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1 **A BLOCK MOMENT METHOD TO HANDLE SPECTRAL**
2 **CONDENSATION PHENOMENON IN PARABOLIC CONTROL**
3 **PROBLEMS**

4 ASSIA BENABDALLAH*, FRANCK BOYER†, AND MORGAN MORANCEY*

5 **Abstract.** This article is devoted to the characterization of the minimal null control time for
6 abstract linear control problem. More precisely we aim at giving a precise answer to the following
7 question: what is the minimal time needed to drive the solution of the system starting from any
8 initial condition in a given subspace to zero? Our setting will encompass a wide variety of systems
9 of coupled one dimensional linear parabolic equations with a scalar control.

10 Following classical ideas we reduce this controllability issue to the resolution of a moment problem
11 on the control and provide a new block resolution technique for this moment problem. The obtained
12 estimates are sharp and hold uniformly for a certain class of operators. This uniformity allows various
13 applications for parameter dependent control problems and permits us to deal naturally with the
14 case of algebraically multiple eigenvalues in the underlying generator.

15 Our approach sheds light on a new phenomenon: the condensation of eigenvalues (which can cause
16 a non zero minimal null control time in general) can be somehow compensated by the condensation
17 of eigenvectors. We provide various examples (some are abstract systems, others are actual PDE
18 systems) to highlight those new situations we are able to manage by the block resolution of the
19 moment problem we propose.

20 **Key words.** Control theory; parabolic partial differential equations; minimal null control time;
21 block moment method

22 **AMS subject classifications.** 93B05; 93C20; 93C25; 30E05; 35K90; 35P10

23 **1. Introduction.**

24 **1.1. Problem under study and state of the art.**

25 This paper is concerned with the following abstract linear control system

26 (1)
$$\begin{cases} y'(t) + \mathcal{A}y(t) = \mathcal{B}u(t), \\ y(0) = y_0. \end{cases}$$

27 We are more precisely interested in the minimal time issue for null-controllability,
28 which can be roughly expressed as follows: what is the smallest time $T_0 \geq 0$ such
29 that, for any $T > T_0$, for any initial condition y_0 , there exists a control u such that
30 the associated solution of (1) satisfies $y(T) = 0$? Under quite general assumptions,
31 we shall give formulas (that are reasonably explicit and usable) for such a minimal
32 control time. The precise notion of solution as well as the wellposedness result for
33 such system will be detailed below (see Propositions 1.1 and 1.2).

34 We will consider assumptions on the operator \mathcal{A} that will relate (1) to parabolic
35 evolution equations. Thus, due to regularizing properties, one cannot expect to reach
36 any target in the state space and should rather try to reach any trajectory. By
37 linearity, this is equivalent to the aforementioned null-controllability property (see for
38 instance [19, Secs. 2.3 and 2.5]).

39 Pioneering works for null-controllability of a scalar one dimensional heat equa-
40 tions are due to H.O. Fattorini and D.L. Russell [26, 27]. For instance, they proved

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41 null-controllability of the one dimensional heat equation using a nonhomogeneous
42 boundary condition as a control. For this purpose, they give a direct strategy re-
43 ducing the null-controllability property to a moment problem that the control should
44 satisfy (see Section 1.4 for a presentation of the moment problem). The moment
45 method they propose consists in solving this problem using a biorthogonal family in
46 $L^2(0, T)$ to the family of exponentials associated to the eigenvalues of \mathcal{A}^* . Let us
47 mention that this idea of reducing a (optimal) control problem to a moment problem
48 is already present in the works [24] by J.V. Egorov and [30] by L.I. Gal'chuk.

49 Later on, A.V. Fursikov, O.Yu. Imanuvilov [29] and G. Lebeau, L. Robbiano [38]
50 used Carleman estimates to solve the boundary and internal null-controllability prob-
51 lem of the heat equation in any space dimension. These two papers have gener-
52 ated plenty of null-controllability results for various parabolic problems. The com-
53 mon qualitative behavior of these results is that for scalar parabolic equations null-
54 controllability holds in arbitrary time (i.e. $T_0 = 0$) and without any restriction on the
55 control domain.

56 Among all of these results let us mention the peculiar work [20] by S. Dolecki.
57 He considered a one dimensional heat equation, with a scalar control located at one
58 point inside the space interval, and proved that choosing suitably the location of this
59 control point one can achieve any value in $[0, +\infty]$ for the minimal null-control time
60 T_0 . Until the years 2000's this work seemed to be considered too peculiar and the
61 possible presence of a positive minimal null-control time (a very natural property
62 in the hyperbolic case) was expected to be generically not possible in a parabolic
63 setting. However, this point of view has been reconsidered recently in various works
64 as for instance: [3] for abstract control problems, [5] for the abstract problem (1) with
65 applications to systems of one dimensional coupled parabolic equations, and [11] for
66 a degenerate parabolic two dimensional equation of Grushin type.

67 Since then, the minimal null-control time property was investigated on various
68 examples, still in the setting of coupled one dimensional parabolic systems [6, 21,
69 42, 46] or in the setting of degenerate parabolic scalar equations [10, 12, 13, 14, 22].
70 For coupled parabolic systems a geometric control condition may also be needed for
71 approximate controllability to hold [18, 41], proving once again that hyperbolic-like
72 behavior can be observed in the parabolic setting.

73 Concerning the study of the abstract control problem (1), some characterizations
74 are provided in the series of works [31, 32, 33, 34, 35] using the formalism of Carleson
75 measures. However the precise question of an abstract characterization of the minimal
76 null-control time has not been much considered. A formula has been given for the
77 minimal null-control time of system (1), in [5], in a setting encompassing coupled
78 one dimensional parabolic equations with a scalar control. Its value depends on the
79 condensation index of the eigenvalues of \mathcal{A}^* (see Section 7.5 for a precise definition)
80 and the observation of the associated eigenvectors. In this work the authors assume
81 that the eigenvectors form a Riesz basis of the state space. Let us also mention the
82 work [7] where the null-control time is studied through a resolvent-like inequality
83 (introduced in [23]) that is a quantification of the well-known Fattorini-Hautus test
84 for approximate controllability (see [25, 41]). It is an abstract characterization that
85 might not be easily computable on actual examples but provides a common setting
86 for all the previous examples exhibiting a positive null-control time. The last two
87 mentioned results also rely on the moment method. Note that, even if the Carleman
88 approach is very powerful, it does not seem to be applicable to all the systems of
89 interest: in many situations (including the ones discussed in Section 5.2) the moment

90 method is still the only successful technique up to now.

91 To highlight the limitations of the existing literature on such problems and the
 92 improvements we propose, let us consider the following control problem

$$93 \quad \begin{cases} \partial_t y + \begin{pmatrix} -\partial_{xx} & 1 \\ 0 & -\partial_{xx} + c(x) \end{pmatrix} y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & (t, x) \in (0, T) \times (0, 1), \\ y(t, 0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, & y(t, 1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & t \in (0, T), \\ y(0, x) = y_0(x), & x \in (0, 1), \end{cases}$$

94 where $c \in L^2(0, 1; \mathbb{R})$ is a given potential. We insist on the fact that our goal is
 95 not to study this particular example but to develop a general characterization. The
 96 application to this particular example is detailed in Section 5.2. The study of the
 97 minimal null-control time for this system for an arbitrary potential c is not covered
 98 by the literature for several reasons.

- 99 • First, depending on c , the underlying operator can have geometrically double
 100 eigenvalues. This induces (a finite number of) non-observable modes and thus
 101 prevents even approximate controllability. We thus propose to extend the
 102 study of the minimal null-control time to a given subspace of initial conditions.
 103 This allows to still analyze the controllability properties in this case.
- 104 • Even if the potential c is such that the eigenvalues are geometrically simple
 105 it can happen that some of them are algebraically double. In this case, to
 106 the best of our knowledge, the only existing results are [28, 4] which ensures
 107 null-controllability in arbitrary time if the eigenvalues are well separated (i.e.
 108 satisfy the classical gap condition recalled in (29)).
- 109 • Finally, if the potential c is such that the eigenvalues are geometrically and
 110 algebraically simple, to the best of our knowledge, the only existing result
 111 can be found in [5]. Under an extra assumption (on the observability of
 112 eigenfunctions), it provides null-controllability at any time T satisfying

$$113 \quad T > T^* = \limsup_{\lambda \in \sigma(\mathcal{A}^*)} \frac{-\ln \text{dist}(\lambda, \sigma(\mathcal{A}^*) \setminus \{\lambda\})}{\lambda}.$$

114 However, their arguments to disprove null-controllability at time $T < T^*$
 115 cannot be applied in this example when the potential c is such that the family
 116 of eigenvectors forms a complete family but is not a Riesz basis. Therefore, the
 117 above formula for T^* may dramatically overestimate the actual null-control
 118 time for the system. We will see in Section 5.2.1 that it may happen that
 119 $T^* = +\infty$ whereas the system is actually null-controllable at any time $T > 0$.

120 We will use quite general assumptions in our analysis answering all these concerns
 121 in the case of scalar control (see [17] for an extension to non-scalar control). Doing so,
 122 we will prove that the difference between the Riesz basis assumption and the complete
 123 family assumption for the eigenvectors is not only technical and a new phenomenon
 124 can appear: the condensation of eigenvalues can be compensated by the condensation
 125 of eigenvectors.

126 We continue this introduction by stating more precisely the problem under con-
 127 sideration and the obtained results.

128 1.2. Functional setting.

129 Let X be a separable Hilbert space, whose inner product and norm are denoted
130 by $\langle \bullet, \bullet \rangle$ and $\|\bullet\|$ respectively. We shall systematically identify X to its dual through
131 the Riesz theorem. Let $(\mathcal{A}, D(\mathcal{A}))$ be an unbounded operator in X such that $-\mathcal{A}$
132 generates a C^0 -semigroup in X and $(\mathcal{A}^*, D(\mathcal{A}^*))$ its adjoint in X . Up to a suitable
133 translation, we can assume that 0 is in the resolvent set of \mathcal{A} . We denote by X_1 (resp.
134 X_1^*) the Hilbert space $D(\mathcal{A})$ (resp. $D(\mathcal{A}^*)$) equipped with the norm $\|x\|_1 := \|\mathcal{A}x\|$
135 (resp. $\|x\|_{1^*} := \|\mathcal{A}^*x\|$). We define X_{-1} as the completion of X with respect to the
136 norm

$$137 \quad \|y\|_{-1} := \sup_{z \in X_1^*} \frac{\langle y, z \rangle}{\|z\|_{1^*}}.$$

138 Notice that X_{-1} is isometrical to the topological dual of X_1^* using X as a pivot space
139 (see for instance [49, Proposition 2.10.2]); the corresponding duality bracket will be
140 denoted by $\langle \bullet, \bullet \rangle_{-1, 1^*}$.

141 Let U be an Hilbert space (that we will identify to its dual) and $\mathcal{B} : U \rightarrow X_{-1}$ be
142 a linear continuous control operator. We denote by $\mathcal{B}^* : X_1^* \rightarrow U$ its adjoint in the
143 duality described above.

144 **PROPOSITION 1.1.** *Under the above assumptions, for any $T > 0$, any $y_0 \in X_{-1}$,
145 and any $u \in L^2(0, T; U)$, there exists a unique $y \in C^0([0, T]; X_{-1})$ solution to (1) in
146 the sense that it satisfies for any $t \in [0, T]$ and any $z_t \in X_1^*$,*

$$147 \quad (2) \quad \langle y(t), z_t \rangle_{-1, 1^*} - \langle y_0, e^{-t\mathcal{A}^*} z_t \rangle_{-1, 1^*} = \int_0^t \left(u(s), \mathcal{B}^* e^{-(t-s)\mathcal{A}^*} z_t \right)_U ds.$$

148 Moreover there exists $C_T > 0$ such that

$$149 \quad \sup_{t \in [0, T]} \|y(t)\|_{-1} \leq C_T (\|y_0\|_{-1} + \|u\|_{L^2(0, T; U)}).$$

150 The proof of this result is recalled in Appendix 7.1. Let us mention that this notion of
151 solution is very weak. In most works concerning controllability properties for abstract
152 systems like (1), an extra admissibility assumption is made on the control operator \mathcal{B}
153 to ensure more regularity for the solutions. Note however that this is not mandatory
154 to prove wellposedness of the system in the weak sense above. We will discuss below
155 the regularity properties of the control problem.

156 Let $(X_\diamond^*, \|\cdot\|_{\diamond^*})$ be an Hilbert space such that $X_1^* \subset X_\diamond^* \subset X$ with dense and
157 continuous embeddings. We assume that X_\diamond^* is stable by the semigroup generated by
158 $-\mathcal{A}^*$ (see Remark 1.1). We also define $X_{-\diamond}$ as the subspace of X_{-1} defined by

$$159 \quad X_{-\diamond} := \left\{ y \in X_{-1}; \|y\|_{-\diamond} := \sup_{z \in X_1^*} \frac{\langle y, z \rangle_{-1, 1^*}}{\|z\|_{\diamond^*}} < +\infty \right\},$$

160 which is also isometrical to the dual of X_\diamond^* with X as a pivot space. The corresponding
161 duality bracket will be denoted by $\langle \bullet, \bullet \rangle_{-\diamond, \diamond^*}$. Thus, we end up with the following five
162 functional spaces

$$163 \quad X_1^* \subset X_\diamond^* \subset X \subset X_{-\diamond} \subset X_{-1}.$$

164 We say that the control operator \mathcal{B} is an admissible control operator for (1) with
165 respect to the space $X_{-\diamond}$ if for any $T > 0$ there exists $C_T > 0$ such that

$$166 \quad (3) \quad \int_0^T \left\| \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} z \right\|_U^2 dt \leq C_T \|z\|_{\diamond^*}^2, \quad \forall z \in X_1^*.$$

167 Notice that if (3) holds for some $T > 0$ it holds for any $T > 0$. The admissibility
 168 condition (3) implies that, by density, we can give a meaning to the map

$$169 \quad (t \mapsto \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} z) \in L^2(0, T; U),$$

170 for any $z \in X_\diamond^*$.

171 In this setting, following the lines of [19, Theorem 2.37] we obtain the following
 172 regularity result for the solutions.

173 **PROPOSITION 1.2.** *Assume that (3) holds. Then, for any $T > 0$, any $y_0 \in X_{-\diamond}$,
 174 and any $u \in L^2(0, T; U)$, there exists a unique $y \in C^0([0, T]; X_{-\diamond})$ solution to (1) in
 175 the sense that it satisfies for any $t \in [0, T]$ and any $z_t \in X_\diamond^*$,*

$$176 \quad \langle y(t), z_t \rangle_{-\diamond, \diamond} - \langle y_0, e^{-t\mathcal{A}^*} z_t \rangle_{-\diamond, \diamond} = \int_0^t \left(u(s), \mathcal{B}^* e^{-(t-s)\mathcal{A}^*} z_t \right)_U ds.$$

177 Moreover there exists $C_T > 0$ such that

$$178 \quad \sup_{t \in [0, T]} \|y(t)\|_{-\diamond} \leq C_T (\|y_0\|_{-\diamond} + \|u\|_{L^2(0, T; U)}).$$

REMARK 1.1. *Note that a similar regularity result holds if we don't assume that
 X_\diamond^* is stable by the semigroup generated by $-\mathcal{A}^*$ except that we need to restrict our-
 selves to initial data $y_0 \in X$. In that case the solution satisfies for any $t \in [0, T]$ and
 any $z_t \in X_\diamond^*$,*

$$\langle y(t), z_t \rangle_{-\diamond, \diamond} - \langle y_0, e^{-t\mathcal{A}^*} z_t \rangle_{-\diamond, \diamond} = \int_0^t \left(u(s), \mathcal{B}^* e^{-(t-s)\mathcal{A}^*} z_t \right)_U ds,$$

179

$$180 \quad \sup_{t \in [0, T]} \|y(t)\|_{-\diamond} \leq C_T (\|y_0\| + \|u\|_{L^2(0, T; U)}).$$

181 **REMARK 1.2.** *The case where $X_\diamond^* = X_1^*$ means that we do not have any additional
 182 regularity property for \mathcal{B} . Conversely, the case $X_\diamond^* = X$ means that we have the best
 183 regularity we can hope for system (1) (this is the usual definition of an admissible
 184 control operator as in [19, 49]).*

185 To give more accurate results, we aim at analyzing the minimal null-control time
 186 problem for each specified set of initial data. This is the object of the following
 187 definition.

188 **DEFINITION 1.1.** *Let Y_0 be a closed subspace of $X_{-\diamond}$.*

189 *We say that system (1) is null-controllable at time T from Y_0 if for any $y_0 \in Y_0$
 190 there exists $u \in L^2(0, T; U)$ such that the associated solution of (1) satisfies $y(T) = 0$.*

191 As a specific choice of Y_0 one can think of $Y_0 = X_{-\diamond}$, in which case we recover the
 192 classical notion of null-controllability. On the opposite side, if Y_0 is a one dimensional
 193 subspace $Y_0 = \text{Span}\{y_0\}$, then the notion above amounts to consider only the null-
 194 controllability of the system for that particular initial condition y_0 .

195 From now on, we will assume that the space Y_0 is given, and we denote by P_{Y_0} the
 196 orthogonal projection onto Y_0 with respect to $\|\bullet\|_{-\diamond}$ and by $P_{Y_0}^* \in L(X_\diamond^*)$ its adjoint
 197 in the duality $X_{-\diamond}, X_\diamond^*$. Notice that these definitions yield

$$198 \quad (4) \quad \|P_{Y_0}^* z\|_{\diamond^*} \leq \|z\|_{\diamond^*}, \quad \forall z \in X_\diamond^*.$$

199

Notations. We give here some notations that will be used throughout this article.

- For any integers $a, b, c \in \mathbb{N}$, we shall define the following subsets of \mathbb{N} :

$$\llbracket a, b \rrbracket := [a, b] \cap \mathbb{N},$$

$$\llbracket a, b \rrbracket_{\neq c} := \llbracket a, b \rrbracket \setminus \{c\}.$$

200

- For any complex number $\mu \in \mathbb{C}$ we define $e_\mu : (0, +\infty) \rightarrow \mathbb{C}$ to be the exponential function

201

$$(5) \quad e_\mu : s \mapsto e^{-\mu s}.$$

202

203

- We shall denote by $C_{\gamma_1, \dots, \gamma_l} > 0$ a constant possibly varying from one line to another but depending only on the parameters $\gamma_1, \dots, \gamma_l$.

204

205

- For any multi-index $\alpha \in \mathbb{N}^n$, we denote its length by $|\alpha| = \sum_{j=1}^n \alpha_j$ and its maximum by $|\alpha|_\infty = \max_{j \in \llbracket 1, n \rrbracket} \alpha_j$.

206

207

For $\alpha, \mu \in \mathbb{N}^n$, we say that $\mu \leq \alpha$ if and only if $\mu_j \leq \alpha_j$ for any $j \in \llbracket 1, n \rrbracket$.

208

- For any finite subset $A \subset \mathbb{C}$, we will make use of the polynomial P_A defined by

209

$$(6) \quad P_A(x) := \prod_{\mu \in A} (x - \mu).$$

210

It satisfies in particular, for any $\lambda \in A$,

211

$$P'_A(\lambda) = \prod_{\substack{\mu \in A \\ \mu \neq \lambda}} (\lambda - \mu).$$

212

213

1.3. Presentation of the main results.

214

1.3.1. Spectral assumptions. In addition to the hypotheses described in the introduction that are necessary for the well-posedness and regularity of our control problem, we shall make now the following structural assumptions.

215

216

- First of all, we shall only consider scalar controls in this paper, that is $U = \mathbb{R}$.
- We assume that the spectrum of \mathcal{A}^* is only made of a countable number of geometrically simple eigenvalues denoted by Λ . We refer to Section 6.3 for a discussion on this assumption.

217

218

219

220

We shall also assume for simplicity that the eigenvalues are all real (see however the discussion in Section 6.1) and that

221

222

$$(7) \quad \Lambda \subset [1, +\infty).$$

223

Note that, if (7) does not hold, we can replace \mathcal{A} by $\mathcal{A} + \gamma$ for $\gamma > 0$ large enough and find an associated null-control u . A null-control for the original problem is then given by $t \mapsto e^{\gamma t} u(t)$ and we can explicitly bound its cost with respect to the parameters γ and T .

224

225

226

227

- For any eigenvalue $\lambda \in \Lambda$, we denote by $\alpha_\lambda \geq 1$ its algebraic multiplicity and we assume that there exists an integer $\eta \geq 1$ such that $\alpha_\lambda \leq \eta$ for any $\lambda \in \Lambda$.
- The main structural assumptions on the eigenvalues Λ we shall make in this paper are the following:

228

229

230

231

- Asymptotic behavior:

232

$$(8) \quad \sum_{\lambda \in \Lambda} \frac{1}{\lambda} < +\infty.$$

233

234 – **Weak gap condition** with parameters $p \in \mathbb{N}$ and $\rho > 0$:

235 (9)
$$\# \left(\Lambda \cap [\mu, \mu + \rho] \right) \leq p, \quad \forall \mu \in [0, +\infty).$$

236 In the case $p = 1$, the weak gap condition above simply reduces to

237
$$|\lambda - \lambda'| > \rho, \quad \forall \lambda, \lambda' \in \Lambda, \lambda \neq \lambda',$$

238 which is the usual gap condition used for instance in [26]. If the spectrum Λ
 239 is increasingly indexed as $\Lambda = (\lambda_m)_{m \geq 1}$ the weak gap condition (9) reads

240
$$\lambda_{m+p} - \lambda_m > \rho, \quad \forall m \geq 1.$$

241 As we will use a different labelling of the spectrum in this paper we shall not
 242 use these notations anymore in what follows.

243 • We denote by $(\phi_\lambda^0)_{\lambda \in \Lambda}$ an associated family of eigenvectors of \mathcal{A}^* . These
 244 eigenvectors are chosen to be normalized in X_\diamond^* .

245 As we are interested in null-controllability properties of system (1), we will
 246 first assume that

247 (10)
$$\mathcal{B}^* \phi_\lambda^0 \neq 0, \quad \text{for any } \lambda \in \Lambda.$$

248 This is a necessary condition for the approximate controllability of system (1),
 249 and is therefore mandatory if we expect null-controllability to hold. In our
 250 setting, the assumption (10) is also a sufficient condition for approximate
 251 controllability (see [25, 41]).

252 When the considered set of initial data Y_0 is not the whole space $X_{-\diamond}$, the
 253 approximate controllability condition (10) can be too strong and we can relax
 254 it. We will discuss this point in Section 6.2.

255 • For each $\lambda \in \Lambda$, we denote by $(\phi_\lambda^l)_{l \in \llbracket 1, \alpha_\lambda - 1 \rrbracket}$ a Jordan chain associated with
 256 ϕ_λ^0 , that is a family satisfying

257
$$\mathcal{A}^* \phi_\lambda^l = \lambda \phi_\lambda^l + \phi_\lambda^{l-1}, \quad \forall l \in \llbracket 1, \alpha_\lambda - 1 \rrbracket.$$

258 By (10), we may uniquely determine such Jordan chain if we impose in addi-
 259 tion that the generalized eigenvectors satisfy

260 (11)
$$\mathcal{B}^* \phi_\lambda^l = 0, \quad \forall l \in \llbracket 1, \alpha_\lambda - 1 \rrbracket.$$

261 This particular choice of the Jordan chain is not mandatory but will sim-
 262 plify the forthcoming computations. In the case were the eigenvalues are
 263 algebraically simple ($\eta = 1$) we drop the superscript 0 for the eigenvectors.

• We introduce the notation

$$\Phi := \{ \phi_\lambda^l, \lambda \in \Lambda, l \in \llbracket 0, \alpha_\lambda - 1 \rrbracket \},$$

264 for the family of all the (generalized) eigenvectors of \mathcal{A}^* . We assume that Φ
 265 is complete in X_\diamond^* i.e. for any $y \in X_{-\diamond}$, we have

266 (12)
$$\left(\langle y, \phi \rangle_{-\diamond, \diamond} = 0, \quad \forall \phi \in \Phi \right) \implies y = 0.$$

267 We emphasize the fact that we will not make any additional assumptions on the
 268 family Φ . This is a very important difference with related results in the literature
 269 in which, most of the time, it is assumed that Φ forms a Riesz basis of X_\diamond^* . This is
 270 discussed in Sections 1.3.4 and 3.

271 In the forthcoming paper [17], we will study the extension of our analysis to the
 272 case of possibly infinite dimensional controls.

273 **1.3.2. Groupings of eigenvalues.** To introduce our formula for the minimal
 274 null-control time it is convenient to define adapted groupings for the spectrum Λ . We
 275 highlight that this notion does not exactly coincide with the condensation groupings
 276 introduced by Bernstein [15], even though it is closely related.

277 **DEFINITION 1.2.** *Let $p \in \mathbb{N}^*$ and $r, \rho > 0$. A sequence of sets $(G_k)_{k \geq 1} \subset \mathcal{P}(\Lambda)$
 278 is said to be a grouping for Λ with parameters p, r, ρ , and we will write $(G_k)_{k \geq 1} \in$
 279 $\mathcal{G}(\Lambda, p, r, \rho)$, if it is a covering of Λ*

$$280 \quad \Lambda = \bigcup_{k \geq 1} G_k,$$

281 *with the additional properties that for every $k \geq 1$,*

$$282 \quad g_k := \#G_k \leq p,$$

$$283 \quad \sup(G_k) < \inf(G_{k+1}),$$

$$284 \quad \text{dist}(G_k, G_{k+1}) \geq r,$$

$$285 \quad (13)$$

287 *and*

$$288 \quad (14) \quad \text{diam } G_k < \rho.$$

289 We prove in Appendix (Proposition 7.1) that such a grouping always exists for
 290 any Λ satisfying the weak gap condition (9).

291 Once we are given such a grouping, we shall always adopt the following labelling
 292 of the elements of Λ

$$293 \quad G_k = \{\lambda_{k,1}, \dots, \lambda_{k,g_k}\}$$

294 with $\lambda_{k,1} < \dots < \lambda_{k,g_k}$, and the (generalized) eigenvectors will be relabelled accord-
 295 ingly

$$296 \quad \phi_{k,j}^l := \phi_{\lambda_{k,j}}^l, \quad \forall k \geq 1, \forall j \in \llbracket 1, g_k \rrbracket, \forall l \in \llbracket 0, \alpha_{k,j} - 1 \rrbracket,$$

297 where in the same fashion $\alpha_{k,j} := \alpha_{\lambda_{k,j}}$. For any $k \geq 1$, we gather the multiplicities
 298 associated with the elements of G_k in the multi-index $\alpha_k = (\alpha_{\lambda_{k,1}}, \dots, \alpha_{\lambda_{k,g_k}}) \in \mathbb{N}^{g_k}$.

299 **REMARK 1.3.** *The condition $\lambda_{k,1} < \dots < \lambda_{k,g_k}$ is convenient to treat the abstract
 300 problem (1) but might not be convenient in actual examples. As all the estimates in
 301 our analysis will depend on the parameters p and ρ , the eigenvalues inside a same
 302 group are mostly interchangeable and thus the increasing labelling is not needed.*

303 **1.3.3. Minimal control time definition.** From now on, we assume given a
 304 grouping $(G_k)_k$ in $\mathcal{G}(\Lambda, p, r, \rho)$. Thanks to the assumption (10), we can define the
 305 following family of elements in X_\diamond^*

$$306 \quad (15) \quad \psi_{k,j}^l := \frac{P_{Y_0}^*(\phi_{k,j}^l)}{\mathcal{B}^* \phi_{k,j}^0}, \quad \forall k \geq 1, \forall j \in \llbracket 1, g_k \rrbracket, \forall l \in \llbracket 0, \alpha_{k,j} - 1 \rrbracket.$$

307 Let

$$308 \quad (16) \quad T_0(Y_0) := \limsup_{k \rightarrow \infty} \frac{\ln \left(\max_{\mu \leq \alpha_k} \left\| \psi[\lambda_{k,1}^{(\mu_1)}, \dots, \lambda_{k,g_k}^{(\mu_{g_k})}] \right\|_{\diamond^*} \right)}{\lambda_{k,1}}$$

309 where the notation $\psi[\dots]$ stands for the generalized divided differences (see Section
310 7.3.2, in particular Proposition 7.7). From Proposition 7.11, notice that the quantity
311 $\psi[\lambda_{k,1}^{(\mu_1)}, \dots, \lambda_{k,g_k}^{(\mu_{g_k})}]$ appearing in the previous definition is a linear combination of the
312 elements

$$313 \quad \left\{ \psi_{k,j}^l ; j \in \llbracket 1, g_k \rrbracket, l \in \llbracket 0, \alpha_{k,j} - 1 \rrbracket \right\}$$

314 whose coefficients can be explicitly computed on actual control problems (see Sec-
315 tion 5) and that only depends on the group G_k and on the multiplicity multi-index
316 μ .

317 In the simpler case where the eigenvalues are assumed to be algebraically simple
318 (i.e. $\eta = 1$) we can immediately give a more explicit formula for $T_0(Y_0)$. Indeed, in this
319 case one recovers the standard divided differences (whose definition and properties are
320 recalled in Section 7.3.1) and thus

$$321 \quad (17) \quad T_0(Y_0) = \limsup_{k \rightarrow \infty} \frac{\ln \left(\max_{\substack{m, l \in \llbracket 1, g_k \rrbracket \\ m \leq l}} \left\| \psi[\lambda_{k,m}, \dots, \lambda_{k,l}] \right\|_{\diamond^*} \right)}{\lambda_{k,1}}.$$

322 Then, using Corollary 7.1 and (14) it comes that the computation of all those divided
323 differences is not needed and the formula reduces to

$$324 \quad (18) \quad T_0(Y_0) = \limsup_{k \rightarrow \infty} \frac{\ln \left(\max_{l \in \llbracket 1, g_k \rrbracket} \left\| \psi[\lambda_{k,1}, \dots, \lambda_{k,l}] \right\|_{\diamond^*} \right)}{\lambda_{k,1}}$$

$$325 \quad (19) \quad = \limsup_{k \rightarrow \infty} \frac{\ln \left(\max_{l \in \llbracket 1, g_k \rrbracket} \left\| \sum_{j=1}^l \frac{\psi_{k,j}}{\prod_{i \in \llbracket 1, l \rrbracket, i \neq j} (\lambda_{k,j} - \lambda_{k,i})} \right\|_{\diamond^*} \right)}{\lambda_{k,1}}$$

327 where the last equality comes from the use of Newton formula (see Proposition 7.3).

328 **REMARK 1.4.** *The definition above corresponds to a given grouping of the spec-*
329 *trum, however the minimal null-control result stated in Theorem 1.1 will show that its*
330 *value does not depend on this particular choice of a grouping. As a consequence, for*
331 *specific examples, one can compute the minimal null control time $T_0(Y_0)$ using any*
332 *convenient such grouping in a class $\mathcal{G}(\Lambda, p, r, \rho)$.*

333 For the sake of simplicity, for any $y_0 \in X_{-\diamond}$ we denote by $T_0(y_0)$ the quantity
334 $T_0(\text{Span}(y_0))$. Of course, we have the following proposition relating $T_0(Y_0)$ and $T_0(y_0)$
335 for $y_0 \in Y_0$.

336 **PROPOSITION 1.3.** *For any closed subspace $Y_0 \subset X_{-\diamond}$,*

$$337 \quad \sup_{y_0 \in Y_0} T_0(y_0) = T_0(Y_0).$$

338 This assertion is proved in Section 7.4.

339 REMARK 1.5. Let us discuss the sign of $T_0(Y_0)$.

- 340 • In the case $Y_0 = X_{-\diamond}$ (the operator $P_{Y_0}^*$ thus reduces to the identity), the
- 341 minimal time $T_0(Y_0)$ is always non-negative. Indeed, from the case $\mu =$
- 342 $(1, 0, \dots, 0)$ in the definition (16) of T_0 we have that

$$343 \quad T_0(X_{-\diamond}) \geq \limsup_{k \rightarrow \infty} \frac{\ln \frac{\|\phi_{k,1}^0\|_{\diamond^*}}{|\mathcal{B}^* \phi_{k,1}^0|}}{\lambda_{k,1}}.$$

344 From the admissibility condition (3) applied to $z = \phi_{k,1}^0$, we deduce the fol-

345 lowing upper bound $|\mathcal{B}^* \phi_{k,1}^0| \leq C \sqrt{\lambda_{k,1}} \|\phi_{k,1}^0\|_{\diamond^*}$. Thus,

$$346 \quad \limsup_{k \rightarrow \infty} \frac{\ln \frac{\|\phi_{k,1}^0\|_{\diamond^*}}{|\mathcal{B}^* \phi_{k,1}^0|}}{\lambda_{k,1}} \geq 0,$$

347 which proves that $T_0(X_{-\diamond}) \in [0, +\infty]$.

- 348 • In the general case where Y_0 is a strict closed subspace of $X_{-\diamond}$, it may happen
- 349 that $T_0(Y_0) < 0$.

