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Characteristics of pre-service primary teachers' noticing of students' thinking related to fraction concept sub-constructs

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The aim of this research is to characterise how pre-service primary teachers notice students' reasoning related to the fraction concept sub-constructs: part-whole, measure, quotient, ratio, operator and reasoning up and down. 82 pre-service teachers analysed primary school students' answers to five fraction problems. Each student's answer shows different characteristics of students' reasoning in each sub-construct of the fraction concept. Five profiles of pre-service primary teachers have been identified according to how they used the mathematical elements to recognise students' reasoning.

Keywords: Fraction, students' reasoning, noticing.

Introduction and theoretical background

The study reported here is part of a larger study focused on how pre-service primary school teachers notice characteristics of students' proportional reasoning (Buform, & Fernández, 2014). Several studies have indicated that the development of primary school students' fraction concept is important in order to develop relational thinking and proportional reasoning (Empson, & Levi, 2011; Lamon, 2007; Naik, & Subramaniam, 2008). However, the fraction concept is complex since it consists of multiple sub-constructs: part-whole, measure, quotient, ratio and operator (Behr, Harel, Post, & Lesh, 1992). In this paper, we are going to focus on how pre-service primary teachers notice students' reasoning related to the fraction concept sub-constructs. We also include the sub-construct *reasoning up and down* since it is an important component to develop proportional reasoning (Lamon, 2007; Pitta-Pantazi & Christou, 2011).

The skill of noticing students' mathematical reasoning

Recent research has shown that being able to identify relevant aspects of teaching and learning situations and interpret them to take instructional decisions (Mason, 2002) is an important teaching skill (professional noticing). Focusing on the skill of noticing students' mathematical thinking, Jacobs, Lamb and Philipp (2010) characterise this teaching competence as three interrelated skills: (1) attending to students' strategies that implies identifying important mathematical details in students' strategies; (2) interpreting students' mathematical reasoning taking into account the mathematical details previously identified; and (3) deciding how to respond on the basis of students' reasoning.

Studies, in this line of research, have indicated that identifying the relevant mathematical elements of the problem plays an important role to recognise characteristics of students' mathematical reasoning and also to take instructional decisions (Bartell, Webel, Bowen, & Dyson, 2013; Callejo, & Zapatera, 2016; Sánchez-Matamoros, Fernández, & Llinares, 2015). In the last years, researchers

have focused on different mathematical domains such as the derivative concept (Sánchez-Matamoros et al., 2015), classification of quadrilaterals (Llinares, Fernández, & Sánchez-Matamoros, 2016), algebra (Magiera, van den Kieboom, & Moyer, 2013) and ratio and proportion (Son, 2013) showing that the development of the noticing skill is not easy for pre-service teachers during teacher education programs.

Our study is embedded in this line of research and focuses on analysing how pre-service teachers interpret students' reasoning related to the fraction concept and how they use their interpretation of students' reasoning to propose new activities to help students progress in their reasoning.

Sub-constructs of the fraction concept

In our study, we consider the following sub-constructs of the fraction concept:

- Part-whole: it is defined as a situation in which a continuous quantity or a set of discrete objects is partitioned into parts of equal size (Lamon, 2005).
- Measure: it can be considered as a number which expresses the quantitative character of fractions, its size; or the measure assigned to some interval (Behr, Lesh, Post, & Silver, 1983; Pitta-Pantazi & Christou, 2011).
- Quotient: it can be seen as a result of a division situation (Pitta-Pantazi & Christou, 2011) and interprets a rational number as an indicated quotient (it is exemplified by sharing contexts).
- Operator: it is seen as a function applied to a number, an object or a set (Behr et al., 1992).
- Reasoning up and down: it is a particular case of the part-whole sub-construct where the unit in a task is implicitly defined (Lamon, 2005) and students need to reason up from a rational number to the unit and then back down from the unit to another rational number.

Participants and the task

The participants in this study were 82 pre-service primary teachers (PTs) during their third year in an initial teacher education program at the University of Alicante (Spain). In previous years, pre-service teachers had attended a subject focused on numerical sense (first year) and a subject focused on geometrical sense (second year). In the third year, they were attending a subject related to the teaching and learning of mathematics in primary school. One of the units of this subject was about teaching and learning of the fraction concept and proportional reasoning. The aim of this unit is focusing pre-service teachers' attention on how primary school students learn the fraction concept including features of students' understanding of the different sub-constructs. Data were collected after this unit.

Pre-service teachers solved a professional task focused on interpreting three primary school students' answers to five primary school problems related to the five sub-constructs of the fraction concept (part-whole, measure, quotient, operator, and reasoning up and down) (Table 1).

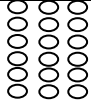

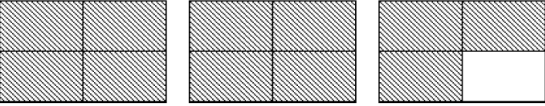
Problems	Characteristics
<p>1. How many spots are in $\frac{2}{3}$ of the set? Explain your answer.</p> 	<p><u>Part-whole</u>. Partitioning the set in 3 equal groups and selecting 2.</p>
<p>2. Indicate which number is X in the following number line. Explain your answer.</p> 	<p><u>Measure</u>. Identifying a unit fraction (for instance $\frac{1}{10}$) and iterating it to find X.</p>
<p>3. Four people are going to share three identical pepperoni pizzas. How much pizza will each person get?</p>	<p><u>Quotient</u>. Result of a division situation in which it is required the division of 3 pizzas between 4 people.</p>
<p>4. The teacher asked Nicolas to make some photocopies. Nicholas made a mistake and pressed the button that reduce the size of each copy by $\frac{3}{4}$. By how much should Nicholas increase each of the reduced copies to reproduce the original size?</p>	<p><u>Inverse operator</u>. Inverse function has to be applied: $\frac{3}{4} \cdot x = 1$.</p>
<p>5. The shaded portion of this picture represents $3 + \frac{2}{3}$. How much do the 4 small rectangles represent?</p> 	<p><u>Reasoning up and down</u>. Reasoning that implies identifying the unit “3 small rectangles” and then, representing a fraction.</p>

Table 1: Problems related to the five sub-constructs of the fraction concept considered in the task

Each student's answer shows different characteristics of students' reasoning in each sub-construct of the fraction concept. In Figure 1, the three primary school students' answers to the part-whole problem presented to pre-service teachers are given. To interpret students' answers, pre-service teachers answered the following four questions (Table 2).

Questions	Aim
a) What mathematical concepts must a primary school student know to solve this problem? Explain your answer.	Identifying the learning objective of the primary school problem
b) What are the characteristics of students' mathematical reasoning involved in each student's answer? Explain your answer.	Recognising characteristics of students' mathematical reasoning
c) How would you change the problem to help students progress in their mathematical reasoning if they have had difficulties solving the problem? Explain your answer.	Responding on the basis of students' mathematical reasoning, supporting (question c) or extending (question d).
d) How would you change the problem to help students progress in their mathematical reasoning if they have not had difficulties solving the problem? Explain your answer.	

Table 2: Questions of the task

Answer the following question using this picture:

How many spots are in $\frac{2}{3}$ of the set? Explain your answer.

Answer 1:

- Dividimos el todo en tres grupos de 6 canicas cada uno.
 - De esos tres grupos cogemos 2.
 - La suma de los puntos de estos dos grupos es 12. (6 puntos \times 2 = 12 puntos)

- We divide the whole in three groups of 6 spots each one.
 - We have three groups and we choose 2.
 - The addition of spots of the two groups is 12
 (6 spots \times 2 = 12 spots)

Answer 2:

$\frac{2}{3}$ of 18 spots are 12

$\frac{2}{3}$ de 18 = $\frac{18 \times 2}{3} = 12$
 $\frac{2}{3}$ de 18 puntos son 12

Answer 3:

dos grupos de 3 bolas

Two groups of 3 spots

Figure 1: Primary students' answers to the part-whole problem

Analysis

Data of this study are pre-service teachers' answers to the first two questions (a and b) of the professional task (Table 2). Therefore, we focus on how pre-service teachers interpret students' reasoning related to the fraction concept in this paper. The answers to each question were analysed individually by three researchers and agreements and disagreements were discussed. We observed how pre-service teachers identified the mathematical elements involved in each problem and how they used them to recognise characteristics of students' mathematical reasoning.

From this analysis, we have identified six different profiles of pre-service teachers considering how they used the mathematical elements of the problem to recognise students' reasoning (Table 3).

Results

Results show that 41 out of 82 pre-service teachers had difficulties in recognizing characteristics of students' reasoning (Profiles 0 and 1). However, 19 out of these 41 pre-service teachers identified the mathematical elements involved in each problem. This data suggests that recognising the important mathematical elements of the problem is not enough to recognise characteristics of students' reasoning.

How pre-service teachers identified and used the mathematical elements of the problem to recognise students' reasoning	Number of PT's
Profile 0: They do not identify the mathematical elements and do not recognise characteristics of students' reasoning in any task	22
Profile 1: They identify the mathematical elements related to all sub-constructs of fraction concept but do not recognise characteristics of students' reasoning in any task	19
Profile 2: They identify the mathematical elements and recognise characteristics of students' reasoning related to part-whole, measure, quotient, and operator	8
Profile 3a: They identify the mathematical elements related to all sub-constructs of fraction concept and recognise characteristics of students' reasoning related to part-whole, measure, quotient, operator and reasoning up and down (but not related to the inverse operator)	25
Profile 3b: They identify the mathematical elements related to all sub-constructs of fraction concept and recognise characteristics of students' reasoning related to part-whole, measure, quotient, operator and inverse operator (but not related to reasoning up and down)	5
Profile 4: They identify the mathematical elements related to all sub-constructs of fraction concept and recognise characteristics of students' reasoning related to all sub-constructs of the fraction concept	3

Table 3: Profiles of pre-service teachers identified

Pre-service teachers of Profile 0 did not identify the mathematical elements and used general expressions such as “*fractions and operations with fractions*”. Pre-service teachers of Profile 1 were more specific, identifying the mathematical elements implied in all the problems. For example, pre-service teachers of Profile 1 indicated: “*In problem 1, the mathematical element involved is part-whole. In problem 2, the idea of measure or number line. In problem 3, quotient. In problem 4, the idea of operator. In problem 5, part-whole and unit*”. However, pre-service teachers in these both profiles did not recognise characteristics of students' reasoning. These pre-service teachers provided general comments based on the correctness of the answer: “*answer 1 is correct; answer 2 is correct; answer 3 is not correct, the student doesn't understand the concept*”; gave a description of the student answer “*the student 1 divides in 3 groups and choices 2 groups, student 2 makes a multiplication and then a division, and student 3 doesn't understand the problem*”; or interpreted incorrectly students' answers “*the three students solved the problem correctly but using different strategies*”.

Pre-service teachers of profiles 2, 3a, and 3b identified the mathematical elements involved in each problem and recognised evidence of students' reasoning in some sub-constructs. Particularly, pre-service teachers of Profile 2 recognised characteristics of students' reasoning related to the sub-constructs part-whole, measure, quotient and operator. For instance, the next excerpt is a pre-service teacher's answer to the part-whole problem (problem 1): “*Answer 1: the student shows the understanding of the part-whole concept because identifies the whole and re-group the spots in equal groups (dividing the whole in equal parts). Answer 2: the student identifies the total of spots (whole) and selects $\frac{2}{3}$. He interprets the fraction as an operator. Answer 3: He doesn't identify the whole and doesn't re-group in equal groups*”; to the measure problem (problem 2): “*Answer 1: he*

solves the problem correctly because he identifies the unit fraction ($1/5$) in the number line. Answer 2: he solves the problem iterating $2/5$ and then uses the idea of operator to obtain $1/2$ of the interval. Answer 3: he doesn't identify the unit fraction and doesn't take into account what means $2/5$ in the number line"; and to the quotient problem (problem 3): "In answers 1 and 2, the student understands the fraction as a quotient because he divides the pizzas in equal parts. Answer 3: he doesn't understand the meaning of quotient because he divides the pizzas in different parts".

Pre-service teachers of Profile 3a identified the mathematical elements and recognised characteristics of students' reasoning related to the sub-constructs part-whole, measure, quotient, operator and reasoning up and down (but not related to the inverse operator). The difference with pre-service teachers of Profile 2 is that pre-service teachers of Profile 3a recognised characteristics of students' reasoning related to the reasoning up and down sub-construct: "*In answer 1, the student doesn't identify the unit and the unit fraction. In answer 2, the student identifies the unit but doesn't identify the fraction that represents 4 small rectangles. In answer 3, the student identifies the unit and identifies correctly which fraction represents 4 small rectangles*"; and pre-service teachers of Profile 3b recognised characteristics of students' reasoning related to the inverse operator instead of the reasoning up and down sub-construct "*A1: he uses an additive wrong strategy. A2: he doesn't know how to make the reduction and the enlargement. A3: he knows how to obtain the original paper multiplying by the inverse fraction of $3/4$* ".

Finally, only 3 pre-service teachers (Profile 4) identified the mathematical elements and recognised characteristics of students' reasoning in all the sub-constructs of the fraction concept.

The different sub-constructs of the fraction concept were used by pre-service teachers to recognise characteristics of students' reasoning in different ways. The way in which pre-service teachers used the sub-constructs operator (and its inverse) and the reasoning up-and-down promoted the emergence of different pre-service teachers' profiles.

Conclusions

The five pre-service teachers' profiles show characteristics of the way in which pre-service teachers notice students' fractional reasoning. The difference between profile 0 and profile 1 is that pre-service teachers start to identify the mathematical elements of the problems but continue giving general comments based on the correctness of answers. The difference between profile 1 and 2 is that pre-service teachers of profile 2 are able to recognise characteristics of students' reasoning related to part-whole, measure, quotient, and operator sub-constructs. However, these pre-service teachers were not able to recognise characteristic of students' reasoning in problems where the unit was implicit (inverse operator and reasoning up and down). The difference between profile 2 and profile 4 is the fact that pre-service teachers of profile 4 recognise characteristics of students' reasoning in all the sub-constructs. However, there are two possible profiles between the profile 2 and profile 4 characterised by: recognising characteristics of students' reasoning related to the inverse operator (but not related to the reasoning up and down, Profile 3a), and recognising characteristics related to the reasoning up and down sub-construct (but not related to the inverse operator, Profile 3b).

These results provide information about different pre-service teachers' stages in the development of the skill of interpreting students' mathematical reasoning related to some sub-constructs of the

fraction concept. This information provides data to conjecture a pre-service primary teacher's hypothetical learning trajectory of noticing students' mathematical reasoning related to those sub-constructs (Figure 2). This hypothetical learning trajectory could inform us about the pre-service teachers' learning process of the skill of interpreting students' mathematical reasoning in the particular mathematical domain of the fraction concept.

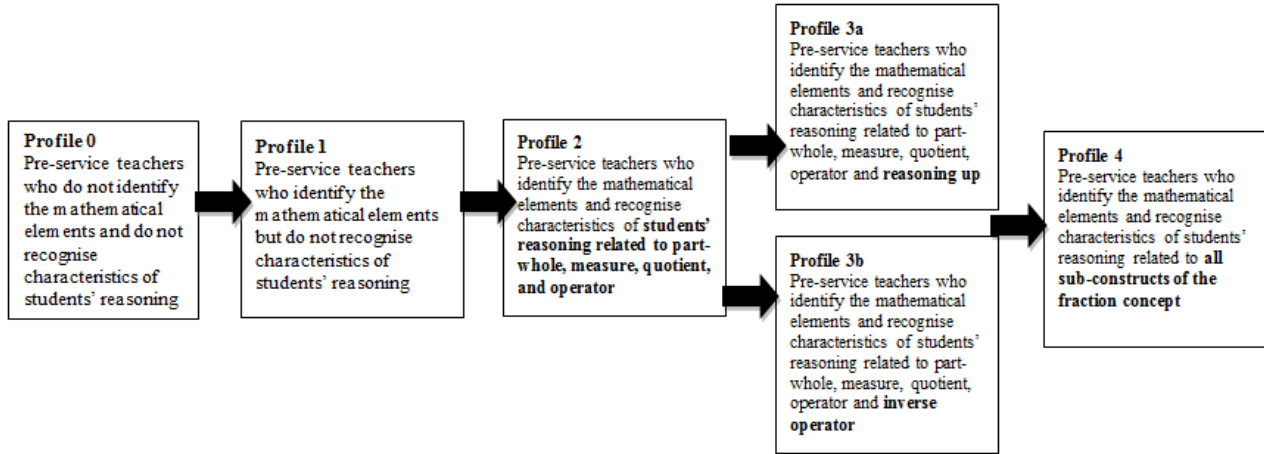


Figure 2: A pre-service primary teacher's hypothetical learning trajectory of noticing students' mathematical reasoning related to the fraction concept

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Analysing mathematics teacher learning in Lesson Study - a proposed theoretical framework

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The purpose of this paper is to analyse mathematics teacher knowledge incorporated during one cycle of lesson study. Analysis is undertaken utilising an extended framework which combines both the theoretical frameworks of Mathematical Knowledge for Teaching (Ball et al., 2008) and Levels of Teacher Activity (Margolinas et al., 2005). The proposed framework is situated as a tool to detail and analyse the use of mathematics teacher knowledge in planning, conducting, and reflecting on research lessons in a lesson study cycle in a primary-school case study in Switzerland.

Keywords: Teacher collaboration, Lesson Study, professional development, mathematical knowledge for teaching, levels of teacher activity.

Lesson Study and models of knowledge for teaching

Lesson study is a collaborative model of professional development which supports teacher learning (Huang & Shimizu (Eds.), 2016). Originating in Japan, this model has grown in international popularity over the past two decades, particularly in the field of mathematics education, and much research has detailed evidence of mathematics teacher learning through lesson study (e.g. Lewis et al., 2009; Murata et al., 2012; Ni Shuilleabhain, 2016).

Lesson study provides teachers with opportunity to contextualize representations of their classroom activities, while also making their implicit knowledge and practices explicit through their conversations within the group (Fujii, 2016). Each lesson study cycle consists of a number of steps where teachers begin by studying the curriculum and deciding on a research theme, planning a research lesson according to that theme, conducting and observing the live research lesson, and reflecting on student learning within the lesson (see Fig. 1) (Lewis 2016; Lewis et al., 2009)

With increased international educational research on lesson study, there have been calls to deepen the knowledge base of the development of teacher knowledge within this model in order to provide a solid theoretical foundation for its use in teacher education (Clivaz, 2015; Miyakawa & Winsløw, 2009). In this paper, we hope to contribute to the literature on professional development for mathematics teachers by analysing the mathematical knowledge utilized by teachers in their participation in lesson study, utilizing our proposed theoretical framework.

The two authors of this paper, in their analysis of teacher knowledge and learning in lesson study, seek to deliberately build on previous existing frameworks of teacher knowledge: Mathematical Knowledge for Teaching (Ball et al., 2008) and the Levels of Teacher Activity (Margolinas et al., 2005). Through analysis utilizing a combination of these frameworks (Prediger et al., 2008), we will detail features of the mathematical knowledge for teaching utilized by teachers in their participation in lesson study and will also track the movement of this knowledge.

MKT/ levels in LS: Towards a coordinated model

In this paper, we propose a framework which was developed based on data generated in two case study sites - with eight participating primary (grade 3-4) teachers in Switzerland and five lower secondary (middle school, grade 7) teachers in the Republic of Ireland. Analysis began by utilizing the Mathematical Knowledge for Teaching framework (Ball et al. 2008) to investigate the contributions by teachers in a lesson study cycle. However, we found that this model did not fully incorporate all the elements of teacher knowledge included in the lesson study cycle, particularly in capturing the educational values and conceptions of teaching and delineating between the layers of planning sequenced content of instruction, while also attending to students' thinking during the lesson. At this point in the analysis, Margolinas et al.'s (2005) Levels of Teacher Activity was identified as a framework which could encapsulate these elements of teachers' knowledge. Building on qualitative data generated through audio/video recordings of teacher conversations during lesson study meetings, teacher notes from lesson study meetings, researcher field notes, and selected samples of student work from research lessons, we present an extended model of the categorization of knowledge required for the teaching of mathematics. Our first example of analysis presented is from the Swiss case study and future work will present further analysis from the Irish case study data. In analysing and comparing these sets of local data, we attempt to demonstrate a more global sense of this proposed framework of mathematics teacher knowledge in lesson study.

Mathematical Knowledge for Teaching

In their ground-breaking work in 2008, Ball, Thames and Phelps addressed the concepts of content and pedagogical content knowledge in their model of Mathematical Knowledge for Teaching (MKT - see upper part of Figure 1). In this paper, they identified domains of Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK) used in teaching, which further defined the knowledge and skills required of mathematics teachers in relation to student learning and to mathematics content.

Research on these different categories of MKT has demonstrated direct links between teacher knowledge and high-level teaching practices (Clivaz, 2014; Hill, Ball, & Schilling, 2008) and with subsequent student learning outcomes (Hill, 2010).

Incorporating this model of MKT with teacher learning in lesson study, research has shown that Knowledge of Content and Students and Knowledge of Content and Teaching (features of PCK as defined by Ball et al. (2008)) are important elements of teacher knowledge utilized in lesson study cycles (Leavy, 2015; Ni Shuilleabhain, 2015b; Tepylo & Moss, 2011). However, considering the multitude of teacher knowledge and practices incorporated within each lesson study cycle in planning, conducting, and reflecting on a mathematics lesson, this model may not capture all the decisions, actions, practices, and skills required of mathematics teachers participating in lesson study.

Levels of teacher activity and MKT

To describe teacher activity, both in and outside of the classroom, Margolinas developed a model of the mathematics teacher's milieu based on Brousseau (1997). This model was designed to take into account the complexity of teachers' actions and to capture the broad range of activities contained in teaching and learning (Margolinas et al., 2005, p. 207).

+3	<i>Values and conceptions about learning and teaching</i>
+2	<i>The global didactic project</i>
+1	<i>The local didactic project</i>
0	<i>Didactic action</i>
-1	<i>Observation of pupils' activity</i>

Table 1: Levels of a teacher's activity (Margolinas et al., 2005, p. 207)

At every level of environment (or milieu) the teacher must consider all that is occurring at the current level as well as those levels that are directly above and below. These multidimensional tensions relate to a non-linear and non-hierarchical interpretation of teacher's work (Margolinas et al., 2005, p. 208). In addition to commonplace professional opportunities where teachers speak about their beliefs and experiences on general educational concepts or about teaching and learning mathematics (level +3), about teaching and learning of a particular mathematical subject (level +2), or about the lesson they are preparing (level +1), during the phases of planning and reflection in LS teachers also have opportunity to discuss their classroom activities (level 0) or observations of student activity from a lesson (level -1).

This activity model was used by Clivaz (2014) and aligned with the MKT model in order to capture the movement of didactical situations, beyond the possible static characterisation which may be interpreted in the MKT model (Ball et al., 2008, p. 403). The combination of these frameworks allowed teacher knowledge to not only be analysed in terms of mathematical knowledge for teaching, but also mathematical knowledge in teaching (Rowland & Ruthven, 2011). Similarly, Ni Shuilleabhain (2015a) used the MKT model to analyse teacher learning in lesson study, but combined this with the idea of the 'student lens' (as suggested by Fernandez, Cannon, & Chokshi, 2003, p. 180), in proposing an additional layer of the model put forward by Ball et al. (2008). This concept of a 'student lens' incorporated the PCK a teacher utilises in seeing mathematics "through the eyes of their students" (Fernandez et al., 2003, p. 179).

When aligned with Margolinas et al.'s (2005) model, this layer of teacher knowledge relates partly to the -1 'Observation of pupils' activity' which can be anticipated and interpreted, but extends this observation to thinking of the mathematical content from the students' perspective. In our proposed framework, we therefore see this view of the mathematics through the eyes of the student as a layer below the observation of a students' work and include a new level of -2 level relevant to teacher knowledge titled the "student lens" (see Figure 1).

Proposed theoretical framework

Explicitly combining these two approaches to analyse the knowledge utilized by mathematics teachers during lesson study, the authors here present a new theoretical framework (see Figure 1). This framework attempts to capture the knowledge required of mathematics teachers, in the broad and complex range of teaching and learning activities, and represents teacher knowledge and activities incorporated during each phase of a lesson study cycle (see Lewis & Hurd, 2006, p. 4).

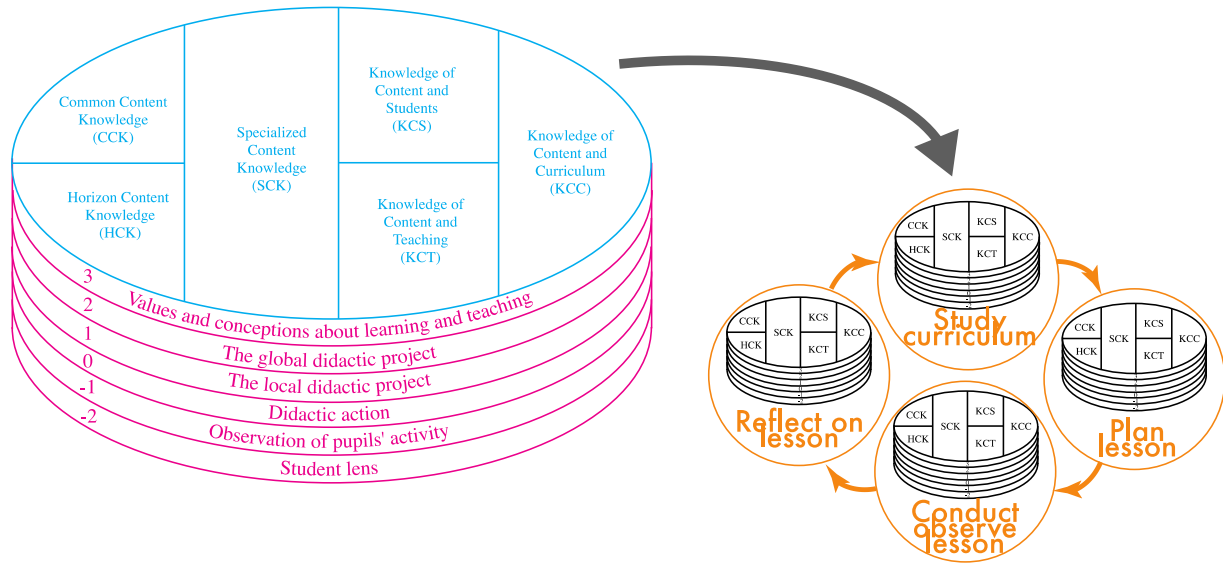


Figure 1: MKT and levels of teacher activity at lesson study phases

We first utilize the model to categorize the knowledge (MKT and levels of activity) appearing during the lesson study cycle. The knowledge about a particular mathematical topic will then be tracked over each phase of lesson study and the relations between the occurrence of this knowledge examined. At this stage, ‘knowledge’ is considered as collective (e.g. Ni Shuilleabhain, 2016).

Analysis

In this paper, data generated through video recordings of the Swiss case study are analysed utilising the proposed framework. Eight primary generalist teachers, new to lesson study, and two facilitators (one specialist in teaching and learning and the other a specialist in mathematics didactic (first author of this paper)) participated in the research which occurred over two academic years. Four cycles of lesson study were undertaken in this time, with a meeting held on average every two weeks during the school year (Clivaz, 2016). Each of these 37 meetings (about 90’ each) were videotaped and transcribed and form the base of the analysis utilizing the framework outlined above (Figure 1) and incorporating defined features of KCS and KCT as utilized in lesson study (Ni Shuilleabhain, 2015b).

We present analysis of the first lesson study cycle where teachers chose to focus on the topic of integers and place value. The main reason for choosing this subject was the difficulty students had with whole numbers. In the first session, teachers discussed a particular difficulty their students had with counting through to new groups in base 10:

Océane: The counting through to the next ten.

Caroline: But each time they have to count through to (tens, hundreds, ...)

Stéphane (facilitator): What’s happening with counting through to the next ten?

Caroline: It’s... that we have no more to write here! We have to use the digits which already exist. So, we count through to come back to one... In fact... Yes, it is the abacus, in fact, we need to move by one each time we arrive at a nine at the end. We need to move by one.

Océane: We exchange one packet of ten.

In this passage, during the *study curriculum* phase, teachers are at level of the *global didactic project* (+3) and this unpacking of mathematical knowledge is a Specialised Content Knowledge (SCK) i.e. the mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students (Ball et al., 2008, p. 399). At this stage, the place aspect of number system was predominant in teachers' discourse and, when the value aspect appeared, it was linked with the value. To further address this knowledge, the facilitators suggested working on students' actual mistakes. Teachers and facilitators proposed mistakes like:

$$5 \text{ hundreds} + 12 \text{ tens} + 3 \text{ units} = 515$$

This work prompted teachers to do the task as if they themselves were students. At some moments during the activity teachers even spoke like students - placing them at the level of *student lens* (-2). This allowed the teachers to go deeper into potential difficulties for students and by further studying curriculum materials (referred to as *kyozai kenkyu* by Takahashi & McDougal, 2016), teachers had opportunity to clarify this aspect for the research lesson. This passage is situated at the same phase, level and type of MKT as the previous excerpt above.

Anne (facilitator): [...] It's a particular type of exchange since it's in the place value system. So, we can distinguish the two dimensions: the dimension of the place and the dimension of the decimal value which is revealed in the exchanges.

Stéphane: In fact, I prefer to talk about grouping/ungrouping instead of exchanging.

Océane: Oh, I see!

Following these two excerpts, we will briefly summarize the work undertaken by these teachers planning the second research lesson and focus on this phase for analysis. The group chose a task in the form of a board game involving the exchange of "1 hundred", "1 ten" and "1 unit" cards. Following a planning exploration of the task, this research lesson was taught by one of the group and, during the post lesson discussion, teachers agreed that the task should be modified to allow students practice the exchange of values and relate these to aspects of the number system. This revised lesson was taught by another member of the group to a different group of students.

At the beginning of the game a student, Julie, arrived on the square "give 35". She had three cards of "1 unit", three cards of "1 ten" and four cards of "1 hundred". In order to get three cards of "1 ten" and two cards of "1 unit", Julie wanted to exchange two "1 hundred" cards. The teacher, Edith, wanted to explain to Julie that two "1 hundred" cards were worth more than these three cards of "1 ten" and two cards of "1 unit".

Edith: So, two hundreds - that's how many?

Julie: Two hundred.

Edith: That's two hundreds. If you tell me: "I want three tens and two units." Three tens, how many is that?

Julie: Thirty.

Edith: You told me three tens makes thirty. And what about two units?

Julie: Two.

Edith: If you put the thirty and the two together? How many is that?

Julie: Thirty-two.

Edith: So you swap two-hundred for thirty-two! You're very generous!

In this passage situated during the *conduct lesson* phase, at level 0 (*didactic action*), the teacher converted all cards into numbers to compare them, instead of doing direct exchanges. Julie followed the teacher without expressing her own way of reasoning (which can be observed in another passage and demonstrates a 'direct exchange' way of thinking). In this case, we categorize the MKT in two ways. First as a KCS, where Edith did not notice or interpret Julie's mathematical thinking or strategies. Second as a SCK, related to the unpacking of mathematical knowledge, as detailed in the following excerpt.

During her dialogue with the class, Edith had to explain that one hundred is the same as ten tens. Here, again, her argument is to convert to units - which requires students to already understand place value. This argument can be summarized as follows:

$$\begin{aligned} 1 \text{ hundred} &= 100 \text{ units} \\ \text{and } 10 \text{ tens} &= 100 \text{ units} \\ \text{therefore, } 1 \text{ hundred} &= 10 \text{ tens} \end{aligned}$$

In the final lesson plan, the group of teachers reflected on this strategy and argued against it:

"Often exchanges are not really carried out and we go through the number. For example, when asked to exchange 12 hundreds into tens, many students (and adults) will go through the number 1200, namely 1200 units, to say that that 1200 is 120 tens, without being able to make a direct exchange from hundred to tens. Teachers also often explain this exchange in this way. In this case, we are in a type of vicious circle, since it means that it is necessary to have understood number system to understand the grouping/ungrouping in the place value system."

This episode appears in our data in the research lesson (*conduct and observe lesson* phase, level 0), in the notes of the observing teachers (*conduct and observe lesson* phase, level 0), in the post lesson discussion (*reflect on lesson* phase, level 0) and, in the above extract, in the lesson plan (*reflect on lesson* phase, level +2) where observations and analysis of the group were generalized and decontextualized from the particular lesson to the level of a global didactic project. In each case the knowledge represents a typical SCK.

The final example of this knowledge was found at the end of the *reflect on the lesson* phase. After discussing the lesson and the mathematical difficulty of directly converting hundreds into tens, Valentine (a teacher with over 30 years of teaching experience) realized she had observed a similar difficulty her own students in this topic, outside of the lesson study group. As a result of their collaborative reflection conversations, she began to realize that her students' errors were likely due to her use of only one strategy in teaching this topic:

Valentine: But, I've got a question. For example, in nine-hundred-sixty-three - how many tens are there? Ninety-six. But my students, they learned a trick - they write the number 963 and just go to the tens digit and write what is left: 96. I'm convinced they just use this trick. I probably didn't know how to explain that to them! Myself... I always convert in money! You will have nine hundred and sixty three one-franc coins. If you need to only have ten-francs notes... then you will have ninety-six ten-francs notes.

Although this observation was not directly related to observations during the research lesson, we still categorize it as level -1 since Valentine put herself in the position of observing her students converting 963 into tens. This conversation incorporates teacher KCS in interpreting students' responses and is situated at level -1 (*observation of pupils' activity*).

Utilising our proposed framework and building on our analysis of teachers' collective conversations, we can detail the types and levels of knowledge incorporated by mathematic teachers in their participation of lesson study. Utilising this framework provides us with opportunity to track the knowledge included in the planning and reflection of mathematics research lesson over various phases of lesson study.

Conclusion

This paper proposes an extended theoretical framework of mathematics teacher learning in lesson study combining the existing frameworks of Mathematical Knowledge for Teaching (Ball et al., 2008) with Levels of Teacher Activity (Margolinas et al., 2005). In this paper the proposed framework is situated as a tool used to detail and analyse the use and movement of mathematics teacher knowledge in planning, conducting, and reflecting on research lessons. Based on case study data generated through mathematics teachers' participation in lesson study, we have analysed teachers' qualitative conversations and considered the potential evolution of mathematics teacher knowledge over a cycle of lesson study. Analysis to date has demonstrated that in planning and reflecting on research lessons, teacher knowledge of various forms (e.g. SCK and KCS (Ball et al., 2008)) and across varying levels of activity (Margolinas et al., 2005) are incorporated in these separate phases of lesson study.

We hope this model will contribute to the literature on professional development of mathematics teachers and may serve to underpin further evidence of teacher learning in lesson study.

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Innovating teachers' practices: Potentiate the teaching of mathematics through experimental activities

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This paper aims to potentiate the teaching of mathematics through hands-on experimental activities at the primary school level and by promoting teachers' professional development, using innovative practices. A Teacher Design Research cycle involving a group of primary school teachers during one school year was performed. This cycle consists in teacher training sessions, including an introduction to science and mathematics content, hands-on workshops for teachers and also classroom interventions in order to promote experimental activities and observe teachers in action. A particular case study of teacher Luísa will be presented. It was found that she gained motivation and self-confidence to innovate her practices, showing enhanced ability to perform experimental activities with her students.

Keywords: Teacher professional development, training practices, hands-on, teacher knowledge, primary school.

Introduction

This paper aims to potentiate the teaching of mathematics, through hands-on student-centered experimental activities at the primary school level and by promoting teachers' professional development using innovative practices.

Several studies show the importance of teaching mathematics and science through experimental activities in the early years of schooling to motivate the younger generations to scientific and technological areas, considered crucial for economic development and scientific literacy (Coll, Dahsah, & Faikhamta, 2010; Hallstrom, Hulten & Lovheim, 2014; Osborne, 2009; Perera, 2014).

Treacy and O'Donoghue (2014) also refer to scarce research about the integration of mathematics and science in classroom contexts as well as the lack of a widely-adopted teaching model. These authors advocate that "hands-on, practical, student-centered tasks should form a central element when designing an effective model for the integration of mathematics and science" (p. 1).

This study is part of a first round of a Teacher Design Research programme involving a group of primary school teachers in a cluster of schools of a region of central Portugal, held during the 2015/2016 school year. This experiment was carried out in the framework of a pedagogic intervention project that aims at introducing new methodologies to promote the learning of mathematics and science through hands-on experimentation and using the "inquiry" method.

To achieve this purpose, a lifelong training course was designed, with the collaboration of university researchers and a formation center, to include mathematics and science hands-on workshops, in order to help teachers develop their teaching skills and update their knowledge on these topics.

In this study, a particular case of a teacher, who used one of the proposed science experimental activities, to explore mathematical concepts and student-centered tasks, using the inquiry-based and problem-solving approach, will be presented.

Literature review

The great lack of professionals in the STEM (Science, Technology, Engineering and Mathematics) areas must be countered with an intervention at the level of the early years of schooling, being crucial to provide quality scientific practices (DeJarnette, 2012; Eshach & Fried, 2005; Johnston, 2005). The incorporation of hands-on experimental activities into the classroom, with scientifically well-prepared adults, lead to significant improvements in performance and produce positive attitudes towards science and learning (Mody, 2015; Myers Spencer & Huss, 2013).

The inquiry approach calls on the natural curiosity of children and develop their creativity and critical questioning at an age when they have the urge to discover the world around them (Alake-Tuenter et al., 2012; Krogh & Morehouse, 2014; Rocard et al., 2007).

Teachers are the cornerstone of any renewal of science education and being part of a network motivates them, contributes to improve the quality of teaching and promotes the sustainability of their professional development (Abell & Lederman, 2007; Rocard et al., 2007; Zehetmeier & Krainer, 2011). Martins (2006) claims to be a priority to strengthen investment in scientific research in the field of science education in the early years of schooling and continuing teacher training. Murphy, Varley and Veale (2012) recommend a professional development for teachers that will allow them to enhance their conceptual and pedagogic knowledge on the inquiry-based approach. Ball (2003) says that an intervention to combat failure in math's performance will only be effective if it is focused on teaching methods: "No curriculum teaches itself and standards do not operate independently of professionals' use of them" (Ball, 2003, p. 1).

Ball, Thames and Phelps (2008) investigated competences that are required to teach and developed an empirical approach to determine the content knowledge needed for teaching (figure 1).

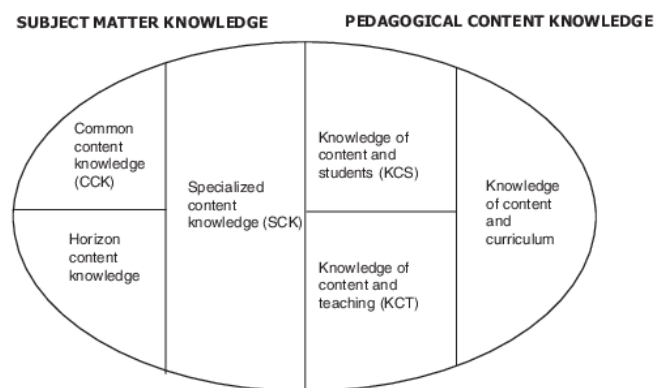


Figure 1: Mathematical knowledge needed for effective instruction (Ball, Thames, & Phelps, 2008)

Ball (2003) concludes that to improve children's mathematical learning, it is crucial to provide learning opportunities to teachers such as tailored courses, workshops and well-designed and taught materials. Afonso, Neves and Morais (2005) recommend that teachers should be given the opportunity to explore and experiment the contents to be developed in class in a reflective, collaborative environment where they feel supported. Also, Kuzle and Biehler (2015) sustain the

importance of “stimulate cooperation among the participants, and between the participants and the professional developer” (p. 2849).

Methodology

Teacher Design Research

Teacher Design Research (TDR) (Kelly, Lesh, & Baek, 2014) aims to promote the development of teachers as adaptive experts, using inquiry. This approach involves a collaborative work with research teams and teachers participating in the process, with the main objective to promote their professional development, leading them to innovate their practices and improve the whole teaching and learning process. The TDR premise is that the involvement of teachers in long-term (e.g. one year) design research periods can promote in-depth content learning and improve their ability to adapt to classroom environment and rethink their teaching beliefs and practices.

The pilot experiment: First cycle of the TDR

A pilot experiment with primary school teachers has been conducted in the academic year 2015/2016. This experiment included continuous training sessions where proposals of new content and experimental mathematics and science hands-on activities, to be used in the classroom, have been put forward. The teachers had the opportunity to explore the content and manipulate the materials to be able to apply them later with their students. In addition, the instructors also visited the trainees' own classroom to carry out experimental activities to exemplify them, and observe the teachers in action with their students. The teachers have also been encouraged to develop their autonomy by creating and implementing their own experimental activities.

Focus Group (FG) (Williams & Katz, 2001) was one of the working methodologies used throughout the training sessions to support the teachers and improve their practices. The last session was mainly focused on FG to promote reflection on the practices developed and make proposals for the following cycles of TDR. “Innovations need to be owned by the person implementing them on a personal level and transformed into their own practice in order to have practical effect” (Zehetmeier, Andreitz, Erlacher, & Rauch, 2015, p. 168).

This paper describes the case study of teacher Luísa (fictitious name) who participated in the study, proposed and carried out math-based activities using the inquiry method.

Participants

The participants in the pilot project comprised 14 teachers of 5 primary schools. These teachers participated in a first round of design research beginning in September 2015 and ending in July 2016. In this paper, we will study teacher Luísa who is 56 years old, has 37 years of service and is in charge of a third-year grade class with 25 students aged 8 and 9 years.

Data collection

Data collection consisted in observations (first author of the paper was a participant observer), semi-structured interviews, written records, and video (Cohen, Lawrence, & Keith, 2007). The action took place in two main moments: workshops with the teachers (to learn and practice what they are expected to implement) and at their classrooms (to support and observe them in action). At the end

of the training action, the participant teachers presented a written report, with a critical account on the pilot experiment and their proposals of innovative practices.

Data analysis and discussion

Teacher Luísa participated in the first TDR round. For about nine months, she attended the continuing training programme that consisted of seven workshops with, 3 to 4-hours duration, which introduced content, hands-on activities and methods of implementation in the classroom.

The first experimental activities carried out on the workshops was about electricity (<http://www.academiacap.ipt.pt/pt/atividades/ciencia/fisica/77/>). Before the intervention, Luísa completed a questionnaire that characterized her and her class. On the questionnaire, she refers that "the experimental aspect was not addressed in my graduation course" and "in the complementary training program (BA-equivalent degree) I attended it was dealt with only too briefly". Although she has attended several training sessions throughout her career, none of them was about electricity.

In the course of the training programme, Luísa was very participatory showing a great interest in the tasks performed. However, on several occasions she stated that: "I'm not comfortable to teach some of the content because I don't have full mastery of concepts and techniques and don't know how to apply them". She also admitted that: "I'm not able to handle some of the materials used in the experimental activities". As early as in the first sessions she posed a series of questions such as: "What if the students ask me a question about this theme and I don't know the answer?" or "And what if an experiment does not yield the expected results?"

In addition to concerns about specialized content knowledge, the teacher also reveals concerns about pedagogical content knowledge. These insecurities have led us to rethink the approach to content and experimental activities, because we realized that it was very important to adapt training to the knowledge and the needs of the teacher, to make her feel secure and motivated to implement the tasks. We also realized that she gives great importance to specialized content knowledge and that she hardly will perform experiments that involve concepts she does not fully understand.

Given the great commitment of teacher Luísa to learn and her pedagogical concerns, she has been selected to receive the trainers in her own classroom to carry out experimental activities with her students, to exemplify the experiments, support the teacher and observe her in action.

During the intervention in the classroom, we have observed that teacher Luísa had a posture of inquiry, making questions to her students to guide them through the tasks, leading them to investigate, in order to find answers to the questions. It was interesting to observe Luísa making a reflection with her students, questioning them about the classroom hands-on activities, what they had learned, and what they would like to explore in the next experiments. Observations and interviews revealed she has a good knowledge of her students and knows how to introduce and adequate the content to each of them, according to their individual needs (KCS and KCT).

Two more sessions with teacher Luísa's students have been held which included tasks not covered by the training course with the other teachers. She again felt insecure and reported that she wouldn't be able to implement it without the support of the instructors. This shows the importance of the training workshops with the teachers before going to their classroom.

It has been suggested to her that she should propose activities involving mathematical content. Due to the difficulty shown by the teacher to achieve this objective, the researchers proposed a worksheet in which, based on experimental records like weigh, fruit diameter, potential difference measurements, it was possible to address the topic “organization and processing of data” that is part of the primary school syllabus. With this proposal, the teacher created some tasks (figures 2, 3 and 4). Figure 2 shows the method used by the teacher to propose mathematical problems, based on the classroom experiment performed by the team of instructors. In writing "electromagnet" she is showing that she knows the content acquired in training but she chooses to present mainly math specific content: problems involving operations, specially multiplication.

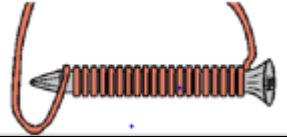
When you used the nail in the electromagnet it was wrapped in copper wire. I unwrapped one and measured the wire. It measured 40 cm.		
1. How many meters of wire were needed for the whole class?	2. One meter of wire costs 5 euros. How much was spent to wrap all the nails?	

Figure 2: Math exercises suggested and implemented by teacher Luísa inspired by the activity performed in the classroom by the instructors

Based on the same experiment Luísa was invited to suggest tasks for data handling and processing, but she decided to collect data with her students (Figure 3) for further processing (Figure 4).

OCTOBER 2015			APRIL 2016		
NAME	WEIGHT	HEIGHT	NAME	WEIGHT	HEIGHT
Adriana	30,7 Kg	129 cm	Adriana	32,2Kg	133 cm
André	31 Kg	139 cm	André	31,9 Kg	142 cm
António	36,4 Kg	139 cm	António	38,4 Kg	142 cm

Figure 3: Recording of heights and weights of the students in teacher Luísa's class

I grew _____ kg In October , the tallest in my class was _____ In May , the heaviest in my class was _____ Which student grew the most ? _____ Which student gained more weight between October and May ? _____ Build a chart with the weights of the students in the class (October-May)	I gained _____ kg
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Figure 4: Math activity suggested and implemented by teacher Luísa

This attitude shows some autonomy on the part of the teacher to propose activities that are not provided for, in the school books. It also shows ability to adapt content to the specific needs of students. On the other hand, this could mean some resistance in using an experiment that was not familiar, preferring to use a context where she felt more comfortable, i.e. collecting data from the students. A possible explanation can be drawn from her report, where she shows lack of SCK:

However, and given the nature of the subject matter addressed and the tools used, I do not feel comfortable to implement, in a natural/individual and consolidated process, many of the tasks proposed. (Teacher Luísa final report)

It also appears that she realized the importance of finding new ways of teaching math as she says in her report that math is part of day-to-day life: “The math activities performed in the class gained a new meaning as it was applied to practical real-life situations of individual students to complete the tasks proposed” (Teacher Luísa final report).

Luísa gained SCK and KCT, showing ability to do research, particularly on the internet, collecting information that she uses to make new approaches on teaching:

(...) finding new ways of teaching math so that people understand that we think mathematically all the time and solve problems at several moments during the day (...) Math is thus part of our life and can be learned in a dynamic, challenging and funny way. (Teacher Luísa final report)

Finally, the teacher recognizes the impact of the project on her students:

The class revealed very motivated when completing the tasks proposed by the instructors. The students adopted a cooperative, experimental attitude in which failure was regarded as a part of the scientific process. (Teacher Luísa final report)

In fact, student’s comments such as “this is the best experience of my life” “this is awesome”, “you should come more often”, among others, had impact on teacher Luísa motivation, contributing to make her recognize the importance of implementing hand-on experiments.

Final considerations

Visits to the classroom to support the teacher during the implementation of the experimental activities revealed very useful to improve teacher knowledge and confidence. Also, the enthusiasm, involvement and participation of the students in the classroom activities (mentioned in the teacher's final report) served to raise her awareness to the importance/relevance of these approaches. Such motivation has been observed in teacher Luísa who gained confidence to innovate her practices after receiving training and guidance. This teacher developed and implemented hands-on experimental activities with her students in classroom context, exploring their curiosity (using inquiry) and proposing problems requiring the use of math.

Although she created mathematics hands-on experiments, there still was, on the part of the teacher, some lack of confidence to innovate without the support of instructors. Like her, almost all teachers who participated in the continuous professional course suggested, during the final FG, that some of the experimental activities should be carried out by the instructors in their classrooms. All teachers were reluctant to propose innovative activities promoting by resorting to the inquiry method. It was noted that strong encouragement and responses on the part of the instructors were required to make the teachers change habitual teaching practices. However, throughout the sessions increased trust of teachers on their instructors and a better response to the tasks proposed has been observed.

It follows therefore that it is necessary to invest more in training and monitoring of teachers to further engage them in these approaches and improve their confidence and autonomy. Special mention should be made to the importance of getting the teachers to work out the activities before implementing them. Finally, it is concluded that this is a process that takes some time to be implemented and further work is needed to achieve the desired results.

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From emergency sirens to birdsong – Narratives of becoming a mathematics teacher

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As part of a larger research project, we asked third-year PSTs to reflect on what they had learned about being mathematics teachers of the teacher education programme. The reflections were intended as presentations for first-year PSTs. In this article, we analyse the films by the third-year PSTs to understand the messages more experienced PSTs choose to communicate to novices. Concepts from Gert Biesta are the framework for the content analysis, and we find a complex picture of how qualification, socialization and subjectification interact in the narratives.

Keywords: *Preservice teacher education, mathematics education, mentors.*

Introduction

This article presents partial results from a research project examining pre-service teachers' (PSTs) developing identities as mathematics teachers, and, in particular, their experiences of mathematics in school placement. Previously published results from the project report how first-year PSTs value what they learn from mentors in practice more than the 'theoretical' input of the university based courses, not seeing the theoretical knowledge as transferable into teaching practices (Bjerke, Eriksen, Rodal, Smestad, & Solomon, 2013). In one intervention addressing the challenge, third-year PSTs presented to the first-year PSTs films describing experiences of becoming mathematics teachers in the course of the first three years of the programme. In this article, we analyze the third-year PSTs' presentations to understand how they view their own emerging professional identities.

Our research question is: What domains of their educational experiences do PSTs highlight in their presentations of their first three years in mathematics teacher education?

Research background and theoretical underpinnings

While learning to teach is about acquiring professional knowledge and skills, it is also about developing a teacher identity (Haniford, 2010). Adding to the identity work of experienced teachers, PSTs have the responsibility of successfully positioning themselves in relation to their teacher education programmes and cooperating teachers (Haniford, 2010). Identity formation is driven by the individual's goal state of what he/she wants to become (Smeby, 2007). Biesta (2012) discusses these processes under the headings of qualification, socialisation and subjectification, and we choose his concepts as the starting point of our analyses of PSTs' narratives.

For Biesta (2012), all education (including teacher education) is a question of judgement, because educators' decisions about the purpose of what they do occur within domains that may be in synergy with each other, but may also be in conflict. He notes three such domains, which interact and overlap: the domains of qualification, socialisation and subjectification. Qualification is about knowledge, skills and dispositions; socialisation and subjectification can be seen as opposites: while

socialisation applies to the induction of novices into existing practices, subjectification denotes how education contributes to a process of individuation, of becoming an independent subject. Educational judgements are underpinned by an understanding of the interdependencies between the three domains. In teacher education, the situation is more complex still: the purpose should not just be PSTs own qualification, socialisation and subjectification, but also to enable PSTs to become ‘educationally wise’, aspiring towards virtuosity in making educational judgements themselves. Such judgements are situated: they are made during the practice of teaching, and cannot be set out in advance, or in general - they are rooted in concrete situations and relate to the need to handle tensions and see possible synergies. To become ‘educationally wise’, one needs experience, together with opportunities to see more experienced others in action, and to discuss those actions in terms of the virtuosity of judgement which underpins them.

