we have $B^* \phi_{\lambda_1}^0 \neq 0$, $B^* \phi_{\lambda_2}^0 \neq 0$ and there is no generalized eigenvector associated to this group G . Therefore, the only new quantity we need to estimate is the contribution of the following divided difference

$$\|\psi[\lambda_1, \lambda_2]\|_{\diamond^*}^2 \leq \frac{1}{|\lambda_1 - \lambda_2|^2} \left\| \frac{\phi_{\lambda_1}^0}{B^* \phi_{\lambda_1}^0} - \frac{\phi_{\lambda_2}^0}{B^* \phi_{\lambda_2}^0} \right\|_{\diamond^*}^2.$$

1597

Using formulas (97) and (100), we find

1598

$$\|\psi[\lambda_1, \lambda_2]\|_{\diamond^*}^2 \leq \frac{1}{|\lambda_1 - \lambda_2|^2} \left\| \frac{1}{B^* \xi_{\lambda_1}} \begin{pmatrix} \varphi_{\lambda_1}^{c_1} \\ \xi_{\lambda_1} \end{pmatrix} - \frac{1}{B^* \varphi_{\lambda_2}^{c_2}} \begin{pmatrix} 0 \\ \varphi_{\lambda_2}^{c_2} \end{pmatrix} \right\|_{\diamond^*}^2.$$

1599

1600

Since λ_1 and λ_2 can be arbitrarily close it is not clear that this estimate does not blow up. In particular, if we use the triangle inequality, we will not be able to take benefit of compensations that occur in the divided difference.

1601

1602

We will thus make appear from (101) the principal part of ξ_{λ_1} as follows

$$\xi_{\lambda_1} = \frac{\beta_{\lambda_1, \lambda_2}}{\lambda_1 - \lambda_2} (\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1}),$$

with $\beta_{\lambda_1, \lambda_2} := (\varphi_{\lambda_1}^{c_1}, \varphi_{\lambda_2}^{c_2})$ and

$$\zeta_{\lambda_1} := \frac{\lambda_1 - \lambda_2}{\beta_{\lambda_1, \lambda_2}} \sum_{\substack{\mu \in \Lambda^{c_2} \\ \mu \neq \lambda_2}} \frac{(\varphi_{\lambda_1}^{c_1}, \varphi_{\mu}^{c_2})}{\lambda_1 - \mu} \varphi_{\mu}^{c_2}.$$

1603 Thus,

$$\begin{aligned} 1604 \quad \frac{\xi_{\lambda_1}}{B^* \xi_{\lambda_1}} - \frac{\varphi_{\lambda_2}^{c_2}}{B^* \varphi_{\lambda_2}^{c_2}} &= \frac{\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1}}{B^* (\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1})} - \frac{\varphi_{\lambda_2}^{c_2}}{B^* \varphi_{\lambda_2}^{c_2}} \\ 1605 \quad (104) \quad &= \left(\frac{1}{B^* (\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1})} - \frac{1}{B^* \varphi_{\lambda_2}^{c_2}} \right) \varphi_{\lambda_2}^{c_2} + \frac{\zeta_{\lambda_1}}{B^* (\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1})} \\ 1606 \end{aligned}$$

1607 Since we are only interested in the asymptotic behavior when λ_1 and λ_2 are
1608 large, we see that we can assume from (93) that the following properties hold

$$1609 \quad (105) \quad |\beta_{\lambda_1, \lambda_2}| \geq 1/2, \quad \sqrt{\lambda_2} \geq \sqrt{\lambda_1}/2 \geq 1.$$

1610 Using that $|\lambda_1 - \mu| \geq r$ for all $\mu \in \Lambda^{c_2}$, with $\mu \neq \lambda_2$, we can find with (105)
1611 and (96) the following bound

$$1612 \quad (106) \quad (A^{c_2} \zeta_{\lambda_1}, \zeta_{\lambda_1}) \leq |\lambda_1 - \lambda_2|^2 \frac{(A^{c_2} \varphi_{\lambda_1}^{c_1}, \varphi_{\lambda_1}^{c_1})}{r^2 |\beta_{\lambda_1, \lambda_2}|^2} \leq C^* |\lambda_1 - \lambda_2|^2 \lambda_1.$$

1613 Moreover, we have

$$\begin{aligned} |B^* \zeta_{\lambda_1}| &\leq \frac{|\lambda_1 - \lambda_2|}{|\beta_{\lambda_1, \lambda_2}|} \sum_{\substack{\mu \in \Lambda^{c_2} \\ \mu \neq \lambda_2}} \frac{|(\varphi_{\lambda_1}^{c_1}, \varphi_{\mu}^{c_2})|}{|\lambda_1 - \mu|} |B^* \varphi_{\mu}^{c_2}|, \\ &\leq C \tilde{C} |\lambda_1 - \lambda_2| \sum_{\substack{\mu \in \Lambda^{c_2} \\ \mu \neq \lambda_2}} \frac{|(\varphi_{\lambda_1}^{c_1}, \varphi_{\mu}^{c_2})|}{|\lambda_1 - \mu|} \sqrt{\mu} \\ 1614 &\leq C \tilde{C} |\lambda_1 - \lambda_2| \left(\sum_{\substack{\mu \in \Lambda^{c_2} \\ \mu \neq \lambda_2}} (\varphi_{\lambda_1}^{c_1}, \varphi_{\mu}^{c_2})^2 \mu \right)^{1/2} \left(\sum_{\substack{\mu \in \Lambda^{c_2} \\ \mu \neq \lambda_2}} \frac{1}{|\lambda_1 - \mu|^2} \right)^{1/2}. \end{aligned}$$

We use Parseval's identity and (96) to bound the second factor by $C\sqrt{\lambda_1}$.
Moreover, by using (103), we have for any $\mu \in \Lambda^{c_2}$, $\mu \neq \lambda_2$,

$$|\lambda_1 - \mu| \geq |\lambda_2 - \mu| - |\lambda_1 - \lambda_2| \geq \rho(\sqrt{\mu} + \sqrt{\lambda_2}) - \rho \geq \rho\sqrt{\mu},$$

1615 so that the value of the series in the last factor is uniformly bounded. Hence,
1616 we have proved

$$1617 \quad (107) \quad |B^* \zeta_{\lambda_1}| \leq C_1 |\lambda_1 - \lambda_2| \sqrt{\lambda_1}.$$

1618 From this last estimate, (102) and (95), we deduce that

$$\begin{aligned}
1619 \quad |B^*(\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1})| &\geq \left(|B^*\varphi_{\lambda_2}^{c_2}| - C_1|\lambda_1 - \lambda_2|\sqrt{\lambda_1} \right) \\
1620 \quad &\geq \bar{C}\sqrt{\lambda_1} - C_1|\lambda_1 - \lambda_2|\sqrt{\lambda_1}.
\end{aligned}$$

1622 Recall that $\rho < \bar{C}/(2C_1)$. Since λ_1 and λ_2 belong to the same group G , we
1623 have $|\lambda_1 - \lambda_2| \leq \rho$ and thus, we obtain the estimate

$$1624 \quad (108) \quad |B^*(\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1})| \geq \frac{\bar{C}}{2}\sqrt{\lambda_1}.$$

1625 Coming back to the definition of $\psi[\lambda_1, \lambda_2]$ and using (104) and the triangle
1626 inequality, we write

$$\begin{aligned}
1627 \quad \|\psi[\lambda_1, \lambda_2]\|_{\diamond^*}^2 &= \frac{1}{|\lambda_1 - \lambda_2|^2} \left\| \frac{1}{B^*(\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1})} \begin{pmatrix} \frac{\lambda_1 - \lambda_2}{\beta_{\lambda_1, \lambda_2}} \varphi_{\lambda_1}^{c_1} \\ \varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1} \end{pmatrix} - \frac{1}{B^*\varphi_{\lambda_2}^{c_2}} \begin{pmatrix} 0 \\ \varphi_{\lambda_2}^{c_2} \end{pmatrix} \right\|_{\diamond^*}^2 \\
1628 \quad &\leq \frac{\left\| \begin{pmatrix} \varphi_{\lambda_1}^{c_1} \\ 0 \end{pmatrix} \right\|_{\diamond^*}^2}{\beta_{\lambda_1, \lambda_2}^2 |B^*(\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1})|^2} + \frac{2 \left\| \begin{pmatrix} 0 \\ \zeta_{\lambda_1} \end{pmatrix} \right\|_{\diamond^*}^2}{|B^*(\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1})|^2 |\lambda_1 - \lambda_2|^2} \\
1629 \quad &\quad + \frac{2 \left\| \begin{pmatrix} 0 \\ \varphi_{\lambda_2}^{c_2} \end{pmatrix} \right\|_{\diamond^*}^2}{|\lambda_1 - \lambda_2|^2} \left(\frac{1}{B^*(\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1})} - \frac{1}{B^*\varphi_{\lambda_2}^{c_2}} \right)^2 \\
1630 \quad &=: S_1 + S_2 + S_3.
\end{aligned}$$

1632 We now analyze each of the three terms.

– Using (96), (105) and (108), we can obtain

$$S_1 \leq \frac{16}{\bar{C}^2}.$$

– Using (106) and (108), we get

$$S_2 \leq \frac{8C^*}{\bar{C}^2}.$$

– Finally, with (96), we write

$$S_3 = \frac{\lambda_2}{|\lambda_1 - \lambda_2|^2} \frac{(B^*\zeta_{\lambda_1})^2}{(B^*(\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1}))^2 (B^*\varphi_{\lambda_2}^{c_2})^2},$$

so that, with (95), (108) and (107), we get

$$S_3 \leq \frac{C_1^2}{\bar{C}^4}.$$

1633 All in all, we have obtained a uniform bound for $\|\psi[\lambda_1, \lambda_2]\|_{\diamond^*}$, which is exactly
1634 the compensation phenomenon we were expecting for this particular system.