For instance, if we choose $y_0 \in X_1$ to be an eigenvector of \mathcal{A} for an eigenvalue $\lambda \in \Lambda$, then we have $T_0(y_0) = -\infty$. Indeed, we first observe that

$$\langle y_0, \phi_{\lambda'}^0 \rangle_{-\diamond, \diamond} = (y_0, \phi_{\lambda'}^0) = 0, \quad \forall \lambda' \in \Lambda, \lambda' \neq \lambda,$$

350 which implies, with $Y_0 = \text{Span}(y_0)$ that $P_{Y_0}^* \phi_{\lambda'}^0 = 0$ for any $\lambda' \neq \lambda$. We

351 deduce that the logarithms in the definition of $T_0(y_0)$ are all equal to $-\infty$ for

352 k large enough.

353 **1.3.4. Null-controllability result.** The main result of this paper reads as fol-

354 lows (see also the extension discussed in Section 6.1).

355 THEOREM 1.1. Assume that the operators \mathcal{A} and \mathcal{B} satisfy the assumptions given

356 in Section 1.3.1. Let $T > 0$ and $T_0(Y_0)$ be defined by (16). Then,

- 357 i. If $T_0(Y_0) < +\infty$ and $T > T_0(Y_0)$, the system (1) is null-controllable from Y_0
- 358 at time T .
- 359 ii. If $T_0(Y_0) > 0$ and $T < T_0(Y_0)$, the system (1) is not null-controllable from Y_0
- 360 at time T .

361 Let us briefly mention that our strategy of proof relies on an adapted block resolu-

362 tion of the associated moment problem (see Theorems 2.1 and 4.1). In the case of

363 spectral condensation this new method of resolution ensures sharper results than the

364 one given by standard biorthogonal families. However, as a by-product, in the case

365 of algebraically simple eigenvalues we recover the known optimal estimates for such

366 biorthogonal families (see Corollary 2.1). In the case of algebraically multiple eigenval-

367 ues we provide new estimates for such biorthogonal families (see Corollary 4.1). Before

368 describing with more details this strategy of proof let us make some comments.

- 369 • There are settings in which formulas for the minimal null-control time are
- 370 already known in the literature for instance when the eigenvalues are alge-
- 371 braically simple and:
 - 372 – when the condensation index of Λ (see Appendix 7.5 for a precise defi-
 - 373 nition) is equal to 0 (see [7, Remark 1.15]);
 - 374 – or when the family $(\phi_\lambda)_{\lambda \in \Lambda}$ of eigenvectors forms a Riesz basis of X_\diamond^*
 - 375 (see [5]).

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Obviously, in those settings we recover the known expressions. This is discussed in Section 3.1 and Section 3.2.

- However, we also prove that the Riesz basis assumption considered in [5] is not only technical. More precisely, we show in Proposition 3.2, that if the Riesz basis assumption does not hold, then the actual minimal control time is less or equal than the value T^* given by the formula in this reference (see (61)). Moreover, we present in Section 5.1, a few examples that are built such that the value of T^* is any chosen element of $[0, +\infty]$ whereas the minimal null-control time $T_0(X_{-\infty})$ is in fact 0.

This highlights a new phenomenon: when $(\phi_\lambda)_{\lambda \in \Lambda}$ does not form a Riesz basis, it may happen that the eigenvectors condensate (or more precisely the eigenvectors normalized with respect to the observation i.e. $\frac{\phi_\lambda}{B^* \phi_\lambda}$) and this condensation can compensate for the condensation of eigenvalues.

- The weak gap condition (9) is particularly well adapted to the applications we have in mind, namely coupled one dimensional parabolic equations in which case the spectrum is given by a finite union of sequences satisfying a classical gap condition (see for instance Lemma 2.1).

The restriction to the one dimensional case in those applications comes from the assumption (8). Although this assumption can be seen as a restriction due to the use of moment method, as we are considering scalar controls ($U = \mathbb{R}$) it is also a necessary null-controllability condition (see for instance [40, Appendix A]).

- As we precised the space of initial conditions in this study of minimal null-control time, it directly comes that finite linear combination of eigenvectors are null-controllable in arbitrary small time: the existence of positive minimal null-control time is definitely a high-frequency phenomenon as already observed in Remark 1.5.

- In this article we not only prove Theorem 1.1 but we also develop a new strategy to solve moments problem: the block moment method presented in Section 1.4. The resolution of these problems (see Theorems 2.1 and 4.1) is done with precise estimates. This not only leads to the construction of a control but also to uniform estimates, with respect to Λ in a certain class, on this control (see Corollaries 2.2 and 2.3). Those uniform estimates are important in various contexts when one wants to achieve bounds on the control for parameter-dependent problems (see for instance [1, 2] for an example in numerical analysis of null-controllability problems, or [39] for an example in oscillating coefficient problems). Thus, the block moment method can be of interest to study parameter-dependent problems even in the presence of a strong gap condition, in particular in the case where the strong gap badly behaves with respect to the parameter. Actually, this strategy has already been applied in [16] to study the boundary controllability of a coupled parabolic system with Robin boundary conditions, uniformly with respect to the Robin parameters. Moreover, this uniformity property will be crucial in Section 4 to infer the results on multiple eigenvalues from the ones on simple eigenvalues.

1.3.5. Structure of the article.

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We end this introduction by describing the global strategy used to prove Theorem 1.1 and giving some further bibliographical comments. Section 2 is dedicated to the proof of Theorem 1.1 in the case of algebraically simple eigenvalues. We provide in Section 3 a comparison of our results with available results of the literature. In

425 Section 4 we prove that the uniform estimates obtained in Section 2 allow to prove
 426 Theorem 1.1 in the general case of algebraically multiple eigenvalues. To highlight
 427 the new cases and phenomenon covered by our analysis we present different examples
 428 in Section 5. Then we propose some extensions in Section 6. To ease the reading we
 429 gather various technical results in Section 7.

430 1.4. Strategy of proof.

431 The proof of the positive controllability result (that is point i. of Theorem 1.1)
 432 relies on a block resolution of the moment problem. Let us give more details about
 433 this strategy.

434 Let $y_0 \in Y_0$ and $u \in L^2(0, T; \mathbb{R})$ given. Using Proposition 1.2, it comes that
 435 $y(T) = 0$ if and only if the control u satisfies

$$436 \quad (20) \quad - \left\langle y_0, e^{-T\mathcal{A}^*} \phi \right\rangle_{-\diamond, \diamond} = \int_0^T u(t) \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} \phi dt, \quad \forall \phi \in X_\diamond^*.$$

437 As the family Φ of (generalized) eigenvectors is assumed to form a complete family
 438 in X_\diamond^* (see (12)) it is in fact sufficient to test (20) against the elements of Φ . Therefore
 439 a null control u is characterized by the following countable set of equations

$$440 \quad (21) \quad - \left\langle y_0, e^{-T\mathcal{A}^*} \phi_\lambda^l \right\rangle_{-\diamond, \diamond} = \int_0^T u(t) \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} \phi_\lambda^l dt, \quad \forall \lambda \in \Lambda, \forall l \in \llbracket 0, \alpha_\lambda - 1 \rrbracket.$$

441 Using the formalism of generalized divided differences, we can give a convenient ex-
 442 pression of the action of the semi-group on the generalized eigenvectors as follows

$$443 \quad (22) \quad \begin{aligned} e^{-t\mathcal{A}^*} \phi_\lambda^l &= e^{-\lambda t} \sum_{p=0}^l \frac{(-t)^p}{p!} \phi_\lambda^{l-p} \\ &= \sum_{p=0}^l \frac{e_t^{(p)}(\lambda)}{p!} \phi_\lambda^{l-p} \\ &= \sum_{p=0}^l e_t[\lambda^{(p+1)}] \phi[\lambda^{(l-p+1)}] \\ &= (e_t \phi)[\lambda^{(l+1)}], \end{aligned}$$

444 this last equality coming from Definition 7.3. Then, $y(T) = 0$ if and only if for any
 445 $\lambda \in \Lambda$ and any $l \in \llbracket 0, \alpha_\lambda - 1 \rrbracket$,

$$446 \quad \int_0^T u(T-t) \mathcal{B}^* ((e_t \phi)[\lambda^{(l+1)}]) dt = - \left\langle y_0, (e_T \phi)[\lambda^{(l+1)}] \right\rangle_{-\diamond, \diamond}.$$

447 By (11), and since $y_0 \in Y_0$, this reduces to find $u \in L^2(0, T; \mathbb{R})$ such that for any
 448 $\lambda \in \Lambda$ and any $l \in \llbracket 0, \alpha_\lambda - 1 \rrbracket$,

$$449 \quad (\mathcal{B}^* \phi_\lambda^0) \int_0^T u(T-t) \frac{(-t)^l}{l!} e^{-\lambda t} dt = - \left\langle y_0, P_{Y_0}^* (e_T \phi)[\lambda^{(l+1)}] \right\rangle_{-\diamond, \diamond},$$

450 that is, using (10) and (15),

$$451 \quad (23) \quad \int_0^T u(T-t) \frac{(-t)^l}{l!} e^{-\lambda t} dt = - \left\langle y_0, (e_T \psi)[\lambda^{(l+1)}] \right\rangle_{-\diamond, \diamond}, \quad \forall \lambda \in \Lambda, \forall l \in \llbracket 0, \alpha_\lambda - 1 \rrbracket,$$

452 To solve this so-called moment problem the classical strategy introduced in [26]
 453 consists in designing a biorthogonal family in $L^2(0, T)$ to

$$454 \quad \{t \mapsto t^l e^{-\lambda t}; \lambda \in \Lambda, l \in \llbracket 0, \alpha_\lambda - 1 \rrbracket\}$$

455 with associated estimates. Then, thanks to these estimates, a suitable control is de-
 456 fined. Usually in this procedure each biorthogonal element is estimated separately.
 457 Thus, this method is somehow inopert to analyse the possible condensation of eigen-
 458 vectors (which is related to possible cancellations in linear combinations of right-hand
 459 sides of (23)). We will thus propose to solve this moment problem using the grouping
 460 introduced in Section 1.3.2, in order to cope with such possible compensations. We
 461 then look for a solution u in the form

$$462 \quad (24) \quad u(t) = - \sum_{k \geq 1} q_k(T - t)$$

463 where each q_k will solve the moment problem corresponding to the group G_k . More
 464 precisely, such a control will formally solve (23) if

$$465 \quad (25) \quad \begin{cases} \int_0^T q_k(t) \frac{(-t)^{l'}}{l'!} e^{-\lambda_{k',j'} t} dt = 0, \quad \forall k' \neq k, \forall j' \in \llbracket 1, g_{k'} \rrbracket, \forall l' \in \llbracket 0, \alpha_{k',j'} - 1 \rrbracket, \\ \int_0^T q_k(t) \frac{(-t)^l}{l!} e^{-\lambda_{k,j} t} dt = \langle y_0, (e_T \psi)[\lambda_{k,j}^{(l+1)}] \rangle_{-\diamond, \diamond}, \\ \forall k \geq 1, \forall j \in \llbracket 1, g_k \rrbracket, \forall l \in \llbracket 0, \alpha_{k,j} - 1 \rrbracket. \end{cases}$$

466 Then the proof of point i. of Theorem 1.1 reduces to the resolution of such a
 467 block moment problem with suitable estimates (see Theorem 4.1). First, we solve
 468 in Theorem 2.1 the block moment problem in the case where the eigenvalues are
 469 algebraically simple i.e.

$$470 \quad (26) \quad \begin{cases} \int_0^T q_k(t) e^{-\lambda_{k',j'} t} dt = 0, \quad \forall k' \neq k, \quad \forall j' \in \llbracket 1, g_{k'} \rrbracket, \\ \int_0^T q_k(t) e^{-\lambda_{k,j} t} dt = e^{-\lambda_{k,j} T} \langle y_0, \psi_{k,j} \rangle_{-\diamond, \diamond}, \quad \forall k \geq 1, \forall j \in \llbracket 1, g_k \rrbracket. \end{cases}$$

471 This construction uses a Laplace transform isomorphism together with a suitable
 472 restriction argument (Proposition 2.4). The obtained estimates on q_k will allow to
 473 prove convergence of the series (24) when $T > T_0(Y_0)$. Those estimates are uniform
 474 with respect to Λ in a certain class (see Definition 2.1) which will allow in Section 4
 475 to infer the resolution of (25) in the general case.

476 **REMARK 1.6.** *Contrarily to the classical strategy, notice that the sequence $(q_k)_k$*
 477 *is not a biorthogonal family to*

$$478 \quad \{t \mapsto t^l e^{-\lambda t}; \lambda \in \Lambda, l \in \llbracket 0, \alpha_\lambda - 1 \rrbracket\}.$$

479 *The function q_k is only orthogonal to those functions corresponding to groups other*
 480 *than G_k . Inside the group G_k its definition is adapted to solve the moment prob-*
 481 *lem (23). Through the right-hand side (adapted to each initial condition) we will*
 482 *possibly take into account the unsufficient observation of eigenvectors, the conden-*
 483 *sation of eigenvalues but also the condensation of eigenvectors. This construction*

484 can thus be seen as a block moment method. As we consider at the same time the
 485 eigenvalues associated to a same group this will lead to sharper estimates than the one
 486 coming from the design of a biorthogonal family (i.e. when considering each eigenvalue
 487 individually).

488 However, as already mentioned, our strategy still allows to prove the existence and
 489 sharp estimates on biorthogonal families (see Corollary 2.1 and Corollary 4.1). Let
 490 us mention that, to the best of our knowledge, the estimate we obtain in Corollary 4.1
 491 for a biorthogonal family in presence of algebraic multiplicity of eigenvalues without
 492 the standard gap condition was not known. Even though these biorthogonal families
 493 are not always suitable to deal with controllability properties in presence of spectral
 494 condensation (this is why we designed this block resolution of the moment problem)
 495 they can be useful for other problems.

496 Let us mention that, in the context of control problems with a spectrum satisfying
 497 the weak-gap condition, divided differences were already used for instance in [8, 9].
 498 Among other things, in these works, the authors give a necessary and sufficient
 499 condition for the family of (generalized) divided differences

$$500 \quad \{t \mapsto e_t[i\lambda_{k,1}], \dots, t \mapsto e_t[i\lambda_{k,1}, \dots, i\lambda_{k,g_k}], k \geq 1\}$$

501 to form a Riesz basis of $L^2(0, T; \mathbb{C})$. The possible condensation of eigenvalues then
 502 appears to deduce properties on the original family of exponentials $(t \mapsto e^{i\lambda t})_{\lambda \in \Lambda}$.
 503 Their results are then applied to hyperbolic control problems.

504 The results presented in our work can be seen as the ‘parabolic’ equivalent (or
 505 the ‘real-valued’ equivalent if one focuses on the exponentials) of these results. Never-
 506 theless, the control problems have really different behaviours as well as the families of
 507 exponentials. Indeed, in our setting the considered families never form a Riesz basis.
 508 Thus, neither work can be deduced from the other.

509 The proof of point ii. of Theorem 1.1, relies on the optimality of the bounds
 510 proved in Theorem 2.1 for the resolution of the block moment problem.

511 As dealing with null-controllability from a proper subspace of initial conditions
 512 is not classical let us recall the following lemma that characterizes this controllability
 513 through an observability inequality.

514 LEMMA 1.1 (see for instance [1, Lemma 2.1]). *Let $M > 0$.*

515 *The following two propositions are equivalent.*

516 1. *For any $y_0 \in Y_0$ there exists a $u \in L^2(0, T; U)$ such that $y(T) = 0$ and*

$$517 \quad \|u\|_{L^2(0, T; U)} \leq M \|y_0\|_{-\diamond}.$$

518 2. *For any $z_T \in X_{\diamond}^*$, the following partial observability holds:*

$$519 \quad (27) \quad \left\| P_{Y_0}^* \left(e^{-T\mathcal{A}^*} z_T \right) \right\|_{\diamond^*}^2 \leq M^2 \int_0^T \left\| \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} z_T \right\|_U^2 dt.$$

520 *In this case, the best constant M satisfying those properties is called the cost of con-*
 521 *trollability from Y_0 at time T and is denoted $M(Y_0, T)$.*

522 2. The case of simple eigenvalues.

523 **2.1. Null-controllability in large time.**

524 The goal of this section is to prove point **i.** of Theorem 1.1 in the case of algebraically simple eigenvalues. Thus, in all this section we assume that $\eta = 1$.

525 As explained in Section 1.4, we will now focus on the construction of a solution
 526 to (26). Of course as we want to design a control $u \in L^2(0, T; U)$ the estimate of
 527 $\|q_k\|_{L^2(0, T; \mathbb{R})}$ will play a crucial role to prove that the series (24) makes sense. Actually
 528 we will prove sharp estimates that are uniformly valid for Λ in a certain class. These
 529 uniform estimates can be used for various applications and will be crucial to deal with
 530 algebraic multiplicity of eigenvalues in Section 4. We start by precisizing the class of Λ
 531 we will consider.

532 DEFINITION 2.1. Let $p \in \mathbb{N}^*$, $\rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$. We say that a countable family Λ belongs to the class $\mathcal{L}_w(p, \rho, \mathcal{N})$ if Λ satisfies the weak-gap condition (9) with parameters p and ρ and if for any $\varepsilon > 0$ we have

536 (28)
$$\sum_{\substack{\lambda \in \Lambda \\ \lambda \geq \mathcal{N}(\varepsilon)}} \frac{1}{\lambda} < \varepsilon.$$

537 This definition is directly inspired by the pioneering work [27]. More precisely,
 538 the class of sequences used in [27] is similar to $\mathcal{L}_w(1, \rho, \mathcal{N})$, but it is however slightly
 539 different since in (28) the summation condition is given on the value of λ itself whereas
 540 in the above reference the condition is on the index of the eigenvalue in Λ (which is
 541 supposed to be sorted increasingly). Despite this small difference (whose aim is to
 542 simplify some computations) the results we shall take from [27] that use this definition
 543 are also valid with this alternative definition and thus we set $\mathcal{L}(\rho, \mathcal{N}) := \mathcal{L}_w(1, \rho, \mathcal{N})$.

544 REMARK 2.1 (The usual gap condition). With our definition, a sequence Λ belongs to $\mathcal{L}(\rho, \mathcal{N})$ if it satisfies the classical gap condition

546 (29)
$$|\lambda' - \lambda| > \rho, \quad \forall \lambda, \lambda' \in \Lambda, \quad \lambda \neq \lambda',$$

547 and the asymptotic behavior estimate (28).

548 As we will see in the examples (Section 5), the typical situation where sequences
 549 satisfying the weak gap condition appear is when one glues a finite number of sequences,
 550 each of them satisfying a standard gap condition as in Remark 2.1. This is
 551 formalized in the following lemma.

LEMMA 2.1. Let $p, \tilde{p} \in \mathbb{N}^*$, $\rho, \tilde{\rho} > 0$ and $\mathcal{N}, \tilde{\mathcal{N}} : (0, +\infty) \rightarrow \mathbb{R}$ given. Then, for any $\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$ and $\tilde{\Lambda} \in \mathcal{L}_w(\tilde{p}, \tilde{\rho}, \tilde{\mathcal{N}})$, we have

$$\Lambda \cup \tilde{\Lambda} \in \mathcal{L}_w(\bar{p}, \bar{\rho}, \bar{\mathcal{N}}),$$

552 with $\bar{p} = p + \tilde{p}$, $\bar{\rho} = \min(\rho, \tilde{\rho})$ and $\bar{\mathcal{N}}(\varepsilon) = \max(\mathcal{N}(\varepsilon/2), \tilde{\mathcal{N}}(\varepsilon/2))$.

553 *Proof.* Let us first prove the weak gap condition. For any $\mu \geq 0$, we have

554
$$\begin{aligned} [\mu, \mu + \bar{\rho}] \cap (\Lambda \cup \tilde{\Lambda}) &= ([\mu, \mu + \bar{\rho}] \cap \Lambda) \cup ([\mu, \mu + \bar{\rho}] \cap \tilde{\Lambda}) \\ &\subset ([\mu, \mu + \rho] \cap \Lambda) \cup ([\mu, \mu + \tilde{\rho}] \cap \tilde{\Lambda}), \end{aligned}$$

555
556

and taking the cardinal, we get

$$\#[\mu, \mu + \bar{\rho}] \cap (\Lambda \cup \tilde{\Lambda}) \leq p + \tilde{p} = \bar{p}.$$

557 For the asymptotic behavior of the sequences, we have

$$558 \quad \sum_{\substack{\lambda \in \Lambda \cup \tilde{\Lambda} \\ \lambda > \mathcal{N}(\varepsilon)}} \frac{1}{\lambda} \leq \sum_{\substack{\lambda \in \Lambda \\ \lambda > \mathcal{N}(\varepsilon)}} \frac{1}{\lambda} + \sum_{\substack{\lambda \in \tilde{\Lambda} \\ \lambda > \mathcal{N}(\varepsilon)}} \frac{1}{\lambda} \leq \sum_{\substack{\lambda \in \Lambda \\ \lambda > \mathcal{N}(\varepsilon/2)}} \frac{1}{\lambda} + \sum_{\substack{\lambda \in \tilde{\Lambda} \\ \lambda > \mathcal{N}(\varepsilon/2)}} \frac{1}{\lambda} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

559 The claim is proved. \square

560 The following straightforward facts will also be useful.

561 **REMARK 2.2.** Let $\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$.

- 562 • Any $\tilde{\Lambda} \subset \Lambda$ also satisfies $\tilde{\Lambda} \in \mathcal{L}_w(p, \rho, \mathcal{N})$.
- 563 • For any $h \in (0, \inf \Lambda)$, $\Lambda + h \in \mathcal{L}_w(p, \rho, \frac{\mathcal{N}}{2})$.

564 Using this class we prove the following theorem.

565 **THEOREM 2.1.** Let $T \in (0, +\infty]$. Let $p \in \mathbb{N}^*$, $r, \rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$.

566 Assume that $\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$ and let $(G_k)_k \in \mathcal{G}(\Lambda, p, r, \rho)$ be an associated grouping.

567 For any $\varepsilon > 0$, there exists a constant $C_{\varepsilon, T, p, r, \rho, \mathcal{N}} > 0$ such that for any $k \geq 1$,
 568 for any $\omega_{k,1}, \dots, \omega_{k,g_k} \in \mathbb{C}$, there exists $q_k \in L^2(0, T; \mathbb{C})$ satisfying

$$569 \quad (30a) \quad \int_0^T q_k(t) e^{-\lambda_{k',j'} t} dt = 0, \quad \forall k' \neq k, \forall j' \in \llbracket 1, g_{k'} \rrbracket,$$

$$570 \quad (30b) \quad \int_0^T q_k(t) e^{-\lambda_{k,j} t} dt = \omega_{k,j}, \quad \forall j \in \llbracket 1, g_k \rrbracket,$$

571 and

$$573 \quad (31) \quad \|q_k\|_{L^2(0, T; \mathbb{C})} \leq C_{\varepsilon, T, p, r, \rho, \mathcal{N}} e^{\varepsilon \lambda_{k,1}} \max_{i \in \llbracket 1, g_k \rrbracket} \left| \omega[\lambda_{k,1}, \dots, \lambda_{k,i}] \right|.$$

574 Moreover, up to the factor $e^{\varepsilon \lambda_{k,1}}$, this last estimate is sharp: any solution $q_k \in$
 575 $L^2(0, T; \mathbb{C})$ of (30b) satisfies

$$576 \quad (32) \quad \|q_k\|_{L^2(0, T; \mathbb{C})} \geq \tilde{C}_p \max_{i \in \llbracket 1, g_k \rrbracket} \left| \omega[\lambda_{k,1}, \dots, \lambda_{k,i}] \right|,$$

577 for some $\tilde{C}_p > 0$.

578 The proof of Theorem 2.1 is conducted all along Sections 2.1.1 and 2.1.2.

579 Before going on with the proof, let us notice that the resolution of the block
 580 moment problem (30) for a specific choice of $\omega_{k,j}$ allows to prove, as a by-product,
 581 the existence and uniform estimates of a biorthogonal family to the exponentials
 582 $(e_\lambda)_{\lambda \in \Lambda}$ where e_λ is defined by (5).

583 **COROLLARY 2.1.** Let $T \in (0, +\infty]$. Let $p \in \mathbb{N}^*$, $r, \rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$.

584 Assume that $\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$ and let $(G_k)_k \in \mathcal{G}(\Lambda, p, r, \rho)$ be an associated grouping.

585 For any $k \geq 1$, for any $j \in \llbracket 1, g_k \rrbracket$, there exists $q_{k,j} \in L^2(0, T; \mathbb{R})$ satisfying

$$586 \quad (33) \quad \int_0^T q_{k,j}(t) e^{-\lambda_{k',j'} t} dt = \delta_{k,k'} \delta_{j,j'}, \quad \forall k, k' \geq 1, \forall j \in \llbracket 1, g_k \rrbracket, \forall j' \in \llbracket 1, g_{k'} \rrbracket,$$

587 where δ denotes the Kronecker symbol. Moreover, for any $\varepsilon > 0$, there exists a
 588 constant $C_{\varepsilon, T, p, r, \rho, \mathcal{N}} > 0$ such that for any $k \geq 1$ and for any $j \in \llbracket 1, g_k \rrbracket$,

$$589 \quad \|q_{k,j}\|_{L^2(0, T; \mathbb{R})} \leq C_{\varepsilon, T, p, r, \rho, \mathcal{N}} \frac{e^{\varepsilon \lambda_{k,1}}}{|P'_{G_k}(\lambda_{k,j})|},$$

590 where P_{G_k} is defined in (6).

Moreover, up to the factor $e^{\varepsilon\lambda_{k,1}}$, this estimate is optimal since any function $q_{k,j}$ satisfying (33), satisfies the lower bound

$$\|q_{k,j}\|_{L^2(0,T;\mathbb{R})} \geq \tilde{C}_p \frac{1}{|P'_{G_k}(\lambda_{k,j})|}$$

591 for some $\tilde{C}_p > 0$.

592 *Proof.* Let $k \geq 1$ and $j \in \llbracket 1, g_k \rrbracket$. Let $q_{k,j} \in L^2(0, T; \mathbb{C})$ be the solution of the
 593 block moment problem (30) given by Theorem 2.1 associated with the right-hand side
 594 $\omega_{k,j'} = \delta_{j,j'}$ for any $j' \in \llbracket 1, g_k \rrbracket$. Since those values of ω are real we can change $q_{k,j}$ in
 595 its real part without changing its properties. Then, the equalities (33) follow directly.
 596 Moreover we have

$$597 \quad \|q_{k,j}\|_{L^2(0,T;\mathbb{R})} \leq C_{\varepsilon,T,p,r,\rho,\mathcal{N}} e^{\varepsilon\lambda_{k,1}} \max_{i \in \llbracket 1, g_k \rrbracket} |\omega[\lambda_{k,1}, \dots, \lambda_{k,i}]|.$$

598 From the Newton formula (see Proposition 7.3) it comes that for any $i \in \llbracket 1, g_k \rrbracket$,

$$599 \quad \omega[\lambda_{k,1}, \dots, \lambda_{k,i}] = \begin{cases} 0, & \text{if } i < j, \\ \frac{1}{\prod_{m \in \llbracket 1, i \rrbracket \neq j} (\lambda_{k,j} - \lambda_{k,m})}, & \text{if } i \geq j. \end{cases}$$

600 To conclude the proof of Corollary 2.1 we prove that there exists $C_{p,\rho} > 0$ such that
 601 for any $k \geq 1$, $j \in \llbracket 1, g_k \rrbracket$ and any $i \in \llbracket j, g_k \rrbracket$,

$$602 \quad (34) \quad \prod_{m \in \llbracket 1, i \rrbracket \neq j} |\lambda_{k,j} - \lambda_{k,m}| \geq C_{p,\rho} |P'_{G_k}(\lambda_{k,j})|.$$

603 Indeed, we have

$$604 \quad \frac{\prod_{m \in \llbracket 1, i \rrbracket \neq j} |\lambda_{k,j} - \lambda_{k,m}|}{|P'_{G_k}(\lambda_{k,j})|} = \frac{\prod_{m \in \llbracket 1, i \rrbracket \neq j} |\lambda_{k,j} - \lambda_{k,m}|}{\prod_{m \in \llbracket 1, g_k \rrbracket \neq j} |\lambda_{k,j} - \lambda_{k,m}|} = \frac{1}{\prod_{m \in \llbracket i+1, g_k \rrbracket} |\lambda_{k,j} - \lambda_{k,m}|}.$$

605 By (14), we get

$$606 \quad |\lambda_{k,j} - \lambda_{k,m}| \leq \rho, \quad \forall m \in \llbracket i+1, g_k \rrbracket.$$

607 Thus,

$$608 \quad (35) \quad \frac{\prod_{m \in \llbracket 1, i \rrbracket \neq j} |\lambda_{k,j} - \lambda_{k,m}|}{|P'_{G_k}(\lambda_{k,j})|} \geq \left(\frac{1}{\rho}\right)^{g_k - i}.$$

609 As the right-hand side only takes a finite number of values, inequality (35) proves (34)
 610 and ends the proof of Corollary 2.1.

611 The lower bound directly follows from (32) and the inequality

$$612 \quad \max_{i \in \llbracket 1, g_k \rrbracket} |\omega[\lambda_{k,1}, \dots, \lambda_{k,i}]| \geq |\omega[\lambda_{k,1}, \dots, \lambda_{k,g_k}]| = \frac{1}{|P'_{G_k}(\lambda_{k,j})|}. \quad \square$$

613 **2.1.1. Resolution of block moment problems in infinite time.**

614 In this section, we start by proving Theorem 2.1 in the case of simple eigenvalues
615 and with $T = +\infty$. More precisely, we prove the following proposition.

616 PROPOSITION 2.1. *Let $p \in \mathbb{N}^*$, $r, \rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$. Assume that
617 $\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$ and let $(G_k)_k \in \mathcal{G}(\Lambda, p, r, \rho)$ be an associated grouping.*

618 *For any $\varepsilon > 0$, there exists a constant $C_{\varepsilon, p, r, \rho, \mathcal{N}} > 0$ such that for any $k \geq 1$, for
619 any $\omega_{k,1}, \dots, \omega_{k,g_k} \in \mathbb{C}$, there exists $\tilde{q}_k \in L^2(0, +\infty; \mathbb{C})$ satisfying*

$$620 \quad (36) \quad \begin{cases} \int_0^{+\infty} \tilde{q}_k(t) e^{-\lambda_{k',j'} t} dt = 0, & \forall k' \neq k, \forall j' \in \llbracket 1, g_{k'} \rrbracket, \\ \int_0^{+\infty} \tilde{q}_k(t) e^{-\lambda_{k,j} t} dt = \omega_{k,j}, & \forall j \in \llbracket 1, g_k \rrbracket, \end{cases}$$

621 and

$$622 \quad \|\tilde{q}_k\|_{L^2(0, +\infty; \mathbb{C})} \leq C_{\varepsilon, p, r, \rho, \mathcal{N}} e^{\varepsilon \lambda_{k,1}} \max_{i \in \llbracket 1, g_k \rrbracket} \left| \omega[\lambda_{k,1}, \dots, \lambda_{k,i}] \right|.$$

623 The proof relies on the construction of an holomorphic function satisfying suitable
624 properties and estimates. The resolution of the block moment problem (36) then
625 comes from the isomorphism induced by the Laplace transform.

626 *Proof.* Let us start by recalling classical properties of the Laplace transform (see
627 for instance [47, pp. 19-20] and the references therein). Let $H^2(\mathbb{C}^+)$ the space of
628 holomorphic functions F on $\mathbb{C}^+ = \{z \in \mathbb{C}; \Re(z) > 0\}$ such that

$$629 \quad \sup_{\sigma > 0} \|F(\sigma + i\bullet)\|_{L^2(\mathbb{R}; \mathbb{C})} < +\infty,$$

630 endowed with the norm

$$631 \quad \|F\|_{H^2(\mathbb{C}^+)}^2 := \sup_{\sigma > 0} \|F(\sigma + i\bullet)\|_{L^2(\mathbb{R}; \mathbb{C})}^2.$$

632 From the properties of $H^2(\mathbb{C}^+)$, it comes that

$$633 \quad \|F\|_{H^2(\mathbb{C}^+)}^2 = \int_{\mathbb{R}} |F(i\tau)|^2 d\tau, \quad \forall F \in H^2(\mathbb{C}^+).$$

634 Then the Laplace transform

$$635 \quad \mathsf{L} : f \in L^2(0, +\infty; \mathbb{C}) \mapsto \left(F : \lambda \in \mathbb{C}^+ \mapsto \int_0^{+\infty} e^{-\lambda t} f(t) dt \right) \in H^2(\mathbb{C}^+)$$

636 is an isomorphism.