Method

The 32 PSTs in this study were enrolled in a four-year programme for primary school teachers (grades 1-7, ages 6-13) in Norway. They had chosen to continue with mathematics beyond the compulsory course spanning the two first years in teacher education, and were asked to look back on their mathematics education course and four school placements and to prepare group presentations describing their development as mathematics teachers in grades 1-7. They were asked to reflect on what they know now which would have been good to know in their first year; how they plan their mathematics teaching now compared to in their first year; what they have learned along the way; and what are the pitfalls and experiences to bear in mind for future mathematics teachers.

Six presentations were made - all short films (F1-F6) incorporating line drawings and sometimes photographs, with voice-over commentary and music. Five of the six films were organised as developmental stories from their novice anticipation and preparation of their first placement, to their reflective stance as third-year PSTs. The remaining film focused on four pitfalls for novice PSTs.

To analyse the data, we operationalised Biesta’s (2012) concepts in terms of: descriptions of their knowledge, skills and dispositions; processes and accounts of making educational judgements and justification of judgement; and accounts of being and becoming a mathematics teacher, as exemplified below:

Qualification: References to knowledge, skills and dispositions, processes and practices from the teaching profession.

Socialisation: References to learning and to expectations in the school context.

Subjectification: References to inner feelings, identity and becoming a teacher and to perception of self (as a teacher).

The presentations were transcribed and analysed in several steps. First, they were coded according to Biesta’s terms *qualification*, *socialisation* and *subjectification*, synergies and conflicts between these domains, and professional judgement. Then disagreements among the researchers were resolved, and all presentations were re-read and coded by another member of the group. A decision was made to organise the analysis in two parts - one around PSTs’ novice anticipation, and one around their reflective stance in third year. Finally, we re-read each presentation to make sure that important longitudinal messages were not lost in our attempt to organise the analysis in two parts.

Looking back at early experiences in school placement - Emergency sirens

Qualification

In the films, there are a few examples of what PSTs learned in the first year, for instance pieces of ‘theory’, such as the importance of using multiple representations, giving feedback, and using the “didaktisk relasjonsmodell” (F1) (a hexagon connecting elements relevant for lesson planning: topic, learning objectives, the pupils in that class, etc.). The first-year PSTs are, however, uncertain about how to put the knowledge into practice. The uncertainty about effective use of manipulatives is most usual (F1, F4, F5), but other dilemmas are also identified:

I also thought about all the theory I learned at university: Piaget's theory of stages of development, Bruner's representation theory and theory of scaffolding, Vygotsky's focus on cooperation, Bandura and his theories on motivation and self-efficacy. How should I use these theories to plan a lesson on fractions? [...] Which pupils should work together: the ones that are on the same level in terms of subject knowledge, or should the strong ones help the weak? (F1)

Several of the films show students meeting concepts as an overwhelming mass of words (Figure 1). At the same time, there are many statements about elements of qualification the PSTs perceive as lacking, both in terms of knowledge (“Do I know enough about this topic?” (F5)) and of processes:

Just think if I have to explain several ways of doing something, to support conceptual understanding! It won't work. It just won't work. Manipulatives, manipulatives. (F5)

It's only natural to carry on from where we left off [in the textbook]. It's not like I have other suggestions on what to do from here onwards. (F2)

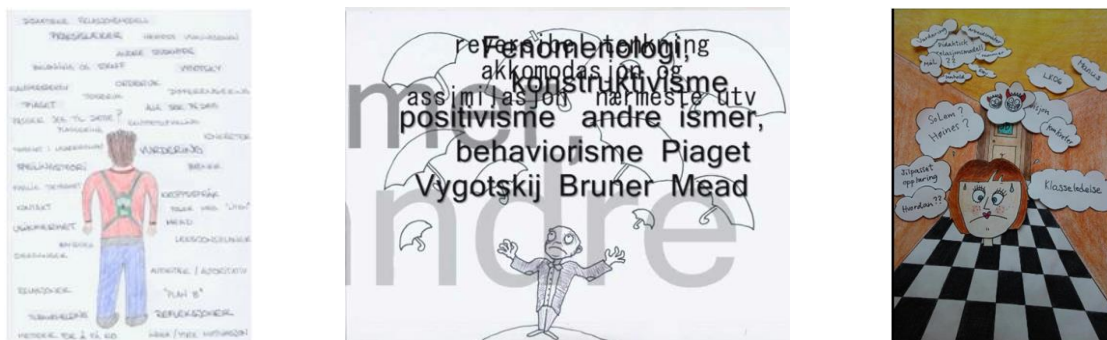


Figure 1: Knowledge in overwhelming amounts in the first year (F1, F4, F5)

Socialisation

The mentor is, naturally, the main role model for students in their first year of teacher education, and the mentor features in most of the (few) examples we find of socialisation when describing the first year. The PSTs are uncertain about what the mentor expects of them, other than using the “didaktisk relasjonsmodell” (mentioned above) which is common in Norway:

As first-year students we used it [the didaktisk relasjonsmodell] slavishly (F6)

...I have to carry on from here [in the textbook], that must be what [the mentor] expects (F2)

The mentor can be viewed as an evaluator:

Shit! This took the entire hour. The mentor glares at me. It didn't go as planned. (F3)

However, the anxiety of seeing the mentor write “like there was no tomorrow” is followed by “I had so many questions” - suggesting that the mentor is regarded as a person to ask for advice, as well.

Subjectification

A basis for subjectification is the development of a certain degree of self-confidence. In the description of the first year, we see little self-confidence - uncertainty and fear dominates:

What if I don't succeed? (F5)

I went from being one of the best in mathematics to being perplexed when the pupils asked me questions about the subject. (F1)

Conflicts and synergies between domains

The three domains of education overlap. These non-empty intersections are implicitly present in the films. There are clear examples that a perceived lack of qualification (being overwhelmed by new concepts and by making sense of these in practice) leads to a lack of self-confidence - “I felt unsure and very, very small” (F1) - which we regard as part of subjectification. This can also work the other way: lack of confidence leads to lack in qualification:

In the first school placement, I struggled a lot with getting the class to settle down. Later on I came to think it was because I did not feel like a confident and clear classroom manager. (F1)

There can also be a conflict between qualification and subjectification, in the sense that learning more makes you aware of your shortcomings:

The more I learnt, the more I discovered what I didn't know. [...] Based on Piaget's theory I knew most of the pupils were at the concrete-operational stage. But which of Bruner's representations should I use? [...] Or should I use the strange Cuisenaire rods that I still haven't really gotten to grips with? (F1)

Drowned in the curriculum he feels puzzled. What is most important? (F4)

Inside the domain of qualifications there are interactions between elements. In one case, the confidence in mathematics is shaken by the practice of teaching:

I went from being one of the best in mathematics to being perplexed when pupils asked me questions. (F1)

At the same time, during the first year the process of lesson planning is weighed down by the awareness that there are many considerations to be taken. This is visible in form of the time that goes into writing a lesson plan (shown with clocks in the films), and the number of books that fill the desk in the process (Figure 2).

There can be a tension between socialisation and subjectification in meeting the mentor: in one example, the role model (supposed to provide socialisation) is so impressive that the PST's self-confidence suffers:

The meeting with the mentor was scary. I saw him as a Superman who really knew his work. He was confident, clear and, not the least, had strong subject knowledge. (F1)



Figure 2: Lesson planning during first-year school placements (F1, F4, F6)

In another example, a PST's attempt at making a choice outside of the textbook is struck down by the mentor:

Hmmm....I think maybe we should stick to the textbook. (F3)

With an emerging sense of agency, the PST questions the mentor's view and asks herself: "Should we always stick to the textbook?" (F3).

To conclude, there are synergies between (a lack of) qualification and (a lack of) subjectification, but also a conflict between qualification and subjectification, as well as between socialization and subjectification.

Practicing educational judgement

Judgement is difficult. A lack of self-confidence leads to a very detailed plan with little room for judgement on-the-fly.

As a first-year PST the plan for the lesson was a long script. We had written down word for word what to say during the lesson. We were dependent on this script and could not improvise along the way. We even planned how to explain simple mathematical things that we actually knew well. This is also about lack of experience and confidence as a teacher. (F6)

At the same time, a lack of qualification translates into constraints on opportunities for judgement in the process of lesson planning:

It's natural to continue from where we left off, it's not like I have other suggestions. (F2)

Looking at their recent experiences in school placement - Birdsong

Qualification

Changes from first to third year are visible in all aspects of qualification, from subject knowledge and knowledge of students and teaching, to the practices of teaching. In terms of knowledge, some films refer to knowing more mathematics, but, in terms of mathematics pedagogy, the films stress that the understanding is deeper, the knowledge can be operationalised to a greater extent.

The process of lesson planning during the first year involved long hours dedicated to the task (F1, F4, F6), and resulted in long scripts produced for each lesson (F4, F6). The films highlight, in comparison, how much quicker lesson planning goes (F1, F6), and how much shorter the scripts become (F1, F4) by third year, but the films give different suggestions on how to take advantage of the reduced burden, from watching TV and playing with the dog (F6) to investing time and energy on the 'frills' of differentiation and using a variety of teaching methods (F1).

Teaching practices out of reach during the first year are now on the agenda (F1): motivating pupils, providing them with opportunities to feel both confident and challenged, seeing the individuals as well as the class as a whole, giving more room to children's contributions, and encouraging enquiry.

Socialisation

The main presence that embodies the socialisation component is the teacher mentor, although some PSTs also mention peers and other colleagues playing a role. At this stage the mentor has transitioned from a feared judge to a colleague (F1), in some cases a role model (F2, F4), although disagreements between the views of PSTs and their mentor may occur, for example regarding the role of textbooks (F3). However, adopting established practices of the teaching community, such as body language (F1) or ways of saying or doing things in the classroom, seems to be perceived by PSTs as a sign of having become teacher-like:

I've even put together extra handouts [for those who might need another type of challenge]. (F2)

Subjectification

Through the journey from first to third year, the PSTs have grown into teachers who are aware that teaching is not just about what you know, it is about making choices about complex situations. As there are no deterministic answers to these dilemmas, neither objectively speaking nor in terms of what is the established way of the teaching community, these choices come down to the individual, they are drawing on the domain of subjectification: "We're more aware that there should be a reason behind our choices" (F4), "I understand my own thoughts" (F5). In their third year, we hear the PSTs stress the importance of trusting their own choices (F1, F2, F4), and being yourself (F2).

Planning lessons is now an altogether more positive experience, described with attributes such as joy, and belief in oneself. Importantly, some of the PSTs realize that becoming a teacher is a continuous process, and experimenting is a part of it:

Don't be afraid to try out new things. (F2)

A lesson plan can never be too good. It's like a piece of silverware that you take out and polish from time to time. (F4)

Conflicts and synergies between domains

As PSTs become more comfortable as teachers (subjectification), some find reassurance in their theoretical knowledge (qualification) as well as their awareness of what is acceptable among teachers (socialisation):

Not everything has to be perfect [...]. The theory I used to think about while planning lessons in my first year is now under my skin. (F1)

The routines of teachers (socialisation) also contribute to being more successful in the practices of teaching, such as lesson planning (qualification): "You don't have to reinvent the wheel (F1)". There is an aspect of growing confidence (subjectification) when the PSTs reuse lesson plans they have had positive experiences with (F1).

Unlike in their first year, lesson planning in the third year takes less time (F1, F6) and the scripts for the lesson are shorter (F1, F4) or even disappear altogether ("We've thrown out the script", F6). The

change is attributed in general to an increase in confidence (subjectification) but in some cases also to an aggregated influence of all three domains:

... more confident in myself and the mathematics, I know more about the pupils' level in mathematics, I have become a clear leader, I dare to make mistakes, I am better at dealing with things as they happen. (F1)

In another film, the three domains come together in synergy to express the PSTs' development:

By contrast with first year, when we used the the syllabus for the course a lot, we now have more knowledge of the subject and of pedagogy. We've become better at making use of our own knowledge, we cooperate more closely with colleagues. (F6)

The way these sentences are linked, makes it possible to interpret it as meaning that better qualification leads to better self-confidence (subjectification) which again leads to better cooperation with others (socialization).

Practicing educational judgement

Increased self-confidence by the third year is not synonymous with knowing just what to do:

How can I connect algorithms and conceptual understanding? I need to be able to show them different strategies, to be sure as many as possible understand. How many strategies for division are there? Maybe they come with some I haven't thought of? Maybe some misconceptions will surface during the lesson? How can I then, in the best possible way, deal with this? (F4)

The difference from the first year is being able to deal with dilemmas, to practice professional judgement, guided by what they see as the goals of teaching:

There's still a lot to think about, but I understand my own thoughts now, I know where I'm heading (F5)

In the third year practicing educational judgement features as a defining factor of the PST-mentor relationship at this stage: the detailed scripts for lesson plans that were in the first year in part written for the sake of the mentor (and in part to boost one's confidence, to feel prepared) are now shorter. A mentor's voice sets expectations:

Just show me that you are aware of the choices you make, and that you can argue for them (F2)

During the third year, educational judgement is visible in reflections on one's own teaching:

I've become better at assessing myself, and I can more readily explain what went well and what could have been done better in class (F1)

Teaching analysis draws on and at the same time feeds into the domain of qualification and perhaps also socialization. This way of assessing oneself - and the knowledge that you do it well - can be regarded as an engine for development also after graduation, it feeds into subjectification.

Concluding remarks

The titles of two subsections of the analysis reflect the soundtrack of a film where the experiences of first- and third-year school placements are introduced with emergency sirens and birdsong respectively. In terms of Biesta's (2012) framework, the overall picture the presentations paint is

that, looking back on their first-year school placement, PSTs remember a combination of a lack of qualifications, unclear expectations from mentors and low self-confidence. The fear many PSTs report on, seems rooted in their low self-confidence and the unclear expectations. Although first-year PSTs are allowed to try different approaches and to fail, the same combination of a lack of repertoire, uncertainty of the mentor's role and lack of self-confidence holds them back. By the third year their qualifications have increased, their role as PSTs is clearer and their self-confidence has grown. Because of this, they also find themselves practicing educational judgement more often.

Such narratives, perhaps in combination with logs from early placements, could be part of educational experiences, supporting PSTs' identity work. First-years might also benefit from watching the films, as not all challenges discussed can be dealt with by the teacher educators. While teacher education can and should make explicit what is expected of PSTs in their school placement, it cannot rush becoming educationally wise. However, we hypothesise that creating spaces where first- and third-year PSTs can discuss their experiences would contribute to the domain of socialisation and subjectification for both groups. Analyzing the students' contributions in terms of Biesta's concepts reveal the complex relationships between qualification, socialization and subjectification in teacher education. The three domains are interdependent, with conflicts and synergies which influences PSTs overall experience. More insight into these conflicts and synergies may contribute to better understanding of PSTs' experiences of their school placements.

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Core practices and mathematical tasks of teaching in teacher education: What can be learned from a challenging case?

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In order to successfully carry out ambitious mathematics teaching, awareness about the underlying mathematical tasks of teaching involved is necessary. This paper presents the case of Martin, who started mathematics teacher education because he likes mathematics and feels that he knows the subject. We observe him in a period of field practice where he is supervised by an experienced mathematics mentor teacher. While planning, conducting and reflecting on a lesson on multiplication of fractions, neither Martin nor his mentor teacher focus on core tasks of teaching mathematics. We use this case as a starting point for discussing challenges and possibilities of increasing the emphasis on core practices and the embedded mathematical tasks of teaching mathematics in teacher education.

Keywords: Core practices, mathematical tasks of teaching, teacher education.

Introduction and theoretical background

Mathematics teaching is complex, and different attempts have been made to decompose it. Some have tried to identify the most critical practices involved in the work of teaching mathematics, and they describe them as “core practices” (e.g., McDonald, Kazemi, & Kavanagh, 2013) or “high-leverage practices” (Ball & Forzani, 2009; Forzani, 2014). These practices are fundamental for supporting students’ learning. Others emphasize the mathematical tasks of teaching that are embedded in the professional work of teaching mathematics (e.g., Ball, Thames, & Phelps, 2008; Hoover, Mosvold, & Fauskanger, 2014). In this paper, we focus on skills necessary to carry out the work of teaching mathematics by emphasizing mathematical tasks of teaching that are embedded in core teaching practices in teacher education. High-leverage practices and the underlying mathematical tasks of teaching “are essential for skillful beginning teachers to understand, take responsibility for, and be prepared to carry out in order to enact their core instructional responsibilities” (Ball & Forzani, 2009, p. 504). It is thus important to develop these practices during teacher education (TeachingWorks, 2015). The mathematical tasks included in these practices are instructional tasks that are mathematical and not pedagogical. When referring to “mathematical tasks of teaching”, we follow the conceptualization by Ball et al. (2008).

Researchers have made various attempts to categorize core practices (Forzani, 2014). In the TeachingWorks project (2015), a register of high-leverage practices is presented to serve as the basis for a core curriculum for the professional training of teachers. For instance, the first two of the nineteen high-leverage practices are: 1) “leading a group discussion”, and 2) “explaining and modeling content, practices, and strategies.” Both of these practices contain some mathematical tasks of teaching. For instance, the second practice obviously involves the task of “presenting mathematical ideas.” It may also involve “finding an example to make a specific mathematical point” (Ball et al., 2008, p. 400), a mathematical task of teaching that has proven to be difficult for pre-service teachers (Zodik & Zaslavsky, 2008). This practice may also involve the tasks of “recognizing what is involved

in using a particular representation” and “linking representations to underlying ideas and to other representations” (Ball et al., 2008, p. 400). Other mathematical tasks of teaching—some are embedded in more than one high-leverage practice—include using language carefully, highlighting core mathematical ideas while sidelining potentially distracting ones, and make their own thinking visible while modeling and demonstrating. The third high-leverage practice presented by TeachingWorks (2015) is “eliciting and interpreting individual students’ thinking.” This practice may involve asking and responding to questions, or presenting the students with exercises that provoke or allow them to share their mathematical thinking in order to evaluate student understanding, guide instructional decisions, and surface ideas that will benefit other students. When engaging in these practices, the teacher is faced with several mathematical tasks of teaching. For instance, teachers are challenged to ask “productive mathematical questions” and to evaluate “the plausibility of students’ claims” (Ball et al., 2008, p. 400). The mathematical tasks of teaching included in various core practices may vary depending on the context.

In Norway, the new national guidelines for primary and lower secondary teacher education use “core practices” as a term of reference (Ministry of Education and Research, 2016). Our study aims at discussing challenges and possibilities of implementing core practices and identifying underlying mathematical tasks of teaching mathematics in mathematics teacher education.

The Norwegian teacher education context

Norwegian teacher education is politically controlled (Hammerness, 2013), and national curriculum guidelines direct the focus and content of all teacher education programs. There are differentiated teacher education programs for primary (years 1–7) and lower secondary (years 5–10) levels; both are four-year bachelor programs. In the primary teacher education program, a mathematics course of 30 ECTS is compulsory for all pre-service teachers, whereas the lower secondary teacher education program requires 60 ECTS in mathematics for pre-service mathematics teachers.

Field practice is a compulsory part of teacher education, but studies indicate that pre-service teachers’ opportunities to learn in this context are not sufficiently utilized (Hammerness, 2013). Pre-service teachers are required to complete 100 days of field practice at partner schools. The aim is that field practice should focus on the subject that pre-service teachers study on campus the current year. In field practice, pre-service teachers normally work in groups that are supervised by an experienced mentor teacher. Prospective mentor teachers are required to take a training course of 15 ECTS, and they are employed by the universities as teacher educators. From 2017, the Norwegian teacher education will be a five-year master program, and the field practice component will be extended to a minimum of 115 days spread across five years (Ministry of Education and Research, 2016).

Methods

In order to discuss challenges and possibilities of implementing high-leverage practices (e.g., TeachingWorks, 2015) and identify underlying mathematical tasks of teaching (e.g., Ball et al., 2008), we consider an empirical case from a cross-disciplinary project in Norwegian teacher education entitled Teachers as Students (TasS). The TasS project has a focus on pre-service teachers’ learning in field practice.

Data collection includes video recordings of group interviews held with each group of pre-service teachers before and after their period of field practice. Based on analyses of these interviews, Martin

(pseudonym) stood out as a special case. He was one of only two pre-service teachers in the project who selected mathematics because they liked it and were good at it. A recent literature review suggests that being good at mathematics is important for pre-service teachers, for instance in perceiving and interpreting students' work (Stahnke, Schueler, & Roesken-Winter, 2016). In this case we only know that Martin sees himself as good, and this was a criterion for selecting him as a case. We observed Martin in a period of field practice in his fourth semester. In the previous semester, he had completed the 60 study points (ECTS) in mathematics/mathematics education that is required to teach mathematics in grades 5–10. Even though Martin enjoyed mathematics, his teaching practices cannot be seen as ambitious teaching practices (Lampert et al., 2010) fundamental for supporting students' learning (e.g., McDonald et al., 2013). In the following, we first show some glimpses from a lesson where Martin teaches multiplication of fractions in grade 7, followed by his discussions with the mentor teacher in the post-lesson mentoring session. The selected episode was typical for Martin's lessons in his three weeks of field practice. The lesson lasted for 38 minutes (we only focus on the whole class teaching in the selected episode), whereas the mentoring session lasted for 13 minutes.

The case of Martin

After a brief repetition from the previous lesson, Martin introduces multiplication of fractions as the focus of this lesson. He writes $\frac{1}{3}$ of $\frac{3}{4} = \frac{1}{3} \times \frac{3}{4}$ on the blackboard and emphasizes that we say “one third of three quarters” when we write an expression like this. “What this means,” he continues, “is that we first have a fraction of one third and split it into three by using two horizontal lines.” He draws a quadrilateral on the blackboard, partitions it and shades the top third part. He draws another quadrilateral, partitions it vertically into fourths and shades three of these (see Figure 1). He then draws a third quadrilateral and says, “If we now want to take a third of this, we partition it into three [draws two horizontal lines in the figure]. How much is one third now?”

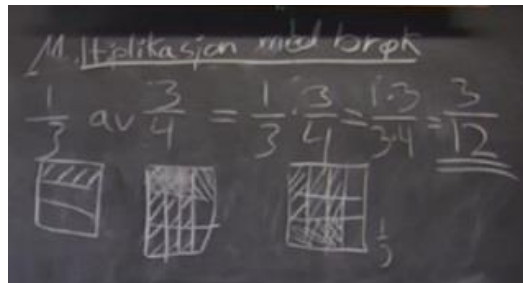


Figure 1. Martin's illustration of $\frac{1}{3} \times \frac{3}{4}$

When the students struggle to respond, Martin points at one of the parts in the figure. “It is one of these. And then we only have a third of what is shaded—look at this one! Did you understand that?” Some students say no, whereas others shake their heads. Martin tries again: “What we can also do, is to say that we put the two fractions on top of each other.” He points at the first figure, pretends to move it over to the second figure and draws two horizontal lines in the middle figure. “What we focus on,” he continues, “is that which has been shaded twice.” When noticing that the students still do not seem to understand, he writes the expression. “It is as simple as taking this one [pointing at the first numerator] times this one [pointing at the second numerator], and then we take the first denominator times the second denominator.” While saying this, he writes it out on the blackboard. “Do you see

that $\frac{3}{12}$ is the answer?” When the students are still hesitant, he quickly wipes everything out. “Let’s take one more example,” he continues.

In the next example, Martin writes $\frac{1}{2}$ of $\frac{1}{3}$ on the blackboard and draws two figures that he partitions and shades—this time by using colored crayons. The figures are of different sizes. “How much of this is both red and blue?” he asks. A girl presents $\frac{4}{6}$ as an answer, whereupon Martin repeats, “That is both blue AND red?” When a boy provides the answer, $\frac{1}{6}$, Martin confirms. “We have one part that is both blue and red, meaning both fractions. There are six parts altogether, and then we get one sixth,” he continues. Martin tries to point the students’ attention to the procedural approach. “If I want to solve this expression [pointing at $\frac{1}{2} \times \frac{1}{3}$ on the blackboard], the operation, how do I do it?” A boy mumbles that you are supposed to multiply, and Martin continues, “We are going to multiply denominator by denominator [pointing at the numerators (!) in the expression], and numerator by numerator [pointing at the denominators (!)].” He writes it out on the blackboard, seemingly ignorant about the fact that he has just mixed numerators with denominators. “Do you think you can make it if you try the tasks for yourselves now?” He then turns to the blackboard again and emphasizes how important it is to remember that although we write $\frac{1}{2} \times \frac{1}{3}$, we say one half of one third. “Important to remember,” he says. “If we don’t remember this, it will be very, very hard to solve the word problems!” Then he writes down $\frac{3}{7} \times \frac{10}{2}$ as another example. “How many of you know how to solve this one?” he asks. “The very operation,” he continues, “forget about the figures!” After this introduction (14 minutes into the lesson) the students start working on similar tasks from their textbooks. When realizing that many students still have problems, Martin presents another example on the blackboard (after 18 minutes). Towards the end of the lesson, when summing up, he tells the students that he forgot to mention that the quadrilaterals (in Figure 1) are supposed to be equal in size.

Martin introduces the post-lesson mentoring session by saying that he can see what went wrong. When asked to elaborate, he points out that he should have used different colors from the beginning, and that he should have presented more examples before letting the students work individually on tasks. He continues to say that in his figures (e.g., Figure 1) he should have “mentioned that they [the quadrilaterals] were equally big”. The mentor teacher supports this by saying: “When you don’t show on the blackboard that they are equal, you cannot expect the students to understand it.” In the next part of the mentoring session, they discuss what different students managed to do during the part of the lesson that involved individual work on textbook tasks. This part of the lesson is not the focus of attention in this paper. Towards the end of the mentoring session, the mentor teacher ends the discussion about the figures used to illustrate fraction multiplication by saying, “I think they [the quadrilaterals] would have worked out very well, as you are pointing out, if you had used colors. And if you had thought about making them equal in size, it would have worked out very well.” Martin agrees, and adds that it is important to be careful about how you draw such figures.

Learning from the case of Martin

The case of Martin illustrates the core practice of “explaining and modeling content, practices, and strategies” (TeachingWorks, 2015). A simplified response to the presented episodes and vignettes could be that Martin does not carry out the core practice of explaining and modeling content, practices, and strategies well, and he needs more practice. Based on research indicating that pre-

service teachers do not necessarily learn from their field practice (Hammerness, 2013), we believe there is more to it than this. Although Martin and his mentor teacher discuss his explanations and modeling of the content, they do not appear to get to the heart of the issue. Ball and Forzani (2009) suggest that certain mathematical tasks of teaching are embedded in these core practices (e.g., Ball et al., 2008; Hoover et al., 2014). In our discussion of challenges and possibilities of implementing core practices, we identify four challenges and discuss the possibilities for highlighting some of the embedded mathematical tasks of teaching.

First, and most importantly, Martin is faced with the mathematical task of “recognizing what is involved in using a particular representation” (Ball et al., 2008, p. 400). In his attempt to use the area model to represent multiplication of fractions, Martin draws three quadrilaterals (Figure 1). The area model for multiplication of fractions requires use of one rectangle only as unit. Martin does, however, say that the third rectangle illustrates the two others “on top of each other”, and he points at the first rectangle and pretends to drag it over the second rectangle. Still, this use of the model appears to confuse the students, and it would have been natural to focus on this mathematical task of teaching in a post-lesson mentoring session. In the given example one can draw three horizontal lines to show four equally large horizontal strips and then divide an area of three of these in three columns by drawing two vertical lines. Then there is still some work to do in order to understand that numerators can be multiplied as well as denominators. Martin tries to help the students develop this understanding by drawing two horizontal lines in the middle figure to illustrate how the four parts have now been divided into three and saying: “what we focus on, is that which has been shaded twice.”

In the post-lesson mentoring session, Martin starts by stating that he can see what went wrong. He points out that he should have used different colors (instead of double shading) and that he should have presented more examples. He continues to say that he should have “mentioned that they [the rectangles] were equally big.” The mentor teacher expresses his agreement. The area model requires use of one rectangle only as unit, but this is not discussed. “Recognizing what is involved in using a particular representation” (Ball et al., 2008, p. 400) is a mathematical task of teaching which might be fruitful in order to facilitate student teachers’ learning to carry out the core practice of explaining and modeling content in teacher education. This mathematical task of teaching is, however, not discussed in the post-lesson mentoring session.

Second, and related to the first, Martin consistently reads the product as “ $\frac{1}{3}$ of $\frac{3}{4}$ ” indicating another model for multiplication of fractions than the area model: the multiplicative comparison model. In this model, one of the fractions is an operator, one is represented as a portion of the area of one rectangle, and the result is represented as another portion of the area of the same rectangle. In this case the rectangles might well be drawn separately, but Martin’s comment that the two first rectangles in Figure 1 should be placed on “top of each other” to make the third indicates that he does not have this model in mind. This is also not discussed in the post-lesson mentoring session.

Third, and related to the mathematical task of selecting appropriate examples “to make a specific mathematical point” (Ball et al., 2008, p. 400), Martin could have selected more appropriate examples when presenting the students with the area model for multiplication of fractions. In using this area model, it is necessary to not only use simple fractions as one-third and three-fourth. One can clarify much better that numerators can be multiplied as well as denominators with say three-fifth of four-

seventh. This illustrates an embedded mathematical task of teaching related to finding examples to make specific a certain mathematical point. In the post-lesson mentoring session, only the number of examples is discussed. For instance, they discuss that Martin should have presented more examples. Selection of examples, which has proven to be difficult for pre-service teachers (Zodik & Zaslavsky, 2008), is not discussed. One way to meet this challenge of randomly generated examples, when careful choices should be made, is to exemplify and discuss carefully selected examples in mathematics teacher education

Fourth, Martin is faced with the mathematical task of using correct mathematical language when presenting the mathematical idea of multiplication of fractions. On a couple of occasions, we observe that he mixes numerator with denominator. This might be regarded as a minor mistake of speaking mathematics, and we do not believe it is the most critical issue in the case of Martin. Correct use of mathematical language and notation is still important, however, and we suggest that this is a mathematical task of teaching that the mentor teacher could have discussed with Martin.

Martin is using a model for representing multiplication of fractions (Figure 1), but the students struggle to understand it. The high-leverage practice of eliciting and interpreting students' thinking (TeachingWorks, 2015) focuses on teachers' practice related to posing questions or tasks that provoke or allow students to share their mathematical thinking in order to evaluate student understanding and guide instructional decisions. To do this effectively, a teacher needs to draw out a student's thinking through carefully selected questions and tasks and to consider and check alternative interpretations of the student's ideas and methods. Although Martin knows the mathematical content himself, he struggles to understand the problems faced by the students. This illustrates the importance of pre-service teachers' mathematical knowledge in order to perceive and interpret students' work (Stahnke et al., 2016). In his communication with the students, Martin does not invite the students to engage in mathematical discussions and reasoning. Instead, the students are invited to give short and confirmative responses only. During the 14-minute whole-class introduction, as well as in the brief wrapping up of the lesson, the students were mostly invited to answer yes or no questions. Two examples of such questions are: "Did you understand that?" and "Do you think you can make it if you try the tasks for yourselves now?" Some questions are asked when the answer is already visible through the example presented on the board, like: "How much is one third now?" and "How much of this is both red and blue?" These questions invite students to answer by single words. Only once were the students invited to answer a how-question, and this question was related to how to carry out a routine procedure. Based on our analyses of how Martin invites the students to participate, we conclude that they are not invited to speak or reason mathematically. This kind of communication does not allow Martin to elicit students' thinking. We thus suggest that the core practice of eliciting and interpreting students' thinking is important to develop in teacher education.

Conclusion

Learning to successfully carry out high-leverage practices in mathematics teaching requires awareness about the underlying mathematical tasks of teaching involved (Ball et al., 2008), and mathematics teaching is a professional practice that requires training (Hoover et al., 2014). In this paper, we have used illustrative data from field practice in Norwegian mathematics teacher education as a starting point for our discussion. From this challenging case, we observe that neither the pre-service mathematics teacher (Martin) nor his mentor teacher appear to be conscious about the

mathematical tasks of teaching that are embedded in the high-leverage practices that have been analyzed and discussed in this paper—all of which are involved in the planning, conducting and reflecting on a lesson on multiplication of fractions.

What challenges and possibilities of implementing core practices and identifying underlying mathematical tasks of teaching mathematics in mathematics teacher education can be identified by analyzing Martin's lesson? In this lesson, the pre-service teacher was challenged to carry out the practice of "Explaining and modeling content, practices, and strategies" (TeachingWorks, 2015). Previous research indicates that strong teacher knowledge supports teachers in using representations to attach meaning to mathematical procedures (e.g., Charalambous, 2010). Although he has completed all of his coursework in mathematics, Martin does not appear to be prepared to carry out this core practice. Martin is also challenged to elicit students' thinking (TeachingWorks, 2015), but he does not seem prepared to carry out this core practice of teaching mathematics either, at least not in an ambitious way (Lampert et al., 2010). From these illustrative data, it appears that there is a lack of awareness about the underlying mathematical tasks of teaching, and the post-lesson mentoring session includes little discussion of these underlying mathematical tasks of teaching.

At least four lessons can be learned from the case of Martin. First, the task of recognizing what is involved in using a particular representation challenges teacher education to include detailed discussions of the area model for representing multiplication of fractions, highlighting the importance of using only one rectangle and how this model relates to numerators are multiplied as well as denominators. Second, selecting appropriate examples using carefully chosen numbers is important in order to clarify that numerators can be multiplied as well as denominators, and therefore important to discuss in teacher education. Third, Martin could have selected better examples when presenting the students with the area model for multiplication of fractions. Fourth, and finally, the case of Martin illustrates the mathematical task of presenting mathematical ideas using correct mathematical language. These mathematical tasks of teaching all seem related to the high-leverage practice of explaining and modeling content (TeachingWorks, 2015).

Mathematics teacher education needs to focus more on preparing prospective teachers to carry out high-leverage practices. Discussions of the embedded mathematical tasks of teaching are then necessary. The vignettes and episodes discussed in this paper indicate that the "unnatural" and complicated work of teaching needs to be explicitly taught in teacher education. One way of approaching this is to practice carrying out the mathematical tasks of teaching on campus as well as in field practice. The national guidelines for current primary and lower secondary teacher education that are now being developed focus on core or high-leverage practices (Ministry of Education and Research, 2016). More research and development efforts are needed to ensure a high-leverage implementation of these ideas in teacher education.

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Characteristics of a learning environment to support prospective secondary mathematics teachers' noticing of students' thinking related to the limit concept

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The aim of this study is to describe changes in the way that prospective secondary school teachers notice students' mathematical thinking related to the limit concept in a learning environment designed ad hoc. The learning environment progressively nests the skills of attending to, interpreting and deciding as three interrelated skills of professional noticing. Results show characteristics of how prospective teachers gained expertise in the three skills since four out of five groups of prospective teachers interpreted students' mathematical reasoning attending to the mathematical elements of the dynamic conception of limit. The links between attending to and interpreting helped prospective teachers justify the teaching activities proposed to support the progression of students' mathematical reasoning: from a mathematical point of view or considering mathematical cognitive processes involved.

Keywords: Noticing, prospective teachers' learning, learning environment.

Introduction and theoretical background

Research has shown that noticing is an important component of teaching expertise (Mason, 2002). Teachers need to attend to students' mathematical reasoning and make sense of it in order to teach in ways that build on students' thinking (Choy, 2016; Sherin, Jacobs, & Philipp, 2011). Noticing has been conceptualised from different perspectives. One of them consists of two main processes: attending to particular teaching events and making sense of these events (Sherin et al., 2011). Jacobs, Lamb, and Philipp (2010) particularise the notion of noticing to children's mathematical thinking, conceptualising this notion as a set of three interrelated skills: attending to children's strategies, interpreting children's mathematical thinking, and deciding how to respond on the basis of children's mathematical thinking.

Previous research has focused on pre-service teachers' ability to interpret students' mathematical thinking (Bartell, Webel, Bowen, & Dyson, 2013; Callejo, & Zapatera, 2016; Fernández, Llinares, & Valls, 2012; Llinares, Fernández, & Sánchez-Matamoros, 2016; Magiera, van den Kieboom, & Moyer, 2013; Sánchez-Matamoros, Fernández, & Llinares, 2015) showing that the identification of the mathematical elements involved in the problem (mathematical content knowledge) plays a significant role in interpreting students' mathematical reasoning. Furthermore, previous research has shown that some contexts can help pre-service or prospective teachers develop the noticing skill: watching video clips (Coles, 2012; van Es, & Sherin, 2002), participating in online debates (Fernández et al., 2012) or participating in learning environments (interventions) designed considering specific mathematical topics. For example, Schack et al. (2013) in the area of early

numeracy; Magiera et al. (2013) in algebra; Callejo and Zapatera (2016) in pattern generalization; Llinares et al. (2016) in classification of quadrilaterals; Sánchez-Matamoros et al. (2015) in the derivative concept; and Son (2013) in the concepts of ratio and proportion. These previous studies underline that the skill of deciding how to respond on the basis of children's mathematical thinking is the most difficult one to develop in teacher education programs. As Choy (2013) pointed "*the specificity of what teachers notice while necessary, is not sufficient for improved practices*" (p. 187). In other words, teachers can be very specific about what they notice without having a teaching decision in mind. So, the relation between how prospective teachers develop the skills of interpreting students' mathematical thinking and deciding how to respond on the basis of students' mathematical thinking deserves further research.

On the other hand, the concept of limit of a function is a difficult notion for high school students (16-18 years old) and is a key concept in the Spanish curriculum (Contreras, & García, 2011). Cottrill and colleagues (1996) indicated that the difficulty of students' understanding of the limit concept could be the result of a limited understanding of the dynamic conception. A way of overcoming this difficulty is by coordinating the processes of approaching in the domain and in the range in different modes of representation. Knowing these characteristics of students' understanding could provide prospective teachers with information to interpret students' mathematical thinking and to make instructional decisions based on students' reasoning.

Therefore, our study analyses changes in the way that prospective teachers notice students' mathematical thinking (attending to, interpreting and deciding) in relation to the limit concept when they participate in a learning environment designed ad hoc. The learning environment designed progressively nests the skills of attending to, interpreting and deciding and its relations. We hypothesise that the structure of the learning environment help prospective teachers to decide how to respond taking into account their previous interpretations of students' mathematical thinking.

Method

Participants and the learning environment

The participants were 25 prospective secondary school teachers (mathematics, physics and engineering) who were enrolled in an initial secondary mathematics teacher training program. One of the subjects of this program is focused on developing the skill of noticing students' mathematical thinking in different mathematical topics and on planning the instruction attending to students' mathematical thinking. One of the mathematical topics considered was the limit concept.

The learning environment consisted of 5 sessions of two hours each and was designed taking into account the nested nature of the skills of attending to, interpreting and deciding (Jacobs et al., 2010). Prospective teachers were divided into five groups of 5 persons to perform the tasks of the learning environment. Firstly, prospective teachers solved three problems related to the limit concept selected from high school textbooks (Figure 1) in order to unpack the important mathematical elements of the limit concept (session 1). Then, prospective teachers had to anticipate hypothetical students' answers to these problems reflecting different characteristics of conceptual development (session 2). That is, they had to anticipate what students are likely to do. Prospective teachers had a document with the definition of the dynamic conception of limit and its mathematical elements (Pons, 2014): (i) approaches from the right and from the left (in the domain and in the range), and

(ii) coordination of the processes of approaching in the domain and in the range considering different modes of representation (graphical, algebraic and numerical).

Problem 1
Consider the function:

$$f(x) = \begin{cases} 2x + 1 & \text{si } x \leq 1 \\ 4 & \text{si } 1 < x \leq 2 \\ x^2 & \text{si } x > 2 \end{cases}$$

Calculate the limit of $f(x)$ when
a) x approaches 1
b) x approaches 2

Problem 2
Consider the next tables:

x_1	0,8	0,9	0,99	...	1,2	1,1	1,01
$f(x_1)$	1,64	1,81	1,9201	...	2,44	2,21	2,0201

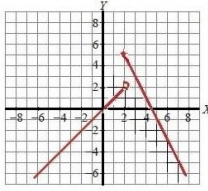
x_2	0	0,9	0,99	...	1,1	1,01	1,001
$g(x_2)$	0	-0,99	-0,9999	...	2,3	2,03	2,003

a) What is the value to which
1. x_1 and x_2 are approaching from the right and from the left?
2. The images of $f(x_1)$ are approaching from the right and from the left?
3. The images of $g(x_2)$ are approaching from the right and from the left?

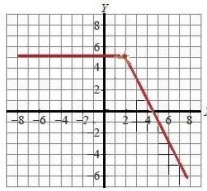
b) What is the value to which
1. The images of $f(x_1)$ are approaching in relation to the value that x_1 approaches?
2. The images of $g(x_2)$ are approaching in relation to the value that x_2 approaches?

Problem 3
Relate the following graphs with the statements a, b and c. Justify your answer

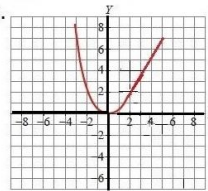
1.



2.



3.



a) The limit of the function in $x = 2$ is 2
b) The limit of the function in $x = 2$ is 5
c) There is not limit of the function in $x = 2$

Figure 1: The three problems related to the limit concept

The aim of the tasks of identifying the mathematical elements in the resolution of the problems and anticipating hypothetical students' answers was to help prospective teachers focus their attention on the relationship between the specific mathematical content and students' mathematical thinking. We conjecture that focusing on this relationship is needed to develop the skill of noticing in a first step. Next, prospective teachers analysed a set of four high school students' answers (Pablo, Rebecca, Luigi and Jorge) to the same problems. Prospective teachers had to attend to students' strategies, interpret students' mathematical thinking and propose new activities (or modify them) to help students progress in their conceptual reasoning (according to their previous interpretations of students' mathematical thinking) (session 3 and 4). The high school students' answers, provided to prospective teachers, reflected different levels of high school students' reasoning of the limit concept (Table 1; Pons, 2014). We also provided prospective teachers with theoretical information that summarise the characteristics of high school students' reasoning of the limit concept from previous research to solve the task (Cornu, 1991; Cottrill et al., 1996; Swinyard, & Larsen, 2012). In figure 2, the answers of Pablo to the three problems are given.

Prospective teachers had to answer the next three questions: (i) which mathematical elements has the student used in each problem? Indicate if he/she has had difficulties with them; (ii) identify some characteristics of how the student understands the limit of a function. Explain your answer using the mathematical elements identified before; (iii) considering the student reasoning, propose an activity that helps the student progress in their conceptual reasoning of the limit concept. Therefore, the objective of sessions 3 and 4 was that prospective teachers focus their attention on the relation between identifying-interpreting and between interpreting-deciding. We conjecture that these relationships are necessary to develop the skill of noticing. Finally, in the session 5, prospective teachers had to answer a similar task individually.

High school students	Level	Levels of students' reasoning about the dynamical conception of the limit concept
Pablo and Luigi	High	Pablo and Luigi coordinate the processes of approaching in the domain and in the range in the three modes of representation
Rebecca	Low	Rebecca coordinates the processes of approaching in the domain and in the range in the graphical mode of representation when limits coincide
Jorge	Intermediate	Jorge coordinates the processes of approaching in the domain and in the range in the algebraic and graphical mode of representation (when limits coincide in this last mode of representation)

Table 1: Characteristics of high school students' answers

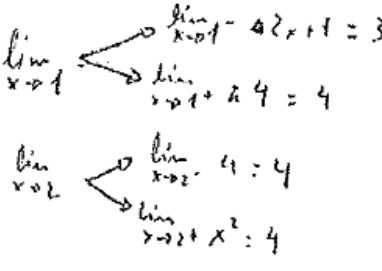
<u>Answer to Problem 1</u>	<u>Answer to Problem 2</u>	<u>Answer to Problem 3</u>
	<p>a1) x_1 is approaching to 1 from the left and from the right and x_2 is approaching to 1 from the left and from the right</p> <p>a2) The images of $f(x_1)$ are approaching to 2 from the left and from the right</p> <p>a3) The images of $g(x_2)$ are approaching to 2 from the right and is approaching to -1 from the left</p> <p>b1) when x_1 is approaching to 1, images of $f(x_1)$ tend to 2</p> <p>b2) when x_2 is approaching to 1, the images of $g(x_2)$ tend to -1 from the left and tend to 2 from the right.</p>	<p>a) Graph 3 because the limit of the function in $x=2$ from the right and from the left is 2.</p> <p>b) Graph 2 because the limit of the function in $x=2$ from the left and from the right is 5.</p> <p>c) Graph 1 because the limit of the function in $x=2$ is not the same from the right and from the left.</p>

Figure 2: Pablo's answers to the three problems

Data and analysis

Data of this study are prospective teachers' answers to the tasks of session 2 (anticipation) and sessions 3 and 4 (interpretation). Through an inductive analysis (Strauss & Corbin, 1994), we generated similarities and differences about how prospective teachers conceived high school students' reasoning of the limit concept and the type of activities they provided to help students progress in their conceptual reasoning. To carry out this analysis, five researchers analysed individually prospective teachers' answers to the anticipation and interpretation tasks and then, the agreements and disagreements were discussed to reach a consensus on these issues.

This analysis let us identify two ways of how prospective teachers conceived high school students' reasoning: as dichotomous (right or wrong) and as a progression (identifying different levels of students' reasoning). The type of activities that prospective teachers provided were categorised in three categories: general decisions, decision based on curricula contents and decisions based on cognitive processes. Examples of these categories are presented in the results section.

Finally, we compared categories obtained in the anticipation task with the categories obtained in the interpretation task to identify changes in the way of how prospective teachers conceived high school students' reasoning and proposed activities to help students progress in their conceptual reasoning.

Results

Our results show that prospective teachers changed the way that they conceived students' reasoning from a dichotomous to a progression way and this shift influenced the type of activities that they proposed to help students progress in their conceptual reasoning.

Changes in the way that prospective teachers conceived students' reasoning: From a dichotomous to a progression

In the anticipation task, three out of five groups conceived students' reasoning as dichotomous (right or wrong). For example, the group of prospective teachers G2 anticipated that a high school student with high level of reasoning of the limit concept (Maria) would coordinate in all modes of representation. For example, this group of prospective teachers anticipated the next answer for the algebraic representation:

The handwritten work shows a piecewise function $f(x)$ defined on the domain \mathbb{R} :

$$f(x) = \begin{cases} 2x+1 & \text{if } x \leq 1 \\ 4 & \text{if } 1 < x \leq 2 \\ x^2 & \text{if } x > 2 \end{cases}$$

Below the function, the student calculates limits at $x=1$ and $x=2$:

a) $x \rightarrow 1$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (2x+1) = 3$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 4 = 4$$

Since the left and right limits are not equal, the limit does not exist at $x=1$.

b) $x \rightarrow 2$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} 4 = 4$$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} x^2 = 4$$

Since the left and right limits are equal, the limit exists at $x=2$ and is equal to 4.

Maria understands the limit concept. The idea of approximation in the domain corresponds to the fact that she properly selects the branch of the function and uses the notion of approximation in the range adequately. It is demonstrated when she replaces on the limit the approach of the independent variable. This student also coordinates the approximations to establish the value of the limit according to the branch.

Furthermore, these prospective teachers (G2) anticipated that a high school student with a not suitable level of reasoning of the limit concept (Pedro) would not coordinate in any mode of representation pointing out: “Pedro only approximates (from the left or from the right) when the function is defined”.

Then (in the interpretation task), four out of five groups of prospective teachers were able to interpret students' mathematical reasoning. They linked students' reasoning with the mathematical elements of the dynamic conception of limit: the approaches from the right and from the left (in the domain and in the range), and the coordination of the approaches in the domain and in the range considering different modes of representation (graphical, algebraic and numerical). For instance, the group of prospective teachers (G2) interpreted the student's answer of problem 1 (Figure 2) as:

The resolution of the student is correct (Pablo). We can notice that the student has identified the kind of function (piecewise function) since he (the student) has approximated in the range (he has calculated the approximation to $x=1$ from the left and from the right and the approximation to $x=2$ from the left and from the right) and in the domain (taking the correct definition of function in each interval). Furthermore, he has coordinated the processes of approaching in the domain and in the range since he has written, for example, that when x tends to 1 from the left, the image of the function tends to 3 (using the function $2x+1$).

This group gave similar comments for the student's answers to the other two problems linking students' reasoning with the important mathematical elements in the other two modes of representations (problems 2 and 3). Afterwards, they wrote a summary about this student level of reasoning:

This student understands the limit concept since he approaches from the right and from the left (in the domain and in the range), and coordinates the processes of approaching in the domain and in the range in the three modes of representation (graphical, algebraic and numerical). This student would be in the high level of reasoning.

These prospective teachers were able to identify different levels of students' reasoning. Therefore, they conceived students' reasoning as gradual.

Changes in the type of activities they proposed to help students progress in their conceptual reasoning

Prospective teachers who conceived students' reasoning as dichotomous did not propose specific activities to help students progress in their reasoning. These prospective teachers gave general comments about teaching as instructional actions. For instance, the group of prospective teachers G2 proposed to Maria (in the anticipation task) the representation of the graph of the function of problem 1. This decision was not based on the conceptual progression of the student.

When prospective teachers interpreted students' mathematical thinking identifying different levels of students' reasoning (linking students' mathematical reasoning with the important mathematical elements), they were able to provide specific activities to help students progress in their conceptual reasoning. For the students who only coordinate the approaches in the domain and in the range in one mode of representation, they proposed new activities to integrate these mathematical elements gradually in the different modes of representation. The proposed activities required a coordination of approaches in the domain and in the range in the different modes of representation. For the students who coordinate the approximations in the domain and in the range in all modes of representation (such as the student of Figure 2), they also provided activities to help students progress in their reasoning.

We have identified two ways in which they justified their new activities: some justifications were based on the mathematical elements and others on the cognitive processes involved. In the first case, prospective teachers focused their attention on introducing a new mathematic content. In the following example, they introduced a new type of discontinuity – an avoidable discontinuity. The justification of this type of activities was based on the use of new mathematical elements (in this case, introducing other type of functions).

The activity: We would modify the function of problem 1 and we would use:

$$f(x) = \begin{cases} 2x & \text{si } x > 2 \\ 5 & \text{si } x = 2 \\ 2x & \text{si } x < 2 \end{cases}$$

Our justification: The student (Figure 2) seems to understand the limit concept in the three modes of representation, so we would provide him a more difficult function with an avoidable discontinuity.

In the second case, prospective teachers focused their attention on the cognitive processes involved to understand the limit concept. In the next example, prospective teachers focused on the reversal as a cognitive mechanism that leads students to a new reasoning level. That is to say, prospective teachers justified the proposed activity by the need of generating learning opportunities to develop the reverse mechanism that allows the construction of cognitive objects.

The activity: Represent a graph of a function which limit in $x=-1$ is 4 and that there is not limit in $x=1$.

Our justification: with this activity students need to do the inverse process that is, they need to use all the important mathematical elements to build that function.

Discussion and conclusions

Results show that after the participation in a learning environment that progressively nests the three interrelated skills of professional noticing (attending to, interpreting and deciding), prospective secondary school teachers gained expertise in noticing. Four out of five groups of prospective teachers were able to interpret students' mathematical thinking linking students' reasoning with the important mathematical elements of the dynamic conception of limit. These findings support the claim that some characteristics of the learning environment such as considering the nested nature of the skills help prospective teachers develop the skill of noticing (Sánchez-Matamoros et al, 2015; Schack et al., 2013).