1635 As a conclusion, we finally proved that, whatever the group G is, all the divided
1636 differences $\psi[\lambda]$, $\psi[\lambda, \lambda]$ or $\psi[\lambda_1, \lambda_2]$ remain bounded uniformly. It follows from (16)
1637 that $T_0(Y_0) \leq 0$, so that our main Theorem 1.1 show that (91) is null-controllable at
1638 any time $T > 0$ for any initial data $y_0 \in Y_0$. \square

1639 It remains to prove the lemma.

1640 *Proof (of Lemma 5.1).* By definition, the function ξ_λ satisfies

1641 (109)
$$-\partial_{xx}\xi_\lambda + c_2(x)\xi_\lambda = \lambda\xi_\lambda - \varphi_\lambda^{c_1}, \quad \text{in } (0, 1).$$

Using [2, Lemmas 2.2 and 2.3], and the fact that $\varphi_\lambda^{c_1}$ is normalized in $L^2(0, 1; \mathbb{R})$, we have

$$|\xi_\lambda(x)|^2 + \frac{1}{\lambda}|\partial_x\xi_\lambda(x)|^2 \leq C \left(|\xi_\lambda(y)|^2 + \frac{1}{\lambda}|\partial_x\xi_\lambda(y)|^2 \right) + \frac{C}{\lambda}, \quad \forall x, y \in [0, 1].$$

We take $y = 0$ in this inequality and we integrate with respect to x to obtain

$$\|\xi_\lambda\|^2 \leq \frac{C}{\lambda}|\partial_x\xi_\lambda(0)|^2 + \frac{C}{\lambda} = \frac{C}{\lambda}|B^*\xi_\lambda|^2 + \frac{C}{\lambda}.$$

It remains to bound from below the L^2 norm of ξ_λ . To this end, we multiply (109) by $\varphi_\lambda^{c_1}$ and integrate over $(0, 1)$. After integration by parts, and using the equation satisfied by $\varphi_\lambda^{c_1}$, we get

$$-1 = \int_0^1 (c_2 - c_1)\xi_\lambda\varphi_\lambda^{c_1}.$$

The Cauchy-Schwarz inequality gives

$$1 \leq \|c_1 - c_2\| \|\xi_\lambda\| \|\varphi_\lambda^{c_1}\|_{L^\infty},$$

1642 and since by (93), we have a uniform L^∞ bound on $\varphi_\lambda^{c_1}$, the proof is complete. \square

1643 **5.2.2. A system with different diffusions and a non constant coupling**
 1644 **term.** Let us briefly describe another example of a boundary controlled coupled
 1645 parabolic system which has motivated our study. This example is analyzed in details
 1646 in [42]. We consider the following system

1647 (110)
$$\begin{cases} \partial_t y + \begin{pmatrix} -\partial_{xx} & q(x) \\ 0 & -\nu\partial_{xx} \end{pmatrix} y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & (t, x) \in (0, T) \times (0, 1), \\ y(t, 0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad y(t, 1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & t \in (0, T), \\ y(0, x) = y_0(x), \end{cases}$$

1648 where $q \in L^\infty(0, 1)$ and $\nu > 0$.

1649 The spectrum of $\mathcal{A}^* = \begin{pmatrix} -\partial_{xx} & 0 \\ q(x) & -\nu\partial_{xx} \end{pmatrix}$ is $\Lambda = \{k^2, \nu k^2, \quad k \geq 1\}$.

- 1650 • System (110) in the case where $q(x) = 1$ and $\nu \neq 1$ was studied in [5] where
- 1651 the influence of the condensation of eigenvalues in the system was first pointed
- 1652 out. It was proved that the minimal null-control time was exactly the con-
- 1653 densation index of Λ , provided that $\sqrt{\nu} \notin \mathbb{Q}$.
- 1654 • System (110) with a non constant q but with the same diffusions, that is
- 1655 $\nu = 1$, was studied in [6]. The picture is different since in that case, there is
- 1656 no condensation of eigenvalues but there may however exist a minimal null-
- 1657 control time (depending on the coupling term q) due to very weak observation
- 1658 properties of the eigenfunctions.

1659 • In the general case, assuming that $\sqrt{\nu} \notin \mathbb{Q}$, the eigenvalues are algebraically
1660 and geometrically simple and it is proved in [42] that the associated family of
1661 eigenfunctions is complete in $X_{\diamond}^* = (H^{-1}(0, 1))^2$, and that, moreover, there
1662 exist functions q and values of ν , $\sqrt{\nu} \notin \mathbb{Q}$, such that this family (properly
1663 normalized) is not a Riesz basis of X_{\diamond}^* . Therefore the abstract results in [5, 6]
1664 do not apply.
1665 Inspired by the block moment method presented in the present paper, a
1666 suitable value of $T_0^{q,\nu}$ is defined in [42] (taking into account both effects of
1667 condensation of eigenvalues and weak observation of eigenfunctions) such that
1668 $T_0^{q,\nu}$ is the minimal null-control time of (110).

1669 6. Extensions, comments and open problems.

1670 6.1. Dealing with complex valued eigenvalues.

1671 In the previous sections, we decided to state our results in the framework of
1672 real eigenvalues to simplify the presentation. However, most of them still hold for
1673 complex eigenvalues satisfying assumptions largely inspired from [5]. More precisely,
1674 for a function $\mathcal{N} : \mathbb{R}^+ \rightarrow \mathbb{R}$, we will consider the class $\mathcal{L}_w(\delta, p, \rho, \mathcal{N})$ of the families
1675 $\Lambda \subset \mathbb{C}$ satisfying

- Parabolicity condition:

$$\Re \lambda \geq \delta |\lambda|, \quad \forall \lambda \in \Lambda.$$

- Asymptotic behavior : for any $\varepsilon > 0$, we have

$$\sum_{\substack{\lambda \in \Lambda \\ |\lambda| > \mathcal{N}(\varepsilon)}} \frac{1}{|\lambda|} \leq \varepsilon.$$

- Weak gap condition with parameters $\rho > 0$ and $p \in \mathbb{N}^*$:

$$\#\Lambda \cap ([\mu, \mu + \rho] + i\mathbb{R}) \leq p, \quad \forall \mu > 0.$$

In that case, a grouping $(G_k)_k$ should satisfy

$$\Lambda = \bigcup_{k \geq 1} G_k, \quad \#G_k \leq p, \quad \text{diam}(G_k) < \rho, \quad \inf(\Re G_{k+1}) - \sup(\Re G_k) > r.$$

The corresponding formula the minimal time $T_0(Y_0)$ will be now given by

$$T_0(Y_0) := \limsup_{k \rightarrow \infty} \frac{\ln \left(\max_{\substack{\mu \in \mathbb{N}^{g_k} \\ \mu \leq \alpha_k}} \left\| \psi[\lambda_{k,1}^{(\mu_1)}, \dots, \lambda_{k,g_k}^{(\mu_{g_k})}] \right\|_{\diamond^*} \right)}{\Re \lambda_{k,1}},$$

1676 and our results (namely Theorems 1.1, 2.1 and 4.1) still hold in that case.

1677 Most of the proofs are very similar by taking care of the following points:

- 1678 • The divided differences associated with pairwise distinct points x_0, \dots, x_n
1679 in the complex plane do not satisfy the Lagrange theorem but instead the
1680 following slightly weaker result, due to Jensen [32].

1681
1682

PROPOSITION 6.1. *Let $U \subset \mathbb{C}$ be a convex open set and $x_0, \dots, x_n \in U$ be pairwise distinct. For any holomorphic function $f : U \rightarrow \mathbb{C}$,*

– *there exists a $z \in \text{Conv}(\{x_0, \dots, x_n\})$ such that*

$$|f[x_0, \dots, x_n]| \leq \frac{f^{(n)}(z)}{n!}.$$

– *For any $z \in U$, we have*

$$\left| f[x_0, \dots, x_n] - \frac{f^{(n)}(z)}{n!} \right| \leq C_{f,n} \text{diam}(U).$$

- The Blaschke product W_k should be replaced by

$$W_k(z) = \prod_{j=1}^p \prod_{\lambda \in \Lambda_j} \frac{\lambda - z}{\bar{\lambda} + z}.$$

1683
1684

- Finally, in the restriction argument of Section 2.1.2, the holomorphy domain $\mathbb{C}_{2\varepsilon}^+$ should be replaced by a sector $\{z \in \mathbb{C}, \Re z > 2\varepsilon, |\Im z| \leq \frac{\delta}{2}|\lambda|\}$.

1685

6.2. Weakening the assumptions on the control operator.

1686

In this article, we not only study the classical null-controllability property (i.e. $Y_0 = X_{-\circ}$), we also provide a more accurate description depending on the space of initial conditions Y_0 one wants to drive to 0. In this setting, the assumption (9) can be too strong.

1690

It is easily seen than a necessary approximate null-controllability condition in that case is the following: for any $\lambda \in \Lambda$ and any $l \in \llbracket 0, \alpha_\lambda - 1 \rrbracket$ we have

1691

$$(111) \quad \left(\mathcal{B}^* \phi_\lambda^j = 0, \quad \forall j \in \llbracket 0, l \rrbracket \right) \Rightarrow \left(P_{Y_0}^* \phi_\lambda^j = 0, \quad \forall j \in \llbracket 0, l \rrbracket \right),$$

1693

where, in this formula, $(\phi_\lambda^j)_j$ is a Jordan chain associated with the eigenvalue λ . Note that such a Jordan chain is not unique but (111) does not depend on the particular chain we choose. Note also that the assumption (11) can be verified using any Jordan chain.

1697

From now on, we assume that (111) holds. For any $\lambda \in \Lambda$, two cases have to be considered:

1698

- Case 1 : We have

1700

$$\mathcal{B}^* \phi_\lambda^j = 0, \text{ for all } j \in \llbracket 0, \alpha_\lambda - 1 \rrbracket.$$

1701

From (111), it follows that for any $y_0 \in Y_0$, any $T > 0$, all the moment equation (19) corresponding to this eigenvalue are automatically satisfied. It follows that we can simply remove this eigenvalue from the family Λ when studying the control problem at time T from Y_0 .