637 We shall construct for each k , a function $J_k \in H^2(\mathbb{C}^+)$ satisfying

$$638 \quad (37) \quad J_k(\lambda) = 0, \quad \forall \lambda \in \Lambda \setminus G_k,$$

$$639 \quad (38) \quad J_k(\lambda_{k,j}) = \omega_{k,j}, \quad \forall j \in \llbracket 1, g_k \rrbracket,$$

641 and such that for any $\varepsilon > 0$, there exists $C_{\varepsilon, p, r, \rho, \mathcal{N}} > 0$ such that

$$642 \quad \int_{\mathbb{R}} |J_k(i\tau)|^2 d\tau \leq C_{\varepsilon, p, r, \rho, \mathcal{N}} e^{\varepsilon \lambda_{k,1}} \max_{i \in \llbracket 1, g_k \rrbracket} \left| \omega[\lambda_{k,1}, \dots, \lambda_{k,i}] \right|, \quad \forall k \geq 1.$$

643 Taking advantage of the isomorphism property of the Laplace transform we will then
644 set $\tilde{q}_k := \mathsf{L}^{-1}(J_k)$, to conclude the proof.

645 *Construction of J_k .*
 646 We define J_k as

$$647 \quad J_k : z \in \mathbb{C}^+ \mapsto \frac{P_k(z)}{(1+z)^p} W_k(z)$$

648 where P_k is a polynomial of degree less than p which is precised below and W_k is the
 649 following Blaschke-type product

$$650 \quad W_k(z) = \prod_{j=1}^p \left(\prod_{\lambda \in \Lambda_j \setminus G_k} \frac{\lambda - z}{\lambda + z} \right),$$

651 where

$$652 \quad (39) \quad \Lambda_j := \{\lambda_{l, \min(j, g_l)}, \quad l \geq 1\}.$$

653 The sequence Λ_j contains the j -th element of each group G_l , except if this group
 654 contains less than j elements, in which case, we replace it by the largest element of
 655 G_l that is λ_{l, g_l} . In particular, we observe that Λ_j is a subsequence of Λ .

656 From (8), we deduce that $\sum_{\lambda \in \Lambda_j} \frac{1}{\lambda} < +\infty$, so that for any j , the associated
 657 infinite product uniformly converges on any compact of \mathbb{C}^+ . As a consequence, W_k
 658 is well-defined and holomorphic in \mathbb{C}^+ (see for instance [45, Chapter 15]). It follows
 659 that J_k is also holomorphic on \mathbb{C}^+ .

660 We shall need the following property, whose proof is technical and postponed to
 661 Section 2.1.4.

662 PROPOSITION 2.2. *There exists a constant $C_{\varepsilon, p, r, \rho, \mathcal{N}} > 0$ such that for any $k \geq 1$,
 663 any $l \in \llbracket 0, p \rrbracket$, and any $\theta \in \text{Conv}(G_k)$,*

$$664 \quad (40) \quad \left| \left(\frac{1}{W_k} \right)^{(l)}(\theta) \right| \leq C_{\varepsilon, p, r, \rho, \mathcal{N}} e^{\varepsilon \lambda_{k, 1}}.$$

665 From the definition of W_k it comes that (37) is satisfied. Next, it comes that (38)
 666 is equivalent to

$$667 \quad P_k(\lambda_{k, j}) = \frac{(1 + \lambda_{k, j})^p}{W_k(\lambda_{k, j})} \omega_{k, j}, \quad \forall j \in \llbracket 1, g_k \rrbracket.$$

668 Let

$$669 \quad f : s \in \mathbb{R} \mapsto (1 + s)^p, \quad \text{and} \quad f_k : s \in \mathbb{R} \mapsto \frac{f(s)}{W_k(s)}.$$

670 To satisfy (38), we define P_k as the Lagrange interpolating polynomial at points $\lambda_{k, j}$
 671 with values $(f_k \omega)[\lambda_{k, j}] := f_k(\lambda_{k, j}) \omega_{k, j}$ that is, in Newton form,

$$672 \quad P_k(z) := \sum_{j=1}^{g_k} (f_k \omega)[\lambda_{k, 1}, \dots, \lambda_{k, j}] \prod_{i=1}^{j-1} (z - \lambda_{k, i}).$$

673 Thus, to conclude it remains to estimate $\int_{\mathbb{R}} |J_k(i\tau)|^2 d\tau$.

674 *Estimate of J_k .*

675 Notice that since the eigenvalues in Λ are real, for any $k \geq 1$ and any $\tau \in \mathbb{R}$, we
 676 have $|W_k(i\tau)| = 1$. This implies

$$677 \quad (41) \quad |J_k(i\tau)| \leq g_k \frac{(|\tau| + \lambda_{k,g_k})^{p-1}}{(1 + \tau^2)^{p/2}} \max_{j \in \llbracket 1, g_k \rrbracket} |(f_k \omega)[\lambda_{k,1}, \dots, \lambda_{k,j}]|$$

678 and thus $J_k \in H^2(\mathbb{C}^+)$.

679 Using Leibniz formula (see Proposition 7.6),

$$680 \quad (42) \quad |(f_k \omega)[\lambda_{k,1}, \dots, \lambda_{k,j}]| \leq \sum_{i=1}^j |f_k[\lambda_{k,i}, \dots, \lambda_{k,j}]| |\omega[\lambda_{k,1}, \dots, \lambda_{k,i}]|.$$

681 Using again Leibniz formula (see Proposition 7.6),

$$682 \quad (43) \quad f_k[\lambda_{k,i}, \dots, \lambda_{k,j}] = \sum_{m=i}^j f[\lambda_{k,i}, \dots, \lambda_{k,m}] \left(\frac{1}{W_k} \right) [\lambda_{k,m}, \dots, \lambda_{k,j}].$$

683 The two factors in each term of this sum are estimated using Lagrange theorem (see
 684 Proposition 7.4):

685 • First, we have

$$686 \quad f[\lambda_{k,i}, \dots, \lambda_{k,m}] = \frac{f^{(m-i)}(\theta_k)}{(m-i)!}$$

687 with $\theta_k \in [\lambda_{k,i}, \lambda_{k,m}]$. It comes that there exists $C_p > 0$ such that

$$688 \quad (44) \quad |f[\lambda_{k,1}, \dots, \lambda_{k,m}]| \leq C_p (1 + \lambda_{k,m})^p \leq C_p (1 + \lambda_{k,g_k})^p.$$

689 • Second, we have

$$690 \quad \left(\frac{1}{W_k} \right) [\lambda_{k,m}, \dots, \lambda_{k,j}] = \frac{1}{(j-m+1)!} \left(\frac{1}{W_k} \right)^{(j-m+1)} (\tilde{\theta}_k)$$

691 with $\tilde{\theta}_k \in [\lambda_{k,m}, \lambda_{k,j}]$.

692 By using (40), it follows that

$$693 \quad (45) \quad \left| \left(\frac{1}{W_k} \right) [\lambda_{k,m}, \dots, \lambda_{k,j}] \right| \leq C_{\varepsilon, p, \rho, \mathcal{N}} e^{\varepsilon \lambda_{k,g_k}}.$$

694 Recall that (14) implies $\lambda_{k,g_k} - \lambda_{k,1} < \rho$. Then, using (44) and (45) into the identity
 695 (43) proves that there exists $C_{\varepsilon, p, r, \rho, \mathcal{N}} > 0$ such that for any $i, j \in \llbracket 1, g_k \rrbracket$, $i \leq j$, we
 696 have

$$697 \quad |f_k[\lambda_{k,i}, \dots, \lambda_{k,j}]| \leq C_{\varepsilon, p, r, \rho, \mathcal{N}} e^{\varepsilon \lambda_{k,1}}.$$

698 Plugging it in (42) we obtain

$$699 \quad \max_{j \in \llbracket 1, g_k \rrbracket} |(f_k \omega)[\lambda_{k,1}, \dots, \lambda_{k,j}]| \leq C_{\varepsilon, p, r, \rho, \mathcal{N}} e^{\varepsilon \lambda_{k,1}} \max_{i \in \llbracket 1, g_k \rrbracket} |\omega[\lambda_{k,1}, \dots, \lambda_{k,i}]|.$$

700 Finally, getting back to estimate (41), and using the isomorphism property of \mathbf{L}
 701 ends the proof of Proposition 2.1. \square

702 **2.1.2. From infinite time horizon to finite time horizon.**

703 In this section we first prove that the estimates on the solution on $(0, +\infty)$ of the
704 block moment problem (36) for simple eigenvalues given in Proposition 2.1 implies
705 the resolution on $(0, T)$ of the similar block moment problem (30). More precisely we
706 prove the following.

707 PROPOSITION 2.3. *Let $p \in \mathbb{N}^*$, $\rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$. Assume that*
708 *$\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$. For any $T > 0$, there exists a constant $C_{T,p,\rho,\mathcal{N}} > 0$ such that for*
709 *any $\tilde{q} \in L^2(0, +\infty; \mathbb{C})$ there exists $q \in L^2(0, T; \mathbb{C})$ satisfying*

$$710 \quad \int_0^T q(t)e^{-\lambda t} dt = \int_0^{+\infty} \tilde{q}(t)e^{-\lambda t} dt, \quad \forall \lambda \in \Lambda,$$

711 and

$$712 \quad \|q\|_{L^2(0,T;\mathbb{C})} \leq C_{T,p,\rho,\mathcal{N}} \|\tilde{q}\|_{L^2(0,+\infty;\mathbb{C})}.$$

713 For any $T \in (0, +\infty]$, we set

$$714 \quad A(\Lambda, T) := \overline{\text{Span}\{e_\lambda; \lambda \in \Lambda\}}^{L^2(0,T;\mathbb{C})},$$

715 where e_λ is defined in (5). The proof of Proposition 2.3 mainly relies on the following
716 proposition that gives an estimate on the inverse of the restriction operator.

717 PROPOSITION 2.4. *Let $p \in \mathbb{N}^*$, $\rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$. Assume that*
718 *$\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$. Let $T > 0$ be fixed. Then, the restriction operator*

$$719 \quad (46) \quad R_{\Lambda,T} : q \in A(\Lambda, +\infty) \mapsto q|_{(0,T)} \in A(\Lambda, T)$$

720 *is an isomorphism. Moreover there exists a constant $C_{T,p,\rho,\mathcal{N}} > 0$ such that*

$$721 \quad (47) \quad \|R_{\Lambda,T}^{-1}\| \leq C_{T,p,\rho,\mathcal{N}}.$$

722 In the case $p = 1$, this result is due to Fattorini and Russell [27, Theorem 1.3]. Our
723 proof follows closely the strategy developed in this reference and takes advantage of
724 the uniform estimates we established in the previous sections.

725 *Proof.* The fact that $R_{\Lambda,T}$ is an isomorphism is proved in [47] under the sole
726 assumption (8). The only thing to prove is thus the bound (47).

727 The proof is done by contradiction. Assume that the estimate does not hold for
728 given T, p, ρ , and \mathcal{N} , then there exists a sequence $(\Lambda^m)_{m \geq 1}$ belonging to the same
729 class $\mathcal{L}_w(p, \rho, \mathcal{N})$, such that

$$730 \quad (48) \quad \|R_{\Lambda^m, T}^{-1}\| \xrightarrow{m \rightarrow \infty} +\infty.$$

731 For each m , by Proposition 7.1, we consider a grouping $(G_k^m)_k \in \mathcal{G}(\Lambda^m, p, \rho/p, \mathcal{N})$,
732 and from (48) we know that there exists coefficients $a_{k,j}^m$ such that the finite linear
733 combination

$$734 \quad P^m : t \mapsto \sum_{k=1}^{K^m} \sum_{j=1}^{g_k^m} a_{k,j}^m e^{-\lambda_{k,j}^m t},$$

735 satisfies

$$736 \quad \|P^m\|_{L^2(0,+\infty;\mathbb{C})} = 1, \quad \text{and} \quad \|P^m\|_{L^2(0,T;\mathbb{C})} \xrightarrow{m \rightarrow \infty} 0.$$

737 Let $0 < \varepsilon < \frac{T}{2}$ be fixed and let $\mathbb{C}_{2\varepsilon}^+ = \{z \in \mathbb{C} ; \Re(z) > 2\varepsilon\}$. We prove that the
 738 sequence $z \mapsto P^m(z)$ is uniformly bounded on any compact of $\mathbb{C}_{2\varepsilon}^+$.

739 Let $m \geq 1$ and $z \in \mathbb{C}_{2\varepsilon}^+$. Then for any $k \in \{1, \dots, K^m\}$ the application of Propo-
 740 sition 2.1 to the sequence Λ^m yields the existence of $\tilde{q}_k^{m,z} \in L^2(0, +\infty; \mathbb{C})$ satisfying

$$741 \quad (49) \quad \begin{cases} \int_0^{+\infty} \tilde{q}_k^{m,z}(t) e^{-\lambda_{k',j'}^m t} dt = 0, & \forall k' \neq k, \quad \forall j' \in \llbracket 1, g_{k'}^m \rrbracket, \\ \int_0^{+\infty} \tilde{q}_k^{m,z}(t) e^{-\lambda_{k,j}^m t} dt = e^{-\lambda_{k,j}^m z}, & \forall j \in \llbracket 1, g_k^m \rrbracket, \end{cases}$$

742 and

$$743 \quad \|\tilde{q}_k^{m,z}\|_{L^2(0,+\infty;\mathbb{C})} \leq C_{\varepsilon,p,\rho,\mathcal{N}} e^{\varepsilon \lambda_{k,1}^m} \left(\max_{j \in \llbracket 1, g_k^m \rrbracket} |e_z[\lambda_{k,1}^m, \dots, \lambda_{k,j}^m]| \right),$$

744 where e_z is defined in (5).

745 The previous right-hand side is estimated using Lagrange theorem (see Propo-
 746 sition 7.4). As the function e_z is complex-valued we apply it on both its real and
 747 imaginary parts. This yields

$$748 \quad \left(\max_{j \in \llbracket 1, g_k^m \rrbracket} |e_z[\lambda_{k,1}^m, \dots, \lambda_{k,j}^m]| \right) \leq C_{p,\rho} |z|^p e^{-\Re(z) \lambda_{k,1}^m}.$$

749 Thus,

$$750 \quad \|\tilde{q}_k^{m,z}\|_{L^2(0,+\infty;\mathbb{C})} \leq C_{\varepsilon,p,\rho,\mathcal{N}} |z|^p e^{-(\Re(z)-\varepsilon) \lambda_{k,1}^m}.$$

751 Then, using (49) it comes that, for m sufficiently large,

$$752 \quad \langle P^m, \overline{\tilde{q}_k^{m,z}} \rangle_{L^2(0,+\infty;\mathbb{C})} = \sum_{k'=1}^{K^m} \sum_{j=1}^{g_{k'}^m} a_{k',j}^m \int_0^{+\infty} e^{-\lambda_{k',j}^m t} \tilde{q}_k^{m,z}(t) dt = \sum_{j=1}^{g_k^m} a_{k,j}^m e^{-\lambda_{k,j}^m z}.$$

753 From Cauchy-Schwarz inequality we deduce that

$$754 \quad \left| \sum_{j=1}^{g_k^m} a_{k,j}^m e^{-\lambda_{k,j}^m z} \right| \leq \|P^m\|_{L^2(0,+\infty;\mathbb{C})} \|\tilde{q}_k^{m,z}\|_{L^2(0,+\infty;\mathbb{C})} \leq C_{\varepsilon,p,\rho,\mathcal{N}} |z|^p e^{-(\Re(z)-\varepsilon) \lambda_{k,1}^m}.$$

755 Summing these inequalities we obtain that for any $z \in \mathbb{C}_{2\varepsilon}^+$,

$$756 \quad |P^m(z)| \leq \sum_{k=1}^{K^m} \left| \sum_{j=1}^{g_k^m} a_{k,j}^m e^{-\lambda_{k,j}^m z} \right| \leq C_{\varepsilon,p,\rho,\mathcal{N}} |z|^p \sum_{k \geq 1} e^{-(\Re(z)-\varepsilon) \lambda_{k,1}^m},$$

757 From the properties of the groupings (see Definition 1.2), it comes that $\lambda_{k,1}^m \geq \frac{\rho}{p}(k-1)$.

758 Thus, for any $z \in \mathbb{C}_{2\varepsilon}^+$,

$$759 \quad (50) \quad |P^m(z)| \leq C_{\varepsilon,p,\rho,\mathcal{N}} |z|^p e^{-\frac{\rho}{p}(\Re(z)-\varepsilon)} \sum_{k \geq 1} e^{-\varepsilon \frac{\rho}{p}(k-2)} \leq C_{\varepsilon,p,\rho,\mathcal{N}} |z|^p e^{-\frac{\rho}{p} \Re(z)}.$$

760 This gives that $(P^m)_m$ is a sequence of holomorphic functions uniformly bounded
 761 on any compact of $\mathbb{C}_{2\varepsilon}^+$. From Montel's theorem it comes that we can extract a
 762 subsequence converging uniformly on any compact of $\mathbb{C}_{2\varepsilon}^+$ to an holomorphic function
 763 P .

764 Now recall that $\|P^m\|_{L^2(0,T;\mathbb{C})}$ goes to 0 as m goes to infinity. This implies that
 765 $P(t) = 0$ for any $t \in (2\varepsilon, T)$. The function P being holomorphic it comes that it
 766 vanishes on $\mathbb{C}_{2\varepsilon}^+$. Using (50) and the Lebesgue dominated-convergence theorem yields

$$767 \quad \|P^m\|_{L^2(0,+\infty;\mathbb{C})} \xrightarrow{m \rightarrow \infty} 0.$$

768 This is in contradiction with $\|P^m\|_{L^2(0,+\infty;\mathbb{C})} = 1$ and ends the proof of Proposi-
 769 tion 2.4. \square

770 We now have all the ingredients to prove Proposition 2.3.

771 *Proof (of Proposition 2.3).* This proof follows closely the one of [4, Section 4]
 772 and [5, Lemma 4.2]. From [4, Corollary 4.3], as Λ satisfies (8), it comes that $A(\Lambda, +\infty)$
 773 is a proper subspace of $L^2(0, +\infty; \mathbb{C})$. Let Π_Λ the associated orthogonal projection.
 774 Let $\tilde{q} \in L^2(0, +\infty, \mathbb{C})$. Then, by construction, we have

$$775 \quad (51) \quad \int_0^{+\infty} \Pi_\Lambda \tilde{q}(t) e^{-\lambda t} dt = \int_0^{+\infty} \tilde{q}(t) e^{-\lambda t} dt, \quad \forall \lambda \in \Lambda.$$

776 From Proposition 2.4, the restriction operator $R_{\Lambda,T}$ defined by (46) is an isomorphism.
 777 Thus, setting $q := (R_{\Lambda,T}^{-1})^* \Pi_\Lambda \tilde{q}$ ends the proof of Proposition 2.3. Indeed, there exists
 778 $C_{T,p,\rho,\mathcal{N}} > 0$ such that

$$779 \quad \|q\|_{L^2(0,T;\mathbb{C})} \leq C_{T,p,\rho,\mathcal{N}} \|\tilde{q}\|_{L^2(0,+\infty;\mathbb{C})},$$

780 and, using (51), for every $\lambda \in \Lambda$,

$$781 \quad \int_0^T q(t) e^{-\lambda t} dt = \langle (R_{\Lambda,T}^{-1})^* \Pi_\Lambda \tilde{q}, e_\lambda \rangle_{L^2(0,T)} = \langle \Pi_\Lambda \tilde{q}, R_{\Lambda,T}^{-1} R_{\Lambda,T} e_\lambda \rangle_{L^2(0,+\infty)}$$

$$782 \quad = \int_0^{+\infty} \tilde{q}(t) e^{-\lambda t} dt. \quad \square$$

784 We can now conclude the proof of Theorem 2.1 for simple eigenvalues.

785 *Proof (of Theorem 2.1).* The resolution of the block moment problem (30) as well
 786 as the estimate (31) follow directly from Propositions 2.1 and 2.3.

787 The only thing left to prove is the lower bound (32). Let $q_k \in L^2(0, T; \mathbb{C})$ be any
 788 solution of (30b). Using the linearity of divided differences, equalities (30b) imply
 789 that for any $i \in \llbracket 1, g_k \rrbracket$

$$790 \quad (52) \quad \omega[\lambda_{k,1}, \dots, \lambda_{k,i}] = \int_0^T q_k(t) e_t[\lambda_{k,1}, \dots, \lambda_{k,i}] dt$$

791 where e_t is defined by (5). From the Lagrange theorem (see Proposition 7.4) and the
 792 fact that $e_t^{(i-1)}$ is decreasing on $(0, +\infty)$, it comes that for any $t \in (0, T)$, we have

$$793 \quad |e_t[\lambda_{k,1}, \dots, \lambda_{k,i}]| \leq t^p e^{-\lambda_{k,1} t} \leq t^p e^{-t},$$

794 since we assumed that $\Lambda \subset [1, +\infty)$. Thus applying the Cauchy-Schwarz inequality
 795 to (52) gives

$$796 \quad |\omega[\lambda_{k,1}, \dots, \lambda_{k,i}]| \leq \|q_k\|_{L^2(0,T;\mathbb{C})} \left(\int_0^T t^{2p} e^{-t} dt \right)^{\frac{1}{2}} \leq (2p)! \|q_k\|_{L^2(0,T;\mathbb{C})}$$

797 which proves (32) and ends the proof of Theorem 2.1. \square

798 **2.1.3. Construction of the control.**

799 In this section we gather the previous ingredients to prove the positive controlla-
800 bility result. We also give an upper bound and a lower bound on the cost of control-
801 lability from Y_0 at time T as defined by Lemma 1.1.

802 *Proof (of point i. of Theorem 1.1 for simple eigenvalues).*

803 Assume that $T_0(Y_0) < +\infty$ and let us consider an initial data $y_0 \in Y_0$. Without
804 loss of generality we assume that $\|y_0\|_{-\diamond} = 1$.

805 Let $T \in (T_0(Y_0), +\infty)$ and $\varepsilon > 0$ be such that $T > T_0(Y_0) + 2\varepsilon$. From Proposi-
806 tion 1.3, it comes that $T > T_0(y_0) + 2\varepsilon$.

807 For any $k \geq 1$ and $j \in \llbracket 1, g_k \rrbracket$ we set

808
$$\omega_{k,j} := e^{-\lambda_{k,j}T} \langle y_0, \psi_{k,j} \rangle_{-\diamond, \diamond}.$$

809 Let $(q_k)_{k \geq 1}$ be the solution of the block moment problem (30) given in Theorem 2.1.

810 There exists a constant $C_{\varepsilon, T, p, r, \rho, \mathcal{N}} > 0$ such that

811
$$\|q_k\|_{L^2(0, T; \mathbb{R})} \leq C_{\varepsilon, T, p, r, \rho, \mathcal{N}} e^{\varepsilon \lambda_{k,1}} \max_{i \in \llbracket 1, g_k \rrbracket} \left| \omega[\lambda_{k,1}, \dots, \lambda_{k,i}] \right|, \quad \forall k \geq 1.$$

812 Let $\xi_{k,j} := \langle y_0, \psi_{k,j} \rangle_{-\diamond, \diamond}$. Notice that $\omega_{k,j} = e_T(\lambda_{k,j}) \xi_{k,j}$, where e_T is defined in (5).

813 From Leibniz formula (see Proposition 7.6),

814 (53)
$$\omega[\lambda_{k,1}, \dots, \lambda_{k,i}] = \sum_{m=1}^i e_T[\lambda_{k,m}, \dots, \lambda_{k,i}] \xi[\lambda_{k,1}, \dots, \lambda_{k,m}].$$

815 In this expression, $\xi[\dots]$ stands for the divided differences associated with the values
816 $(\lambda_{k,1}, \xi_{k,1}), \dots, (\lambda_{k,g_k}, \xi_{k,g_k})$. From Lagrange theorem (see Proposition 7.4) it comes
817 that

818
$$e_T[\lambda_{k,m}, \dots, \lambda_{k,i}] = \frac{e_T^{(i-m)}(\theta_k)}{(i-m)!} = \frac{(-T)^{i-m}}{(i-m)!} e^{-\theta_k T}$$

819 with $\theta_k \in [\lambda_{k,m}, \lambda_{k,i}]$. Using the definition (18) of $T_0(y_0)$, it comes that,

820 (54)
$$\left| \xi[\lambda_{k,1}, \dots, \lambda_{k,m}] \right| = \left| \langle y_0, \psi[\lambda_{k,1}, \dots, \lambda_{k,m}] \rangle_{-\diamond, \diamond} \right| \leq C e^{\lambda_{k,1}(T_0(y_0) + \varepsilon)}.$$

821 Thus, there exists $C_{T,p} > 0$ such that

822
$$\left| \omega[\lambda_{k,1}, \dots, \lambda_{k,i}] \right| \leq C_{T,p} e^{\lambda_{k,1}(T_0(y_0) + \varepsilon - T)}.$$

823 Then, as $T > T_0(y_0) + 2\varepsilon$, the series (24) is convergent in $L^2(0, T; \mathbb{R})$ and defines
824 a control u that solves the moment problem (26), which implies that the associated
825 solution of (1) satisfies $y(T) = 0$. \square

826 With the same strategy we can prove a more accurate result. Namely we get the
827 following uniform bound for the cost of controllability.

828 **COROLLARY 2.2.** *Let $p \in \mathbb{N}^*$, $r, \rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$. Assume that*
829 *$\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$ and let $(G_k)_k \in \mathcal{G}(\Lambda, p, r, \rho)$ be an associated grouping. Assume that*
830 *$\eta = 1$ and $T_0(Y_0) < +\infty$. For any $T > T_0(Y_0)$, let $C^* > 0$ be such that*

831 (55)
$$\max_{l \in \llbracket 1, g_k \rrbracket} \|\psi[\lambda_{k,1}, \dots, \lambda_{k,l}]\|_{\diamond^*} \leq C^* e^{\lambda_{k,1} \frac{T_0(Y_0) + T}{2}}, \quad \forall k \geq 1.$$

832 Then, there exists a constant $C_{T_0(Y_0),T,p,r,\rho,\mathcal{N}} > 0$ such that for any $y_0 \in Y_0$, there
833 exists a control $u \in L^2(0,T;\mathbb{R})$ such that the associated solution y of (1) satisfies
834 $y(T) = 0$ and

$$835 \quad \|u\|_{L^2(0,T;\mathbb{R})} \leq C_{T_0(Y_0),T,p,r,\rho,\mathcal{N}} \mathcal{C}^* \|y_0\|_{-\diamond}.$$

836 *Proof.* We follow the same strategy as in the proof of point i. of Theorem 1.1
837 with $\varepsilon = \frac{T-T_0(Y_0)}{4}$ but we do not use (54). Instead notice that using (55) we have

$$838 \quad \left| \xi[\lambda_{k,1}, \dots, \lambda_{k,m}] \right| = \left| \langle y_0, \psi[\lambda_{k,1}, \dots, \lambda_{k,m}] \rangle_{-\diamond, \diamond} \right| \leq \mathcal{C}^* e^{\lambda_{k,1} \frac{T_0(Y_0)+T}{2}} \|y_0\|_{-\diamond}.$$

839 From (53) it comes that

$$840 \quad \left| \omega[\lambda_{k,1}, \dots, \lambda_{k,m}] \right| \leq C_{T,p,r,\rho,\mathcal{N}} e^{-\lambda_{k,1}T} \mathcal{C}^* e^{\lambda_{k,1} \frac{T_0(Y_0)+T}{2}} \|y_0\|_{-\diamond}.$$

841 Thus, writing that $\|u\| \leq \sum_{k \geq 1} \|q_k\|$, we get

$$842 \quad \|u\|_{L^2(0,T;\mathbb{R})} \leq C_{T,p,r,\rho,\mathcal{N}} \mathcal{C}^* \|y_0\|_{-\diamond} \sum_{k \geq 1} e^{\lambda_{k,1} \frac{T_0(Y_0)-T}{2}}.$$

843 From Definition 1.2 it comes that $\lambda_{k,1} \geq r(k-1)$ which ends the proof of Corol-
844 lary 2.2. \square

845 We also provide the following lower bound for the cost of controllability.

846 **COROLLARY 2.3.** *Let $p \in \mathbb{N}^*$, $r, \rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$. Assume that*
847 $\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$ and let $(G_k)_k \in \mathcal{G}(\Lambda, p, r, \rho)$ be an associated grouping. Let $T > 0$
848 and $Y_0 \subset X_{-\diamond}$. Assume that $\eta = 1$. If system (1) is null-controllable from Y_0 at time
849 T then,

$$850 \quad M(Y_0, T) \geq \sup_{k \geq 1} \max_{l \in \llbracket 1, g_k \rrbracket} \frac{l! \sqrt{\lambda_{k,1}}}{T^l} \|(e_T \psi)[\lambda_{k,1}, \dots, \lambda_{k,l}]\|_{\diamond^*}$$

851 where e_T is defined by (5), ψ is defined by (15) and $M(Y_0, T)$ the cost of controllability
852 from Y_0 at time T is defined in Lemma 1.1.

853 *Proof.* Let $k \geq 1$ and $l \in \llbracket 1, g_k \rrbracket$. If system (1) is null-controllable at time T from
854 Y_0 , we apply Lemma 1.1 with

$$855 \quad z_T := \sum_{j=1}^l \frac{\frac{\phi_{k,j}}{\mathcal{B}^* \phi_{k,j}}}{\prod_{i \in \llbracket 1, l \rrbracket \neq j} (\lambda_{k,j} - \lambda_{k,i})}.$$

856 By definition of z_T we have

$$857 \quad e^{-T\mathcal{A}^*} z_T = \sum_{j=1}^l e^{-\lambda_{k,j}T} \frac{\frac{\phi_{k,j}}{\mathcal{B}^* \phi_{k,j}}}{\prod_{i \in \llbracket 1, l \rrbracket \neq j} (\lambda_{k,j} - \lambda_{k,i})},$$

858 and thus

$$859 \quad M(Y_0, T)^2 \geq \frac{\|P_{Y_0}^*(e^{-T\mathcal{A}^*} z_T)\|_{\diamond^*}^2}{\int_0^T \left| \sum_{j=1}^l \frac{e^{-\lambda_{k,j}t}}{\prod_{i \in \llbracket 1, l \rrbracket \neq j} (\lambda_{k,j} - \lambda_{k,i})} \right|^2 dt}$$

860 From Newton formula (see Proposition 7.3) and Lagrange theorem (see Proposi-
 861 tion 7.4), for any $t \in [0, T]$, there exists ξ_t in $[\lambda_{k,1}, \lambda_{k,l}]$ such that

$$862 \quad \left| \sum_{j=1}^l \frac{e^{-\lambda_{k,j}t}}{\prod_{i \in [1,l] \neq j} (\lambda_{k,j} - \lambda_{k,i})} \right| \leq \frac{t^l}{l!} e^{-\xi_t t} \leq \frac{T^l}{l!} e^{-\lambda_{k,1}t}.$$

863 Using Newton formula (see Proposition 7.3), we have

$$864 \quad P_{Y_0}^*(e^{-TA^*} z_T) = (e_T \psi)[\lambda_{k,1}, \dots, \lambda_{k,l}],$$

865 which ends the proof of Corollary 2.3. \square

866 2.1.4. Estimates on Blaschke products.

867 This aim of this section is to prove the technical estimate stated in Proposition 2.2.
 868 This relies on an extension of the following result by Fattorini and Russell.

869 LEMMA 2.2 (see [27, Theorem 1.1]). *Let $\gamma > 0$ and $\mathcal{J} : \mathbb{R}^+ \rightarrow \mathbb{R}$. Let $\mathcal{L}(\gamma, \mathcal{J})$ be
 870 the class introduced in Remark 2.1. For any $\Sigma \in \mathcal{L}(\gamma, \mathcal{J})$ and $\sigma \in \Sigma$ we define*

$$871 \quad \mathcal{W}_\sigma^\Sigma : z \in \mathbb{C}^+ \mapsto \prod_{\substack{\sigma' \in \Sigma \\ \sigma' \neq \sigma}} \frac{\sigma' - z}{\sigma' + z}.$$

872 Then, for any $\varepsilon > 0$, there exists $C_{\varepsilon, \gamma, \mathcal{J}} > 0$ such that

$$873 \quad |\mathcal{W}_\sigma^\Sigma(\sigma)| \geq C_{\varepsilon, \gamma, \mathcal{J}} e^{-\varepsilon \sigma}.$$

874 REMARK 2.3. *To be completely accurate let us precise that [27, Theorem 1.1] does
 875 not exactly state such estimate since this theorem only deals with the estimate of a
 876 biorthogonal family. However, the estimate given in this theorem together with the
 877 [27, equality (2.1)] given during its proof directly yield Lemma 2.2.*

878 The generalisation we propose is the following.

879 LEMMA 2.3. *Let $\gamma > 0$ and $\mathcal{J} : \mathbb{R}^+ \rightarrow \mathbb{R}$. For any $\varepsilon > 0$, there exists $C_{\varepsilon, \gamma, \mathcal{J}} > 0$
 880 such that, for any sequence $\Sigma \in \mathcal{L}(\gamma, \mathcal{J})$, for any $\sigma \in \Sigma$, we have*

$$881 \quad |\mathcal{W}_\sigma^\Sigma(z)| \geq C_{\varepsilon, \gamma, \mathcal{J}} e^{-\varepsilon \sigma}, \quad \forall z \in \mathbb{C}^+, \text{ s.t. } |z - \sigma| \leq \frac{\gamma}{2}$$

882 *Proof.* For any $\sigma' > 0$, since $(\sigma' - \Re(z))^2 \leq (\sigma' + \Re(z))^2$, it comes that

$$883 \quad \left| \frac{\sigma' - z}{\sigma' + z} \right|^2 = \frac{(\sigma' - \Re(z))^2 + \Im(z)^2}{(\sigma' + \Re(z))^2 + \Im(z)^2} \geq \frac{(\sigma' - \Re(z))^2}{(\sigma' + \Re(z))^2},$$

884 and thus,

$$885 \quad (56) \quad |\mathcal{W}_\sigma^\Sigma(z)| \geq |\mathcal{W}_\sigma^\Sigma(\Re(z))|.$$

We introduce the family $\tilde{\Sigma}$ obtained from Σ by replacing σ by $\Re(z)$, that is

$$\tilde{\Sigma} := (\Sigma \setminus \{\sigma\}) \cup \{\Re(z)\}.$$

Since only one value has been modified, $\tilde{\Sigma}$ also satisfies

$$\sum_{\tilde{\sigma} \in \tilde{\Sigma}} \frac{1}{\tilde{\sigma}} < +\infty.$$

886 As,

$$887 \quad (57) \quad |\Re(z) - \sigma| \leq |z - \sigma| \leq \frac{\gamma}{2},$$

888 it comes that $\tilde{\Sigma}$ satisfies the gap condition (29) with ρ replaced by $\frac{\gamma}{2}$. Notice that
889 $\{\Re(z)\} \in \mathcal{L}(1, \varepsilon \mapsto \frac{1}{\varepsilon})$. Thus using Remark 2.2 and the arguments of the proof of

890 Lemma 2.1 it comes that $\tilde{\Sigma} \in \mathcal{L}(\frac{\gamma}{2}, \tilde{\mathcal{J}})$ with $\tilde{\mathcal{J}}$ depending only on \mathcal{J} .