Furthermore, prospective teachers were able to provide specific activities to help students progress in their conceptual reasoning. Therefore, the characteristics of the learning environment in which prospective teachers were engaged in the analysis of mathematical elements of limit problems, in the analysis of students' reasoning and in proposing new activities to support students' conceptual development enable them to gain more accurate understanding of the relation between the mathematical content and students' mathematical thinking. This new understanding provides prospective teachers with the needed knowledge to give their teaching decisions based on the progression of students' reasoning: from a mathematical point of view or considering the cognitive processes involved.

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Evaluation of a discovery, inquiry- and project-based learning program

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The program Math.Researchers (“Mathe.Forscher”) funded by the German foundation “Stiftung Rechnen” has the goal to implement discovery, inquiry- and project-based learning in mathematic school lessons. The students are meant to explore and discover mathematics in their every-day-life with the help of their teachers. The teachers are well educated in supporting their students by in-service education organized by the program. In addition to finding out if the in-service education is well established and if any correlation between the students’ or the teachers’ view on mathematics and the positive identification with the program is detectable, an evaluation was planned. The goal is to investigate if while taking part in the program Math.Researchers teachers’ beliefs concerning mathematics are influenced and if an effect on the students’ view on mathematics is detectable.

Keywords: In-service education; secondary school mathematics; discovery, inquiry- and project-based learning; beliefs.

Theoretical framework

The program Math.Researchers (“Mathe.Forscher”) funded in 2010 by the German foundation “Stiftung Rechnen” has the goal to implement discovery, inquiry- and project-based learning in mathematic school lessons. One motivation is that inquiry-based learning may increase the students’ mathematical achievement (Hattie, 2015). Bruner considers the benefit of learning through discoveries that one makes for oneself in the 60’s (Bruner, 1961). His discussed main benefits are similar to Winter’s arguments consisting the thesis that active discoveries by students themselves are the more effective way of especially learning mathematics (Winter, 1989, p. 1–3). The students are meant to explore and discover mathematics in their every-day-life with the help of their teachers. The program Math.Researchers supports the teachers in several sessions, with training, booklets and material for best-practice-activities. In addition a team of scientific assistants and process assistants support the program members. One example for a Math.Researcher activity is “Mathematics at the Zoo” (Ludwig, Lutz-Westphal, 2016). The pupils have to leave their classroom and visit the zoo to develop their own research questions there. The teacher supports the students’ ideas with mathematical knowhow. Back at school they work on their research questions and present their results. A reflection on the presented solutions completes this Math.Researcher activity. Although the teachers were supported in the terms of the program they were insecure whether their lessons were Math.Researchers-lessons. To improve transparency in this respect five special normative program-dimensions were developed by experts of the program in 2014 that can be used as a kind of checklist.

Math.Researcher-dimensions

Derived from the program goals, the following five Math.Researcher-dimensions were determined in 2014: using the Math.Researcher-principles (MRP), opening the classroom (OC), working with researchers’ questions (RQ), acting as a learning guide (LG), and visualizing mathematics (VM).

Each dimension consists of three main elements (Table 1). The more elements considered, the more Math.Researcher-like is the planned unit. The table can be used as a checklist so that the teachers are able to find out whether their planned lessons are conforming to the program. An ideal Math.Researcher-activity contains a minimum one element of each dimension.

Dimension	Main Elements		
Using the Math. Researcher Principles (MRP)	Inquiry-based learning	discovery learning	project-based learning
Opening the Classroom (OC)	interdisciplinary instruction	inviting experts	outdoor lessons
Working with Researcher's Questions (RQ)	including the students living environment	exercising asking questions with the students	allowing multiple approaches
Acting as a Learning Guide (LG)	enable active students role	constructive handling with students ideas	working out milestones together
Visualizing and presenting Mathematics (VM)	documentation of mathematics	using mathematical language	talking about ideas

Table 1: The five dimensions and its main elements

Considering the above-mentioned Math.Researcher activity's example "Mathematics at the Zoo" elements of all dimensions can be found here. The activity is project-based (MRP) and includes outdoor lessons at the zoo (OC). The pupils develop their own mathematical questions in an environment they usually do not connect to an educational context (RQ). The teacher enables student activities and supports them in mathematizing their ideas (LG). The phase of presentation and reflection at the end of this project represents the fifth dimension (VM).

To establish the dimensions a special PD course was conducted (figure 1). At a kick-off-meeting in December 2014 with all participating teachers the dimensions were introduced. The following months the teachers were meant to implement the dimensions in their teaching to get more familiar with them. So called process-assistance visited the teachers at school, watched the lessons, gave feedback. At the Math.Researchers' camp in May 2015 concrete activities were planned. 17 teachers met with program's scientific and process assistance to develop units focusing the dimensions. These units were conducted in May, June and July 2015. Every unit should be documented by a uniform document tool provided on the internet platform. The assistants analyzed these documentations and gave individual feedback.

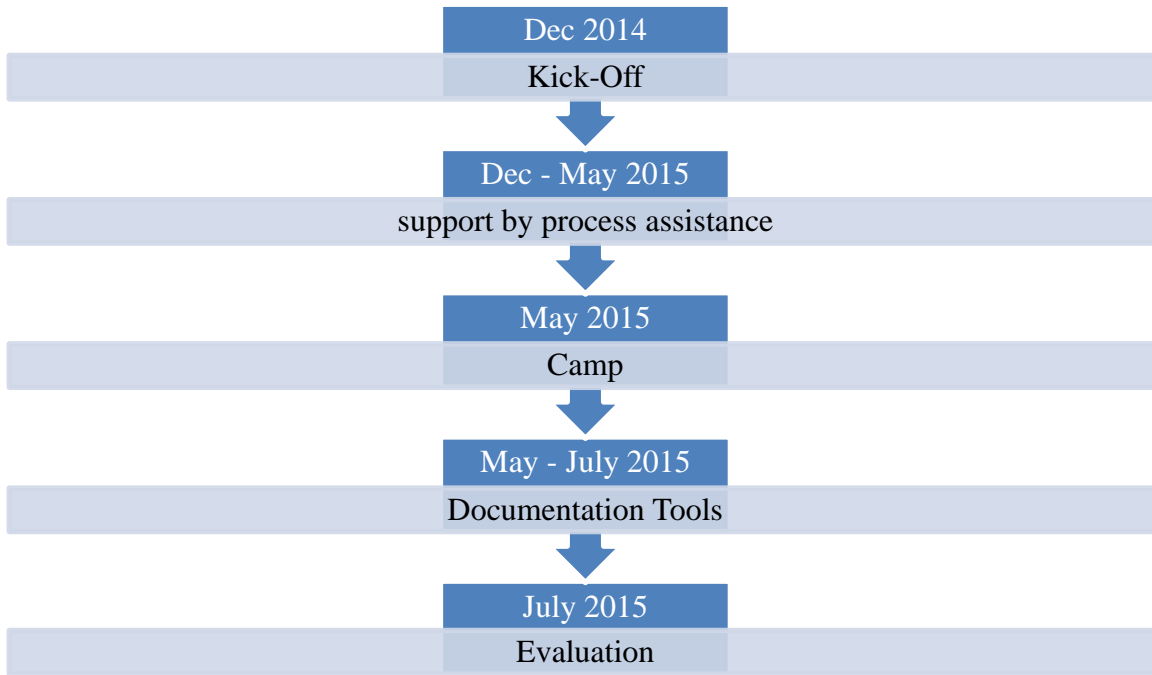


Figure 1: Timeline of treatments

While carrying out the Math.Researcher program with a special focus on the dimensions it was not sure whether and how the terms and ideas of the program were implemented in mathematical school lessons after May 2015. The dimensions can be seen as a kind of program characteristics. So a natural research question came up: Did the development and application of these dimensions succeed sustainable? That means on the one side that they are helpful for the teachers to plan their Math.Researcher lessons, and on the other side that the included main elements are recognizable by the students in the way the teachers planned it. Furthermore, we wanted to know if there is any effect on the view on mathematics of the teachers and also the students. By taking part in the program the students are meant to explore and discover mathematics in their every-day-life. Does this goal reach the students, or is the main opinion after they have taken part in the program that mathematics is still a school topic where you have to learn formulas and not necessary for every-day-life?

Presenting the design principles of in-service education by the German Centre for Mathematics Teacher Education (DZLM) Barzel and Selter (2015) reviewed the current state of research on in-service education with a special focus on mathematics. The effectiveness of in-service education is possible on different levels: Among other things there can be an impact on the acceptance of the training, an impact on the professional competencies and an impact on the classroom teaching (Barzel & Selter, 2015, p. 266). To improve transparency in the respect which impact can be found with the Math.Researcher-program an evaluation was planned with a focus on the following research questions:

- Are the dimensions well established, do the teachers identify with them and integrate belonging elements in their classroom teaching? **(1)**
- Which view on mathematics has been built up by the participants identifying with the Math.Researcher program? **(2)**

Therefore the dimensions were related to the mathematical beliefs. The mathematical world view of teachers (Törner, 1997) can be described by a belief system including four main aspects: the aspect of formalism (F), the schematic aspect (S), aspect of process (P) and the aspect of application (A). The aspect of formalism and the schematic aspect can be interpreted as a static view on mathematics education: mathematics as a system. The dynamic process character allows understanding facts, recognizing connections and building knowledge: mathematics is an activity. This belief system is also part of the competencies framework for in-service education of the DZLM (Barzel & Selter, 2015, p. 263).

The pilot study

A pilot study was conducted from May to July 2015 in the region Rhein-Neckar to find out whether the dimensions are well established and if any correlation between the students' or the teachers' view on mathematics and the positive identification with the dimensions are detectable. The program started in this region in 2012 in ten secondary schools. All respondents – students and teachers – of the pilot study have taken part in at least one Math.Researcher-activity within the last two years and especially in the PD course focusing the dimensions listed above.

Methods

The questionnaire contained belief-questions (taken from surveys by Grigutsch/Raatz/Törner, 1998), questions belonging to the program and its dimensions and some general questions (mostly developed by ourselves and taken from an earlier Math.Researcher evaluation in the region Nord conducted by the TU Berlin, Lubke et al., 2011). The teachers (n=20) answered an online-questionnaire containing 139 questions. The students (n=168) answered a paper print-version with 79 questions. The items were scaled with 1="totally don't agree" to 5="totally agree". The higher the scale of questions belonging to the dimensions the higher is the identification with the program. All questions were formulated to be understandable for students not familiar with the dimensions. For example the students were asked whether they liked talking about their own mathematical ideas, which can be related to the dimension visualizing and presenting mathematics (VM). Or if they sometimes do outdoor mathematics, which is related to the dimension opening the classroom (OC). Compared to that the teachers were asked whether their students' have to present own results at the end of a project (VM). Or if they sometimes do outdoor mathematics with their students (OC). The items were related to the dimensions by an experts' rating.

Additionally, some teachers (n=14, 8 congruent with the respondents of the questionnaire) and some of the students (n=31, all of them also filled a questionnaire) were interviewed. The interviews only contained questions about the program. For example both groups were asked to characterize Math.Researcher lessons by identifying features or describing which special Math.Researcher-activity they did.

Results

The analysis of the data (factor analysis and experts' rating) showed that the identification of the separate Math.Researcher dimensions was not possible. So in the following results the dimensions are considered in total, the belief aspects are separate. The five resulting factors formalism (F), scheme (S), process (P), application (A) and Math.Researcher-Dimension (Dim) can be seen as reliable (Cronbachs Alpha ,728 to ,800, Table 2).

	F	S	P	A	Dim
Number of Items	11	8	10	7	25
Cronbachs α	,728	,764	,798	,800	,769

Table 2: Number of items and Cronbachs Alpha of each main factor

Figure 2 shows the means of the four belief-aspects and the mean of the Math.Researcher-dimensions of all teachers in total. Comparing them, application (A), process (P) and the Math.Researcher-Dimensions (Dim) are rated similarly positive.

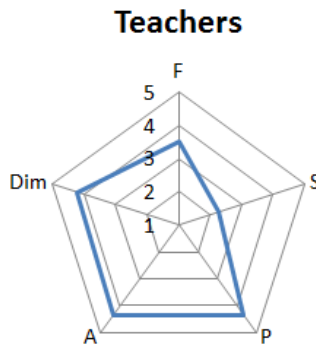


Figure 2: Means of the teachers' belief aspects and the M.R-dimensions in total (1= do not agree at all, 5= totally agree)

The exact scales can be seen in Table 2. The aspect of formalism (F) is rated not as positive as these, but considerably better than the scheme aspect (S). Scheme (S) is rated worse. (P) and (Dim) show a significant correlation of $r = .59$ so 35 % of the processes variance can be explained by the dimensions. The other beliefs cannot be explained significantly by the dimensions. Comparing the standard deviations of the beliefs and the dimensions in total shows that the teachers have rated the single items reasonably homogeneously (Table 2). The scheme aspect rated average worse is perceived the most ambivalent.

n=20	F	S	P	A	M.R-Dim
M	3,51	2,26	4,35	4,34	4,22
SD	0,44	0,56	0,42	0,51	0,35

Table 3: Mean and standard deviation of the teachers' belief aspects and the M.R-dimensions in total (1= do not agree at all, 5= totally agree)

Comparing the teachers' results with the students' a decline in the values of the scheme aspect is not recognizable (Figure 3). The five means do not differ that noticeably. Comparing the standard deviations (Table 4) shows that the students' answers differ a lot. The process aspect has the highest standard deviation of nearly 1.

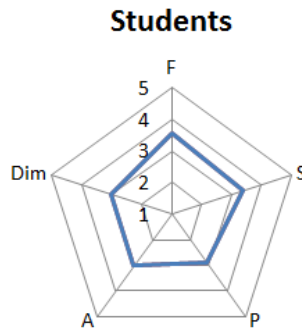


Figure 3: Means of the students' belief aspects and the M.R-dimensions in total (1= do not agree at all, 5= totally agree)

The process aspect has the highest standard deviation of nearly 1. The students' values are much more ambivalent than the values of the teacher's evaluation.

n=164	F	S	P	A	M.R-Dim
M	3,55	3,39	2,91	3,02	3,01
SD	0,69	0,68	0,91	0,85	0,47

Table 4: Mean and standard deviation of the students' belief aspects and the M.R-dimensions in total (1= do not agree at all, 5= totally agree)

The students were also asked whether they would recommend the Math.Researcher program. It is noticeable that the students who would recommend the program to friends and other students (values of 4 and 5) show a significantly higher identification with the Math.Researcher dimensions. The correlation of students' (P) and (Dim) is low but highly significant ($r = .43$).

In order to answer the question whether the dimensions are well established (1) the dimensions have also been considered separately. The analysis revealed that the dimension "working with researchers questions" (RQ) could not be identified separately. Items of this dimension (RQ) were assigned to the dimension "using the Math.Researcher principles" (MRP) or the dimension "acting as a learning guide" (LG). That's why in the following (RQ) is not included.

N=20	MRP	OC	LG	VM	RQ
M	4,26	3,56	3,99	4.43	-
SD	0,43	0,43	0,33	0,41	-

Table 5: Mean and standard deviation of the teachers' M.R dimensions separated (1= do not agree at all, 5= totally agree)

The dimensions "using the Math.Researcher principles" (MRP) and "visualizing mathematics" (VM) were high and homogeneously rated (Table 5). The identification with the dimension "opening the classroom" (OC) is the lowest. This assumption gets strengthened by the students' data. In interviews they were asked to characterize Math.Researcher lessons by conspicuous features. Compared to the teachers who are familiar with the dimensions, the students have not been educated in the Math.Researcher dimensions. Features belonging to the dimension (OC) with its main elements

“interdisciplinary instruction”, “inviting experts” and “outdoor lessons” were only rated by 10 % of the interviewed students. Features belonging to the dimensions (MRP), (LG) or (VM) were more often mentioned noteworthy.

Discussion

In summarizing these results we can say that the dimensions are well established (1). During the interviews the teachers often answered that the Math.Researcher program became more structured and concrete since the dimensions were introduced. They reported that the dimensions made them more certain about their planned lessons.

We can also say that a dynamic view on mathematics is more predominant if the participants identify with the Math.Researcher program (2).

However, the pilot study gives no data about a sustainable impact of the Math.Researcher program. So the researched questions of the evaluation of the pilot study were reused and extended into a main study. The main study still wants to investigate whether the dimensions are well established (1) and which view on mathematics is connected to the participants’ identification with the Math.Researcher program (2). Additionally, the main study wants to find out whether taking part in the program influences the students’ or the teachers’ view on mathematics (3). This third question is similar to previous research questions, for example of Cooper and Touitou (Cooper, Touitou, 2013). They wanted to find out, which beliefs can be found and how these identified beliefs change during a one-year PD course.

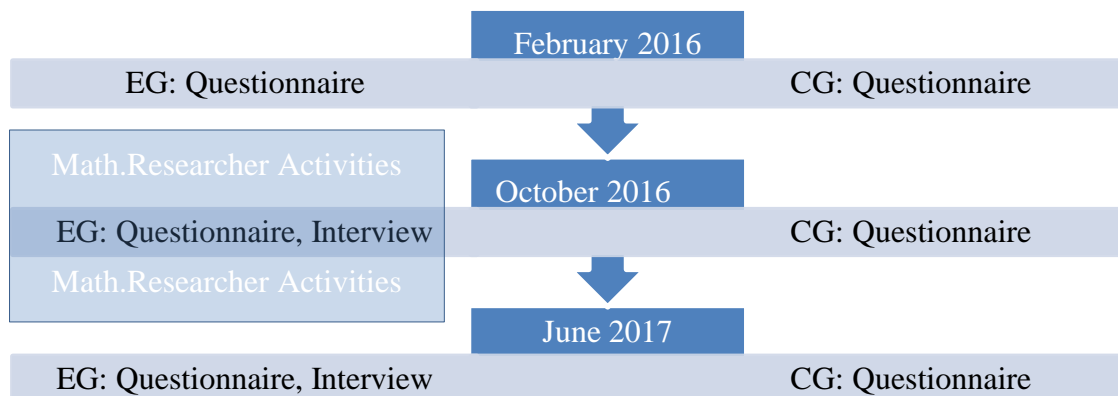


Figure 4: Three survey periods

The program-region Heilbronn-Franken joined in the Math.Researchers program in December 2015 and started activities in February and March 2016. Three survey periods from 02/2016 to 06/2017 are planned: one before, one during and one after taking part in the program (figure 4). The experimental group (EG) fills in a questionnaire in all of the three periods. In addition, teachers and students of the experimental group get interviewed in the second and third survey period. The control group (CG) gets the same questionnaire as the experimental group in all the three survey periods, but the control group will not be interviewed. If the control group and the experimental group show the same results at all of the three times of measurement, it is not possible to attribute any influence on the view on mathematics to the Math.Researcher program.

The questionnaires of the pilot study were shortened with the help of factor analysis and expert ratings. Items that were rated clearly by the experts, items with loadings exceeding ,39 and some

items of special interest stayed in the questionnaires. For example, the students' questionnaire had 79 content-related questions in the pilot study. Their questionnaire in the main study only has 29 content-related questions.

The next step is the second survey period in October 2016. First comparisons to the results of the first survey period and further details of the pilot study will be presented at CERME 10 in Dublin.

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Factors influencing the developing effectiveness of early career teachers' mathematics teaching – initial findings

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Early career primary school teachers in the UK enter a varied and complex situation in terms of professional development opportunities and support for their ongoing career progression as teachers of mathematics. Literature suggests that the professional development a teacher receives impacts on both their subject knowledge for teaching and their beliefs and attitudes to the subject. This multiple case study looks to fill gaps in the research undertaken in this area through detailed analysis of the personal perspectives of the participants over a two year period. This paper reports on the initial findings of a comparative analysis of the trajectory of two teachers to date. These teachers were in seemingly similar contexts for their first year of teaching, yet had very different experiences and held very different perspectives on their development.

Keywords: Teacher characteristics, professional development, elementary school mathematics.

Introduction

Students start a primary teacher education course in England with a range of qualifications in mathematics, a range of experiences in school mathematics and a range of attitudes and beliefs about the subject. During the course student teachers develop their subject and pedagogical knowledge and gain experience in mathematics teaching. They then enter a complex and changeable situation in schools in terms of provision for their ongoing development (Advisory Committee on Mathematics Education, 2013). Using a multiple case study approach, the aim of this study is to gain a deeper understanding of how the effectiveness of early career primary school teachers' mathematics teaching develops and what impacts on this, with a particular focus on each teacher's own perspective.

Theoretical background

The notion of effectiveness as applied to mathematics teaching is complex and ideas vary about what this looks like and even what its impact should be (Cai, 2007). Although generally agreed that teaching can only be considered effective if there is an impact on those being taught, i.e. effective teaching leads to effective learning and gains in understanding (Bryan, Wang, Perry, Wong, & Cai, 2007), the notion of 'understanding' is complex. Skemp (1976) made a clear distinction between instrumental understanding, simply knowing rules and procedures at a shallow level, and relational understanding which enables pupils to build conceptual schemas. Most mathematics educators agree that this second type of understanding is the most desirable and teachers internationally agree that an indicator of mathematical understanding is that pupils can use this understanding to problem solve flexibly in a range of situations (Byran et al., 2007). The significance of understanding the connected nature of mathematics is very apparent (e.g. Askew, Brown, Rhodes, Johnson, & Wiliam, 1997) and is consistent with the notion of relational understanding. The concept of effective learning in mathematics being based on this type of understanding within the context of appropriate intellectual challenge is endorsed by current policy makers and teacher inspection systems in the UK (Department for Education (DfE), 2013; Ofsted, 2012) and is the definition used in this study.

The literature suggests that teachers' effectiveness is impacted by their subject knowledge and by affective dispositions including beliefs, attitudes, motivations and emotions. Definition of these constructs are debated in the literature and indeed there are a range of ideas as to how they overlap and directions of influence (Lewis 2016). In agreement with Hannula (2011) I consider beliefs are both cognitively and affectively situated. As a starting point for discussions of attitude and emotion, I am adopting the three dimensional model of attitude from Di Martio and Zan (2010): emotional disposition towards mathematics, vision of mathematics (relational/instrumental) and perceived competence in mathematics.

Many studies, e.g. Ernest (1989) and Askew et al. (1997), have looked at teachers' beliefs about the nature of mathematics and the impact these beliefs have on their teaching. Ernest (1989), for example, argues the most effective teachers see mathematics as a dynamic, creative, problem solving subject and adopt a 'Facilitator' teaching approach.

Teachers' attitudes to the subject, including their emotional disposition, also impact on effectiveness. Particularly in primary teaching, teachers and student teachers find themselves teaching subjects that they have not necessarily enjoyed or been particularly successful at learning themselves. Many feel insecure with their subject knowledge in mathematics and recount their experience of mathematics at school as a subject that caused difficulties and even "real emotional turbulence" (Brown, 2005 p. 21). For many teachers therefore, mathematics is linked with negative emotions, particularly anxiety, and this can impact on their wider attitudes to it. It seems that teachers even protect pupils from mathematics and, in seeking to simplify it, emphasise the step by step procedures that are likely to lead pupils into developing instrumental understanding and potentially negative attitudes (Hodgen & Askew, 2007), reducing their effectiveness.

A further influence on the effectiveness of teaching of mathematics is subject knowledge. Indeed, evidence suggests that the impact of beliefs and attitudes and subject knowledge are interdependent (Askew et al., 1997). Most recent research on teachers' subject knowledge for mathematics uses Shulman's (1986; 1987) seminal papers as a starting point. Others have applied his ideas to mathematics teaching, debated which aspects are most essential and relevant, and sought to measure or evaluate them. Ball, Thames & Phelps (2008) argue that there is knowledge that is specific to teachers of mathematics and that might have an identifiable impact on the effectiveness of their teaching; Specialised Content Knowledge. This includes, for example, the understanding needed to be able to explain procedures, to analyse errors and strategies, and to consider appropriate examples. Baumert et al. (2010) conclude that pedagogical content knowledge (PCK) makes the greatest contribution to pupil progress, but weaknesses in mathematical content knowledge are not offset by greater PCK. Askew et al. (1997) found that it was not the formal qualifications or the amount of subject knowledge that the teachers had which was significant in the effectiveness of their teaching, but rather the connectedness of their subject knowledge "in terms of the depth and multi-faceted nature" of the meanings and uses of concepts in mathematics (p. 69). Other authors highlight the importance of pupil voice and the ability of the teacher to choreograph classroom discourse as key characteristics of effective teaching in mathematics (e.g. Schoenfeld, 2013), and Barwell's (2013) discursive psychology perspective emphasises that knowledge is contextual and can be changed or reconstructed accordingly. There is much critiquing of Shulman's and Ball's ideas but there seems

to be general agreement that the subject knowledge needed for effective teaching goes further than just having a strong conceptual knowledge of the subject being taught.

There is considerable evidence that a teacher's depth of reflection and their beliefs and attitudes towards mathematics are a crucial influence over their trajectory of development (Di Martino & Zan, 2010; Turner, 2008). Hodgen and Askew (2007) and Schoenfeld (2013) suggest that the development of some teachers is hindered by their linking of mathematics with emotion and that, for some teachers, professional development activities and goals should be much more about changes in their beliefs and attitudes than about improving their subject knowledge.

Teachers' work is within a social context, (Levine, 2010), and this can have a significant impact on the nature of a teacher's professional development. The community within the school might have a range of different foci and agendas and there seems to be a wide variation in practice between schools. The very structured collaborative approach to teacher development in China seems high effective although dependent on the considerable time given to teachers to discuss, prepare and analyse their work (Paine & Ma, 1993). In contrast Ball, Ben-Peretz & Cohen (2014) consider that in the US most teachers work in isolation and the potential benefits of sharing good practice are lost. In England, although the need for quality on-going professional development opportunities for primary teachers is recognised and highlighted by Ofsted (2012), the current context is of variable provision in formal professional development (ACME, 2013).

In summary, the literature suggests that a teacher's trajectory as a teacher of mathematics is influenced by the interaction of their beliefs and attitudes, their subject knowledge and the professional development they receive, through both formal education opportunities and personal reflection, and these factors therefore influence the effectiveness of their teaching and the effectiveness of their pupils' learning. My study sits within this theoretical framework and seeks to extend the existing literature particularly though highlighting teachers' own perspectives on this process.

Methodology

A multiple case study approach is being employed to follow the trajectories of a small sample of teachers as they progress into their first two years as a qualified teacher. Four participants were chosen for the pilot study, two with mathematics qualifications beyond the minimum required for primary school teachers in England. An initial interview at the end of their one year postgraduate teaching course focused on their relationship with, and attitude to, mathematics and their progress in teaching the subject as a student teacher. To facilitate discussion, Lewis's (2016) idea of a graphic display was adopted; participants were asked to draw and explain their relationship with mathematics over time. This gave insights into participants' attitudes to mathematics and their perspectives as learners of the subject as well as in their student teacher role. Participants were able to reflect on how their relationship with the subject influenced their current teaching approaches. Twice yearly interviews, including further graphing at the end of each year, and discussion of documentation related to their progress as early career teachers, such as formal observation feedback, provide evidence of their ongoing development as teachers of mathematics. Interview questions have been designed to probe about the participants' beliefs, attitudes and subject knowledge for teaching mathematics, what they perceive to be the characteristics of effective teachers of mathematics and their perspectives on their development as teachers of the subject. Within each interview they also describe two particular

lessons: their chosen ‘best’ and ‘most challenging’ lessons since the previous interview, providing insights into what they consider to be effective and ineffective teaching and their subject knowledge. In addition, they keep records of their professional development in teaching mathematics.

To begin to analyse the data, a mind map has been drawn of each data collection point for each participant. The mind map contains evidence of the participants’ changing context and development as a teacher of mathematics, including both factual and non-factual information, and also the participant’s interpretation and my interpretation of this evidence. The mind maps allow for a thematic approach, enabling identification of concepts and themes such as ‘awareness of self as a mathematician’ and ‘priorities in teaching mathematics’. In regard to their relationship with mathematics and their development as teachers of the subject, the participants have stories to tell and thus narrative analysis techniques are being used to support analysis of these stories. Whilst these techniques have enabled useful analysis of the early findings, the next stage is to further research and develop these analysis methods so that data can be analysed increasingly effectively and efficiently.

Results and discussion

In this paper the focus is on the early findings of a comparative analysis of the trajectory in the first year of the study of two of the teachers, Gina and Rama (pseudonyms), who in their first year as qualified teachers were in seemingly similar contexts: they both taught children aged 5-6 years in schools with a two class entry, working alongside more experienced colleagues and teaching mathematics to their own, mixed attainment, classes. Rama has a stronger mathematics background and also chose to study a mathematics specialism as part of the teaching course, but both students achieved the highest teaching course grade in all areas of the Teachers’ Standards (DfE, 2011). In their interviews, both were able to identify many ways in which they had evolved as teachers of mathematics to date, but they answered questions in ways that revealed very different perspectives on their development. In addition, their professional development records indicated that they received very different opportunities and approaches to their professional development. This raises questions about why these are so different and the extent to which this is school dependent or dependent on the approach and philosophy of the teachers themselves.

Descriptions of each participant’s relationship with mathematics and how this had evolved over time revealed interesting and complex relationships (Figure 1) and it was clear that past experiences impacted their current thinking. Whilst the graphs drawn are not directly comparable as the interpretation of the vertical scale was left open, the trajectories illustrated and discussion of these provide some scope for comparison. The two peaks in Gina’s relationship with mathematics indicate two different perceived aspects of success in mathematics. Firstly she recalls being successful in the subject in Year 9 (aged 13-14) when she felt she responded well to the high expectations her teachers had of her. The second period of success, during the teaching course, was due to her own development of the conceptual understanding that the literature suggests is essential for learners to gain for long term, secure understanding of mathematics. This enabled her to reflect on her previous period of ‘success’ which she then realised was based on superficial learning:

Gina: I could do the methods, but I didn’t understand them.

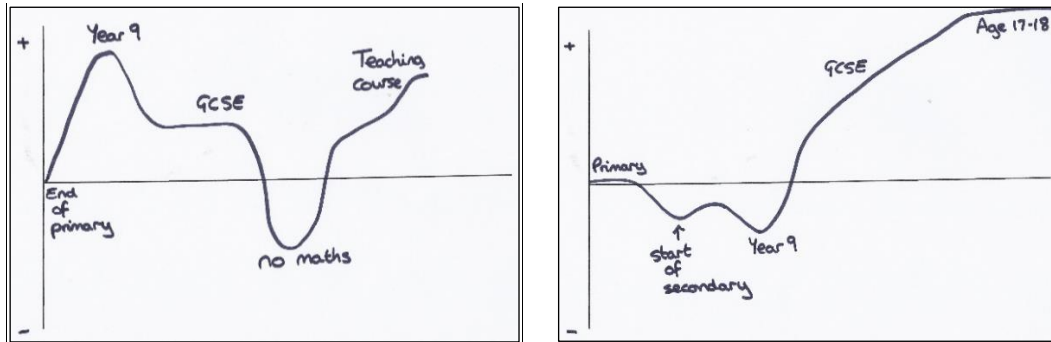


Figure 1: Gina and Rama's relationship with mathematics, drawn at end of teaching course

Gina's personal experience seems to have given her a particular focus in her own teaching of mathematics, particularly how strategies she is aiming to use in her teaching could prevent the children taking the path of superficial learning that she had followed. Discussing how she developed her teaching in her first few months as a qualified teacher, for example, she identified the use of representations as significant and something that she perceived as missing in her own learning:

Gina: And the pictorial things, definitely. I don't think I've ever done that as a maths learner myself and I think maybe if I had it would have been much more easy.

Rama's mathematical background also seems to have impacted on her priorities in teaching mathematics. Although she finished her studies with a degree where she used mathematics, she finished primary school not enjoying the subject and lacking in confidence. She seems to have been a shy and hesitant learner, fearful of getting the wrong answer in a subject she saw as right or wrong. At secondary school she was initially placed in lower sets but talked with great enthusiasm about her Year 11 class (aged 15/16), a top set, where the environment of the classroom was such that she felt she could make mistakes without fear. She described too a change at this point from memorising how to do certain methods to taking ownership of her learning and finding her own ways of solving problems. Priorities identified by Rama in her teaching include the importance of children enjoying mathematics alongside gaining conceptual understanding, both aspects that for her were missing until the later stages of her schooling.

It seems that Gina and Rama entered two very different teaching communities with different agendas which also impacted on the nature of their professional development (Levine, 2010), and graphs drawn at the end of their first year of teaching revealed very different perspectives on their year (see Figure 2). Rama's saw her journey as a teacher of mathematics as a smooth upward trend, with occasional dips when she taught poorer lessons. Whilst having few formal professional development opportunities, she indicated that she is confident in her own subject knowledge and in independently planning and teaching the subject. Gina described a much more structured and intensive programme with regards to the support she received. However, her graph illustrates her perspective that she has had a sometimes difficult year and narrative analysis revealed, interestingly, that the intense involvement of her school in her professional development seems to have led to a crisis in her confidence and a turbulent second half of her first year of teaching.

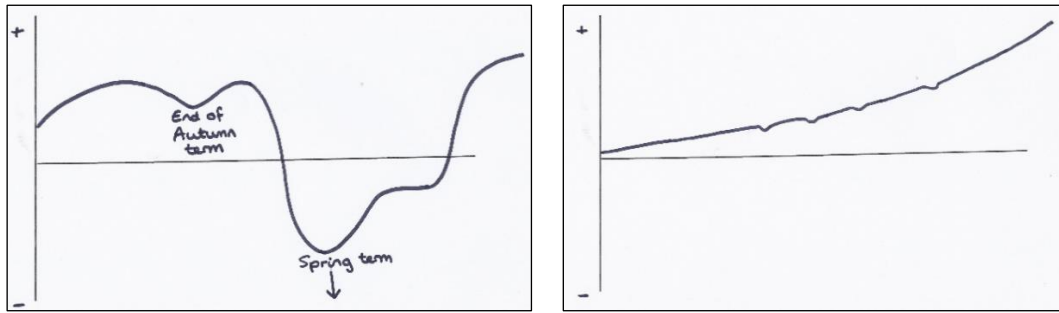


Figure 2: Gina and Rama’s relationship with mathematics during their first year of teaching

This analysis highlights the emotional impact on Gina of the series of intervention events that followed on from a formal observation of her teaching that she labelled several times as a ‘disaster’, and then her subsequent recovery. In her story particular emphasis is evident of the impact on both the perceived competence and emotional disposition aspects of her attitude (Di Martino and Zan, 2010), with a consequent impact on her confidence in her ability to teach mathematics:

Gina: It seemed no matter what I was doing to change, I was still getting negative feedback and it was disheartening. There was a big period of time when I literally dreaded every single lesson, because you just think “What could happen? I don’t know what I’m doing”

The records of professional development kept by Gina and Rama over the year confirm that Rama’s in school support has been much more informal and she places a high priority on targets she generally sets for herself, seeking to improve the quality of her teaching through a reflective approach (Turner, 2008). In contrast to Gina’s record of events, her record is a series of targets. Contrasts in their priorities in planning and teaching of mathematics, and indications of their subject knowledge have been evident through different ideas about the characteristics of effective teachers and descriptions of their best and most challenging lessons. It seems that both teachers are seeking to teach with a relational approach, with Rama at this stage in a stronger position regarding subject knowledge. It is interesting that end of year assessment information indicates that children in both Rama and Gina’s classes progressed well in their mathematical understanding over the year; this seems to have contributed to Gina finishing the year in a more confident frame of mind.

Conclusion

The findings of these case studies to date confirm my analysis of the literature in that there are two related and interwoven, but distinct, categories of factors that might influence an early career teacher’s trajectory in relation to the effectiveness of their mathematics teaching - those that are related to the teacher themselves, in terms of beliefs, attitudes and subject knowledge and those that are related to their teaching context. Gina’s experience during her first year of teaching provides a particularly interesting example of professional development that impacts in complex ways on beliefs and attitudes to the subject. Although it is too early to draw more than very tentative conclusions, it is interesting to ponder to what extent Gina and Rama’s trajectories would have been different had they been in one another’s school, and this leads to potential implications for initial teacher education.

These case studies align with current professional documentation (ACME, 2013) to suggest that there is a lack of uniformity in teachers’ professional development experiences despite there being systems in place in the UK promoting uniformity. The existence of these issues has implications for the

preparation of student teachers in Initial Teacher Education (ITE) and suggest that ITE providers could seek to more explicitly discuss differences and similarities in primary school approaches to teacher development and further strengthen their policies of students gaining experience in a range of contrasting placements, within which they can not only participate in whole school professional development events, but also talk with teachers about their development experiences.

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A theoretical framework for analyzing training situations in mathematics teacher education

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The thoughts on the primary schoolteacher training have led to the production of many resources for primary schoolteachers. Faces of the abundance of such documents, teacher educators need some tools to identify the knowledge potentially at stake in training situations and to allow them to implement such situations according to their own objectives and context. We present a five-level analysis framework that characterizes the training tasks, taking account of the activities induced by the task, according to the expected posture of the prospective teacher, to the type of the knowledge at stake and to possible degrees of decontextualization. We illustrate this analysis framework by presenting an example of a training scenario based on the principle of role-play.

Keywords: Teacher education, professional development, primary education, knowledge for teaching, analysis framework.

Introduction

The research about primary schoolteacher education in mathematics and professional development has led to the production of many resources for educators. In France, the COPIRELEM¹ group produced many of them. These resources provide “training situations” based on various training strategies (Houdement & Kuzniak, 1996), and are generally accompanied with information about their implementation (phases, steps, instructions, elements of institutionalisation) with regard to the stakes of the training. But their quality does not guarantee an accurate appropriation by teacher educators. Our questioning is: how is it possible to help teacher educators to exploit any “training situations” in a relevant way, according to their objectives?

The research literature usually provides studies about knowledge for teaching, teacher conceptions, and their evolution (Shulman, 1986; Houdement & Kuzniak, 1996; Ball, Thames, & Phelps, 2008). Other studies present one training situation, and generally focus on its effect on the prospective teachers. For example Horoks and Grugeon (2015) “analyse the contents and methods of an initiation course in research in mathematics education, and [...] how it can influence the beginner teachers’ practises” (p.2811). To our knowledge, no study focuses on the characteristics of training situations nor provides specific framework in order to analyse any training situation. This led us to develop an analysis framework for training situations. The paper presents this COPIRELEM’s work in progress.

¹ The COPIRELEM is a commission dedicated to the education to the primary school. It is stemming from the network of IREM (French institute of research on mathematical education).

Presentation of the analysis framework

We define a *training situation* as a situation that involves prospective teachers (students, pre-service or in-service teachers) and educators within an institution of teacher education. It is composed of a set of tasks that could be conducted by a teacher educator. We take into account all the tasks proposed by the teacher educator. From a training situation, the educator may elaborate a training scenario that is to say a set of chronologically organized tasks chosen among all the tasks that constitute the training situation. We voluntarily distinguish situations from “scenarios” because we intend to underline the dynamic aspect of the scenario.

In response to each task of a training situation, prospective teachers develop an *activity* that corresponds to “what [they engage] in during the completion of the task” (Rogalski, 2013, p.4). We distinguish five different types of activity: “mathematical activity” (doing maths during the completion of a mathematical task), “mathematical analysis activity” (analysing the maths at stake in a mathematical task), “didactical and/or pedagogical activity” (highlighting didactical and/or pedagogical choices related to the mathematical task), “didactical and/or pedagogical analysis activity” (analysing didactical and/or pedagogical choices related to the mathematical task), “problematisation activity” (identifying and investigating professional issues by mobilizing mathematical, didactical and pedagogical concepts). For each type of activity we take into account three dimensions (Fig. 1): the type of knowledge at stake; the degree of decontextualization of this knowledge; the posture of the prospective teachers expected by the teacher educator. These dimensions are specified in the next sections.

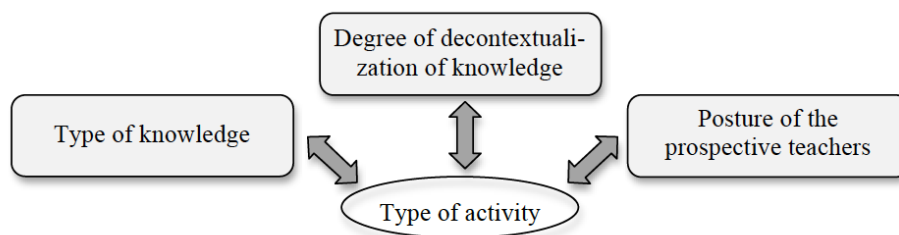


Figure 1: Three dimensions for characterizing a type of activity

Three types of knowledge

We rely on the three types of knowledge for teaching mathematics identified by Houdement and Kuzniak (1996): mathematical knowledge, pedagogical knowledge and didactical knowledge. “Mathematical knowledge corresponds to mathematics that a teacher needs to know in order to prepare, regulate and evaluate his lesson and his students” (Houdement, 2013, p.12). It “includes and specifies the content knowledge” identified by Shulman (1986). It roughly can be related to Subject Matter Knowledge (Ball and *al.*, 2008), and the specific didactical nature of mathematical knowledge can be identified to the Specialized Content Knowledge (SCK). According to (Houdement, 2013), didactical knowledge is linked to the mathematical content and fed by research in the field of mathematics didactics. It corresponds to analysis of teaching and learning phenomenon and to propositions of engineering. Therefore it can be associated with at least two categories (Ball and *al.*, 2008): Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT).

Pedagogical knowledge² is characterised as “knowledge of experience” (Portugais, 1995). It is related to teaching and learning conceptions and to the organisation and management of the class. It is less dependent of the mathematical content than other types of knowledge. It is important to take this knowledge into account because schoolteachers deal with various school subjects.

Three degrees of decontextualization

Brousseau (1997) and Douady (1985) identify three degrees of decontextualization of a mathematical knowledge: implicitly mobilized, explicitly mobilized in context or decontextualized (to become available in other contexts). We extend this notion to didactical and pedagogical knowledge. A mathematical knowledge is (*implicitly*) *mobilized in context* (in act) if it is used as tool (Douady, 1985) in a mathematical task. This task can be carried out: what is asked is effectively achieved (manipulation, elaboration and writing a solution for example). But the task can only be evoked: it is mentally achieved. A mathematical knowledge is *explicit in context* if its use (as tool) is identified and formulated. At least, a mathematical knowledge is *decontextualized* if it is identified as an object of learning: a status of object is given (by the educator) to the concept used previously as tool, usually during an institutionalisation phase³ (Brousseau, 1997). The didactical/pedagogical knowledge is *mobilized in context* when the didactical/pedagogical choices are made for the considered mathematical task. It is *explicit in context* during the analysis about the consequences of these choices. At least, it is *decontextualized* when the underlying didactical/pedagogical concepts are highlighted.

Four postures of the prospective teachers

In conjunction with the teacher trainer’s relationship to the prospective teachers identified by Sayac (2008), we define four specific postures of prospective teachers, which are expected by the educator during a training situation⁴. Prospective teachers are in a posture of *student* relatively to the mathematical knowledge when they have to perform mathematical activity or when they are concerned with the mathematical knowledge of this activity. They are in a *student/teacher* posture when they investigate mathematical tasks for students or students’ works, or when they analyse the conditions of implementation of a task in the classroom. They are in a *teacher* posture when entering in a broader questioning on classroom practices and issues of mathematical learning. Finally, they are in a *practitioner/researcher* posture when they problematize a professional issue related to mathematical learning or teaching.

Five study levels

In order to analyse a training situation, we define five study levels. To each level corresponds a type of activity, that induces (implicitly or explicitly) a posture of the prospective teacher (expected by

² According to (Houdement, 2013), Ball’s, Phelps’ and Thames’ typology doesn’t seem to take into account this type of knowledge.

³ In institutionalisation phase (Brousseau 1997), the teacher gives a cultural (mathematical) status to some knowledge emerging from students’ actions during the situation.

⁴ We notice that prospective teachers are not always aware of these postures.

the educator), and that involves different types of knowledge in a certain degree of decontextualization (see Fig. 2).

<i>Study levels</i>	<i>Type of activity</i>	<i>Posture of prospective teachers</i>	<i>Type of knowledge</i>	
			<i>Mathe- matical</i>	<i>Didac- Pedago- tical gical</i>
Level 4	Problematisation activity, related to professional issues classroom practices, learning issues and/or didactical analysis tools)	<i>Practitioner/ researcher</i>	Decontextualized	Decontextualized
Level 3	Didactical and/or pedagogical analysis activity, related to the activity of level 2	<i>Teacher</i>		
Level 2	Didactical and/or pedagogical activity, related to the activity of level 0	<i>Student/teacher</i>		Explicit in context
Level 1	Mathematical analysis activity, related to the activity of level 0	<i>Student/teacher Student</i>	Mobilized implicitly or explicitly in context	Mobilized implicitly in context
Level 0	Mathematical activity (action performed effectively or mentally)	<i>Student</i>		

Figure 2: Characteristics of the five study levels

Level 0. A task may induce a mathematical activity. This activity can be performed or evoked (mentally performed). The mathematical knowledge is mobilized (implicitly or explicitly) in context. The prospective teachers are in a posture of student (relatively to the mathematical knowledge).

Level 1. A task may induce a mathematical analysis activity related to the activity of level 0 when it highlights decontextualized mathematical knowledge and the prospective teachers are in a posture of (learning mathematics) student. In this task, the didactical and/or pedagogical knowledge can be implicitly mobilized in context and then initiates the change toward a student/teacher posture of prospective teachers.

Level 2. A task may induce a didactical and/or pedagogical activity related to the activity of level 0 when it corresponds to the analysis of implementation conditions - actual or anticipated only of the mathematical task. The didactical and pedagogical knowledge is explicit in context. The prospective teachers are in a student/teacher posture.

Level 3. A task may induce a didactical and/or pedagogical analysis activity related to the activity of level 2 when it is for example a questioning on classroom practice (specific learning tasks, professional actions...) or on issues of mathematical learning for one or several contents (curriculum, progressions...), or even a highlighting of didactical analysis concepts (didactic situation phases, types of tasks...). This analysis leads to the decontextualization of didactical and/or pedagogical knowledge. The prospective teachers are in a posture of teacher.

Level 4. A task may induce a problematisation activity when it corresponds to the problematisation of professional issues related to classroom practices, learning issues and/or didactical analysis tools.

The prospective teachers are in a posture of practitioner/researcher, especially when it comes to developing an analysis methodology of this issue and to infer results.

Each study level is based on the study of the activity of previous levels and involves some mathematical, didactical and/or pedagogical knowledge. The change from study level n to study level $n + 1$ is linked either to a change of the prospective teachers' posture or to a change of degree of decontextualization for at least one type of knowledge (from implicitly mobilized in context to explicit in context, from explicit in context to decontextualized). But the different activities induced by a training situation don't usually appear in a chronological order (from level 0 to level 4). For examples, see the analysis of various training situations developed in French context by the COPIRELEM group (Guille-Biel Winder, Petitfour, Masselot & Girmens, 2015; Bueno-Ravel and al., 2017). We think that the analysis could be extended to situations based on different training strategies. That is why we present here the analysis of a training scenario based on the principle of role-play developed in an international context (Lajoie and Pallascio, 2001; Lajoie and Maheux, 2013; Lajoie, accepted).

An example of use of the analysis framework

Definition of role-play

As Lajoie and Pallascio (2001) state "role-play involves staging a problematic situation with characters taking roles". It is used over many years in mathematics education course in UQAM (University of Québec in Montréal) and is organized as follows:

First, the 'theme' on which students will need to role-play is introduced (introduction time). Second, students then have about 30 minutes to prepare in small groups (preparation time). Third comes the play itself (play time), where students chosen by the educator come in front of the classroom and improvise a teacher-student(s) interaction (sometimes, like in the case reported here, involving the whole class). Finally, we have a whole classroom discussion on the play (discussion time). (Lajoie, accepted)

We designed a role-play on the teaching of proportions based on a problem from a textbook. We use the analysis framework to illustrate an example of analysis aimed at highlight the potential of this situation.

An example of role-play

The role-play presented below is intended for pre-service schoolteacher education. We describe the different phases.

Introduction time. The educator distributes to prospective teachers an excerpt from a fifth grade (10-11 year old pupils) handbook presenting a problem of proportions (Fig. 3), and various productions of pupils. The teaching issue announced by the educator is the following: to manage a class discussion about the pupils' strategies and about their ideas and solutions, in order to share them in the class community and to determine their validity and efficiency.


Preparation time. The prospective teachers have to prepare the discussion class about the pupils' strategies.

Play time. At the end of the preparation time, the educator chooses prospective teachers to play the game: some of them play pupils, one of them plays the teacher, while the others are watching the discussion class and taking notes.


Discussion time. The debate intends to highlight and to analyse the choices of the « teacher » during the play game: what worked well during the implementation of the discussion class? What was difficult? What seemed to be important? What alternative implementations could be realized?

*Institutionalisation time*⁵. The educator institutionalizes the knowledge at stake: he generalizes some elements about how to manage a discussion class or about proportion problems solving.

Research




Some chocolate for everybody




Here is the part of chocolate taken by each child:


Figurine




Millie



Logix



Decimus



1. How much weights the part :

a. of Figurine?
b. of Millie?
2. How much weights the part :

a. of Logix?
b. of Decimus?
3. How much weights the remaining part of chocolate ?

Figure 3: A proportion problem

Analysis of this role-play

The initiating task is a professional situation and corresponds to a level 2 activity: the prospective teachers are initially in a student/teacher posture. But they will need « to go down » to a student posture and « to go up » to a teacher posture during the phases of the scenario. The preparation time of the discussion class leads the activity of the prospective teachers “to go back and forth” to the study levels 0, 1 and 2. The problem solving corresponds to a level 0 activity and the mathematical analysis of the problem solving to a level 1 activity. Moreover there are various strategies to solve this proportion problem. Preparing the discussion class of the pupils’ strategies (level 2 activity) hence needs to analyse and rank them (from the least to the most elaborate). This analysis corresponds to a level 2 activity. The prospective teachers don’t have the same activity during the playtime. The study level is different according to the role to play: mostly levels 0 and 1 for the students’ roles and level 2 for the teacher’s role. The discussion time corresponds to a level 2 activity when the prospective teachers analyse how the discussion class has been managed. But it can also correspond to lower levels activities, when they discuss about pupils’ strategies, difficulties, mistakes and their exploitation during the discussion class. Various institutionalisations can be considered, according to the knowledge that was developed at different study levels. The institutionalized elements will be more or less developed according to the teacher educator’s objectives and progression, the prospective teachers’ knowledge, etc. Here are some propositions organized in ascending order of study levels. The teacher educator can institutionalize some

⁵ We add this new time to the four ones proposed by Lajoie and Pallascio (2001).

mathematical knowledge at stake (level 1) and related to the proportionality field: various methods to solve a proportion problem, mathematical justifications and mathematical theories they are relied on. He can situate the proportion problems in the more general category of multiplicative problems, or he can explicit some didactical variables usually at stake in proportion problems (level 3). He also can identify some difficulties or mistakes revealed by the pupils' productions as « usual » and highlight mistaken conceptions: identification of quantities, choice of an adapted strategy, persistence of an « additive model », etc. At least, in regard of the announced objective of the role-play, the teacher educator also can institutionalize some didactical knowledge, relatively to the organization of a discussion class (level 3): formulation and validation in mathematics; teacher's tasks before, during and after the discussion class...