1702

1703

1704

1705

- Case 2 :

1706

$$(112) \quad \text{there exists } j^* \in \llbracket 0, \alpha_\lambda - 1 \rrbracket \text{ s.t. } \mathcal{B}^* \phi_\lambda^j = 0, \forall j < j^*, \text{ and } \mathcal{B}^* \phi_\lambda^{j^*} \neq 0.$$

In that case, for $j > j^*$ we set

$$\beta_j := -\frac{\mathcal{B}^* \phi_\lambda^j}{\mathcal{B}^* \phi_\lambda^{j^*}},$$

1707 and then by induction, we define

$$1708 \quad (113) \quad \tilde{\phi}_\lambda^j = \begin{cases} \phi_\lambda^j, & \text{for } j \leq j^*, \\ \phi_\lambda^j + \sum_{k=j^*}^{j-1} \beta_{j+j^*-k} \tilde{\phi}_\lambda^k, & \text{for } j > j^*. \end{cases}$$

1709 This construction ensures that $(\tilde{\phi}_\lambda^j)_j$ and $(\phi_\lambda^j)_j$ span the same space, that

$$1710 \quad (114) \quad \mathcal{B}^* \tilde{\phi}_\lambda^j = 0, \text{ if and only if } j \neq j^*,$$

and finally satisfy the equations

$$\mathcal{A}^* \tilde{\phi}_\lambda^j = \lambda \tilde{\phi}_\lambda^j + \tilde{\phi}_\lambda^{j-1} + \gamma_j \phi_\lambda^{j^*-1},$$

1711 for some $\gamma_j \in \mathbb{R}$ whose precise value is unimportant in the sequel.

A straightforward computation shows that the semi-group generated by $-\mathcal{A}^*$ satisfy

$$e^{-t\mathcal{A}^*} \tilde{\phi}_\lambda^j \in (e_t \tilde{\phi})[\lambda^{(j+1)}] + V^{j^*},$$

1712 where $V^{j^*} := \text{Span}(\phi_\lambda^0, \dots, \phi_\lambda^{j^*-1})$. We shall prove that the term in V^{j^*} does
1713 not contribute to the moment problem. Indeed, from (111) and (112), we
1714 have $V^{j^*} \subset \text{Ker } \mathcal{B}^* \cap \text{Ker } P_{Y_0}^*$. Thus:

– Concerning the control term, we have

$$\mathcal{B}^* e^{-t\mathcal{A}^*} \tilde{\phi}_\lambda^j = \mathcal{B}^*(e_t \tilde{\phi})[\lambda^{(j+1)}],$$

and by (114), it simply remains

$$\mathcal{B}^* e^{-t\mathcal{A}^*} \tilde{\phi}_\lambda^j = \begin{cases} 0, & \text{if } j < j^*, \\ (\mathcal{B}^* \tilde{\phi}_\lambda^{j^*}) e_t[\lambda^{(j-j^*)}], & \text{if } j \geq j^*. \end{cases}$$

1715 – Concerning the contribution of the source term, we have

$$1716 \quad P_{Y_0}^* e^{-T\mathcal{A}^*} \tilde{\phi}_\lambda^j = P_{Y_0}^*(e_T \tilde{\phi})[\lambda^{(j+1)}]$$

$$1717 \quad = P_{Y_0}^* \sum_{k=1}^{j+1} e_T[\lambda^{(j+2-k)}] \tilde{\phi}[\lambda^{(k)}]$$

$$1718 \quad = P_{Y_0}^* \sum_{k=j^*+1}^{j+1} e_T[\lambda^{(j+2-k)}] \tilde{\phi}[\lambda^{(k)}],$$

1719 with the convention that the sum is 0 as soon as $j < j^*$.

1720 We may now adapt the definition of our null-control time by setting

$$1722 \quad (115) \quad \psi_\lambda^l = \frac{P_{Y_0}^* \tilde{\phi}_\lambda^{j^*+l}}{\mathcal{B}^* \tilde{\phi}_\lambda^{j^*}}, \quad \forall l \in \llbracket 0, \alpha_\lambda - 1 - j^* \rrbracket,$$

1723 so that the moment problem associated with this eigenvalue becomes

$$1724 \quad \int_0^T u(T-t) \frac{(-t)^l}{l!} e^{-\lambda t} dt = - \left\langle y_0, (e_T \psi)[\lambda^{(l+1)}] \right\rangle_{-\diamond, \diamond}, \quad \forall l \in \llbracket 0, \alpha_\lambda - 1 - j^* \rrbracket.$$

1725 This is formally exactly the same as (21) except that the multiplicity of the
1726 eigenvalue have been changed into $\alpha_\lambda - j^*$ and the associated values of ψ_λ^\bullet
1727 have been constructed as explained above by (113) and (115).

1728 As a conclusion, to obtain the definition of the minimal null-control time from Y_0
 1729 assuming that (111) holds, we simply need to ignore the eigenvalues corresponding
 1730 to case 1, and to modify the multiplicity and the *Jordan chain* as explained above
 1731 for the eigenvalues that are in case 2. Then, we define formally $T_0(Y_0)$ by the same
 1732 formula as (15) and we obtain exactly the same result as Theorem 1.1.

1733 Moreover, it clearly appears from the proof that (111) is actually a necessary and
 1734 sufficient condition to solve the moment problem associated to any finite number of
 1735 eigenvalues. Thus (111) is a necessary and sufficient condition for the approximate
 1736 null-controllability from Y_0 .

1737 7. Appendices.

1738 We gather in this final section some of the definition or intermediate results that
 1739 we used in this paper.

1740 7.1. Wellposedness.

1741 This section is dedicated to the proof of Proposition 1.1.

1742 First of all, let us notice that the problem (2) admits at most one solution $y \in$
 1743 $C^0([0, T]; X_{-1})$ and that the continuous dependency directly follows from (2). Thus,
 1744 it remains to prove the existence of a function $y \in C^0([0, T]; X_{-1})$ satisfying (2).

1745 From [45, Propositions 2.10.3] it comes that \mathcal{A} can be uniquely extended to an
 1746 operator $\tilde{\mathcal{A}} \in \mathcal{L}(X, X_{-1})$. Moreover it comes from [45, Propositions 2.10.4] that $-\tilde{\mathcal{A}}$
 1747 generates an C^0 -semigroup in X_{-1} satisfying

$$1748 \quad e^{-t\tilde{\mathcal{A}}} = \tilde{\mathcal{A}}e^{-t\mathcal{A}}\tilde{\mathcal{A}}^{-1}, \quad \forall t \geq 0.$$

1749 Thus, for any $T > 0$, any $y_0 \in X_{-1}$ and any $u \in L^2(0, T; U)$ the problem

$$1750 \quad \begin{cases} y'(t) + \tilde{\mathcal{A}}y(t) = \mathcal{B}u(t), \\ y(0) = y_0 \end{cases}$$

1751 admits a unique *mild* solution $y \in C^0([0, T], X_{-1})$ given by

$$1752 \quad (116) \quad y(t) = e^{-t\tilde{\mathcal{A}}}y_0 + \int_0^t e^{-(t-s)\tilde{\mathcal{A}}}\mathcal{B}u(s)ds.$$

1753 We prove now that this function satisfies (2). To do so, we simply prove that the
 1754 semigroup $e^{-t\tilde{\mathcal{A}}}$ is the adjoint of $e^{-t\mathcal{A}^*}$ in the duality between X_1^* and X_{-1} .

1755 Let $x \in X$ and $z \in X_1$ such that $x = \mathcal{A}z$. As $\tilde{\mathcal{A}}$ is an extension of \mathcal{A} it also comes
 1756 that $x = \tilde{\mathcal{A}}z$. Then, as $e^{-t\mathcal{A}}(X_1) \subset X_1$ it comes that

$$1757 \quad e^{-t\tilde{\mathcal{A}}}x = \tilde{\mathcal{A}}e^{-t\mathcal{A}}\tilde{\mathcal{A}}^{-1}\tilde{\mathcal{A}}z = \tilde{\mathcal{A}}e^{-t\mathcal{A}}z = \mathcal{A}e^{-t\mathcal{A}}z = e^{-t\mathcal{A}}\mathcal{A}z = e^{-t\mathcal{A}}x.$$

1758 Then, for any $x \in X$ and any $z \in X_1^*$

$$1759 \quad \left\langle e^{-t\tilde{\mathcal{A}}}x, z \right\rangle_{-1,1^*} = \left\langle e^{-t\mathcal{A}}x, z \right\rangle_{-1,1^*} = \left\langle e^{-t\mathcal{A}}x, z \right\rangle = \left\langle x, e^{-t\mathcal{A}^*}z \right\rangle_{-1,1^*}.$$

1760 Thus, the density of X in X_{-1} implies

$$1761 \quad (117) \quad \left\langle e^{-t\tilde{\mathcal{A}}}y, z \right\rangle_{-1,1^*} = \left\langle y, e^{-t\mathcal{A}^*}z \right\rangle_{-1,1^*}, \quad \forall y \in X_{-1}, \forall z \in X_1^*.$$

1762 Finally, the duality pairing of (116) with any $z_t \in X_1^*$ with the computation rule (117)
 1763 directly gives (2).

1764 **7.2. Existence of a grouping for sequences satisfying the weak gap con-**
 1765 **dition.**

1766 PROPOSITION 7.1. *For any Λ satisfying (8), there exists at least one grouping in*
 1767 $\mathcal{G}\left(\Lambda, p, \frac{\rho}{p}, \rho\right)$.