891 Obviously, as the terms $\sigma' \in \Sigma$ that are different from σ have not been modified
892 it comes that

$$893 \quad \mathcal{W}_{\sigma}^{\Sigma} = \mathcal{W}_{\Re(z)}^{\tilde{\Sigma}}.$$

894 Applying Lemma 2.2 it comes that for any $\varepsilon > 0$, there is $C_{\varepsilon, \gamma, \mathcal{J}} > 0$ such that

$$895 \quad |\mathcal{W}_{\Re(z)}^{\tilde{\Sigma}}(\Re(z))| \geq C_{\varepsilon, \gamma, \mathcal{J}} e^{-\varepsilon \Re(z)}.$$

896 Finally, recalling (56) and (57), we obtain

$$897 \quad |\mathcal{W}_{\sigma}^{\Sigma}(z)| = |\mathcal{W}_{\Re(z)}^{\tilde{\Sigma}}(z)| \geq |\mathcal{W}_{\Re(z)}^{\tilde{\Sigma}}(\Re(z))| \geq C_{\varepsilon, \gamma, \mathcal{J}} e^{-\varepsilon \Re(z)} \geq C_{\varepsilon, \gamma, \mathcal{J}} e^{-\varepsilon \frac{\gamma}{2}} e^{-\varepsilon \sigma}$$

898 which ends the proof of Lemma 2.3. \square

899 We now turn to the estimates we need for the derivatives of $\frac{1}{\mathcal{W}_{\sigma}^{\Sigma}}$.

900 PROPOSITION 2.5. *Let $\gamma > 0$ and $\mathcal{J} : \mathbb{R}^+ \rightarrow \mathbb{R}$.*

901 *Then, for any $l \geq 0$, for any $\varepsilon > 0$, there exists $C_{l, \varepsilon, \gamma, \mathcal{J}} > 0$ such that for any*
902 $\Sigma \in \mathcal{L}(\gamma, \mathcal{J})$,

$$903 \quad \left| \left(\frac{1}{\mathcal{W}_{\sigma}^{\Sigma}} \right)^{(l)}(\sigma) \right| \leq C_{l, \varepsilon, \gamma, \mathcal{J}} e^{\varepsilon \sigma}, \quad \forall \sigma \in \Sigma.$$

904 *Proof.* The case $l = 0$ is nothing but the estimate given in Lemma 2.3.

905 Let

$$906 \quad D_{\sigma, \gamma} := \left\{ z \in \mathbb{C}^+ ; |z - \sigma| \leq \frac{\gamma}{2} \right\}, \quad \mathcal{C}_{\sigma, \gamma} := \left\{ z \in \mathbb{C}^+ ; |z - \sigma| = \frac{\gamma}{2} \right\}.$$

908 As $\mathcal{W}_{\sigma}^{\Sigma}$ does not vanish in an open neighbourhood of $D_{\sigma, \gamma}$ it comes that $\frac{1}{\mathcal{W}_{\sigma}^{\Sigma}}$ is
909 holomorphic on this domain. Thus applying Cauchy formula yields

$$910 \quad \left(\frac{1}{\mathcal{W}_{\sigma}^{\Sigma}} \right)^{(l)}(\sigma) = \frac{l!}{2i\pi} \int_{\mathcal{C}_{\sigma, \gamma}} \frac{\frac{1}{\mathcal{W}_{\sigma}^{\Sigma}}(z)}{(z - \sigma)^{l+1}} dz.$$

911 From Lemma 2.3 it comes that for any $\varepsilon > 0$ there exists $C_{\varepsilon, \gamma, \mathcal{J}} > 0$ such that

$$912 \quad \left| \left(\frac{1}{\mathcal{W}_{\sigma}^{\Sigma}} \right)(z) \right| \leq C_{\varepsilon, \gamma, \mathcal{J}} e^{\varepsilon \sigma}, \quad \forall z \in \mathcal{C}_{\sigma, \gamma}.$$

913 This directly implies

$$914 \quad \left| \left(\frac{1}{\mathcal{W}_{\sigma}^{\Sigma}} \right)^{(l)}(\sigma) \right| \leq C_{\varepsilon, \gamma, \mathcal{J}} \frac{l!}{\gamma^l} e^{\varepsilon \sigma}$$

915 and ends the proof of Proposition 2.5. \square

916 We shall now move to the proof of Proposition 2.2 which is the main objective of
 917 this section.

918 *Proof (of Proposition 2.2).* Recall that the function $\mathcal{N} : \mathbb{R}^+ \rightarrow \mathbb{R}$ is the one
 919 appearing in (28) and that the subsequences Λ_j are defined in (39).

920 We recall that the index k is fixed, as well as the value $\theta \in \text{Conv}(G_k)$. We intro-
 921 duce the new sequence $\tilde{\Lambda}_j$ obtained from Λ_j by replacing the k -th value $\lambda_{k, \min(j, g_k)}$
 922 by θ i.e.

$$923 \quad \tilde{\Lambda}_j := (\Lambda_j \setminus \{\lambda_{k, \min(j, g_k)}\}) \cup \{\theta\}.$$

924 Notice that, using Proposition 7.1, the fact that Λ_j is a subsequence of Λ such that
 925 each term belong to a different group, and by the assumption on θ , we obtain that
 926 $\tilde{\Lambda}_j$ satisfies the gap condition (29) with ρ replaced by $\gamma = \frac{\rho}{p}$. Notice that $\{\theta\} \in$
 927 $\mathcal{L}(1, \varepsilon \mapsto \frac{1}{\varepsilon})$. Thus using Remark 2.2 and the arguments of the proof of Lemma 2.1
 928 it comes that $\tilde{\Lambda}_j \in \mathcal{L}(\gamma, \tilde{\mathcal{J}})$ with $\tilde{\mathcal{J}}$ depending only on \mathcal{N} .

929 With these notations and Proposition 7.1 it comes that

$$930 \quad \frac{1}{W_k(z)} = \prod_{j=1}^p \frac{1}{W_{\theta}^{\tilde{\Lambda}_j}(z)}.$$

931 Finally, using Leibniz rule (for derivatives), evaluating the result at $z = \theta$ and using
 932 Proposition 2.5 yield the claim. \square

933 2.2. Lack of null-controllability in small time.

934 The goal of this section is to prove the point ii. of Theorem 1.1 in the case of
 935 algebraically simple eigenvalues. Thus, in all this section we assume that $\eta = 1$. The
 936 proof mainly relies on the optimality on the bound obtained in the resolution of the
 937 block moments problem in (32).

938 *Proof.* Let $T > 0$ and assume that null controllability from Y_0 in time T of (1)
 939 holds.

940 Thus, there exists $C_T > 0$ such that: for any $y_0 \in X_{-\diamond}$, there exists $u \in$
 941 $L^2(0, T; \mathbb{R})$ such that the associated solution of (1) with initial condition $P_{Y_0} y_0$ satis-
 942 fies $y(T) = 0$ and $\|u\|_{L^2(0, T; \mathbb{R})} \leq C_T \|y_0\|_{-\diamond}$.

943 Due to the equivalence between null controllability and the moment problem (23)
 944 it comes that

$$945 \quad \int_0^T e^{-\lambda(T-t)} u(t) dt = -\langle y_0, e^{-\lambda T} \psi_\lambda \rangle_{-\diamond, \diamond}, \quad \forall \lambda \in \Lambda.$$

946 Recall that ψ_λ is defined by (15). Thus, for any $k \in \mathbb{N}^*$, the control $u(T - \bullet)$
 947 solves (30b). Using the lower bound (32) with $\omega_{k, j} := \langle y_0, e^{-\lambda_{k, j} T} \psi_{k, j} \rangle_{-\diamond, \diamond}$, it comes
 948 that

$$949 \quad |\omega[\lambda_{k, 1}, \dots, \lambda_{k, l}]| \leq C_p \|u\|_{L^2(0, T; \mathbb{R})} \leq C_{p, T} \|y_0\|_{-\diamond}, \quad \forall k \in \mathbb{N}^*, \forall l \in \llbracket 1, g_k \rrbracket.$$

950 Due to the definition of $\omega_{k, j}$, this can be rewritten as

$$951 \quad |\langle y_0, (e_T \psi)[\lambda_{k, 1}, \dots, \lambda_{k, l}] \rangle_{-\diamond, \diamond}| \leq C_{p, T} \|y_0\|_{-\diamond}, \quad \forall k \in \mathbb{N}^*, \forall l \in \llbracket 1, g_k \rrbracket,$$

952 where e_t is defined by (5). By the dual characterization of the norms this implies

$$953 \quad (58) \quad \|(e_T \psi)[\lambda_{k, 1}, \dots, \lambda_{k, l}]\|_{\diamond^*} \leq C_{p, T}, \quad \forall k \in \mathbb{N}^*, \forall l \in \llbracket 1, g_k \rrbracket,$$

954 Notice that $\psi_{k,j} = e^{\lambda_{k,j}T} e^{-\lambda_{k,j}T} \psi_{k,j} = e_{-T}[\lambda_{k,j}](e_T\psi)[\lambda_{k,j}]$. Thus, using Leibniz
 955 formula (see Proposition 7.6), we obtain,
 (59)

$$956 \quad \psi[\lambda_{k,1}, \dots, \lambda_{k,l}] = \sum_{j=1}^l (e_T\psi)[\lambda_{k,1}, \dots, \lambda_{k,j}] e_{-T}[\lambda_{k,j}, \dots, \lambda_{k,l}], \quad \forall k \in \mathbb{N}^*, \forall l \in \llbracket 1, g_k \rrbracket.$$

957 From Proposition 7.4, it comes that for any $j, l \in \llbracket 1, g_k \rrbracket$, there exists $z \in [\lambda_{k,j}, \lambda_{k,l}]$
 958 such that

$$959 \quad |e_{-T}[\lambda_{k,j}, \dots, \lambda_{k,l}]| = \left| \frac{e^{(l-j)}(z)}{(l-j)!} \right| \leq C_p e^{\lambda_{k,j}T} \leq C_{p,\rho,T} e^{\lambda_{k,1}T}.$$

960 Finally, plugging this estimate and (58) into (59) we obtain,

$$961 \quad \|\psi[\lambda_{k,1}, \dots, \lambda_{k,l}]\|_{\diamond^*} \leq C_{p,\rho,T} e^{\lambda_{k,1}T}.$$

962 Due to the definition of $T_0(Y_0)$, this implies that $T \geq T_0(Y_0)$ and ends the proof on
 963 the lack of null-controllability at time $T < T_0(Y_0)$. \square

964 3. Comparison with some already known results.

965 In this section, we prove that we actually recover the known formulas for the
 966 minimal null-control time when there is no condensation of eigenvalues or when the
 967 eigenvectors are assumed to form a Riesz basis of X_\diamond^* . Doing so we will highlight in
 968 Proposition 3.2 that the actual minimal null-control time is always smaller than the
 969 value predicted by the formula that would be valid under the Riesz basis assumption.
 970 As all these results were proved for algebraically simple eigenvalues we assume in all
 971 this section that $\eta = 1$.

972 Notice that the proofs in all this section only rely on the definition of the minimal
 973 null-control time (19) and thus do not depend on Theorem 1.1.

974 3.1. When there is no condensation of eigenvalues.

975 In this section we prove that, if the condensation index of the sequence Λ vanishes
 976 (the definition of $c(\Lambda)$ is recalled in Appendix 7.5) then the expression (19) coincides
 977 with the known expression relating the minimal time for null-controllability to the
 978 observation of the eigenvectors ϕ_λ through the operator \mathcal{B}^* .

979 PROPOSITION 3.1. *Assume that \mathcal{A} and \mathcal{B} satisfy the assumptions of Theorem 1.1*
 980 *with $\eta = 1$. If $c(\Lambda) = 0$, then, we have*

$$981 \quad T_0(X_{-\diamond}) = \limsup_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Lambda}} \frac{-\ln |\mathcal{B}^* \phi_\lambda|}{\lambda}.$$

982 This result was already proved in [5] with the additional assumption that the
 983 family of eigenvectors $\Phi = (\phi_\lambda)_{\lambda \in \Lambda}$ forms a Riesz basis of X_\diamond^* or in [7, Remark 1.15]
 984 in a more general framework encompassing the one studied here.

985 *Proof.* Notice that when $Y_0 = X_{-\diamond}$, the operator $P_{Y_0}^*$ reduces to the identity.
 986 Thus, considering $l = 1$ in (19) always lead to

$$987 \quad T_0(X_{-\diamond}) \geq \limsup_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Lambda}} \frac{-\ln |\mathcal{B}^* \phi_\lambda|}{\lambda}.$$

988 We assume that

$$989 \quad (60) \quad T_0(X_{-\diamond}) > \limsup_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Lambda}} \frac{-\ln |\mathcal{B}^* \phi_\lambda|}{\lambda},$$

990 and we will prove that $c(\Lambda) > 0$.

991 We shall reason as in the proof of point **ii.** of Theorem 1.1 (see Section 2.2) but
 992 starting with the formula (17) instead of (18). We can find an integer $l^* \geq 1$, an
 993 extraction $(\kappa_n)_{n \geq 1}$ and integers m_n such that $1 \leq m_n \leq m_n + l^* - 1 \leq g_{\kappa_n}$ and such
 994 that if

$$995 \quad x_n := \psi[\lambda_{\kappa_n, m_n}, \dots, \lambda_{\kappa_n, m_n + l^* - 1}],$$

996 we have

$$997 \quad \lim_{n \rightarrow \infty} \frac{\ln \|x_n\|_{\diamond^*}}{\lambda_{\kappa_n, 1}} = T_0(X_{-\diamond}).$$

998 Moreover, we can assume that for any $l \in \llbracket 1, l^* - 1 \rrbracket$, we have for some $\varepsilon > 0$

$$999 \quad \tilde{T}(l) := \limsup_{n \rightarrow +\infty} \frac{\ln \left(\max_{\substack{m, r \in \llbracket 1, g_{\kappa_n} \rrbracket \\ m \leq r \\ r - m < l}} \|\psi[\lambda_{\kappa_n, m}, \dots, \lambda_{\kappa_n, r}]\|_{\diamond^*} \right)}{\lambda_{\kappa_n, 1}} < T_0(X_{-\diamond}) - \varepsilon,$$

1000 since, if it is not the case, we can reduce the value of l^* accordingly. Note that, as
 1001 $\|P_{Y_0}^* \phi_\lambda\|_{\diamond^*} \leq 1$, by (60), we know that $l^* > 1$.

1002 From the definition of divided differences (see Definition 7.1), it comes that

$$1003 \quad x_n = \frac{\psi[\lambda_{\kappa_n, m_n + 1}, \dots, \lambda_{\kappa_n, m_n + l^* - 1}] - \psi[\lambda_{\kappa_n, m_n}, \dots, \lambda_{\kappa_n, m_n + l^* - 2}]}{\lambda_{\kappa_n, m_n + l^* - 1} - \lambda_{\kappa_n, m_n}}.$$

1004 For n sufficiently large, we have

$$1005 \quad \|x_n\|_{\diamond^*} \geq e^{\lambda_{\kappa_n, 1}(T_0(X_{-\diamond}) - \varepsilon/2)}.$$

1006 Using the definition of $\tilde{T}(l^* - 1)$ it comes that, for n large enough,

$$1007 \quad \|\psi[\lambda_{\kappa_n, m_n + 1}, \dots, \lambda_{\kappa_n, m_n + l^* - 1}]\|_{\diamond^*} + \|\psi[\lambda_{\kappa_n, m_n}, \dots, \lambda_{\kappa_n, m_n + l^* - 2}]\|_{\diamond^*} \\ 1008 \quad \leq e^{\lambda_{\kappa_n, 1}(\tilde{T}(l^* - 1) + \varepsilon/2)}.$$

1009 Thus, since $l^* \geq 2$, we can combine the last two estimates to obtain

$$1010 \quad |\lambda_{\kappa_n, m_n + 1} - \lambda_{\kappa_n, m_n}| \leq |\lambda_{\kappa_n, m_n + l^* - 1} - \lambda_{\kappa_n, m_n}| \\ 1011 \quad \leq e^{-\lambda_{\kappa_n, 1}(T_0(X_{-\diamond}) - \varepsilon - \tilde{T}(l^* - 1))} \\ 1012 \quad \leq e^{\rho(T_0(X_{-\diamond}) - \varepsilon - \tilde{T}(l^* - 1))} e^{-\lambda_{\kappa_n, m_n}(T_0(X_{-\diamond}) - \varepsilon - \tilde{T}(l^* - 1))}.$$

1013 In particular, we have

$$1014 \quad \limsup_{n \rightarrow \infty} \frac{-\ln |P'_{G_{\kappa_n}}(\lambda_{\kappa_n, m_n})|}{\lambda_{\kappa_n, m_n}} \geq \limsup_{n \rightarrow \infty} \frac{-\ln |\lambda_{\kappa_n, m_n + 1} - \lambda_{\kappa_n, m_n}|}{\lambda_{\kappa_n, m_n}} \\ 1015 \quad \geq T_0(X_{-\diamond}) - \varepsilon - \tilde{T}(l^* - 1) > 0.$$

1016 Using Proposition 7.12, we conclude that $c(\Lambda) > 0$, and the claim is proved. \square

1018 **3.2. When there is a Riesz basis of eigenvectors.**

1019 As already mentioned the null-control problem for (1) has been considered in [5]
 1020 with the additional assumption that the family $(\phi_\lambda)_{\lambda \in \Lambda}$ forms a Riesz basis of X_\diamond^* .
 1021 Observe that it is equivalent to ask that $(\phi_\lambda / \|\phi_\lambda\|)_{\lambda \in \Lambda}$ is a Riesz basis of X .

1022 With this additional assumption, the minimal null-control time from $Y_0 = X_{-\diamond}$
 1023 was proved to be equal to

$$1024 \quad (61) \quad T^* := \limsup_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Lambda}} \frac{\ln \frac{1}{|\mathcal{B}^* \phi_\lambda| |E'_\Lambda(\lambda)|}}{\lambda},$$

1025 where the interpolating function E_Λ is defined in (126).

1026 **REMARK 3.1.** Notice that, since ϕ_λ is normalized in X_\diamond^* , there exists $C > 0$ such
 1027 that

$$1028 \quad \frac{1}{C\lambda} \leq \|\phi_\lambda\| \leq C, \quad \forall \lambda,$$

1029 so that the value of T^* in (61) does not change if one considers the normalization of
 1030 eigenvectors in X instead of in X_\diamond^* .

1031 In our setting, we prove that the formula above for T^* is always an upper bound
 1032 of the actual minimal null-control time, without assuming the Riesz basis condition.

1033 **PROPOSITION 3.2.** Assume that \mathcal{A} and \mathcal{B} satisfy the assumptions of Theorem 1.1
 1034 with $\eta = 1$. Then, $T_0(X_{-\diamond}) \leq T^*$ where T^* is defined by (61).

1035 *Proof. First step:* we begin by proving that the grouping designed in Proposi-
 1036 tion 7.1 ensures a simpler expression for T^* . Let $(G_k)_{k \geq 1} \in \mathcal{G}(\Lambda, p, r, \rho)$ be a grouping
 1037 as introduced in Section 1.3.2. For each $\lambda \in \Lambda$, we denote by $G^{[\lambda]}$ the unique group
 1038 in $(G_k)_{k \geq 1}$ that contains λ . Then, we have

$$1039 \quad (62) \quad T^* = \limsup_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Lambda}} \frac{\ln \frac{1}{|\mathcal{B}^* \phi_\lambda| |P'_{G^{[\lambda]}(\lambda)}|}}{\lambda},$$

1040 where, for each group G , the polynomial P_G is defined by (6).

1041 • Let G be a group of eigenvalues and $\lambda \in G$. We prove that, for any finite subset M
 1042 of $\Lambda \setminus G$, whose cardinal is denoted by $n := \#M$, we have

$$1043 \quad (63) \quad \prod_{\mu \in M} |\lambda - \mu| \geq r^n \left[\frac{n}{2p} \right]!$$

1044 where $r := \frac{\rho}{p}$. To this end, for any $j \in \llbracket 1, p \rrbracket$, we define

$$1045 \quad M_j := \{\mu \in M ; \exists k \geq 1 \text{ such that } \mu = \lambda_{k,j}\}.$$

1046 Since the groups are covering Λ , we have a disjoint union $M = \bigcup_{j=1}^p M_j$. It follows that

1047 there exists $j_0 \in \llbracket 1, p \rrbracket$ such that $\#M_{j_0} \geq \left\lfloor \frac{n}{p} \right\rfloor$. From (13) it comes that

$$1048 \quad |\lambda - \mu| \geq r, \quad \forall \mu \in M,$$

1049 and

$$1050 \quad |\mu - \mu'| \geq r, \quad \forall j \in \llbracket 1, p \rrbracket, \forall \mu, \mu' \in M_j, \mu \neq \mu'.$$

1051 Then,

$$\begin{aligned}
1052 \quad \prod_{\mu \in M} |\lambda - \mu| &= \left(\prod_{\substack{j=1 \\ j \neq j_0}}^p \prod_{\mu \in M_j} |\lambda - \mu| \right) \left(\prod_{\mu \in M_{j_0}} |\lambda - \mu| \right) \\
1053 \quad &\geq \left(r^{\#(M \setminus M_{j_0})} \right) \left(r^{\#(M_{j_0})} \right) \left[\frac{\#M_{j_0}}{2} \right]! \\
1054
\end{aligned}$$

1055 This proves (63).

1056 • From (63) we apply [5, Theorem 3.8] to obtain that for any subsequence $(\lambda_n)_{n \geq 1} \subset$
1057 Λ ,

$$1058 \quad (64) \quad \lim_{n \rightarrow \infty} \left(\frac{\ln \frac{1}{|E'_\Lambda(\lambda_n)|}}{\lambda_n} - \frac{\ln \frac{1}{|P'_{G[\lambda_n]}(\lambda_n)|}}{\lambda_n} \right) = 0.$$

1059 This directly implies (62).

1060 REMARK 3.2. Notice that (63) is not the exact assumption required in [5, Theo-
1061 rem 3.8]. For this result the authors assumed

$$1062 \quad (65) \quad \prod_{\mu \in M} |\lambda - \mu| \geq r^n n!,$$

1063 with the same notation as in the proof above. We claim that with the exact same proof
1064 it is sufficient to assume (63). Indeed, in the proof of [5, Theorem 3.8], the only point
1065 were assumption (65) is used is the Second step in the middle of page 2097. Then
1066 the term $n!$ is estimated asymptotically using Stirling formula to prove that the term
1067 $\Gamma_{k,1}$ goes to 0 as k goes to ∞ . As the rest of the proof is long, technical and remains
1068 unchanged when replacing (65) by (63) we do not reproduce it here for the sake of
1069 brevity.

1070 *Second step:* we end the proof of Proposition 3.2. Recall that from (34) we have
1071 that there exists $C_{p,\rho} > 0$ such that for any $k \geq 1$, $l \in \llbracket 1, g_k \rrbracket$ and any $j \in \llbracket 1, l \rrbracket$,

$$1072 \quad \prod_{i \in \llbracket 1, l \rrbracket \neq j} |\lambda_{k,j} - \lambda_{k,i}| \geq C_{p,\rho} |P'_{G_k}(\lambda_{k,j})|.$$

1073 As we have considered normalized eigenvectors, and by (4), for any $k \geq 1$ and
1074 any $l \in \llbracket 1, g_k \rrbracket$, we have

$$\begin{aligned}
1075 \quad \left\| \sum_{j=1}^l \frac{\psi_{k,j}}{\prod_{i \in \llbracket 1, l \rrbracket \neq j} (\lambda_{k,j} - \lambda_{k,i})} \right\|_{\diamond^*} &\leq \left\| \sum_{j=1}^l \frac{\frac{\phi_{k,j}}{\mathcal{B}^* \phi_{k,j}}}{\prod_{i \in \llbracket 1, l \rrbracket \neq j} (\lambda_{k,j} - \lambda_{k,i})} \right\|_{\diamond^*} \\
1076 \quad &\leq l \max_{j \in \llbracket 1, l \rrbracket} \left\| \frac{\frac{\phi_{k,j}}{\mathcal{B}^* \phi_{k,j}}}{\prod_{i \in \llbracket 1, l \rrbracket \neq j} (\lambda_{k,j} - \lambda_{k,i})} \right\|_{\diamond^*} \\
1077 \quad &\leq l \max_{j \in \llbracket 1, l \rrbracket} \frac{1}{|\mathcal{B}^* \phi_{k,j}| \prod_{i \in \llbracket 1, l \rrbracket \neq j} |\lambda_{k,j} - \lambda_{k,i}|}. \\
1078
\end{aligned}$$

1079 Using (34) this leads to

$$1080 \quad \left\| \sum_{j=1}^l \frac{\psi_{k,j}}{\prod_{i \in \llbracket 1, l \rrbracket \neq j} (\lambda_{k,j} - \lambda_{k,i})} \right\|_{\diamond^*} \leq Cl \max_{j \in \llbracket 1, l \rrbracket} \frac{1}{|\mathcal{B}^* \phi_{k,j}| |P'_{G_k}(\lambda_{k,j})|}.$$

1081 Thus,

$$1082 \quad \frac{\ln \max_{l \in \llbracket 1, g_k \rrbracket} \left(\left\| \sum_{j=1}^l \frac{\psi_{k,j}}{\prod_{i \in \llbracket 1, l \rrbracket \neq j} (\lambda_{k,j} - \lambda_{k,i})} \right\|_{\diamond^*} \right)}{\lambda_{k,1}} \leq \max_{j \in \llbracket 1, g_k \rrbracket} \frac{\ln \frac{1}{|\mathcal{B}^* \phi_{k,j}| |P'_{G_k}(\lambda_{k,j})|} \lambda_{k,j}}{\lambda_{k,j}} \frac{\lambda_{k,j}}{\lambda_{k,1}} + \frac{\ln(Cl)}{\lambda_{k,1}}.$$

1084 Then, using (62), we obtain

$$1085 \quad T_0(X_{-\diamond}) \leq T^*. \quad \square$$

1086 We now prove that we indeed recover exactly the expression of the minimal
1087 time (61) (or (62)) when we assume that the eigenvectors form a Riesz basis.

1088 **PROPOSITION 3.3.** *Assume that \mathcal{A} and \mathcal{B} satisfy the assumptions of Theorem 1.1*
1089 *with $\eta = 1$ and that $(\phi_\lambda)_{\lambda \in \Lambda}$ forms a Riesz basis of X_\diamond^* . Then, $T_0(X_{-\diamond}) = T^*$ where*
1090 *T^* is defined by (62).*

REMARK 3.3. It will appear clearly in the proof that the Riesz basis assumption is much stronger than what we really need. The only thing that we actually use at the very beginning of the proof, is that the spectral radius of the inverse of the Gram matrix $M_k := \text{Gram}_{X_\diamond^}(\phi_{k,1}, \dots, \phi_{k,g_k})$ satisfies*

$$\sup_{k \geq 1} \rho(M_k^{-1}) < +\infty.$$

A careful inspection of the proof shows that it is in fact sufficient to assume that

$$\limsup_{k \rightarrow \infty} \frac{\ln \rho(M_k^{-1})}{\lambda_{k,1}} = 0.$$

1091 *Note in particular that, in practice, estimating such a spectral radius in each group is*
1092 *much simpler than proving that the whole family is a Riesz basis.*

1093 *Proof.* As we assumed that $(\phi_\lambda)_{\lambda \in \Lambda}$ is a Riesz basis of X_\diamond^* it comes that there
1094 exists $C > 0$ such that for any $k \geq 1$, for any $\alpha_{k,1}, \dots, \alpha_{k,g_k} \in \mathbb{R}$,

$$1095 \quad \max_{j \in \llbracket 1, g_k \rrbracket} |\alpha_{k,j}| \leq \left(\sum_{j=1}^{g_k} \alpha_{k,j}^2 \right)^{\frac{1}{2}} \leq C \left\| \sum_{j=1}^{g_k} \alpha_{k,j} \phi_{k,j} \right\|_{\diamond^*},$$

1096 and thus

$$1097 \quad \max_{j \in \llbracket 1, g_k \rrbracket} |\alpha_{k,j}| \leq C \max_{l \in \llbracket 1, g_k \rrbracket} \left\| \sum_{j=1}^l \alpha_{k,j} \phi_{k,j} \right\|_{\diamond^*}.$$

1098 Setting

$$1099 \quad \alpha_{k,j} := \frac{1}{\mathcal{B}^* \phi_{k,j} \prod_{i \in \llbracket 1, g_k \rrbracket \neq j} (\lambda_{k,j} - \lambda_{k,i})},$$

1100 yield

$$\begin{aligned}
1101 \quad \max_{j \in \llbracket 1, g_k \rrbracket} \frac{1}{|\mathcal{B}^* \phi_{k,j}| \prod_{i \in \llbracket 1, g_k \rrbracket \neq j} |\lambda_{k,j} - \lambda_{k,i}|} &\leq C \left\| \sum_{j=1}^{g_k} \frac{\frac{\phi_{k,j}}{\mathcal{B}^* \phi_{k,j}}}{\prod_{i \in \llbracket 1, g_k \rrbracket \neq j} (\lambda_{k,j} - \lambda_{k,i})} \right\|_{\diamond^*} \\
1102 \quad &\leq C \max_{1 \leq l \leq g_k} \left(\left\| \sum_{j=1}^l \frac{\frac{\phi_{k,j}}{\mathcal{B}^* \phi_{k,j}}}{\prod_{i \in \llbracket 1, l \rrbracket \neq j} (\lambda_{k,j} - \lambda_{k,i})} \right\|_{\diamond^*} \right).
\end{aligned}$$

1104 It follows that for any $j \in \llbracket 1, g_k \rrbracket$,

$$\begin{aligned}
1105 \quad \frac{1}{|\mathcal{B}^* \phi_{k,j}| |P'_{G_k}(\lambda_{k,j})|} &= \frac{1}{|\mathcal{B}^* \phi_{k,j}| \prod_{i \in \llbracket 1, g_k \rrbracket \neq j} |\lambda_{k,j} - \lambda_{k,i}|} \\
1106 \quad &\leq C \max_{1 \leq l \leq g_k} \left(\left\| \sum_{j=1}^l \frac{\frac{\phi_{k,j}}{\mathcal{B}^* \phi_{k,j}}}{\prod_{i \in \llbracket 1, l \rrbracket \neq j} (\lambda_{k,j} - \lambda_{k,i})} \right\|_{\diamond^*} \right).
\end{aligned}$$

1108 Thus, taking the logarithm,

$$\begin{aligned}
1109 \quad \frac{-\ln |\mathcal{B}^* \phi_{k,j}| |P'_{G_k}(\lambda_{k,j})|}{\lambda_{k,j}} &\leq \frac{\ln \max_{l \in \llbracket 1, g_k \rrbracket} \left(\left\| \sum_{j=1}^l \frac{\frac{\phi_{k,j}}{\mathcal{B}^* \phi_{k,j}}}{\prod_{i \in \llbracket 1, l \rrbracket \neq j} (\lambda_{k,j} - \lambda_{k,i})} \right\|_{\diamond^*} \right)}{\lambda_{k,1}} + \frac{\ln C}{\lambda_{k,1}}.
\end{aligned}$$

1111 Since by definition we have $G_k = G^{[\lambda_{k,j}]}$, this ends the proof of Proposition 3.3. \square

1112 4. The case of multiple eigenvalues.

