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Written documentations in final exams with CAS

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This article focuses on students' written solutions from CAS-allowed exams. Based on the analysis of students' solutions a descriptive model is set up. It can be used for assessing students' solution as well as creating exemplary documentations. The purpose of these documentations is to help teachers reflect about their practise of writing down solutions and the norms they set by this for exams. This paper also shows how formative assessment could be a means to help students develop their competencies in communicating mathematics.

Keywords: Upper secondary education, exams, written solutions, Computer Algebra Systems, formative assessment.

Introduction: Challenges in exams with computer algebra systems

Digital technologies like graphical calculators and computer-algebra-systems (CAS) influence various aspects of mathematics education in general and the classroom-practise in particular (cf. Barzel ea. 2005). Most prominently, the way tasks can be solved changes drastically: where once it was necessary to differentiate a function with pen and paper, now every student can use a CAS and by pressing a short series of buttons the result appears on the screen. The new possibilities fuelled the hope that “[t]he new tool provided the chance to concentrate more on central competencies in mathematics education, concept formation, problem solving and modelling competencies, and to outsource algorithmic operations to the machine” (Drijvers & Weigand, 2010). While the shortcutting of the work of calculating with pen and paper enables classes to have more time for the central competencies, it also forces teachers and students to think about how to document the process of “working” on a task. This article focuses on the aspect of how communication in a CAS-environment could be shaped.

In this context, the study aims at identifying problems, difficulties and possibilities concerning students' solutions, categorize elements in those according to their function within the solving process and suggest a possible standard for written solutions with CAS. Against this background, the suggestions can also be interpreted as assessment criteria and, thus, a given students' solution can be assessed. Furthermore, it is a goal to provide a theoretical model that describes the development of students' documenting competency. The following research questions are

1. How do students write down their solutions in final exams? Which different forms of documentations do they use? What kinds of problems or difficulties (if any) are connected with these forms?
2. How can students' written solutions be described by means of a category system?
3. How could a developmental model look like that
 - a. describes how students' competencies in documenting the solving process with CAS can be developed?
 - b. offers learning strategies and exemplary solutions for shape this development?
 - c. encompasses criteria for assessment in final exams?

The major motivation for this study is to create a sound set of suggestions for teachers, for application in the classroom and for preparing students for the final exams. Formative assessment plays a crucial role in developing an adequate documentation competence. This article gives an answer to the second question and presents an outline of ideas to research question (3).

Theoretical framework

The theoretical framework to tackle the questions above encompasses two important fields: written documentations as communicative texts, and formative assessment, which will be the basis for creating didactical material for teachers.

In exams the purpose of documentations is to enable others to understand how the solution has been gained and to evaluate to which degree the solution is correct or incorrect (cf. Ball & Stacey 2003). Thus, the communication of mathematical knowledge is the primary aspect. In terms of the communication model by Jakobson (1960) the communication situation can be described as follows: the learner is the ADDRESSER, the teacher is the ADDRESSEE and the written-down solution is the CONTACT (or channel) for the MESSAGE (1960, p. 353). The CODE in this communication situation can be considered as coming from three different areas: (1) the natural language, (2) the mathematical language, encompassing the symbolic language as well as the mathematical register (cf. Pimm 1987), and (3) the computer world with CAS-commands and also its own register (cf. Siller & Greefrath, 2010).

In exams the written documentation is – according to the communication model above – the only channel by which the message is sent from the addresser to the addressee. Naturally, in such a situation it is neither possible nor allowed for the corrector to inquire in case he or she does not understand a part of the solution. Busse speaks of all written communication as “reduced communication situation[s]” (2015, p 320, translation by the author), arguing that only the text itself and the recipient of the text are present in the situation. As a result, the understanding of texts can be reduced to the allocation of the recipient’s knowledge to elements of the text.