Conclusion

The example of role-play situation shows how the analysis framework can be a tool for an *a priori* analysis. Moreover this example shows that the organization of the study levels is not a chronological but a hierarchical one: the initiating task can induce an activity of level 0, 1, 2, 3 or 4. But the transition to lower levels activity is often necessary. The conceptual maps of the knowledge for teaching developed by Houdement and Kuzniak (1996) or by Ball and *al.* (2008) have a descriptive, predictive and prescriptive dimension (Ball & *al.*, 2008, p.405). But beyond their interest, (Houdement, 2013, p. 21) stressed the importance of the knowledge reconfiguration in connection with the mathematical content. The analysis framework reports how, during a training situation, the types of knowledge for mathematics' teaching are dynamically hinged to one another in connection with the mathematical content. The analysis framework allows teacher educators to identify the potentialities of a full range of training situations. We intend to extend its use to study other types of training situations (for example e-learning situations). By clarifying the stakes of the various phases of the implementation, the analysis framework reveals various possible strategies for the teacher educator. Thereafter it could be a useful tool for elaborating different training scenarios. Hence, the teacher educator should be able to implement situations in a specific context according to his objectives and constraints (time and period of training, place in a progression which take into account the mathematical, didactical and pedagogical knowledge ever studied...). Besides it is possible to consider a sequence of successive scenarios. The analysis framework could also highlight various possible "training paths", which should reveal the educator's training strategy at a more global scale. A perspective is now to study how teacher educators appropriate this framework and how it supports their teaching practises.

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Changes in beliefs of teachers on a PD course after reflecting on students' learning results

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In this paper, we discuss results of a qualitative part of a research project aiming to investigate the impact of a professional development course with a specific focus on reflecting of students' learning results on teachers' beliefs towards teaching and learning of mathematics. We refer briefly to some aspects of teachers' professional development. Afterwards, we discuss teachers' beliefs as the main theoretical construct for our research. Data for this paper about teachers' beliefs were collected by semi-structured interviews with three teachers after the professional development course. Results show that teachers report in interviews changes referring to teaching and learning of mathematics from a transmission-oriented view to a more constructivist-oriented view.

Keywords: Beliefs, belief change, professional development, reflecting students' learning results.

Introduction

There is a consensus that university studies and internships are not enough to prepare future teachers for all challenges in their professional career (Mayr & Neuweg, 2009). For this reason, professional development (PD) is understood as being a key factor for innovating and reforming mathematics teaching in school (Garet et al., 2001). However, every change seems to be dependent on specific characteristics of PD: Desimone (2009) summed up five key features of high quality teacher PD. She described that PD will be effective, if a PD course is content-focused, enables active learning, is coherent, has a critical duration and if teachers take part in a PD course collectively. In addition, Franke et al. (1998) reported that reflecting students' learning results makes teachers' learning sustainable. Reflecting students' learning results means, that teachers collect in distance phases of a PD course students' results when working in a specific learning environment or working on a specific task developed in the PD course. Afterwards these students' learning results were the basis of the next face-to-face meeting in the PD course.

Although research identified several features of PD, in most cases the effectiveness is not clear (Yoon et al., 2007). In addition, it is not clear how they influence teachers' learning. For example, Franke et al. (1998) could show that a mix of several characteristics including reflecting students' learning results is effective for teachers' learning, but they did not focus on reflecting students' learning results as a single variable. In a qualitative design, for example also Strahan (2003) found that teachers reflecting on students' learning results increased students' achievement on elementary school level. In addition, Schorr (2000) showed that students' achievement increases when teachers completed a PD including the analysis of students' problem solving processes. Like a conclusion, Little et al. (2003) state that reflecting students' learning results has the potential to bring students more explicitly into deliberations of teachers. However, more research is needed to understand how teachers learn from reflecting students' learning results and whether it impacts on teachers' professional competence including particularly teachers' beliefs.

To investigate the efficiency of reflecting students' learning results in PD courses on the teachers' knowledge, beliefs or motivation, the study as a whole considers two PD courses with a quasi-

experimental setting. Content, teacher trainer and learning time were mostly the same in both courses. However, we integrated reflecting students' learning results in the first PD course, but not in the second. In this paper we will not refer to quantitative results of our research that we reported elsewhere (e. g. Hahn, & Eichler, 2016). We further do not refer to differences between the effects of the two PD groups, but we emphasize results of a qualitative interview study including three teachers of the PD course with reflecting students' learning results. In these interviews, we primarily refer to the teachers' beliefs concerning the teaching and learning of mathematics.

Beliefs about teaching and learning of mathematics

Beliefs can be defined as “psychologically held understandings, premises, or propositions about the world that are thought to be true. [...] Beliefs might be thought of as lenses that affect one's view of some aspect of the world or as dispositions toward action.” (Philipp, 2007, p. 259).

For our research a crucial question is whether beliefs could be viewed to be stable or changeable. Partly, researchers use stability as a part of their definition of beliefs (Fives & Buehl, 2012; Liljedahl, Oesterle, & Bernèche, 2012). In contradiction to that, there are studies which demonstrate belief change by a special intervention. For example, Decker, Kunter, and Voss (2015) reported changes referring to teaching and learning of preservice teachers and teacher trainees. Liljedahl et al. (2012) analyzed related literature and conclude that there is a different meaning of stability in research studies. As a result, they propose to avoid stability in the definition of beliefs. For them, belief change is a natural process that requires a sufficient extent of time. For this reason, in this research study beliefs are described as changeable.

A further crucial question concerns the definition of the belief object that should be changed by PD. Beliefs can refer to a special subdomain of mathematics (Eichler & Erens, 2015), to mathematics itself, teaching and learning of mathematics or an overachieving orientation that is independent from a subdomain (Staub & Stern, 2002). Beliefs referring to teaching and learning of mathematics can be divided into two paradigms: the transmission-oriented and the constructivist-oriented view of learning (Fives, Lacatena, and Gerard, 2015). Transmission-oriented beliefs of teaching imply that knowledge is directly transmitted from teacher to the learners and learners absorb all information. In this case, learners are passive recipients. For this reason, teachers' role is to prepare all information for students to enable an effective storing and an optimal recall. In contradiction, the constructivist-oriented view of learning reflects the active role of learners as constructors of their own knowledge structures. In this case, students learn new information based on their existing knowledge and beliefs to enable an integration of information in their mental networks (Decker et al., 2015; Staub & Stern, 2002). For this reason, teachers take a role as constructor of learning environments that enable students to learn self-directed. Voss, Kleickmann, Kunter, and Hachfeld (2013) proposed evidence that these two dimensions are not the endpoints of a continuum. Instead of this, the authors proposed to understand these two dimensions as two distinct, negative correlated dimensions. These dimensions can be assessed on different scales. For this reason, it is possible that teachers have a high extent of both views on teaching and learning of mathematics.

Based on the definition of beliefs and the two main aspects of beliefs for our research, we primarily focus on the following research question:

Which influence show the PD course with reflecting on students' learning results on teachers' beliefs referring to teaching and learning of mathematics out of teachers' perspective?

Methods

We regard three groups of teachers in our study as a whole. Two groups of teachers were enrolled in a PD course that focused on problem solving and modelling in secondary school. An ongoing task for both PD courses was to develop tasks that meet different criteria of problem solving and modelling. The first group further was asked to give these tasks to their students and to collect the results of students' work that we call results of students' learning. These results of students' learning were the basis of the next face-to-face phase of the PD course. Teachers of second group were asked to improve problem solving tasks and a third group of teachers did not get any intervention.

We conducted pre- and posttests to measure the efficiency of the mentioned specific aspect of teachers' PD, i.e. reflecting students' learning results. Further we conducted interviews with the teachers. In this paper we regard three teachers of the first group who took part in the interview session voluntarily. The interviews took place about one month after the last meeting of the PD course. In semi-structured interviews the teachers were asked to report about their changes towards beliefs of teaching and learning of mathematics. All teachers are teachers of upper secondary schools and at the age of 40 to 50 and were women. These teachers could be representative for the whole group, because it consists of 21 teachers at the mentioned age. In addition, seventeen of these teachers were women.

We analyzed the interviews with a coding method including deductive and inductive codes (Mayring, 2015). The deductive codes were based on existing research referring to teaching and learning of mathematics considering the transmission-oriented and the constructivist-oriented beliefs. According to both types of beliefs, we created codes for teachers' answers. In this context, we distinguish between teachers' beliefs before and after the professional development course. The distinction is based on hints in teachers' statements which enables us to match beliefs to the appropriate point in time. In addition, we analyzed the role of "reflecting students' learning results" for teachers' learning during the PD course.

Results

The results section is structured into three parts. At first, we want to show how teachers' statements are coded according to beliefs about teaching and learning of mathematics. Second, we sum up beliefs of the three teachers. And third, the effects of "reflecting students' learning results" are considered.

In the interviews teachers were asked to report about their beliefs before and after taking part in the professional development course. In particular, they should consider changes in their statements. For example, Mrs. B states:

Mrs. B: [...] it has changed that the tasks are different. Students should argue more and I do not have to work off stacks of tasks. I can work off all facets more determined and I do not have to say that I must work off a model and then practice, practice, practice
....

Mrs. B reported about changes in her beliefs about teaching and learning mathematics. We interpreted her statement as follows: At the end of her statement, she mentioned that she now does not have to introduce a mathematical model followed by many exercises in the lessons. Furthermore, she reported

on working off stacks of tasks before the PD course. Both parts of the statement are coded as transmission-oriented beliefs, because she emphasized practicing as repetition of information or procedures which is represented in stacks of tasks. In addition, the mentioned parts of the statement were coded as beliefs at the beginning of PD course, since she reported about changes during the PD (see table 1). In the first part of her statement, she considered a change in tasks and argumentation in her classroom. She also mentioned that she can work off the mathematical ideas more determined. These parts of the statement were coded as constructivist-oriented beliefs, because Mrs. B reported a student centered teaching style where students are asked to argue about mathematical concepts and to talk about mathematical problems with each other (see student-oriented perspective in teaching in table 2). In addition, these parts of the statement were coded as beliefs after taking part in the PD course, because in the whole statement Mrs. B reported about changes. In conclusion, the statement of Mrs. B shows a belief change from a transmission-oriented view to a constructivist-oriented view of teaching.

The three teachers showed similarities in reporting aspects of a transmission orientation and a constructivists orientation. A code that seemed to be crucial for all three teachers, but was not included in the quotation of Mrs. B, was expressed by Mrs. C:

Mrs. C: Students should do more and I have to restrain myself a little bit more.

We interpret the statement as following: Mrs. C also reported changes in her beliefs. The whole statement was coded as constructivist-oriented belief, because she wanted the students to be more active in her classroom when they are learning mathematics. In addition, Mrs. C reports on restraining herself in lessons. This is in line with the constructivist view, because on this perspective, teachers are creators of learning environments and students shall learn self-directed. For this reason, it is necessary that teachers shall restrain themselves in lessons. This statement was coded as beliefs after taking part in PD course, since “more” indicates that she had other beliefs at the beginning of PD. In particular, the beliefs at the beginning compared to those reported in the statement included that Mrs. C was more in the center of the lesson and students were more passive which are in line with transmission-oriented beliefs.

The results of the analysis of the three teachers’ beliefs showed that they were transmission-oriented before they took part in the PD course. However, they seem to change their beliefs towards a constructivist-orientation during the PD course. The teachers also reported existing constructivist-beliefs they had at the beginning of PD course, but they stated changes towards more constructivist beliefs while they reduced the strength of transmission-oriented beliefs. Table 1 shows the results of coding for beliefs before teachers take part in the PD course:

	Mrs. A	Mrs. B	Mrs. C
repetition of information (practicing)	X	X	X
frontal teaching	X	X	X
exact instruction	X	X	
teacher is in center of lesson	X	X	X

Table 1: Predominant beliefs before taking part in PD course (mentioned by teachers)

Teachers reported in the interviews that they changed their beliefs referring to teaching and learning from a teacher-centered perspective to a more constructivist-oriented teaching perspective. The following table shows statements of teachers referring to beliefs of teaching and learning of mathematics after taking part in the PD course.

	Mrs. A	Mrs. B	Mrs. C
active role of students	X	X	X
knowledge construction	X		
teacher withdrawing in lessons	X	X	X
students' discussions are important	X		X
students should analyze own mistakes	X		X
tasks with a meaningful context within real life			X
cooperative learning (group work, ...)	X		X
student-oriented perspective in teaching		X	
pool of teaching methods		X	X

Table 2: Predominant beliefs after taking part in PD course (mentioned by teachers)

Both tables show different beliefs of teachers before and after the PD. These tables do not imply that all teachers had only teacher-centered beliefs at the beginning of the PD. They reported also that they had constructivist beliefs. For example, the statement of Mrs. C shows that she has more constructivist beliefs about teaching. This does not imply that she has not had constructivist-oriented beliefs at the beginning of the PD course. In her statement, Mrs. C only reported about more constructivist-oriented beliefs after taking part in PD. In addition, all the coded statements show that they also have transmission-oriented beliefs after PD course, but they report to think of teaching with more student-centered beliefs. Note, both tables only show predominant beliefs of teachers before and after PD. For this reason, peripheral beliefs about teaching and learning of mathematics are left out.

The results about beliefs show changes. For this reason, we want to know how reflection of students' learning results impact on teachers learning. In this context, Mrs. A and Mrs. C state:

Mrs. A: [...] I have learnt a lot about my students and I have also learnt a lot about myself and for this reason I have tested some things.

Mrs. C: I think it was good. The students' learning results show how other teachers proceed in teaching, which approach they use and how they describe. Within this action, you could take new ideas that I found in students' solutions. [...].

The statements of the three teachers show that "reflecting students learning results" were used to reflect the own practice of teaching. In this context, these teachers learnt about characteristics of their students and about themselves as teachers. As a consequence, Mrs. A tested some ideas contained in the PD course. In addition, the three teachers recognized other teaching styles when they reflected on students' learning results. For this reason, they also reflected about the practice of other teachers and their teaching approaches. In particular, they also reflected about their own practice compared to the

one of others to get new ideas (Mrs. C). These statements are typical for all three teachers. According to this, the reflection of teaching style could be understood as one factor that result in changes of beliefs about teaching and learning.

Discussion

The data from the interviews indicate that the three teachers changed their beliefs from a more transmission orientation to a more constructivist orientation of teaching and learning of mathematics. This is shown by statements of teachers after taking part in the PD course. The phrases in both tables consider the aspects of both beliefs about teaching and beliefs about learning. Based on the analysis in the results section, statements of table 1 refer to transmission-oriented beliefs and statements of table 2 to constructivist-oriented beliefs (e.g. Fives et al., 2015). In fact, teachers focused on constructivist-oriented beliefs, but they still expressed some transmission-oriented beliefs. This is in line with research findings of Voss et al. (2013), who supposed that both types of beliefs can co-exist. The statement of Mrs. C supports this assumption. Although she expressed transmission-oriented beliefs before the PD, she said that students should be more active, which indicates that she has had constructivist-oriented beliefs at the beginning of the PD.

Belief changes in teacher education are also shown in the study of Franke et al. (1998) for teachers on primary level who examined a student-centered framework in their PD. In this context, our results are similar to those of Franke et al. on secondary level, because we also used reflecting students' learning results to emphasize student-centered teaching. Furthermore, the teachers reported that they used students' solution to reflect their own teaching and to get to know information of other teachers' approaches and also about their students. This is in line with the results of Little et al. (2003), because teachers consider the ideas of their students in their deliberations more strongly after taking part in the PD course. In addition, belief change caused by reflection is reported by Decker et al. (2015) in the way that there is a relationship between the extent of reflection and teachers' belief change. For this reason, it is possible that all teachers of this study changed beliefs because they reflect intensively about their own practice. Skott (2015) stated that substantial new experiences are necessary to change beliefs. Concerning our research, we hypothesize that the intensive reflection of the own classroom practice includes the mentioned new experiences.

There are some limitations of this research. At first the interviews took place after the whole PD course. For this reason, teacher reports about the practice before the PD course can be influenced by the experiences of the PD. For a more precise research approach it would have been necessary to interview the teachers at the beginning of the PD. Considering the external circumstances in this research project, it was not possible to interview teachers, because they took part in PD voluntarily and the first meetings last about the whole day. In addition, these qualitative data provide no proof for "reflecting students learning results" as a feature of effective PD, but there are hints that beliefs can change and teacher reflect their own practice during this part of PD. Furthermore, the teachers of the second group were not interviewed. For this reason, it is not possible to indicate whether the teachers of the second group also changed their beliefs. In addition, we do not know whether second group teachers reflect on their own practice as deep as teachers of the first group.

Future research

This part of the whole research project considers qualitative data of the first PD course. Future research should use also quantitative data (measured by items of Staub & Stern (2002)) to support the results of teachers' belief change during PD. This could also show whether reflecting students learning results is one element that is connected with teachers' belief change empirically. In addition, it is necessary to link the qualitative data analyzed in this paper and the quantitative data which will we analyzed in the future. Combination of both resources can show the positive impact of reflecting students' learning results on teachers' knowledge, beliefs and motivation.

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Shifting frames - A teacher's change towards explorative instruction

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The case of one middle school teacher's change in practice is examined through the lens of "shifting frames" from ritual to explorative instruction. These frames are collection of coherent mathematical as well as subjectifying (people-related) meta-rules. The teacher, who participated in a year-long PD program, started out in a ritual frame and gradually shifted to a more explorative frame. The shift was not uniform, and could be seen first in the subjectifying meta-rules, and only later (and very partially) in mathematical meta-rules of exploring objects. In addition, newly learned practices were partially distorted through the old ritual frame.

Keywords: Mathematics instruction, professional development, frames, explorative practice.

Classrooms where students engage in explorative mathematical learning produce more robust, conceptual learning, and more positive mathematical identities (Schoenfeld, 2014). Despite these findings, teaching in the US and worldwide still provides mostly opportunities for ritual participation, where learning is made up of reciting procedures and facts (McCloskey, 2014; Resnick, 2015). Efforts at changing mathematical instruction, have shown ritual instruction is difficult to change (e.g. Santagata, Kersting, Givvin, & Stigler, 2011). However, the complex processes of change that do occur in teachers' practice, in the context of certain professional development (PD) programs, have largely remained obscure, perhaps because of the lack of theoretical frameworks for examining such complex processes. In this study, I offer the concept of "frames" for tracking subtle changes in one teacher's practice over a period of a school year.

Theoretical background

I take as a starting point Sfard's (2008) view, that the gist of mathematical activity, both in the classroom and over history, is the construction and exploration of "mathematical objects." These discursive objects, like numbers, triangles, and functions are not existing objects in the physical world, yet the "metaphor of object" is used by any skilled mathematician to talk about them as though they were indeed such physical objects. Mathematical objects can have many realizations, or physical symbolic representations. A function, for instance, can be represented as a graph, a table, an algebraic expression or a verbal statement. In the process of learning, according to Sfard, students come to see all these realizations as signifying one object. Once they do that, they start talking about it as an object existing of itself, a process termed "objectification."

Within this view, Sfard and Lavie (2005) theorize the process of mathematical learning as moving from a ritual, peripheral phase, where activity is first and foremost aimed at pleasing the experts of the discourse (e.g. the teacher), to an explorative phase where new mathematical narratives are produced by oneself for the sake of the activity itself. In contrast to explorative participation, the focus of ritual participation is activity itself, not the mathematical narrative produced by it. Ritual participation is often characterized by *syntactic mediation*, where instead of using mathematical signs as signifiers of objects, these signs are manipulated according to prescribed (often memorized) rules.

Ritual and explorative participation are governed by certain meta-rules or “patterns in the activity of the discursants” (Sfard, 2008, p. 201). These rules can be divided into mathematical meta-rules, which dictate how mathematical narratives are to be derived from each other (for example, by proof or by computation) and *subjectifying* meta-rules which govern the actions of people (e.g. asking questions, giving directions, talking with each other). To capture the fact that meta-rules have a certain structure and coherence, Heyd-Metzuyanim, Munter & Greeno (in review) suggested the term “frames”, borrowed from socio-linguistic and socio-cognitive research to describe “a set of expectations an individual has about the situation in which she finds herself that affect what she notices and how she thinks to act” (Hammer, Elby, Scherr, & Redish, 2005, p. 9). Accordingly, they defined “frames” in mathematical classrooms as a set of meta-rules, both mathematical and social, which includes appropriate questions, answers, justifications and other discursive actions in a situation of solving a mathematical problem or performing a mathematical task. Whereas frames of explorations would be sets of meta-rules that cohere around the goal of producing mathematical narratives based on logical justifications, ritual frames would be more aligned with goals such as performing a procedure accurately according to a prescribed set of steps, and adhering to external authority.

Method

The case of Mr. M is taken from a larger study, where we followed 7 teachers and 5 teacher leaders throughout 8 months of professional development (PD) during 2014 – 2015. The PD was led by Margaret Smith and Victoria Bill from the University of Pittsburgh. It centered on Smith & Stein’s (2011) “5 Practices for Orchestrating Productive Mathematics Discussions” and on Accountable Talk™ (see http://ifl.pitt.edu/index.php/educator_resources/accountable_talk). Teachers were supported via four full-day PD sessions and in-school individual coaching sessions.

Mr. M was chosen for closer inspection because he showed a steady growth in the “implementation” score of the Instructional Quality Assessment tool (Boston, 2012), used in the larger study (Heyd-Metzuyanim, Smith, Bill, & Resnick, 2016) to score lessons for cognitive demand. Mr. M’s lesson 1 scored a 1, lesson 2: 2, lesson 3 & 4: 3 (where 4 is highest). Though his lessons never achieved the highest level of cognitive demand, in the last two lessons cognitive demand was partially preserved.

Data collection included four cycles (September, December, February and May), each containing: 1. Pre-lesson and post-lesson interviews with the teacher; 2. Lesson recording and students’ worksheets. Frames of mathematical instruction were searched in four data categories:

(1) The potential of the task to engage students with different realizations of mathematical objects and with mathematical meta-rules of justification and generalizations. (2) Teacher’s plans as collected from the pre-lesson interviews, as well as his post-lesson reflections. In these we looked for evidence of mathematical and subjectifying aspects of explorative frames. (3) Students’ work: evidence of engagement with different realizations of objects and with connecting between them. Evidence for justifications and explanations based on mathematical logic. (4) Whole-classroom discussions: evidence for ritual vs. explorative meta-rules. Is there only talk about routines for manipulating symbols detached from their meaning, or are the symbols also related to mathematical objects?

Findings

During the first, introductory interview, Mr. M declared himself to be aligned with what may seem to be explorative instruction. Describing quality instruction, he said:

... where groups get together in giving a situation in which they come up with strategies on their own, as opposed to a teacher standing up in front of the room and just saying “okay, this is how you do this, this is how you do this, now just take this problem and solve it”. So ... I keep trying to look for my own stuff that I’m doing, so I would look to see that in other peoples.

Notably, in remarking that such instruction was “my own stuff that I’m doing”, Mr. M attested for practicing quality instruction. To this, he added:

one of my philosophies (is that) there’s never, ever, only one way to solve a problem, that there’s many different strategies and many different pathways you can take to find different problems ...

Mr. M also described his challenges in encouraging students to take on agency and not rely ritually on his guidance.

I would say a majority of the students I’ve worked with over the last 7, 8 years pretty much when they were on a challenging problem, their first shot is to ask me to give them help and guidance along the way. I used to do that more often, ..., so I tried to step back away from that for a moment, getting to know the kids a little bit better helps me understand those that are just being lazy and those that truly don’t know it.

Thus, in his introductory interviews, Mr. M seemed to be well aligned with “reform” mathematical pedagogy, especially with regard to letting students struggle and encouraging them to look for “different strategies” for solving problems.

Lesson 1

Despite these declarations, the first lesson revealed Mr. M’s actual practices were not well aligned with the explorative frame. For the main task of the lesson, Mr. M chose the task seen in Figure 1:

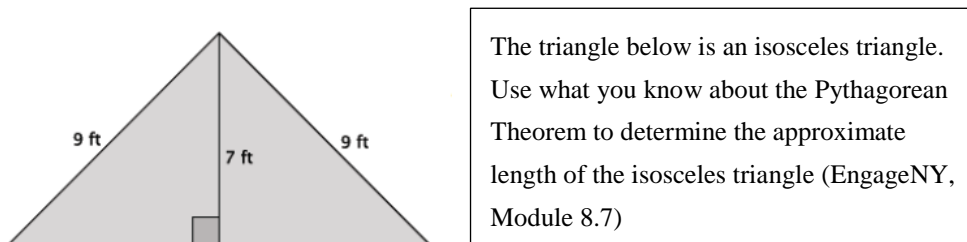


Figure 1 - Pythagorean Task given in lesson no. 1

The task was part of a module that introduced irrational numbers through the Pythagorean Theorem. The EngageNY¹ teacher guide encourages explicitly to present the task only after students had become familiar with this Theorem. The task was taken verbatim from the module, which assumed this previous experience, thus including the “use what you know” phrase. However, Mr. M. was well aware of the fact that his students were probably not familiar with the Pythagorean Theorem. In the pre-conference of the lesson he said:

¹ See <https://www.engageny.org/resource/grade-8-mathematics-module-7>

The first challenge that I'm expecting is that we've not introduced Pythagorean Theorem at all last year. Um, I'd like to hear where their conversation goes and what they're focusing on –... when they're trying to find an opposite base, ... given the $a^2+b^2=c^2$ idea, do they put the numbers in the right place – do they remember the square numbers, do they remember to take the square roots of numbers to get themselves back to what the – the single variable would be? ... So I'm looking to see – what kind of challenges working with square roots and squares creates.

Though Mr. M was anticipating “challenges” in working with the problem (aligned with the PD encouraging letting students struggle), the way he talked about students coping with these challenges was through “remembering”. Thus, meta-rules belonging to the explorative frame (student agency) were distorted through the ritual frame into meta-rules of fact-retrieval, and memorizing procedures.

During the lesson, the launch of the task proceeded as follows:

90. T Okay, so when we take a look at this (*pointing to a right triangle on the board, the sides of which are labeled 'a', 'b,' and 'c'*) I'm going to give you a little bit of a formula. And it may help you today in doing some of your work (*writes $a^2+b^2=c^2$ on the board*) ... Who can read this out loud for me, please?
- 91.-95. *Three students are prompted to read the formula "in different ways". One reads "a two plus b two equals c two". The second "a to the second power, b to the second power, and b to the second power", the third "a squared plus b squared equals c squared"*
96. T You see, all three of those ways that they said say the exact same thing. ... So when we're looking at this, can you now start to see that knowing two of the three sides will allow me to figure out the third side? ... I'm gonna have you work in groups today to see if you can't use ...any knowledge that you have plus anything that you've seen today. Alright?

The subjectifying meta-rules enacted in this excerpt bore resemblance to explorative frames. Students were asked to provide “different ways” of saying the “exact same thing”, and were directed to “figure out” and “work in groups” to “use any information” they have to solve the problem. Yet the mathematical meta-rules, that is, the ways to derive one mathematical narrative from another, were obscure at best. $a^2+b^2=c^2$ was written on the board, detached from the right triangle, with no indication of a^2 , b^2 , and c^2 signifying the geometrical squares adjacent to the triangles sides. By that, Mr. M was treating the $a^2+b^2=c^2$ Pythagorean Theorem syntactically, as a series of signifiers, detached from their meaning as signifying geometrical objects.

Not surprisingly, since students had no access to the meaning of the Pythagorean Theorem mediated by geometrical objects, they struggled over where to place the a, b, and c labels given by the formula, on the newly presented triangle. After some leading questions from Mr. M. students concluded they should label the base of the triangle as ‘C’ or ‘C²’. They subtracted $9^2-7^2=32$, but then were unsure as to what to do with that result. Some students thought it should simply be divided between the two halves of the base, labeling the whole base as ‘32’ and halves of it with ‘16’. Others simply labeled the whole base as ‘16’ or went another step and labeled the halves with ‘8’. Only 3 or 4 students, out of 24, figured out that the 32 should be square-rooted and multiplied by 2. However, even they did not label the triangle’s base with the resulting number 11.4 (or estimated 12). The mathematical meta-rules governing the classroom activity were thus primarily ritual. They could be summarized as “apply a set of symbols, somehow related to a right-triangle, to a new triangle”.

It was not that Mr. M was intending students to apply the 'a', 'b', and 'c' symbols randomly. He did have a certain type of reasoning he was looking for. After having asked a student to come to the board and present her solution, and while the student was labeling the triangle sides, he said:

289. T Yeah, please notice - ... if we looked at that formula - $A^2 + B^2 = C^2$, and then I asked this question of some of you: Does it make a difference where C is? ... (*Student answers C has to be bigger than A and B*)
291. T So if it's bigger than all the other ones, then the question becomes where does the C have to go - which side?
292. Student On the longest side.
293. T The longest side. Now the one misconception I saw that I didn't expect was that some of you thought that this (the base) was the longest side, the whole way across. Just be careful about that, you're focusing on just one of those two triangles, okay?

The meta-rules of the activity, as gleaned from Mr. M's words were thus "figure out where the C is according to it being bigger than $a + b$. Then figure out the longest side and label it as C ". However, this was detached from the physical meaning of the Pythagorean Theorem. In relation to the confusion or "misconception" the students had, regarding where to place the C , Mr. M did not have much advice besides "be careful about that" [293]. There was no other indication why they should be focusing on one of the right triangles, and not the whole triangle.

In the post conference, Mr. M seemed reasonably content with the results of the activity. He explained that since students were unfamiliar with the Pythagorean Theorem, he expected them to struggle, but that that wasn't the focus of the lesson. Rather the focus was discussing imperfect squares, to which they got at the end of the discussion, when one student said 32 should be rooted and Mr. M led students to estimate the root between 5 and 6. Yet having detached the geometric, physical realization of the Pythagorean Theorem from its algebraic formula, the rationale behind the existence of imperfect squares had no way of being foregrounded. The fact that Mr. M was not concerned with students struggling considerably with something that was not "the focus of the lesson", indicates that through his frame, the whole process of working in groups and discussing solutions was not a central measure for achieving the mathematical goal of the lesson. Rather, it was a sort of a "side effect", performed for the sake of the lesson recording or for students to practice working in groups.

I now move to a similar examination of the last lesson. A description of the two middle lessons is out of the scope of this paper. However, evidence for movement from a ritual frame to a more explorative frame were starting to show during the 3rd lesson, where Mr M received, together with the rest of the teachers of the PD, a high-level task selected by the PD leaders and directed explicitly on the different solution paths that should be sought in the lesson. Lesson no. 4 seems to have reaped the benefits of this process, though as we shall see, the shift between frames was still very fragile.

Lesson 4

The first indicator that Mr. M's practice was starting to align with an explorative frame could be seen in his choice of a task. This task involved modeling of real-life processes with a quadratic equation. Following is the task as presented on the worksheet (taken from [EngageNY, Algebra 1](#)):

The baseball team pitcher was asked to participate in a demonstration for his math class. He took a baseball to the edge of the roof of the school building and threw it up into the air at a slight angle

so that the ball eventually fell all the way to the ground. The class determined that the motion of the ball from the time it was thrown could be modeled closely by the function: $h(t) = -16t^2 + 64t + 80$, where $h(t)$ represents the height of the ball in feet after t seconds.

Students were then asked to find the behavior of the function (maximum, minimum, vertex), to graph it and to indicate how many minutes passed until the ball fell to the ground and what was the meaning of $h(0)$. The task thus had ample potential for using different realizations for a mathematical object – namely the quadratic formula.

The next indication for Mr. M's movement between frames was his talk about expectations for students' work during the pre-lesson interview. This is how he presented the choice for the task:

I've been working on questions ... like "how do you know that that's where your graph crosses the X axis?" So more than just– what is that value, how do you know what it is. How did you figure it out. Um, why does your graph continue past zero? In the context of the problem, what does that mean? ... And try to gain some understanding of their understanding.

Instead of "remembering", Mr. M now talked about students' "understanding". He was also working hard on preparing questions that would assess students' understanding, a skill specifically taught in the PD sessions.

Examination of students' worksheets showed that most students performed the calculations involved in the problem correctly. 4 out of the 27 students wrote that $h(0)$ indicated the height in which the ball was thrown was 80 feet, and 3 more related $h(0)$ to the initial stage, before the ball was thrown in the air. Other students, however, either left this question blank, or stated that $h(0)$ means "the ball is on the ground" indicating incongruence between the physical situation and the graphical realization. Thus, in contrast to the first lesson, there was evidence that a small part of the classroom was able to flexibly move between different realizations of the quadratic function.

During the whole classroom discussion, students' explanation on the board were mostly concentrated on the calculations involved in the problem. Mr. M encouraged this by calling on students to present "different ways to solve the problem" which referred to factoring by dividing the expression by 16 or by 8. He also referred to finding the value of $h(0)$ by substituting $t=0$ or by just "looking" at the last term in the equation, as two different ways to find $h(0)$. Yet these two "ways" were, in fact, both tending to the algebraic realization and could be considered as "different" only when looking at the function's expression syntactically. There was no mentioning in the whole classroom discussion of the graphical realization of the function or the physical "real life" story modeled by it. It was thus still mostly characterized by meta-rules of carrying out prescribed procedures using syntactic mediation, rather than meta-rules of exploring mathematical objects. However, some slight changes could be observed. One of the students who talked in the whole classroom discussion did, even if very briefly, mention the physical "real life" situation that the function was modelling. He talked about the ball ("the ball hasn't been thrown yet") and hinted at the height of the roof in "throw the ball at 80". Also, there was brief mentioning of the physical realization when Mr. M elicited from one student that the maximum was at "2 minutes".

In terms of social meta-rules, there were many more prompts for students to "restate" what other students have done. Mr. M made every effort to use this talk move, taught in the PD, albeit somewhat inflexibly, whenever a student made a mathematical statement he deemed as important.

Discussion

In this paper, I have used the concept of “framing” to examine subtle changes in one teacher’s practice over a period of one school year, through which the teacher was receiving support both through PD sessions and through in-school coaching. These changes, including the different aspects of frames, are summarized in table 1.

Table 1 - Social and Mathematical aspects of frames in two lessons

	Social (subjectifying) meta-rules	Mathematical meta-rules
Lesson 1	Beginnings of explorative social meta-rules through teacher’s declarations (students expected to persevere, work together). In practice, most of the talking is done by the teacher.	Ritual: Meta-rules only have to do with recalling facts and procedures from memory. Mathematical signifiers are detached from the object they are representing. Students’ work indicates syntactic mediation with no connection to geometrical realizations.
Lesson 4	Explorative social meta-rules are more dominant. Students are asked to restate each other’s ideas. Two students explain their thinking to the whole class. Evidence that some student listen to each other.	Beginnings of explorative mathematical meta-rules: Teacher seeks students “understanding”, not just rule following. A small portion of students’ work indicates connections between multiple realizations of the quadratic function. Still ritual meta-rules in teacher’s discourse still dominate instruction.

As can be seen from Table 1, the shift from ritual to explorative frames in Mr. M’s case was not uniform. The shift occurred first in the social meta-rules and only later in mathematical meta-rules. Also, both shifts seemed to occur first at the level of declarations and only later in practice. Thus, talk about social meta-rules aligned with explorative participation was evident already in the first interview, but enacted mostly in the last lesson. Talk about expectations for explorative mathematical meta-rules was evident in the last lesson pre-conference interviews, and only very slightly evident in the enactment of the lesson. This finding corroborates earlier findings showing that teachers are quicker to adopt declarations about explorative instruction than they are to enact it (Cohen, 2001) as well as our previous findings regarding social meta-rules of teachers trying to enact “dialogic” or “reform” instruction being more aligned with explorative instruction than mathematical meta-rules (Heyd-Metzuyanim et al., in review). Most importantly, the findings reveal that teachers’ learning of new practices is not a matter of acquiring new practices on a “tabula rasa” of non-existing former practices. Rather, at first, new practices are seen through the old frame. As such, they gain unpredictable “twists”, such as perceiving different syntactic procedures as “different solutions” sought after, or encouraging students to restate unimportant mathematical ideas.

Acknowledgments

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Situation-specific diagnostic competence of mathematics teachers – a qualitative supplementary study of the TEDS-follow-up project

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Abstract: One main facet of teachers' professional competences is diagnostic competence. While diagnostic competence of teachers becomes relevant in several situations within teaching and learning, this paper focuses on situation-specific diagnostic competence required by teachers within class. In a qualitative supplementary study of the TEDS-Follow-Up project, this situation-specific diagnostic competence is analysed using the video instrument of the TEDS-Follow-Up study. 131 primary level mathematics teachers participated in the primary study of this project and examined specific learning situations in the video instrument. These instances were analysed using qualitative text analysis (Mayring 2015). Results indicate that teachers notice very different aspects in those teaching situations. Two different diagnostic types can be differentiated: content-related and judging diagnostic type versus student-related and action-oriented diagnostic type.

Keywords: Professional competences of mathematics teachers, diagnostic competence, mathematics teachers' knowledge, situation-specific skills of mathematics teachers, video study.

Introduction

Teachers are faced with many challenges in the context of teaching and learning. In order to plan and to conduct teaching sequences that enable students to achieve their best possible learning results, teachers need several characteristics such as professional knowledge and situation-specific skills. According to Weinert et al. (1990, p.172), diagnostic knowledge is one of the “four areas of knowledge [...] as constituting the cognitive components of teacher expertise” in addition to classroom management subject matter knowledge and instructional competence. With this perspective, this article focuses on the diagnostic competence of primary school mathematics teachers, more precisely on their diagnostic competence that becomes relevant during class – the so-called situation-specific diagnostic competence.

Theoretical background

Theoretical background of the study is the discussion on mathematics teachers' professional competences, which will be described specifically focusing on their diagnostic competence.

Professional competences of teachers

According to Weinert (2001), competence involves two main facets: a cognitive facet as well as an affect-motivational facet. The cognitive facet is often differentiated following Shulman (1986, 1987) into (Mathematics) Content Knowledge (MCK), (Mathematics) Pedagogical Content Knowledge (MPCK) and General Pedagogical Knowledge (GPK). Several empirical studies that deal with

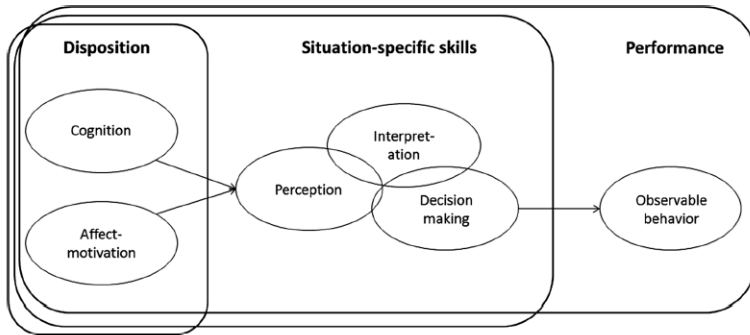


Figure 1: Modeling competence as a continuum (Blömeke et al. 2015, p. 7)

teachers' professional competences use this conceptualization of competence (see for example Mathematics Teaching in the 21st Century (MT21); Blömeke et al. 2008, Teacher Education and Development Study in Mathematics (TEDS-M); Blömeke et al. 2014 and Cognitive Activation in the classroom (COACTIV) Kunter et al. 2011). In addition to teachers' knowledge, these studies assess affective-motivational aspects such as epistemological beliefs,

motivational aspects and those about the teaching profession (cf. Blömeke et al. 2008, Blömeke et al. 2014, Baumert and Kunter 2011). Thus, cognition and affect-motivation are hypothesized to build the basis of competent performance in classroom situations.

More situated approaches to assess teachers' professional competences are based on models of competence that also include more situated facets such as teachers' perception, interpretation and decision-making that become especially relevant during classroom interaction. According to Blömeke et al. (2015), competence is a continuum from personal traits such as cognition and affect-motivation that underlie and affect situation-specific skills which again determine the actual performance or behaviour in specific situations. Here, the perception, interpretation and decision-making in concrete situations "mediate between disposition and performance" (ibid., p. 7). A content-based specification of this broader concept of competence is the diagnostic competence (Abs 2007).

Diagnostic competence

Diagnosis and diagnostic tests are usually associated with medicine. Doctors need to diagnose illnesses on the basis of symptoms. However, teachers need diagnosis in their profession as well. They may use diagnostic (clinical or standardized) tests to detect learning disabilities (Ketterlin-Geller and Yovanoff 2009) but they also need to diagnose students' achievements and learning processes during class without using standardized educational or psychological tests. Both described instances require teachers' diagnostic competence. However, we describe the process of diagnosing in the course of teaching and learning as teachers' situation-specific diagnostic competence as opposed to the facet of diagnostic competence that becomes relevant in adequately choosing, using and evaluating diagnostic tests (see Hoth et al. 2016).

For this situation-specific diagnostic competence, teachers' situation-specific skills become relevant as proposed by Blömeke et al. (2015) in their model of competence (see figure 1). On the basis of this theoretical background, the study presented in this paper focuses on the following research questions:

1. How do the perception, interpretation and decision-making of primary mathematics teachers differ?
2. Can different diagnostic types be reconstructed?
3. How do these diagnostic types relate to their professional knowledge regarding the three knowledge facets mathematics content knowledge (MCK), mathematics pedagogical content knowledge (MPCK), general pedagogical content knowledge (GPK)?

To address those research questions, the methodological approach that was used to analyze the data, will be described.

Methodological approach

The study presented in this paper is a qualitative supplementary study of the TEDS-Follow-Up project (Follow-Up to the international Teacher Education and Development Study in Mathematics, TEDS-M). For the specific purpose of analysing teachers' situation-specific diagnostic competence, the focus of the supplementary study lay on the video instrument of the TEDS-FU study. TEDS-FU, its conceptualisation and design will be described in the following, prior to the outline of the methodological approach that was realised in this specific supplementary study.

The TEDS-FU study

The TEDS-FU study is the German Follow-Up-study of the international comparative study about mathematics teacher education TEDS-M. A subsample of primary and secondary school mathematics teachers who participated in TEDS-M was reassessed after four years of work experience. A total of 300 mathematics teachers participated in the primary school study, including 131 primary school teachers who are at the focus of this paper. Therefore, the TEDS-FU study analyses the teachers' development in their first years of work experience.

The study is based on an understanding of competence as a continuum (Blömeke et al. 2015, figure 1) and closely refers to research in the field of teachers' expertise (cf. Li & Kaiser 2011) and the concept of 'Teacher Noticing' (cf. Sherin, et al. 2011). In order to assess more situated facets of teachers' professional competences, three situated facets are distinguished in TEDS-FU in addition to knowledge-based facets of teachers' professional competencies (MCK, MPCK, GPK):

“(a) *Perceiving* particular events in an instructional setting, (b) *Interpreting* the perceived activities in the classroom (c) *Decision making*, either as anticipating a response to students' activities or as proposing alternative instructional strategies” (Kaiser et al. 2015, p. 373).

Therefore, different test instruments are used in the study. An online-survey assessed different contextual components such as beliefs, working conditions and school characteristics, a newly developed video analysis instrument assessed teachers' situation-specific skills and a shortened version of the TEDS-M proficiency test was used to assess teachers' MCK, MPCK and GPK. In addition, a time-limited test was included where teachers had to identify typical student errors (see Pankow et al. 2016).

With the aim to analyse teachers' situation-specific diagnostic competence, all tasks were focused from the TEDS-FU primary school study that required situation-based diagnostic competence. This was ensured by the video analysis test instrument as well as in some verbally described situations of the reduced proficiency test. The video test consisted of three short video clips of a primary school

mathematics classroom and corresponding questions concerning didactical and pedagogical aspects of the teaching sequence. A total of 19 questions were selected for the analyses, 14 questions from the video analysis test and five questions concerning verbally described situations in the MPCK proficiency test. Teachers' answers to the selected questions were analysed using qualitative text analysis (Mayring 2015; Kuckartz 2014). All answers of all teachers to all 19 selected questions were analysed using reducing and structural procedures (Mayring 2015).

To exemplify the coding process, one example item will be introduced as well as teachers' responses to the item and the result of their analysis. The example item refers to the video analysis instrument, more specific, it refers to the video 'real world problem' that shows a third grade mathematics classroom in Germany dealing with a real world mathematics problem that is shown in figure 2. In the video, three students' working groups are shown who discuss their working results. The students use very different approaches, one student produces a symbolic result while another student solves the task using a visual drawing of the situation. Referring to this scene, the teachers were asked to characterise the two solution approaches contrastingly. Teachers' responses to this task were coded

Figure 2: Real world mathematics problem that is discussed in the video vignette

Anna and Felix want to go to the movies. They take the bus together. Felix pays the bus tickets for both of them. One ticket costs 2€. Anna pays both cinema tickets. One ticket costs 7€. At the end of the day, the children want to share the costs equitably. How can they do that?

using reducing processes of qualitative context analysis, capturing the content of the teachers' answers. In this regard, the following teacher's answer was categorised as 'contrasting the students' form of representation':

Teacher 1: *"Lea does mental arithmetic and uses a symbolic approach. She does not have to use an iconic or enactive solution while Kim needs an iconic approach. She should try an enactive approach to see that Lea's approach is correct as well."*

This coding approach categorized each of the teachers' answers with regard to their content. In another coding process using structuring procedures, the teachers' answers were coded with regard to judgments. A teacher's response that contained a judgment of the two students' solutions is the following:

Teacher 2: *"Lea's approach is more practical and more advanced than Kim's approach because she already subtracted the amount that the girl has to give to the boy."*

Finally, in another structuring coding procedure, all teachers' responses to the selected questions were coded with regard to proposed alternatives or continuations. A teacher's response that proposes a possible continuation of the presented situation is the following:

Teacher 3: *"Lea does mental arithmetic and uses a symbolic approach. She does not have to use a visual or acting solution while Kim needs a visual approach. She should try an acting approach to see that Lea's approach is correct as well."*

These coding processes resulted in a category system that built the basis for type-building text analysis (Kuckartz 2014). Here, three dimensions emerged which further constructed a feature space to generate diagnostic types: the perspective that teachers' chose on the teaching sequences, their tendency to judge and their tendency to propose alternatives and continuations. In order to reconstruct ideal diagnostic types, connections between these three dimensions were analysed and idealised. Finally, these diagnostic types were interrelated using contingency analyses in a Mixed-Method-Design (Kelle & Buchholtz 2015) with the teachers' knowledge scores that resulted from the reduced

	Didactical perspective	Didactical AND mathematical perspective	Mathematical perspective
Average MCK	519	567	597
Average MPCK	536	552	543
Average GPK	647	640	683

Table 1: Contingency analysis between teachers' professional knowledge and their perspective

teachers' situation-specific skills and their professional knowledge. Table 1 shows this contingency analysis between the perspective chosen and the professional knowledge for the example item. However, since the perspectives were coded in every selected question, these connections were also analyzed independent of specific teaching sequences.

Results

Resulting from the reducing and structuring procedures, teachers' responses differed with regard to several aspects. On the one hand, teachers chose different perspectives on the classroom incidents. While some teacher focused on the content, other teachers focused on the students, their understanding, motivation, behaviour etc. On the other hand, it became obvious that teachers had varying tendencies (a) to judge the classroom events which they observed and analyzed and (b) to anticipate teaching alternatives or continuations.

Relating the three dimensions that resulted from the coding processes—(1) the perspective chosen (2) the tendency to judge (3) the tendency to anticipate teaching alternatives or continuations and—showed that teachers who often chose a content-related mathematical perspective in classroom situations also often judged these incidents. In addition, the more often teachers chose a student-related perspective, the more teaching alternatives and continuations were anticipated. These connections between the three dimensions formed the basis for building diagnostic types. In this regard, the following two ideal diagnostic types could be identified:

“Content-related and judging: This diagnostic type is characterized by a content-related perspective in the phases of perceiving and interpreting relevant incidents. These noticed criteria are subsequently used to judge the relevant incidents. The phase of decision-making is also characterized by a content-related focus. Here, the teaching continuation is conducted by the subject's (here mathematical) content.

Student-related and action-oriented: This diagnostic type is characterized by a student-related perspective in the phases of perceiving and interpreting relevant incidents. This means that the students, their learning processes, understanding, motivation and behaviour are the noticing focus. If classroom situations show deficits with regard to students' understanding and learning, this phase is automatically followed by a phase of deciding on alternatives that improve the given situation or possibilities to optimally continue the situations. The phase of decision making is primarily characterized by considering teaching methods and the instructional organization.” (Hoth et al. 2016, p.50)

Connecting to these results concerning different diagnostic types, a Mixed-Methods Design was realised that interrelated the different perspectives chosen with the teachers' knowledge that was assessed by the proficiency test in the TEDS-FU study. In this regard, connecting the perspectives chosen by the teachers to their professional knowledge showed the following results:

- Teachers who often choose a mathematical perspective on teaching situations have average or above average mathematics content knowledge while their mathematics pedagogical content knowledge is below average.
- The more often teachers choose a didactical perspective on teaching situations, the higher is their mathematics pedagogical content knowledge.
- The general pedagogical knowledge of teachers who often choose a pedagogical perspective exceeds their content specific knowledge.
- Teachers who often judge teaching instances have high mathematics content knowledge.
- Teachers who often propose teaching alternatives and continuations possess high mathematics and mathematics pedagogical content knowledge while teachers who seldom do this have below average general pedagogical knowledge.

With regard to specific teaching instances, the results indicate that teachers with comparatively high content-related knowledge (MCK and MPCK) plan their teaching with regard to the content while teachers with comparatively high general pedagogical knowledge focus to a greater extent on pedagogical facets while planning their teaching. Furthermore, other connections indicate that teachers with only little content knowledge more often miss aspects in teaching and learning that are relevant for the students' learning processes but focus on behavioural aspects if they are very striking. Teachers who focus on aspects of understanding and learning despite of the striking student behaviour have above-average mathematics content knowledge.

Summary and discussion

In this paper, mathematics teachers' situation-specific diagnostic competence is analysed. This is the diagnostic competence that teachers require during class. For this purpose, the answers of 131 mathematics teachers are analysed who took part in the TEDS-Follow-Up study. In this video-based study, the teachers are asked to answer questions referring to video scenes of mathematics classroom. Analyses showed that teachers focus on very different aspects in the same teaching scene and two diagnostic types were differentiated: the content-related and judging type on the one hand and the student-related and action-oriented type on the other hand. In addition, contingency analyses showed that there are connections between the teachers' professional knowledge and their focus on and analysis of teaching sequences.

The results enrich the already existing findings in the field of teacher noticing (cf. Sherin et al. 2011). The proposed connection between teachers' noticing and their knowledge is verified empirically. As a consequence, teachers' practice is essentially influenced by their professional knowledge which in turn emphasizes the importance of mathematics teacher education. However, following questions arise about further connections between teachers' identification, their beliefs and the perspectives that were distinguished in this paper. In addition, developing and implementing teacher education courses to foster teachers' situation-specific diagnostic competence may give further insight into the development of this specific facet of teachers' diagnostic competence.

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Pre-service mathematics teachers: How to make them ready to be ready

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The changes concerning the final state examination, determined by the novelty of Slovak university law, stimulated us to make serious changes in the examination model of Didactics of Mathematics at our university. Moreover, we have observed significant gaps in pedagogical content knowledge (PCK) of our secondary pre-service teachers during the last years. These two stimuli led us to an improvement of the course of Didactics of Mathematics. In this paper, we present our new approach which was mainly focused on the assessment of the lesson plans presented by the pre-service teachers. The assessment was based on the rubrics which were developed through the course. After the application of the new approach, we have observed growth in PCK. We have partially confirmed reliability of the rubrics as well.

Keywords: Secondary pre-service teacher, pedagogical content knowledge, lesson plan, rubrics for PCK assessment.

Introduction

Before we introduce our new approach to secondary pre-service mathematics teacher PCK development, we bring a short description of the system of pre-service teachers' preparation at our university. First, all the secondary pre-service teachers (PSTs) study two disciplines in various combinations, including combinations of natural and humanistic disciplines. Second, pedagogical, psychological and didactics studies are reduced to the minimum for the bachelor level. Therefore, we speak about the joint degree study of the particular two subjects (e.g. mathematics and physics) at the bachelor level. Most of the pedagogy is concentrated at the master level of university studies, where we talk about the teacher preparation program. Third, complex state exam from both disciplines and their didactics, Pedagogy and Psychology and the thesis defense are required for successful finish of the study. The aim of our university was to make state exams more efficient and foster mutual relationships of all parts of state exams. Moreover, we had found out serious gaps in the PCK of our PSTs, so the bigger pressure for its growth was necessary. Implementation of the PCK assessment into the state examination seemed to be a way, because "what you test is what you get," is the broadly accepted quote (attributed to Lauren Resnick). As we realized, when the assessment is changed, the instructions should be changed as well. Therefore, we have prepared a new model of state exams and apply inevitable changes at the classes of Didactics of Mathematics. To be sure that the new approach has the positive impact on the PSTs' learning, we posed following research questions: **(1)** Are the rubrics reliable tool to assess PSTs? **(2)** Was the pre-service teachers' PCK developed effectively? Which of its components were developed?