1113 In this section we prove Theorem 1.1 in the case where we allow algebraic multi-
1114 plicity for the eigenvalues i.e. $\eta \geq 2$. As previously, the main issue is the resolution
1115 of the block moment problem given in (25). This is detailed in the next subsection.

1116 4.1. Resolution of block moment problems.

1117 We prove in this subsection the following theorem which is the generalization of
1118 Theorem 2.1.

1119 **THEOREM 4.1.** *Let $T \in (0, +\infty]$. Let $p \in \mathbb{N}^*$, $r, \rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$.
1120 Assume that $\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$ and let $(G_k)_k \in \mathcal{G}(\Lambda, p, r, \rho)$ be an associated grouping.
1121 We also consider an integer $\eta \geq 1$.*

1122 *For any $\varepsilon > 0$, there exists a constant $C_{\varepsilon, T, p, r, \rho, \eta, \mathcal{N}} > 0$ such that for any $k \geq 1$,
1123 for any multi-index $\alpha_k \in \mathbb{N}^{g_k}$ with $|\alpha_k|_\infty \leq \eta$, any set of values $\omega_{\alpha_k} \in \mathbb{C}^{|\alpha_k|}$, there
1124 exists $q_k \in L^2(0, T; \mathbb{C})$ satisfying*

$$1125 \quad (66a) \quad \int_0^T q_k(t) \frac{(-t)^{l'}}{l'!} e^{-\lambda_{k',j'} t} dt = 0, \quad \forall k' \neq k, \forall j' \in \llbracket 1, g_{k'} \rrbracket, \forall l' \in \llbracket 0, \eta \rrbracket,$$

$$1126 \quad (66b) \quad \int_0^T q_k(t) \frac{(-t)^l}{l!} e^{-\lambda_{k,j} t} dt = \omega_{k,j}^l, \quad \forall j \in \llbracket 1, g_k \rrbracket, \forall l \in \llbracket 0, \alpha_{k,j} - 1 \rrbracket,$$

1127

1128 and the bound

$$1129 \quad (67) \quad \|q_k\|_{L^2(0,T;\mathbb{C})} \leq C_{\varepsilon,T,p,r,\rho,\eta,\mathcal{N}} e^{\varepsilon\lambda_{k,1}} \max_{\substack{\mu \in \mathbb{N}^{g_k} \\ \mu \leq \alpha_k}} \left| \omega[\lambda_{k,1}^{(\mu_1)}, \dots, \lambda_{k,g_k}^{(\mu_k)}] \right|.$$

1130 Moreover, up to the factor $e^{\varepsilon\lambda_{k,1}}$, this last estimate is sharp: any solution q_k of (66b),
1131 satisfy

$$1132 \quad (68) \quad \|q_k\|_{L^2(0,T;\mathbb{C})} \geq \tilde{C}_{p,\eta} \max_{\substack{\mu \in \mathbb{N}^{g_k} \\ \mu \leq \alpha}} \left| \omega[\lambda_{k,1}^{(\mu_1)}, \dots, \lambda_{k,g_k}^{(\mu_k)}] \right|,$$

1133 for some $\tilde{C}_{p,\eta} > 0$.

1134 In the case $p = 1$ (usual gap condition), a solution to (66) is given by the biorthogonal
1135 family built in [4]. Here, we extend this resolution using a weak gap condition (9) and
1136 we prove that the obtained estimates are uniform with respect to Λ in a given class
1137 $\mathcal{L}_w(\bullet, \bullet, \bullet)$.

1138 COROLLARY 4.1. Let $T \in (0, +\infty]$. Let $p \in \mathbb{N}^*$, $r, \rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$.
1139 Assume that $\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$ and let $(G_k)_k \in \mathcal{G}(\Lambda, p, r, \rho)$ be an associated grouping.

1140 We consider an integer $\eta \geq 1$ and for any k we suppose given a multi-index
1141 $\alpha_k \in \mathbb{N}^{g_k}$ such that $|\alpha_k|_\infty \leq \eta$.

1142 Then, for any $k \geq 1$, for any $j \in \llbracket 1, g_k \rrbracket$ and any $l \in \llbracket 0, \alpha_{k,j} - 1 \rrbracket$ there exists
1143 $q_{k,j,l} \in L^2(0, T; \mathbb{C})$ satisfying

$$1144 \quad \int_0^T q_{k,j,l}(t) \frac{(-t)^{l'}}{l'!} e^{-\lambda_{k',j'} t} dt = \delta_{k,k'} \delta_{j,j'} \delta_{l,l'},$$

1145 for any $k, k' \geq 1$, any $j \in \llbracket 1, g_k \rrbracket$, $j' \in \llbracket 1, g_{k'} \rrbracket$ and any $l \in \llbracket 0, \alpha_{k,j} - 1 \rrbracket$, $l' \in$
1146 $\llbracket 0, \alpha_{k',j'} - 1 \rrbracket$. Moreover, for any $\varepsilon > 0$, there exists a constant $C_{\varepsilon,T,p,r,\rho,\eta,\mathcal{N}} > 0$ such
1147 that for any $k \geq 1$, any $j \in \llbracket 1, g_k \rrbracket$ and any $l \in \llbracket 0, \alpha_{k,j} - 1 \rrbracket$, we have

$$1148 \quad \|q_{k,j,l}\|_{L^2(0,T;\mathbb{C})}$$

$$1149 \quad \leq C_{\varepsilon,T,p,r,\rho,\eta,\mathcal{N}} \frac{e^{\varepsilon\lambda_{k,1}}}{\prod_{i \in \llbracket 1, g_k \rrbracket \neq j} |\lambda_{k,j} - \lambda_{k,i}|^{\alpha_{k,i}}} \frac{1}{\left(\min_{i \in \llbracket 1, g_k \rrbracket \neq j} |\lambda_{k,j} - \lambda_{k,i}| \right)^{\alpha_{k,j} - l - 1}}.$$

1152 The proof of Corollary 4.1 is left to the reader: it follows closely the one of Corollary 2.1
1153 and makes use of the estimate given in Proposition 7.11 instead of the Newton formula
1154 for standard divided differences.

1155 REMARK 4.1. Contrary to the estimate in Corollary 2.1, the above estimate is
1156 not optimal in general, even if we do not consider the exponential factor. Indeed,
1157 some cancellations can occur depending on the relative positions and multiplicities of
1158 the eigenvalues that are not taken into account in the above general bound. In actual
1159 examples, one needs to compute carefully the coefficients of the generalized divided
1160 differences introduced in Proposition 7.11 to see whether or not a sharper estimate
1161 can be obtained.

1162 Here also, the proof of Theorem 4.1 relies on the resolution of the block moment
1163 problem (66) with $T = +\infty$ and then on a restriction argument. For pedagogical
1164 reasons (the proof being less technical) let us present first this restriction argument
1165 (which is the generalization of Proposition 2.3).

1166 PROPOSITION 4.1. Let $p \in \mathbb{N}^*$, $r, \rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$. Assume that
 1167 $\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$ and let $(G_k)_k \in \mathcal{G}(\Lambda, p, r, \rho)$ be an associated grouping. We also
 1168 consider an integer $\eta \geq 1$.

For any $T > 0$, there exists a constant $C_{T,p,r,\rho,\eta,\mathcal{N}} > 0$ such that for any $\tilde{q} \in L^2(0, +\infty; \mathbb{C})$, there exists $q \in L^2(0, T; \mathbb{C})$ satisfying

$$\int_0^T q(t) \frac{(-t)^l}{l!} e^{-\lambda t} dt = \int_0^{+\infty} \tilde{q}(t) \frac{(-t)^l}{l!} e^{-\lambda t} dt, \quad \forall \lambda \in \Lambda, \forall l \in \llbracket 0, \eta \rrbracket,$$

and the estimate

$$\|q\|_{L^2(0,T;\mathbb{C})} \leq C_{T,p,r,\rho,\eta,\mathcal{N}} \|\tilde{q}\|_{L^2(0,+\infty;\mathbb{C})}.$$

Proof. For any $h > 0$, we define

$$\Lambda_h := \bigcup_{l=0}^{\eta} (\Lambda + lh).$$

1169 Using Remark 2.2 and Lemma 2.1 we have that $\Lambda_h \in \mathcal{L}_w(p\eta, \rho, \tilde{\mathcal{N}})$ for some $\tilde{\mathcal{N}}$ which
 1170 does not depend on h . We suppose given a fixed \tilde{q} and, for any $h > 0$, we can
 1171 apply Proposition 2.3 with the sequence Λ_h and obtain the existence of a function
 1172 $q_h \in L^2(0, T; \mathbb{C})$ such that

$$1173 \quad (69) \quad \int_0^T q_h(t) e^{-(\lambda+lh)t} dt = \int_0^{+\infty} \tilde{q}(t) e^{-(\lambda+lh)t} dt, \quad \forall \lambda \in \Lambda, \forall l \in \llbracket 0, \eta \rrbracket,$$

and satisfying moreover the uniform estimate

$$\|q_h\|_{L^2(0,T;\mathbb{C})} \leq C_{T,p\eta,r,\rho,\tilde{\mathcal{N}}} \|\tilde{q}\|_{L^2(0,+\infty;\mathbb{C})}, \quad \forall h > 0.$$

1174 We can then find a subsequence $(q_{h_n})_n$ that weakly converges towards some $q \in$
 1175 $L^2(0, T; \mathbb{C})$ such that $\|q\|_{L^2(0,T;\mathbb{C})} \leq C_{T,p\eta,r,\rho,\tilde{\mathcal{N}}} \|\tilde{q}\|_{L^2(0,+\infty;\mathbb{C})}$. We will show that q
 1176 solves the required equations.

1177 Let $\lambda \in \Lambda$ and $l \in \llbracket 0, \eta - 1 \rrbracket$ be fixed. Combining the equations (69) to make
 1178 appear divided differences, we have the equality

$$1179 \quad (70) \quad \int_0^T q_{h_n}(t) e_t[\lambda, \dots, \lambda + lh_n] dt = \int_0^{+\infty} \tilde{q}(t) e_t[\lambda, \dots, \lambda + lh_n] dt,$$

where e_t is defined in (5). The Lagrange theorem (see Proposition 7.4) implies that, for any t and any n , there is a $\xi_{t,n} \in [\lambda, \lambda + lh_n]$ such that

$$e_t[\lambda, \dots, \lambda + lh_n] = \frac{(-t)^l}{l!} e^{-\xi_{t,n} t},$$

which implies that $|e_t[\lambda, \dots, \lambda + lh_n]| \leq \frac{t^l}{l!} e^{-\lambda t}$ and

$$e_t[\lambda, \dots, \lambda + lh_n] \xrightarrow{n \rightarrow \infty} \frac{(-t)^l}{l!} e^{-\lambda t}.$$

1180 By the Lebesgue dominated-convergence theorem we deduce the strong convergence
 1181 in $L^2(0, +\infty; \mathbb{C})$ of $t \mapsto e_t[\lambda, \dots, \lambda + lh_n]$ towards $t \mapsto \frac{(-t)^l}{l!} e^{-\lambda t}$ and the claim follows
 1182 by weak-strong convergence in (70). \square

1183 Let us now turn to the resolution of the block moment problem (66) for $T = +\infty$.
 1184 The next proposition is the generalization of Proposition 2.1.

1185 PROPOSITION 4.2. Let $p \in \mathbb{N}^*$, $r, \rho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$. Assume that
 1186 $\Lambda \in \mathcal{L}_w(p, \rho, \mathcal{N})$ and let $(G_k)_k \in \mathcal{G}(\Lambda, p, r, \rho)$ be an associated grouping. We also
 1187 consider an integer $\eta \geq 1$.

1188 For any $\varepsilon > 0$, there exists a constant $C_{\varepsilon, p, r, \rho, \eta, \mathcal{N}} > 0$ such that for any $k \geq 1$, for
 1189 any multi-index $\alpha_k \in \mathbb{N}^{g_k}$ with $|\alpha_k|_\infty \leq \eta$, and any set of values $\omega_{\alpha_k} \in \mathbb{C}^{|\alpha_k|}$, there
 1190 exists $q_k \in L^2(0, +\infty; \mathbb{C})$ satisfying

$$1191 \quad \begin{cases} \int_0^{+\infty} q_k(t) \frac{(-t)^{l'}}{l'!} e^{-\lambda_{k', j'} t} dt = 0, & \forall k' \neq k, \forall j' \in \llbracket 1, g_{k'} \rrbracket, \forall l' \in \llbracket 0, \eta \rrbracket, \\ \int_0^{+\infty} q_k(t) \frac{(-t)^l}{l!} e^{-\lambda_{k, j} t} dt = \omega_{k, j}^l, & \forall j \in \llbracket 1, g_k \rrbracket, \forall l \in \llbracket 0, \alpha_{k, j} - 1 \rrbracket, \end{cases}$$

1192 and the bound

$$1193 \quad (71) \quad \|q_k\|_{L^2(0, +\infty; \mathbb{C})} \leq C_{\varepsilon, p, r, \rho, \eta, \mathcal{N}} e^{\varepsilon \lambda_{k, 1}} \max_{\substack{\mu \in \mathbb{N}^{g_k} \\ \mu \leq \alpha_k}} \left| \omega[\lambda_{k, 1}^{(\mu_1)}, \dots, \lambda_{k, g_k}^{(\mu_k)}] \right|.$$

1194 Before getting to the proof let us mention that Propositions 4.1 and 4.2 imply
 1195 Theorem 4.1. The lower bound (68) is proved in the exact same way as (32) and is
 1196 thus left to the reader.

Proof. As in the previous proof, for $h > 0$, we define

$$\Lambda_h := \bigcup_{l=0}^{\eta} (\Lambda + lh),$$

that belongs to the class $\mathcal{L}_w(p\eta, \rho, \tilde{\mathcal{N}})$. For any $k \geq 1$, we set

$$G_{k, h} := \bigcup_{l=0}^{\eta} (G_k + lh).$$

1197 For any $h < r/(2\eta)$, the family $(G_{k, h})_k$ is a grouping in $\mathcal{G}(\Lambda_h, p\eta, r/2, \rho + r/2)$.

1198 Now, we are given a fixed index k . We observe that, there exists a $h_0 \in (0, r/(2\eta))$
 1199 (possibly depending on k) such that, for any $h < h_0$, the sets $G_k, G_k + h, \dots, G_k + \eta h$
 1200 are pairwise disjoint.

Since we need to take into account precisely the multiplicities we are interested in, encoded in the multi-index α_k , we introduce the modified k -th group

$$\tilde{G}_{k, h} = \bigcup_{j=1}^{g_k} \{\lambda_{k, j}, \lambda_{k, j} + h, \dots, \lambda_{k, j} + (\alpha_{k, j} - 1)h\} \subset G_{k, h},$$

and the new family

$$\tilde{\Lambda}_h = \left(\bigcup_{\substack{l \geq 1 \\ l \neq k}} G_{l, h} \right) \cup \tilde{G}_{k, h},$$

1201 which satisfies $\tilde{\Lambda}_h \subset \Lambda_h$ and therefore also belongs to the class $\mathcal{L}_w(p\eta, \rho, \tilde{\mathcal{N}})$.

1202 By construction, the family of points in $\tilde{G}_{k,h}$, that we denote by $\mu_{k,h,1} < \dots <$
 1203 $\mu_{k,h,|\alpha_k|}$ is an approximation of the weighted family $((\lambda_{k,1}, \dots, \lambda_{k,g_k}), \alpha_k)$ in the sense
 1204 of Definition 7.2. Let $F : \mathbb{R} \rightarrow \mathbb{C}$ be a smooth function satisfying the conditions

$$1205 \quad (72) \quad \frac{1}{l!} F^{(l)}(\lambda_{k,j}) = \omega_{k,j}^l, \quad \forall j \in \llbracket 1, g_k \rrbracket, \forall l \in \llbracket 0, \alpha_{k,j} - 1 \rrbracket.$$

1206 For each $h > 0$, we apply Proposition 2.1 to the family $\tilde{\Lambda}_h$ to find a solution $q_{k,h} \in$
 1207 $L^2(0, +\infty; \mathbb{C})$ to the following moment problem

$$1208 \quad (73) \quad \begin{cases} \int_0^{+\infty} q_{k,h}(t) e^{-(\lambda_{k',j'} + hl)t} dt = 0, & \forall k' \neq k, \forall j' \in \llbracket 1, g_{k'} \rrbracket, \forall l \in \llbracket 0, \eta \rrbracket, \\ \int_0^{+\infty} q_{k,h}(t) e^{-(\lambda_{k,j} + hl)t} dt = F(\lambda_{k,j} + hl), & \forall j \in \llbracket 1, g_k \rrbracket, \forall l \in \llbracket 0, \alpha_{k,j} - 1 \rrbracket, \end{cases}$$

and satisfying the following bound, with a constant uniform with respect to h ,

$$\|q_{k,h}\|_{L^2(0, +\infty; \mathbb{C})} \leq C_{\varepsilon, \eta, p, r, \rho, \tilde{N}} e^{\varepsilon \lambda_{k,1}} \max_{i \in \llbracket 1, |\alpha_k| \rrbracket} |F[\mu_{k,h,1}, \dots, \mu_{k,h,i}]|.$$

1209 By Proposition 7.7, we know that the right-hand side in the above estimate converges
 1210 when $h \rightarrow 0$ towards a similar quantity with generalized divided differences instead
 1211 of the usual divided differences. It follows that we can extract a subsequence $(q_{k,h_n})_n$
 1212 that weakly converges in $L^2(0, +\infty; \mathbb{C})$ towards a function q_k that satisfies the bound
 1213 (71).

1214 Finally, by the same argument as in the proof of Proposition 4.1 above, we can
 1215 combine the equations (73) to make appear divided differences on both side and pass
 1216 to the weak-strong limit in the integral to finally get

$$1217 \quad \begin{cases} \int_0^{+\infty} q_k(t) \frac{(-t)^l}{l!} e^{-\lambda_{k',j'} t} dt = 0, & \forall k' \neq k, \forall j' \in \llbracket 1, g_{k'} \rrbracket, \forall l \in \llbracket 0, \eta \rrbracket, \\ \int_0^{+\infty} q_k(t) \frac{(-t)^l}{l!} e^{-\lambda_{k,j} t} dt = F[\lambda_{k,j}^{(l)}], & \forall j \in \llbracket 1, g_k \rrbracket, \forall l \in \llbracket 0, \alpha_{k,j} - 1 \rrbracket, \end{cases}$$

1218 which is exactly our claim since, by the computation rule (121) and by (72), we have
 1219 $F[\lambda_{k,j}^{(l)}] = \omega_{k,j}^l$. □

1220 4.2. Proof of the minimal null-control time property.

1221 In this section we end the proof of Theorem 1.1. The extension of Corollaries 2.2
 1222 and 2.3 as well as their proofs to the case $\eta \geq 2$ are straightforward and left to the
 1223 reader.

1224 *Controllability in large time: proof of point i. of Theorem 1.1.*

1225 Let $T > T_0(Y_0)$ and $y_0 \in Y_0$. For any $k \geq 1$, let $q_k \in L^2(0, T; \mathbb{C})$ be given by
 1226 Theorem 4.1 with

$$1227 \quad \omega_{k,j}^l := \left\langle y_0, (e_T \psi)[\lambda_{k,j}^{(l+1)}] \right\rangle_{-\diamond, \diamond}.$$

1228 As in Section 2.1.3, since $T > T_0(Y_0)$, the estimates (67) imply that

$$1229 \quad u := - \sum_{k \geq 1} q_k(T - \bullet) \in L^2(0, T; \mathbb{C}).$$

1230 Moreover, as q_k solves the block moment problem (66) it comes that u solves the
 1231 moment problem (23) and thus $y(T) = 0$.

1232 *Lack of null-controllability in small time: proof of point ii. of Theorem 1.1.*
 1233 The proof follows exactly the lines of Section 2.2 and relies on the lower bound (68)
 1234 given for the solution of the block moments problem (66).

1235 5. Examples.

1236 In this section we study various examples. In Section 5.1, we design ‘abstract
 1237 examples’ to highlight the phenomenon described in Section 1.3.4: the condensation
 1238 of eigenvectors can compensate the condensation of eigenvalues. More precisely we
 1239 design an example which is null-controllable in arbitrary time but with an arbitrary
 1240 condensation of the eigenvalues. We also give examples to illustrate the new settings
 1241 covered by our analysis when the eigenvalues are algebraically multiple in the absence
 1242 of a gap condition. The interest of these abstract examples is to highlight the different
 1243 phenomena as the computations are straightforward.

1244 Finally, we provide in Section 5.2, actual examples of one dimensional coupled
 1245 parabolic control systems that have motivated the present study. The precise analysis
 1246 of null-controllability for those systems was not possible using existing results in the
 1247 literature.

1248 5.1. Abstract examples: a possible compensation of condensation of 1249 eigenvalues.

1250 The design of these abstract examples is inspired from the work [3]. Our goal is
 1251 to illustrate, in particular, the fact that, even if the control operator has no influence
 1252 on the minimal null-control time, the knowledge of the condensation index of the
 1253 eigenvalues of the operator \mathcal{A} is not sufficient to understand the null-controllability
 1254 properties of system (1).

1255 Let A be a positive definite self-adjoint operator with compact resolvent in a
 1256 Hilbert space H whose eigenvalues $(\mu_k)_{k \geq 1}$ are assumed to be sorted in increasing
 1257 order. One can think of A , for instance, as the Laplace operator $-\partial_{xx}$ or any Sturm-
 1258 Liouville operator with homogeneous Dirichlet boundary conditions.

1259 If we denote by $(\varphi_k)_{k \geq 1}$ a corresponding Hilbert basis of eigenvectors, A may be
 1260 written

$$1261 \quad A = \sum_{k \geq 1} \mu_k (\bullet, \varphi_k)_H \varphi_k, \quad D(A) = \left\{ x \in H; \sum_{k \geq 1} \mu_k^2 (x, \varphi_k)_H^2 < +\infty \right\},$$

1262 where $(\bullet, \bullet)_H$ denotes the scalar product in H . We assume that $(\mu_k)_{k \geq 1}$ satisfies (8)
 1263 and (9) with $p = 1$, i.e., satisfies the so-called gap property. Let $\rho > 0$ be such that

$$1264 \quad (74) \quad 0 < \rho < \inf_{k \geq 1} (\mu_{k+1} - \mu_k)$$

1265 and $f : \sigma(A) \rightarrow \mathbb{R}$ a positive function defined on $\sigma(A)$ the spectrum of A satisfying

$$1266 \quad (75) \quad 0 < f(\mu_k) < \rho, \quad \forall k \geq 1.$$

1267 Let $f(A)$ be the operator defined on $D(A)$ by

$$1268 \quad f(A) := \sum_{k \geq 1} f(\mu_k) (\bullet, \varphi_k)_H \varphi_k.$$

1269 Let $x_0 \in H$ fixed satisfying

$$1270 \quad (76) \quad |(x_0, \varphi_k)_H| \geq e^{-\sqrt{\mu_k}}, \quad \forall k \geq 1.$$

1271

1272 **REMARK 5.1.** *This vector x_0 will be used to design the control operator \mathcal{B} . This*
 1273 *assumption will ensure that the terms $\mathcal{B}^* \phi_\lambda$ appearing in the definition (19) have*
 1274 *no influence. This will allow us to really emphasize the role of the condensation of*
 1275 *eigenvectors.*

1276 **5.1.1. Perturbation of a 2×2 Jordan block.**

1277 Let $X = H \times H$,

$$1278 \quad (77) \quad \mathcal{A} = \begin{pmatrix} A & I \\ 0 & A + f(A) \end{pmatrix}, \quad D(\mathcal{A}) = D(A) \times D(A),$$

1279 and

$$1280 \quad (78) \quad \mathcal{B} : u \in \mathbb{R} \mapsto u \begin{pmatrix} 0 \\ x_0 \end{pmatrix} \in X.$$

1281 It is easy to see that $(-\mathcal{A}, D(\mathcal{A}))$ generates a \mathcal{C}_0 -semigroup on X and that $\mathcal{B} : \mathbb{R} \rightarrow X$
 1282 is bounded. Thus we consider for this example that $X_\diamond^* = X = X_{-\diamond}$ and $Y_0 = X$.

1283 The spectrum of $(\mathcal{A}^*, D(\mathcal{A}))$ is given by

$$1284 \quad \Lambda = \{\mu_k, \mu_k + f(\mu_k); k \geq 1\}.$$

1285 **PROPOSITION 5.1.** *Let us consider the control system (1) with \mathcal{A} and \mathcal{B} given*
 1286 *by (77)-(78).*

1287 *i. For any function f satisfying (75), null-controllability from X holds in any*
 1288 *time i.e. $T_0(X) = 0$.*

1289 *ii. For any $\tau \in [0, +\infty]$, there exists a function f satisfying (75) such that $c(\Lambda) =$*
 1290 *τ .*

1291 This gives a first example in this setting where the minimal time is not related to
 1292 the condensation index. As it will appear from the proof, see (79), this is due to a
 1293 condensation of eigenvectors compensating the condensation of eigenvalues.

1294 *Proof.* The proof of point *ii.* directly follows from straightforward computations
 1295 using Proposition 7.12 with the explicit choices $f : s \mapsto \rho e^{-\sqrt{s}}$, $f : s \mapsto \rho e^{-cs}$ with
 1296 $c > 0$ or $f : s \mapsto \rho e^{-s^2}$.

1297 We now turn to the computation of the minimal null-control time. Using (74)
 1298 and (75), it comes that (8) and (9) are satisfied with $p = 2$. We define our grouping
 1299 by setting $\lambda_{k,1} := \mu_k$ and $\lambda_{k,2} := \mu_k + f(\mu_k)$. The associated normalized eigenvectors
 1300 are

$$1301 \quad \phi_{k,1} := \frac{1}{\sqrt{1 + f(\mu_k)^2}} \begin{pmatrix} -f(\mu_k) \\ 1 \end{pmatrix} \varphi_k, \quad \phi_{k,2} := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k,$$

1302 which do form a complete family in X . Moreover, for all $k \geq 1$,

$$1303 \quad \mathcal{B}^* \phi_{k,1} = \frac{1}{\sqrt{1 + f(\mu_k)^2}} (x_0, \varphi_k)_H, \quad \text{and} \quad \mathcal{B}^* \phi_{k,2} = (x_0, \varphi_k)_H,$$

1304 so that, with (15), we have

$$1305 \quad \psi_{k,1} = \frac{1}{(x_0, \varphi_k)_H} \begin{pmatrix} -f(\mu_k) \\ 1 \end{pmatrix} \varphi_k, \quad \psi_{k,2} = \frac{1}{(x_0, \varphi_k)_H} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k.$$

1306 From Definition (18), we have

$$1307 \quad T_0(X) = \limsup_{k \rightarrow \infty} \frac{\ln \left(\max \left\{ \|\psi_{k,1}\|, \frac{\|\psi_{k,2} - \psi_{k,1}\|}{f(\mu_k)} \right\} \right)}{\mu_k}.$$

1308 One has

$$1309 \quad (79) \quad \frac{\|\psi_{k,2} - \psi_{k,1}\|}{f(\mu_k)} = \frac{1}{|(x_0, \varphi_k)_H|} \left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k \right\| = \frac{1}{|(x_0, \varphi_k)_H|}.$$

1310 Using (76) and (75) we easily deduce that

$$1311 \quad T_0(X) = \limsup_{k \rightarrow \infty} \frac{\ln \frac{1}{|(x_0, \varphi_k)_H|}}{\mu_k} = 0. \quad \square$$

1312 **REMARK 5.2.** Notice that,

$$1313 \quad \|\phi_{k,2} - \phi_{k,1}\|^2 = \frac{2 \left(1 + f(\mu_k)^2 - \sqrt{1 + f(\mu_k)^2} \right)}{1 + f(\mu_k)^2} \xrightarrow{k \rightarrow \infty} 0,$$

1314 thus the eigenvectors of \mathcal{A}^* do not form a Riesz basis of X . If this family were a
1315 Riesz basis, then we would deduce from [5] that the minimal null-control time would
1316 be equal to the condensation index $c(\Lambda)$.

1317 **REMARK 5.3.** Let us consider in the same setting the evolution problem (1) given
1318 by

$$1319 \quad \mathcal{A} = \begin{pmatrix} A & I \\ 0 & A \end{pmatrix}.$$

1320 In this case, the operator \mathcal{A}^* has spectrum $\sigma(\mathcal{A}^*) = \{\mu_k ; k \geq 1\}$ with algebraically
1321 double eigenvalues satisfying the gap property and an associated Hilbert basis of (gen-
1322 eralized) eigenvectors given by

$$1323 \quad \phi_k^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k, \quad \text{and} \quad \phi_k^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k.$$

1324 Notice that from (79) one has

$$1325 \quad \frac{\psi_{k,2} - \psi_{k,1}}{f(\mu_k)} = \frac{1}{(x_0, \varphi_k)_H} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k = \frac{\phi_k^1}{\mathcal{B}^* \phi_k^0}.$$

1326 Thus, the analysis of (77)-(78), is unchanged if ones sets $f = 0$.

1327 **5.1.2. Algebraically multiple eigenvalues.**

1328 Let $X = H \times H \times H$. Let $\beta > 0$ and $g : \sigma(A) \rightarrow \mathbb{R}$ be such that

1329 (80)
$$g(\mu_k) = \rho e^{-\beta \mu_k},$$

1330 with ρ satisfying (74). Let

1331 (81)
$$\mathcal{A} = \begin{pmatrix} A & I & 0 \\ 0 & A & 0 \\ 0 & 0 & A + g(A) \end{pmatrix}, \quad D(\mathcal{A}) = D(A) \times D(A) \times D(A),$$

1332 and

1333 (82)
$$\mathcal{B} : u \in \mathbb{R} \mapsto u \begin{pmatrix} 0 \\ x_0 \\ x_0 \end{pmatrix}.$$

1334 Again \mathcal{B} is a bounded control operator and we also set for this example $X_\diamond^* = X = X_{-\diamond}$
 1335 and $Y_0 = X$.

1336 The spectrum of $(\mathcal{A}^*, D(\mathcal{A}))$ is given by

1337
$$\Lambda = \{\mu_k, \mu_k + g(\mu_k); k \geq 1\}.$$

1338 PROPOSITION 5.2. *Let us consider the control system (1) with \mathcal{A} and \mathcal{B} given*
 1339 *by (81)-(82). Then,*

1340 (83)
$$T_0(X) = 2\beta = 2c(\Lambda).$$

1341 REMARK 5.4. *In this case, the family of (generalized) eigenvectors do form a*
 1342 *Hilbert basis in X . However due to the presence of algebraically multiple eigenvalues*
 1343 *one cannot compute the value of the minimal null-control time using [5]. Its value*
 1344 *is still related to the condensation index of Λ but also depends on the multiplicity of*
 1345 *each eigenvalue in the system.*

1346 *Proof.* From (80), we see that the eigenvalues are geometrically simple. Then, it
 1347 comes that (8) and (9) are satisfied with $p = 2$. We define our grouping by setting
 1348 $\lambda_{k,1} := \mu_k$ and $\lambda_{k,2} := \mu_k + g(\mu_k)$.

1349 In this setting, the eigenvalue $\lambda_{k,1}$ is algebraically double and $\lambda_{k,2}$ is algebraically
 1350 simple. The associated eigenvectors and generalized eigenvectors of \mathcal{A}^* are

1351
$$\phi_{k,1}^0 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \varphi_k, \quad \phi_{k,1}^1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \varphi_k, \quad \phi_{k,2}^0 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \varphi_k,$$

1352 which obviously form a complete family in X . Moreover, for all $k \geq 1$,

1353
$$\mathcal{B}^* \phi_{k,1}^0 = \mathcal{B}^* \phi_{k,2}^0 = (x_0, \varphi_k)_H,$$

1354 leading to

1355
$$\psi_{k,1}^0 = \frac{1}{(x_0, \varphi_k)_H} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \varphi_k, \quad \psi_{k,2}^0 = \frac{1}{(x_0, \varphi_k)_H} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \varphi_k,$$

1356 and

$$1357 \quad \psi_{k,1}^1 = \frac{1}{(x_0, \varphi_k)_H} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \varphi_k.$$

1358 To compute the minimal time $T_0(X)$, let us estimate the different terms appearing
1359 in (18). We have $\psi[\lambda_{k,1}] = \psi_{k,1}^0$ and $\psi[\lambda_{k,1}, \lambda_{k,1}] = \psi_{k,1}^1$ implying

$$1360 \quad \|\psi[\lambda_{k,1}]\| = \|\psi[\lambda_{k,1}, \lambda_{k,1}]\| = \frac{1}{|(x_0, \varphi_k)_H|}.$$

1361 Using Proposition 7.10, it only remains to compute and estimate the generalized
1362 divided difference $\psi[\lambda_{k,1}, \lambda_{k,1}, \lambda_{k,2}]$. This comes from (121) and (122) as follows

$$1363 \quad \psi[\lambda_{k,1}, \lambda_{k,2}] = \frac{1}{(x_0, \varphi_k)_H} \frac{1}{\lambda_{k,2} - \lambda_{k,1}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \varphi_k = \frac{1}{(x_0, \varphi_k)_H} \frac{1}{g(\mu_k)} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \varphi_k,$$

1364 and

$$1365 \quad \psi[\lambda_{k,1}, \lambda_{k,1}, \lambda_{k,2}] = \frac{\psi[\lambda_{k,1}, \lambda_{k,2}] - \psi[\lambda_{k,1}, \lambda_{k,1}]}{\lambda_{k,2} - \lambda_{k,1}}$$

$$1366 \quad = \frac{1}{(x_0, \varphi_k)_H} \frac{1}{g(\mu_k)^2} \begin{pmatrix} -g(\mu_k) \\ -1 \\ 1 \end{pmatrix} \varphi_k.$$

1367

1368 Thus, using (80) we obtain

$$1369 \quad \|\psi[\lambda_{k,1}, \lambda_{k,1}, \lambda_{k,2}]\| = \frac{1}{(x_0, \varphi_k)_H} \frac{1}{g(\mu_k)^2} \sqrt{g(\mu_k)^2 + 2}$$

$$1370 \quad = \frac{1}{(x_0, \varphi_k)_H} \rho^{-2} e^{2\beta\mu_k} \sqrt{\rho^2 e^{-2\beta\mu_k} + 2}.$$

1371

1372 Then, for k large enough, we have

$$1373 \quad \max \left\{ 1, \rho^{-2} e^{2\beta\mu_k} \sqrt{\rho^2 e^{-2\beta\mu_k} + 2} \right\} = \rho^{-2} e^{2\beta\mu_k} \sqrt{\rho^2 e^{-2\beta\mu_k} + 2},$$

1374 and, using (76), this leads to (83). □

1375 5.1.3. Competition between different perturbations.