The second part of the theoretical background is about formative assessment and how it might be used to develop the documentation competence of students over a longer period. “Assessment for learning”, as formative assessment is sometimes called, can be outlined as “the process of seeking and interpreting evidence for use by learners and their teachers to decide where the learners are in their learning, where they need to go and how best to get there” (ARG 2002, p. 2). Black & Wiliam (2009) describe how five key strategies constitute formative assessment:

1. Clarifying and sharing learning intentions and criteria for success;
2. Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding;
3. Providing feedback that moves learners forward;
4. Activating students as instructional resources for one another; and
5. Activating students as the owners of their own learning (Black & Wiliam 2009).

The crucial and most difficult point here is to have criteria for good written solutions. As Weigand

points out “there are no algorithmic rules or norms how to document a solution on paper” (Weigand 2013, p. 2772). He reports from a long-term project in Bavaria (a part of Germany) that students have “difficulties in using SC [scientific calculator] and (problem-)adequate representations especially, as well as the documentation of the solution with paper and pencil” (Weigand 2013, p. 2763). Therefore, teachers need to focus on the development of the competence to document adequately over a longer time. Students have to reflect about documentations and grow into the communication practices of the mathematics community. In Germany, the most important framework of mathematical competencies is the one by the KMK (cf. KMK 2012). The KMK distinguishes three requirement-levels to describe the requirements that can be addressed in tasks in relation to six central mathematical competencies and five central mathematical guiding ideas. This framework is not made for the development of the competence to document (which I see as only a part of the competence of communicating mathematically) but for describing and testing the competencies. Thus, a model was created that focuses more on the development and tries to reflect the difficulties of handling the CAS, too. The competence model by Dreyfus & Dreyfus (1991) is insofar an important reference work that it describes how the development of competencies from a novice stage to an expert stage happens in five steps. According to Dreyfus & Dreyfus, novices act according to rules very explicitly while experts have internalized the rules so much that the behaviour has become part of them. The model for the development of the documentation competence distinguishes only three stages. For teachers it is thus easier to think of the stages as of the three consecutive years (10th to 12th grade) when CAS-classes can be allowed permanently in special classes. The model tries to reflect some difficulties teachers reported in CAS-classes (cf. Beck 2015, Weigand 2013) by shifting the focus of each of the three stages:

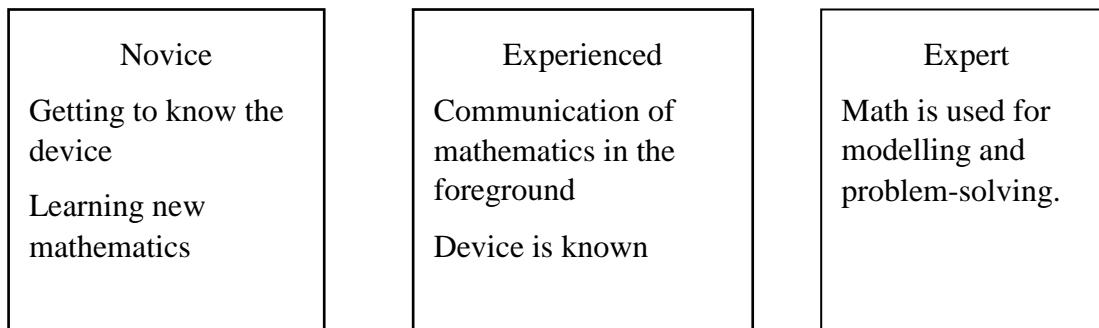


Figure 1 – Development model

1. Novice: Focus on technology-use: Learning to deal with already known and new mathematical content *with the new tool*, in order to get accustomed to it.
2. Experienced: Focus on communication: Reflecting about the use of the tool and the communication of mathematical content.
3. Expert: Focus on modelling and problem-solving: Applying the mathematics to modelling problems with the help of the CAS.

Although high-school CAS-classes only use one device (like the TI-nspire or the CASIO Class Pad II) it is not the aim to restrict the mathematical competence to just this one device. Therefore, also the competence to document should not be chained to the tool that is used but be applicable to all kinds

of tasks and problems. To achieve such a flexible competence teachers could use formative assessment as shown below.