The novelty of the approach can be understood geographically - there are no serious studies on PCK development in Slovakia or even in neighbor countries (Depaepe, 2013). Moreover, we have not found any study which would consider utilization of general – not thematically specific – rubrics for

development of PCK of secondary PSTs. The new approach is described and discussed below; however, some theoretical underpinnings are necessary to be stated first.

Theoretical background

Knowledge areas, in which should a good teacher of mathematics systematically grow and achieve a certain standard, are named for several decades. Shulman (1986) explained pedagogical content knowledge (PCK) which is beyond subject matter knowledge and

„includes an understanding of what makes the learning of specific topics easy or difficult. Teachers need knowledge of the strategies most likely to be fruitful in reorganizing the understanding of learners, because those learners are unlikely to appear before them as blank slates” (p. 9).

Furthermore, Shulman (1986) emphasized the importance of curricular knowledge, familiarity with the instructional materials and tools available for teaching distinct concepts at different levels. PCK in the domain of mathematics comprises mathematical knowledge a different kind to that used in everyday life and in other professions which need mathematics (Ball & Bass, 2000).

Hill et al. (2008) built on Shulman’s (1986) definitions of different types of knowledge and proposed the model with three types of content knowledge and three types of PCK:

Subject matter (content) knowledge

(1) **Common Content Knowledge (CCK)** – knowledge which is used in the classroom in ways common with how it is used in other professions that also use mathematics. (2) **Specialized Content Knowledge (SCK)** – knowledge how to accurately represent mathematical ideas, provide mathematical explanations for common rules, procedures, understand unusual methods of solution. (3) **Knowledge at the mathematical horizon** – awareness of the large mathematical landscape in which the present experience and instruction is situated (Zazkis & Mamolo, 2011).

Pedagogical content knowledge

(1) **Knowledge of Content and Students (KCS)** – knowledge how students in general learn a concept, what mistakes and misconceptions are common, it involves understanding of students’ thinking and what makes the learning of particular concepts easy or difficult. (2) **Knowledge of Content and Teaching (KCT)** – knowledge about how to develop students’ thinking and how to deal with student errors effectively, “knowledge of teaching moves” (p. 378). (3) **Knowledge of Curriculum (KC)** – knowledge about the content of curriculum and knowledge how to utilize the content of curriculum to present.

In our approach we suggest an authentic assessment in order to make bigger pressure for the growth of each component of PCK and to move forward PST beliefs about mathematics. There are four features of authentic assessment (Darling-Hammond, Aness & Falk, 1995): (1) “they are designed to be truly representative of performance in the field” (p. 11); (2) “the criteria used in the assessment seek to evaluate “essentials” of performance against well-articulated performance standards.” (p.12); (3) “self-assessment plays an important role in authentic tasks” (p.12), (4) “the students are often expected to present their work publicly and orally.” (p.12).

The course design

Ten PSTs attended mandatory class in Didactics of Mathematics led by the second author and assisted by the first author. Total time of every week meeting was two hours and 15 minutes, the course lasted for 10 weeks, 5 of them were designed as follows:

(1) All PSTs picked randomly one of the topics which were prescribed by the second author. They were asked to create a lesson plan due to a particular date. The lesson plan had to be focused on the mastering a new curriculum. The template they were asked to fill as the front page of the preparation contained following items: topic, grade, goals, necessary entry knowledge, didactics problem and misconceptions, tools, methods and forms. Only the front page was fixed, the rest of the preparation was in the hands of the PSTs. (2) The PST who was up to present his/her preparation the next week, sent his/her lesson plan in one week advance to the whole group. Everybody was expected to raise some questions. (3) The PST had 30 minutes for the presentation of the lesson plan. Next 20 minutes were reserved for discussion. The PST explained the main line of the new curriculum mastering, underlined the important connections between the information from the front page and the tasks and subsequently, in the discussion, clarified inconsistencies pointed out by the teachers or the PSTs. (4) Everybody who was actually present at the class was asked to assess the presentation and provide the feedback and score for the PST who presented the preparation. To minimize subjectivism of the assessment, following rubrics (Table 1) were created and used for the scoring. The PSTs were instructed how to use it, examples for the particular levels were supplied.

Rubrics for lesson plan assessment

The base for the initial rubric version was the observations which were conducted by the second author in the previous academic year. The PSTs were asked to fulfil the similar assignment. Nevertheless, the scoring was not conceptualized; everybody was expected to assess presenting PST an appropriate number of points. Such assessment did not produce pressure for specific part of PCK growth. That was the reason why we, for the next academic year, developed five rubrics that we believe to enhance PCK of PSTs during the preparation of the lesson.

(1) Rubric **Learning Objectives** is defined as the ability to formulate and clarify essential objectives of the unit and to link them with the rest of the preparation. Learning objectives result from curriculum. The rubric is connected with KC. We assess whether PST applies this curriculum content to appropriate learning activities for students. (2) Rubric **Motivation** shows PSTs' potential to present mathematical ideas in an attractive way, to provide reasons and mathematical explanations for the topic. It is linked with SCK. (3) Third rubric – **Correctness** - pertains to mathematical correctness of the lesson plan. It is associated with CCK and SCK. (4) Rubric **Didactic Means (Tools)** includes chosen didactic method, tasks and materials. Using this rubric we assess PST knowledge about how to build on students' thinking and how to address students' errors effectively. The aim is to identify the level of KCT. (5) Last Rubric we called **Didactic Problems** and it contains assessment of the level of PST knowledge of how students think about the topic, how students typically learn a concept from the topic, what mistakes and misconceptions are common. Rubric is designed to determine the level of KCS.

Level	Learning objectives	Motivation	Thematic Working Group 18 Correctness	Didactic Means	Didactic problem and misconceptions
0	A PST cannot explain the objectives of the unit, or the explanation is only formal (there are only weak connection between the objectives and the preparation and/or the objectives are not appropriate for the students age group).	There is no explicit motivation within the lesson plan, or the stated motivation is very formal (neither students' activity nor questions cannot be expected).	The unit curriculum is introduced incorrectly (mathematics terms definitions, mathematics statements or tasks assignments are not formulated correctly or comprehensively or they are not appropriate for the students age group,) and/or the learning trajectory is not respected at all.	Chosen didactic means clearly support instructive approach to mathematics education, there is no prompt for students' activity and/or the tasks are chosen superficially.	A PST does not realize didactic problem and misconceptions connected to the unit topic or he/she realizes only marginal or general problems and misconceptions.
1	A PST explains the objectives of the unit partially, and/or he/she does not propose certain of the important goals, and/or certain objectives are not appropriate for the students' age group.	The motivation stated in the lesson plan probably would be interesting only for a few students, and/or there are no tight relations between the motivation and the objectives.	Language inaccuracies in oral or written communication are observable and/or the learning trajectory is respected only partially.	Chosen didactic means lead to rather instructive approach, there is small prompt for students' activity. The structure of the teaching unit is not well thought out and/or some important cognitive phase is missing and/or selection of the tasks is only partly thought out.	A PST can name the didactics problem and misconceptions connected to the unit topic and he/she resolve them within the preparation just partially.
2	A PST formulates and clarifies essential objectives of the unit, links them with the rest of the preparation and the objectives are appropriate for the students' age group.	The motivation stated in the preparation probably engages most of the students and it is linked with the objectives. If possible, the motivation suggests connection of mathematics and everyday life.	The unit curriculum is introduced correctly (mathematics terms definitions, mathematics statements and tasks assignments are formulated correctly and comprehensively and they are appropriate for the students age group,) and the learning trajectory is respected.	Chosen didactic means leads to a creative environment where activity of students dominates and the proposed teaching unit has a coherent structure, no important cognitive phase is missing. Task selection is thought out.	A PST can name the key didactics problem and misconceptions connected to the unit topic and he/she resolve them within the preparation.

Table 1: Rubrics for PCK assessment

In the rubric Correctness, the term learning trajectory is not conceptualized, it is used in simplified meaning concerning the important entry knowledge before the new one is going to be taught and learnt.

The presented lesson plans were not the first ones created by the PST. They had prepared and taught 18 lessons during their practice teaching before the course started. Additionally, they developed 15 lessons plans for the state examination, which was scheduled 5 months after the course finished. Some of PSTs worked on these lesson plans in groups. All these lessons plans were taken in account to track the PCK development.

Each of the rubrics also provides three general development levels: Beginning, Developing, and Advancing (see table 1). In order to explain more precisely authors' approach to assessment, we present two examples for the rubric Motivation from level 0 and level 1.

Level 0 - Example for "the stated motivation is very formal (neither students' activity nor questions cannot be expected)": Lesson plan for the Topic: How to multiply decimal by decimal numbers

PST: For the motivation, I chose the following task: A farmer stored fuel for the tractor in canisters of 0.5 hl. He has a) 10, b) 5, c) 2, d) 1, e) 0.5 canisters full of fuel. How much fuel does he have for the tractor (write result in hl)?

T (teacher – the author): Explain the reasons why you consider this task as motivating one.

PST: I think, in fact, this is not very motivating task. Maybe students could solve the last case of the task and found out how to multiply decimal by decimal number.

Level 1 - Example for "there are no tight relations between the motivation and the objectives": Lesson plan for the Topic: Definition of the concept Limit of a sequence

PST: Motivation task: In the hotel we have an infinite number of single rooms. Rooms are sequentially numbered by natural numbers. The hotel is fully booked. To the hotel, however, added three other tourists who would like to stay. Is it possible to accommodate them?

T: What is the connection between this task and the definition of the concept Limit of a sequence?

PST: The task is about infinity and when we count limit of a sequence we work with infinity.

Preliminary results and discussion

We try to answer the research questions mentioned in the Introduction.

(1) The concept of inter-rater reliability was used to find the answer to the first research question. The Table 2 depicts numbers of consistent and inconsistent decisions for each particular rubric.

Only the authors' assessment was taken in account because the PSTs' assessment was obviously loaded by the social norms and the relationships within the group. Some inconsistencies between the authors were caused by the usage of the rule, that for one deficiency only one point should be get off and the authors included the same mistake in the different rubrics. As we can see, the rubrics worked well and after precise preparation of their user, they can be considered as reliable tool. The certain vagueness of the developing levels formulations does not seem to be a problem when comes to its identification within the actual presentation.

The rubric	Match	One level differences	Two level differences
Learning objectives	9	1	0
Motivation	9	1	0
Correctness	9	1	0
Didactics Means	7	3	0
Didactics Problem	8	2	0

Table 2: Inter rater reliability

(2) At the beginning, PSTs filled the front page formally (see The course design). Most of the PSTs had no idea about how the front page tailors to remaining part of lesson plan. As the course continued, we could see how PSTs were moving forward in the development of their PCK. We explain it using examples from 3 rubrics (Learning objectives, Didactic Means, Didactics Problem).

(2a) Development of PCK within the group of PSTs - Learning objectives

At the beginning of the course, PSTs did not formulate any learning objectives or formally mention some objectives of prepared lesson but were not able to explain how to achieve it. Also, some of PSTs did not meet the learning objectives they have formulated.

Example from the lesson plan for the topic: The Binomial Theorem

PST: The student is able to formulate the binomial theorem and to write the binomial theorem by using the summation operator.

T: Formulate the binomial theorem and write it by using the summation operator.

PST: *(Started to write on the blackboard, but did not remember correct formulation, then started to look for the correct formulation in the hard copy of the lesson plan.)*

T: The student should be able to formulate the theorem and the teacher is not?

At the end of the course, PSTs started to formulate learning objectives in connection with chosen tasks and their solutions and they also explain more precisely what they expect.

Learning goals for the topic: Increasing and decreasing function

The student is able to identify whether the function is increasing or decreasing from the graph, from the table. The student is able to draw graph of increasing and decreasing function. The student is able to define increasing and decreasing function. The student is able to prove from the formula whether the function is increasing, decreasing or neither one nor the other.

Additionally, in the prepared lesson plan we can found tasks as means to meet the formulated objectives.

(2b) Development of PCK within the group of PSTs - Didactic Means

In the first stage, PSTs chose the tasks superficially, e.g. the PST prepared lesson plan for the Topic: Law of sines and he explain the task selection as follows:

PST: (*Presents some task where students have to calculate side or angle of triangle using law of sines.*)

T: Explain us, why did you choose exactly these tasks.

PST: I found them in the textbook.

T: Did you solve them?

PST: Only the last one because it looked hard.

T: Do you think these tasks will help your students to understand deeply methods of solution using Law of sines?

PST: Maybe some types of tasks are missing. (He started to draw on the blackboard.)

At the last stage the PST who prepared for the topic Definition of the concept Limit of a sequence explained precisely the reasons for each selected task.

T: Why did you solve the absolute-value inequality: $|x - 4| \leq 2$ at the beginning of the lesson?

PST: I chose this task because I wanted to recall the geometric properties of absolute-value which students meet in the definition of the limit.

T: You suggest dividing students into 6 groups and each group will work with different sequence. Explain your reason.

PST: Two groups will get increasing sequence, two groups decreasing and two groups oscillating sequence. I chose these sequences in order to prevent the following misconception: Only decreasing sequence has limit. Oscillating sequence cannot have a limit.

During the presentation of the lesson plans on the state exams the most of students showed that they are better able to intertwine all the items from the front page of the lesson plan with the tasks and their solutions, activities, mathematical explanations. Students were also able to explain better their reasons for selecting the particular tasks and activities for the lesson.

(2c) Development of PCK within the group of PSTs - Didactics Problem

Firstly, most of PSTs did not see any didactic problem and misconception with the most of the topics, some PSTs wrote the most common misconceptions, such as problem with negative sign during working with algebraic expression, or they formulated didactic problem very generally such as students have problems while working with fractions. Later, when PSTs were trying to identify didactics problems and misconceptions, they started to utilize experience from their life (this task/concept/method was a problem for my sibling, friend, me) and also from their teaching practice. The next example illustrates formulation of the misconception based on PST experience.

PST: Students think that the following scalar products $(1,1) \cdot (3,2)$ & $(2,2) \cdot (3,2)$ are equal. To prevent this misconception I formulated the following task: Find out if following scalar products have the same value:

A. $\vec{u} = (1,1); \vec{v} = (3,2); \varphi = 60^\circ$

B. $\vec{u} = (2,2); \vec{v} = (3,2); \varphi = 60^\circ$

Although the task is incorrect it shows PST's effort not only to formulate the misconception but also to look for the way how to prevent it.

Our experience indicates that the authentic assessment focused on PSTs' lesson plans and objectivized by five rubrics, tied with the content knowledge and three types of PCK, can help PSTs to develop their PCK. Presented examples demonstrate development of KCS (identification of didactic problems and misconceptions), KCT (precise selection of tasks) and KC (formulation of learning objectives).

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Measuring beliefs concerning the double discontinuity in secondary teacher education

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In this paper, we discuss a research project to measure prospective mathematics teachers' beliefs towards their perceived disagreements between mathematics at university level and school mathematics that is known as double discontinuity. Firstly, we introduce the double discontinuity problem. Secondly, we refer to the construct of beliefs as the main part of our theoretical framework. Afterwards, we outline our method including a brief discussion on our approach of bridging the double discontinuity problem with so-called teacher-oriented tasks that are appropriate to illustrate connections between university mathematics and school mathematics. Furthermore, our results of pilot studies aiming to measure prospective teachers' beliefs are provided.

Keywords: Beliefs, double discontinuity, prospective teachers, teacher education.

Introduction

Over 100 years ago, Felix Klein coined in the preface of his textbook “Elementary Mathematics from a Higher Standpoint” the term of a “double discontinuity” (Klein, 1908, p. 1). This notion embodies the challenges of transitions in the mathematical socialization of mathematics teachers. The first discontinuity, i.e. the transition from secondary to tertiary education, became a main topic in research on university mathematics education during the last decades (e.g. Gueudet, 2008; Thomas et al., 2015). A challenge that accompanies the second discontinuity is the transformation of academic mathematics gained at university into educational forms of school mathematics (Prediger, 2013; Winsløw & Grønbaek, 2014). As a consequence of both discontinuities, teachers may lose sight of academic mathematics after university studies and, thus, teach on the basis of experiences from their own schooldays (Bauer & Partheil, 2009; Hefendehl-Hebeker, 2013).

Although mathematics instruction in schools as well as teacher education in universities changed considerably since Klein's claim, the phenomenon of a double discontinuity still seems to exist and prospective teachers nowadays frequently believe that the topics of university mathematics do not meet the demands of their later profession in school (cf. Ableitinger, Kramer, & Prediger, 2013; Hefendehl-Hebeker, 2013). The prospective teachers' perception of the mentioned relationships between school mathematics and university mathematics could be regarded as part of teachers' beliefs (Eichler & Isaev, 2016). Considering this background, we primarily rely on teachers' beliefs as the main construct of our theoretical framework in the next paragraph. Subsequently, we outline the method for our research project with a specific focus on pilot studies. Finally, we use the results of these pilot studies to explain the development of our instrument aiming to measure prospective teachers' beliefs concerning the double discontinuity.

Theoretical framework

Prospective teachers' beliefs

On the construct of beliefs we refer to the definition of Philipp (2007, p. 259), who defines beliefs as “psychologically held understandings, premises, or propositions” affecting an individual’s view of the world or a special part of it, such as filters for receiving information. Beliefs can be considered as a component of teachers’ mathematics related affect (Hannula, 2012), which itself can be defined as “a disposition or tendency or an emotion or feeling attached to an idea or object” (Philipp, 2007, p. 259). Thus, in our research we regard prospective teachers’ beliefs as their propositions concerning the relationships between school mathematics and university mathematics. Following Calderhead (1996), teachers’ beliefs are understood not only to impact on teachers’ professional knowledge but also to potentially have an effect on teachers’ classroom practices (cf. also Skott, 2009). Therefore, it seems essential to tackle the issue early and to investigate, how beliefs concerning the double discontinuity develop in secondary teacher education.

In related literature, beliefs are partially seen and sometimes even defined as relatively stable dispositions. In contrast, several studies provided a change of beliefs due to interventions, especially concerning prospective teachers (e. g. Decker, Kunter, & Voss, 2015). In this paper, we aim to provide an instrument in order to investigate, whether and how prospective mathematics teachers change their beliefs concerning the double discontinuity phenomenon based on our intervention project. From this background, we address beliefs being changeable and consider change as “a natural part of the development of beliefs and the reaction of beliefs in the face of experiences” (Liljedahl, Oesterle, & Bernèche, 2012, p. 35).

Related Research

Although numerous projects and institutions across the world aimed to overcome the perceived gap between school mathematics and university mathematics in recent years, not much research has been done on the perception of prospective teachers’ concerning the double discontinuity problem. Winsløw and Grønbæk (2014) distinguished three dimensions of Klein’s double discontinuity, which are not independent but important to separate: the institutional context (i.e. school vs. university), the difference in the subject’s role within the institution (i.e. student at university or school vs. teacher of mathematics), and the difference in mathematical contents (i.e. elementary vs. advanced). In our research, we primarily refer to the content aiming to figure out possibilities of “building bridges” (Winsløw & Grønbæk, 2014, p. 64) and to investigate the effect of bridging activities on the prospective teachers’ beliefs referring to the double discontinuity.

Becher and Biehler (2016) used narratives in order to ask prospective secondary teachers in their third year of university studies about what benefits they see in learning university mathematics for their future career as a school teacher and which aspects are articulated by the prospective teachers in their evaluation of benefits of university mathematics. The results revealed a wide range of prospective teacher’s beliefs on benefits of learning university mathematics with regard to school mathematics. Most of the statements can be matched with one of four levels of mathematical content knowledge based on Krauss et al. (2013), i.e. “A deep understanding of the content of the secondary school mathematics curriculum (e.g., ‘elementary mathematics from a higher standpoint,’ as taught at university)” (Krauss et al., 2013, p. 155). Taking these studies into account, we developed a

questionnaire for measuring prospective mathematics teachers' beliefs towards their perceived disagreements between mathematics at university level and school mathematics.

Method

The institutional frame

Referring to this theoretical framework, the main target of our project is to investigate changes in prospective teachers' perception of a double discontinuity that could be explained by our approach of building bridges. Prospective secondary mathematics teachers in Germany are usually enrolled in the same mathematics courses as mathematics majors (e.g. analysis), particularly in the first semesters. A big challenge for all students in the initial phase of the studies is the task to complete a range of exercises every week as homework (Ableitinger & Herrmann, 2013). These tasks can be solved on the basis of the plenary lectures (by usual four hours per week) and are reviewed in additional small courses (by two hours per week) which are organized by student assistants. Our aim is to develop and establish the desired bridges in these introductory mathematics courses for prospective secondary teachers. Our focus is here to enrich the set of tasks for homework with so-called "teacher-oriented tasks" that are appropriate to illustrate connections between university mathematics and school mathematics to prospective secondary teachers.

Teacher-oriented tasks

We conceptualize specific tasks which potentially demonstrate bridges between school mathematics and university mathematics to a model of domains of teacher knowledge according to Ball, Thames, and Phelps (2008). More precise, we differentiate our teacher-oriented tasks referring to the subdomains of specialized content knowledge (SCK), knowledge of content and students (KCS), knowledge of content and teaching (KCT) as well as curriculum knowledge. One example that represents specialized content knowledge (SCK) within the notion of mathematical tasks for teaching is provided below. In this exercise, "Evaluating the plausibility of students' claims" and "Giving or evaluating mathematical explanations" (Ball et al., 2008, p. 400) are requirements which can be used well to describe the setting.

In the subsequent task from a mathematics contest for students ("Känguru der Mathematik 2009") the participants were asked to solve which of the following figures is the greatest one.

$$(A) \sqrt{2} - \sqrt{1} \quad (B) \sqrt{3} - \sqrt{2} \quad (C) \sqrt{4} - \sqrt{3} \quad (D) \sqrt{5} - \sqrt{4} \quad (E) \sqrt{6} - \sqrt{5}$$

A student in grade 12 chose answer (E) and stated:

" $\sqrt{6} - \sqrt{5}$ is the greatest figure, because roots are monotone. So, the greater is x, the greater is $f(x)$. Thus, their difference is the greatest, as well (by going more to the right)."

1. Analyze the student's answer. Where do you see problems with the argumentation?
2. Provide an own student-oriented answer to this topic.
3. Show in general: $\lim_{n \rightarrow \infty} \sqrt{n} - \sqrt{n-1} = 0$

Figure 1: Task "roots" for prospective secondary mathematics teachers

The design

In order to gain empirical evidence for the efficiency of our method, prospective teachers in the relevant mathematics courses are assigned at random to a treatment group and a control group. While the control group is taught traditionally, the treatment group gets an extra teacher-oriented task for homework every week that focuses on bridging mathematics at university level and school mathematics. Our main research question is whether and how prospective teachers change their beliefs about the double discontinuity phenomenon based on our intervention in the introductory mathematics courses. In this paper we only refer to pilot studies where prospective teachers got homework including an extra teacher-oriented task on a trial basis. The main aim of these pilot studies was to develop an instrument for measuring teachers' perception of a double discontinuity.

During the winter semester 2015/16, two basic mathematics courses at the University of Kassel were selected in which the mentioned teacher-oriented tasks bridging mathematics and mathematics education was administrated: "principles of mathematics" and "analysis". Prospective mathematics teachers attend these courses usually in the first or in the third semester of their university studies. Three prospective teachers from the third semester were interviewed at the end of the semester. The data was analyzed by qualitative content analysis. We also developed a questionnaire with 16 items for measuring prospective teachers' beliefs concerning their perceived disagreements between mathematics at university level and school mathematics. The questionnaire was piloted in a mathematics course for prospective secondary teachers ($N = 60$) and is outlined in the following paragraph.

Discussion of results

The analyses of the interviews revealed that all prospective teachers took the contents dimension into account when reflecting their university studies. For example, the first prospective teacher (PT_1), believes, that in university mathematics, there are too little relations to school with regard to the content.

PT_1: "There are not many connections - direct contents connections, so - as well as in the other lectures. And also in analysis I notice, that it - that the university stuff of the mathematics lecture - almost simply goes beyond school and it is more or less by chance, if there are contents, which fall together with school mathematics - I have the feeling."

Further, two of the prospective teachers also mentioned another aspect of the double discontinuity problem, i.e. their beliefs concerning the relevance of learning university mathematics with regard to school mathematics, such as can be found in the proposition of PT_2.

PT_2: „If I finished school, I would have the same status, which I want to teach the students. And if now deeper questions arise, I would not be able then to answer them, for instance, because I myself have never had this and then - such a teacher one also did not want formerly, who could only tell, what he has just done.“

Indeed, Klein (1908) addresses both aspects in the double discontinuity phenomenon since on the one hand the problems at university may not suggest the things at school, and on the other hand, university studies may remain only a memory with no relevance upon teaching.

The collected data from the interviews also supported the development of our current questionnaire. Our first version (cf. Eichler & Isaev, 2016) contained 9 items using a 6-point Likert scale to assess students' beliefs about the double discontinuity problem. The questionnaire was piloted in a mathematics course for prospective secondary teachers ($N = 60$) and seemed to provide good internal consistency (Cronbach's alpha 0.782). Interestingly, a higher reliability (Cronbach's alpha 0.831) was achieved when regarding only the prospective teachers in the course ($N = 35$) and not all university students in the mathematics course. As a possible reason for this phenomenon, we identified two theoretical discernable domains in our questionnaire. On the one hand, we asked items which contained a personal statement including the words “I” or “me” like “I think that I require a deep understanding of mathematics in order to teach mathematics in school.” On the other hand, we provided a few statements such as “University mathematics has mostly little relation to school mathematics”, which were rather matter-of-fact. These items might have led to a different extent of identification within the different groups of university students. To provide useful information, we applied both dimensions in our current questionnaire containing 16 items.

Taking into account the four levels of mathematical content knowledge based on Krauss et al. (2013), we derived further items from the interviews such as “By the use of university mathematics, gaps are filled in the mathematical knowledge that is required in school”. We grouped all items into three subscales which we identified through our development process: “contents relationship”, “relevance for profession” and “higher standpoint”. In a further pilot study with prospective secondary teachers in higher semesters ($N = 24$), we approved these subscales to be internal consistent¹ with a total reliability value of Cronbach's alpha .911. Moreover, we used our qualitative approach in order to validate our survey afterwards.

Contents relationship – beliefs concerning the connections between university mathematics and school mathematics on the contents dimension (4 items; Cronbach's alpha .821):	
4.	University mathematics offers many parallels to school mathematics with regard to contents.
9.	School mathematics and university mathematics are applied to each other in contents.
11.*	In university mathematics, there are too little relations in contents to school.
12.*	School mathematics and university mathematics are two different worlds with regard to contents.
Relevance for profession – beliefs concerning the relevance of university mathematics for the later profession as a school teacher (6 items; Cronbach's alpha .814):	

¹ 7 Personal items with Cronbach's alpha .845 and 12 non- personal items with Cronbach's alpha .858.

1.	University mathematics is very useful for the teaching profession.
3.*	I will hardly ever need university mathematics after studying.
5.	By the use of university mathematics, I am well prepared to the job profile of a mathematics teacher.
10.	Without university mathematics, I could hardly teach mathematics in school.
15.*	Learning mathematics at university is not so important for the teaching profession.
19.	The relevance of university mathematics for the teaching profession is ²
Higher standpoint – beliefs concerning the usefulness of university mathematics as a higher standpoint for elementary mathematics (6 items; Cronbach's alpha .818):	
2.	University mathematics helps me to get deeper into school mathematics.
6.	By the use of university mathematics, gaps are filled in the mathematical knowledge that is required in school.
7.	By the use of university mathematics, I gain a deeper understanding of concepts in school.
8.	By the use of university mathematics, I understand relationships within school mathematics much better.
14.	Learning mathematics at university promotes me to be in thinking “one step ahead” of the students.
16.	As a mathematics teacher, an in-depth mathematical content knowledge is required.

Table 1: questionnaire for measuring prospective teachers' beliefs concerning a double discontinuity

The possible range of scores for each component is between 1 and 6. Higher scores correspond to more positive beliefs (by reversing the responses to the negatively formulated items indicated with an asterisk*).

Concluding remarks

The main topic of this paper was to discuss our approach of measuring prospective mathematics teachers' beliefs towards their perceived disagreements between mathematics at university level and school mathematics that is known as double discontinuity. In order to be able to investigate changes in prospective teachers' beliefs referring to the double discontinuity problem, we chose a mixed methods design and developed a questionnaire including 16 items that actually seems to measure these beliefs. Grounded on our preliminary results, the following steps of our research will be a comparison between two groups of prospective teachers - one group in a traditional course and one group in a course using the mentioned teacher-oriented tasks to prove if the type of the course has an effect of the prospective teachers' beliefs. Since a variety of other factors may be related to our outcomes, we also collect among others data to study interest (Schiefele, Krapp, Wild, & Winteler,

² Whereas in the previous items the prospective teachers may choose an option in a scale from “strongly disagree” to “strongly agree”, the last item refers to a scale from “very low” to “very high”.

1993) and study satisfaction (Dargel, 2005) in a pretest and a posttest as well as additional items referring to relevant demographic and academic background information.

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Using a fraction learning trajectory as a tool to develop pre-service primary teachers' noticing of students' fractional reasoning

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Since noticing has been identified as a critical skill that teachers must develop, research on how pre-service teachers develop this skill in teacher education programs has emerged. In this study, we focus on how pre-service teachers notice students' fractional reasoning through a task designed taking into account a students' Learning Trajectory of fractional reasoning. Our results show that pre-service teachers' learning of the Learning Trajectory helped them to notice students' fractional reasoning in a structured way: identifying important mathematical elements of the problems and, establishing relationships between the mathematical elements and students' fractional reasoning levels of the Learning Trajectory to help students progress in their fractional reasoning.

Keywords: Noticing, fractional reasoning, learning trajectories.

Noticing and learning trajectories

Noticing has been shown as an important skill for teachers. This skill has been conceptualised from different perspectives (Jacobs, Lamb, & Philipp, 2010; Mason, 2002, 2011; Sherin, Jacobs, & Philipp, 2011) but all of them emphasise the importance of identifying the relevant aspects in teaching and learning situations and interpreting them to make teaching decisions. Mason stated that “noticing is a movement or shift of attention” (Mason, 2011, p. 45) and identified different ways in which people can attend (p.47):

Holding wholes is attending by gazing at something without particularly discerning details.

Discerning details is picking out bits, discriminating this from that, decomposing or subdividing and so distinguish and, hence, creating things.

Recognizing relationships is becoming aware of sameness and difference or other relationships among the discerned details in the situation.

Perceiving properties is becoming aware of particular relationships as instances of properties that could hold in other situations.

Reasoning on the basis of agreed properties is going beyond the assembling of things you think you know, intuit, or induce must be true in order to use previously justified properties as the basis for convincing yourself and others, leading to reasoning from definitions and axioms.

This perspective emphasises the importance of identifying the relevant aspects of the teaching-learning situations (discerning details) and interpreting them (recognising relationships) to support instructional decisions (perceiving properties).

On the other hand, research has shown that when pre-service teachers attend to students learning progressions in a particular mathematical domain, they are better able to make decisions about next

instructional steps (Wilson, Mojica, & Confrey, 2013). In this context, students' learning trajectories (Battista, 2012) can assist pre-service teachers in identifying learning goals for their students, in anticipating and interpreting students' mathematical reasoning and in responding with appropriate instruction (Sztajn, Confrey, Wilson, & Edgington, 2012). Our study is embedded in this line of research and analyse how pre-service teachers' learning of a fractional reasoning Learning Trajectory supports their development of noticing students' fractional reasoning. Our research question is: how do pre-service teachers interpret student' fractional reasoning and respond with instructional actions using a learning trajectory of fractional reasoning?

A learning trajectory of fractional reasoning

A Learning Trajectory consists of three components: a learning goal, learning activities, and a hypothetical learning process (Battista, 2011). A Learning Trajectory includes descriptions of learning activities that are designed to support students in the transition through intermediate stages to a more sophisticated level of reasoning.

The learning goal of the fractional reasoning Learning Trajectory used in this study is derived from the Spanish Primary Education's curriculum: the meaning of fraction and its different representations and, the meaning of fractions operations. This learning goal highlights two key aspects: a) the transition from an intuitive meaning of splitting into equal parts to the idea of fraction as part-whole taking into account different representations, and b) the construction of the meaning of operations with fractions.

The student's learning process takes into account how the student reasoning about fractions develops over time (Battista, 2012; Steffe, 2004; Steffe, & Olive, 2009). We have considered six different levels of students' fractional reasoning (learning trajectory proficiency levels): **at level 1**, students have difficulties in recognising that the parts of the whole must be congruent; **at level 2**, students recognise that the parts could be different in form but congruent in relation to the whole. This allows them to identify and represent fractions in a continuous context but they have difficulties with discrete contexts. They also begin to use unit fractions as an iterative unit (i) to represent proper fractions (although they have difficulties with improper fractions) and (ii) to solve some fraction addition problems with the same denominator; **at level 3**, students identify and represent fractions in discrete contexts recognising that the groups must be equal. They also recognise that a part could be divided into other parts. When comparing fractions, they recognise that the size of a part decreases when the number of parts increases. They can use a part (not necessarily the unit fraction) as an iterative unit to represent proper ($f < 1$) and improper ($f > 1$) fractions. They can also reconstruct the whole using any fraction as an iterative unit (continuous and discrete contexts). In addition, they use intuitive graphical representations to add/ subtract fractions with different denominators; **at level 4**, students can solve simple arithmetic problems with the help of a guide or support. They can do equivalent fractions so that operations can be graphically represented. When they add or subtract fractions with different denominators, they understand that the parts must be congruent to join/separate although they need a guide that allows them to choose the unit correctly. When they multiply, they understand the fraction as an operator " a/b of c/d " and when they divide, they develop two types of reasoning; (i) division as a measure and (ii) division as a partition; **at level 5**, students can operate and solve arithmetic problems symbolically, identifying patterns. They can graphically justify what they do but only in simple situations. At this level, they

are able to interpret the remainder of a division of fractions; **at level 6**, students can explain operations graphically. They do not need a guide to represent fraction operations.

Method

Participants and context

Participants were 31 pre-service primary school teachers (PT) enrolled in a degree to become primary school teachers. They were enrolled in a subject of 150 hours (60/90 attendance/nonattendance) related to teaching and learning of mathematics in primary school. In previous courses, these pre-service teachers had participated in a subject related to Numerical Sense and in a subject related to Geometrical Sense.

Instrument: The task

The task consists of three pairs of primary school students answers, with different learning trajectory proficiency levels of fractional reasoning, to a problem that implies the identification of a fraction (adapted from Battista, 2012) (Figure 1). These answers reflect characteristics of the first three levels of the Learning Trajectory. The answers of Xavi and Victor show characteristics of the level 1 since they are not able to identify that the parts of a whole must be congruent. The answers of Joan and Tere reflect characteristics of the second level since they are able to identify that the parts of a whole must be congruent in continuous contexts but they still do not recognise that a part can be divided into other parts. This last characteristic is evidenced when they say that Figure E is not three quarters because it is divided into 24 equal parts and there are 18 shaded. Finally, the answers of Álvaro and Félix show that not only they are able to recognise that the whole must be divided into congruent parts but also they acknowledge that a part could be divided into other parts.

Pre-service teachers had to answer the next four questions. To answer them, we provided pre-service teachers with theoretical information about the mathematical elements of the fraction concept and about the Learning Trajectory of fractional reasoning used in this study.

Q1- Describe the problem taking into account the learning objective: what are the mathematical elements that the student needs to know to solve it?

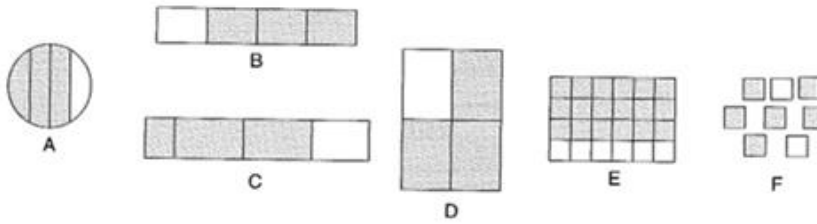
Q2- Describe how each pair of students has solved the problem identifying how they have used the mathematical elements involved and the difficulties they have had with them.

Q3- What are the characteristics of students' reasoning (Learning Trajectory) that can be inferred from their responses? Explain your answer.

Q4- How could you respond to these students? Propose a learning objective and a new activity to help students progress in their fractional reasoning.

These questions and the theoretical information given (Learning Trajectory of fractional reasoning) focus pre-service teachers' attention on relevant aspects of students' answers (discerning details) identifying the relevant mathematical elements; on interpreting these answers (recognising relationships between the mathematical elements and students' reasoning) and on supporting instructional decisions (attending students' mathematical reasoning).

1. Choose the figures below that show $\frac{3}{4}$. Explain your answers.



Xavi and Víctor's answers

Víctor: Mmmm, well we think Figures A, B, C and D represent three-quarters.

Teacher: Xavi, do you agree with Víctor?

Xavi: Yes, A, B, C and D are divided in 4 parts, and 3 are shaded.

Joan and Tere's answers

Tere: We believe that Figures B and D are three quarters because they are divided into four equal parts and three are shaded. Figures A and C have 3 parts of 4 shaded, but the parts are not equal...

Teacher: And Figure E? What do you think about Figure E?

Joan: Figure E is not three quarters because it is divided into 24 equal parts and there are 18 shaded.

Tere: Sure, it is not three-quarters.

Teacher: And the F?

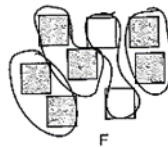
Both: It is not a fraction. In figure F, there are only 6 shaded squares.

Félix and Álvaro's answers

Félix: Well ... yes. We agree with Joan and Tere answers related to figures A, B, C, and D but we think differently about figure E...

Teacher: What do you think? Could you explain your answer?

Álvaro: Well ... mmm sure. If you look each line of Figure E, each line has 6 squares, and as there are 3 lines shaded of the 4 total lines then it is three quarters. In addition, Figure F also represents three quarters because if you group the squares in groups of 2, you get 4 groups of 2, and there are three groups shaded.



Álvaro and Félix answer to Figure F

Figure 1: Task to support pre-service teachers' learning of a fractional reasoning Learning Trajectory to notice students' mathematical reasoning

Analysis

Taking into account Mason's work and the Learning Trajectory of fractional reasoning, we analysed pre-service teachers' answers according to if they had (i) identified relevant elements of fractional reasoning in the student's answers (discerning details); (ii) interpreted the student's reasoning considering the characteristics of students' fractional reasoning from the Learning Trajectory (recognising relationships between the elements identified and the different levels of students' learning progress of fractional reasoning); (iii) made instructional decisions (reasoning about next steps providing different activities that promote students' progression in the Learning Trajectory).

To carry out the analysis, initially a subset of pre-service teachers' answers was analysed by three researchers independently considering the points mentioned above. Then, we put together our respective analyses and compared and discussed our discrepancies until reaching an agreement. Afterwards, new data samples were added to review our allocation.

Results

From the analysis, we have identified three groups of pre-service primary school teachers according to the way that they used the Learning Trajectory to interpret students' fractional reasoning and make teaching decisions. These results show that 20 pre-service teachers were able to use the Learning Trajectory to interpret students' fractional reasoning, while the other pre-service teachers (group 1) had difficulties in using the Learning Trajectory to interpret students' answers. The characteristics of the different groups of pre-service teachers are:

- Group 1. Pre-service teachers who used some mathematical elements of the Learning Trajectory but in rhetoric way or without sense (11 PT).
- Group 2. Pre-service teachers who used the mathematical elements of the Learning Trajectory to recognise different levels of students' fractional reasoning, but they were not able to propose new activities considering the learning trajectory proficiency levels (11 PT)
- Group 3. Pre-service teachers who used the mathematical elements of the Learning Trajectory to recognise different levels of students' fractional reasoning, and proposed new activities to help students progress in their fractional reasoning taking into account the learning trajectory proficiency levels (9 PT)

Group 1: Pre-service teachers who used some mathematical elements of the Learning Trajectory but in rhetoric way or without sense

Pre-service teachers of this group used the mathematical elements implied in the problem (the parts of the whole must be congruent and a part can be divided in other parts) in a rhetoric way when they described students' answers but they did not recognise characteristics of the different Learning Trajectory proficiency levels in students' answers. For instance, the pre-service teacher E27 answered question 3 of the task, pointing out (emphasis has been added underlying the mathematical elements):

Víctor and Xavi: They are at Level 1 of the Learning Trajectory because they do not know the concept of congruence and they do not know that a part could be divided in other parts

Joan and Tere: They are at Level 1 because they have difficulties in recognising that the part must be congruent and they do not recognise that a part could be divided in other parts.

Félix and Álvaro: They are at Level 1 because, related to congruence they know the same that Joan and Tere, although they recognise that a part could be divided in other parts in continuous and discrete contexts.

This pre-service teacher did not recognise differences between students' fractional reasoning saying that all pairs of students have difficulties with the mathematical element *the parts of the whole must be congruent* although he used the mathematical elements to describe students' answers.

Group 2: Pre-service teachers who used the mathematical elements of the Learning Trajectory to recognise different levels of students' fractional reasoning, but they were not able to propose new activities considering the Learning Trajectory proficiency levels

Pre-service teachers of this group used the mathematical elements of the Learning Trajectory that correspond with the problem (the parts of the whole must be congruent and a part can be divided in other parts) to recognise the different levels of students' fractional reasoning. However, these pre-service teachers did not justify a new activity taking into account the students' fractional reasoning. For instance, the pre-service teacher E09 answered to question 2 and 3 for each pair of students (emphasis has been added underlying the mathematical elements):

Víctor and Xavi have difficulties in recognising that the parts must be congruent as they identify as a $\frac{3}{4}$ figures A and C whose parts are not equal. Another characteristic that we can identify is that they have difficulties in recognising that a part could be divided in other parts. They do not notice that figures E and F are divided in 4 parts, maybe they notice that E has 24 squares and F has 8 squares. Thus they do not realise that both are equivalents. So, these students are at Level 1.

Joan and Tere are able to identify and represent fractions in a continuous context recognising that the parts must be congruent as they recognise that, although figures A and C are divided in 4 parts and 3 are shaded they do not represent $\frac{3}{4}$ because the parts are not congruent. They also identify that B and D are $\frac{3}{4}$. They are not able to recognise that a part could be divided in other parts/consider a group of parts as a part since they do not identify that even though E and F are divided in more parts, they represent $\frac{3}{4}$. So, these students are at Level 2.

Félix and Álvaro agree with Joan and Tere about figures A, B, C, and D, thus they recognise that the parts must be congruent. Furthermore they recognise that a part could be divided in other parts and they identify fractions in discrete contexts since for figure E they say that, although it is divided in 24 squares, it represents $\frac{3}{4}$ because there are 4 lines with 6 squares each and 3 of those 4 are shaded (they recognise the equivalence $18/24=3/4$). Besides of that, in figure F they group in pairs the eight squares of the whole to represent the $\frac{3}{4}$. So, these students are at Level 3.

Nevertheless, this pre-service teacher was not able to propose a specific activity considering the Learning Trajectory in order to help students progress in their conceptual reasoning. For instance, this pre-service teacher proposed for the first pair of students: *“With Víctor and Xavi we would work with the recognition that the parts must be congruent. To do that, we could propose the same task but with other figures and they (students) could represent $4/6$ ”*.

The answers of this group of pre-service teachers indicated the difficulty of making instructional decisions considering the Learning Trajectory proficiency levels.

Group 3: Pre-service teachers who used the mathematical elements of the Learning Trajectory to recognise different levels of students' fractional reasoning, and proposed new activities to help students progress in their fractional reasoning taking into account the learning trajectory proficiency levels

Pre-service teachers of this group, after using the mathematical elements (the parts of the whole must be congruent and a part can be divided in other parts) to recognise different levels of students' fractional reasoning, proposed new activities focused on helping students progress in their fractional

reasoning according to the learning trajectory proficiency levels. For example the pre-service teacher E25 proposed the next objective and activity to help Victor and Xavi progress in their fractional reasoning:

Objective: In order to progress from Level 1 to Level 2, students have to recognise that the parts of a whole must be congruent (although they could be different in form).

Activity: Represent in the following figure (square) $\frac{2}{4}$ in three different ways

This group of pre-service teachers used their knowledge of the Learning Trajectory to interpret students' fractional reasoning, and proposed new activities to help students develop their fractional reasoning.

Discussion and conclusions

The aim of this research was to analyse how pre-service teachers' learning of a Learning Trajectory of fractional reasoning supports their development of noticing students' fractional reasoning. We focus on how pre-service teachers interpret student' fractional reasoning and respond with instructional actions using a learning trajectory of fractional reasoning.

Twenty out of thirty-one pre-service teachers who participated in the task were able to use the mathematical elements to interpret students' fractional reasoning considering the characteristics of the students learning progression of fractional reasoning and identifying different levels of students reasoning. This result indicates that the information about a Learning Trajectory of a particular mathematic topic can be used by pre-service teachers to begin to notice features of students' mathematical thinking in a particular domain and therefore, to develop the skill of noticing. The Learning Trajectory can be seen as a powerful tool that help pre-service teachers focus their attention on important mathematical aspects of the problem, on the students' mathematical reasoning and on making instructional decisions on the basis of students' mathematical reasoning. The other eleven pre-service teachers had difficulties in using the Learning Trajectory to interpret students' answers. This result is in line with other studies that have shown that interpreting students' mathematical reasoning is a challenging task for some pre-service teachers (Llinares, Fernández, & Sánchez-Matamoros, 2016; Sánchez-Matamoros, Fernández, & Llinares, 2015).

However, only nine out of these twenty pre-service teachers could use their interpretations of students' fractional reasoning to propose new activities according to the Learning Trajectory in order to help students progress in their fractional reasoning. Previous research has pointed out that the skill of making instructional decisions is the most difficult one to develop in teacher education programs (Callejo & Zapatera, 2016; Ivars & Fernández, 2016; Llinares, Fernández, & Sánchez-Matamoros, 2016; Sánchez-Matamoros, Fernández, & Llinares, 2015). Nevertheless, approximately one third of the participants, in our task, were able to design an activity to promote students' progressions of fractional reasoning according to the Learning Trajectory. Therefore, we think that the task of our study, designed according to a Learning Trajectory, seems to have a relevant paper in the development of the skill of providing activities that could help students progress in their learning. The Learning Trajectory could be seen as a referent or guide for pre-service teachers that could help them to link the mathematical domain (mathematical elements), the student's reasoning and the instruction that considers students' learning progressions.

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Collaborative research in in-service teacher professional development

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Teachers in Iceland are faced with challenges to differentiate teaching as they implement a policy of inclusive education. This collaborative inquiry into teaching of mathematics aims at learning to understand how teachers develop their mathematics teaching through participating in a developmental research. Seven primary teachers worked at improving their mathematics teaching and researched their practice together with a teacher educator for three years. Narrative inquiry was used as an analytical tool to study the teachers' learning. In this paper the focus is on one of the teachers and her learning from participating in the project. The results indicate that she gained confidence in teaching mathematics in diverse classrooms while participating in workshops and that collaborative research can support teachers in developing their practice when meeting new challenges in their work.

Keywords: In-service teacher education, developmental research, collaborative inquiry.

Introduction

This paper reports on findings from a three-year qualitative collaborative inquiry into mathematics teaching and learning with the purpose of deepening our understanding of how teachers meet new challenges in their classrooms. The aim was to learn about the processes that emerge through collaborative inquiry between classroom teachers and a teacher educator. In this paper the focus is on one of the teachers', Pála, and her development in teaching mathematics while participating in the project. The research question that will be answered is:

In what way did Pála affect the learning developed within the project and how is her participation reflected in her mathematics teaching?

The study built on earlier research on teacher development in mathematics teaching in Iceland that revealed that teachers take a passive role in their mathematics teaching and lack experience in creating meaningful learning environments for all children (Guðjónsdóttir, & Kristinsdóttir, 2011; Savola, 2010). They have particularly focused on instrumental understanding as opposed to relational understanding (Skemp, 1976) and emphasised that their pupils learn to carry out the steps of the 'traditional' algorithm (Fosnot & Dolk, 2005). My fellow teacher educators and I have found that if teachers are given opportunities to collaboratively investigate 'with' mathematics and solve mathematical problems, they discover how the different experiences they bring into the community can contribute to their own understanding of the mathematics involved, as well as how individuals learn mathematics (Guðjónsdóttir & Kristinsdóttir, 2011; Gunnarsdóttir, Kristinsdóttir, & Pálsdóttir, 2013). In our work with pre- and in-service teachers, we found that they must be offered opportunities to experience learning that enhances inclusive education. Our results correspond with those of Bredcamp (2004) and Moore (2005), who emphasised that if teachers' work is expected to be aimed at diversity and mutual understanding, they require the opportunity to develop and enhance their knowledge about teaching and learning in an environment that reflects the very same aspects that they are expected to foster in their own classrooms.

Teacher development in inclusive settings

Teaching children mathematics requires teachers to understand how their students learn mathematics and they need to be skilled both in mathematics and pedagogy as well as the knowledge that combines knowing about teaching and knowing about mathematics. In recent years the Nordic countries have emphasized mathematical competences of which eight specific mathematical competences were identified. These form two clusters; the ability to ask and answer questions in and with mathematics; and the ability to deal with mathematical language and tools (Niss & Højgård, 2011). Niss & Højgård also outlined a model for mathematics teacher competency where the ability to develop one's competency as a mathematics teacher as well as the competencies of working with students and others towards professional development were identified. It is important to note that development of teaching in classrooms is dependent both on the teachers' knowledge and their ability to learn together with others, both their students and colleagues.

Attention and awareness are important features of mathematics learning. Mason, (1998) holds that teaching is fundamentally about attention and teachers can enhance pupils' attention by attending to their own awareness. When someone else points something out to us our awareness changes slightly; we become more explicitly aware of some features, and less aware of others. Thus in collaborating with colleagues, teachers are afforded the ideal conditions in which work on their own awareness, which can provide conditions for their students to experience them too.

When gaining competence in teaching mathematics teachers build on their knowledge and experience and an essential factor in this process is the participation in learning communities. In order to be able to support learners in their classrooms in acquiring mathematics competence, teachers need to urge their pupils' to ask probing questions, take risks and learn from their mistakes.

In communities of learning the individual learner draws on knowledge in the community as well as on personal knowledge. Nevertheless the main emphasis has been on individualized learning in response to diversity in classrooms. Schools have thus adopted what Ainscow (1995) called integration by making only a limited number of arrangements for including all learners in classroom activities. Askew (2015) argued that learning communities are more inclusive than taking the individual as the starting point for planning learning experiences. In these communities teachers work with the collective construction of mathematical knowledge while still ultimately addressing the needs of the individuals within that community. This is the same position I took in working with teachers, attending to their diverse needs for improving their teaching and finding ways to work in inclusive ways with diverse groups of learners.

Through collaborative activity a community learns from the thinking, practices, and development of the individual. Important features of such communities are discussions about the mathematics attended to in the classroom. In the communities learners listen to each other's solutions and think about connections to their solutions while helping each other refine their methods and explanations. When learners participate in mathematical practices in whatever way they can diversity is no longer an obstacle to classroom talk. It is thus being enriched through the diversity of learners' contributions (Askew, 2015).