Proof. Let $r = \rho/p$. We set $\mu_1 = \inf \Lambda$. and we consider the p disjoint sets

$$\Lambda \cap (\mu_1, \mu_1 + r], \Lambda \cap (\mu_1 + r, \mu_1 + 2r], \dots, \Lambda \cap (\mu_1 + (p-1)r, \mu_1 + pr].$$

By (8), we know that one of this sets is empty since if it not the case, there is at least $p+1$ elements in $\Lambda \cap [\mu_1, \mu_1 + \rho]$ because $\mu_1 \in \Lambda$ and $pr = \rho$. Let $j \in \llbracket 1, p \rrbracket$ such that $\Lambda \cap (\mu_1 + (j-1)r, \mu_1 + jr] = \emptyset$. We define

$$G_1 := \Lambda \cap [\mu_1, \mu_1 + (j-1)r],$$

whose cardinal is, by (8), less or equal than p and diameter is less than ρ . Moreover, by construction, we have

$$(\inf(\Lambda \setminus G_1)) - \sup G_1 > r.$$

1768 This allows to build G_2 by the same construction applied on $\Lambda \setminus G_1$ while ensuring
 1769 the required properties, and following this process we construct the sequence $(G_k)_k$. \square

1770 **7.3. About divided differences.**

1771 In this section we give all the properties concerning divided differences that are
 1772 used all along this article. This notion is a key technical tool in our analysis as it
 1773 drastically ease the computations and the formulation of the results. The definition
 1774 and results given in Sec. 7.3.1 are classical in the field of interpolation (see for in-
 1775 stance [39, Chap. 5]). To deal with algebraic multiplicity we use a generalization of
 1776 divided differences where the ‘interpolation points’ are not necessarily distincts. Let
 1777 us mention that there exists generalizations in this direction (see for instance [39,
 1778 Chap. 5]) in the context of Hermite interpolation. However as we are not directly
 1779 dealing with interpolation, we propose such a generalization adapted to our purposes.
 1780 This is detailed in Sec. 7.3.2.

1781 **7.3.1. Definitions and basic properties.**

1782 Let V be a real vector space, $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathbb{R}$. Assume that x_1, \dots, x_n
 1783 are pairwise distinct (see Sec. 7.3.2 for a generalization). Let $f_1, \dots, f_n \in V$ be given.

DEFINITION 7.1. *The divided differences are defined by*

$$f[x_i] := f_i, \quad \forall i \in \llbracket 1, n \rrbracket,$$

1784 *and then recursively for any $k \in \llbracket 2, n \rrbracket$, for any pairwise distinct $i_1, \dots, i_k \in \llbracket 1, n \rrbracket$, by*

$$1785 \quad f[x_{i_1}, \dots, x_{i_k}] := \frac{f[x_{i_1}, \dots, x_{i_{k-1}}] - f[x_{i_2}, \dots, x_{i_k}]}{x_{i_1} - x_{i_k}}.$$

1786 In all what follows, if $f : \mathbb{R} \rightarrow V$ is a given function it will be implicitly assumed
 1787 that $f_i = f[x_i] = f(x_i)$.

1788 PROPOSITION 7.2. *The divided differences are symmetric with respect to their ar-*
 1789 *guments : for any $k \in \llbracket 1, n \rrbracket$, for any pairwise distinct $i_1, \dots, i_k \in \llbracket 1, n \rrbracket$ and any*
 1790 *$\sigma \in \mathfrak{S}(\{i_1, \dots, i_k\})$,*

$$1791 \quad f[x_{\sigma(i_1)}, \dots, x_{\sigma(i_k)}] = f[x_{i_1}, \dots, x_{i_k}].$$

1792 The following property states another (equivalent) definition of divided differences
 1793 known as Newton formula.

1794 PROPOSITION 7.3. *For any $k \in \llbracket 1, n \rrbracket$, for any pairwise distinct $i_1, \dots, i_k \in \llbracket 1, n \rrbracket$*

$$1795 \quad f[x_{i_1}, \dots, x_{i_k}] = \sum_{j=1}^k \frac{f[x_{i_j}]}{\prod_{l \in \llbracket 1, k \rrbracket, l \neq j} (x_{i_j} - x_{i_l})}.$$

1796 The next result about divided differences is crucial to obtain the different esti-
 1797 mates we need. It is known as Lagrange theorem.

1798 PROPOSITION 7.4. *Assume that $V = \mathbb{R}$ and that $f \in C^{n-1}(\text{Conv}\{x_1, \dots, x_n\})$.*
 1799 *For any $k \in \llbracket 1, n \rrbracket$, for any pairwise distinct $i_1, \dots, i_k \in \llbracket 1, n \rrbracket$, there exists a $z \in$*
 1800 *$\text{Conv}\{x_{i_1}, \dots, x_{i_k}\}$ such that*

$$1801 \quad f[x_{i_1}, \dots, x_{i_k}] = \frac{f^{(k-1)}(z)}{(k-1)!}.$$

1802 The divided differences naturally appear in polynomial interpolation problems as
 1803 recalled in the following classical result.

1804 PROPOSITION 7.5. *The polynomial function $P : \mathbb{R} \rightarrow V$ defined by*

$$1805 \quad (118) \quad P(x) := f[x_1] + (x - x_1)f[x_1, x_2] + \dots + \left(\prod_{i=1}^{n-1} (x - x_i) \right) f[x_1, \dots, x_n],$$

1806 *is the unique polynomial of degree less than $n - 1$ such that*

$$1807 \quad (119) \quad P(x_i) = f[x_i], \quad \forall i \in \llbracket 1, n \rrbracket.$$

1808 We recall a simple way to compute divided differences of a product of functions
 1809 which is known as the Leibniz rule.

1810 PROPOSITION 7.6. *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ and $(gf)[x] := g(x)f[x]$. For any $k \in \llbracket 1, n \rrbracket$, for*
 1811 *any pairwise distinct $i_1, \dots, i_k \in \llbracket 1, n \rrbracket$,*

$$1812 \quad (gf)[x_{i_1}, \dots, x_{i_k}] = \sum_{j=1}^k g[x_{i_1}, \dots, x_{i_j}] f[x_{i_j}, \dots, x_{i_k}].$$

1813 Finally, we deduce from the results above the following useful corollary.

1814 COROLLARY 7.1. *Assume that V is equipped with a norm $\|\bullet\|_V$. For any $k \in$*
 1815 *$\llbracket 1, n \rrbracket$ and any pairwise distinct $i_1, \dots, i_k \in \llbracket 1, n \rrbracket$, we have*

$$1816 \quad \|f[x_{i_1}, \dots, x_{i_k}]\|_V \leq n2^{n-1}(1+R)^{n-1} \max_{j \in \llbracket 1, n \rrbracket} \|f[x_1, \dots, x_j]\|_V,$$

1817 *where $R = \text{diam}(\{x_1, \dots, x_n\})$.*

1818 *Proof.* Let P be the Lagrange interpolation polynomial defined in (118) and let
 1819 i_1, \dots, i_k be fixed.

1820 By the Hahn-Banach theorem, there exists $\phi \in V'$, such that $\|\phi\|_{V'} = 1$ and

$$1821 \quad \langle \phi, f[x_{i_1}, \dots, x_{i_k}] \rangle_{V', V} = \|f[x_{i_1}, \dots, x_{i_k}]\|_V.$$

1822 Additionally, by (119) and by linearity of ϕ , we know that

$$1823 \quad \langle \phi, f[x_{i_1}, \dots, x_{i_k}] \rangle_{V', V} = \langle \phi, P \rangle_{V', V} [x_{i_1}, \dots, x_{i_k}].$$

1824 Applying Proposition 7.4 to $x \mapsto \langle \phi, P(x) \rangle_{V', V} \in \mathbb{R}$ we find that for some $z \in$
 1825 $\text{Conv}\{x_1, \dots, x_n\}$, we have

$$1826 \quad \langle \phi, P \rangle_{V', V} [x_{i_1}, \dots, x_{i_k}] = \frac{1}{(k-1)!} (\langle \phi, P \rangle_{V', V})^{(k-1)}(z)$$

$$1827 \quad = \frac{1}{(k-1)!} \left\langle \phi, P^{(k-1)}(z) \right\rangle_{V', V}.$$

1829 Combining those identities, we arrive at

$$1830 \quad \|f[x_{i_1}, \dots, x_{i_k}]\|_V = \frac{1}{(k-1)!} \left\langle \phi, P^{(k-1)}(z) \right\rangle_{V', V} \leq \frac{1}{(k-1)!} \|P^{(k-1)}(z)\|_V.$$

1831 Let us compute the derivatives of P . Let \mathcal{C} be the circle of center z and radius R in
 1832 the complex plane. The Cauchy formula leads to

$$1833 \quad \frac{1}{(k-1)!} P^{(k-1)}(z) = \frac{1}{2i\pi} \int_{\mathcal{C}} \frac{P(w)}{(z-w)^k} dw,$$

1834 so that

$$1835 \quad \frac{1}{(k-1)!} \|P^{(k-1)}(z)\|_V \leq R^{1-k} \max_{w \in \mathcal{C}} \|P(w)\|_V.$$

1836 Then, the triangle inequality implies that for any $w \in \mathcal{C}$,

$$1837 \quad \|P(w)\|_V \leq \|f[x_1]\|_V + (2R) \|f[x_1, x_2]\|_V + \dots + (2R)^{n-1} \|f[x_1, \dots, x_n]\|_V,$$

1838 which finally gives the result. \square

1839 7.3.2. Generalization of divided differences.