It is the teacher's task to initiate a discussion about how the digital tool changes the nature of the solving process and, as a consequence, the written solution (strategy 2, see above). This is also a reflection of the benefits of a CAS. As already mentioned, it is not necessary anymore to write down every step in the solving process in order to perform it. A lot of algorithmic procedures can be outsourced to the CAS. This reflection might help students to progress further on their way from novices to experts.

Furthermore, as Black & Wiliam state "peer- and self-assessment are activities that might be used to pursue the fourth and fifth [key strategy] respectively" (p. 8). As a useful activity students (and teachers as well) could check and discuss whether their own documentation and the documentation of classmates meet the criteria. Students can discuss problems regarding the understandability of solutions amongst themselves and with their teacher. From a theoretical perspective, it seems to be most promising to apply these activities with experienced students (Fig. 1), after the students are used to work with the CAS but before complex (modelling-)problems are treated. Yet, this assumption has not been tested empirically. Regarding the documentation of solutions one aim is a high level of language use (most prominently: mathematical terminology). This is part of the competences communicating mathematically and mathematical reasoning (cf. KMK-Bildungsstandards, K6 and K1; Blum 2010).

Methodology

As a first step, a descriptive model has been developed from students' authentic solutions from high-stake final exams. These exams are the last time in the students' life and therefore these reflect (to a certain degree) the knowledge and the practise of the students. From a linguistic perspective, it is one aim to identify which elements students use in their documentations and which function is connected with each form. This is a typical pragmatic approach (cf. Meibauer 2008). The underlying question of this form-function-analysis can be formulated as follows: With which forms of representation do students document each step of the solving process? From a mathematical perspective, it is the aim to identify difficulties and problems in the students' solution. One problem is that traditional mathematical notation is mixed with computer language with the result that the created expressions do not fit the requirements of "the community of mathematicians".

Bavarian teachers of CAS-classes have been asked to send in nine written solutions each from the final exams. The students have three groups of tasks to solve (calculus, geometry, data & statistics) and have 180 minutes time. Three solutions came from students who have been average, three from students who have been above average, and three from students who have been below average in the preceding semester. Similar data has been collected every year (starting with 2014) for further evaluation and research. Four to five teachers answer this request every year.

The first research question is how students document their solving process in exams. So far, in Bavaria (Germany) only little official advice about documentations of solving processes is given. Normally, the Institute for School Quality and Educational Research (ISB) provides such material and official notes in addition to the curriculum. In order to develop such advice, it is a very valuable first step for researchers to analyse authentic documentations and to develop a descriptive model with which

problems and difficulties can be identified and categorized.

The representational dimension describes with which forms of representation students document. There might be expressions, which use some kind of formulaic symbols (traditional mathematical, computer-syntax, mixed-forms), verbalisations (both natural language and the special mathematical vernacular) and graphic representations. In the latter category, mixed forms (such as graphs, tables, sketches, etc.) are also counted.