1376 Let $X = H \times H \times H$. Let $\alpha, \beta > 0$ with $\alpha \neq \beta$ and $f, g : \sigma(A) \rightarrow \mathbb{R}$ be such that

$$1377 \quad f(\mu_k) = \rho e^{-\alpha\mu_k}, \quad g(\mu_k) = \rho e^{-\beta\mu_k},$$

1378 with ρ satisfying (74). Let

$$1379 \quad (84) \quad \mathcal{A} = \begin{pmatrix} A & I & 0 \\ 0 & A + f(A) & 0 \\ 0 & 0 & A + g(A) \end{pmatrix}, \quad D(\mathcal{A}) = D(A) \times D(A) \times D(A),$$

1380 and

$$1381 \quad (85) \quad \mathcal{B} : u \in \mathbb{R} \mapsto u \begin{pmatrix} 0 \\ x_0 \\ x_0 \end{pmatrix}.$$

1382 Again \mathcal{B} is a bounded control operator and we still set for this example $X_\diamond^* = X = X_{-}$
 1383 and $Y_0 = X$.

1384 PROPOSITION 5.3. Let us consider the control system (1) with \mathcal{A} and \mathcal{B} given
 1385 by (84)-(85). Then,

$$1386 \quad T_0(X) = \beta + \min\{\alpha, \beta\}.$$

1387 *Proof.* The spectrum of $(\mathcal{A}^*, D(\mathcal{A}))$ is given by

$$1388 \quad \Lambda = \{\mu_k, \mu_k + f(\mu_k), \mu_k + g(\mu_k); k \geq 1\}.$$

1389 By construction, these eigenvalues are geometrically simple. Then, it comes that (8)
 1390 and (9) are satisfied with $p = 3$. We define our grouping by setting

$$1391 \quad \lambda_{k,1} := \mu_k, \quad \lambda_{k,2} := \mu_k + f(\mu_k), \quad \text{and} \quad \lambda_{k,3} = \mu_k + g(\mu_k).$$

1392 Notice that the eigenvalues are not necessarily increasingly sorted inside the k^{th} group
 1393 depending on the relative positions of α and β but, due to the invariance of divided
 1394 differences with respect to permutations, this does not change our analysis.

1395 These eigenvalues are algebraically and geometrically simple and the associated
 1396 eigenvectors are

$$1397 \quad \phi_{k,1} := \frac{1}{\sqrt{1 + f(\mu_k)^2}} \begin{pmatrix} -f(\mu_k) \\ 1 \\ 0 \end{pmatrix} \varphi_k, \quad \phi_{k,2} := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \varphi_k, \quad \phi_{k,3} := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \varphi_k,$$

1398 which do form a complete family in X . Moreover, for all $k \geq 1$,

$$1399 \quad \mathcal{B}^* \phi_{k,1} = \frac{1}{\sqrt{1 + f(\mu_k)^2}} (x_0, \varphi_k)_H, \quad \text{and} \quad \mathcal{B}^* \phi_{k,2} = \mathcal{B}^* \phi_{k,3} = (x_0, \varphi_k)_H,$$

1400 leading to

$$1401 \quad \psi_{k,1} = \frac{1}{(x_0, \varphi_k)_H} \begin{pmatrix} -f(\mu_k) \\ 1 \\ 0 \end{pmatrix} \varphi_k, \quad \psi_{k,2} = \frac{1}{(x_0, \varphi_k)_H} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \varphi_k,$$

1402 and

$$1403 \quad \psi_{k,3} = \frac{1}{(x_0, \varphi_k)_H} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \varphi_k.$$

1404 To compute the minimal time $T_0(X)$, let us determine the different terms appearing
 1405 in (18). We have $\psi[\lambda_{k,1}] = \psi_{k,1}$,

$$1406 \quad \psi[\lambda_{k,1}, \lambda_{k,2}] = \frac{1}{(x_0, \varphi_k)_H} \frac{1}{\lambda_{k,2} - \lambda_{k,1}} \begin{pmatrix} f(\mu_k) \\ 0 \\ 0 \end{pmatrix} \varphi_k = \frac{1}{(x_0, \varphi_k)_H} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \varphi_k,$$

1407

$$1408 \quad \psi[\lambda_{k,2}, \lambda_{k,3}] = \frac{1}{(x_0, \varphi_k)_H} \frac{1}{\lambda_{k,3} - \lambda_{k,2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \varphi_k = \frac{1}{(x_0, \varphi_k)_H} \frac{1}{g(\mu_k) - f(\mu_k)} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \varphi_k,$$

1409 and finally

$$\begin{aligned}
 1410 \quad \psi[\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}] &= \frac{\psi[\lambda_{k,1}, \lambda_{k,2}] - \psi[\lambda_{k,2}, \lambda_{k,3}]}{\lambda_{k,1} - \lambda_{k,3}} \\
 1411 \quad &= \frac{1}{(x_0, \varphi_k)_H} \frac{1}{g(\mu_k)(g(\mu_k) - f(\mu_k))} \begin{pmatrix} g(\mu_k) - f(\mu_k) \\ -1 \\ 1 \end{pmatrix} \varphi_k.
 \end{aligned}$$

Since $\lim_{k \rightarrow +\infty} g(\mu_k) = 0$, we immediately see that, for k large enough, we have

$$\max \{ \|\psi[\lambda_{k,1}]\|, \|\psi[\lambda_{k,1}, \lambda_{k,2}]\|, \|\psi[\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}]\| \} = \|\psi[\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}]\|,$$

1413 so that using (76) and (18) we get

$$1414 \quad (86) \quad T_0(X) = \limsup_{k \rightarrow \infty} \frac{\ln \|\psi[\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}]\|}{\mu_k} = \limsup_{k \rightarrow \infty} \frac{-\ln |g(\mu_k)(g(\mu_k) - f(\mu_k))|}{\mu_k}.$$

1415 The analysis is now split into two cases:

1416 1. Assume first that

$$1417 \quad (87) \quad \beta < \alpha.$$

1418 We deduce from (86) that

$$1419 \quad T_0(X) = \limsup_{k \rightarrow \infty} \frac{-\ln e^{-2\beta\mu_k} (1 - e^{-(\alpha-\beta)\mu_k})}{\mu_k} = 2\beta.$$

1420 2. Assume now that

$$1421 \quad (88) \quad \beta > \alpha.$$

1422 We deduce from (86) that

$$1423 \quad T_0(X) = \limsup_{k \rightarrow \infty} \frac{-\ln e^{-(\alpha+\beta)\mu_k} (1 - e^{-(\beta-\alpha)\mu_k})}{\mu_k} = \alpha + \beta. \quad \square$$

1424 **REMARK 5.5.** *As previously the family of eigenvectors does not form a Riesz basis*
 1425 *since for instance we have $\|\phi_{k,1} - \phi_{k,2}\| \xrightarrow[k \rightarrow \infty]{} 0$. Thus, one cannot apply the results*
 1426 *of [5] that would give that the minimal null-control time is (see Appendix 7.5)*

$$1427 \quad c(\Lambda) = \limsup_{k \rightarrow \infty} \frac{-\ln (f(\mu_k)g(\mu_k))}{\mu_k} = \alpha + \beta.$$

1428 *Yet, in the case (88) we still have $T_0(X) = c(\Lambda)$. However, in the case (87) we have*
 1429 *$0 < T_0(X) = 2\beta < c(\Lambda)$. Notice that, in this case, setting $f = 0$ one recovers the*
 1430 *system studied in subsection 5.1.2 for which the minimal time is exactly 2β .*

1431 5.2. Condensation in partial differential equations.

1432 We provide in this section actual PDE examples covered by our analysis. First
 1433 of all, let us emphasize that our setting naturally covers a wide range of coupled one
 1434 dimensional parabolic equations. Indeed if there exists $p \in \mathbb{N}^*$ such that the spectrum
 1435 of \mathcal{A} is given by the union of p families

$$1436 \quad \Lambda^j = \left\{ \lambda_k^j ; k \geq 1 \right\}$$

1437 such that each family satisfies (9) and (8), then the structural assumptions on Λ are
 1438 automatically satisfied (see Lemma 2.1).

1439 **5.2.1. A system with two different potentials.**

1440 Let us consider the following boundary control system

$$1441 \quad (89) \quad \begin{cases} \partial_t y + \begin{pmatrix} -\partial_{xx} + c_1(x) & 1 \\ 0 & -\partial_{xx} + c_2(x) \end{pmatrix} y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & (t, x) \in (0, T) \times (0, 1), \\ y(t, 0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad y(t, 1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & t \in (0, T), \\ y(0, x) = y_0(x), \end{cases}$$

1442 where $c_1, c_2 \in L^2(0, 1; \mathbb{R})$. Without loss of generality we assume that c_1 and c_2 are non-
 1443 negative. The operator \mathcal{A} appearing in this system is defined in $X = (L^2(0, 1; \mathbb{R}))^2$
 1444 with domain $X_1^* = D(\mathcal{A}) = (H^2(0, 1; \mathbb{R}) \cap H_0^1(0, 1; \mathbb{R}))^2$. The control operator \mathcal{B} is
 1445 defined in a weak sense as in [49]. The expression of its adjoint is easier to rule out
 1446 and is given by

$$1447 \quad \mathcal{B}^* : \begin{pmatrix} f \\ g \end{pmatrix} \in X_1^* \mapsto \begin{pmatrix} 0 \\ B^*g \end{pmatrix} = \begin{pmatrix} 0 \\ -g'(0) \end{pmatrix}.$$

1448 Here we denoted by B^* the (scalar) normal derivative operator at $x = 0$ defined
 1449 on $H^2(0, 1; \mathbb{R})$. Standard parabolic regularity properties show that, if we define
 1450 $X_\diamond^* = (H_0^1(0, 1; \mathbb{R}))^2$, then the operator \mathcal{B} is admissible with respect to $X_{-\diamond} =$
 1451 $(H^{-1}(0, 1; \mathbb{R}))^2$, in the sense of (3). Therefore, for any $u \in L^2(0, T; \mathbb{R})$, (89) is well-
 1452 posed in $\mathcal{C}^0([0, T], (H^{-1}(0, 1; \mathbb{R}))^2)$.

1453 For any non-negative potential $c \in L^2(0, 1; \mathbb{R})$, we denote by A^c the definite posi-
 1454 tive self-adjoint operator in $L^2(0, 1; \mathbb{R})$ with domain $H^2(0, 1; \mathbb{R}) \cap H_0^1(0, 1; \mathbb{R})$ defined
 1455 by $A^c y = -\partial_{xx} y + c(x)y$. Its spectrum is denoted by $\Lambda^c \subset (\pi^2, +\infty)$ and satisfies

$$1456 \quad (90) \quad \inf_{\substack{\lambda, \mu \in \Lambda^c \\ \lambda \neq \mu}} |\sqrt{\lambda} - \sqrt{\mu}| > 0.$$

1457 We choose associated eigenfunctions denoted by φ_λ^c that are normalized in $L^2(0, 1; \mathbb{R})$
 1458 and that satisfy (see for instance [37, Theorem 4.11])

$$1459 \quad (91) \quad \varphi_\lambda^c(x) = \sqrt{2} \sin(\sqrt{\lambda}x) + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right), \quad \text{uniformly in } x,$$

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$$1461 \quad (92) \quad \partial_x \varphi_\lambda^c(x) = \sqrt{2}\sqrt{\lambda} \cos(\sqrt{\lambda}x) + \mathcal{O}(1), \quad \text{uniformly in } x.$$

1462 In particular, there exist $\bar{C}, \tilde{C} > 0$ such that,

$$1463 \quad (93) \quad \bar{C}\sqrt{\lambda} \leq |B^* \varphi_\lambda^{c_i}| = |\partial_x \varphi_\lambda^{c_i}(0)| \leq \tilde{C}\sqrt{\lambda}, \quad \forall \lambda \in \Lambda^{c_i}, \forall i = 1, 2.$$

The analysis will be based on the careful inspection of spectral properties of the
 adjoint operator

$$\mathcal{A}^* = \begin{pmatrix} A^{c_1} & 0 \\ 1 & A^{c_2} \end{pmatrix}.$$

1464 It is easily seen that the spectrum of \mathcal{A}^* is given by $\Lambda = \Lambda^{c_1} \cup \Lambda^{c_2}$. We will often use
 1465 the following straightforward property

$$1466 \quad (94) \quad (A^{c_i} \varphi_\lambda^{c_j}, \varphi_\lambda^{c_j}) \leq C\lambda, \quad \forall \lambda \in \Lambda^{c_j}, \forall i, j \in \{1, 2\},$$

1467 where C depends only on $\|c_1\|$ and $\|c_2\|$.

1468 Our controllability result concerning system (89) is the following.

1469 THEOREM 5.1. For any non-negative potentials c_1, c_2 , there exists a closed sub-
 1470 space Y_0 of $(H^{-1}(0, 1; \mathbb{R}))^2$ of finite codimension such that:

- 1471 • For any $y_0 \notin Y_0$, system (89) is not approximately controllable.
- 1472 • For any $y_0 \in Y_0$, system (89) is null-controllable at any time $T > 0$.

1473 REMARK 5.6. The set Y_0 can be equal to the whole space $(H^{-1}(0, 1; \mathbb{R}))^2$, for
 1474 instance if c_1 and c_2 are close enough.

1475 Before proving this theorem, we would like to emphasize the fact that for a system
 1476 like (89), the condensation index of its spectrum can be arbitrary. Therefore, Theorem
 1477 5.1 gives another example of a system which is null-controllable at any time $T > 0$
 1478 (for well-prepared initial data) despite the fact that the condensation index of the
 1479 spectrum is non zero.

1480 PROPOSITION 5.4. For any $\tau \in [0, +\infty]$ there exist $c_1, c_2 \in L^2(0, 1; \mathbb{R})$ such that
 1481 the condensation index of the spectrum Λ of the operator \mathcal{A}^* satisfies $c(\Lambda) = \tau$.

1482 *Proof.* This follows from inverse spectral theory. Indeed, it is proven in [44,
 1483 Chapter 3] for instance, that for any $\alpha \in \mathbb{R}$ and any sequence $(\nu_k)_{k \geq 1} \in l^2$, one can find
 1484 a potential $c \in L^2(0, 1; \mathbb{R})$ such that the spectrum of A^c is given by $(k^2\pi^2 + \alpha + \nu_k)_k$.
 1485 It is then clear that we can choose c_1 and c_2 such that the spectrums of A^{c_1} and A^{c_2}
 1486 are asymptotically as close as we want and then generate an arbitrary condensation
 1487 index for the spectrum of \mathcal{A}^* . Note that such potentials are not necessarily non-
 1488 negative, but this is actually not really needed in our analysis (we simply need that
 1489 the spectrum of A^c is made of positive eigenvalues).

1490 In the context of parabolic control problems, this was already noticed and used
 1491 in [42]. □

1492 We can now move to the proof of the main result of this section.

1493 *Proof (of Theorem 5.1).* The first part of the proof consists in a precise descrip-
 1494 tion of the spectral properties of \mathcal{A}^* .

- 1495 • For any $\lambda \in \Lambda^{c_2}$, we have a first eigenfunction of \mathcal{A}^* given by

$$1496 \quad (95) \quad \phi_\lambda^0 := \begin{pmatrix} 0 \\ \varphi_\lambda^{c_2} \end{pmatrix}.$$

Moreover, by (93), we have

$$\mathcal{B}^* \phi_\lambda^0 = B^* \varphi_\lambda^{c_2} \neq 0,$$

1497 so that all those eigenfunctions are observable.

- 1498 – If $\lambda \notin \Lambda^{c_1}$, this eigenfunction is algebraically and geometrically simple.
- 1499 – However, if $\lambda \in \Lambda^{c_2} \cap \Lambda^{c_1}$, this eigenvalue is (algebraically or geometri-
 1500 cally) double. As detailed in Section 6.3, we can deal with geometric
 1501 multiplicity of eigenvalues with an adequate choice of the space of initial
 1502 conditions Y_0 . This choice will be precised in (101).

1503 Let us define

$$1504 \quad (96) \quad \beta_\lambda := (\varphi_\lambda^{c_1}, \varphi_\lambda^{c_2}).$$

1505 By (91), we see that there exists λ_0 such that

$$1506 \quad (97) \quad \frac{1}{2} \leq \beta_\lambda \leq 1, \quad \forall \lambda \in \Lambda^{c_2} \cap \Lambda^{c_1}, \text{ s.t. } \lambda > \lambda_0.$$

* If $\beta_\lambda = 0$ then there exists a solution of

$$(A^{c_2} - \lambda)\vartheta_\lambda = -\varphi_\lambda^{c_1},$$

that we can choose to satisfy $B^*\vartheta_\lambda = 0$ in such a way that

$$\tilde{\phi}_\lambda^0 := \begin{pmatrix} \varphi_\lambda^{c_1} \\ \vartheta_\lambda \end{pmatrix},$$

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is another independent eigenfunction of \mathcal{A}^* associated with λ that satisfy $B^*\tilde{\phi}_\lambda^0 = 0$. Note that, by (97), we know that β_λ can vanish only for a finite number of values of λ .

* Assume now that $\beta_\lambda \neq 0$. In that case, λ is geometrically simple but there exists a generalized eigenfunction ϕ_λ^1 associated with ϕ_λ^0 of the following form

$$\phi_\lambda^1 := \frac{1}{\beta_\lambda} \begin{pmatrix} \varphi_\lambda^{c_1} \\ \chi_\lambda \end{pmatrix},$$

where χ_λ is the unique solution of

$$(A^{c_2} - \lambda)\chi_\lambda = \beta_\lambda\varphi_\lambda^{c_2} - \varphi_\lambda^{c_1},$$

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that satisfy $B^*\chi_\lambda = 0$.

We can express χ_λ in the basis $\varphi_\bullet^{c_2}$ as follows

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$$\chi_\lambda = a_\lambda\varphi_\lambda^{c_2} - \sum_{\substack{\mu \in \Lambda^{c_2} \\ \mu \neq \lambda}} \frac{(\varphi_\lambda^{c_1}, \varphi_\mu^{c_2})}{\lambda - \mu} \varphi_\mu^{c_2},$$

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with

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$$a_\lambda = \frac{1}{B^*\varphi_\lambda^{c_2}} \sum_{\substack{\mu \in \Lambda^{c_2} \\ \mu \neq \lambda}} \frac{(\varphi_\lambda^{c_1}, \varphi_\mu^{c_2})}{\lambda - \mu} B^*\varphi_\mu^{c_2}.$$

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- Consider now $\lambda \in \Lambda^{c_1} \setminus \Lambda^{c_2}$. We obtain another family of eigenfunctions given by

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$$(98) \quad \phi_\lambda^0 := \begin{pmatrix} \varphi_\lambda^{c_1} \\ \xi_\lambda \end{pmatrix},$$

where ξ_λ satisfies

$$(A^{c_2} - \lambda)\xi_\lambda = -\varphi_\lambda^{c_1}.$$

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This last equation has a unique solution since $\lambda \notin \Lambda^{c_2}$ and it can be expressed as follows

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$$(99) \quad \xi_\lambda = \sum_{\mu \in \Lambda^{c_2}} \frac{(\varphi_\lambda^{c_1}, \varphi_\mu^{c_2})}{\lambda - \mu} \varphi_\mu^{c_2}.$$

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We now state the following lemma, whose proof is postponed at the end of this subsection.

1523 LEMMA 5.1. *There exists $C_1, C_2 > 0$ depending only on c_1, c_2 such that*

1524 (100) $|B^*\xi_\lambda|^2 \geq C_1\lambda - C_2, \quad \forall \lambda \in \Lambda^{c_1} \setminus \Lambda^{c_2}.$

1525 This lemma shows in particular that $B^*\xi_\lambda$ can only vanish for a finite number
1526 of values of λ .

1527 It is straightforward to prove that the family of (generalized) eigenfunctions we
1528 just computed is complete in X . We can now define Y_0 to be the set of initial data
1529 $y_0 \in X_{-\diamond}$ such that

1530 (101)
$$\begin{cases} \langle y_0, \tilde{\phi}_\lambda^0 \rangle_{-\diamond, \diamond} = 0, & \forall \lambda \in \Lambda^{c_1} \cap \Lambda^{c_2}, \text{ s.t. } \beta_\lambda = 0, \text{ see (96),} \\ \langle y_0, \phi_\lambda^0 \rangle_{-\diamond, \diamond} = 0, & \forall \lambda \in \Lambda^{c_1} \setminus \Lambda^{c_2}, \text{ s.t. } B^*\xi_\lambda = 0, \text{ see (99).} \end{cases}$$

1531 By construction, this set is closed and of finite codimension, moreover it is clear that
1532 initial data not belonging to this set are not approximately controllable. Note that this
1533 definition actually excludes the influence of the possible presence of a geometrically
1534 double eigenvalue in the system.

We will now endow the space X_\diamond^* with the following norm

$$\left\| \begin{pmatrix} f \\ g \end{pmatrix} \right\|_{\diamond^*}^2 := \langle A^{c_1} f, f \rangle_{H^{-1}, H_0^1} + \langle A^{c_2} g, g \rangle_{H^{-1}, H_0^1},$$

1535 which is equivalent to the usual H^1 -norm and more comfortable for the following
1536 computations. Note that, if $f, g \in H^2(0, 1; \mathbb{R})$, this quantity is simply equal to
1537 $(A^{c_1} f, f) + (A^{c_2} g, g)$.

From (90), we can find a $\rho > 0$ such that

$$|\sqrt{\lambda} - \sqrt{\mu}| > \rho, \quad \forall \lambda \neq \mu \in \Lambda^{c_i}, \forall i = 1, 2.$$

1538 This implies, in particular, that

1539 (102) $|\lambda - \mu| > \rho(\sqrt{\lambda} + \sqrt{\mu}) > \rho, \quad \forall \lambda \neq \mu \in \Lambda^{c_i}, \forall i = 1, 2.$

1540 Without loss of generality, we can assume that $\rho < \frac{\bar{C}}{2C_1}$ where \bar{C} and C_1 are respec-
1541 tively defined in (93) and (100).

1542 It follows that Λ satisfies the summability condition (8), as well as the weak gap
1543 condition (9) with $p = 2$. We can thus consider a grouping $(G_k)_k \in \mathcal{G}(\Lambda, 2, r, \rho)$ for
1544 a suitable $r > 0$. We will now use the formula (17) we obtained for $T_0(Y_0)$ to prove
1545 that the system is null-controllable from Y_0 at any time $T > 0$. For that we will
1546 consider one of the groups G (we drop the index k which is not important here) and
1547 give estimates of the corresponding divided differences.

- 1548 • Case 1: $G = \{\lambda\}$ is of cardinal 1.
 - If $\lambda \in \Lambda^{c_2}$ we need to estimate the quantity

$$\|\psi[\lambda]\|_{\diamond^*}^2 := \left\| \frac{P_{Y_0}^* \phi_\lambda^0}{B^* \phi_\lambda^0} \right\|_{\diamond^*}^2 \leq \left\| \frac{\phi_\lambda^0}{B^* \phi_\lambda^0} \right\|_{\diamond^*}^2$$

except if $\lambda \in \Lambda^{c_1} \cap \Lambda^{c_2}$ and $\beta_\lambda = 0$. The computations above, and (93),
show that

$$\|\psi[\lambda]\|_{\diamond^*}^2 \leq \frac{1}{|B^* \phi_\lambda^{c_2}|^2} (A^{c_2} \phi_\lambda^{c_2}, \phi_\lambda^{c_2}) = \frac{\lambda}{|B^* \phi_\lambda^{c_2}|^2} \leq \frac{1}{\bar{C}^2}.$$

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- If $\lambda \in \Lambda^{c_1} \setminus \Lambda^{c_2}$, recall that ϕ_λ^0 is given by (98) and that we need to estimate the same quantity $\|\psi[\lambda]\|_{\diamond^*}$, in the case where $B^*\xi_\lambda \neq 0$. Since λ is the only element in the group G , we know that $|\lambda - \mu| \geq r$ for any other eigenvalue μ . With this remark, we can deduce that

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$$\begin{aligned} \|\psi[\lambda]\|_{\diamond^*}^2 &\leq \frac{1}{|B^*\xi_\lambda|^2} ((A^{c_1}\varphi_\lambda^{c_1}, \varphi_\lambda^{c_1}) + (A^{c_2}\xi_\lambda, \xi_\lambda)) \\ &= \frac{1}{|B^*\xi_\lambda|^2} \left(\lambda + \sum_{\mu \in \Lambda^{c_2}} \frac{\mu(\varphi_\lambda^{c_1}, \varphi_\mu^{c_2})^2}{(\lambda - \mu)^2} \right) \\ &\leq \frac{1}{|B^*\xi_\lambda|^2} \left(\lambda + \frac{1}{r^2} \sum_{\mu \in \Lambda^{c_2}} (\varphi_\lambda^{c_1}, \varphi_\mu^{c_2})(\varphi_\lambda^{c_1}, A^{c_2}\varphi_\mu^{c_2}) \right). \end{aligned}$$

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Using Parseval's identity, (94) and then (100), we finally obtain

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$$\begin{aligned} \|\psi[\lambda]\|_{\diamond^*}^2 &\leq \frac{1}{|B^*\xi_\lambda|^2} \left(\lambda + \frac{1}{r^2} (\varphi_\lambda^{c_1}, A^{c_2}\varphi_\lambda^{c_1}) \right) \\ &\leq \frac{\lambda}{|B^*\xi_\lambda|^2} \left(1 + \frac{C}{r^2} \right) \\ &\leq C. \end{aligned}$$

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- Finally, if $\lambda \in \Lambda^{c_1} \cap \Lambda^{c_2}$, then we need to estimate the contribution of the generalized eigenvector $\|\psi[\lambda, \lambda]\|_{\diamond^*}^2 := \|P_{Y_0}^* \phi_\lambda^1 / (\mathcal{B}^* \phi_\lambda^0)\|_{\diamond^*}^2$. A computation similar to the one above, for $\lambda > \lambda_0$, leads to

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$$\begin{aligned} \|\psi[\lambda, \lambda]\|_{\diamond^*}^2 &\leq \frac{1}{|\mathcal{B}^*\phi_\lambda^0|^2 \beta_\lambda^2} ((A^{c_1}\varphi_\lambda^{c_1}, \varphi_\lambda^{c_1}) + (A^{c_2}\chi_\lambda, \chi_\lambda)) \\ &= \frac{1}{|B^*\varphi_\lambda^{c_2}|^2 \beta_\lambda^2} \left(\lambda(1 + a_\lambda^2) + \sum_{\substack{\mu \in \Lambda^{c_2} \\ \mu \neq \lambda}} \frac{(\varphi_\lambda^{c_1}, \varphi_\mu^{c_2})^2}{(\lambda - \mu)^2} \mu \right) \\ &\leq \frac{1}{|B^*\varphi_\lambda^{c_2}|^2 \beta_\lambda^2} \left(\lambda(1 + a_\lambda^2) + \frac{1}{r^2} (A^{c_2}\varphi_\lambda^{c_1}, \varphi_\lambda^{c_1}) \right) \\ &\leq \frac{\lambda}{|B^*\varphi_\lambda^{c_2}|^2 \beta_\lambda^2} \left(1 + a_\lambda^2 + \frac{C}{r^2} \right) \\ &\leq C(1 + a_\lambda^2). \end{aligned}$$

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Here, we have used (97) to bound from below the term β_λ . It remains

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to bound a_λ . We proceed as follows, by using (92), (102), and (94)

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$$|a_\lambda| \leq \frac{1}{|B^* \varphi_\lambda^{c_2}|} \sum_{\substack{\mu \in \Lambda^{c_2} \\ \mu \neq \lambda}} \left| \frac{(\varphi_\lambda^{c_1}, \varphi_\mu^{c_2})}{\lambda - \mu} \right| |B^* \varphi_\mu^{c_2}|$$

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$$\leq \frac{C}{\sqrt{\lambda}} \sum_{\substack{\mu \in \Lambda^{c_2} \\ \mu \neq \lambda}} |(\varphi_\lambda^{c_1}, \varphi_\mu^{c_2})| \frac{\sqrt{\mu}}{\sqrt{\lambda} + \sqrt{\mu}}$$

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$$\leq \frac{1}{\sqrt{\lambda}} \left(\sum_{\substack{\mu \in \Lambda^{c_2} \\ \mu \neq \lambda}} |(\varphi_\lambda^{c_1}, \varphi_\mu^{c_2})|^2 \mu \right)^{\frac{1}{2}} \left(\sum_{\mu \in \Lambda^{c_2}} \frac{1}{\mu} \right)^{\frac{1}{2}}$$

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$$\leq \frac{C}{\sqrt{\lambda}} (A^{c_2} \varphi_\lambda^{c_1}, \varphi_\lambda^{c_1})^{\frac{1}{2}}$$

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$$\leq C.$$

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This concludes the proof of the uniform bound of $\|\psi[\lambda, \lambda]\|_{\diamond^*}$.

- Case 2: $G = \{\lambda_1, \lambda_2\}$ is of cardinal 2. Since the diameter of G is smaller than ρ , we can choose the numbering such that $\lambda_1 \in \Lambda^{c_1} \setminus \Lambda^{c_2}$ and $\lambda_2 \in \Lambda^{c_2} \setminus \Lambda^{c_1}$. In particular, we have $B^* \phi_{\lambda_1}^0 \neq 0$, $B^* \phi_{\lambda_2}^0 \neq 0$ and there is no generalized eigenvector associated to this group G . Therefore, the only new quantity we need to estimate is the contribution of the following divided difference

$$\|\psi[\lambda_1, \lambda_2]\|_{\diamond^*}^2 \leq \frac{1}{|\lambda_1 - \lambda_2|^2} \left\| \frac{\phi_{\lambda_1}^0}{B^* \phi_{\lambda_1}^0} - \frac{\phi_{\lambda_2}^0}{B^* \phi_{\lambda_2}^0} \right\|_{\diamond^*}^2.$$

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Using formulas (95) and (98), we find

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$$\|\psi[\lambda_1, \lambda_2]\|_{\diamond^*}^2 \leq \frac{1}{|\lambda_1 - \lambda_2|^2} \left\| \frac{1}{B^* \xi_{\lambda_1}} \begin{pmatrix} \varphi_{\lambda_1}^{c_1} \\ \xi_{\lambda_1} \end{pmatrix} - \frac{1}{B^* \varphi_{\lambda_2}^{c_2}} \begin{pmatrix} 0 \\ \varphi_{\lambda_2}^{c_2} \end{pmatrix} \right\|_{\diamond^*}^2.$$

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Since λ_1 and λ_2 can be arbitrarily close it is not clear that this estimate does not blow up. In particular, if we use the triangle inequality, we will not be able to take benefit of compensations that occur in the divided difference.