1840 Assume that V is a normed vector space.

Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ be pairwise distinct real numbers and let $\alpha \in \mathbb{N}^n$ a multi-index such that $\alpha > 0$. To such a multi-index we associate elements of V that we gather in a $f_\alpha \in V^{|\alpha|}$ and that are indexed as follows

$$f_j^l, \quad j \in \llbracket 1, n \rrbracket, l \in \llbracket 0, \alpha_j - 1 \rrbracket.$$

1841 DEFINITION 7.2. We set $N = |\alpha|$. We say that a family of points $(y_p^h)_{p \in \llbracket 1, N \rrbracket}$,
 1842 depending on a small parameter $h > 0$, is an approximation of the weighted family
 1843 (x, α) if

- 1844 • For each $h > 0$, the points y_1^h, \dots, y_N^h are pairwise distinct.
- There exist disjoint subsets $P_j \subset \llbracket 1, N \rrbracket$ such that for any $j \in \llbracket 1, n \rrbracket$,

$$\#P_j = \alpha_j, \quad \text{and} \quad y_p^h \xrightarrow{h \rightarrow 0} x_j, \quad \forall p \in P_j.$$

1845 PROPOSITION 7.7. *With the notation above, let $F : \mathbb{R} \rightarrow V$ be any smooth function*
 1846 *satisfying*

$$1847 \quad (120) \quad \frac{1}{l!} F^{(l)}(x_j) = f_j^l, \quad \forall j \in \llbracket 1, n \rrbracket, \quad \forall l \in \llbracket 0, \alpha_j - 1 \rrbracket.$$

1848 *For any approximation of the weighted family (x, α) , the (usual) divided difference*
 1849 *$F[y_1^h, \dots, y_N^h]$ weakly converges when $h \rightarrow 0$ towards an element in V that depends*
 1850 *only on x, α and f_α . In particular it does not depend on the particular choice of F*
 1851 *nor on the approximation families $(y_p^h)_p$.*

1852 *This limit is called the generalized divided difference associated with the points x ,*
 1853 *the multi-index α and the values f_α and is denoted by*

$$1854 \quad f[x_1^{(\alpha_1)}, \dots, x_n^{(\alpha_n)}], \quad \text{or } f[\underbrace{x_1, \dots, x_1}_{\alpha_1 \text{ times}}, \underbrace{x_2, \dots, x_2}_{\alpha_2 \text{ times}}, \dots],$$

1855 *or, in a more compact way, $f[x^{(\alpha)}]$.*

1856 *Moreover, we extend this definition if some of the α_j are 0, simply by not consid-*
 1857 *ering the corresponding points.*

1858 REMARK 7.1. *If the function F is chosen to take its values in a finite dimension*
 1859 *space then the above convergence is actually strong. It is always possible to make*
 1860 *this assumption, for instance by choosing F that takes its values in the subspace of V*
 1861 *spanned by the elements f_α .*

1862 *Proof (of Proposition 7.7).* The proof is done by recurrence on N .

1863 • If $N = 1$, then we necessarily have $n = 1$ and $\alpha_1 = 1$. The result is just a
 1864 consequence of the continuity of F and we simply have $f[x_1] = f_1^0$.
 1865 • Assume that the result holds for a given value of N and let us prove it for the value
 1866 $N + 1$.

1867 – *First case :* If there is only one point x_1 . It means that $n = 1$ and $\alpha_1 = N + 1$.
 1868 In this case, for any $h > 0$, and any $\psi \in V'$, we use the Lagrange theorem to
 1869 get the existence of a $z^{\psi, h} \in \text{Conv}(\{y_1^h, \dots, y_{N+1}^h\})$ such that

$$1870 \quad \langle \psi, F[y_1^h, \dots, y_{N+1}^h] \rangle_{V', V} = \langle \psi, F \rangle_{V', V} [y_1^h, \dots, y_{N+1}^h]$$

$$1871 \quad = \frac{1}{N!} \langle \psi, F \rangle_{V', V}^{(N)}(z^{\psi, h}).$$

1872
 1873 Since, by assumption, all the points y_p^h converge to the same point x_1 , we
 1874 have $z^{\psi, h} \rightarrow x_1$ and thus

$$1875 \quad \langle \psi, F[y_1^h, \dots, y_{N+1}^h] \rangle_{V', V} \xrightarrow{h \rightarrow 0} \frac{1}{N!} \langle \psi, F^{(N)}(x_1) \rangle_{V', V} = \langle \psi, f_1^N \rangle_{V', V},$$

1876 – *Second case :* We assume that $n > 1$. By assumption there exists two distinct
 1877 indices $j_1, j_2 \in \llbracket 1, n \rrbracket$ and two distinct indices $p_1, p_2 \in \llbracket 1, N + 1 \rrbracket$ such that
 1878 $y_{p_1}^h \rightarrow x_{j_1}$ and $y_{p_2}^h \rightarrow x_{j_2}$. By symmetry of the usual divided differences, we
 1879 can always assume that $p_1 = N$ and $p_2 = N + 1$. It follows that we can write

$$1880 \quad F[y_1^h, \dots, y_{N+1}^h] = \frac{F[y_1^h, \dots, y_{N-1}^h, y_{N+1}^h] - F[y_1^h, \dots, y_N^h]}{y_{N+1}^h - y_N^h}.$$

1881 The recurrence assumption shows that the two terms in the numerator have
 1882 weak limits that only depends on the points x , the multiplicities α and on the
 1883 values f_α , whereas the denominator $y_{N+1}^h - y_N^h$ converges to $x_{j_2} - x_{j_1}$ which
 1884 is not zero. The result follows. \square

1885 The above construction also shows, as a by-product, the following rules to com-
 1886 pute the generalized divided differences: for any $\mu \in \mathbb{N}^n$ such that $\mu \leq \alpha$

$$1887 \quad (121) \quad f[x_1^{(\mu_1)}, \dots, x_n^{(\mu_n)}] = f_j^{\mu_j-1}, \quad \text{if } \mu_{j'} = 0 \text{ for any } j' \neq j,$$

1888 and for all $j_1 \neq j_2$ and $\mu_{j_1} > 0, \mu_{j_2} > 0$

$$1889 \quad (122) \quad f[x_1^{(\mu_1)}, \dots, x_n^{(\mu_n)}] = \frac{f[\dots, x_{j_1}^{(\mu_{j_1}-1)}, \dots, x_{j_2}^{(\mu_{j_2})}, \dots] - f[\dots, x_{j_1}^{(\mu_{j_1})}, \dots, x_{j_2}^{(\mu_{j_2}-1)}, \dots]}{x_{j_1} - x_{j_2}}.$$

1890 Let us now give some useful properties that are the extension of the classical
 1891 properties recalled in Sec. 7.3.1.

1892 DEFINITION 7.3. Let $\alpha \in \mathbb{N}^n$ be a multi-index, $f_\alpha \in \mathbb{R}^{|\alpha|}$ a set of real values
 1893 associated with α and $g_\alpha \in V^{|\alpha|}$ a set of elements of V associated with α .

We define $(fg)_\alpha \in V^{|\alpha|}$ to be the product set of values as follows :

$$(fg)_j^l := \sum_{k=0}^l f_j^k g_j^{l-k}, \quad \forall j \in \llbracket 1, n \rrbracket, \forall l \in \llbracket 0, \alpha_j - 1 \rrbracket.$$

1894 PROPOSITION 7.8 (Leibniz formula). Let $x \in \mathbb{R}^n$ pairwise distinct points, $\alpha \in \mathbb{N}^n$,
 1895 $f_\alpha \in \mathbb{R}^{|\alpha|}$ a set of real values, and $g_\alpha \in V^{|\alpha|}$ a set of values in V .

1896 Then, for any family of multi-indices $(\mu^p)_{p \in \llbracket 0, |\alpha| \rrbracket} \subset \mathbb{N}^n$ satisfying

$$1897 \quad (123) \quad \begin{cases} \mu^{p-1} \leq \mu^p, & \forall p \in \llbracket 1, |\alpha| \rrbracket, \\ |\mu^p| = p, & \forall p \in \llbracket 0, |\alpha| \rrbracket, \\ \mu^{|\alpha|} = \alpha, \end{cases}$$

we have the Leibniz formula

$$(fg)[x^{(\alpha)}] = \sum_{p=1}^{|\alpha|} f[x^{(\mu^p)}]g[x^{(\alpha-\mu^{p-1})}].$$

1898 *Proof.* By assumption, for each $p \in \llbracket 1, |\alpha| \rrbracket$, the multi-index μ^p is obtained from
 1899 μ^{p-1} by incrementing exactly one of its element. We denote by $i_p \in \llbracket 1, n \rrbracket$ this index,
 1900 and we define $y_p^h := x_{i_p} + ph$. It is easily seen that, for $h > 0$ small enough, those
 1901 points are pairwise distinct.

Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying (120) and $G : \mathbb{R} \rightarrow V$ be a function
 satisfying (120) but with the values g_α instead of f_α . The usual Leibniz formula as
 well as the Definition 7.3 shows that the product function FG exactly satisfies

$$\frac{1}{l!} (FG)^{(l)}(x_j) = (fg)_j^l, \quad \forall j \in \llbracket 1, n \rrbracket, \forall l \in \llbracket 0, \alpha_j - 1 \rrbracket.$$

We can thus apply the Leibniz formula from Proposition 7.6 as follows

$$(FG)[y_1^h, \dots, y_{|\alpha|}^h] = \sum_{p=1}^{|\alpha|} F[y_1^h, \dots, y_p^h]G[y_p^h, \dots, y_{|\alpha|}^h],$$

1902 and then pass to the limit as $h \rightarrow 0$ to obtain the claim. \square

PROPOSITION 7.9 (Lagrange theorem). *Let x, α as before. We set $N = |\alpha|$. With any $f : \mathbb{R} \rightarrow \mathbb{R}$ of class C^{N-1} , we associate the set of values $f_\alpha \in \mathbb{R}^N$ by*

$$f_j^l := \frac{1}{l!} f^{(l)}(x_j), \quad \forall j \in \llbracket 1, n \rrbracket, \forall l \in \llbracket 0, \alpha_j - 1 \rrbracket.$$

Then, there exists a $z \in \text{Conv}(\{x_1, \dots, x_n\})$ such that the generalized divided difference built on these data satisfy

$$f[x^{(\alpha)}] = \frac{1}{(N-1)!} f^{(N-1)}(z).$$

Proof. Let y_1^h, \dots, y_N^h be an approximation of the weighted family of points (x, α) as in Definition 7.2. By definition, the generalized divided difference $f[x^{(\alpha)}]$ is the limit as h goes to 0, of the usual divided difference $f[y_1^h, \dots, y_N^h]$. For this last divided difference, we can apply Lagrange theorem (see Proposition 7.4) to get the existence of a point $z^h \in \text{Conv}(\{y_1^h, \dots, y_N^h\})$ such that

$$f[y_1^h, \dots, y_N^h] = \frac{1}{(N-1)!} f^{(N-1)}(z^h).$$

1903 It is clear that $(z^h)_h$ is contained in a compact set so that, up to a subsequence,
 1904 we may find a limit z of $(z^h)_h$ that belongs to $\text{Conv}(\{x_1, \dots, x_n\})$ and satisfies the
 1905 required property. \square

PROPOSITION 7.10. *Let $(\mu^p)_{p \in \llbracket 0, |\alpha| \rrbracket} \subset \mathbb{N}^n$ be a family of multi-indices satisfying (123). For any multi-index μ such that $\mu \leq \alpha$ we have*

$$\|f[x^{(\mu)}]\| \leq N 2^{N-1} (1+R)^{N-1} \max_{p \in \llbracket 1, |\alpha| \rrbracket} \|f[x^{(\mu^p)}]\|.$$

1906 *Proof.* We proceed as in the proof of Proposition 7.8 by passing to the limit in
 1907 the similar result for standard divided differences (Corollary 7.1). \square

1908 For generalized divided differences, there is no simple equivalent to the Newton
 1909 formula (Proposition 7.3). However, we can state the following result.

1910 PROPOSITION 7.11. *For any multi-index $\mu \leq \alpha$, there exists coefficients $(\theta_{j,l}^\mu)_{j,l}$
 1911 depending only on x and μ , such that*

$$1912 \quad f[x^{(\mu)}] = \sum_{j=1}^n \sum_{l=0}^{\alpha_j-1} \theta_{j,l}^\mu f_j^l,$$

and which satisfy the following estimates

$$|\theta_{j,l}^\mu| \leq \begin{cases} 0 & \text{if } \mu_j < l+1, \\ \frac{C_{|\mu|}}{\left(\prod_{i \in \llbracket 1, n \rrbracket \neq j} |x_i - x_j|^{\mu_i}\right)} \frac{1}{(\min_{i \in \llbracket 1, n \rrbracket \neq j} |x_i - x_j|)^{\mu_j - l - 1}}, & \text{if } \mu_j \geq l+1. \end{cases}$$

1913 *Proof.* Since the divided difference are clearly linear with respect to the data
 1914 f_α , the existence of the coefficients $\theta_{j,l}^\mu$ is straightforward. Let us prove the claimed
 1915 estimates. From now on we assume that l is fixed. Moreover, for any $j \in \llbracket 1, n \rrbracket$ we
 1916 introduce the notation $d_j := \min_{i \in \llbracket 1, n \rrbracket \neq j} |x_i - x_j|$ and we define $\delta^j \in \mathbb{N}^n$ to be the
 1917 Kronecker multi-index, that is $\delta_i^j = 0$ for $i \neq j$ and $\delta_j^j = 1$.

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1919
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1921
1922

- When $\mu_j < l + 1$, it is clear from the recurrence formulas (121) and (122) that the value of $f[x^{(\mu)}]$ does not depend on the value f_j^l , and therefore $\theta_{j,l}^\mu = 0$.
- Let us show by induction on $N = |\mu|$ that, for all $j \in \llbracket 1, n \rrbracket$, with $\mu_j \geq l + 1$, we have

1923

$$(124) \quad |\theta_{j,l}^\mu| \leq \frac{C_{|\mu|}}{\left(\prod_{i \in \llbracket 1, n \rrbracket \neq j} |x_j - x_i|^{\mu_i}\right)} \frac{1}{d_j^{\mu_j - l - 1}}.$$

- Assume first that $N = l + 1$ and let μ such that $|\mu| = N$. If $\mu_j < l + 1$ we have already seen that $\theta_{j,l}^\mu = 0$ which obviously implies (124). If $\mu_j = l + 1$, since $|\mu| = l + 1$, we necessarily have $\mu_i = 0$ for any $i \neq j$, so that (121) gives

$$f[x^{(\mu)}] = f_j^l,$$

1924
1925
1926
1927

- which implies that $\theta_{j,l}^\mu = 1$, that is exactly (124) with $C_{|\mu|} = 1$ in that case.
- Assume now that, for some $N \geq l + 1$, (124) holds and let μ such that $|\mu| = N + 1$.
If $\mu_i = 0$ for any $i \neq j$, then we have

$$f[x^{(\mu)}] = f_j^{\mu_j - 1},$$

1928

- which implies that $\theta_{j,l}^\mu = 0$ since $l \neq \mu_j - 1$ and (124) is obvious.
If there is a $i_0 \neq j$ such that $\mu_{i_0} \geq 1$ then we use (122) to get

$$f[x^{(\mu)}] = \frac{f[x^{(\mu - \delta^j)}] - f[x^{(\mu - \delta^{i_0})}]}{x_{i_0} - x_j},$$

which implies the formula

$$\theta_{j,l}^\mu = \frac{\theta_{j,l}^{\mu - \delta^j} - \theta_{j,l}^{\mu - \delta^{i_0}}}{x_{i_0} - x_j},$$

and thus

$$|\theta_{j,l}^\mu| \leq \frac{|\theta_{j,l}^{\mu - \delta^j}|}{|x_{i_0} - x_j|} + \frac{|\theta_{j,l}^{\mu - \delta^{i_0}}|}{|x_{i_0} - x_j|}.$$

1929
1930
1931

- Since $|\mu - \delta^j| = |\mu - \delta^{i_0}| = N$, we can apply the induction hypothesis to bound the two terms in the right-hand side as follows

1932
1933

$$\frac{|\theta_{j,l}^{\mu - \delta^j}|}{|x_{i_0} - x_j|} \leq \frac{C_{N-1} |\theta_{j,l}^{\mu - \delta^j}|}{d_j} \leq \frac{C_{N-1}}{\left(\prod_{i \in \llbracket 1, n \rrbracket \neq j} |x_j - x_i|^{\mu_i}\right)} \frac{1}{d_j^{\mu_j - l - 2}} \frac{1}{d_j},$$

1934
1935

and

1936

$$\begin{aligned} \frac{|\theta_{j,l}^{\mu - \delta^{i_0}}|}{|x_i - x_j|} &\leq \frac{C_{N-1} |\theta_{j,l}^{\mu - \delta^{i_0}}|}{|x_j - x_{i_0}|} \\ &\leq \frac{C_{N-1}}{\left(\prod_{i \in \llbracket 1, n \rrbracket \neq j} |x_j - x_i|^{\mu_i - \delta_i^{i_0}}\right)} \frac{1}{d_j^{\mu_j - l - 1}} \frac{1}{|x_j - x_{i_0}|}. \end{aligned}$$

1937
1938
1939

- Summing those two inequalities give (124) with $C_N = 2C_{N-1}$. \square

1940 **7.4. The supremum of $T_0(y_0)$.**

1941 We prove here Proposition 1.3, that is

1942
$$\sup_{y_0 \in Y_0} T_0(y_0) = T_0(Y_0).$$

1943

1944 *Proof.* Since by definition $T_0(y_0)$ only depends on $\text{Span}(y_0)$, it is actually equiv-
1945 alent to prove

1946
$$\sup_{\substack{y_0 \in Y_0 \\ \|y_0\|_{-\diamond} = 1}} T_0(y_0) = T_0(Y_0).$$

1947 To ease the reading let us do the computations in the simpler case $\eta = 1$; the
1948 extension to the case $\eta \geq 2$ being straightforward. Let us introduce

1949
$$x_{k,l} := \sum_{j=1}^l \frac{\psi_{k,j}}{\prod_{1 \leq i \neq j \leq l} (\lambda_{k,j} - \lambda_{k,i})}, \quad \forall k \geq 1, \forall l \in \llbracket 1, g_k \rrbracket.$$

1950 with $\psi_{k,j} := \frac{P_{Y_0}^* \phi_{k,j}}{\mathcal{B}^* \phi_{k,j}}$ as defined in (14).