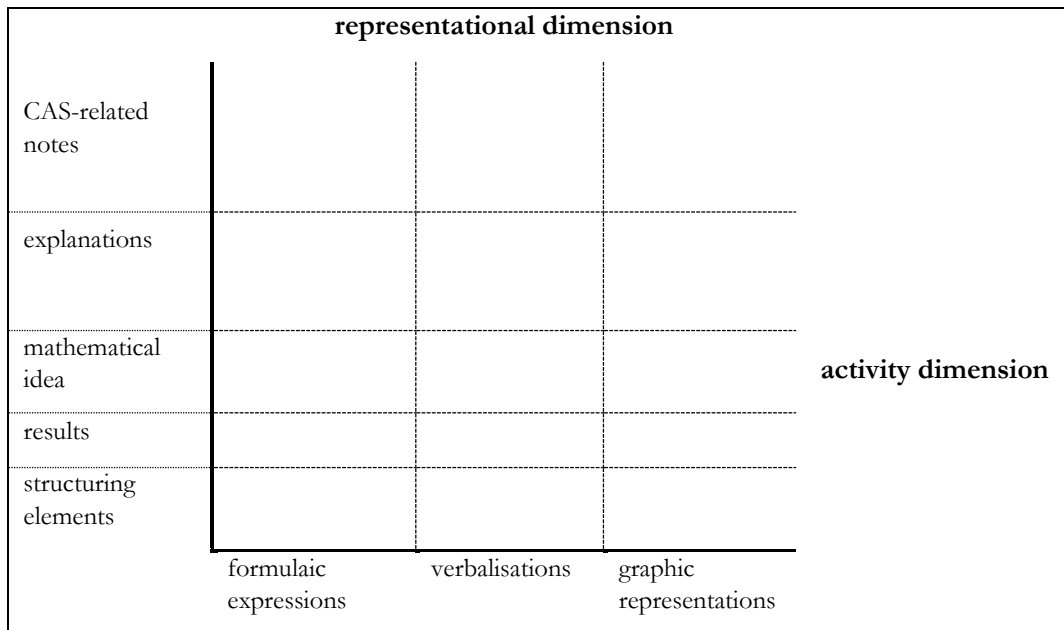


Figure 2 – Category system for description of students' solutions

The second, *activity dimension* describes which purpose an element has, that is what actually is documented with it and which step, or activity, in the solving process it is related to. Central categories are:

- *CAS-related notes* make the use of CAS explicit, either by stating the CAS command (input), by writing down the output (e.g. “false”, which is odd in a German text), or by unspecifically writing – in short form – that the CAS was used (e.g. “CAS: ...”).
- According to Wagner and Wörn (2011) *explanations* comprise three different facets: concepts and ideas (what-explanation), algorithms and procedures (how-explanation), argumentations and logical connections (why-explanation). They often focus on:
 - *mathematizations*, which show that information given in the task-description is translated into mathematical notation or terminology;
 - *interpretations*, which are translations of computer-output and the construction of meaning in relation to the task.

The terms *mathematizations* and *interpretations* are related to the respective activities in the extended modelling cycle by Siller & Greefrath (2010).

- Furthermore, there are elements which refer to the underlying *mathematical idea*, e.g. in order to find the maximum of a function f it is necessary to solve the equation $f'(x)=0$. From this element, the mathematical idea can be reconstructed.

- Every mathematical activity leads to some *result*. This might be the answer to a posed question or a step that takes one closer to the final result.
- *Structuring elements* are used to structure the text on the surface (the layout) and the way the information is presented. They can also be used to set up links between pieces of information such as single steps in the solving process and the chronological order in which they were performed.

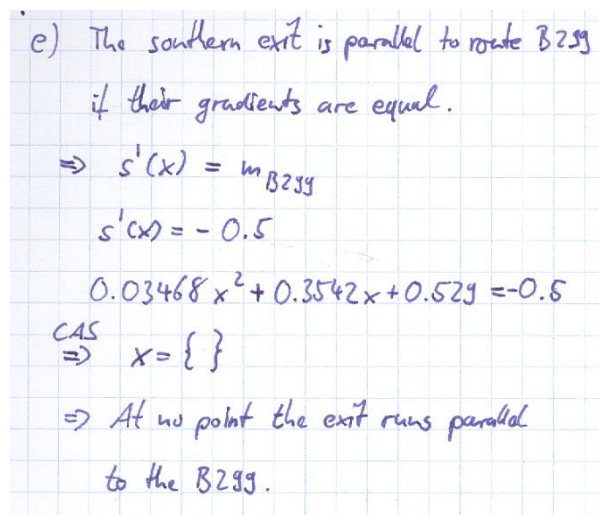
Results

The first result is that the category system above (Fig. 2) is suitable to describe students solutions. It can be observed that in regard to the documentation of CAS-commands the style was very homogenous throughout each class. In one of the classes, CAS-commands have been documented. In the second class, the CAS-use was indicated by writing “CAS” either over an equation or at the beginning of a line. In the third and fourth class there were no CAS-commands at all. This phenomenon can be explained by the normative standards that the respective teacher had set in the preceding year. Secondly, students who had a correct solution always showed the necessary mathematical ideas. Fig. 3 shows two different ways how mathematical ideas can be presented: either in verbalized form (line 1) or encapsulated in a formulaic expression (line 3).