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We will thus make appear from (99) the principal part of ξ_{λ_1} as follows

$$\xi_{\lambda_1} = \frac{\beta_{\lambda_1, \lambda_2}}{\lambda_1 - \lambda_2} (\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1}),$$

with $\beta_{\lambda_1, \lambda_2} := (\varphi_{\lambda_1}^{c_1}, \varphi_{\lambda_2}^{c_2})$ and

$$\zeta_{\lambda_1} := \frac{\lambda_1 - \lambda_2}{\beta_{\lambda_1, \lambda_2}} \sum_{\substack{\mu \in \Lambda^{c_2} \\ \mu \neq \lambda_2}} \frac{(\varphi_{\lambda_1}^{c_1}, \varphi_\mu^{c_2})}{\lambda_1 - \mu} \varphi_\mu^{c_2}.$$

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Thus,

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$$\begin{aligned} \frac{\xi_{\lambda_1}}{B^* \xi_{\lambda_1}} - \frac{\varphi_{\lambda_2}^{c_2}}{B^* \varphi_{\lambda_2}^{c_2}} &= \frac{\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1}}{B^* (\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1})} - \frac{\varphi_{\lambda_2}^{c_2}}{B^* \varphi_{\lambda_2}^{c_2}} \\ (103) \quad &= \left(\frac{1}{B^* (\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1})} - \frac{1}{B^* \varphi_{\lambda_2}^{c_2}} \right) \varphi_{\lambda_2}^{c_2} + \frac{\zeta_{\lambda_1}}{B^* (\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1})} \end{aligned}$$

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1590 Since we are only interested in the asymptotic behavior when λ_1 and λ_2 are
 1591 large, we see that we can assume from (91) that the following properties hold

$$1592 \quad (104) \quad |\beta_{\lambda_1, \lambda_2}| \geq 1/2, \quad \sqrt{\lambda_2} \geq \sqrt{\lambda_1}/2 \geq 1.$$

1593 Using that $|\lambda_1 - \mu| \geq r$ for all $\mu \in \Lambda^{c_2}$, with $\mu \neq \lambda_2$, we can find with (104)
 1594 and (94) the following bound

$$1595 \quad (105) \quad (A^{c_2} \zeta_{\lambda_1}, \zeta_{\lambda_1}) \leq |\lambda_1 - \lambda_2|^2 \frac{(A^{c_2} \varphi_{\lambda_1}^{c_1}, \varphi_{\lambda_1}^{c_1})}{r^2 |\beta_{\lambda_1, \lambda_2}|^2} \leq C^* |\lambda_1 - \lambda_2|^2 \lambda_1.$$

1596 Moreover, we have

$$\begin{aligned} |B^* \zeta_{\lambda_1}| &\leq \frac{|\lambda_1 - \lambda_2|}{|\beta_{\lambda_1, \lambda_2}|} \sum_{\substack{\mu \in \Lambda^{c_2} \\ \mu \neq \lambda_2}} \frac{|(\varphi_{\lambda_1}^{c_1}, \varphi_{\mu}^{c_2})|}{|\lambda_1 - \mu|} |B^* \varphi_{\mu}^{c_2}|, \\ &\leq C\tilde{C} |\lambda_1 - \lambda_2| \sum_{\substack{\mu \in \Lambda^{c_2} \\ \mu \neq \lambda_2}} \frac{|(\varphi_{\lambda_1}^{c_1}, \varphi_{\mu}^{c_2})|}{|\lambda_1 - \mu|} \sqrt{\mu} \\ &\leq C\tilde{C} |\lambda_1 - \lambda_2| \left(\sum_{\substack{\mu \in \Lambda^{c_2} \\ \mu \neq \lambda_2}} (\varphi_{\lambda_1}^{c_1}, \varphi_{\mu}^{c_2})^2 \mu \right)^{1/2} \left(\sum_{\substack{\mu \in \Lambda^{c_2} \\ \mu \neq \lambda_2}} \frac{1}{|\lambda_1 - \mu|^2} \right)^{1/2}. \end{aligned}$$

We use Parseval's identity and (94) to bound the second factor by $C\sqrt{\lambda_1}$.
 Moreover, by using (102), we have for any $\mu \in \Lambda^{c_2}$, $\mu \neq \lambda_2$,

$$|\lambda_1 - \mu| \geq |\lambda_2 - \mu| - |\lambda_1 - \lambda_2| \geq \rho(\sqrt{\mu} + \sqrt{\lambda_2}) - \rho \geq \rho\sqrt{\mu},$$

1598 so that the value of the series in the last factor is uniformly bounded. Hence,
 1599 we have proved

$$1600 \quad (106) \quad |B^* \zeta_{\lambda_1}| \leq C_1 |\lambda_1 - \lambda_2| \sqrt{\lambda_1}.$$

1601 From this last estimate, (100) and (93), we deduce that

$$\begin{aligned} |B^* (\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1})| &\geq \left(|B^* \varphi_{\lambda_2}^{c_2}| - C_1 |\lambda_1 - \lambda_2| \sqrt{\lambda_1} \right) \\ &\geq \bar{C} \sqrt{\lambda_1} - C_1 |\lambda_1 - \lambda_2| \sqrt{\lambda_1}. \end{aligned}$$

1605 Recall that $\rho < \bar{C}/(2C_1)$. Since λ_1 and λ_2 belong to the same group G , we
 1606 have $|\lambda_1 - \lambda_2| \leq \rho$ and thus, we obtain the estimate

$$1607 \quad (107) \quad |B^* (\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1})| \geq \frac{\bar{C}}{2} \sqrt{\lambda_1}.$$

1608 Coming back to the definition of $\psi[\lambda_1, \lambda_2]$ and using (103) and the triangle

1609 inequality, we write

$$\begin{aligned}
1610 \quad \|\psi[\lambda_1, \lambda_2]\|_{\diamond^*}^2 &= \frac{1}{|\lambda_1 - \lambda_2|^2} \left\| \frac{1}{B^*(\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1})} \begin{pmatrix} \frac{\lambda_1 - \lambda_2}{\beta_{\lambda_1, \lambda_2}} \varphi_{\lambda_1}^{c_1} \\ \varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1} \end{pmatrix} - \frac{1}{B^* \varphi_{\lambda_2}^{c_2}} \begin{pmatrix} 0 \\ \varphi_{\lambda_2}^{c_2} \end{pmatrix} \right\|_{\diamond^*}^2 \\
1611 \quad &\leq \frac{\left\| \begin{pmatrix} \varphi_{\lambda_1}^{c_1} \\ 0 \end{pmatrix} \right\|_{\diamond^*}^2}{\beta_{\lambda_1, \lambda_2}^2 |B^*(\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1})|^2} + \frac{2 \left\| \begin{pmatrix} 0 \\ \zeta_{\lambda_1} \end{pmatrix} \right\|_{\diamond^*}^2}{|B^*(\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1})|^2 |\lambda_1 - \lambda_2|^2} \\
1612 \quad &\quad + \frac{2 \left\| \begin{pmatrix} 0 \\ \varphi_{\lambda_2}^{c_2} \end{pmatrix} \right\|_{\diamond^*}^2}{|\lambda_1 - \lambda_2|^2} \left(\frac{1}{B^*(\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1})} - \frac{1}{B^* \varphi_{\lambda_2}^{c_2}} \right)^2 \\
1613 \quad &=: S_1 + S_2 + S_3.
\end{aligned}$$

1615 We now analyze each of the three terms.

– Using (94), (104) and (107), we can obtain

$$S_1 \leq \frac{16}{C^2}.$$

– Using (105) and (107), we get

$$S_2 \leq \frac{8C^*}{C^2}.$$

– Finally, with (94), we write

$$S_3 = \frac{\lambda_2}{|\lambda_1 - \lambda_2|^2} \frac{(B^* \zeta_{\lambda_1})^2}{(B^*(\varphi_{\lambda_2}^{c_2} + \zeta_{\lambda_1}))^2 (B^* \varphi_{\lambda_2}^{c_2})^2},$$

so that, with (93), (107) and (106), we get

$$S_3 \leq \frac{C_1^2}{C^4}.$$

1616 All in all, we have obtained a uniform bound for $\|\psi[\lambda_1, \lambda_2]\|_{\diamond^*}$, which is exactly
1617 the compensation phenomenon we were expecting for this particular system.

1618 As a conclusion, we finally proved that, whatever the group G is, all the divided
1619 differences $\psi[\lambda]$, $\psi[\lambda, \lambda]$ or $\psi[\lambda_1, \lambda_2]$ remain bounded uniformly. It follows from (17)
1620 that $T_0(Y_0) \leq 0$, so that our main Theorem 1.1 show that (89) is null-controllable at
1621 any time $T > 0$ for any initial data $y_0 \in Y_0$. \square

1622 It remains to prove the lemma.

1623 *Proof (of Lemma 5.1).* By definition, the function ξ_λ satisfies

$$1624 \quad (108) \quad -\partial_{xx} \xi_\lambda + c_2(x) \xi_\lambda = \lambda \xi_\lambda - \varphi_\lambda^{c_1}, \quad \text{in } (0, 1).$$

Using [2, Lemmas 2.2 and 2.3], and the fact that $\varphi_\lambda^{c_1}$ is normalized in $L^2(0, 1; \mathbb{R})$, we have

$$|\xi_\lambda(x)|^2 + \frac{1}{\lambda} |\partial_x \xi_\lambda(x)|^2 \leq C \left(|\xi_\lambda(y)|^2 + \frac{1}{\lambda} |\partial_x \xi_\lambda(y)|^2 \right) + \frac{C}{\lambda}, \quad \forall x, y \in [0, 1].$$

We take $y = 0$ in this inequality and we integrate with respect to x to obtain

$$\|\xi_\lambda\|^2 \leq \frac{C}{\lambda} |\partial_x \xi_\lambda(0)|^2 + \frac{C}{\lambda} = \frac{C}{\lambda} |B^* \xi_\lambda|^2 + \frac{C}{\lambda}.$$

It remains to bound from below the L^2 norm of ξ_λ . To this end, we multiply (108) by $\varphi_\lambda^{c_1}$ and integrate over $(0, 1)$. After integration by parts, and using the equation satisfied by $\varphi_\lambda^{c_1}$, we get

$$-1 = \int_0^1 (c_2 - c_1) \xi_\lambda \varphi_\lambda^{c_1}.$$

The Cauchy-Schwarz inequality gives

$$1 \leq \|c_1 - c_2\| \|\xi_\lambda\| \|\varphi_\lambda^{c_1}\|_{L^\infty},$$

1625 and since by (91), we have a uniform L^∞ bound on $\varphi_\lambda^{c_1}$, the proof is complete. \square

1626 **5.2.2. A system with different diffusions and a non constant coupling**
 1627 **term.** Let us briefly describe another example of a coupled parabolic system with
 1628 boundary control which has motivated our study. This example is analyzed in details
 1629 in [46]. We consider the following system

$$1630 \quad (109) \quad \begin{cases} \partial_t y + \begin{pmatrix} -\partial_{xx} & q(x) \\ 0 & -\nu \partial_{xx} \end{pmatrix} y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & (t, x) \in (0, T) \times (0, 1), \\ y(t, 0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad y(t, 1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & t \in (0, T), \\ y(0, x) = y_0(x), \end{cases}$$

1631 where $q \in L^\infty(0, 1)$ and $\nu > 0$.

1632 The spectrum of $\mathcal{A}^* = \begin{pmatrix} -\partial_{xx} & 0 \\ q(x) & -\nu \partial_{xx} \end{pmatrix}$ is $\Lambda = \{k^2, \nu k^2, \quad k \geq 1\}$.

- 1633 • System (109) in the case where $q(x) = 1$ and $\nu \neq 1$ was studied in [5] where
 1634 the influence of the condensation of eigenvalues in the system was first pointed
 1635 out. It was proved that the minimal null-control time was exactly the con-
 1636 densation index of Λ , provided that $\sqrt{\nu} \notin \mathbb{Q}$.
- 1637 • System (109) with a non constant q but with the same diffusions, that is
 1638 $\nu = 1$, was studied in [6]. The picture is different since in that case, there is
 1639 no condensation of eigenvalues but there may however exist a minimal null-
 1640 control time (depending on the coupling term q) due to very weak observation
 1641 properties of the eigenfunctions.
- 1642 • In the general case, assuming that $\sqrt{\nu} \notin \mathbb{Q}$, the eigenvalues are algebraically
 1643 and geometrically simple and it is proved in [46] that the associated family of
 1644 eigenfunctions is complete in $X_\diamond^* = (H^{-1}(0, 1))^2$, and that, moreover, there
 1645 exist functions q and values of ν , $\sqrt{\nu} \notin \mathbb{Q}$, such that this family (properly
 1646 normalized) is not a Riesz basis of X_\diamond^* . Therefore the abstract results in [5, 6]
 1647 do not apply.

1648 Inspired by the block moment method presented in the present paper, a
 1649 suitable value of $T_0^{q, \nu}$ is defined in [46] (taking into account both effects of
 1650 condensation of eigenvalues and weak observation of eigenfunctions) such that
 1651 $T_0^{q, \nu}$ is the minimal null-control time of (109).

6. Extensions.

6.1. Dealing with complex eigenvalues.

In the previous sections, we decided to state our results in the framework of real eigenvalues to simplify the presentation. However, most of them still hold for complex eigenvalues satisfying assumptions largely inspired from [5]. More precisely, for a function $\mathcal{N} : \mathbb{R}^+ \rightarrow \mathbb{R}$, we will consider the class $\mathcal{L}_w(\delta, p, \rho, \mathcal{N})$ of the families $\Lambda \subset \mathbb{C}$ satisfying

- Parabolicity condition:

$$\Re \lambda \geq \delta |\lambda|, \quad \forall \lambda \in \Lambda.$$

- Asymptotic behavior: for any $\varepsilon > 0$, we have

$$\sum_{\substack{\lambda \in \Lambda \\ |\lambda| > \mathcal{N}(\varepsilon)}} \frac{1}{|\lambda|} \leq \varepsilon.$$

- Weak gap condition with parameters $\rho > 0$ and $p \in \mathbb{N}^*$:

$$\#\Lambda \cap ([\mu, \mu + \rho] + i\mathbb{R}) \leq p, \quad \forall \mu > 0.$$

In that case, a grouping $(G_k)_k$ should satisfy

$$\Lambda = \bigcup_{k \geq 1} G_k, \quad \#G_k \leq p, \quad \text{diam}(G_k) < \rho, \quad \inf(\Re G_{k+1}) - \sup(\Re G_k) > r.$$

The corresponding formula the minimal time $T_0(Y_0)$ will be now given by

$$T_0(Y_0) := \limsup_{k \rightarrow \infty} \frac{\ln \left(\max_{\substack{\mu \in \mathbb{N}^{g_k} \\ \mu \leq \alpha_k}} \left\| \psi[\lambda_{k,1}^{(\mu_1)}, \dots, \lambda_{k,g_k}^{(\mu_{g_k})}] \right\|_{\mathcal{O}^*} \right)}{\Re \lambda_{k,1}},$$

and our results (namely Theorems 1.1, 2.1 and 4.1) still hold in that case.

Most of the proofs are very similar by taking care of the following points:

- The divided differences associated with pairwise distinct points x_0, \dots, x_n in the complex plane do not satisfy the Lagrange theorem but instead the following slightly weaker result, due to Jensen [36].

PROPOSITION 6.1. *Let $U \subset \mathbb{C}$ be a convex open set and $x_0, \dots, x_n \in U$ be pairwise distinct. For any holomorphic function $f : U \rightarrow \mathbb{C}$,*

- *there exists a $z \in \text{Conv}(\{x_0, \dots, x_n\})$ such that*

$$|f[x_0, \dots, x_n]| \leq \frac{f^{(n)}(z)}{n!}.$$

- *For any $z \in U$, we have*

$$\left| f[x_0, \dots, x_n] - \frac{f^{(n)}(z)}{n!} \right| \leq C_{f,n} \text{diam}(U).$$

- The Blaschke product W_k should be replaced by

$$W_k(z) = \prod_{j=1}^P \prod_{\lambda \in \Lambda_j} \frac{\lambda - z}{\bar{\lambda} + z}.$$

- Finally, in the restriction argument of Section 2.1.2, the holomorphy domain $\mathbb{C}_{2\varepsilon}^+$ should be replaced by a sector $\{z \in \mathbb{C}, \Re z > 2\varepsilon, |\Im z| \leq \frac{\delta}{2}|\lambda|\}$.

6.2. Weakening the assumptions on the control operator.

In this article, we not only study the classical null-controllability property (i.e. $Y_0 = X_{-\diamond}$), we also provide a more accurate description depending on the space of initial conditions Y_0 one wants to drive to 0. In this setting, the assumption (10) can be too strong.

It is easily seen that a necessary approximate null-controllability condition in that case is the following: for any $\lambda \in \Lambda$ and any $l \in \llbracket 0, \alpha_\lambda - 1 \rrbracket$ we have

$$(110) \quad \left(\mathcal{B}^* \phi_\lambda^j = 0, \quad \forall j \in \llbracket 0, l \rrbracket \right) \Rightarrow \left(P_{Y_0}^* \phi_\lambda^j = 0, \quad \forall j \in \llbracket 0, l \rrbracket \right),$$

where, in this formula, $(\phi_\lambda^j)_j$ is a Jordan chain associated with the eigenvalue λ . Note that such a Jordan chain is not unique but (110) does not depend on the particular chain we choose. Note also that the assumption (12) can be verified using any Jordan chain.

From now on, we assume that (110) holds. For any $\lambda \in \Lambda$, two cases have to be considered:

- Case 1: We have

$$1683 \quad \mathcal{B}^* \phi_\lambda^j = 0, \text{ for all } j \in \llbracket 0, \alpha_\lambda - 1 \rrbracket.$$

From (110), it follows that for any $y_0 \in Y_0$, any $T > 0$, all the moment equation (21) corresponding to this eigenvalue are automatically satisfied. It follows that we can simply remove this eigenvalue from the family Λ when studying the control problem at time T from Y_0 .

- Case 2:

$$1689 \quad (111) \quad \text{there exists } j^* \in \llbracket 0, \alpha_\lambda - 1 \rrbracket \text{ s.t. } \mathcal{B}^* \phi_\lambda^j = 0, \forall j < j^*, \text{ and } \mathcal{B}^* \phi_\lambda^{j^*} \neq 0.$$

In that case, for $j > j^*$ we set

$$\beta_j := -\frac{\mathcal{B}^* \phi_\lambda^j}{\mathcal{B}^* \phi_\lambda^{j^*}},$$

and then by induction, we define

$$1691 \quad (112) \quad \tilde{\phi}_\lambda^j = \begin{cases} \phi_\lambda^j, & \text{for } j \leq j^*, \\ \phi_\lambda^j + \sum_{k=j^*}^{j-1} \beta_{j+j^*-k} \tilde{\phi}_\lambda^k, & \text{for } j > j^*. \end{cases}$$

This construction ensures that $(\tilde{\phi}_\lambda^j)_j$ and $(\phi_\lambda^j)_j$ span the same space, that

$$1693 \quad (113) \quad \mathcal{B}^* \tilde{\phi}_\lambda^j = 0, \text{ if and only if } j \neq j^*,$$

and finally satisfy the equations

$$\mathcal{A}^* \tilde{\phi}_\lambda^j = \lambda \tilde{\phi}_\lambda^j + \tilde{\phi}_\lambda^{j-1} + \gamma_j \phi_\lambda^{j^*-1},$$

1694 for some $\gamma_j \in \mathbb{R}$ whose precise value is unimportant in the sequel.

A straightforward computation shows that the semi-group generated by $-\mathcal{A}^*$ satisfy

$$e^{-t\mathcal{A}^*} \tilde{\phi}_\lambda^j \in (e_t \tilde{\phi})[\lambda^{(j+1)}] + V^{j^*},$$

1695 where $V^{j^*} := \text{Span}(\phi_\lambda^0, \dots, \phi_\lambda^{j^*-1})$. We shall prove that the term in V^{j^*} does
1696 not contribute to the moment problem. Indeed, from (110) and (111), we
1697 have $V^{j^*} \subset \text{Ker } \mathcal{B}^* \cap \text{Ker } P_{Y_0}^*$. Thus:

– Concerning the control term, we have

$$\mathcal{B}^* e^{-t\mathcal{A}^*} \tilde{\phi}_\lambda^j = \mathcal{B}^* (e_t \tilde{\phi})[\lambda^{(j+1)}],$$

and by (113), it simply remains

$$\mathcal{B}^* e^{-t\mathcal{A}^*} \tilde{\phi}_\lambda^j = \begin{cases} 0, & \text{if } j < j^*, \\ (\mathcal{B}^* \tilde{\phi}_\lambda^{j^*}) e_t[\lambda^{(j-j^*)}], & \text{if } j \geq j^*. \end{cases}$$

1698 – Concerning the contribution of the source term, we have

$$\begin{aligned} 1699 \quad P_{Y_0}^* e^{-T\mathcal{A}^*} \tilde{\phi}_\lambda^j &= P_{Y_0}^* (e_T \tilde{\phi})[\lambda^{(j+1)}] \\ 1700 &= P_{Y_0}^* \sum_{k=1}^{j+1} e_T[\lambda^{(j+2-k)}] \tilde{\phi}[\lambda^{(k)}] \\ 1701 &= P_{Y_0}^* \sum_{k=j^*+1}^{j+1} e_T[\lambda^{(j+2-k)}] \tilde{\phi}[\lambda^{(k)}], \\ 1702 \end{aligned}$$

1703 with the convention that the sum is 0 as soon as $j < j^*$.

1704 We may now adapt the definition of our null-control time by setting

$$1705 \quad (114) \quad \psi_\lambda^l = \frac{P_{Y_0}^* \tilde{\phi}_\lambda^{j^*+l}}{\mathcal{B}^* \tilde{\phi}_\lambda^{j^*}}, \quad \forall l \in \llbracket 0, \alpha_\lambda - 1 - j^* \rrbracket,$$

1706 so that the moment problem associated with this eigenvalue becomes

$$1707 \quad \int_0^T u(T-t) \frac{(-t)^l}{l!} e^{-\lambda t} dt = - \left\langle y_0, (e_T \psi)[\lambda^{(l+1)}] \right\rangle_{-\diamond, \diamond}, \quad \forall l \in \llbracket 0, \alpha_\lambda - 1 - j^* \rrbracket.$$

1708 This is formally exactly the same as (23) except that the multiplicity of the
1709 eigenvalue have been changed into $\alpha_\lambda - j^*$ and the associated values of ψ_λ^\bullet
1710 have been constructed as explained above by (112) and (114).

1711 As a conclusion, to obtain the definition of the minimal null-control time from Y_0
1712 assuming that (110) holds, we simply need to ignore the eigenvalues corresponding
1713 to case 1, and to modify the multiplicity and the *Jordan chain* as explained above
1714 for the eigenvalues that are in case 2. Then, we define formally $T_0(Y_0)$ by the same
1715 formula as (16) and we obtain exactly the same result as Theorem 1.1.

1716 Moreover, it clearly appears from the proof that (110) is actually a necessary and
1717 sufficient condition to solve the moment problem associated to any finite number of
1718 eigenvalues. Thus (110) is a necessary and sufficient condition for the approximate
1719 null-controllability from Y_0 .

1720 **6.3. Dealing with geometrical multiplicities.**

1721 In the considered setting of scalar controls, if one wants to be able to control
 1722 every initial data, that is to take $Y_0 = X_{-\diamond}$, then it is absolutely necessary to assume,
 1723 as we did in Section 1.3.1, that all the eigenvalues are geometrically simple. Indeed,
 1724 as soon as $\dim \ker(\mathcal{A}^* - \lambda) \geq 2$, there necessarily exists an eigenfunction that belongs
 1725 to $\ker \mathcal{B}^*$ and the system is thus not even approximately controllable.

1726 However, if we are willing to restrict the space of initial data we consider, then our
 1727 result can apply when some eigenvalues are not geometrically simple. More precisely,
 1728 we will show that our results can be adapted under the condition

1729 (115)
$$\ker(\mathcal{A}^* - \lambda) \cap \ker \mathcal{B}^* \subset \ker P_{Y_0}^*, \quad \forall \lambda \in \Lambda,$$

1730 replacing the geometrical simplicity condition and the observation condition (10).
 1731 We will also assume, for simplicity, that geometrically multiple eigenvalues are alge-
 1732 braically simple. Let us give the main arguments.

- 1733 • First, it is clear that (115) is a necessary condition for the null-controllability
 1734 from initial data in Y_0 . Indeed, if this condition does not hold, there exists
 1735 a $\phi \in \ker(\mathcal{A}^* - \lambda) \cap \ker \mathcal{B}^*$ and a $y_0 \in Y_0$ such that $\langle y_0, \phi \rangle_{-\diamond, \diamond} \neq 0$. For
 1736 this particular y_0 , it cannot exist a u such that $y(T) = 0$ because it would
 1737 contradict the equality (20).
 1738 • Assume now that (115) holds. For each $\lambda \in \Lambda$, there are two cases:
 1739 – If $\ker(\mathcal{A}^* - \lambda) \subset \ker \mathcal{B}^*$, then by (115), we also have $\ker(\mathcal{A}^* - \lambda) \subset$
 1740 $\ker P_{Y_0}^*$. In that case, the controllability condition (20) automatically
 1741 holds for any $\phi \in \ker(\mathcal{A}^* - \lambda)$ and any control u since both terms in the
 1742 equality are equal to 0.
 1743 Hence, everything happens as if λ were not an eigenvalue of \mathcal{A}^* , and we
 1744 can essentially not consider it in the moment problem to be solved.
 – If $\ker(\mathcal{A}^* - \lambda) \not\subset \ker \mathcal{B}^*$, we fix any ϕ_λ^0 in $\ker(\mathcal{A}^* - \lambda)$ such that $\mathcal{B}^* \phi_\lambda^0 \neq 0$.
 As \mathcal{B}^* is a linear form, we observe that

$$\ker(\mathcal{A}^* - \lambda) = (\text{Span } \phi_\lambda^0) + (\ker(\mathcal{A}^* - \lambda) \cap \ker \mathcal{B}^*).$$

1745 It follows from (115) that (20) holds for any $\phi \in \ker(\mathcal{A}^* - \lambda)$ if and only
 1746 if (20) holds for $\phi = \phi_\lambda^0$.
 1747 Therefore, everything happens as if λ were geometrically simple with
 1748 ϕ_λ^0 as a unique (up to a scalar) eigenvector. Our analysis can then be
 1749 pushed forward without change.

1750 **7. Appendices.**

1751 We gather in this final section some definitions or intermediate results that we
 1752 used in this paper.

1753 **7.1. Wellposedness.**

1754 This section is dedicated to the proof of Proposition 1.1.

1755 First of all, let us notice that the problem (2) admits at most one solution $y \in$
 1756 $C^0([0, T]; X_{-1})$ and that the continuous dependency directly follows from (2). Thus,
 1757 it remains to prove the existence of a function $y \in C^0([0, T]; X_{-1})$ satisfying (2).

1758 From [49, Proposition 2.10.3] it comes that \mathcal{A} can be uniquely extended to an
 1759 operator $\tilde{\mathcal{A}} \in \mathcal{L}(X, X_{-1})$. Moreover it comes from [49, Proposition 2.10.4] that $-\tilde{\mathcal{A}}$
 1760 generates a C^0 -semigroup in X_{-1} satisfying

$$1761 \quad e^{-t\tilde{\mathcal{A}}} = \tilde{\mathcal{A}}e^{-t\mathcal{A}}\tilde{\mathcal{A}}^{-1}, \quad \forall t \geq 0.$$

1762 Thus, for any $T > 0$, any $y_0 \in X_{-1}$ and any $u \in L^2(0, T; U)$ the problem

$$1763 \quad \begin{cases} y'(t) + \tilde{\mathcal{A}}y(t) = \mathcal{B}u(t), \\ y(0) = y_0 \end{cases}$$

1764 admits a unique *mild* solution $y \in C^0([0, T], X_{-1})$ given by

$$1765 \quad (116) \quad y(t) = e^{-t\tilde{\mathcal{A}}}y_0 + \int_0^t e^{-(t-s)\tilde{\mathcal{A}}}\mathcal{B}u(s)ds.$$

1766 We prove now that this function satisfies (2). To do so, we simply prove that the
 1767 semigroup $e^{-t\tilde{\mathcal{A}}}$ is the adjoint of $e^{-t\mathcal{A}^*}$ in the duality between X_1^* and X_{-1} .

1768 Let $x \in X$ and $z \in X_1$ such that $x = \mathcal{A}z$. As $\tilde{\mathcal{A}}$ is an extension of \mathcal{A} it also comes
 1769 that $x = \tilde{\mathcal{A}}z$. Then, as $e^{-t\mathcal{A}}(X_1) \subset X_1$ it comes that

$$1770 \quad e^{-t\tilde{\mathcal{A}}}x = \tilde{\mathcal{A}}e^{-t\mathcal{A}}\tilde{\mathcal{A}}^{-1}\tilde{\mathcal{A}}z = \tilde{\mathcal{A}}e^{-t\mathcal{A}}z = \mathcal{A}e^{-t\mathcal{A}}z = e^{-t\mathcal{A}}\mathcal{A}z = e^{-t\mathcal{A}}x.$$

1771 Then, for any $x \in X$ and any $z \in X_1^*$

$$1772 \quad \left\langle e^{-t\tilde{\mathcal{A}}}x, z \right\rangle_{-1,1^*} = \left\langle e^{-t\mathcal{A}}x, z \right\rangle_{-1,1^*} = \left\langle e^{-t\mathcal{A}}x, z \right\rangle = \left\langle x, e^{-t\mathcal{A}^*}z \right\rangle_{-1,1^*}.$$

1773 Thus, the density of X in X_{-1} implies

$$1774 \quad (117) \quad \left\langle e^{-t\tilde{\mathcal{A}}}y, z \right\rangle_{-1,1^*} = \left\langle y, e^{-t\mathcal{A}^*}z \right\rangle_{-1,1^*}, \quad \forall y \in X_{-1}, \forall z \in X_1^*.$$

1775 Finally, the duality pairing of (116) with any $z_t \in X_1^*$ with the computation rule (117)
 1776 directly gives (2).

1777 **7.2. Existence of a grouping for sequences satisfying the weak gap con-** 1778 **dition.**

1779 PROPOSITION 7.1. *For any Λ satisfying (9), there exists at least one grouping in*
 1780 $\mathcal{G}\left(\Lambda, p, \frac{\rho}{p}, \rho\right)$.

Proof. Let $r = \rho/p$. We set $\mu_1 = \inf \Lambda$. and we consider the p disjoint sets

$$\Lambda \cap (\mu_1, \mu_1 + r], \Lambda \cap (\mu_1 + r, \mu_1 + 2r], \dots, \Lambda \cap (\mu_1 + (p-1)r, \mu_1 + pr].$$

By (9), we know that one of this sets is empty since if it not the case, there is at least
 $p+1$ elements in $\Lambda \cap [\mu_1, \mu_1 + \rho]$ because $\mu_1 \in \Lambda$ and $pr = \rho$. Let $j \in \llbracket 1, p \rrbracket$ such that
 $\Lambda \cap (\mu_1 + (j-1)r, \mu_1 + jr] = \emptyset$. We define

$$G_1 := \Lambda \cap [\mu_1, \mu_1 + (j-1)r],$$

whose cardinal is, by (9), less or equal than p and diameter is less than ρ . Moreover, by construction, we have

$$(\inf(\Lambda \setminus G_1)) - \sup G_1 > r.$$

1781 This allows to build G_2 by the same construction applied on $\Lambda \setminus G_1$ while ensuring
1782 the required properties, and following this process we construct the sequence $(G_k)_k$. \square

1783 7.3. About divided differences.

1784 In this section we give all the properties concerning divided differences that are
1785 used all along this article. This notion is a key technical tool in our analysis as it
1786 drastically eases the computations and the formulation of the results. The definition
1787 and results given in Section 7.3.1 are classical in the field of interpolation (see for
1788 instance [43, Chap. 5]). To deal with algebraic multiplicity we use a generalization
1789 of divided differences where the ‘interpolation points’ are not necessarily distincts.
1790 Let us mention that there exists generalizations in this direction (see for instance [43,
1791 Chap. 5]) in the context of Hermite interpolation. However as we are not directly
1792 dealing with interpolation, we propose such a generalization adapted to our purposes.
1793 This is detailed in Section 7.3.2.

1794 7.3.1. Definitions and basic properties.

1795 Let V be a real vector space, $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathbb{R}$. Assume that x_1, \dots, x_n
1796 are pairwise distinct (see Section 7.3.2 for a generalization). Let $f_1, \dots, f_n \in V$ be
1797 given.

DEFINITION 7.1. *The divided differences are defined by*

$$f[x_i] := f_i, \quad \forall i \in \llbracket 1, n \rrbracket,$$

1798 and then recursively for any $k \in \llbracket 2, n \rrbracket$, for any pairwise distinct $i_1, \dots, i_k \in \llbracket 1, n \rrbracket$, by

$$1799 \quad f[x_{i_1}, \dots, x_{i_k}] := \frac{f[x_{i_1}, \dots, x_{i_{k-1}}] - f[x_{i_2}, \dots, x_{i_k}]}{x_{i_1} - x_{i_k}}.$$

1800 In all what follows, if $f : \mathbb{R} \rightarrow V$ is a given function it will be implicitly assumed
1801 that $f_i = f[x_i] = f(x_i)$.