1951 Notice that, since $\|y_0\|_{-\diamond} = 1$, for any $z \in X_{\diamond}^*$,

1952
$$\begin{aligned} \left\| P_{\text{Span}(y_0)}^* z \right\|_{\diamond^*} &= \sup_{\substack{y \in X_{-\diamond} \\ \|y\|_{-\diamond} = 1}} \left| \left\langle y, P_{\text{Span}(y_0)}^* z \right\rangle_{-\diamond, \diamond} \right| \\ 1953 &= \sup_{\substack{y \in X_{-\diamond} \\ \|y\|_{-\diamond} = 1}} \left| \left\langle P_{\text{Span}(y_0)} y, z \right\rangle_{-\diamond, \diamond} \right| \\ 1954 &= \sup_{\substack{y \in X_{-\diamond} \\ \|y\|_{-\diamond} = 1}} \left| (y, y_0)_{-\diamond} \langle y_0, z \rangle_{-\diamond, \diamond} \right| \\ 1955 &= \left| \langle y_0, z \rangle_{-\diamond, \diamond} \right|. \\ 1956 \end{aligned}$$

1957 Thus, with those notations, we have

1958
$$T_0(y_0) = \limsup_{k \rightarrow +\infty} \frac{\ln \left(\max_{l \in \llbracket 1, g_k \rrbracket} \left| \langle y_0, x_{k,l} \rangle_{-\diamond, \diamond} \right| \right)}{\lambda_{k,1}},$$

1959
$$T_0(Y_0) = \limsup_{k \rightarrow +\infty} \frac{\ln \left(\max_{l \in \llbracket 1, g_k \rrbracket} \|x_{k,l}\|_{\diamond^*} \right)}{\lambda_{k,1}}.$$

1960

1961 • Since y_0 is normalized, we have $\left| \langle y_0, x_{k,l} \rangle_{-\diamond, \diamond} \right| \leq \|x_{k,l}\|_{\diamond^*}$, for any k and l , it
1962 immediately comes that $T_0(y_0) \leq T_0(Y_0)$ and thus

1963
$$\sup_{y_0 \in Y_0} T_0(y_0) \leq T_0(Y_0).$$

1964 • Conversely, let T be such that

1965
$$\sup_{\substack{y_0 \in Y_0 \\ \|y_0\|_{-\diamond} = 1}} T_0(y_0) < T.$$

1966 Setting $\tilde{x}_{k,l} := e^{-\lambda_{k,1}T} x_{k,l}$, it comes that for any $y_0 \in Y_0$, we have

$$1967 \quad \sup_{\substack{k \geq 1 \\ l \in \llbracket 1, g_k \rrbracket}} \left| \langle y_0, \tilde{x}_{k,l} \rangle_{-\diamond, \diamond} \right| < +\infty,$$

and this property is in fact true for any $y_0 \in X_{-\diamond}$ since $P_{Y_0}^* \tilde{x}_{k,l} = \tilde{x}_{k,l}$, so that we have

$$\langle y_0, \tilde{x}_{k,l} \rangle_{-\diamond, \diamond} = \langle y_0, P_{Y_0}^* \tilde{x}_{k,l} \rangle_{-\diamond, \diamond} = \langle P_{Y_0} y_0, \tilde{x}_{k,l} \rangle_{-\diamond, \diamond},$$

1968 and $P_{Y_0} y_0 \in Y_0$.

1969 Applying the Banach-Steinhaus theorem, this implies that

$$1970 \quad \sup_{\substack{k \geq 1 \\ l \in \llbracket 1, g_k \rrbracket}} \|\tilde{x}_{k,l}\|_{\diamond^*} < +\infty.$$

1971 Thus there exists $C > 0$ such that $\|x_{k,l}\|_{\diamond^*} \leq C e^{\lambda_{k,1}T}$ for any $k \geq 1$ and any
1972 $1 \leq l \leq g_k$. Finally this yields,

$$1973 \quad T_0(Y_0) = \limsup_{k \rightarrow \infty} \frac{\ln \left(\max_{l \in \llbracket 1, g_k \rrbracket} \|x_{k,l}\|_{\diamond^*} \right)}{\lambda_{k,1}} \leq T.$$

1974 This ends the proof of Proposition 1.3. □

1975 7.5. On the condensation index.

1976 In this appendix we give some useful properties concerning the condensation index
1977 of a sequence. Let Σ be a family of positive real numbers. We start by recalling the
1978 definition of $c(\Sigma)$.

1979 DEFINITION 7.4. *Assume that Σ satisfies*

$$1980 \quad \sum_{\sigma \in \Sigma} \frac{1}{\sigma} < +\infty.$$

1981 *The interpolating function is defined by*

$$1982 \quad (125) \quad E_\Sigma : z \in \mathbb{C} \mapsto \prod_{\sigma \in \Sigma} \left(1 - \frac{z^2}{\sigma^2} \right).$$

1983 *The condensation index $c(\Sigma)$ is defined by*

$$1984 \quad c(\Sigma) = \limsup_{\substack{\sigma \in \Sigma \\ \sigma \rightarrow \infty}} \frac{-\ln |E'_\Sigma(\sigma)|}{\sigma}.$$

1985 This definition (and also its extension to complex sequences) is given in [44].

1986 In the case where the considered sequence satisfies the weak gap condition (8) the
1987 computation of the condensation index can be simplified: the grouping introduced in
1988 Proposition 7.1 is an optimal condensation grouping in the following sense.

1989 PROPOSITION 7.12. Assume that Σ satisfies the assumptions of Definition 7.4 as
 1990 well as the weak gap condition (8). Denote by $(G_k)_{k \geq 1}$ a grouping satisfying the
 1991 conditions of Definition 1.2. Then,

$$1992 \quad c(\Sigma) = \limsup_{\substack{\sigma \in \Sigma \\ \sigma \rightarrow \infty}} \frac{-\ln |P'_{G^{[\sigma]}}(\sigma)|}{\sigma}.$$

1993 Recall that $G^{[\sigma]}$ is the element of $(G_k)_{k \geq 1}$ containing σ .

1994 *Proof.* The proof follow directly from (65). \square

1995 Using this result, we compute easily the condensation index of the particular
 1996 sequence used in Sec. 5.1.

1997 PROPOSITION 7.13. Let $(\mu_k)_{k \geq 1}$ be a real increasing sequence such that

$$1998 \quad \sum_{k \geq 1} \frac{1}{\mu_k} < +\infty.$$

1999 Let $\alpha > \beta > 0$ and $\Theta = \{\mu_k, \mu_k + e^{-\alpha\mu_k}, \mu_k + e^{-\beta\mu_k}; k \in \mathbb{N}^*\}$. Then,

$$2000 \quad c(\Theta) = \alpha + \beta.$$

2001 *Proof.* One can directly verify that the grouping defined by

$$2002 \quad G_k := \{\mu_k, \mu_k + e^{-\alpha\mu_k}, \mu_k + e^{-\beta\mu_k}\}$$

2003 satisfies the requirements given in Proposition 7.1. Then, direct computations lead to

$$2004 \quad |P'_{G_k}(\mu_k)| = |\mu_k - (\mu_k + e^{-\alpha\mu_k})| |\mu_k - (\mu_k + e^{-\beta\mu_k})| = e^{-(\alpha+\beta)\mu_k},$$

2005

$$2006 \quad |P'_{G_k}(\mu_k + e^{-\alpha\mu_k})| = |\mu_k + e^{-\alpha\mu_k} - \mu_k| |\mu_k + e^{-\alpha\mu_k} - (\mu_k + e^{-\beta\mu_k})| \\ 2007 \quad = e^{-(\alpha+\beta)\mu_k} (1 - e^{-(\alpha-\beta)\mu_k}),$$

2008

2009 and

$$2010 \quad |P'_{G_k}(\mu_k + e^{-\beta\mu_k})| = |\mu_k + e^{-\beta\mu_k} - \mu_k| |\mu_k + e^{-\beta\mu_k} - (\mu_k + e^{-\alpha\mu_k})| \\ 2011 \quad = e^{-2\beta\mu_k} (1 - e^{-(\alpha-\beta)\mu_k}).$$

2012

2013 Thus, as $2\beta < \alpha + \beta$, we obtain $c(\Theta) = \alpha + \beta$. \square

2014

REFERENCES

- 2015 [1] D. ALLONSIUS AND F. BOYER, *Boundary null-controllability of semi-discrete coupled*
 2016 *parabolic systems in some multi-dimensional geometries*, preprint, (2018). [https://hal.](https://hal.archives-ouvertes.fr/hal-01827044)
 2017 [archives-ouvertes.fr/hal-01827044](https://hal.archives-ouvertes.fr/hal-01827044).
 2018 [2] D. ALLONSIUS, F. BOYER, AND M. MORANCEY, *Spectral analysis of discrete elliptic opera-*
 2019 *tors and applications in control theory*, Numerische Mathematik, 140 (2018), pp. 857–911,
 2020 [doi:10.1007/s00211-018-0983-1](https://doi.org/10.1007/s00211-018-0983-1).
 2021 [3] F. AMMAR KHODJA, A. BENABDALLAH, C. DUPAIX, AND I. KOSTIN, *Null-controllability of some*
 2022 *systems of parabolic type by one control force*, ESAIM Control Optim. Calc. Var., 11 (2005),
 2023 pp. 426–448, [doi:10.1051/cocv:2005013](https://doi.org/10.1051/cocv:2005013).