Methodology and methods

The study is a collaborative inquiry into mathematics teaching and learning (Goos, 2004), and the aim is to build a co-learning partnership between teachers and a researcher in order to support classroom inquiry (Jaworski, 2006). The methodology of developmental research (Gravemeijer, 1994) and the ‘developmental research cycle’ (Goodchild, 2008) guided the cyclic process of the research.

In an attempt to make explicit the ‘practice’ in which teachers and researchers participate when collaborating, Jaworski (2003) suggested shifting from the notion of community of practice (Wenger, 1998) to that of ‘community of inquiry’, where teaching is seen as learning-to-develop-learning. In such a community, teachers and researchers both learn about teaching through inquiring into it. In this project the vision was that all the participants would learn about teaching mathematics in diverse classrooms.

For three years I worked with seven teachers at 90-minute workshops on monthly basis. They taught 10 to 12 years old pupils in two neighbouring schools, four were homeroom teachers and three were support teachers that joined them in mathematics classes. The focus of the workshops was on reflection on mathematics, and on mathematics teaching and learning. To help the teachers develop their own understanding of mathematics, we worked with problems that had the potential to promote mathematical activity and thinking as well as to stimulate collaboration where discussions and sharing thinking were meaningful. We also discussed new research on mathematics education and stories from the teachers’ classrooms, reflected on their pupils’ mathematics learning and considered how their mathematical thinking developed. To learn about the teachers’ visions for the project and the cultures in their mathematics classrooms I interviewed them and observed their classrooms at the outset of the project, after the first year, and one year after the last workshop. Data was collected of videotapes from workshops, audiotapes from interviews and notes from classroom observations.

Narrative inquiry was used as an analytical tool to study the teachers’ learning in participating in this project. It is a way of understanding and researching experience through collaboration between a researcher and participants and to research with practitioners their lived experience as a source of their knowledge and understanding (Clandinin, 2013). The stories the teachers told about their work, at the workshops and in interviews, are the basis of the narrative inquiry. The teachers read the drafts of their narratives and commented on them, and then on the final version.

Findings

Pála had been a general classroom teacher over 30 years in grades 5-7 when she participated in the collaborative project. In her teacher education her focus was on language skills and she had attended many in-service courses about language teaching but only a few about mathematics teaching. As a classroom teacher she taught mathematics to 10-12 year old children.

Emphasis on instrumental understanding

When I observed Pála’s classroom at the outset of the study her emphasis on carefully describing the steps of algorithms was dominant. She started the lesson by reviewing homework and then

discussed the content of the lesson. She described carefully to her pupils how to work through the problems in their textbook that she wanted them to solve.

Pála worked closely together with her colleague Dóra, at teaching pupils in their 5th grade classrooms. At our first workshop Dóra wanted to discuss the teaching of ‘traditional’ algorithms (Fosnot & Dolk, 2005). She had discussed the algorithm of long division with Pála and questioned her belief that is necessary for their pupils to learn the steps of the algorithm. Pála added:

What we have been reflecting on is, is it bad, does it spoil anything for them? Does it destroy their thinking process, does it stop anything?

Pála was eager to learn more about how to teach children to calculate. She had emphasised the memorising of facts and at her school children were regularly tested on multiplication facts. Dóra had also questioned this tradition and Pála was starting to review her beliefs about instrumental understanding (Skemp, 1976).

Reviewing her own way of calculating

Pála was eager from the beginning to improve her own way of solving mathematical tasks. When we at our forth workshop discussed how many cans there were needed to build a ten storey tower of cans she said:

There would be 10 here [points to the bottom row of the 10 storey tower she drew]. Then I would count 9 and 1, 8 and 2, 7 and 3, 6 and 4. Then I have 10, 20, 30, 40, 50 and then add these 5 [points to her drawing for each step] and have got 55. I do this to be quick at counting.

Pála was reflecting on her own way of calculating when she said that she did this to be quick at calculating thus attending to her own awareness of learning (Mason, 1998).

As the project developed Pála brought in problems she had been solving with her pupils and wanted to discuss her understanding of the problems with us. At Workshop 15 she told us about her discussions with her pupils about how many handshakes there would be in their class if they all shook hands with each other. The children decided to try this and were quick to realise that they would only shake hands once with each person. They developed a rule that could be used to calculate the handshakes in their group of 15 pupils: $14+13+12+ \dots +2+1$. They then split into smaller groups to test if their rule could be applied to a group of any size. Pála had not thought about the solution of this problem before it was discussed in her class and therefore took an active part in the solution process. By comparing the total handshakes for different number of pupils, they then had developed a formula together. Pála was keen to discuss with us whether the formula $n(n-1)/2$ could be applied to calculate the handshakes for a group of any size. Pála said:

I do not understand why this equation works, why this connection. I know it works, we have tried it for many cases. Can you help me to understand why it works? I would like to proceed to work with the children in this way.

By asking us to discuss her experience with us Pála was adding to her competence of learning together with colleagues and in discussing with her pupils she was developing her competence in learning with them. She was also supporting her pupils in developing their the ability to ask and answer questions in and with mathematics (Niss & Højgård, 2011).

I reminded Pála on her earlier addition of consecutive numbers in relation to the tower of cans. Pála said that she remembered it but she still could not understand why the formula she had developed with her pupils worked. We then discussed their formula and why it could be used to calculate the handshakes and in doing so we were inquiring into our own mathematics learning (Goos, 2004) and cultivating our learning community (Jaworski, 2003). I pointed out that she took an active part in the learning process in the classroom. Not only did she learn about the children's thinking but also about her own thinking about the problem. She had given them a problem that neither she nor they knew beforehand how to approach. Then they all started to investigate and look for patterns and developed a rule together. Through these discussions our co-learning partnership was cultivated as we focused on classroom inquiry (Jaworski, 2006).

Learning together with her pupils

Pála was starting to learn together with her pupils by exploring with them in the classroom as opposed to the beginning of our collaboration when she had carefully explained to her pupils, how to solve problems. At our final workshop she shared with us her discussions with her pupils. They had worked with different kinds of word-problems in their textbook. They were required to write their solutions to the problems with algebraic expressions. She gave examples of the pupils' discussions about the problems and how they wrote the expressions. She had recorded these examples in her notebook and now wrote on the whiteboard to show us how the pupils calculated and how she interpreted their thinking about the problems.

We discussed two of the problems:

Klara is 4 years younger than her brother Kári. Their total age is 18 years. How old is Kári?

A large apple costs 11 ISK more than a small apple. The total price of a small apple and a large apple is 59 ISK. What is the price of a large apple?

Pála had solved the problems herself and her thinking was different from her pupils' but they all came to the same conclusions. She wanted to discuss this experience with us and hear my interpretation of the different ways they solved the problems. She was particularly keen to hear my opinion with regard to the way she had accepted her pupils' way of solving a problem instead of telling them to think about it in the same terms she did.

Jónína: Pála, you said that the children wrote $x+x+4=18$ and you wrote $x+x-4=18$.

Pála: Yes. And for the apples they wrote $x+x+11=59$ and I wrote $x+x-11=59$.

We discussed how the value of the unknown variable in Pála's equation was different from the value in the children's equation. Still in both cases they came to the same conclusion about the age of the siblings and the price of the apples. Pála said that all the children in her class were able to solve the word problems by first trying some numbers and then adjusting them until they found the right numbers. Many of them could write the equations and they then supported each other in doing so. Finally Pála concluded: "These were just my thoughts. I found it interesting to see how they understood and thought about this".

When Pála shared this story with us she was cultivating our learning community (Askew, 2015; Jaworski, 2006). But she had also attended to her pupils' way of learning and was now focusing on their way of expressing themselves instead of describing carefully to them the steps they needed to

take as she did to begin with thus making herself aware of her pupils diverse ways of learning (Mason, 1998).

Grouping pupils into ability groups

In Pála's school it had been the custom for many years to group children into groups in mathematics classes based on the outcomes of an end of term test. When the project began Pála and Dóra had divided the 43 children in fifth grade into three groups in mathematics classes. A special education teacher taught the pupils who got the lowest grades, Dóra taught those who got the highest grades and Pála taught the middle group. With this arrangement they were responding to diversity by making only a limited number of arrangements for including all learners in classroom activities (Ainscow, 1995). To begin with Pála was concerned that the pupils in her group were not capable of solving problems without her leading them step by step. Gradually as she became more confident with exploring with mathematical problems herself she started to listen to them and allow herself to join them in their explorations with problems as discussed above.

When Pála shared her experiences of working with her pupils with us we discussed how her approach supported the children's learning like the case with the handshake problem. She told us that some of her pupils understood why the formula could be applied to solve this problem and others did not. They though all understood that they could calculate the total number of handshakes by adding $(n-1) + \dots + 1$. We then related to our former discussions of tasks that can be solved at many levels and are therefore suitable to work with in diverse classrooms. Pála was satisfied with this experience and found that she was beginning to trust that all her pupils were capable of more in-depth learning than she had realized before thus acknowledging that diversity is no longer an obstacle (Ainscow, 1995; Askew, 2015).

The final year the project was running Pála and her close colleague, Dóra, had decided not to group their pupils into ability groups any more. They had become confident in investigating in mathematics with their pupils and found that all the children in their classes were capable of learning together and gained from sharing experiences with each other.

Professional development and influence on our project

Pála took an active part in using the tools for professional development that I offered the teachers in our learning community. She visited her colleague's classrooms and discussed with them what they learned from their visits and she recorded her lessons to learn from her communication with her pupils. She also shared her experience from her learning in the classroom with us and gradually started to lead what to focus on at our workshops. Not only did she share this experience with us she also brought in problems she had found elsewhere and asked us to solve them with her.

The project was only planned for one year to begin with. As we approached the end Pála expressed her wish to meet for a second year. She felt that she and the other teachers were just starting to develop their teaching and could not stop when they felt that they were gaining so much from our collaboration. The other teachers agreed with her and our project ran for three years as the teachers wished to extend it for the third year. With her willingness to share her thinking with us and take lead in what to focus on at the workshops Pála shaped the developmental process of the project and affected the 'developmental research cycle' (Goodchild, 2008).

Conclusions

Based on the narratives of Pála's participation in the collaborative project I have concluded that she gained confidence in teaching mathematics in diverse classrooms and that collaborative research can support teachers in developing their practice when meeting new challenges in their work. The sketches from our collaboration are representative for the learning that emerged during our collaboration. In the communities of inquiry we managed to build at the workshops we supported each other in learning-to-develop-learning (Jaworski, 2003) by reflecting collectively on the stories the teachers told of their classroom experiences. From the stories Pála told us we learned how her pupils' competences in dealing with mathematical language and tools were developing as well as their ability to learn about their own learning in working with their pupils (Niss & Højgård, 2011).

By offering the teachers opportunities to experience learning that enhances inclusive education (Bredcamp, 2004; Moore, 2005), the teachers were empowered to develop their teaching as was reflected in Pála's learning.

During our three years of collaboration I, as a teacher educator and a researcher learned about teachers' capabilities to develop their own teaching if they are supported in reflecting on their learning of mathematics as well as their pupils' learning. In reflecting on their learning about mathematics teaching my understanding has deepened of the opportunities and challenges teachers meet when including all learners in meaningful mathematics learning.

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Integrating teachers institutional and informal mathematics education: The case of ‘Project ArAl’ group in Facebook

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After a sketch of our ArAl project devoted to teaching/learning early algebra, we introduce our ‘Progetto ArAl’ group in Facebook, conceived not only to share and discuss among teachers didactical experiences, theoretical questions and materials but, more in general, to educate in informal way teachers in early algebra. For its features it can be said a non standard group (NSG) in Fb. The main question we put ourselves is: may a NSG become a community of practice? To find an answer to this question we compared our group with a larger Italian group devoted to mathematics at primary school analyzing the interactions in the two groups launched by some common members. On the base of this comparison we delineate some hypotheses for the management of a NSG as a community of practice, where well known mentors and transparent theoretical guidelines allow the teachers consciously to approach the theory for the practice.

Keywords: Early algebra, community of practice, informal on line education, teachers’ professional development.

Introduction

The ArAl project belongs to the stream of studies devoted to the renewal of the teaching in the arithmetic-algebraic area in the perspective of early algebra. It is characterized by the intertwining among: a) the activation in the classes of innovative didactical paths on early algebra; b) educational processes of teachers based on the critical analysis of the mathematical discussions developed inside the didactical paths. It promotes a relational approach to arithmetic of linguistic and metacognitive type, to be realized through socio-constructive modalities. The classroom activities are based on the negotiation of a didactical contract for the solution of problems according to the principle: “first represent, then solve”. For room questions we cannot discuss deeply our theoretical frame (we refer to Cusi et al. 2011 and related references), here we simply recall some key aspects of it: (i) the plurality of representations of a given quantity, beyond the canonical decimal representation¹; (ii) the identification and making explicit algebraic relationships and structures underpinning concepts and representations in arithmetic²; (iii) the initiation to the essential algebraic cycle: *representing*,

¹ For instance, the number twelve, has a canonical representation in base 10, i.e. 12, but expressions such as 3×4 , $(2+2) \times 3$, $36/3$, $10+2$, 3×2^2 are other ways to express the same quantity, we call them non-canonical representations of 12, each has its sense related to the process that characterizes it and offers pieces of information about the number. Being able to fluently shift among these forms allows pupils to easier recognize structural similarity among different numbers and to build the basis for understanding scriptures as $a.b$, $-4p$, x^2y , $k/3$).

² For instance to see the equal sign, in writings such as $3+4=7$, not only in its *procedural* sense of connection between an operation and a result, but in its *relational* sense, as an indicator of equivalence between *two different representations of the same number*).

transforming, interpreting (Bell 1996) through *the devolution to the students of: a) the formalization of verbal relationships* individuated during explorative numerical activities (process named by us *algebraic babbling*); (b) *the interpretation of simple algebraic sentences* both in themselves and with reference to a given context³; (v) the stress on natural language as didactical mediator in the slow construction of syntactic and semantic aspects of algebraic language⁴.

Our work with and for teachers has always been realized in a community of practice, or better of inquiry in Jaworski sense⁵, where the practice of the researchers and the one of the teachers meet, compare and develop in co-partnership and where, in addition to theory, methods and aims, values and expectations are shared (Cusi & Malara 2015). Because of the teaching in an early algebra perspective requires in the teachers, mainly the ones of primary school, a deep rebuilding of knowledge, beliefs, behaviours, and manners in the class, we have conceived specific modalities and apposite tools for teachers education. We simply recall here our *Multicommented Transcripts Methodology*, we have enacted to promote in teachers awareness of their own ways of being in the class and to guide them in managing mathematical discussions. Key tools of this methodology are the teacher's transcriptions of the classroom discussions enriched by written multiple comments (by tutors, maths educators and other teachers), *the MTs*. The joint reflections on each MT attain a shared development of the theoretical frame, of the classroom methodologies and of the teaching materials that shall create the basis for the teachers' professional evolution. The productions of MTs became in the time a distinctive character of the Project Aral membership.

The ArAl Project Group in Facebook

Along the years many times we have been asked to make available ArAl materials to a greater number of teachers; for this in 2014 we opened the 'ArAl Project' group in Facebook. Fb is mainly used as a way to share experiences, practices and materials among teachers and other professionals (see for

³ For instance, to recognize that the sentence $85=4 \times 21+1$ represents 85 through the quotient and the remainder of its division by 4 (or by 21), and in the same time - looking for the letter which stays at the 85° place in a sequence generated by the ABCD module – to recognize that the same sentence allows to understand what the letter is (the pupils have to interpret the term 4×21 as the part of the sequence done repeating 21 times the module and the remainder as the number of place of the letter in the successive module).

⁴ For instance, two pupils express in natural language, and then translate in mathematical language, their different ways to calculate the number of pearls in the necklace ○○●●●○○●●●○○●●●○○●●●○○●●●○○●●●: the first pupil says “I counted white and black pearls and I added them” and translates: $2 \times 6 + 3 \times 6$; the second says: “I saw that there are 6 groups, each group has 2 white and 3 black pearls and I multiplied 2 plus 3 by 6 and wrote $(2+3) \times 6$. The comparison of the two sentences allow the pupils to gain experience about the distributive property.

⁵ We recall that a community of practice (CP) is constituted by a group of people who share a craft and/or a profession. The group can evolve because of the members' common interest in a particular domain or area, or it can be created specifically with the goal of gaining knowledge related to their field (Lave and Wenger, 1991). Jaworski (2003), referring to the joint work developed between maths educators and in service teachers about classroom teaching-learning processes, introduces the construct of Inquiry Community (IC) and underline that what distinguish a CI from a CP is that all the participants engage with inquiry as a tool to develop meta-knowing, a form of critical awareness that manifests itself in inquiry as a way of being.

example Bodell & Hook 2011, Manca & Ranieri 2014), but in the last years it has also been used in educational activities for teachers (Staudt et al. 2013, Van Bommel & Liljekvist 2016). In our case, the initial idea was to spread themes and principles of early algebra among teachers and to motivate and help them in approaching it in the class but also to observe new, spontaneous didactical experiences, arising under the stimuli offered by the ArAl institutional courses. We believe that the Fb group can be a way to integrate institutional and informal education offering the teachers new occasions to promote their professional development. We started inviting expert teachers collaborating since long time in our project to become supporters of the group and to share their experiences with the teachers, recently involved in ArAl courses promoted by the schools, who have been invited to become followers of the Fb group. The fundamental methodological choices in managing the group are: our daily on line presence and prompt reactions to the teachers posts; the stimuli offered by the expert teachers posts through videos or pictures of classroom activities; our periodical posts about: mathematics questions and related theoretical references; examples of innovative activities, equipped by MTs, papers, powerpoint presentations for deepening the discussed questions and stimulating free experiments among the followers. The posts in Fb are classified in: ‘like-agree’ interventions; ‘propositive-constructive interventions’, doubtful-skeptical interventions; moreover meaningful sets of interventions related to interesting mathematical teaching questions are collected and commented in files put in our website. Periodical analysis of the data allowed us to highlight the interplay between our interventions and the teachers’ ones, and to reflect on the teachers change. We discuss their evolution according to three temporal phases.

First phase (scholastic year 2014–15). In this first period, in front of a small group of teachers (in most part coordinators in the schools of the ArAl project activities), who were very active in posting documents related to their class activities as well as in commenting other posts, the other members were not so active, and their comments often were short and superficial. These teachers appeared *awed*: the most part of them had a feeble or null control over the early algebra topics and the strong difference among the competence of the expert teachers in the group and their knowledge in the arithmetic-algebra area did not encourage them to do more ‘important’ interventions. At the same time every day new members enrolled to the group. Some more expert teachers, members both in our group and in other groups for maths teaching, suggested us to visit them and in particular invited us to take part into the group ‘Mathematics at primary school’ (one of the most numerous and active Italian groups on maths teaching in the web, more than 5000 members), to offer our interventions whenever we seemed appropriate to do so. We call this last group a Standard Group (SG), in the sense that there are not pre-established leaders and that the exchange takes place freely through the sharing and negotiation of the individuals’ knowledge. The comparison with the SG and other groups dealing with teaching issues brings in evidence that ArAl Project group is different from them, mainly for two reasons: (a) it deals with a well defined subject area, early algebra, it is structured according to a clear theoretical perspective for facing it, and it proposes methodologies, problematic situations, tools fitting with this framework; (b) it is daily supported by us and it is animated by experts teachers who may act as mediators among the members. Therefore we call it a non-Standard Group (NSG).

Second phase (scholastic year 2015-16): In this period we had continued to enter, as previously, examples of didactical activities, MTs, papers, powerpoint presentations but, at the same time - on our initiative or invited by the teachers - we had become more active in intervening on posts both in SG and in NSG. By way of example of this change in our strategy we focus on an episode: a post

inserted by a teacher which received great attention (154 likers and 75 comments), started in the SG and developed, through reciprocal sharings, also in the NSG. The initial post contained a link to a note inserted in the Unit 12 of ArAl project and presented in the form of FAQ in www.progettoaral.it site. In this post it is developed a critical analysis of a typical Italian school practice, supported also by many textbooks, for introducing in primary school the decimal system of representation of the natural numbers: the indication of the units with the letter 'u', the tens with 'da', the hundreds with 'h' and so on (the so-called 'marks'); thus there follow improper equalities such as $653=6h+5da+3u$. Because of the impasse generated in the SG, a follower - a member of both groups - asked us for an intervention on this topic. In a comment of a theoretical and linguistic type Malara wrote:

“The symbols h, da, u represent words of Italian language. They are categorical terms that refer to orders of magnitude and they are used as ‘indicators of quantity’. They are useful for bringing the pupils to shift from the experience with the abacus - where an assigned quantity is split into opportune multiple of powers of 10, operating for successive groupings of 10 – to the representation of the result of this operation through a string of symbols, each between 0 and 9 (extremes included), from which the name of the given quantity was born. This means that, for example the string 6h, 5da, 3u synthetizes the verbal sentence ‘the quantity is constituted by six hundreds, 5 tens and 3 units’ which generates the name of the number 653. The translation into the arithmetical language of this verbal sentence requires the conversion of the term ‘hundreds’ in the arithmetic operator ‘ $\times 100$ ’, the term ‘tens’ in ‘ $\times 10$ ’ and the ‘units’ in ‘ $\times 1$ ’ and the conversion of the connector ‘and’ in the operation of addition ‘+’. So, the total verbal sentence is translated into ‘ $3 \times 100 + 5 \times 10 + 3 \times 1$ ’. The sentence $653=6h+5da+3u$ is improper because it mixes the two languages, verbal and arithmetic, and confuses the metacognitive plan with the operational one”.

While the debate on this issue was developing in the GS, many teachers did not understand why in the ArAl project sentences as $653=60+50+3$, $653=6 \times 100 + 5 \times 10 + 3 \times 1$, $653=6 \times 10^2 + 5 \times 10^1 + 3 \times 10^0$ were proposed as correct and not the one they used, and opposed resistance to accept the explanation that $653=6h+5da+3u$ is to be discarded because it is not a correct representation in mathematical language. To facilitate this understanding, the improper mingling between verbal and arithmetic languages has been pointed also using examples of verbal sentences with words in two languages; the discussion then focused on the correct and incorrect representations of a natural number, the concept of ‘equality’ and on the meanings of the symbol ‘=’.

Third phase (June 2016 to now) The analysis of the dynamics arisen and the kind of the comments posted in NSG and SG led us to the identification of some thematic questions who have given us valuable indications on a question that we did with increasing frequency: may a group with the characteristics of ArAl group become a *community of practice*? If the answer is yes, in which ways may this happen? How may a gradual constitution of a library of shared knowledge be put in place? This leads us to identify some answers to these questions concerning the prevailing attitudes of teachers who enroll in these groups. We discuss them articulating in the following points.

Features of a SG and of a NSG

Members of a SG feel all *equal*: they exchange information, questions, requests without demanding to receive in-depth and substantive answers; they hope to share with their peers working suggestions which are at the level of their knowledge and of their willingness to get involved. Individual growth stems from the strength of exchanges and the wealth of experiences put into circulation. Internal

leaders emerge, who often are recognizable more by the diligence than by the quality of interventions; they often are the most convincing not for their knowledge but because they expose themselves more than others, writing frequently comments. Members may find appealing ideas for new activities but their enthusiasm is not supported by an adequate knowledge; they express insecurity when discussing their colleagues' proposals of those embryos of new ideas. Everyone feels free to comment on impulse. On the contrary, a NSG as 'Progetto ArAl' gives the majority of subscribers some (cultural and psychological) constraints that limit them in exposing their contributions. The same dynamics occur in a working-group in which an expert is present. But then: if it is understandable, for the reasons explained, that a GS exceeds 5000 subscribers, how has to be interpreted the success of our NSG that in two years is approaching 1000 members? The answer could be given with a metaphor: the members *have the impression of living a moment of institutional training*. They know that in the NSG there are experts involved in the discussions, extemporary comments should be avoided and the participants are invited to put forth questions and to interact with others. At the same time they know that there are not 'free rounds' (as often happens in the SGs, where a rich variety of cues are offered but they often remain at a messy, unspecified, superficial level) and should deal with the theoretical aspects through an *individual study*. In fact, at the base of ArAl Project there is an organic vision that aims to propose a framework on early algebra, offering the participants opportunities to reflect on knowledge, beliefs, stereotypes. They accept a commitment which attracts them: to avoid free, trivial conversations or Pindaric flights.

How can personal experiences, beliefs, inclinations be influenced by interventions based on strong theoretical references?

The interventions on SG highlight different objectives between mathematics educators and teachers: basically, specialists focus their interest on the discipline, the teachers on their pupils. These different perspectives can create misunderstandings or misinterpretations. Then, in the NSG, mediations between them are necessary, that is: on one side the founding principles of mathematical knowledge – in our case of early algebra - have to be respected, but on the other side, at the same time, it has to be offered to the teachers a certain 'serenity' about the fact that deepening and changes of perspectives in teaching do not affect learning, but on the contrary pave the way for subsequent extensions of mathematical concepts. There is a strongly felt concern that pupils do not understand or that a concept is too difficult or inappropriate (of course this concern is correct because teachers have the responsibility of the learning of their students, so they constantly consider the difficulty and feasibility of new proposals).

Limited capacity to distinguish between different types of knowledge

The posts and comments put in evidence that most teachers, along the years, reach their convictions grounding them more on the accumulation of heterogeneous strategies, methods, tools than on their consistency. One of the consequences of this behavior is that teachers confront themselves superficially with the theoretical references. For example:

Elena: I think sometimes that famous 'didactic contract', of which we all partake the negative effects on pupils, has been moved up on teachers: "It is so, Tom said, Dick reiterated it"; someone makes it [i.e. the didactical contract] arguing and expressing his/her opinion (experience counts,

anyway!); someone else makes it 'getting on the chair'. But: be they teachers or pupils or propagandists or colleagues, always 'didactic contract' is.

To what is Elena referring when she writes "the negative effects on pupils"? Her so peremptory statement was not reconsidered in the later comments: what does this mean? It could indicate that it has not been understood, or that it has been read superficially, or that it is not shared, or that it is an unfamiliar concept and no colleague wants to explore it. This short episode shows indeed that there are interactions between members, but in general they go on without reaching a real conclusion; at most, members achieve a superficial agreement, or a generic praise, or they remain on their positions. It would have been important to ask Elena what she means with this term (originally it is a theoretical construct by Guy Brousseau). Probably such statements would not have been made in our NSG. This might be a limit for the group because many convictions would not be expressed for a kind of compliance towards the coordinators experts. A low understanding of the key aspects of mathematics education (at the primary-secondary school level) favours the choice of cues - references, materials, paths, methods - that fit with the convictions and the personality of the teachers more than with the organicity of the knowledge taught. In this way, those facilitators that favour the perspective of *making* are privileged. The weak capacity to connect effectively the suggestions of experts and mentors implies that one prefers a 'do-it-by yourself' shared with those who are felt as fellow-travellers: if an activity, a text, a method are exalted or defended by other members of the group, they may be adopted, or at least tested. Often, the length of an experiments is short because the activities are heterogeneous, have 'little oxygen' (the interest on them goes out early); almost immediately they are put aside without any reflection in general terms, mostly on the basis of local success achieved by pupils (or, more trivially, because they appear 'nice').

New characters emerging in NSG

The dense interactions developed in the NSG together with the offered theoretical and practical supports brought some new attitudes and awarenesses in the members of NSG. The members begin to understand that a new approach to the arithmetic and algebra teaching lies on a different role of the teacher. As to this a decisive importance assumes what J. Mason has called *the art of noticing* the classroom micro-situations for being ready to adopt the opportune micro-decisions (Mason 2002), intertwined with the attention to the *languages* and to the continuous recourse to the *argumentation*. Thanks to our frequent interventions where we underline that: a) a math teacher has to control a *plurality* of languages and that also a formal language must be monitored at two levels, the *semantic* one about the meanings and the *syntactic* one about the structure of the sentences into play; b) the weak control over grammatical/syntactical aspects of a sentence in mathematical language leads to temporary and unstable jargons in which the meanings assigned to the symbols are dictated by an apparent common sense that reduces the difficulties, promotes an immediate but feeble understanding that leaves the problem unsolved; we observed in the activities posted by many members a bigger attentions forwards the translations questions between verbal and fromal languages and the increasing use of argumentations in their students. From a methodological point of view, thaks to our suggestions, when the teachers publish at the NSG the post of an activity, they begin to understand that it is not enough to insert some captivante images, but that it is necessary to equip them with a presentation that synthetically shows the activated competencies and that includes the most meaningful protocols, the path in which the activity is inserted, how it develops in the next steps, the

theoretical references (ArAl Units, items on the website, Powerpoint presentations, papers). **Our idea is to slowly bring them to approach the MTs methodology.** An important contribution in this sense is offered by an increasing number of members the NSG, who are not involved in ArAl experimentations but following the project in a convincing way (teachers educators, mentors, collaborators of publishing houses, members of other research groups). Thanks to this people the posted comments begin to be richer and meaningful; the authors express their ideas also asking for experts' suggestions aimed at promoting new and more adequate behaviours for teaching arithmetic/algebra in a *relational* perspective. So, posts and comments begin to produce virtuous relations which gradually enhance the system: the posts induce comments of increasing quality, which generate important feedbacks in the organization of the successive posts.

The recent mutations observed in the NSG members' posts delineates a new character of their participation which appears in tune with our aim to build a shared identity in the NSG and effective in offering contributions which can bring it to become *a community of di practice*. As to this, particularly meaningful appear the recent initiatives generated by the NSG discussions concerning the publication in the ArAl project website (<http://www.progettoaral.it/>) of two documents, respectively devoted to: (1) the most interesting classroom episodes presented by the NSG members, with the main related comments; (2); the early algebra papers written by members external to the project and inspired by our previous productions. Next to this we have to consider the request expressed on the web by several members of NSG to organize some ArAl meetings of one or few days to allow the participant know themselves *de visu* and to plan some common work. It seem us that these new tends in NSG may generate inside the group, mainly with the more sensitive and expert members, an embryo of a *community of inquiry*. In this frame institutional and informal ArAl educational initiatives are developing important merging points.

Final considerations

A NSG as 'Progetto ArAl' may initially disorient new participants, but its own structure can be considered its force because many of them declare that they appreciate the possibility to join to a group where experts favor an organization of knowledge according to transparent and shared principles. On the base of the observations made, we formulate some key points related to early algebra for the management of the NSG so that it can become a significant community of practice in this field: (a) to help teachers understand not only merits and limitations of instruments and didactical strategies that they implement along the years, but above all the importance of their coherence and adherence to a set of theoretical principles, such as: the importance of languages and, consequently, of the translation between them; (b) to bring teachers to consider the perspective of the generalization since the first years of primary school, highlighting the structural analogy between representations of the various occurrences of a phenomenon and guiding their modeling; (c) to propose any time, during the discussion on the issues raised by the members, gradual general frameworks, accompanying them with clarifications, insights, extensions which give answers for doubts, perplexities, conflicts emerging from the discussion. The basic idea is that the theory should be gained through a gradual process of refinement of knowledge in a continuous exchange among the members of the group, adapting explanations and deepening to the difficulties or to the resistances and injecting now and then proposals of mini-workshops.

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Prospective teachers interpret student responses: Between assessment, educational design and research

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We present a study about the development of interpretative skills in prospective teachers. In particular we discuss a kind of tasks designed by us for teacher education, containing the request of interpreting students' answers. The task in this study was built on an item concerning the sum of powers of 10 and was proposed to a group of prospective secondary teachers who were attending a Math Education course. The task was first faced individually by them and then discussed in group. We present the interpretations proposed by two prospective teachers before and after the collective discussion, in order to reflect on the differences in terms both of mathematical knowledge put in play and of attitudes exhibited.

Keywords: Teacher education, prospective secondary teachers, interpretation of students' answers, arithmetic and algebra.

Mistakes are the portals of discovery (James Joyce)

Introduction

In their daily practice, teachers are required to continuously interpret students' responses and productions. This not only assists them in evaluating their difficulties and achievements, but also allows to plan the next steps of the teaching activities. Therefore, this "interpretation activity" is one of the most crucial (and often most difficult) tasks teachers perform. Empirical evidence suggests that the true quality of a mathematics teacher stems largely from his/her ability to interpret students' productions, along with a flexible attitude to redesign the teaching approach based on them. However, the ability to make sound and accountable interpretations is rarely recognized as a crucial goal in teacher training. Moreover, in a previous research study (Ribeiro, Mellone, & Jakobsen, 2013), we observed that teachers do not naturally develop this ability as they gain work experience. So, presently, our research questions are the following. Is it possible for a teacher to acquire these interpretation skills or is it a matter of innate talent? In the first case, how can a prospective teacher develop this ability? Should we engage prospective teachers in mathematical discussions concerning the interpretation of students' reasoning?

Guided by our conviction that it is possible for a teacher to develop this ability, as a part of a joint research project, we explored a particular type of tasks we have conceptualized. In one part of these tasks, we asked the prospective teachers to interpret some students' responses to a problem and reflect on possible feedback they could provide to each student. This exercise had a twofold aim: to support prospective teachers in developing the skills required for interpreting and commenting on student work, and to investigate to what extent the particular mathematical knowledge and skills possessed by the (prospective) teachers support or hinder them in their interpretations and

“constructive” reactions. Our analysis shows that the abilities to interpret and to design an educational activity based on students’ productions are inhibited in prospective teachers with a poor mathematical knowledge, due to their limited understanding of the subject and lack of appreciation of various ways that problems can be solved (Ribeiro, Mellone, & Jakobsen, 2013). In particular, this is true for prospective primary teachers, many of whom have poor mathematical knowledge. However, we have also observed that prospective secondary mathematics teachers—who have studied more advanced mathematics during the three years of their Bachelor in Mathematics degree (Jakobsen, Mellone, Ribeiro, & Tortora, 2016)—also struggle with this kind of work. This led us to posit that the ability and knowledge to interpret student work depends not only on (prospective) teachers’ mathematical knowledge and its components, but also on their attitudes and beliefs toward mathematics and its teaching.

The tasks described herein are presently used in our Mathematics Education courses in three different modalities. First, we ask prospective teachers to individually solve the problem, before interpreting and reflecting on some selected students’ productions, and finally engaging in group discussions on the mathematical aspects involved in these students’ productions. Given that the prospective teachers (both primary and secondary) have difficulties in interpreting and in giving meaning to some students’ answers, it was necessary to first assess their ability to solve problems that these students are given. This was informative, as some prospective teachers struggle with providing constructive feedback to the students even when they do not encounter any difficulties in solving the given problems for themselves. The findings yielded by this first phase of our research were utilized in the subsequent mathematical discussions of students’ solutions and corresponding teachers’ interpretations. These group discussions were helpful to most prospective teachers, as they were able to gain new perspectives on students’ work and strategies that can be employed in teaching.

In this paper, we present the interpretations, given by a group of Italian prospective secondary teachers, of students’ responses concerning a problem where sums of powers of 10 are involved. (see Jakobsen et al. (2016) for details). Here, we present analysis of the interpretations teachers gave before and after the mathematical discussions, in order to document their progress, as most demonstrated more sophisticated attitudes and greater mathematical knowledge following group discussions.

Theoretical framework

In order to characterize and study the features of teachers’ interpretations of students’ productions, in some of our previous work (see for example Ribeiro, Mellone, & Jakobsen, 2016), we have introduced the notion of *interpretative knowledge*, framed within the general Mathematical Knowledge for Teaching (MKT) framework (Ball, Thames, & Phelps, 2008). We define interpretative knowledge as the knowledge that allows teachers to give sense to pupils’ answers, in particular to “non-standard” ones, i.e., adequate answers that differ from those teachers would give or expect, or answers that contain errors. We posit that interpretative knowledge is closely related to the ability of teachers to support the development of pupils’ mathematical knowledge, starting from their own reasoning, even if students’ ideas are incomplete or non-standard. Some similar ideas are implied in the notion of *discipline of noticing* (Mason, 2002). In particular, our construct encompasses the idea of teachers working “on becoming more sensitive to notice opportunities in

the moment, to be methodical without being mechanical” (Mason, 2002, p. 61). The development of pupils’ mathematical knowledge starting from their own reasoning is, in our view, only possible if the teacher activates a real process of interpretation, shifting from a simple evaluative listening to a more careful hermeneutic listening (Davis, 1997).

In this sense, the notion of interpretative knowledge incorporates into the MKT framework the idea that errors and non-standard reasoning are considered as learning opportunities (Borasi, 1996). Moreover, the content of interpretative knowledge shapes teachers’ ability to make informed choices in contingency moments (as defined by Rowland, Huckstep, & Thwaites, 2005), in order to respond to and deal with non-planned situations. In that sense, we felt the need to incorporate the role of beliefs and attitudes pertinent to the use of mathematical knowledge (Carrillo, Climent, Contreras, & Muñoz-Catalán, 2013).

With the goal of better understanding (prospective) mathematics teachers’ act of interpretation, we characterized their interpretations of students’ productions and attitudes, using the following three categories: (i) Evaluative interpretation: a process through which the teacher determines congruence between pupils’ productions and the mathematical scheme of correct answers he/she has; (ii) Interpretation for the educational design: the manner in which the teacher designs educational steps based on the work produced by the students; (iii) Interpretation as research: teacher’s willingness and ability to revise his/her mathematical formalization in order to ensure that it is coherent with students’ productions (even when these seem in conflict with the traditional mathematics taught in school).

In Webster dictionary, “interpretation” is defined as “The act of interpreting, explanation of what is obscure”; however, it is also defined as “An artist's way of expressing his thought or embodying his conception of nature.” This last definition stresses the potential creative nature of the act of interpreting that is in our context perceived as the potential new mathematical knowledge that can be developed owing to the process of analyzing students’ productions.

Context and method

For several years, we have been studying the nature of (prospective) teachers’ interpretative knowledge (e.g., Ribeiro et al., 2013; Jakobsen et al., 2016) by exploring the manner in which prospective teachers respond to specific interpretation tasks. In this design study the tasks are developed after the typical cycles of redesign of the design study method (Cobb et al., 2003). In their present form, essentially consist of three steps: (i) the teachers are initially required to solve a mathematical problem by themselves; (ii) they are given several students’ productions in response to the same problem, some containing errors and some mathematically valid but following less standard procedures, which they are asked to interpret; and (iii) teachers are prompted to provide what they deem would be appropriate feedback to these students based on their solutions. The teacher trainees are asked to address these requests individually and in paper format (they are usually given 90 minutes to complete all three steps). In the next phase of the study, the educator engages all prospective teacher participants in a collective mathematical discussion (which again typically lasts about 90 minutes). The framework of the mathematical discussion is based on that proposed by Bussi (1996), as the aim is to allow the group of prospective teachers create a polyphony of articulated voices on the mathematical object starting from the interpretation of a

student's production. Upon completion of the group discussion, the prospective teachers are asked to provide in writing a new individual interpretation of the students' productions, allowing the researchers to determine if any progress has been made.

The task utilized in the present study is depicted in Figure 1, and was adopted from the annual Italian national assessment (2010-2011) for grade 10 released by INVALSI (Istituto Nazionale per la VALutazione del Sistema educativo di Istruzione e di formazione). A group of 34 fourth-year master students of mathematics enrolled in a Mathematics Education course took part in this investigation. Since most of these students are going to become secondary school teachers, we consider them prospective secondary teachers.

In our previous study, we focused on the interpretation these prospective secondary teachers gave to their students' productions (Jakobsen et al., 2016). Our analysis revealed that they experienced problems in mobilizing their mathematical knowledge for interpreting students' work. Indeed, while they were able to "see" some of the mathematical aspects involved in the solutions to the problems their students proposed, they seemed unaware of many important aspects relevant to mathematics teaching and problem solving.

The expression $10^{37} + 10^{38}$ is also equal to:

A. 20^{75}

B. 10^{75}

C. $11 \cdot 10^{37}$

D. $10^{37 \cdot 38}$

Figure 1: Item given to students to solve

In the next section, we will present two out of seven students' productions that were included in the task given to the study participants (for their selection, see Mellone, Romano, and Tortora (2013)). This will be followed by the interpretations of these students' productions, provided by two prospective teachers—to whom we refer as Rossella and Gennaro (pseudonyms)—before (BF) and after (AF) the mathematical discussion.

Interpretation of two students' productions

The following brief analysis of the students' productions included in the task aims to elucidate our reasoning behind the decision to deem these two students' productions effective for exploring prospective secondary teachers' ability to interpret the work of others.

Emanuela (Figure 2a) obtained the correct result, despite making three errors in her work: the first and the last can be described as lack of use of parenthesis and the second can be seen as a wrong application of linearity. Ciro (Figure 2b) arrived at the right answer using the arithmetical algorithm of the arrangement of the decimal representation of the numbers in column. In his responses, we can also recognize his perception of the algebraic structure connected with more general ideas implicit in calculus. Indeed, Ciro's use of the ellipses reveals the potentiality of generalization of his

response to other two consecutive powers of ten, and not only to the particular case given (as one of the mathematics students that took part in our study noted during the collective mathematical discussion).

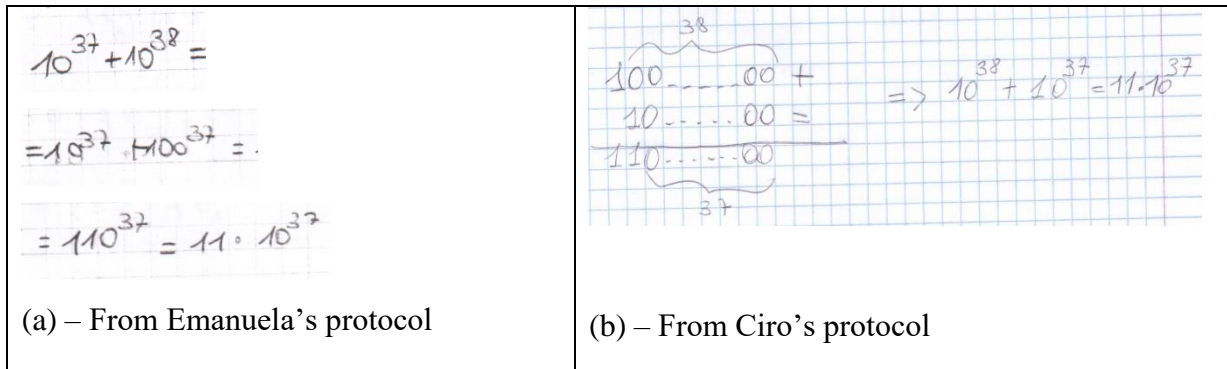


Figure 2: Two out of seven students’ productions

Prospective teachers’ comments on students’ productions before and after mathematical discussion

Reflections on Emanuela’s work

When individually interpreting Emanuela’s response, prospective teachers seemed to experience difficulty in trying to understand the steps she used in arriving at the solution (see Jakobsen et al., 2016). Here, we focus on the interpretations given by a secondary prospective teacher —Rossella— before (BF) and after (AF) the discussion¹:

The second prospective teacher, Rossella, shared the following:

Rossella (BF): There is no application of rules; it is pure invention. I don’t know what I would say to the girl, but I would think that she had copied the solution and then tried to invent a justification.

Rossella (AF): Even if Emanuela’s answer is correct, her arguments are far from being mathematically founded. Still, we can observe an interesting aspect in them, namely that if we repeat the steps with two powers having the base different from 10 and the exponents differing by one, we get the correct result. I followed this approach using different numbers, just to test this reasoning, which allowed me to assert that Emanuela’s thought process appeared to work. More specifically, when you change the bases and use two consecutive numbers as exponents, or even not consecutive, her logic gives the correct result.

$$37^{25} + 37^{26} = 37^{25} + (1369)^{25} = 1406^{25} = (38 \cdot 37)^{25} = 38 \cdot 37^{25}$$

$$\frac{x}{c} - \frac{1}{10} \frac{0^3}{0} + \frac{x}{c} - \frac{1}{10} \frac{0^4}{0} = \frac{x}{c} - \frac{1}{10} \frac{0^3}{0} + \frac{x}{c} - \frac{1}{100} \frac{0^3}{0} = \frac{x}{c} - \frac{9}{100} \frac{0^3}{0} = \frac{x}{c} - \frac{9}{10} \frac{0^3}{0} - \frac{1}{10} \frac{0^3}{0}$$

$$2^{-3} + 2^{-2} = 2^{-3} + 4^{-3} = 6^{-3} = 3 \cdot 2^{-3}$$

$$x^n + x^{n+1} = x^n + (x^2)^n = (x+x^2)^n = ((x+1)x)^n = (x+1)x^n$$

¹ Of course, the students’ words were translated from Italian into English. We tried our best to retain the exact expressions, including some errors, but some nuances are inevitably lost.

In other words, the above three errors, made in sequence, yield a correct result that does not depend on the particular numbers used. This observation has prompted our reflection on how mathematics is usually managed. We are used to judge, without hesitation, arguments such as those mentioned above as wrong and completely invented. However, after our discussions, we were of view that errors should be seriously considered and, if possible, exploited as a stepping stone toward the construction of new knowledge. This is exactly what I tried to do in order to bring out something new from Emanuela's reasoning. Indeed I also tried to build a new system of rules for powers according with her reasoning. Several attempts have convinced me that this is not possible. Thus, my conclusion is that no new knowledge about the rules of powers can be derived from Emanuela's suggestions. However, her errors can be an invaluable tool to stimulate discussions and to highlight the need for a true comprehension of the rules of powers, which are often hard to grasp for students.

Reflecting on Rossella's BF and AF words, we can observe a change in the attitude when interpreting student's work. The first interpretation is an evaluative one—there is no effort to understand the rationality of Emanuela's steps. This results in Rossella's bias towards a solution like that Emanuela offered (referring to an her possible unfair behavior), which can be due to fear of moving toward an educational path she cannot (immediately) control.

The interpretation given by Rossella after the discussion is markedly different. She not only made an effort to derive a generalization from the errors in Emanuela's production, but further concluded that the three steps Emanuela used in solving the problem will give the right answer with other bases and exponents. Rossella thus went beyond the simple observation that Emanuela's steps are not mathematically sound, as she investigated the possibility to build “a new system of rules for powers according with her reasoning.” For this reason, her second interpretation can be considered a form of interpretation as research.

Reflections about Ciro's work

Gennaro (BF): Ciro reached the correct answer by a more practical method than those employed by his peers. In addition, the formalism seems original. He appears to have a strong expertise in the calculations with powers of 10, which highlights their significance and the importance of handling them correctly. Still, his method seems limited to powers of 10. It would be interesting to see how Ciro would proceed if presented with a different base. I think that Ciro's protocol could be used as an opportunity to explore differences between the properties of powers of 10 and those of other bases.

Gennaro (AF): Ciro's argument is of an arithmetic character. Nonetheless, it allows us to appreciate some deep algebraic insights. Moreover, although it seems confined to powers of 10, it can actually be generalized to any base, if one represents the number in the base of the power. Hence, from Ciro's production going further, it would be possible to study the tables of operations in different bases, or even the divisibility rules in bases other than 10.

In his first interpretation of Ciro's protocol, Gennaro appreciates the originality of his method, while noting that it is limited to powers with base 10. Based on this observation, Gennaro proposed possible questions and issues that could be explored with Ciro and the rest of the students in the classroom, starting from his production. For this reason, Gennaro's first interpretation is aimed at educational design.

As with Rossella, whose interpretations we analyzed previously, the comments Gennaro gave on Ciro's protocol after the discussion shifted in focus. First, there is a subtle distinction between arithmetic and algebra that could be investigated and debated endlessly. Moreover, Gennaro's comments reveal his awareness that Ciro's method can be applied to other bases (indeed, $100\dots 0$ always represents the n -th power of the base). This fact was observed during the collective discussion by another prospective teacher, and for Gennaro, this discovery was so important that it became part of his new written interpretation. In other words, Gennaro's knowledge and interpretation benefitted from the mathematical discussion on Ciro's production. He reconceived the systems of representing numbers in different bases, which motivated him to explore the true meaning of digits, as well as of strings of digits. For this reason, Gennaro's second interpretation is perceived as interpretation as research.

Conclusive remarks

We started this paper by asking if mathematics teachers can develop the ability to interpret their students' productions in order to flexibly redraw the mathematical learning path, or if this should be considered as an innate talent. We are convinced that it is not only possible to develop this skill but is highly desirable. The observed difficulties these prospective teachers experienced when giving sense to student productions, along with the findings yielded by extant studies, indicate that the development of this ability requires a special attention in teacher education. These first results about our proposed method of working with prospective teachers appear to support its effectiveness. It stimulates prospective teachers' interpretive and critical skills and increases knowledge they must possess in order to teach effectively, taking into consideration the specificities of such knowledge. The value of our method stems from the nature of interpretive tasks involving student productions, as well as subsequent discussions among peers under the guidance of an expert on these interpretation tasks.

Our analysis of the interpretations given by two prospective teachers, before and after the collective discussion led by the educator, clearly demonstrates changes in terms of both their attitude and mathematical knowledge or awareness. We can hypothesize that the collective discussion mobilized mathematical knowledge that was previously present, but probably not put in play, and it also support the development of new mathematical knowledge, like for Gennaro. However, the improvements we witnessed were also due to the change in attitudes and beliefs supported by the discussion, and of course by the attitudes and beliefs incorporated in the educator's practice.

Still, our work leaves many questions to be answered in future research. It would be interesting to evaluate the sustainability of these changes, for example, by following the work of these prospective teachers in their future educational practices. Moreover, analysis of the mathematical discussion on interpretative task needs to be developed in order to clarify its features and dynamics.

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Investigating potential improvements of mathematics student teachers' instruction from Lesson Study

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This paper reports from a project where Lesson Study (LS) was implemented in the field practice component in four subject areas of two teacher education programs at one Norwegian university. Previous analyses of data from interviews and mentoring sessions indicate that mathematics was a challenging case that makes it interesting to investigate further. In the present study, we analyze classroom observations with the Classroom Assessment Scoring System (CLASS) in order to investigate potential improvements from the intervention. The results indicate that there were no significant differences between the control group and the intervention group. Possible explanations for this are discussed and implications for future implementations of LS in field practice are suggested.

Keywords: Mathematics teacher education, field practice, Lesson Study.

Introduction

This paper has a focus on developing student teachers' ability to teach mathematics through LS. When Stigler and Hiebert (1999) published the results from their comparative study of mathematics teaching in Japan, Germany and the USA, they argued that there was a "teaching gap" among these countries. The teaching quality of the Japanese classrooms appeared significantly higher than in German and US classrooms, and Stigler and Hiebert suggested that a main explanation for this teaching gap could be found in the incremental developments of teaching through LS in Japan. In the aftermath of this study, LS continues to gain popularity as a practice-based approach to professional development outside of Japan. There is also growing interest among researchers to adapt and use LS in teacher education (e.g., Ricks, 2011).

In mathematics education research, numerous studies focus on how LS might increase mathematics student teachers' knowledge and understanding of the mathematical content. For instance, Cavey and Berenson (2005) argue that their adapted version of LS has a potential to increase student teachers' understanding of the mathematical content. Drawing upon the idea that knowledge for teaching must be learned in and from practice (Ball & Cohen, 1999), one might argue that LS has a potential to serve as a "professional development tool when faced with the challenge of providing high-quality learning experiences for student-teachers" (Murata & Pothen, 2011, p. 104).