1802 PROPOSITION 7.2. *The divided differences are symmetric with respect to their ar-*
1803 *guments: for any $k \in \llbracket 1, n \rrbracket$, for any pairwise distinct $i_1, \dots, i_k \in \llbracket 1, n \rrbracket$ and any*
1804 *$\sigma \in \mathfrak{S}(\{i_1, \dots, i_k\})$,*

$$1805 \quad f[x_{\sigma(i_1)}, \dots, x_{\sigma(i_k)}] = f[x_{i_1}, \dots, x_{i_k}].$$

1806 The following property states another (equivalent) definition of divided differences
1807 known as Newton formula.

1808 PROPOSITION 7.3. *For any $k \in \llbracket 1, n \rrbracket$, for any pairwise distinct $i_1, \dots, i_k \in \llbracket 1, n \rrbracket$*

$$1809 \quad f[x_{i_1}, \dots, x_{i_k}] = \sum_{j=1}^k \frac{f[x_{i_j}]}{\prod_{l \in \llbracket 1, k \rrbracket, l \neq j} (x_{i_j} - x_{i_l})}.$$

1810 The next result about divided differences is crucial to obtain the different esti-
 1811 mates we need. It is known as Lagrange theorem.

1812 PROPOSITION 7.4. Assume that $V = \mathbb{R}$ and that $f \in C^{n-1}(\text{Conv}\{x_1, \dots, x_n\})$.
 1813 For any $k \in \llbracket 1, n \rrbracket$, for any pairwise distinct $i_1, \dots, i_k \in \llbracket 1, n \rrbracket$, there exists a $z \in$
 1814 $\text{Conv}\{x_{i_1}, \dots, x_{i_k}\}$ such that

$$1815 \quad f[x_{i_1}, \dots, x_{i_k}] = \frac{f^{(k-1)}(z)}{(k-1)!}.$$

1816 The divided differences naturally appear in polynomial interpolation problems as
 1817 recalled in the following classical result.

1818 PROPOSITION 7.5. The polynomial function $P : \mathbb{R} \rightarrow V$ defined by

$$1819 \quad (118) \quad P(x) := f[x_1] + (x - x_1)f[x_1, x_2] + \dots + \left(\prod_{i=1}^{n-1} (x - x_i) \right) f[x_1, \dots, x_n],$$

1820 is the unique polynomial of degree less than $n - 1$ such that

$$1821 \quad (119) \quad P(x_i) = f[x_i], \quad \forall i \in \llbracket 1, n \rrbracket.$$

1822 We recall a simple way to compute divided differences of a product which is known
 1823 as the Leibniz rule.

1824 PROPOSITION 7.6. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ and $(gf)[x] := g(x)f[x]$. For any $k \in \llbracket 1, n \rrbracket$, for
 1825 any pairwise distinct $i_1, \dots, i_k \in \llbracket 1, n \rrbracket$,

$$1826 \quad (gf)[x_{i_1}, \dots, x_{i_k}] = \sum_{j=1}^k g[x_{i_1}, \dots, x_{i_j}] f[x_{i_j}, \dots, x_{i_k}].$$

1827 Finally, we deduce from the results above the following useful corollary.

1828 COROLLARY 7.1. Assume that V is equipped with a norm $\|\bullet\|_V$. For any $k \in$
 1829 $\llbracket 1, n \rrbracket$ and any pairwise distinct $i_1, \dots, i_k \in \llbracket 1, n \rrbracket$, we have

$$1830 \quad \|f[x_{i_1}, \dots, x_{i_k}]\|_V \leq n2^{n-1}(1+R)^{n-1} \max_{j \in \llbracket 1, n \rrbracket} \|f[x_1, \dots, x_j]\|_V,$$

1831 where $R = \text{diam}(\{x_1, \dots, x_n\})$.

1832 *Proof.* Let P be the Lagrange interpolation polynomial defined in (118) and let
 1833 i_1, \dots, i_k be fixed.

1834 By the Hahn-Banach theorem, there exists $\phi \in V'$, such that $\|\phi\|_{V'} = 1$ and

$$1835 \quad \langle \phi, f[x_{i_1}, \dots, x_{i_k}] \rangle_{V', V} = \|f[x_{i_1}, \dots, x_{i_k}]\|_V.$$

1836 Additionally, by (119) and by linearity of ϕ , we know that

$$1837 \quad \langle \phi, f[x_{i_1}, \dots, x_{i_k}] \rangle_{V', V} = \langle \phi, P \rangle_{V', V} [x_{i_1}, \dots, x_{i_k}].$$

1838 Applying Proposition 7.4 to $x \mapsto \langle \phi, P(x) \rangle_{V', V} \in \mathbb{R}$ we find that for some $z \in$
 1839 $\text{Conv}\{x_1, \dots, x_n\}$, we have

$$1840 \quad \langle \phi, P \rangle_{V', V} [x_{i_1}, \dots, x_{i_k}] = \frac{1}{(k-1)!} (\langle \phi, P \rangle_{V', V})^{(k-1)}(z)$$

$$1841 \quad = \frac{1}{(k-1)!} \left\langle \phi, P^{(k-1)}(z) \right\rangle_{V', V}.$$

1843 Combining those identities, we arrive at

$$1844 \quad \|f[x_{i_1}, \dots, x_{i_k}]\|_V = \frac{1}{(k-1)!} \left\langle \phi, P^{(k-1)}(z) \right\rangle_{V',V} \leq \frac{1}{(k-1)!} \|P^{(k-1)}(z)\|_V.$$

1845 Let us compute the derivatives of P . Let \mathcal{C} be the circle of center z and radius R in
1846 the complex plane. The Cauchy formula leads to

$$1847 \quad \frac{1}{(k-1)!} P^{(k-1)}(z) = \frac{1}{2i\pi} \int_{\mathcal{C}} \frac{P(w)}{(z-w)^k} dw,$$

1848 so that

$$1849 \quad \frac{1}{(k-1)!} \|P^{(k-1)}(z)\|_V \leq R^{1-k} \max_{w \in \mathcal{C}} \|P(w)\|_V.$$

1850 Then, the triangle inequality implies that for any $w \in \mathcal{C}$,

$$1851 \quad \|P(w)\|_V \leq \|f[x_1]\|_V + (2R) \|f[x_1, x_2]\|_V + \dots + (2R)^{n-1} \|f[x_1, \dots, x_n]\|_V,$$

1852 which finally gives the result. \square

1853 7.3.2. Generalization of divided differences.

1854 Assume that V is a normed vector space.

Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ be pairwise distinct real numbers and let $\alpha \in \mathbb{N}^n$ a multi-index such that $\alpha > 0$. To such a multi-index we associate elements of V that we gather in a $f_\alpha \in V^{|\alpha|}$ and that are indexed as follows

$$f_j^l, \quad j \in \llbracket 1, n \rrbracket, l \in \llbracket 0, \alpha_j - 1 \rrbracket.$$

1855 DEFINITION 7.2. We set $N = |\alpha|$. We say that a family of points $(y_p^h)_{p \in \llbracket 1, N \rrbracket}$,
1856 depending on a small parameter $h > 0$, is an approximation of the weighted family
1857 (x, α) if

- 1858 • For each $h > 0$, the points y_1^h, \dots, y_N^h are pairwise distinct.
- There exist disjoint subsets $P_j \subset \llbracket 1, N \rrbracket$ such that for any $j \in \llbracket 1, n \rrbracket$,

$$\#P_j = \alpha_j, \quad \text{and} \quad y_p^h \xrightarrow{h \rightarrow 0} x_j, \quad \forall p \in P_j.$$

1859 PROPOSITION 7.7. With the notation above, let $F : \mathbb{R} \rightarrow V$ be any smooth function
1860 satisfying

$$1861 \quad (120) \quad \frac{1}{l!} F^{(l)}(x_j) = f_j^l, \quad \forall j \in \llbracket 1, n \rrbracket, \forall l \in \llbracket 0, \alpha_j - 1 \rrbracket.$$

1862 For any approximation of the weighted family (x, α) , the (usual) divided difference
1863 $F[y_1^h, \dots, y_N^h]$ weakly converges when $h \rightarrow 0$ towards an element in V that depends
1864 only on x, α and f_α . In particular it does not depend on the particular choice of F
1865 nor on the approximation families $(y_p^h)_p$.

1866 This limit is called the generalized divided difference associated with the points x ,
1867 the multi-index α and the values f_α and is denoted by

$$1868 \quad f[x_1^{(\alpha_1)}, \dots, x_n^{(\alpha_n)}], \quad \text{or} \quad f[\underbrace{x_1, \dots, x_1}_{\alpha_1 \text{ times}}, \underbrace{x_2, \dots, x_2}_{\alpha_2 \text{ times}}, \dots],$$

1869 or, in a more compact way, $f[x^{(\alpha)}]$.

1870 Moreover, we extend this definition if some of the α_j are 0, simply by not consid-
1871 ering the corresponding points.

1872 REMARK 7.1. *If the function F is chosen to take its values in a finite dimension*
 1873 *space then the above convergence is actually strong. It is always possible to make*
 1874 *this assumption, for instance by choosing F that takes its values in the subspace of V*
 1875 *spanned by the elements f_α .*

1876 *Proof (of Proposition 7.7).* The proof is done by recurrence on N .

1877 • If $N = 1$, then we necessarily have $n = 1$ and $\alpha_1 = 1$. The result is just a
 1878 consequence of the continuity of F and we simply have $f[x_1] = f_1^0$.
 1879 • Assume that the result holds for a given value of N and let us prove it for the value
 1880 $N + 1$.

1881 – *First case:* If there is only one point x_1 . It means that $n = 1$ and $\alpha_1 = N + 1$.
 1882 In this case, for any $h > 0$, and any $\psi \in V'$, we use the Lagrange theorem to
 1883 get the existence of a $z^{\psi,h} \in \text{Conv}(\{y_1^h, \dots, y_{N+1}^h\})$ such that

$$\begin{aligned} 1884 \quad \langle \psi, F[y_1^h, \dots, y_{N+1}^h] \rangle_{V',V} &= \langle \psi, F \rangle_{V',V} [y_1^h, \dots, y_{N+1}^h] \\ 1885 &= \frac{1}{N!} \langle \psi, F \rangle_{V',V}^{(N)}(z^{\psi,h}). \end{aligned}$$

1887 Since, by assumption, all the points y_p^h converge to the same point x_1 , we
 1888 have $z^{\psi,h} \rightarrow x_1$ and thus

$$1889 \quad \langle \psi, F[y_1^h, \dots, y_{N+1}^h] \rangle_{V',V} \xrightarrow{h \rightarrow 0} \frac{1}{N!} \langle \psi, F^{(N)}(x_1) \rangle_{V',V} = \langle \psi, f_1^N \rangle_{V',V},$$

1890 – *Second case:* We assume that $n > 1$. By assumption there exists two distinct
 1891 indices $j_1, j_2 \in \llbracket 1, n \rrbracket$ and two distinct indices $p_1, p_2 \in \llbracket 1, N + 1 \rrbracket$ such that
 1892 $y_{p_1}^h \rightarrow x_{j_1}$ and $y_{p_2}^h \rightarrow x_{j_2}$. By symmetry of the usual divided differences, we
 1893 can always assume that $p_1 = N$ and $p_2 = N + 1$. It follows that we can write

$$1894 \quad F[y_1^h, \dots, y_{N+1}^h] = \frac{F[y_1^h, \dots, y_{N-1}^h, y_{N+1}^h] - F[y_1^h, \dots, y_N^h]}{y_{N+1}^h - y_N^h}.$$

1895 The recurrence assumption shows that the two terms in the numerator have
 1896 weak limits that only depends on the points x , the multiplicities α and on the
 1897 values f_α , whereas the denominator $y_{N+1}^h - y_N^h$ converges to $x_{j_2} - x_{j_1}$ which
 1898 is not zero. The result follows. \square

1899 The above construction also shows, as a by-product, the following rules to com-
 1900 pute the generalized divided differences: for any $\mu \in \mathbb{N}^n$ such that $\mu \leq \alpha$

$$1901 \quad (121) \quad f[x_1^{(\mu_1)}, \dots, x_n^{(\mu_n)}] = f_j^{\mu_j - 1}, \quad \text{if } \mu_{j'} = 0 \text{ for all } j' \neq j,$$

1902 and for all $j_1 \neq j_2$ and $\mu_{j_1} > 0, \mu_{j_2} > 0$

$$1903 \quad (122) \quad f[x_1^{(\mu_1)}, \dots, x_n^{(\mu_n)}] = \frac{f[\dots, x_{j_1}^{(\mu_{j_1}-1)}, \dots, x_{j_2}^{(\mu_{j_2})}, \dots] - f[\dots, x_{j_1}^{(\mu_{j_1})}, \dots, x_{j_2}^{(\mu_{j_2}-1)}, \dots]}{x_{j_1} - x_{j_2}}.$$

1904 Let us now give some useful properties that are the extension of the classical
 1905 properties recalled in Section 7.3.1.

1906 DEFINITION 7.3. *Let $\alpha \in \mathbb{N}^n$ be a multi-index, $g_\alpha \in \mathbb{R}^{|\alpha|}$ a set of real values*
 1907 *associated with α and $f_\alpha \in V^{|\alpha|}$ a set of elements of V associated with α .*

We define $(gf)_\alpha \in V^{|\alpha|}$ to be the product set of values as follows:

$$(gf)_j^l := \sum_{k=0}^l g_j^k f_j^{l-k}, \quad \forall j \in \llbracket 1, n \rrbracket, \forall l \in \llbracket 0, \alpha_j - 1 \rrbracket.$$

1908 PROPOSITION 7.8 (Leibniz formula). Let $x \in \mathbb{R}^n$ pairwise distinct points, $\alpha \in \mathbb{N}^n$,
 1909 $g_\alpha \in \mathbb{R}^{|\alpha|}$ a set of real values, and $f_\alpha \in V^{|\alpha|}$ a set of values in V .

1910 Then, for any family of multi-indices $(\mu^p)_{p \in \llbracket 0, |\alpha| \rrbracket} \subset \mathbb{N}^n$ satisfying

$$1911 \quad (123) \quad \begin{cases} \mu^{p-1} \leq \mu^p, & \forall p \in \llbracket 1, |\alpha| \rrbracket, \\ |\mu^p| = p, & \forall p \in \llbracket 0, |\alpha| \rrbracket, \\ \mu^{|\alpha|} = \alpha, \end{cases}$$

we have the Leibniz formula

$$(gf)[x^{(\alpha)}] = \sum_{p=1}^{|\alpha|} g[x^{(\mu^p)}] f[x^{(\alpha - \mu^{p-1})}].$$

1912 *Proof.* By assumption, for each $p \in \llbracket 1, |\alpha| \rrbracket$, the multi-index μ^p is obtained from
 1913 μ^{p-1} by incrementing exactly one of its element. We denote by $i_p \in \llbracket 1, n \rrbracket$ this index,
 1914 and we define $y_p^h := x_{i_p} + ph$. It is easily seen that, for $h > 0$ small enough, those
 1915 points are pairwise distinct.

Let $F : \mathbb{R} \rightarrow V$ be a function satisfying (120) and $G : \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying (120) but with the values g_α instead of f_α . The usual Leibniz formula as well as the Definition 7.3 shows that the product GF exactly satisfies

$$\frac{1}{l!} (GF)^{(l)}(x_j) = (gf)_j^l, \quad \forall j \in \llbracket 1, n \rrbracket, \forall l \in \llbracket 0, \alpha_j - 1 \rrbracket.$$

We can thus apply the Leibniz formula from Proposition 7.6 as follows

$$(GF)[y_1^h, \dots, y_{|\alpha|}^h] = \sum_{p=1}^{|\alpha|} G[y_1^h, \dots, y_p^h] F[y_p^h, \dots, y_{|\alpha|}^h],$$

1916 and then pass to the limit as $h \rightarrow 0$ to obtain the claim. \square

PROPOSITION 7.9 (Lagrange theorem). Let x, α as before. We set $N = |\alpha|$. With any $f : \mathbb{R} \rightarrow \mathbb{R}$ of class C^{N-1} , we associate the set of values $f_\alpha \in \mathbb{R}^N$ by

$$f_j^l := \frac{1}{l!} f^{(l)}(x_j), \quad \forall j \in \llbracket 1, n \rrbracket, \forall l \in \llbracket 0, \alpha_j - 1 \rrbracket.$$

Then, there exists a $z \in \text{Conv}(\{x_1, \dots, x_n\})$ such that the generalized divided difference built on these data satisfies

$$f[x^{(\alpha)}] = \frac{1}{(N-1)!} f^{(N-1)}(z).$$

Proof. Let y_1^h, \dots, y_N^h be an approximation of the weighted family of points (x, α) as in Definition 7.2. By definition, the generalized divided difference $f[x^{(\alpha)}]$ is the limit as h goes to 0, of the usual divided difference $f[y_1^h, \dots, y_N^h]$. For this last divided

difference, we can apply Lagrange theorem (see Proposition 7.4) to get the existence of a point $z^h \in \text{Conv}(\{y_1^h, \dots, y_N^h\})$ such that

$$f[y_1^h, \dots, y_N^h] = \frac{1}{(N-1)!} f^{(N-1)}(z^h).$$

1917 It is clear that $(z^h)_h$ is contained in a compact set so that, up to a subsequence,
 1918 we may find a limit z of $(z^h)_h$ that belongs to $\text{Conv}(\{x_1, \dots, x_n\})$ and satisfies the
 1919 required property. \square

PROPOSITION 7.10. Let $(\mu^p)_{p \in \llbracket 0, |\alpha| \rrbracket} \subset \mathbb{N}^n$ be a family of multi-indices satisfying (123). For any multi-index μ such that $\mu \leq \alpha$ we have

$$\|f[x^{(\mu)}]\| \leq N 2^{N-1} (1+R)^{N-1} \max_{p \in \llbracket 1, |\alpha| \rrbracket} \|f[x^{(\mu^p)}]\|.$$

1920 *Proof.* We proceed as in the proof of Proposition 7.8 by passing to the limit in
 1921 the similar result for standard divided differences (Corollary 7.1). \square

1922 For generalized divided differences, there is no simple equivalent to the Newton
 1923 formula (Proposition 7.3). However, we can state the following result.

1924 PROPOSITION 7.11. For any multi-index $\mu \leq \alpha$, there exists coefficients $(\theta_{j,l}^\mu)_{j,l}$
 1925 depending only on x and μ , such that

$$1926 \quad f[x^{(\mu)}] = \sum_{j=1}^n \sum_{l=0}^{\alpha_j-1} \theta_{j,l}^\mu f_j^l,$$

and which satisfy the following estimates

$$|\theta_{j,l}^\mu| \leq \begin{cases} 0 & \text{if } \mu_j < l+1, \\ \frac{C_{|\mu|}}{\left(\prod_{i \in \llbracket 1, n \rrbracket \neq j} |x_i - x_j|^{\mu_i}\right)} \frac{1}{\left(\min_{i \in \llbracket 1, n \rrbracket \neq j} |x_i - x_j|\right)^{\mu_j - l - 1}}, & \text{if } \mu_j \geq l+1. \end{cases}$$

1927 *Proof.* Since the divided differences are clearly linear with respect to the data
 1928 f_α , the existence of the coefficients $\theta_{j,l}^\mu$ is straightforward. Let us prove the claimed
 1929 estimates. From now on we assume that l is fixed. Moreover, for any $j \in \llbracket 1, n \rrbracket$ we
 1930 introduce the notation $d_j := \min_{i \in \llbracket 1, n \rrbracket \neq j} |x_i - x_j|$ and we define $\delta^j \in \mathbb{N}^n$ to be the
 1931 Kronecker multi-index, that is $\delta_i^j = 0$ for $i \neq j$ and $\delta_j^j = 1$.

- 1932 • When $\mu_j < l+1$, it is clear from the recurrence formulas (121) and (122)
 1933 that the value of $f[x^{(\mu)}]$ does not depend on the value f_j^l , and therefore
 1934 $\theta_{j,l}^\mu = 0$.
- 1935 • Let us show by induction on $N = |\mu|$ that, for all $j \in \llbracket 1, n \rrbracket$, with $\mu_j \geq l+1$,
 1936 we have

$$1937 \quad (124) \quad |\theta_{j,l}^\mu| \leq \frac{C_{|\mu|}}{\left(\prod_{i \in \llbracket 1, n \rrbracket \neq j} |x_j - x_i|^{\mu_i}\right)} \frac{1}{d_j^{\mu_j - l - 1}}.$$

- Assume first that $N = l+1$ and let μ such that $|\mu| = N$. If $\mu_j < l+1$
 we have already seen that $\theta_{j,l}^\mu = 0$ which obviously implies (124). If
 $\mu_j = l+1$, since $|\mu| = l+1$, we necessarily have $\mu_i = 0$ for any $i \neq j$, so
 that (121) gives

$$f[x^{(\mu)}] = f_j^l,$$

1938 which implies that $\theta_{j,l}^\mu = 1$, that is exactly (124) with $C_{|\mu|} = 1$ in that
 1939 case.

1940 – Assume now that, for some $N \geq l + 1$, (124) holds and let μ such that
 1941 $|\mu| = N + 1$.
 If $\mu_i = 0$ for any $i \neq j$, then we have

$$f[x^{(\mu)}] = f_j^{\mu_j - 1},$$

1942 which implies that $\theta_{j,l}^\mu = 0$ since $l \neq \mu_j - 1$ and (124) is obvious.
 If there is a $i_0 \neq j$ such that $\mu_{i_0} \geq 1$ then we use (122) to get

$$f[x^{(\mu)}] = \frac{f[x^{(\mu - \delta^j)}] - f[x^{(\mu - \delta^{i_0})}]}{x_{i_0} - x_j},$$

which implies the formula

$$\theta_{j,l}^\mu = \frac{\theta_{j,l}^{\mu - \delta^j} - \theta_{j,l}^{\mu - \delta^{i_0}}}{x_{i_0} - x_j},$$

and thus

$$|\theta_{j,l}^\mu| \leq \frac{|\theta_{j,l}^{\mu - \delta^j}|}{|x_{i_0} - x_j|} + \frac{|\theta_{j,l}^{\mu - \delta^{i_0}}|}{|x_{i_0} - x_j|}.$$

1943 Since $|\mu - \delta^j| = |\mu - \delta^{i_0}| = N$, we can apply the induction hypothesis to
 1944 bound the two terms in the right-hand side as follows

1945

$$1946 \frac{|\theta_{j,l}^{\mu - \delta^j}|}{|x_{i_0} - x_j|} \leq \frac{C_{N-1} |\theta_{j,l}^{\mu - \delta^j}|}{d_j} \leq \frac{C_{N-1}}{\left(\prod_{i \in \llbracket 1, n \rrbracket \neq j} |x_j - x_i|^{\mu_i}\right)} \frac{1}{d_j^{\mu_j - l - 2}} \frac{1}{d_j},$$

1947

and

1948

$$1949 \frac{|\theta_{j,l}^{\mu - \delta^{i_0}}|}{|x_i - x_j|} \leq \frac{C_{N-1} |\theta_{j,l}^{\mu - \delta^{i_0}}|}{|x_j - x_{i_0}|}$$

$$1950 \leq \frac{C_{N-1}}{\left(\prod_{i \in \llbracket 1, n \rrbracket \neq j} |x_j - x_i|^{\mu_i - \delta_i^{i_0}}\right)} \frac{1}{d_j^{\mu_j - l - 1}} \frac{1}{|x_j - x_{i_0}|}.$$

1951

1952

1953 Summing those two inequalities gives (124) with $C_N = 2C_{N-1}$. \square

1954 7.4. The supremum of $T_0(y_0)$.

1955 We prove here Proposition 1.3, that is

1956
$$\sup_{y_0 \in Y_0} T_0(y_0) = T_0(Y_0).$$

1957

1958 *Proof.* Since by definition $T_0(y_0)$ only depends on $\text{Span}(y_0)$, it is actually equiv-
 1959 alent to prove

1960
$$\sup_{\substack{y_0 \in Y_0 \\ \|y_0\|_{-\diamond} = 1}} T_0(y_0) = T_0(Y_0).$$

1961 To ease the reading let us do the computations in the simpler case $\eta = 1$; the
 1962 extension to the case $\eta \geq 2$ being straightforward. Let us introduce

$$1963 \quad x_{k,l} := \sum_{j=1}^l \frac{\psi_{k,j}}{\prod_{1 \leq i \neq j \leq l} (\lambda_{k,j} - \lambda_{k,i})}, \quad \forall k \geq 1, \forall l \in \llbracket 1, g_k \rrbracket.$$

1964 with $\psi_{k,j} := \frac{P_{Y_0}^* \phi_{k,j}}{\mathcal{B}^* \phi_{k,j}}$ as defined in (15).

1965 Notice that, since $\|y_0\|_{-\diamond} = 1$, for any $z \in X_{\diamond}^*$,

$$\begin{aligned} 1966 \quad \left\| P_{\text{Span}(y_0)}^* z \right\|_{\diamond^*} &= \sup_{\substack{y \in X_{-\diamond} \\ \|y\|_{-\diamond} = 1}} \left| \langle y, P_{\text{Span}(y_0)}^* z \rangle_{-\diamond, \diamond} \right| \\ 1967 &= \sup_{\substack{y \in X_{-\diamond} \\ \|y\|_{-\diamond} = 1}} \left| \langle P_{\text{Span}(y_0)} y, z \rangle_{-\diamond, \diamond} \right| \\ 1968 &= \sup_{\substack{y \in X_{-\diamond} \\ \|y\|_{-\diamond} = 1}} \left| (y, y_0)_{-\diamond} \langle y_0, z \rangle_{-\diamond, \diamond} \right| \\ 1969 \quad &= \left| \langle y_0, z \rangle_{-\diamond, \diamond} \right|. \\ 1970 \end{aligned}$$

1971 Thus, with those notations, we have

$$\begin{aligned} 1972 \quad T_0(y_0) &= \limsup_{k \rightarrow +\infty} \frac{\ln \left(\max_{l \in \llbracket 1, g_k \rrbracket} \left| \langle y_0, x_{k,l} \rangle_{-\diamond, \diamond} \right| \right)}{\lambda_{k,1}}, \\ 1973 \quad T_0(Y_0) &= \limsup_{k \rightarrow +\infty} \frac{\ln \left(\max_{l \in \llbracket 1, g_k \rrbracket} \|x_{k,l}\|_{\diamond^*} \right)}{\lambda_{k,1}}. \\ 1974 \end{aligned}$$

1975 • Since y_0 is normalized, we have $\left| \langle y_0, x_{k,l} \rangle_{-\diamond, \diamond} \right| \leq \|x_{k,l}\|_{\diamond^*}$, for any k and l , it
 1976 immediately comes that $T_0(y_0) \leq T_0(Y_0)$ and thus

$$1977 \quad \sup_{y_0 \in Y_0} T_0(y_0) \leq T_0(Y_0).$$

1978 • Conversely, let T be such that

$$1979 \quad \sup_{\substack{y_0 \in Y_0 \\ \|y_0\|_{-\diamond} = 1}} T_0(y_0) < T.$$

1980 Setting $\tilde{x}_{k,l} := e^{-\lambda_{k,1} T} x_{k,l}$, it comes that for any $y_0 \in Y_0$, we have

$$1981 \quad \sup_{\substack{k \geq 1 \\ l \in \llbracket 1, g_k \rrbracket}} \left| \langle y_0, \tilde{x}_{k,l} \rangle_{-\diamond, \diamond} \right| < +\infty,$$

and this property is in fact true for any $y_0 \in X_{-\diamond}$ since $P_{Y_0}^* \tilde{x}_{k,l} = \tilde{x}_{k,l}$, so that we have

$$\langle y_0, \tilde{x}_{k,l} \rangle_{-\diamond, \diamond} = \langle y_0, P_{Y_0}^* \tilde{x}_{k,l} \rangle_{-\diamond, \diamond} = \langle P_{Y_0} y_0, \tilde{x}_{k,l} \rangle_{-\diamond, \diamond},$$

1982 and $P_{Y_0}y_0 \in Y_0$.
 1983 Applying the Banach-Steinhaus theorem, this implies that

$$1984 \quad \sup_{\substack{k \geq 1 \\ l \in \llbracket 1, g_k \rrbracket}} \|\tilde{x}_{k,l}\|_{\phi^*} < +\infty.$$

1985 Thus there exists $C > 0$ such that $\|x_{k,l}\|_{\phi^*} \leq Ce^{\lambda_{k,1}T}$ for any $k \geq 1$ and any
 1986 $1 \leq l \leq g_k$. Finally this yields,

$$1987 \quad T_0(Y_0) = \limsup_{k \rightarrow \infty} \frac{\ln \left(\max_{l \in \llbracket 1, g_k \rrbracket} \|x_{k,l}\|_{\phi^*} \right)}{\lambda_{k,1}} \leq T.$$

1988 This ends the proof of Proposition 1.3. □

1989 7.5. On the condensation index.

1990 In this appendix we give some useful properties concerning the condensation index
 1991 of a sequence. Let Σ be a family of positive real numbers. We start by recalling the
 1992 definition of $c(\Sigma)$.

1993 DEFINITION 7.4. Assume that Σ satisfies

$$1994 \quad (125) \quad \sum_{\sigma \in \Sigma} \frac{1}{\sigma} < +\infty.$$

1995 The interpolating function is defined by

$$1996 \quad (126) \quad E_\Sigma : z \in \mathbb{C} \mapsto \prod_{\sigma \in \Sigma} \left(1 - \frac{z^2}{\sigma^2} \right).$$

1997 The condensation index $c(\Sigma) \in [0, +\infty]$ is defined by

$$1998 \quad c(\Sigma) = \limsup_{\substack{\sigma \in \Sigma \\ \sigma \rightarrow \infty}} \frac{-\ln |E'_\Sigma(\sigma)|}{\sigma}.$$

1999 This definition (and also its extension to complex sequences) is given in [48]. Notice
 2000 that due to the assumption (125), both functions

$$2001 \quad z \in \mathbb{C} \mapsto \prod_{\sigma \in \Sigma} \left(1 - \frac{z}{\sigma} \right), \quad z \in \mathbb{C} \mapsto \prod_{\sigma \in \Sigma} \left(1 + \frac{z}{\sigma} \right),$$

2002 are entire. Since $\Sigma \subset (0, +\infty)$, the function E_Σ has simple roots corresponding exactly
 2003 to Σ . Thus, the condensation index of such sequences is well defined. The fact that
 2004 it belongs to $[0, +\infty]$ is proved in [5].

2005 In the case where the considered sequence satisfies the weak gap condition (9) the
 2006 computation of the condensation index can be simplified: the grouping introduced in
 2007 Proposition 7.1 is an optimal condensation grouping in the following sense.

2008 PROPOSITION 7.12. Assume that Σ satisfies the assumptions of Definition 7.4 as
 2009 well as the weak gap condition (9). Denote by $(G_k)_{k \geq 1}$ a grouping satisfying the
 2010 conditions of Definition 1.2. Then,

$$2011 \quad c(\Sigma) = \limsup_{\substack{\sigma \in \Sigma \\ \sigma \rightarrow \infty}} \frac{-\ln |P'_{G^{[\sigma]}}(\sigma)|}{\sigma}.$$

2012 Recall that $G^{[\sigma]}$ is the element of $(G_k)_{k \geq 1}$ containing σ .

2013 *Proof.* The proof follows directly from (64). \square

2014 Using this result, we compute easily the condensation index of the particular
2015 sequence used in Section 5.1.

2016 PROPOSITION 7.13. *Let $(\mu_k)_{k \geq 1}$ be a real increasing sequence such that*

$$2017 \quad \sum_{k \geq 1} \frac{1}{\mu_k} < +\infty.$$

2018 *Let $\alpha > \beta > 0$ and $\Theta = \{\mu_k, \mu_k + e^{-\alpha\mu_k}, \mu_k + e^{-\beta\mu_k}; k \in \mathbb{N}^*\}$. Then,*

$$2019 \quad c(\Theta) = \alpha + \beta.$$

2020 *Proof.* One can directly verify that the grouping defined by

$$2021 \quad G_k := \{\mu_k, \mu_k + e^{-\alpha\mu_k}, \mu_k + e^{-\beta\mu_k}\}$$

2022 satisfies the requirements given in Proposition 7.1. Then, direct computations lead to

$$2023 \quad |P'_{G_k}(\mu_k)| = |\mu_k - (\mu_k + e^{-\alpha\mu_k})| |\mu_k - (\mu_k + e^{-\beta\mu_k})| = e^{-(\alpha+\beta)\mu_k},$$

2024

$$2025 \quad |P'_{G_k}(\mu_k + e^{-\alpha\mu_k})| = |\mu_k + e^{-\alpha\mu_k} - \mu_k| |\mu_k + e^{-\alpha\mu_k} - (\mu_k + e^{-\beta\mu_k})| \\ 2026 \quad = e^{-(\alpha+\beta)\mu_k} \left(1 - e^{-(\alpha-\beta)\mu_k}\right),$$

2027

2028 and

$$2029 \quad |P'_{G_k}(\mu_k + e^{-\beta\mu_k})| = |\mu_k + e^{-\beta\mu_k} - \mu_k| |\mu_k + e^{-\beta\mu_k} - (\mu_k + e^{-\alpha\mu_k})| \\ 2030 \quad = e^{-2\beta\mu_k} \left(1 - e^{-(\alpha-\beta)\mu_k}\right).$$

2031

2032 Thus, as $2\beta < \alpha + \beta$, we obtain $c(\Theta) = \alpha + \beta$. \square

2033

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