- 2024 [4] F. AMMAR KHODJA, A. BENABDALLAH, M. GONZÁLEZ-BURGOS, AND L. DE TERESA, *The*
2025 *Kalman condition for the boundary controllability of coupled parabolic systems. Bounds*
2026 *on biorthogonal families to complex matrix exponentials*, J. Math. Pures Appl. (9), 96
2027 (2011), pp. 555–590, doi:10.1016/j.matpur.2011.06.005.
- 2028 [5] F. AMMAR KHODJA, A. BENABDALLAH, M. GONZÁLEZ-BURGOS, AND L. DE TERESA, *Minimal*
2029 *time for the null controllability of parabolic systems: The effect of the condensation index*
2030 *of complex sequences*, Journal of Functional Analysis, 267 (2014), pp. 2077–2151, doi:10.
2031 1016/j.jfa.2014.07.024.
- 2032 [6] F. AMMAR KHODJA, A. BENABDALLAH, M. GONZÁLEZ-BURGOS, AND L. DE TERESA, *New phe-*
2033 *nomena for the null controllability of parabolic systems: minimal time and geometrical de-*
2034 *pendence*, J. Math. Anal. Appl., 444 (2016), pp. 1071–1113, doi:10.1016/j.jmaa.2016.06.058.
- 2035 [7] F. AMMAR KHODJA, A. BENABDALLAH, M. GONZÁLEZ-BURGOS, AND M. MORANCEY, *Quantita-*
2036 *tative fattorini–hautus test and minimal null control time for parabolic problems*, Journal
2037 de Mathématiques Pures et Appliquées, (2018), doi:10.1016/j.matpur.2018.05.006.
- 2038 [8] S. A. AVDONIN AND S. A. IVANOV, *Riesz bases of exponentials and divided differences*, Algebra
2039 i Analiz, 13 (2001), pp. 1–17.
- 2040 [9] K. BEAUCHARD AND P. CANNARSA, *Heat equation on the Heisenberg group: observability and*
2041 *applications*, J. Differential Equations, 262 (2017), pp. 4475–4521, doi:10.1016/j.jde.2016.
2042 12.021.
- 2043 [10] K. BEAUCHARD, P. CANNARSA, AND R. GUGLIELMI, *Null controllability of Grushin-type oper-*
2044 *ators in dimension two*, J. Eur. Math. Soc. (JEMS), 16 (2014), pp. 67–101, doi:10.4171/
2045 JEMS/428.
- 2046 [11] K. BEAUCHARD, J. DARDÉ, AND S. ERVEDOZA, *Minimal time issues for the observabil-*
2047 *ity of Grushin-type equations*. working paper or preprint, Jan. 2018, [https://hal.](https://hal.archives-ouvertes.fr/hal-01677037)
2048 [archives-ouvertes.fr/hal-01677037](https://hal.archives-ouvertes.fr/hal-01677037).
- 2049 [12] K. BEAUCHARD, B. HELFFER, R. HENRY, AND L. ROBBIANO, *Degenerate parabolic operators of*
2050 *Kolmogorov type with a geometric control condition*, ESAIM Control Optim. Calc. Var.,
2051 21 (2015), pp. 487–512, doi:10.1051/cocv/2014035.
- 2052 [13] K. BEAUCHARD, L. MILLER, AND M. MORANCEY, *2D Grushin-type equations: minimal time*
2053 *and null controllable data*, J. Differential Equations, 259 (2015), pp. 5813–5845, doi:10.
2054 1016/j.jde.2015.07.007.
- 2055 [14] V. BERNSTEIN, *Leçons sur les Progrès Récents de la Théorie des Séries de Dirichlet*, Gauthier-
2056 Villars, 1933.
- 2057 [15] F. BOYER AND G. OLIVE, *Approximate controllability conditions for some linear 1D parabolic*
2058 *systems with space-dependent coefficients*, Math. Control Relat. Fields, 4 (2014), pp. 263–
2059 287, doi:10.3934/mcrf.2014.4.263.
- 2060 [16] J.-M. CORON, *Control and nonlinearity*, vol. 136 of Mathematical Surveys and Monographs,
2061 American Mathematical Society, Providence, RI, 2007.
- 2062 [17] S. DOLECKI, *Observability for the one-dimensional heat equation*, Studia Math., 48 (1973),
2063 pp. 291–305.
- 2064 [18] M. DUPREZ, *Controllability of a 2×2 parabolic system by one force with space-dependent cou-*
2065 *pling term of order one*, ESAIM: COCV, (2017), doi:10.1051/cocv/2016061. Forthcoming
2066 paper.
- 2067 [19] M. DUPREZ AND A. KOENIG, *Control of the Grushin equation: non-rectangular control region*
2068 *and minimal time*. working paper or preprint, July 2018, [https://hal.archives-ouvertes.fr/](https://hal.archives-ouvertes.fr/hal-01829294)
2069 [hal-01829294](https://hal.archives-ouvertes.fr/hal-01829294).
- 2070 [20] T. DUYCKAERTS AND L. MILLER, *Resolvent conditions for the control of parabolic equations*, J.
2071 Funct. Anal., 263 (2012), pp. 3641–3673, doi:10.1016/j.jfa.2012.09.003.
- 2072 [21] J. V. EGOROV, *Some problems in the theory of optimal control*, Ž. Vyčisl. Mat. i Mat. Fiz., 3
2073 (1963), pp. 887–904.
- 2074 [22] H. FATTORINI, *Some remarks on complete controllability*, SIAM J. Control, 4 (1966), pp. 686–
2075 694.
- 2076 [23] H. FATTORINI AND D. RUSSELL, *Exact controllability theorems for linear parabolic equations in*
2077 *one space dimension*, Arch. Rational Mech. Anal., 43 (1971), pp. 272–292.
- 2078 [24] H. FATTORINI AND D. RUSSELL, *Uniform bounds on biorthogonal functions for real exponentials*
2079 *with an application to the control theory of parabolic equations*, Quart. Appl. Math., 32
2080 (1974/75), pp. 45–69, doi:10.1090/qam/510972.
- 2081 [25] A. FURSIKOV AND O. IMANUVILOV, *Controllability of evolution equations*, vol. 34 of Lecture
2082 Notes Series, Seoul National University Research Institute of Mathematics Global Analysis
2083 Research Center, Seoul, 1996.
- 2084 [26] L. I. GAL’CHUK, *Optimal control of systems described by parabolic equations*, SIAM J. Control,
2085 7 (1969), pp. 546–558.

- 2086 [27] F. GAUNARD, *Problèmes d'Interpolation dans les Espaces de Paley-Wiener et Applica-*
 2087 *tions en Théorie du Contrôle*, PhD thesis, Université de Bordeaux, 2011. [https://tel.](https://tel.archives-ouvertes.fr/tel-00652210)
 2088 [archives-ouvertes.fr/tel-00652210](https://tel.archives-ouvertes.fr/tel-00652210).
- 2089 [28] B. JACOB, J. R. PARTINGTON, AND S. POTT, *Interpolation by vector-valued analytic functions,*
 2090 *with applications to controllability*, J. Funct. Anal., 252 (2007), pp. 517–549, doi:10.1016/
 2091 [j.jfa.2007.07.008](https://doi.org/10.1016/j.jfa.2007.07.008).
- 2092 [29] B. JACOB, J. R. PARTINGTON, AND S. POTT, *Weighted interpolation in Paley-Wiener spaces*
 2093 *and finite-time controllability*, J. Funct. Anal., 259 (2010), pp. 2424–2436, doi:10.1016/j.
 2094 [jfa.2010.06.014](https://doi.org/10.1016/j.jfa.2010.06.014).
- 2095 [30] B. JACOB, J. R. PARTINGTON, AND S. POTT, *Weighted multiple interpolation and the con-*
 2096 *trol of perturbed semigroup systems*, J. Evol. Equ., 13 (2013), pp. 395–410, doi:10.1007/
 2097 [s00028-013-0184-4](https://doi.org/10.1007/s00028-013-0184-4).
- 2098 [31] B. JACOB, J. R. PARTINGTON, AND S. POTT, *Applications of Laplace-Carleson embeddings*
 2099 *to admissibility and controllability*, SIAM J. Control Optim., 52 (2014), pp. 1299–1313,
 2100 doi:10.1137/120894750.
- 2101 [32] J. B. W. V. JENSEN, *Sur une expression simple du reste dans la formule d'interpolation de*
 2102 *Newton.*, Kjöb. Overs., (1894), pp. 1–7.
- 2103 [33] A. KIRSCH, *An introduction to the mathematical theory of inverse problems*, vol. 120 of
 2104 Applied Mathematical Sciences, Springer, New York, second ed., 2011, doi:10.1007/
 2105 [978-1-4419-8474-6](https://doi.org/10.1007/978-1-4419-8474-6).
- 2106 [34] G. LEBEAU AND L. ROBBIANO, *Contrôle exact de l'équation de la chaleur*, Comm. Partial
 2107 Differential Equations, 20 (1995), pp. 335–356, doi:10.1080/03605309508821097.
- 2108 [35] A. LÓPEZ AND E. ZUAZUA, *Uniform null-controllability for the one-dimensional heat equation*
 2109 *with rapidly oscillating periodic density*, Annales de l'Institut Henri Poincaré (C) Non
 2110 Linear Analysis, 19 (2002), pp. 543–580, doi:10.1016/s0294-1449(01)00092-0.
- 2111 [36] O. LYDIA, *Minimal time of null controllability of two parabolic equations*, preprint, (2018).
 2112 <https://hal.archives-ouvertes.fr/hal-01941299>.
- 2113 [37] L. MILLER, *On the controllability of anomalous diffusions generated by the fractional Laplacian*,
 2114 Math. Control Signals Systems, 18 (2006), pp. 260–271, doi:10.1007/s00498-006-0003-3.
- 2115 [38] G. OLIVE, *Boundary approximate controllability of some linear parabolic systems*, Evol. Equ.
 2116 Control Theory, 3 (2014), pp. 167–189, doi:10.3934/eect.2014.3.167.
- 2117 [39] M. J. D. POWELL, *Approximation theory and methods*, Cambridge University Press,
 2118 Cambridge-New York, 1981.
- 2119 [40] J. PÖSCHEL AND E. TRUBOWITZ, *Inverse spectral theory, volume 130 of Pure and Applied*
 2120 *Mathematics*, Academic Press Inc., Boston, MA, 1987.
- 2121 [41] W. RUDIN, *Real and complex analysis*, McGraw-Hill Book Co., New York, third ed., 1987.
- 2122 [42] E. H. SAMB, *Boundary null-controllability of two coupled parabolic equations: simultaneous*
 2123 *condensation of eigenvalues and eigenfunctions*, preprint, (2018).
- 2124 [43] L. SCHWARTZ, *Étude des sommes d'exponentielles réelles*, NUMDAM, Publications de l'Institut
 2125 de Mathématique de l'Université de Clermont-Ferrand, 1943, [http://www.numdam.org/](http://www.numdam.org/item?id=THESE_1943__259__1_0)
 2126 [item?id=THESE_1943__259__1_0](http://www.numdam.org/item?id=THESE_1943__259__1_0).
- 2127 [44] J. SHACKELL, *Overconvergence of Dirichlet series with complex exponents*, J. Analyse Math.,
 2128 22 (1969), pp. 135–170, doi:10.1007/BF02786787.
- 2129 [45] M. TUCSNAK AND G. WEISS, *Observation and control for operator semigroups*, Birkhäuser
 2130 Advanced Texts: Basler Lehrbücher., Birkhäuser Verlag, Basel, 2009, doi:10.1007/
 2131 [978-3-7643-8994-9](https://doi.org/10.1007/978-3-7643-8994-9).