From an ongoing review of literature on LS in mathematics teacher education, we notice that, while several studies attempt to measure the effects of LS on student teachers' knowledge and understanding, few studies report on effects of LS on the quality of instruction. Chassels and Melville (2009) suggest that LS "provides opportunities for teacher candidates to build professional learning communities, to deepen understanding of curriculum and pedagogy, and to develop habits of critical observation, analysis, and reflection" (p. 734). When investigating mathematics student teachers' development of lesson plans, Fernandez (2010) suggests that implementation of LS appears to influence their development of professional knowledge. Jansen and Spitzer (2009) focus on

mathematics student teachers' reflective thinking, and their study includes analyses of student teachers' own interpretations of their teaching. Although the issues raised in these studies are of importance, neither of them focus directly on effects of LS implementations on the quality of teaching. Leavy's (2010) study includes a focus on observing teaching, but the analyses emphasize student teachers' reflections and development of knowledge rather than their actual teaching. Ricks (2011) reports from an intervention study, but his focus is on mathematics student teachers' reflections rather than on their teaching. With this as a background, the aim of this paper is to investigate possible effects of LS implementation on the quality of student teachers' mathematics teaching. We consider the following research question: What potential influences can be observed from a LS intervention on the quality of classroom interactions in the field practice of mathematics student teachers? In order to approach this research question, we analyze classroom observations from a time-lagged design experiment where LS was implemented in the field practice of two Norwegian teacher education programs. Videos of classroom teaching are analyzed by using the Classroom Assessment Scoring System (CLASS).

The Classroom Assessment Scoring System

CLASS scores are related to students' academic performance (Teachstone, 2012), and research indicates that substantial gains in measured student achievement is mediated by teacher-student interaction qualities (Allen, Pianta, Gregory, Mikami, & Lun, 2011; Teachstone, 2012). An important mediator for academic outcome is the extent to which the students' interactions with their teachers motivate them (Pianta & Allen, 2008). Based on this, student-teacher interactions in the classroom are the focus of attention when observing classrooms using the CLASS instrument. This instrument is designed to assess the fit between teacher-student interactions and students' developmental, intellectual, and social needs, i.e. elements of high-quality teaching that have been identified as central to student achievement (Allen et al., 2011). The CLASS instrument consists of three major domains that provide behavioral anchors for describing and assessing critical aspects of classroom interactions (Teachstone, 2012): 1) Emotional Support, 2) Classroom Organization, and 3) Instructional Support. Student Engagement is also included, due to the importance of observing student behavior in addition to behavioral anchors on the classroom and teacher level.

The first domain, Emotional Support, relates to students' social and emotional functioning in the classroom and is highlighted in the CLASS instrument because "relational supports and connections, autonomy and competence, and relevance are critical to school success" (Teachstone, 2012, p. 2). Second, Classroom Organization is included in the instrument based on research, highlighting the relationship(s) between aspects of organization and students' opportunities to learn. The foundation for the third domain, Instructional Support, is constituted by the following teaching strategies that enhance learning: "consistent, process-oriented feedback, focus on higher-order thinking skills, and presentation of new content within a broader, meaningful context" (Teachstone, 2012, p. 4).

These three domains can be further divided into twelve dimensions or CLASS indicators that are defined in the CLASS manual (see Tables 1, 2 and 3). In addition to these observable indicators of effective interactions, the CLASS manual includes behavioral markers that provide clear examples of how teacher-student interactions in the classroom can be observed and assessed. These descriptions are specified and examples of justifications are provided on the basis of concrete classroom videos, coded by CLASS experts.

Method

This study is situated within the larger, cross-disciplinary project, Teachers as Students (TasS), which involved mathematics, science, physical education, English as a second language, as well as pedagogy. The TasS project (2012–2015), supported by the Norwegian Research Council (grant number 212276), investigated student teachers' learning during field practice, aiming at learning more about how student teachers develop the knowledge and skills required to promote student learning in schools. LS was used in a time-lagged design experiment (Hartas, 2010) in two Norwegian teacher education programs, both four-year integrated programs, one for grades 1–7 and one for grades 5–10. Subject matter and didactics (pedagogy) should thus be integrated in all subjects, and there should be a close relationship between what was taught on campus and in schools when student teachers had field practice (100 days within the four years). The mentor teachers have an important role and are considered teacher educators in field practice.

The student teachers were organized in groups of three or four during a three-week period of field practice both in the Business as Usual condition (BAU) and in the LS intervention (INT). The TasS study recruited student teachers during the spring term of their fourth semester (except the science groups in the BAU condition, who were in their sixth semester). The TasS project includes data with two groups of student teachers from the four subjects in both data collection periods (see Munthe, Bjuland & Helgevold, 2016 for an overview). In this paper, we mainly report from analyses of classroom recordings of lessons taught in mathematics from the BAU and INT condition, using the CLASS (Classroom Assessment Scoring System) observational instrument (Allen et al., 2011; Teachstone, 2012). We also draw upon findings from previous analyses in the discussion section based on conversations in mentoring sessions and pre- and post interviews (before and after the field practice in both conditions).

In the BAU condition, the mentor teachers were asked to conduct their mentoring sessions the way they normally did without any influence from the researchers in the project. In the LS condition (INT), it was crucial that mentor teachers were introduced to essential principles about LS since “they played the role as facilitator and knowledgeable other for the group of student-teachers who made up the Lesson Study group” (Munthe et al., 2016, p. 145). This required another approach. Three afternoon seminars were organized (from November 2012 to January 2013) for mentor teachers and the research group in order to discuss important characteristics within the LS cycle and to establish a shared understanding of implementing LS in student teachers' field practice. An important component of these afternoon sessions was to develop a “Handbook for Lesson Study”, which included a text about important principles in LS and a list of questions which could support both the mentor teachers and the student teachers through pre- and post-lesson mentoring sessions.

Three researchers (the authors of this paper) took part in the coding of videos from the four lessons in both conditions (INT and BAU). We divided the videos into 19-minutes sections. This resulted in 22 sections (12 BAU and 10 INT). After watching a video section, we started the scoring for each dimension individually, using the 7-point range that is described in the CLASS manual, Low (1, 2), Mid (3, 4, 5) and High (6, 7). We made our judgements based on the general scoring guideline. Our scores were then discussed before we started to observe a new video section. The results that are presented in the three tables below illustrate the scores given by the three researchers (see Tables 1, 2 and 3). Where two scores are given, our individual coding differed.

Results

The CLASS domain of Student Engagement intended to capture “the degree to which all students in the class are focused and participating in the learning activity presented or facilitated by the teacher” (Teachstone, 2012, p. 109), was coded as Mid for both BAU and INT. This code means that either the students are listening to, or watching the student teacher, rather than actively engaging in classroom discussions and activities, that there is a mix of student engagement, or they are engaged part of the time and disengaged for the rest of the time. Across all the videos, there is a lack of off-task behavior and the students appear to be engaged.

Emotional support

The domain of Emotional Support is divided into three dimensions (Table 1)¹. Across the first two dimensions, the code Mid was given by all three coders. A Mid score on the first dimension, Positive climate, indicates that the student teacher and students sometimes provide positive comments and appear quite supportive and interested in one another. A Mid score on the second dimension, Teacher sensitivity, indicates that the student teacher sometimes monitors students for cues and generally attempts to help students who need assistance, but these attempts are not always effective in addressing student concerns. This code also indicates that some of the students sometimes seek support, respond to questions and share their ideas.

CLASS dimension	BAU	INT
Positive climate	MID	MID
Teacher sensitivity	MID	MID
Regard for adolescent perspectives	LOW/MID	LOW/MID

Table 1. Results from CLASS analysis of Emotional Support.

The third dimension, Regard for adolescent perspectives, was coded as Low/Mid, illustrating that individual coding differed among the coders. Low to Mid on this dimension indicates that the teaching is the teacher’s show. The students are rarely provided opportunities for autonomy and leadership.

As can be seen from Table 1, the LS intervention did not affect the coding for any of the dimensions included in this domain.

Classroom organization

The domain of Classroom Organization is composed by three dimensions (Table 2). Behavior management encompasses the student teacher’s use of methods to maximize the learning time for the students. The code Mid indicates that there is some evidence that the student teachers encourage desirable behavior and prevent misbehavior. Productivity, the second dimension, does not relate to quality, but rather deals with the students’ opportunity to get involved and the extent to which the teacher makes sure that everyone has something to do. A Mid score on Productivity indicates that most of the time there are tasks for the students and some routines are clearly in place. However, transitions could be more efficient and the student teachers could be better prepared.

¹ The CLASS domains are written with capital letters in both words (e.g., Emotional Support), whereas the CLASS dimensions are written with one capital letter (e.g., Positive climate).

CLASS dimension	BAU	INT
Behavior management	MID	MID
Productivity	MID	MID
Negative climate	LOW	LOW

Table 2. Results from CLASS analysis of Classroom Organization.

The dimension Negative climate is scored in reverse. A low score indicates that the overall level of negativity among student teachers and students is low or absent. This code counts for more than the others in this domain, indicating that the classroom processes related to the organization and management of time, student behavior and attention in the classroom provide the students with opportunities to learn. As can be seen from Table 2, the LS intervention did not affect the coding for any of the dimensions included in this domain.

Instructional support

The domain of Instructional Support is divided into five dimensions as shown in Table 3. The first dimension, Instructional learning formats, was coded Mid in both conditions, illustrating that learning objectives may be discussed, but they are not clearly communicated in order to support student attention which is an indicator for a high score.

It is only in the second dimension, Content understanding, that there are indications of a possible effect, from discrete pieces of depth of lesson content to sometimes finding meaningful discussions in order to help students comprehend the mathematical content. The three last dimensions, which emphasize higher-order thinking among the students with a purposeful use of a content-focused discussion in the classroom, are all coded Low or Low/Mid. The focus on mathematical content is not strong in the teacher-student classroom interactions. A Low score on Analysis and inquiry indicates teaching that does not let the students think, or that students are neither engaged in higher-order thinking, metacognition, nor have opportunities for novel application. The teaching is in a rote manner. A Low/Mid score on the dimension of Quality of feedback indicates that the feedback provided to the students neither expands or extends learning nor encourages student participation. A Low/Mid score on the dimension of Instructional dialogue indicates that the student teachers do not involve the students in content-based discussions in class.

CLASS dimension	BAU	INT
Instructional learning formats	MID	MID
Content understanding	LOW/MID	MID
Analysis and inquiry	LOW	LOW
Quality of feedback	LOW/MID	LOW/MID
Instructional dialogue	LOW/MID	LOW/MID

Table 3. Results from CLASS analysis of Instructional Support.

The results illustrate that our LS intervention had little effect on the quality of classroom interventions (Table 3). Across the videos, and in both BAU and INT, the classes are mostly dominated by student teachers' talk.

Concluding discussion

Many studies suggest that LS has potential to contribute to mathematics student teachers' development, but few studies analyze potential effects of LS interventions in mathematics teacher education on the quality of classroom interactions. CLASS analysis of classroom videos from a control group and an intervention group indicates that our LS intervention did not increase the quality of classroom instruction. One might argue that the challenging results that arise from this study are problematic since one would hardly expect the results to show any variance based on one LS cycle only. However tempting it is to bypass the reporting of such challenging results, we do the opposite. The majority of research reports in mathematics teacher education appear to be success stories, but we suggest that it is also important to discuss results that were not as positive as desired. In the following, we highlight three issues that might have influenced the results of this study: 1) experience and time, 2) lack of focus on critical aspects of LS, and 3) personal factors.

First, we discuss the issues of experience and time. Stigler and Hiebert (1999, p. 109) describe LS as a “system that leads to gradual, incremental improvements in teaching over time.” Japanese improvements in teaching happen through systematic work over several decades, and it is unfair to expect significant improvements in teaching from groups of student teachers who have just been introduced to LS. Most implementations of LS in mathematics teacher education seem to involve participants with little or no previous experience with LS (e.g., Bjuland & Mosvold, 2015; Leavy, 2010). In addition, most of these studies are short-term studies that often report results from participants who have completed one LS cycle only (e.g., Bjuland & Mosvold, 2015; Chassels & Melville, 2009; Leavy, 2010). It is not realistic to expect significant long-term effects from studies like this.

Second, a possible explanation for the challenging results in this study might be that the participants failed to implement some important aspects of LS. Previous analyses of data from mentoring sessions and interviews support this. For instance, Bjuland and Mosvold (2015) identified four indicators of why the implementation was challenging in mathematics. First, the student teachers reported about a lack of emphasis on pedagogical content knowledge on campus before field practice, and they called for more focus on students' difficulties and teaching strategies. A second indicator was related to a lack of formulating a research question. In a LS cycle, the student teachers should collaboratively plan, conduct and evaluate a research lesson with a focus on students' learning, but they should also formulate a research question that focuses on their own learning. No signs of this were found in the mentor sessions in the mathematics groups (Bjuland & Mosvold, 2015). Third, there was little focus on student learning and structured observation – both of which are decisive in LS. The mentor teachers' questions did, however, focus more on planning, observation and student engagement in the LS intervention (Bjuland, Mosvold, & Fauskanger, 2015). A fourth indicator was that student teachers organized research lessons around individual work with textbook tasks – making observation of student learning difficult (Bjuland & Mosvold, 2015). These observations may explain why the LS intervention was not successful and why the quality of classroom interactions did not increase.

Third, other factors like the student teachers' background, motivation and support may have influenced the results of this study. From analyses of data from mentoring sessions as well as focus-group interviews, we learned that one group of student teachers in the intervention may have had a lack of motivation for participating in the study. In the other group of student teachers from the

intervention, the mentor teacher was absent for a period of time, and the resulting lack of support from the mentor teacher might have influenced the quality of classroom interactions. Similar factors were also observed in the BAU groups. For instance, one of those groups struggled to collaborate (Bjuland & Mosvold, 2014), and we cannot revoke the potential influence of such problems. Although many problems can be avoided or taken care of in a research project, there will always be a potential influence of human factors that cannot be controlled by the researcher.

Further long-term studies are called for in mathematics teacher education to investigate participants who have completed more than one LS cycle, emphasizing that teaching develops through incremental improvements over time (Stigler & Hiebert, 1999). We observe that many studies focus on potential effects of a LS intervention on student teachers' understanding of the mathematical content. More studies are called for to investigate effects of LS implementations on classroom instruction.

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Ritual towards explorative classroom participation of pre-service elementary school mathematics teachers

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In this paper we follow preservice elementary school mathematics teachers learning processes in a course that was organized around problem solving and aimed at providing opportunities for students to participate exploratively. The goal of this study is to characterize pre-service teachers' participation on the "ritual-explorative" continuum, to understand better what opportunities for explorative participation are given and taken up by students. Findings show that the request to suggest various solution paths seems to help students focus on the explorative question of "where do I want to get at?" rather than at the ritual question of "how do I proceed?".

Keywords: Rituals, explorations, preservice mathematics teachers.

Introduction

This paper focuses on learning processes of pre-service elementary school mathematics teachers during a course whose goal was to promote students' mathematical thinking by engaging them in a discourse which is closer to that of mathematicians, and thus to provide our pre-service teachers opportunities to participate exploratively in doing mathematics. This resonates with Blanton (2002): "The development of a cadre of classroom mathematics teachers whose practices reflect current research on teaching mathematics rests in part on how pre-service teachers, as students, experience mathematics" (p. 117).

In a former study about opportunities for learning in a prospective mathematics teachers' classroom (Heyd-Metzuyanim, Tabach & Nachlieli, 2015), we found that despite what seemed to be an explorative environment, pre-service students still participated mostly ritually in a mathematics course. Since we believe that the opportunities to learn that preservice teachers provide their future students should be more explorative, it is obvious that they themselves should participate exploratively. We therefore designed a course that would provide opportunities for explorative participation. The course was taught in two separate groups by two instructors simultaneously. We are now starting to learn about what actually happened in those courses - could the participation of students be characterized as, at least sometimes, explorative? What characteristics of the course design and instruction seems to provide explorative learning opportunities? In the current study we focus only on one group of students, working on one type of tasks during two lessons, and follow learning processes in an attempt to characterize pre-service teachers' participation on the "ritual-explorative" continuum.

Theoretical framework

In this study we adopt Sfard's socio-cultural approach to conceptualize and study learning – the communicational framework (Sfard, 2008), and refer to the on-going development and refinement of the discourse on rituals and explorations (Sfard & Lavie, 2005; Nachlieli & Tabach, 2012; Heyd-

Metzuyananim, Tabach & Nachlieli, 2015; Heyd-Metzuyananim & Graven, 2015; Sfard & Lavi, in process).

One of the main characteristics of a discourse is the routines participants perform. Routines are repetitive patterns that are repeated in similar situations. That is, when one views a situation as similar to one he had participated in, and performs the same action. This could be social – e.g. when entering your home and placing the keys at a particular place. It could also be a cognitive action – e.g., in a mathematics classroom, when a student refers to a certain problem as similar to one performed earlier, and adopts the same procedure to solve the problem. The participant is not always aware of this repetitiveness. That is, a routine is such in the eyes of the researcher. Sfard & Lavie (in process) define the term *explorations* as "routines whose success is evaluated by answering the single question of whether a new endorsed narrative has been produced". That is, the task of an exploration is to produce new "historical facts" or a new "truth" about mathematical objects. Exploration is hence an act of production. Performers of explorations focus on the question: "what it is that I want to get?".

Rituals are "routines performed for the sake of social rewards or in an attempt to avoid a punishment." (Sfard & Lavie, in process). Ritual performance is usually initiated by, and addressed at, somebody else. Usually, the performance is an imitation of someone else's former performance. The procedure is rigid and the performer of the ritual never tries to make independent decisions. Performers of rituals ask themselves: "how do I proceed?".

It is important to stress that the same procedure performed (even simply multiplying 26 by 31) could be an exploration or a ritual, depending solely on whether the participant is engaged in trying to produce a new narrative, or, simply socially engaging in class, doing what she is expected to do using given procedures.

Former studies suggest that while explorative participation is desired, rituals are inevitable. Especially in the process of objectification, of developing new mathematical objects (such as the development of numerical discourse by children (Sfard & Lavie, 2005; Sfard & Lavie, in process) or the development of the discourse on function of 7th grade students, (Nachlieli & Tabach, 2012).

In mathematics classrooms, students' participation is usually neither purely ritual nor explorative. Those could be seen as ends of some continuum that differ in the performer's ability to separate the procedure and the task. As long as the performer does not just strive to arrive at a particular outcome but also feels compelled to do this by performing a specific procedure, the routine cannot count as a pure exploration.

The goal of this study is to characterize pre-service teachers' participation on the "ritual-explorative" continuum so that we could understand better what opportunities for explorative participation are given and taken up by students.

Method

Data collection

The data for this paper are taken from a course about "promoting mathematical thinking", for prospective elementary school mathematics teachers studying at a college of education in Israel. The course was a one-semester course which was taught in 2014. It included 13 lessons, each lasting an hour and a half. During the first lesson, the project was described to the students and consent forms

were collected and hence the lesson was not videotaped. The remaining 12 of the 13 lessons were videotaped and transcribed. Lesson plans and all of the students' written work (exams and the planned unit) were collected. The language of the data was Hebrew. This data was analyzed in its original language and parts were translated to English by the authors. To learn about students' ritual and explorative participation, we focus on their studying a specific type of problems - serial tasks (calculating sums of sequences). This topic was discussed in lessons 2 and 3. Assuming that students' participation may change when shifting to a new subject, we chose to focus on one specific topic in its entirety. The data analyzed include all whole class discussions that took place during each of the lessons.

Participants

The research participants include a group of 18 prospective elementary school mathematics teachers. The students are studying at their final academic year in a college of Education in Israel. The course instructor, the first author, has a B.Sc in mathematics and PhD in mathematics education. She had been teaching in this college for 15 years.

The course

Over the past two decades, accumulating evidence has shown that classroom environments that support “explorative” participation, that is, that encourage students' authority (Herbel-Eisenmann, Choppin, & Wagner, 2012); engage students in tasks that are cognitively demanding and are open to different solutions and procedures (Boston & Smith, 2009); and foster a community of learners that listen to each other and build on each other's ideas (Resnick, Michaels, & O'Connor, 2010) promote conceptual understanding. The aim of the designed course was for students to deepen their mathematical thinking by working on high cognitive-demand problems (Smith & Stein, 2011), solving problems in various ways and making connections between the different solutions as well as between the mathematical ideas related to the problems and the solutions. During the lessons, students worked in small groups to solve the problems and were encouraged to come up with as many solutions as possible. Whole-class discussions about the different solutions followed. The instructors chose to provide students with as much time as needed to work on certain problems alone or in groups, focusing entire lessons on discussing different solution paths suggested by the students. The students had to take three exams during the course and were required to plan a 3-lesson unit about any topic for elementary school students, which aim was to promote their students' mathematics thinking. Two groups (of around 20 students each) were taught simultaneously – by each of the two authors. This study refers to Talli's group only.

Data analysis

To identify whether students' participation is more ritual or more explorative, and to identify shifts in participation we followed all whole-class discussion around a specific type of tasks. The discussions were analyzed by addressing the questions in Table1.

		Ritual	Exploration
1.	What is the question the performer is trying to address?	How do I proceed?	What is it that I want to get?

2.	How does the performer evaluate its' success?	Performing a specific task-related procedure	A new narrative had been produced
3.	By whom is the routine initiated? To whom it is addressed?	Initiated by former performer of a similar task. Addressed at the teacher (or other superior).	oneself
4.	Flexibility	Applies a rigid routine. Seldom makes independent decisions	Could consider various routines. Makes independent decisions on the way.
5.	Separation between procedure and task	Not separated as the main task is to perform a (certain) procedure.	Separated
6.	Authority	The teacher	One's own

Table 1: Rituals and explorations

Findings and discussion

To understand what opportunities for explorative participation were given and taken up by students, we present our findings about students' participation while working on a specific type of tasks (calculating sums of sequences). This learning took place during lessons 2 and 3.

The first problem that the students solved was calculating the number of Hanukkah candles one lights during the 8 days of the Holiday (2 candles on the first evening, 3 on the second, ... and 9 candles on the eighth night): $2+3+4+5+6+7+8+9 = (2+9) + (3+8) + \dots = 4 \cdot 11 = 44$ (following Gauss's idea of pairing elements of the sequence: first element with the n^{th} , second element with the one in the $(n-1)$ place, and so on. The sums of each of the pairs are equal.). Then, after quickly calculating the sum of integers between 1 and 99, the students were asked to calculate the following sum: $1+3+5+7+\dots+997+999 =$.

The following conversation took place:

- 1 Maya I remember that there's something, a formula, I don't remember it now. To find the element in the middle.
- 2 Inst. To find a formula for finding the middle element?
- 3 Maya no, no, there is a formula
- 4 [the class discusses a solution path suggested by a student]
- 42 Sonya it could be factorial, right?
- 43 inst. Factorial?
- 44 Sonya yea, factorial, I remember something, I think this could be related.
- 45 inst. [to the class] do you what factorial is? Remember what it is?

The first student's (Maya) saying referred to the existence of a formula that could, perhaps, be helpful in this situation. She also talked of remembering. That is, Maya's first reaction was to seek a formerly learned routine that could be helpful in this situation. Maya followed Gauss's idea to add pairs of

numbers. As the number of elements in this sequence is even (500 numbers), Maya's search for the "middle number" is surprising. It is possible that either Maya mistakenly thought that there actually is an odd number of elements in the sequence. A different interpretation is that as all the problems that the students have worked on in the course so far have been of an odd number of elements, Maya looked for the "middle element" as part of performing a given routine practiced earlier.

After Maya's remark, a whole-class discussion arose about the number of elements in the sequence, whether it was even or odd. After agreeing that there are 500 elements (half of the elements in the sequence 1, 2, 3, ..., 1000), the students used Gauss' idea and calculated: $(1+999) \cdot 250 = 250,000$. The instructor asked for a different solution when Sonya replied: "it could be factorial, right?" she then added: "I remember something, I think this could be related". That is, similar to Maya, Sonya seeks a formerly learned and used routine to be applied here. She does not remember the routine or what the idea behind "factorial" is, but something about this problem reminds her of this formerly learned idea. Considering Maya and Sonya's communication about the given problem, it has strong ritual characteristics:

		Ritual	Exploration
1.	What is the question the performer is trying to address?	How do I proceed? <i>There is evidence of the students' attempts to seek a ready-made formula that could help them proceed. In Maya's case, the formula is not needed as there is an even number of elements. In Sonya's case – the idea she thinks of (factorial) is not relevant. Sonya is not sure what it actually is, but she does find some connection between the procedure and the problem at hand.</i>	What is it that I want to get?
2.	How does the performer evaluate its' success?	Performing a specific task-related procedure <i>Although not yet performing the procedure, Maya & Sonya seek a task-related procedure to follow. Both turn to memory (1, 3, 44).</i>	A new narrative had been produced
3.	By whom is the routine initiated? To whom it is addressed?	Initiated by former performer of a similar task. Addressed at the teacher (or other superior). <i>The routine was learned sometime in the past (not during this course). They do not try to develop a routine for the specific task by analyzing where it is they want to get at (1, 3, 44).</i>	oneself
4.	Flexibility	Applies a rigid routine. Seldom makes independent decisions <i>The students do not make any decisions at this moment. They turn to the instructor to remind them of the formula / concept that they thought was relevant (42).</i>	Could consider various routines. Makes independent decisions.
5.	Authority	The teacher (42)	One's own

Table 2: Analysis of Maya & Sonya's participation

During the rest of the lesson, the students worked in groups and came up with three ways to solve the problem. Later in the second lesson, the students suggested and discussed different solutions paths different series.

During the lesson the students learned: (1) that a problem could be solved by different solution paths; (2) that Gauss' idea could be helpful to solve sums of certain sequences; (3) to apply Gauss' idea, one needs to know the number of elements in the sequence. If the number is odd, then either the element in the middle should be identified or, the first or last element should be sided and later returned to the series. The students were encouraged to come up to the board to suggest solution paths, to make certain that they understand others' ways of solving the problem and ask questions when things were not understood.

At the beginning of the 3rd lesson students were asked to suggest ways to calculate:

$$1 + 2 - 3 - 4 + 5 + 6 - 7 - 8 + \dots + 2009 + 2010 - 2011 - 2012 + 2013$$

Solution paths suggested by two students (Nur and Sara):

- 1 Nur so, I put the 1 aside, and saw that (in) each two pairs, the first gives me minus 1 and the second pair gives me plus 1. So I have 2012 numbers here and then, if I divide this to pairs, I have 1006 pairs. So half of them give me 1, 503 altogether, and the other half gives me minus, so it's (-503) and it cancels and so I have 1 left.
- 2 Inst. Do you see what Nur did? Any questions?
- 3 Ziv no. it's perfectly clear
- 4 Nur now I found another one. After... I left the 1 aside here too, then I saw that if I take the 2 and the 2013, it gives me 2015. If I take then the minus 3 and the minus 2012 it will give me (-2015). So I have the sum of two negative numbers that give me (-2015) and the sum of two positive numbers give me (+2015). So it still cancels out and I have the 1 left.
- 5 inst. What do you all say? yes?
- 6 Sara I have another one. I saw here, my language is not that well, correct me, ok? like a sort of continuum, that every four, every three operations give (-4) each time, then its repeated.
- 7 Inst. Every three operations, you mean, the sum of four numbers?
- 8 Sara yes, the sum is (-4). So it's repeated till number 2012. So 2012 is divided by 4 and I get 503 times that it's repeated. ... then I multiplied 503 by (-4) cause every such part is, mm.... (-4). And this is the result so far [-2012]. Then I have plus 2013 and it's 1.

The analysis of Nur and Sara's suggested solution-paths is in Table3.

		Nur (2 solution paths)	Sara
1.	What is the question the performer is trying to address?	<i>What is it that I want to get?</i> The student is trying to solve the given problem and create a new solution path.	
2.	How does the performer evaluate its' success?	By creating narratives of two types – (1) the sum of the series (equals 1), and (2) new solution paths to	By creating narratives of two types – (1) the solution (the sum is 1). However, this

		solve the problem. In both, the first number of the sequence is left aside. In the 1st solution, adjacent numbers are paired. The sum of each pair is 1 or (-1) alternately. Therefore the solution is 1. In the 2nd solution, Gauss' principle is applied to create pairs of numbers whose sum is 2015 or (-2015) alternately. Therefore, the total sum is 1.	narrative is already known from previous answers. (2) a new solution path to solve the problem: the sum of every 4 adjacent elements is (-4). The sum of all quadruplets is $503 \cdot (-4) = (-2012)$. The last element of the sequence (2012) is added. Therefore, the sum of the sequence is 1.
3.	By whom is the routine initiated? To whom it is addressed?	The routine used by the student to solve the problem (placing the first or last number aside and checking sums of sets of numbers), as well as Gauss's principle were used by the class in the previous lesson. However, this is not simple mimicking of previously performed routines by others, as those routines have not been used together for the same problem yet. Some adaptation needed to be done. Therefore, the routine is initiated by the student.	
4.	Flexibility	The student used previously performed procedures to create a new solution path to solve this problem. Considering the students' decision making – the student made all decisions for adapting previously used routines to this problem.	
5.	Separation between procedure and task	It is not clear whether for the student the procedure is a part-and-parcel of solving this problem or not.	
6.	Authority	One's own	

Table 3: Analysis of Nur and Sara's participation

It seems that Nur and Sara participated exploratively: they produced narratives that include new solution paths for the task and reaching a solution. They articulate their solution path in a way that makes clear that they have made independent decisions while adapting different formerly performed routines to solve the given problem.

Discussion

In this study we tried to characterize students' learning while solving a specific type of tasks on the ritual towards exploration continuum. In the discussed course, in which students were constantly asked to work on problems on their own or in small groups, and suggest various solution paths to each problem, there is evidence of students shifting from more ritual to more explorative participation. It may seem obvious - the course was designed in a way that would invite students to engage exploratively. Yet, studies show that even when teachers design lessons that aim at certain opportunities to learn, this does not always happen. We found that when faced with a task that is of a new type (to the learner), the learner's immediate response is of a ritual type - to seek related procedures that would assist her in solving the task. The request to suggest various solution paths seems to help students focus on the question of "where do I want to get at?" and not remain focused on the question of "how do I proceed?". Also, once a student chooses a certain routine as a solution-path, it is sometimes followed blindly, ritually, thus getting farther away from the task at hand.

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Co-disciplinary mathematics and physics Research and Study Courses (RSC) within two groups of pre-service teacher education

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In this paper, we present results of an inquiry based teaching implementation carried out on a teacher training course in the University. The framework of the Anthropological Theory of Didactics (ATD) is adopted, and a co-disciplinary Research and Study Course (RSC) whose generative question requires studying physics and mathematics together is carried out by N=25 training teachers of Mathematics at University. Some conclusions concerning on the conditions, restrictions and relevance of introducing the RSC in teachers training courses at the university are performed.

Keywords: Inquiry based teaching, pre-service teachers training, modelization; research and study course.

Introduction

The Anthropological Theory of the Didactic (ATD) has proposed the Paradigm of Research and Questioning the World (Chevallard, 2012, 2013 a) advocating an epistemological and didactic revolution (Chevallard, 2012) of the teaching of mathematics and school disciplines, where knowledge should be taught by its usefulness or potential uses in life. The present work shows results obtained in two courses of pre-service mathematics teacher education (N=25) during a teaching inspired in the paradigm of questioning the world, by means of a Research and Study Course (RSC). To learn what an RSC is, and which kind of teaching is involved in, the trainee teachers (TT) must deeply experience a genuine RSC. Thus, a physics and mathematics co-disciplinary RSC is designed, implemented and analyzed with the students. Co-disciplinary means that in this case, physics does not only trigger the study of mathematics, but rather that both disciplines play a central role, being necessary to study both as well. The starting point of the RSC is the question Q₀: Why did the Mucediza stone in Tandil fall? Which, to be answered – in a provisional and unfinished way- needs study Physics and Mathematics jointly. The rationale of the paper is to describe the trainee teachers' activities and their difficulties when they must experience a genuine RSC and to face a strong question. Some reflections on the ecology and economy of this kind of didactic devices for the pre-service teachers training are performed.

The research and study courses (RSC)

The ATD defines the RSC as devices that allow the study of mathematics by means of questions. The RSC establish that the starting points of mathematical knowledge are questions called generative questions, because its study should generate new questions called derivative. Teaching by means of RSC is complex and demands rootle changes in the roles of the teacher and students. The RSC are defined by the *developed Herbartian model* (Chevallard, 2013 b):

$$[S(X;Y;Q) \rightarrow \{R_1^\diamond, R_2^\diamond, R_3^\diamond, \dots, R_n^\diamond, Q_{n+1}, \dots, Q_{m+1}, \dots, O_m, O_{m+1}, \dots, O_p\}] \rightarrow R^\heartsuit$$

Where Q is a certain generative question; S is a didactical system around of the study of Q . S is formed by a group of people trying to answer the question (X) and by people helping the study (Y). In classrooms of mathematics, X represent the students and Y represent the teacher and other instruments helping in the search of answers to Q . S has to build a didactic medium M to study Q , whereas M is composed by different knowledge, expressed by R_i^\diamond , Q_j and O_k . The R_i^\diamond are any existing answer or “socially accepted answer”, the Q_j are derivative questions of Q , and the O_k are any other knowledge that must be studied developing the answers. Finally, R^\heartsuit is some possible and partial response to Q given by S . In the a priori analysis stage, the specific and didactic knowledge which could be involved within an RSC is set up and the Praxeological Reference Model (PRM) is elaborated. The researchers analyze the potential set of questions which the study and the research into Q might encompass together with the knowledge, mathematics and physics in this case, necessary to answer those questions (Chevallard, 2013). The PRM underlies the whole of the teacher, student and researcher’s activity, being always likely and desirable to identify and clarify it, emphasizing the dynamic nature of the PRM.

Methodology

This is a qualitative and exploratory research that aims to carry out inquiry based teaching as it is proposed by the ATD, in a mathematics teacher training course at the University. The RSC was implemented in a state university, in the city of Tandil, Argentina, in a discipline which is part of the didactic studies within the Mathematics Teaching Training Course, in which the researchers are also the teachers, where $N=12$ and $N=13$ students from the last year (4th), aged 21-33 took part in it. The students had studied the ATD in two Didactic courses; however, they had problems to understand what an RSC is, and how it works? To emphasize the inquiry dimension of the RSC, the lessons were carried out in the University Library, given the wide availability of books and internet-based searching, during 10 weeks (the half of the course extension), with a total of 7 weekly hours provided in two lessons. Six work groups were organized with approximately 4 members each. During the lessons students and teachers interacted permanently. In a RSC, the generative question Q_0 has to be pointed out by the teacher, and this was made in the first lesson. Then, the students started their research in the library, by selecting some texts, documents etc. as possible R_i^\diamond . At the end of this class, each group presented and discussed with the teacher and the other groups their findings and possible ways to face Q_0 . In the second class, many emergent questions Q_i were made explicit, and the teacher and students groups selected which questions Q_i and their related knowledge O_k were to be studied. This was the regular dynamic during the RSC. Recordings of each meeting were obtained and the students’ productions were digitalized and returned in the subsequent meeting. The data analysis was performed by using the categories provided by the *developed Herbartian model* (Chevallard, 2013) summarized before, and all derivative questions Q_i , all “socially accepted answers” R_i^\diamond found by the students, together with the O_k studied were described and analyzed.

The Praxeological model of reference (PMR) and the RSC

The starting question Q_0 is: Why did the Movediza Stone in Tandil fall down? This enormous basalt stone has remained the city’s landmark, providing it with a distinctive feature. Many local people

and national celebrities visited the place to watch closely the natural monument. It was a 248-ton rock, sitting on the top of a 300-meter-high hill (above sea level), which presented very small oscillations when disturbed in a non-arbitrary spot, as shown in Figure 1. Unexpectedly, on February 28, 1912, the stone fell down the cliff and fractured into three pieces, filling the town with dismay at the loss of their symbol. For over 100 years the event produced all kinds of conjecture, legends, and unlikely scientific explanations for the causes of the fall. Within the two groups where the RSC has been performed, there existed a certain curiosity and interest in finding a scientific answer to this question.

Once in contact with the available information, the question evolved into: What conjectures are about the causes the Movediza Stone fall, and which is the most likely from a scientific viewpoint? Considering that the fall can be explained by means of the Mechanical Resonance phenomenon, several questions Q_i arose which are linked to the physical and mathematical knowledge necessary to understand and answer Q_0 .

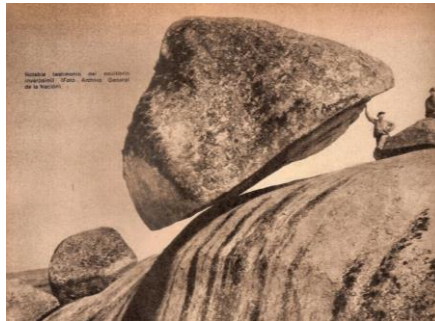


Figure 1: Photography of the Movediza Stone (Photo Archivo General de la Nación Argentina, available in: <http://bibliocicop.blogspot.com.ar/2012/02/piedra-movediza-100-anos-de-su-caida.html>)

If we consider that the real system is an oscillating system, the study can be carried out within the Mechanic Oscillations topic, starting from the spring or pendulum models which are ideal at the beginning. In this case, frictionless systems are used, in which the only force in action is the restoring force depending in a linear way on the deviation respect to the equilibrium position, and which produces oscillating systems known as harmonic, whose motion is described by a second-order linear differential equation, called by the same name. In the case of the pendulum, the restoring force can be considered depending on the oscillating angle (for small angles).

Progressively, the system becomes more complex. If friction-produced damping is considered, it provides a new term to the differential equation connected to the first derivative of the position (speed). Finally, it is possible to study systems that apart from being damped, they are under the action of an external force, and therefore called driven systems. Whenever this given force is periodic and its frequency coincides with the natural (free of external forces) frequency of the oscillating system, a maximum in the oscillation amplitude is produced, generating the phenomenon known as mechanical resonance. By increasing the complexity of the model, it is possible to consider a suspended rotating body, instead of a specific mass. This leads to the study of the torque and the moment of inertia of an oscillating body. Here again, the linear system is for small amplitude oscillations and the damped and driven cases can be also considered, corresponding to the same mathematical model, but in which the parameters have a different physical interpretation.

However, as it refers to a suspended oscillating body, this is not a suitable physical model for the Movediza stone system. Since that the base of the Stone was not flat, it is necessary to consider more precise models of the real situation. This leads to the mechanics of supported (and not hanging) oscillating rigid solids. In this case, we consider a rocker-like model in which the movediza stone base is curved and it lies on a flat surface, where the oscillation is related to a roto-translation motion (Otero, Llanos, Gazzola, Arlego, 2016 a, b). The application of Newton Laws to the rocker model of the movediza stone leads to a differential equation of the type harmonic oscillator, where the parameters are now specific of the movediza system: mass, geometry, inertia moment, friction in the base, external torque, etc. It leads to the following *effective* Harmonic oscillator mathematical model of the movediza physical system:

$$\ddot{\varphi} + \gamma \dot{\varphi} + w_0^2 \varphi = (M_0 / I) \cos(wt) \quad (1)$$

The solution to equation (1) is $\varphi(t) = \varphi_M \cos(wt - \psi)$, being the amplitude φ_M and the phase ψ

$$\varphi_M = \frac{M_0 / I}{\sqrt{(w_0^2 - w^2)^2 + w^2 \gamma^2}} \quad \psi = \operatorname{tg}^{-1} \left(\frac{\gamma w}{w_0^2 - w^2} \right)$$

The maximum of φ_M is for $w = \sqrt{w_0^2 - \gamma^2}$. The parameters: M_0 (external torque), I (inertia moment), w_0 (natural oscillation system frequency) and γ (friction coefficient), must be estimated. Detailed data about the shape, dimensions and center of mass position of the movediza stone are available (Peralta, et al 2008) after a replica construction and its relocation in 2007 on the original place (although fixed to the surface and without possibility to oscillate). These data bring us the possibility to fix some parameters in our model, as e.g. mass, inertia moment, and the distance of 7.1 m, from which the external torque could be exerted efficiently by up to five people (per historical chronicles) to start the small oscillation. By using these values, it is possible to study the behavior of the $\varphi_M(w)$ function for w_0 in a range of frequencies between 0,7 Hz and 1 Hz, historically recognized as the natural oscillation frequencies in the movediza stone system and calculate for each case the maximum amplitude $\varphi_M(w_m)$ that occurs for $w_m = \sqrt{w_0^2 - \gamma^2}$. The Stone would fall if $\varphi_c \leq \varphi_M(w_m)$, being $\varphi_M(w_m) = M_0 / w_0 I \gamma$ the maximum value of the amplitude function, that is to say $\varphi_c \leq (M_0 / w_0 I \gamma)$. The value of φ_c can be determined by an elementary stability analysis, which according to the dimensions of the base of the stone and the center of mass position is estimated to be approximately of 6° . In the present model, we cannot estimate γ . If we adopt “ad doc” for this parameter, a magnitude order $\gamma \geq 10^{-2}$, we obtain in the various situations considered with different torques and the interval frequencies previously mentioned, that all the scenarios support the overcoming of the critical angle, i.e., predict the fall. Later, in search of a more appropriate approximation of the physics model for the damping that is clearly not due to air, we can consider the stone as a deformable solid, where the contact in the support is not a point, but a finite extension. Therefore, the normal force is distributed on such a surface, being larger in the motion direction and generating a rolling resistance, manifested through a torque contrary to the motion due to friction. The rolling resistance depends on the speed stone, giving a physical interpretation to the damping term. Therefore, the physics behind the damping is the same that

makes a tire wheel rolling horizontally on the road come to a stop, but in the case of the stone, the deformation is much smaller. Although the deformable rocker model has extra free parameters, tabulated values of rolling resistance coefficient for stone on stone, which are available in the specialized literature, allowed us to estimate and justify the damping values that we incorporate otherwise ad-hoc in the rigid rocket movediza model.

Some results obtained in the two implementations

During the implementations, the students aim at answering how and why the stone fell down the cliff. At the beginning, the TTs searched in the elementary physics textbooks for an “already-made” mathematical and physical model, which allowed them to solve a differential equation in a specific way. In both implementations, several physical and mathematical questions arose; the main preoccupation of the TTs was to study the oscillations subject, because it was a new knowledge to them. The implementation was carried out in parallel with a Differential Equations course, and none of the groups seemed to have difficulties with the underlying mathematics. In both implementations, the TTs tried to find a physical model suitable for the situation and they decided on the physical pendulum model initially, whose mathematical model might be adequate to the problem, although physically inadequate. However, the path performed in the first implementation was different from the second. In the first case, the TTs did not question the physical model, and spent most of the time to the study of inertia moment concept and their calculation for regular solids, which would result in an appropriate model for the irregular shape of the stone. Several interesting questions were therefore generated, for which the answers were provided by the teacher and the students together. After, the students calculated by themselves the characteristic frequency of the system, making use of the moment of inertia previously obtained. Thus, only a parameter resulted undetermined: the damping. But in the end, the physical pendulum model became an obstacle, because the stone was a supported body, and not hanged. On the other hand, the damping they considered was due to air-friction, whereas in the case of the movediza stone the main source of friction is the contact with the support surface.

At this point, the external torque (there were different trials to analyze and estimate it) and the solution of the equation remained unstudied. Until this moment, the solution for the differential equation did not seem to present any obstacles to the students, who considered they were facing an initial value problem. Once they had obtained the parameters, which they considered fixed, the solution seemed simple. However, they had problems to arrive at a final solution, even though this can be found in the physics textbooks (without its deduction). For this reason, it was discarded and they decided to do the calculation on their own. This event complicated the quantification they aimed to obtain, as well as the physical interpretation. Some groups in this cohort removed the term of damping, to reduce degrees of freedom; thus, the stone would have been in perpetual motion. This did not create any contradiction to them. Other ones adopted a damping value due to air-friction, which also led to wrong results. In summary, instead of adopting and adapting the solution that was presented in physics books, the TTs in this cohort dismissed it and did not interpret the answer in the texts concerning the Stone. The decision of the teachers to delay their intervention was with the purpose to make students live the dynamics of progress and drawbacks typical of the research and study courses. In addition the TTs had problems to understand the utility and necessity

of mathematical models, due to an epistemological conception close to pure or formal mathematics. The TTs did not understand how to use the mathematical models, neither the role that the parameters could play, which were considered as fixed and universal. In consequence, they failed to establish different sets of parameters and did not generate the feasible families of functions and values, whose compatibility with the physical situation could have been analyzed. These difficulties were considered for the second implementation of RSC. In the second cohort (TT2), the teachers had already perceived that the fundamental problem seemed to be in the models and in the modelization. For this reason, it was decided to devote 8 sessions to the development of two intra-mathematical RSCs (Chappaz & Michon, 2003; Ruiz, Bosch, Gascón, 2007), that the TTs could experience by themselves, therefore emphasizing the role of the modelization and the use of devices as spreadsheets and plotters. Besides, in this case, the teachers intervened as soon as the students proposed the physical pendulum and spring models. One group studied the AMS for the simple pendulum, the spring, and the physical pendulum, another group studied the spring model in all its possibilities and the third one did not develop further than the AMS in simple pendulum and spring. The synthesis stage corresponding to that class allowed the production of a complete answer for the three models and its possibilities, from which the TTs and the teacher arrived at the conclusion that the same mathematical model represented (9) nine different physical systems (Figure 2). A large amount of time was devoted to pondering on the differences and similarities between the mathematical and physical models and their connection with the real system we aimed at modeling. Then, the answers to the equations presented in the books were checked out.

Existen 10 parámetros (GOTIC)

Lineal

Lineal/rotatoria

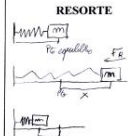
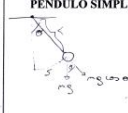

MOVIMIENTO	MAS (Movimiento Armónico Simple)	Movimiento Amortiguado	Movimiento Forzado
MODELO	Ecuación	Ecuación	Ecuación
RESORTE 	$m \frac{d^2 x}{dt^2} + K x = 0$ $\frac{d^2 x}{dt^2} + \frac{K}{m} x = 0 \quad \omega = \sqrt{\frac{K}{m}}$ $\frac{d^2}{dt^2} + \omega^2 x = 0$	$m \ddot{x} = -K x - b \dot{x}$ $\frac{d^2 x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \omega^2 x = 0$ $\omega^2 = \frac{K}{m}$	$m \ddot{x} = -K x - b \dot{x} + f(t)$ $\frac{d^2 x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \omega^2 x = \frac{f(t)}{m}$ $\frac{d^2 x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \omega^2 x = \frac{f(t)}{m}$
Solución: $x(t) = A \cos(\omega t + \phi)$	Solución: $x(t) = A e^{-\frac{b}{2m}t} \cos(\omega t + \phi)$	Solución: $x(t) = x_p + A \cos(\omega t + \phi)$	Análisis de Resonancia
PÉNDULO SIMPLE 	$\frac{d^2 \theta}{dt^2} + \omega^2 \theta = 0$ $\omega = \sqrt{\frac{g}{L}}$	$\frac{d^2 \theta}{dt^2} + \gamma \frac{d\theta}{dt} + \omega^2 \theta = 0$ $\gamma = \frac{b}{mL}$ $\omega = \frac{g}{L}$	$\frac{d^2 \theta}{dt^2} + \gamma \frac{d\theta}{dt} + \omega^2 \theta = \frac{F(t)}{mL}$ $F(t) = F_0 \cos(\omega t + \phi)$
Solución: $\theta(t) = \theta_0 \cos(\omega t + \phi)$	Solución: $\theta(t) = \theta_0 e^{-\frac{\gamma}{2}t} \cos(\omega t + \phi)$	Solución: $\theta(t) = \theta_p + A \cos(\omega t + \phi)$	Análisis de Resonancia
PÉNDULO FÍSICO 	$\frac{d^2 \theta}{dt^2} + \omega^2 \theta = 0$ $\omega^2 = \frac{mgd}{I}$	$\frac{d^2 \theta}{dt^2} + \gamma \frac{d\theta}{dt} + \omega^2 \theta = 0$ $\gamma = \frac{b}{I}$	$\frac{d^2 \theta}{dt^2} + \gamma \frac{d\theta}{dt} + \omega^2 \theta = \frac{F(t)}{I}$
Solución: $\theta(t) = \theta_0 \cos(\omega t + \phi)$	Solución: $\theta(t) = \theta_0 e^{-\frac{\gamma}{2}t} \cos(\omega t + \phi)$	Solución: $\theta(t) = \theta_p + A \cos(\omega t + \phi)$	Análisis de Resonancia

Figure 2: Protocol of the student E17. Implementation 2

In both cohorts, as a fixed route that is inevitably set by the books, the TTs came across the physical pendulum. However, in the second cohort some students presented strong objections to the possibility of using it in the case of the stone, not so much in relation to a body that is supported but as an “inverted” pendulum. This drew the discussion once more towards the real system and the standing point, so that the path went through the models which refer specifically to the system and that are not, usually in elementary books, like a rocker.

Firstly, the equilibrium was analyzed and the critical angle was calculated, and then, the model of the base of the movediza stone was sophisticated. For the study of the rigid solid physical model, the teacher proposed to the students a little text, as a new O_k that could be introduced into the didactic medium M . Finally, the students and the teacher calculated and estimated the parameters of the differential equation solution, and the classroom elaborated an answer that allows the explanation, by means of a model, of the plausibility of the fall.

Conclusions

Despite of the difficulties, the TTs experienced a genuine RSC within its means. There is a visible initial reluctant attitude on the part of the TTs: Why physics should be studied if we are teachers of mathematics? Later, it was gradually understood that the idea was to experience a genuinely co-disciplinary RSC, to analyze it and comprehend the teaching model supporting an RSC.

Even though the TTs had studied the ATD and other didactic theories, they did it in a traditional way comparable to the traditional training they get. This is reflected in the difficulties they have to understand and to use both physics and mathematical models. It was not expected that the TTs developed the models by themselves, but it was expected that they used the mathematical answers presented in the physics textbooks in a pertinent and exoteric manner. This fact did not occur in the first group and improved in the second one from the didactic decision to make a previous incursion into mono-disciplinary RSC particularly suitable for evidencing the role of the functional modelization. In addition, this allowed teachers to discuss the relationship between the mathematical model and the physical model and the meaning and role of the parameters.

The TT's behavior is interpreted from the fact that although they have experienced four years of "hard" university studies, the utility of the science they aim at teaching had never been visible. The epistemological conception about the mathematics produced by the traditional paradigm is so ingrained, that it is complex to reverse it. This would be, in our view, the most relevant drawback to permit the TTS at least understand what an RSC is and how the modelization activity works? However, it is important to notice that the sporadic incursions in the modelization activity do not seem enough to allow the TTs develop such school practices. Although the predominant teaching is mainly traditional, the TTs will face increasing demands for a change to a mathematic teaching based on the research, questioning and modeling. It is unlikely that a teacher whose training has been answers-based teaching can teach by means of questions. Then the training of teachers has to change profoundly.

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Writing fictional mathematical dialogues as a training and professional advancement tool for pre-service and in-service math teachers

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Pre-Service and In-Service Math Teachers are often surprised to find themselves at a loss for words in the mathematics classroom. This feeling is not limited to the first day of class or to beginning teachers. Even experienced teachers describe unexpected classroom situations in which they cannot find the proper words to respond or to explain or mediate ideas. The teaching routine is fraught with on time decisions teachers must make. The objective of the current study was to promote the development of spoken and written mathematical discourse among pre-service and in-service math teachers in the context of classroom scenarios they considered unexpected and complex. The training was directed toward developing argumentative mathematical discourse skills through writing, with emphasis on writing fictional dialogues. The research focuses on the characteristics of fictional mathematical dialogues written by pre-service and in-service math teachers and seeks to show that these dialogues can be used as a professional advancement tool.

Keywords: Writing fictional mathematical dialogues; professional advancement tool; pre-service and in-service math teachers.

Introduction

Teachers make decisions based upon knowledge, goals, beliefs and orientations. Accordingly, developing all of these factors can help promote decision making in the mathematics classroom.

For many years, I have been seeking creative ideas that will enable in-service and pre-service teachers to predict scenarios and unexpected situations in the mathematics classroom. Thus, they will be able to practice mathematical discourse before coming to class and to learn to provide argumentative responses that are quick, accessible and flexible. Teachers' responses in class and their responsibility in developing mathematical discussions and discourse have been the topics of much investigation (e.g., Schoenfeld, 2008-2011). The literature has placed less emphasis on examining training methods for developing discourse management for predicting unexpected classroom situations in advance, particularly training all that through writing.