A further result of the analysis of the students’ documents is that written solutions *without* verbalised explanations were often harder to understand and that the solving process could not be reconstructed that easily.

Authentic and exemplary solutions

As shown above, elements of written solutions can have different functions. Among them explanations can contribute a lot to make students’ documents easily understandable. According to Jörissen and Schmidt-Thieme explanations can be characterised as “primarily verbal statements” with the goal that the reader can understand connections (2015, p. 401, translation by the author).



e) The southern exit is parallel to route B299 if their gradients are equal.

$$\Rightarrow s'(x) = m_{B299}$$

$$s'(x) = -0.5$$

$$0.03468x^2 + 0.3542x + 0.529 = -0.5$$

CAS
 $\Rightarrow x = \{ \}$

\Rightarrow At no point the exit runs parallel to the B299.

Figure 3 – Student’s solution: original and translation

Furthermore, additional explanations extend the transmitted information with the possible consequence of redundancy. However, misunderstandings can possibly be prevented. As already mentioned Wagner & Wörn distinguish three different types of explanations: explain-what, explain-how and explain-why (2011). These sub-categories can be found – rudimentarily – in the students’

solutions, too. It is most important to notice here that students often explained verbally although it was not explicitly asked to do so in the formulation of the task.

The task of the example (Fig. 3) is to check whether there is a point at which the exit of a highway – modelled by a polynomic function s – runs parallel to another road – the route B299.

In the example (Fig. 3) we see that the student explains the mathematical idea of his solution verbally at the beginning. It is a rudimentary how-explanation. The verbal inaccuracy at this point is not that important because the information given in the text is supported by the mathematical formulaic expression, which is the equation. The output (“{}”) follows a CAS-use which is documented *unspecifically* (see above). The student confuses proper mathematical syntax with device-specific CAS-output and mixes both into an incorrect expression. As a concluding answer to the task a verbal interpretation of this output is written down.

The categories from Fig. 1 can be used to describe and explain students’ solutions. But they can also be used to help teachers to reflect written solutions and their own practise of writing mathematical texts. Furthmore, on the basis of the categories exemplary solutions can be created, as shown below (Fig. 4).

Category	Exemplary solution
Explanation	The roads run parallel to each other when there is a point at which \varnothing has the same gradient as $\varnothing(\varnothing) = -0.5\varnothing$.
Mathematical idea	$\varnothing'(\varnothing) = -0.5$
Result	This equation has no solution, therefore,
Expanation	the roads do not run parallel to each other.

Figure 4 – Exemplary solution for teachers

It cannot be expected of students’ solution to show such a degree of verbalisation. It is not the purpose of exemplary solutions to set minimal standards for students but to show teachers how solutions can be documented. The categories help to structure the text and to make the function of single elements more apparent.

Discussion of results and conclusion

Teachers may apply the categories in two ways: firstly, the categories can provide guidelines for documentations in a constructive way. For example, every documentation should make clear what the *mathematical idea* was that lead to the solution. Furthermore, students and teacher could agree in classroom discussions that verbalized *explanations* help to make clear the connections of different (symbolic) elements and, thus, allow the reader(s) to follow the solving process more easily. The notation of *CAS-commands* (if and how) could also be agreed upon in the class or amongst teacher even on a school level. Secondly, they then might be used for assessing students’ solutions for formative purposes as well as summative purposes. The categories might help teachers and researches alike to see documentations more clearly as a set of elements with different functions that make up the whole text. The exemplary solutions illustrate different possibilities of documentations. They can

be used for reflection in pre-service professional development courses as well as in in-service professional development courses.

In conclusion, it is important that students develop the competence to assess by themselves if a solution is acceptable and to create good solutions on their own (formative assessment key strategies 4 and 5). The main determining factors are the purpose of the documentation and the intended addressee, both of which may make an additional verbal explanation necessary. To provide support for this development is a major challenge for modern mathematics education. The combination of “learning to document” with some elements of formative assessment is a promising way to meet this challenge.

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