Zazkis and Koichu (2015) describe a fictional dialogue on infinitude of primes between Euclid and Dirichlet and use this as a research method. The current study focuses on pre-service and in-service math teachers who write "fictional dialogues" as part of their training. The goal of this writing is to develop their ability to explain, respond and engage in argumentative mathematical discourse in a learning situation characterized by unexpected situations. The results of the current study indicate that the task of writing fictional dialogues has several advantages. One advantage relates to professional development and renewal. Veteran teachers tend to feel less challenged and less interested in preparing lessons in advance. Writing fictional dialogues challenges them to formulate unexpected mathematical situations for mathematical topics and ideas that for them are seemingly

simple and trivial. In writing fictional dialogues, they discovered both mathematical and didactic innovations. Another advantage applies to training. In writing the dialogues, beginning teachers learned to develop written mathematical discourse that explains the essence of mathematical terms. Further, they learned to use visual or other representations in context and practiced giving explanations to learners with a variety of learning styles.

Theoretical background

Unexpected situations in the mathematics classroom differ from teacher to teacher due to differences in the extent and depth of their mathematical knowledge, their ability to identify such situations and their ability to make decisions in real time about the didactic concepts appropriate for each situation. Hence, I examined the research literature on major topics related to the current research. These include training pre-service and in-service math teachers by means of writing, the role of the teacher in discourse development and management in the mathematics classroom, mathematical argumentation as a teaching tool and interaction in the mathematics classroom (Malaspina, Mallart, Font, , & Flores, 2016). The conclusions of these studies led me to formulate ideas for a unique intervention "training" program with the potential to promote mathematical discourse in the classroom in general and argumentative mathematical discourse in unexpected situations in particular. In the following sections, I review the relevant literature in these fields and explain how these studies relate to the current research.

Professional development and learning through writing

Teacher training usually incorporates writing through writing assignments about ideas learned in class or as reflection on learning (Korkko et al., 2016). Turning writing into a goal in and of itself is an innovation in the training of mathematics teachers. Therefore, in order to construct an intervention program that emphasizes writing, I surveyed and studied research that examines the advantages of writing in teaching math and of pedagogy based on writing in general.

In the study by Bostiga, Cantin, Fontana and Casa (2016), the students learned by writing diaries on mathematical argumentation. The research indicates that the process of writing develops students' in-depth thinking about mathematical concepts as well as underlining erroneous or other perceptions of concepts or phenomena. The writing process and the accompanying feedback prompted the students to write more precisely about mathematics, directed them to give arguments, explanations and reasoning in their writing and taught them to edit and rethink mathematical ideas. From this study among students, I decided to try to generalize the method for adults and to examine the results. Adults with a common professional interest often write together in a process that advances their shared understanding and learning in the field (Lowry et al., 2004).

Griffin and Beatty (2010) examined the attributes of shared writing among adults with a common professional interest. Their research pointed to several advantages, including professional and personal growth among the writers, a greater degree of creativity, the generation of new ideas and understandings, diversification in areas of specialization, increased documentation and output abilities, and shared knowledge. Shared writing generates a unified voice, increases feelings of satisfaction and pride in integrating the personal voice into the voice of the group and expresses respect for individual knowledge. Therefore, in this study the writing took place in pairs or in small groups as part of the process of developing skills in argumentative mathematical writing.

Read (2010) proposed the IMSCI model for supporting the writing process, with writing serving as a pedagogical tool for assimilating learning. In the IMSCI acronym, "I" stands for inquiry, "M" for modeling, "S" for shared writing, "C" for collaborative writing and "I" for independent writing. This scaffolding model was integrated into the intervention process in the current study.

Spoken or written mathematical discourse

According to Sfard (2008a; 2008b), discourse has four characteristics: vocabulary, visual mediators, unique routines and customary utterances. In the communicative approach, thinking constitutes an individual's discourse with the self. Such a discourse can yield ideas that express the thinking of those participating in the discourse. In contrast to those who talk, some people express themselves through writing and symbolic mathematical representations and have difficulty expressing their ideas verbally. Such individuals may eventually become teachers whose skills in developing and conducting mathematical discourse are not sufficiently developed. In most cases, this does not point to a lack of mathematical knowledge but rather to the difficulty teachers experience in translating this knowledge, which perhaps is represented in their minds through nonverbal symbols, into verbal tools. Mathematics teachers must generate significant discourse in their classrooms. Such discourse constitutes an organized and connected collection of all their students' and their own intellectual ideas. The job of the teacher is to conduct a discourse that reflects ideas and encourages participants to discuss these ideas, to endorse or refute them and to arrive at valid and agreed-upon mathematical rules that can be implemented in new situations that are similar or different. How can we promote and cultivate teachers who have the awareness and skills to cultivate this type of classroom reality?

Waggoner (2015) proposed five strategies for supporting meaningful math talk in class. First, teachers must talk with their students and arrive at common insights regarding the importance of math talk in the classroom. Second, teachers are responsible for teaching their students to listen and respond appropriately to one another. Third, teachers must teach their students to write sentence stems to emphasize their responses. Fourth, teachers must teach and demonstrate the difference between explaining and justifying what someone else says. Finally, teachers must provide examples of all these actions in class. The current study implemented all of Waggoner's ideas with pre-service and in-service teachers in the general context of group mathematical discourse and the particular context of written mathematical discourse in unexpected situations in the mathematics classroom.

Methodology

Participants

The research participants included undergraduate students taking a course that taught didactic and pedagogic skills for teaching math in elementary and junior high school and graduate students in mathematics education who teach math to all ages and at all levels. The two groups together totaled 35 students, as half of them were teachers were in fact teachers.

The research tool

Intervention design –The two courses comprised the same several stages. First, the students read the article by Zazkis and Koichu (2015) about fictional dialogues in order to understand and define fictional dialogues in the context of their unique methodological role in the original article. Next, we adopted the skill of writing fictional dialogues as a tool for developing spoken and written

mathematical discourse in lesson planning for unexpected situations in the math classroom. We embraced the following quote with the understanding that we as students also seek interesting learning methods. "People are eager for stories. Not dissertations. Not lectures. Not informative essays for stories" (Haven, 2007, p. 8).

Third, we defined and formulated conditions determining whether a potential fictional dialogue met the objective. In this stage, we read mathematical dialogues from various sources that resembled fictional mathematical dialogues and we reworked their mathematical discourse so it matched our definition of a fictional dialogue. Fourth, the students independently wrote fictional mathematical dialogues. In the fifth and final stage, the students showed their dialogues to their classmates. This generated an evaluative argumentative discussion and, if necessary, led to redesigning the dialogues. Throughout the course, we documented the sessions and their outcomes *focuses on fictional Mathematical dialogues*.

Definition of "fictional dialogue" in the current study

The definition of fictional dialogue emerged from agreement among all course participants and included the following characteristics: The dialogue must take place between two people with some sort of major gap between them. This gap may be rooted in culture, age, expertise, historical period (e.g., one speaker lives in contemporary times and the other lived 700 years ago), mathematical knowledge and more. One speaker is an expert in the field and should be able to bridge the gap through argumentative dialogue that leads the two speakers to understanding, definition and agreement on the mathematical topic they are discussing. The expert presents the mathematical explanation using formal intra-mathematical tools and extra-mathematical or other simple, practical and concrete examples and explanations. The non-expert participant's dialogue develops in unexpected directions, so that this participant can surprise the expert with questions or examples that seemingly contradict the mathematical concept under discussion or that present a challenge to the clear, simple and popular explanation. In the dialogue, the two participants express their perceptions of the mathematical topic being discussed, and each attempts to enrich the other's world through the mathematical knowledge at his or her disposal. Through the dialogue, the gap between the speakers becomes smaller in that all the relevant mathematical nuances in the field find expression in the dialogue.

Data analysis

The data analysis focused on the process of establishing the conditions for fictional dialogue.

Findings

In this paper, I describe one mathematical event representing two stages of the intervention period. Because the research focuses on the final product—"writing"—I give two examples of writing and discuss the processes involved in creating them. These two examples show that writing fictional mathematical dialogues can serve as a training and professional advancement tool for pre-service and in-service math teachers. The first finding refers to the third stage of the intervention period, in which we redesigned a dialogue and rewrote it as a group fictional dialogue. At this stage, each student individually redesigned the dialogue by writing a new dialogue based on the existing dialogue and thus creating a new personal product that conformed to the required conditions. In the next stage in the joint group work, the students showed their dialogues to their classmates for

evaluation, leading to writing an agreed-upon group product. The dialogue is the unified product after the group discussed their differences and went through the entire learning process.

Group design of a given dialogue and its transformation into a fictional dialogue

The given dialogue is from an Abbott and Costello movie titled *Buck Privates*:

Abbott: You're 40 years old, and you're in love with a little girl, say 10 years old. You're four times as old as that girl. You couldn't marry that girl, could you?

Costello: No.

Abbott: So you wait 5 years. Now the little girl is 15, and you're 45. You're only three times as old as that girl. So you wait 15 years more. Now the little girl is 30, and you're 60. You're only twice as old as that little girl.

Costello: She's catching up?

Abbott: Here's the question. How long do you have to wait before you and that little girl are the same age?

Costello: What kind of question is that? That's ridiculous. If I keep waiting for that girl, she'll pass me up. She'll wind up older than I am. Then she'll have to wait for me!

In order to determine whether this qualifies as a fictional dialogue, we mapped it to see whether it fulfills the conditions for fictional dialogues formulated in the second stage of the course. The mapping results indicate that the dialogue does not meet the conditions to qualify as a fictional dialogue. Hence, we redesigned the dialogue to fulfill the necessary conditions. Each course participant individually designed and wrote a fictional dialogue. In the next stage, the students as a group combined these individual dialogues into a fictional group dialogue. The group dialogue features an expert "player" called Achilles, provides intra- and extra-mathematical explanations, stresses the perceptions of each of the speakers so that it is clear who represents the erroneous perception and who represents the appropriate perception and stresses the unexpected situation. Using the ideas from the individual dialogues, the group wrote an argumentative fictional dialogue that gap the discrepancy between the speakers to the point of generating an unexpected situation in which the speakers "reverse" their roles, so that the rookie, Costello, triumphs over the expert, Achilles. The following lines from the dialogue demonstrate compliance to condition (4) as written by the group.

.....

Achilles: I, Achilles, run at a speed of 10 meters per second. My friend the turtle runs 1 meter per second. I decide to give the turtle a head start of 100 meters at the beginning of the race.

Costello: Wait a minute. This is a fable, right? So I want to convert it to apply to me. I gave the girl a forty-year head start. Wow, that's a lot. I am four times older than she is! And you run ten times faster than the turtle. Great, I get it.

.....

Costello: So let's assume I'm 240 years old. How old will she be??? She will always be 30 years younger than me, so she'll be 210 years old. So her age will be seven-eighths of my age. It appears we are slowly advancing to the point where we're the same age.

Achilles: No. That's not right. Let's go back to my turtle.

Costello: I'm not going back to your turtle because I've discovered the problem and also the solution. The girl and I will never get married because there will always be a fixed difference of 30 years in our ages. But there is not a fixed difference of 100 meters between you and the turtle because around 12 seconds after the beginning of the race you will already catch up with the turtle. In ten seconds you run 100 meters and in another two seconds you run another ten meters, so the race is over because the turtle continues to trail behind you.

Achilles: Does that mean that the age difference problem is not representative of infinity.

Costello: Now we've switched roles. I'm the mathematician. What are you? The concept of infinity can be represented if the beautiful girl and I live forever and do not die. From a situation in which I'm four times her age and then three times her age, we get to a situation where the ratio is 7:8, and we can go on to 8:9 and even further. While the discrepancy in our ages is still thirty years, with time the relative difference in our ages gets smaller. In contrast, when you race against the turtle, a quick calculation tells me that you'll overtake your opponent after 12 seconds. The 100-meter difference between you is not fixed because you "grow at different rates" That is, you each run at a different speed and you are ten times faster than the turtle.

Dialogue analysis "Age difference problem"

In the above dialogue, the students completed all the conditions that were missing from the original given dialogue. They created two fictional characters and delineated a significant historical and mathematical gap between them. They defined an expert speaker who led the dialogue. They formulated intra-mathematical explanations (e.g., speed as the ratio between distance time) and extra-mathematical explanations (e.g., representing the concept of infinity by means of the girl and Costello, who grow forever and never die) for the age problem and for the paradox of Achilles and the tortoise. Furthermore, they created two unexpected situations in the dialogue. One was the comparison between the age problem and the Achilles paradox. The other was that Costello understood the difference between the problems and claimed that the turtle problem differs from the age problem ("Now we've switched roles. I'm the mathematician. What are you?"). They created a specific explanation for the problem and its concepts and accurately differentiated between the two problems. Using the dialogue, they understood that the age problem demonstrates Costello's misconception about the age gap, as he thought the gap would decrease over time.

In contrast, the turtle paradox shows that the gap between the turtle and Achilles is not fixed and that the distance decreases with time. Using numbers, the students demonstrated the two situations, showing that the gap in the age problem remains constant while the distance between the turtle and Achilles continues to diminish. At this stage, they reduced the gap between the speakers' dialogue.

During the group formulation, the students explored ideas and mathematical explanations. They designed and formulated the dialogue as a group exercise, so that in cases of disagreement they stopped and sought a consensus in the group.

Discussion and conclusions

The current study is a pioneer in this field. The research was inspired by studies that examined student writing in math classrooms (Bostiga et al., 2016) and writing-based pedagogies (Korkko et al., 2016; Zazkis, et al 2009). The study implemented Read's (2010) method using the IMSCI

model. Implementing this model one-step at a time was found to be effective and to validate the results of studies claiming that only theories that are practically applied in the training process can be properly implemented in the field (Anderson & Stillman, 2013; Bråten & Ferguson, 2015; Cheng, Tang, & Cheng, 2012; Gomez Zwiép,. & Benken, 2013). That is, it would have been more effective to teach the theory of fictional dialogue in the course and then to practice it step by step (IMSCI) through actual writing.

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Tensions in the role of mathematics coaches

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The use of mathematics coaches as a means of professional development for teachers is an increasing phenomenon in North American schools. The research presented here identifies tensions experienced by mathematics coaches and how they cope with those tensions. Utilizing a framework that characterizes tensions as dichotomous pairings, the results indicate that there are tensions that are unique to mathematics coaches. This adds to a growing body of research into the role of mathematics teachers.

Keywords: Tensions, mathematics coaches, professional development.

Introduction and background

In their search for school-wide models that support improvements in the teaching and learning of mathematics, districts are, “embracing coaching as a model of authentic professional development wherein teachers can learn in the context of their schools and their instructional practice” (Campbell & Malkus, 2014, p. 213). Underlying this development is the recognition that schools need to become places where both students and teachers can learn (Hawley & Valli, 1999). With this in mind, districts have begun placing mathematics specialists in their schools to work directly with practicing teachers (Anstey, 2010).

Bearing a variety of labels, such as mathematics specialist, numeracy specialist, lead teacher or learning coach, a mathematics coach is generally a highly knowledgeable teacher hired to support the improvement of mathematics teaching and learning within a district (Anstey, 2010). An effective mathematics coach would have a deep understanding of mathematical content combined with pedagogical expertise and strong interpersonal skills. Usually they are former classroom teachers, recognized for their abilities and promoted from within (Campbell & Malkus, 2011).

Mathematics coaches are responsible for providing ongoing professional development of the inservice teachers in their districts by “advocating for their change, nurturing their performance, advancing their thinking, increasing their mathematical understanding, and saluting their attempts” (Campbell & Malkus, 2011, p. 459). To reach this goal, mathematics coaches’ work varies from modeling mathematics lessons in a teacher’s classroom to observing and supporting a teacher as they teach (Campbell & Malkus, 2014). This is a varied, demanding role that Campbell and Malkus (2011) suggest “the profession does not understand and is only beginning to examine” (p. 449).

The tensions experienced by mathematics coaches is one such unexamined area. A review of the literature reveals little information about the dilemmas mathematics coaches face. Literature regarding generalist coaches is much richer and suggests several common tensions. For instance, Neufeld (2003) identified a series of tensions experienced by generalist coaches that begins with a lack of time. This is a frequently experienced tension, whether it is a lack of time to conference with teachers or lack of time to prepare for working with teachers. Leaving the classroom environment causes tensions for some generalist coaches, as does the switch to working with adults. Tensions also occur when teachers are slow to uptake change or are actively opposed to its implementation. Finally, Neufeld (2003) suggests tensions for generalist coaches arise from working with uncooperative

school cultures or administration and from a lack of opportunities for personal professional development. While there are perhaps some commonalities between these tensions experienced by learning coaches in general and those experienced by mathematics coaches in particular, it would be of benefit to identify whether there are any tensions specific to mathematics coaches. Jones (1995) suggests that “members of the mathematics education community, whether in schools, colleges, or universities, have a responsibility to help one another recognize and deal with tensions in a productive way” (p. 233). The intent of this paper then, is to identify some of the tensions experienced by mathematics coaches.

Theoretical background

Endemic to the teaching profession, tension encompasses the inner turmoil teachers experience when faced with contradictory alternatives for which there are no clear answers (Adler, 2001; Berry, 2007). Building on the work of Berlak and Berlak (1981) who identified sixteen dilemmas that illuminated the relationship between everyday school events and broader social, economic, and political issues, it was Lampert (1985) who first suggested the notion of teachers as dilemma managers who accept conflict as useful in shaping both identity and practice.

For the purposes of this study, I turn to the work of Berry (2007) whose self-study of tensions in her role as a teacher-educator resulted in a binary categorization of tensions. Seeking to depict the inner turmoil she experienced from the competing pedagogical demands in her practice, she proposed a framework for both identifying and understanding tensions. Isolating the following six pairs of interconnected tensions, Berry used these as a lens to examine her practice: (1) Telling and growth—between informing and creating opportunities to reflect and self-direct (2) Confidence and uncertainty—between exposing vulnerability as a teacher educator and maintaining prospective teachers’ confidence in the teacher educator as a leader (3) Action and intent—between working towards a particular ideal and jeopardising that ideal by the approach chosen to attain it (4) Safety and challenge—between a constructive learning experience and an uncomfortable learning experience (5) Valuing and reconstructing experience—between helping students recognise the ‘authority of their experience’ and helping them to see that there is more to teaching than simply acquiring experience (6) Planning and being responsive—between planning for learning and responding to learning opportunities as they arise in practice (Berry, 2007, pp. 32–33).

Although initially used as a framework to isolate tensions in the work of teacher education of pre-service teachers, Berry’s (2007) framework has been used in other contexts as well. As part of a larger, ongoing project, of which this paper is a part, Liljedahl, Andrà, Di Martino, and Rouleau (2015) applied Berry’s tensions framework to a fictional composite of a mathematics teacher that comprised a collection of data sets. Their work expanded Berry’s framework by identifying new tension pairs and they concluded that some tensions may be the driving force behind a teacher’s pursuit of professional development by fueling a desire for change in practice.

While considered relative newcomers, mathematics coaches are part of a mathematics community that includes both pre-service and inservice teachers. Given that developing a shared understanding of the tensions teachers face gives them the power to shape the course and outcomes of their teaching practice (Adler, 2001); it is likely the same would be true for mathematics coaches. Bringing the

challenging aspects of their work to light would offer mathematics coaches the opportunity to recognise, talk about, and act on the tensions in their practice.

My goal then, in this article, is to isolate some of the tensions experienced by mathematics coaches. Specifically, using Berry's (2007) framework, I will identify and describe the tensions they face and how they cope with them. Thus, my research questions are as follows: (1) What are some of the tensions experienced by mathematics coaches? (2) How do mathematics coaches cope with those tensions?

Context and method

This study is part of an ongoing research project regarding tensions in teaching. In particular, it is the first look into the tensions experienced by mathematics coaches. This is a small scale, qualitative study that involves only three participants. As such, I am focusing on proving the existence of a phenomenon rather than its prevalence. It is important to note, however, that I chose to report only on those tensions that were experienced by more than one participant. While aware that a single instance of a tension can be as revealing as multiple instances, it is less likely to be seen as representative of a generalizable pattern. In keeping with that, the data corpus comprises interviews with three mathematics coaches working in three separate school districts.

Tara is employed by a small urban school district that employs 430 teachers in 17 schools. She has been in the role of K-12 District Numeracy Coordinator for 4 years. Prior to that, she worked as an elementary classroom teacher for 18 years. Having had negative experiences as a learner of mathematics, Tara's interest in math was only ignited 14 years ago after attending a mathematics professional development workshop. During the ensuing years, she attended every mathematics professional development opportunity offered and developed a passion for the teaching and learning of mathematics. She is about to begin a Master's degree program with a numeracy focus.

Pam works for a small rural school district with 23 schools and 320 teachers. Employed as a classroom teacher for over 30 years, she has taught all grades K-7. Pam completed a Master's degree with a numeracy focus in 2011 and left the classroom in 2012 to take on the role of Numeracy Helping Teacher. She has always enjoyed and had a passion for math.

Ray is employed by a large urban school district that employs over 1000 teachers in 49 schools. He worked for his district for 18 years as a secondary math teacher before taking on the role of Math and Science Program Consultant 4 years ago. During his time in the classroom, Ray completed a Master's in secondary mathematics education and was involved with his province's math teachers' association. Ray went on to serve a term as the association's president, while working as his district's math consultant. Like Pam, Ray has always enjoyed math and wants to recreate that experience for the teachers and students in the classrooms he supports.

Data was collected from the participants during semi-structured interviews that were transcribed in their entirety. The data corpus was then scrutinized using Berry's (2007) framework as an a priori frame for identifying and coding tensions. To begin this was done by searching the interview transcripts for evidence of tensions. In particular, I looked for evidence of utterances with negative emotional components such as "I think what's been difficult..." or utterances that conveyed doubt or uncertainty such as "I wasn't 100% sure, but...". The identified tensions were then grouped according

to the pairings described by Berry. Additionally, the framework was extended to encompass a tension that did not fit within her established framework.

Analysis

In the following analysis, Berry's (2007) framework will be used to identify and analyze the tensions experienced by the three participants in their roles as district mathematics coaches. During the analysis, the following four tensions pairs were evident.

Safety and challenge - Unwelcome in the classroom

All three of the participants mentioned the conflict they experienced between their desire to be working with teachers in their classrooms and not having that support seen as threatening. They describe the teachers as uncomfortable in having someone observe them and therefore are unable to utilize this valuable learning opportunity. A tension arises for the mathematics coaches who, like Berry (2007), want the teachers to feel safe, but who also recognize the value in challenging the teachers to open their doors. This is evident in the following excerpts:

Ray: And I think that teachers are a little reluctant to have people in their classroom and do, sort of team teaching or have someone observe them... that hasn't happened as much as I kind of thought it would or as much as I'd sort of like.

Pam: The kind of biggest piece, I think, for us, is how do you support those teachers that are too nervous or too anxious about having someone come in?

Tara: If I get invited in, I'm in, I go. Absolutely. But unless I'm invited in, it doesn't, like, I don't just, well, I shouldn't say I don't just show up.... but to be actually modelling in a classroom and doing observations, that's all by invite.

As a rationale for the teachers' reluctance, both Pam and Ray offer related possibilities. Pam suggests that the teachers' reluctance stems from a fear of being evaluated even though she feels she makes it clear that her role is one of mentorship and has no evaluative elements stating, "They haven't shifted away from that fear yet, that I'm there to judge. I'm not, I'm there to support them." Ray suggests that the teachers are concerned with the overall quality of their lesson, which then becomes a barrier to observation, "When it comes down right to it, you know, we're all a little bit unhappy with every lesson we ever do so I don't really want you seeing me because, you know, it's got its warts and all that stuff. And so, a lot of good people, but not necessarily wanting people in their classrooms."

For all three, this appears to be an unresolved, ongoing tension in that none have successfully found ways to make classroom visits an accepted part of their roles. Pam, in particular, mentions that this tension leads her to consider ways of presenting this opportunity to learn as risk-free noting, "Well, I'd really like to be in more rooms and influencing more teachers. I'm trying to think of ways I can do that to support them."

Valuing and reconstructing experience - Resistance to change

Another of the tensions that was apparent for all three participants was similar to Berry's (2007) tension of valuing and reconstructing experience. The mathematics coaches experienced a dilemma between acknowledging the authority of the teachers' experience and helping them to see that there

is more to teaching than simply having acquired a requisite amount of experience. This is best exemplified by Ray in the following excerpt:

Ray: I think the biggest barrier tends to be, as teachers, we've gone through a system a certain way that we can visualize how it looks in the classroom. We've taught that way and we see successes in that, in either ourselves or some students, and we hang onto those successes as sort of validation for doing what we do. And we tend to say, 'Well, those other kids just aren't being successful. They're not working hard enough. They're not trying hard enough. They need to do things differently. They need to change.' And I don't think a lot of teachers are as good at saying, 'Well, what do I need to do differently? What do I need to do to change?'

The mathematics coaches value the experience their teachers have, but know that experience can always be broadened and improved. None of data from the participants suggest they use a deficit model approach to coaching teachers, but rather they believe there is always room for growth. This belief perhaps stems from their own experience with life-long learning. They want the teachers they work with to consider which areas of their practice would benefit from further learning and support. As Ray suggests, "And, I think if teachers just come out a little bit more with the willingness that, you know, as good as I am, (laugh) I probably could be a lot better. That would be very helpful."

The data suggests that all three mathematics coaches see this tension as a resistance to change and this manifests in different ways. For Tara, who described her own career in terms of ongoing growth and change, the tension stems from the assumption that her colleagues would be open to similar experiences. She finds it difficult to accept that change is slow stating, "So I made the assumption that was once other teachers kind of have these a-ha moments [as she did], they would just fly and I've come to realize that's not the case."

Ray also experiences tension in slow change, but notes that, while "teachers can be very confident about some things and don't necessarily challenge themselves as much as they could", it is possible that "as much as we sometimes want to change, it's a lot of work to change and people only have so much time in the day so they sometimes just don't even get started." Ray's view suggests that outside influences play a role in teachers' readiness or willingness to change.

Pam views the resistance to change as more of a readiness factor. Her tension lies in the fact that the teachers she works with are not as ready to reconstruct their teaching experiences as she would like them to be. She recognizes that she "wants them to try more than they're capable of trying" and is aware that she's "not giving them time to slowly implement what they're comfortable with". She values the experience they bring, but struggles to encourage them to consider new practices.

This too appears as an unresolved tension that all three mathematics coaches deal with on an ongoing basis. Pam was the only participant to offer a partial solution, albeit unsatisfactory to her. She approaches this tension with perseverance tinged by frustration saying, "Well, I think, you kind of got to persist, but it can kind of get a little frustrating at times."

Confidence and uncertainty - Questioning role and ability

A tension that surfaced for both Tara and Pam correlates closely with Berry's (2007) tension pairing of confidence and uncertainty. Both coaches mention having had colleagues question their role and their qualifications. This created a tension between the necessity of exposing their vulnerability and

maintaining the confidence of the teachers they mentor. Tara mentions, “You get the naysayers in the room that might, you know, question you on things. The biggest thing I get is what are your qualifications to do this job. That's what I get all the time.” And Pam adds:

Pam: I've had people that have said to me, I know enough that I don't really need you and I don't understand why the district is wasting money on your job. It's the senior math people, the 10, 11, and 12, that are the hardest to influence and they don't want to be influenced by me. I've been told many times by them that I have not the experience.

For both Pam and Tara, this appears to be a managed tension. Although the questions regarding their qualifications continue to be asked, neither seem to regard it as an ongoing source of tension. Both admit to limiting their role to elementary and junior high school and, for Pam, holding a Master's in Numeracy was perhaps sufficient to manage any remaining tension. Tara chose two other methods, which appear to offer the credibility she needs to answer any questions—she outlines her credentials and acknowledges the research behind best practice in mathematics:

Tara: So what I started to do more of after that was, whatever I was giving a recommendation for, I always had research to back up my recommendation. So I was always presenting what the research was saying. Always. [...] I lay out what courses I've taken, the journey I talked to you about, and why it's a passion. They seem to be a little better once they hear that story.

Initiative and systemic barriers - Working with learning assistants

This is a tension pairing that extends Berry's (2007) framework as it does not have a counterpart within her original set of tensions. It surfaced when the mathematics coaches were asked what they would like to implement in their role but have not been able to as of yet. Both Pam and Ray mentioned working with learning assistants. A strong desire to support every adult involved in the learning of students in their district drives them to want to work more closely with the learning assistants. Yet to do so would disrupt an existing functioning system. Ray expresses this clearly in the following excerpt:

Ray: I've got a few things that are sort of happening, but not as deeply as I'd like. One of them is the learning support group in our district. They all work a little bit differently and it's kind of hard to connect with them the way we're set up in the system.

Their initiative meets with resistant in the form of systemic barriers. In Pam's case, it is a result of an administration system that limits her contact with colleagues to only classroom teachers. For Ray, it is a result of different priorities. Like Pam, the learning assistants in his district provide both numeracy and literacy support — and that support tends more towards literacy. Ray notes, “They tend to be very heavily focused on reading recovery/writing kind of stuff over the years and they just haven't had a lot of time to get together and talk about anything around math.”

Both Pam and Ray mention wanting to circumnavigate the systemic barriers and provide professional development to the assistants, who, in their respective districts, tend to work one-on-one in pullout environments with students. Ray wants the opportunity to offer more effective resources. Pam agrees, adding, “They are sending these support people out to work with students, but they're working same old, same old. The child gets the same kind of repetitive practice over and over again and it never moves them forward.”

This is an unresolved tension that has both mathematics coaches searching out solutions. Ray offers the vague “hope” that he will be able to connect with the learning assistants this coming school year, but does not go deeper into his plan. Pam plans on speaking with her assistant superintendent to seek her assistance in convincing the learning assistant teachers that she is capable of providing them with support.

Discussion and conclusion

The first goal of this study was to identify tensions experienced by mathematics coaches. Three tensions emerged that closely aligned with the tension pairings in Berry’s (2007) framework. The fourth was a tension the mathematics coaches experienced in their desire to support learning assistants. With no obvious parallel in Berry’s work, likely due to her role as a pre-service teacher educator, the presence of this tension requires the framework to be extended when considering tensions experienced by mathematics coaches.

The findings also revealed tensions that could be considered unique to mathematics coaches, as there were two tensions they experienced that were not included in the list of tensions identified by generalist coaches. The first finding suggests that mathematics coaches may experience tension regarding their qualifications. This could be explained by the expectation that a specialist in one subject would be expected to have specific skills that a generalist, who works across all subjects, would not be expected to have. Additionally, given that many of the mathematics coaches are pulled from teaching positions within their districts (Campbell & Malkus, 2011), their former colleagues might question their abilities. Interestingly, Ray, who was a secondary mathematics teacher has never experienced this tension. Despite having no elementary experience, he stated that he has always been “well received” by the elementary staff. His status as a high school mathematics teacher appears to offer him credibility across all grade levels. The second tension experienced by mathematics coaches, but not generalists, was working with learning assistants. This might be the result of the relative newness of the role of mathematics coaches (Anstey, 2010). Districts are still in the process of determining the scope of the responsibility of the mathematics coaches in their employ. As Tara suggests, “As for the job itself, it was pretty much I just had to build the airplane as I was flying it.”

The second goal of this study was to identify how the mathematics coaches coped with the tensions they experienced. The findings suggest that they appear to fit Lampert’s (1985) image of dilemma managers who accept and cope with continuing tension. This means that the mathematics coaches initially manage the tensions that surface while never fully resolving their competing conflicts. What was interesting was the managed tension that Pam and Tara, both of whom are elementary trained teachers, experienced regarding questions about their role and qualifications. Their method for managing this tension was avoidance of interactions at the high school level. Similarly to the finding in Liljedahl et al. (2015), this suggests that this tension is managed on some levels but there is a possibility it could resurface at some point. While both are only required to work with teachers who volunteer and are willing, both of their roles encompass grades K to 12.

While the small number of participants in this study may limit its generalization, the findings do indicate the presence of tensions experienced by mathematics coaches. This language of tensions could be useful as a means for discussion and reflection on the practice of mathematics coaches. Whether managed or unresolved, identifying and describing these tensions will contribute to a small,

but growing body of research into mathematics coaching. If employing mathematics coaches in schools is to be a viable complement to professional development, more study will be necessary.

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How to deal with learning difficulties related to functions – assessing teachers' knowledge and introducing a coaching

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Students often have difficulties with the content area of functions. If their teachers are not aware of these problems and lack of adequate teaching methods, they cannot counteract pointedly in their classrooms. This paper presents a project developing and evaluating a coaching to foster teachers' pedagogical content knowledge about several learning difficulties with functions and about how to respond to them. As this work is still in progress, we here focus on the project description as well as on the development of the survey to measure teachers' corresponding knowledge.

Keywords: Mathematics education, teachers' professional development, functions, learning difficulties, teacher survey.

Introduction

Being able to adequately reason with functions is considered to be a central goal of mathematics education (e.g. Eisenberg, 1992; KMK, 2003; NCTM, 2000). More precisely, reasoning with functions characterizes a specific way of thinking in interdependencies, relationships or changes (Vollrath, 1989) that is especially required when working on inner- and extra-mathematical problems (Hinrichs, 2008; NCTM, 2000).

However, several studies show that learners have particular difficulties in this domain (see for an overview Nitsch, 2015 or Vogel, 2006). For instance, they may experience problems with the meaning of the parameters (e.g. Schoenfeld et al., 1993), conceive graphs as pictures (e.g. Monk, 1992), confound the slope and the height of graphs (e.g. Hadjidemetriou & Williams, 2001) or have difficulties with word problems in the sense of the word-order-matching-process (e.g. Clement, 1982). Often, their teachers are not aware of these difficulties (Hadjidemetriou & Williams, 2002; Sproesser et al., in press) and therefore cannot counteract explicitly. Moreover, the study of Nitsch (ibid.) revealed systematic differences between school classes referring to learning difficulties with functions. She concludes from this finding that some teachers are more successful than others in responding to such difficulties.

Theoretical background

The findings mentioned above raise the assumption that teachers' professional development (TPD) focusing on such typical learning difficulties may enhance teachers' pedagogical content knowledge (PCK, see e.g. Shulman, 1987), their instruction and mediate also students' learning in this field. This is also in line with the general understanding that teachers need TPD in order to meet the challenges that they encounter in their professional lives as university studies cannot satisfy all of demands from practice (cf. Mayr & Neuweg, 2009). To our best knowledge, there is no empirical

evidence about the effects of a TPD related to learning difficulties with functions, especially taking into account the interplay between the teacher and student level, yet.

Particular TPD-characteristics that have already proven to be effective in general can be implemented in a TPD referring to dealing with learning difficulties in the domain of functions. In this context, Lipowsky (2013), for instance, found that TPD should be related to one specific domain instead of focusing on different domains. Furthermore, long-term TPD courses enable to integrate input, practice and reflection phases. The study of Lipowsky (ibid.) additionally confirmed that giving feedback (e.g. Shute, 2008) supports learning also in the context of TPD. Teacher coaching represents a specific form of TPD that can also implement the mentioned characteristics. In adaptive (teacher) coaching (Leutner, 2004), the coach refers back to the teachers' statements and activities. If teacher coaching focuses on a concrete classroom situation, it is, for example, possible to encourage teachers to reflect on this situation (West & Staub, 2003) or to train them in giving supportive feedback to students showing a particular learning difficulty.

In several studies, such a focus on responding to students' difficulties or errors (e.g. through giving feedback) has already shown to be useful in order to measure or promote teachers' PCK concerning different mathematical content areas: For instance Biza et al. (2007) propose to measure (pre-service) teachers' PCK by requesting them to analyze wrong student solutions and to formulate supportive feedback. The study of An & Wu (2012) revealed that teachers' PCK can be fostered through asking them to analyze students' errors and to develop approaches how to correct them.

Research goals

The teacher coaching developed in this project aims at building up teachers' pedagogical content knowledge related to learning difficulties with elementary functions and hence to support also their instruction and students' achievement in reasoning with functions. As a narrow content focus has proven to be a characteristic of effective TPD (Lipowsky, 2013), we here refer to specific PCK components as defined by Ball et al. (2008): In the case of 1) knowledge of content and students (KCS), we focus at fostering teachers' knowledge about typical learning difficulties and about students' thinking related to functions; in the case of 2) their knowledge of content and teaching (KCT), we train them in "adequately" responding to such specific learning difficulties. Within the content area of functions, we concentrate on linear functions and on the understanding of the concept of a bivariate functional relationship in order to assure a narrow content focus. The emphasis on this subdomain also takes into account that viable concepts about elementary functions appear to be crucial for understanding higher-order functional classes later on.

As the majority of existing TPD courses is not carried out in an experimental design, it cannot be clearly identified which of their characteristics would be responsible for a certain effect (e.g. Yoon et al., 2007). Therefore, this teacher coaching is brought out via two variations, namely with and without focus on feedback. This procedure takes into account findings from other studies showing positive effects of giving feedback (see above) but additionally evaluates the effectiveness of this characteristic (explicitly training to give feedback to students showing concrete learning difficulties). In this sense, the main goal of the project described in this paper is to prove what effective aspects of an adaptive teacher coaching are.

More precisely, we evaluate the following research questions:

- What do teachers know about typical learning difficulties in the domain of functions and what ideas do they have how to react to them (pretest)?
- To what extent can teachers' KCS and KCT related to functions be fostered through two variations of teacher coaching (pre- and posttest)?
- Which impact do the coaching treatments have on students' domain-specific competence?

Methods

Pilot study

The content of the coaching was identified via a pilot study in the academic year 2014/15 (see Figure 1 for an overview of the project's structure): Part I of the pilot study revealed that all of the learning difficulties derived from the literature (see Introduction) occurred among students within our learning settings (paper-and-pencil-tests in 4 classes of grade 7 and 8). Moreover, we found that their teachers only knew some of these learning difficulties and that their knowledge about them and about how to respond to them was very heterogeneous (interviews with 4 teachers). Therefore, TPD in this domain appears to be useful. A summary of these results can be found in Sproesser et al. (in press). As to our knowledge there is no consensus about how to "accurately" respond to such learning difficulties or how to largely prevent them, teacher trainers and university educators were interviewed about these issues within part II of the pilot study. Via these expert interviews, we collected and further developed teaching ideas, methods and material for the coaching.

Main study

Within the main study, the teacher coaching (3 modules) accompanies the instructional unit of linear functions in grade 7 or 8, respectively: Module 1 is held before, module 2 during and module 3 after this unit. This structure enables to implement the content of the coaching in the teachers' classroom as well as to reflect on the teachers' experiences within the TPD. About 60 teachers of grade 7 or 8 are assigned to one of two treatment groups or to a control group. Both treatments contain input, reflection and activity phases in order to foster teachers' KCS and KCT related to learning difficulties concerning elementary functions. Only in one of the two treatment groups, teachers are specifically trained in giving supportive feedback to students facing a particular learning difficulty.

In order to gain empirical evidence about effective characteristics of the coaching, the teachers' PCK as well as their students' knowledge related to elementary functions are assessed before and after the coaching / teaching unit. This data structure allows using analysis tools such as multilevel analyses and hence to evaluate the interplay between the two levels. The student survey (pre-, post- and follow-up-test) contains large parts of the test instruments developed by Nitsch (2015): Via a number of tasks referring to elementary functions, several learning difficulties (see above) can be identified. Moreover, covariates such as students' cognitive abilities (Heller & Perleth, 2000) or motivational variables (Pekrun et al., 2002) are gathered.

In order to measure KCT and KCS of the participating teachers, we developed a survey that particularly refers to several tasks of the student test. The development and the structure of the teacher survey will be presented in more detail in the next section.

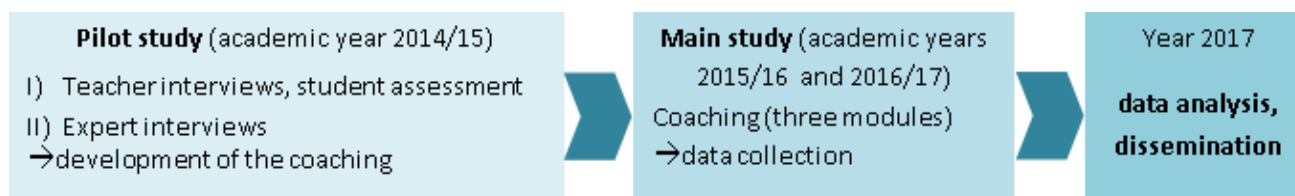


Figure 1: Outline and content of the project

Teacher survey

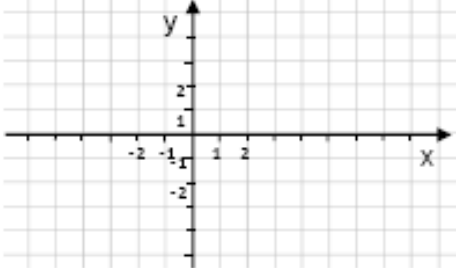
The participating teachers are requested to complete before and after the coaching a structurally identical paper-pencil-survey. This procedure allows directly investigating teachers' KCS and KCT developed in the course of the coaching. The PCK items of the teacher survey are all structured in the same way (see Figure 2 for a sample item): The teachers are shown a task of the student test and they are asked about typical mistakes or learning difficulties referring to this task (questions a) and b) in Figure 2) and how they would respond to them (question c) in Figure 2). Hence, according to the classification of Ball and colleagues (2008) the questions a) and b) are part of the knowledge component KCS as "Teachers must anticipate what students are likely to think and what they will find confusing" (ibid., p. 401). These authors propose to measure teachers' KCS for instance via questions about what students may find difficult or about interpreting students' thinking. Within our survey, teachers in question a) are asked which mistakes and learning difficulties they had already noticed concerning the given type of task; in question b), on the basis of a wrong student solution they have to put themselves in a student's position in order to make transparent his or her thinking process when working on the task. Hence, these tasks require knowledge of typical student (mis)conceptions and errors as well as about students' thinking. The third PCK item (question c) in Figure 2) corresponds to the knowledge component KCT (Ball et al., ibid.): Teachers need to know about mathematics and about teaching in order to sequence their instruction and hence to promote students' understanding. For instance, they need to know different methods and procedures and choose appropriate ones for their instruction. This means that KCT is particularly relevant when teachers respond to students' mistakes and difficulties or when they aim at building up viable concepts through their instruction. Ball et al. (ibid.) propose to measure KCT e.g. by asking for examples for simplifying particular content or how learning of a specific content can be facilitated. As displayed in Figure 2, such KCT items are also included within our test instrument: In question c), teachers are asked to outline how they would react to a concrete student mistake.

Within the whole survey, the sequence of PCK items is always as displayed in Figure 2: The first question a) is open-ended in order to collect the teachers' ideas and experiences without being influenced by specifications of the survey. Afterwards (question b)), teachers are confronted with a concrete students' mistake referring to this task and they are requested which (mis)conception could cause the mistake. As in real classroom situations, responding to a student mistake (cf. KCT) happens after its noticing (cf. KCS), the question sequences are always ended up by the KCT item (question c) in Figure 2).

This sequence of questions (a) open-ended, b) referring to a concrete mistake) was chosen to gather data about teachers' knowledge and experience concerning several student problems in general and related to specific mistakes. Within the teacher interviews of the pilot study, this sequence was also

used and proved to provide essential findings. However, one particular limitation of this sequencing should not be disregarded: Teachers could add the mistakes and learning difficulties displayed in b) to the open-ended question in a) even if they had not thought of them without the indication of the survey. We decided to accept this possible drawback that may occur in field studies as the our rather than in laboratory studies because interviews instead of the paper-pencil-survey would be extremely time-consuming for the numerous participants of the main study and could irritate them; furthermore, a digital survey with time markers could hardly be implemented as the coaching is brought out in different schools where we cannot count on a safe internet-connection. In the teachers writings it can mostly be identified if they have come back to a previous item or not.

We consider the relevance and the validity of these PCK items as relatively high because of several issues: First, empirical studies show that the presented learning difficulties are common among students and hence they are relevant for teachers. In their research Ball et al. (2008) similarly have drawn typical student mistakes and learning difficulties from the literature. Furthermore and as pointed out above, the kind of questions that we use are also proposed by these authors. Hence, our approach is not arbitrary but systematical and can also be applied in other content areas.

<p>Student task</p> <p>Draw the graph according to the functional equation</p> <p>$y = 5x - 2$</p> <p>in the given coordinate system.</p> <p>Explain briefly how you proceeded.</p>	
<p>a) Which typical mistakes or learning difficulties would you expect from your experience in this student task?</p> <p>b) A student solved the task as displayed on the right. Which concept could underlie this solution? Please justify your answer.</p> <p>c) Imagine you would be confronted with this learning difficulty. How would you respond to it in your mathematics classroom?</p>	

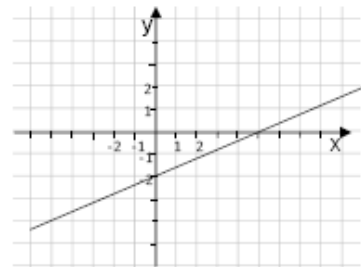


Figure 2: Sample item of the teacher survey

In addition to the mentioned PCK items, the teacher survey contains covariates for instance about their professional background (e.g. university degree, teaching experience), beliefs related to mathematics education (e.g. their constructivist conviction (Stern & Staub, 2002), assumed determinants for mathematical ability (Stipek et al., 2001) or their experience with and motivation for TPD (see several scales in Jerusalem et al., 2007).

Current status and future steps of the project

As mentioned above, the pilot study has already been carried out in the academic year 2014/15 and its evaluation is almost concluded. Student assessment and teacher interviews revealed that a TPD referring to dealing with learning difficulties related to elementary functions would be useful for teachers within our learning settings. Moreover, the expert interviews were helpful to gather “best-practice-methods” and material for the coaching.

Concerning the content of the coaching, both treatments focus on the same learning difficulties (problems with the parameters, graphs-as-picture-mistake, slope-height-confusion, emphasis on the word-order in word problems). Teachers get information about their prevalence in empirical studies. Moreover, we illustrate the best-practice-methods and material how to prevent or overcome them that we have gathered through the pilot study. There are also active phases for the teachers: On the basis of the presented methods / material, they are asked to further develop tasks and material for their own classroom. Moreover, based on described classroom-situations showing concrete student mistakes, they are requested to think up a reaction to support the student to overcome his problem. In these tasks, the variation with / without focus on giving feedback comes into play: In the treatment with focus on feedback, teachers are asked to concretely formulate the feedback and explicitly explain the hints that they would use when being confronted with the corresponding student difficulty (e.g. “How would you respond to this learning difficulty in your mathematics classroom? Please be explicit: Verbalize your feedback and illustrate other ways to support the student.”). In the treatment without focus on feedback, teachers are simply requested to mention adequate ways of responding to these learning difficulties in a more general way (e.g. “How would you respond to this learning difficulty in your mathematics classroom?”). Furthermore, in both treatments teachers’ experiences in the course of the learning unit are discussed and reflected as well as their classes’ results - if the teachers agree with students testing before and after the unit.

The coaching has already been carried out in the academic year 2015/16 and it is still offered in the year 2016/17. Hence, the main study is in progress at the moment and data will be gathered at the student and teacher level. Results are expected from the end of the academic year 2016/17 onwards.

Acknowledgment

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Math centers: A pedagogical tool for student engagement in intermediate math class

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This paper draws from a qualitative exploratory case study that aimed at exploring the learning experiences of teachers as they engage in professional learning project. The case study involved three elementary school teachers' professional learning experiences as they engaged in developing a practical, research-based approach to differentiated instruction using a flipped classroom and student-centered pedagogical approaches that would result in enabling students to be engaged with mathematics..

Keywords: Professional learning, flipped classroom, student-centered, engagement, intermediate mathematics.

Introduction

Research indicates that professional learning, which is job-embedded (Joyce & Showers, 2002), collaborative (Garmston & Wellman, 2003), occurs over time, and is driven by the needs of the teachers involved (Fullan, 1995; Lawler & King, 2000; Little, 2002), is effective. Furthermore, effective professional learning is focused on student outcomes, integrated into the teacher's day-to-day culture, and often tied to the school's improvement process (Way, 2001). The paper draws from a study that aimed at exploring the learning experiences of three teachers engaged in a professional development project in Ontario at an Intermediate level (grade 6, grade 7 and grade 8). The professional development project is part of an initiative of The Elementary Teachers Federation of Ontario (ETFO). ETFO invited and provided support for teams of teachers from the same school or in similar roles at different schools to come together and conduct professional learning projects relevant to their specific professional needs, circumstances and interests. In addition to the three teachers a university researcher was invited to collaborate with the group and conduct a case study of the professional development project. The three teachers in the professional development project collaborated in developing a practical, research-based approach to differentiated instruction *using a flipped classroom approach and student-centered pedagogical approach* that would result in enabling students to be engaged with mathematics. This flipped classroom approach and student-centered approach involved the use of grade-appropriate math centers where students would engage in a variety of math problems and/or topics; have opportunities to practice and consolidate basic facts and operational skills; use technology and manipulative as learning tools; become efficient communicators in math; and develop a sense of self-awareness toward their own math skills. Students would also grow in their ability to work independently and cooperatively as they work through various math centers, allowing the teacher(s) to conference with individuals and small groups of students. In this paper we describe some of the findings from the case study. The case study is guided by two questions: How did the professional development project facilitate teachers' understanding of the use of math centers in a flipped classroom and student-centered approach for

teaching and learning mathematics? How did the teachers negotiate constraints and possibilities as they engaged in their professional development project? .

Research has demonstrated that engaging students in the learning process increases their attention and focus (Jonathan & Aaron, 2012). Further it motivates students to practice in higher-level of critical thinking while promoting meaningful learning experiences. Chickering & Gamson (1987.), states that

“learning is not a spectator sport. Students do not learn much just by sitting in classes listening to teachers, memorizing pre-packaged assignments, and spitting out answers. They must talk about what they are learning, write about it, related it to past experiences and apply it to their daily lives. They must make what they learn part of themselves”.

In this sense, a flipped classroom and student-centered learning is essential. Educators who adopt to a flipped classroom and student-centered approach as a pedagogical method find that it increases student engagement, which allows for learners to successfully achieve the learning objectives (Jonathan & Aaron, 2012). For a flipped classroom and student-centered mathematics class to be effective, a shift in the role of the teacher and students in the classroom must be adapted. On the one hand, teacher’s role is viewed as a guide for students’ constructive processes towards mathematical meanings and mathematical ways of knowing. On the other hand, learning is viewed as an active, constructive activity in which students wrestle through problems that arise as they participate in the mathematical practices in the classrooms (Cobb, Yackel & Wood, 1992). Recently, there has been an upsurge of interest in instruction that focuses on flipped classroom approach for teaching mathematics.

Type of Math Centers

There were various types of math centers that each participating teacher used in their classroom. The choice of a math center was based on teachers’ professional judgment of the students and students’ needs.

Inquiry based Center: - A group of 4-5 students rotated from one station to another to learn about various topics. Examples of these topics included: explore and connect station, “what happens when...”, various word problems, “Reflection on this”, “Test your knowledge.” Each station had the option of students either working by themselves and/or in their respective groups. Teachers used as one or more of inquiry based centers to develop curiosity of a given topic among the students.

Resource Center: - This center was available for students all the time. This station consisted of graph papers, blank papers, mathematics dictionaries, mathematics textbook (e.g. *Math on call*, *Math on hand*, and *Math makes sense* 8). This center also included measuring tools such as meter -sticks, rulers, weight measuring scale and measuring tape. This center gave student the opportunity to select an appropriate tool for themselves in order to learn a topic/concept at hand.

Online Research Center: - Students had the opportunity to use their own technology or/and the computer station located in class to deepen their understanding of topic/concept at hand. Students also had the option of exploring a given topic at home via online research.

Debriefing Center: - This is usually available at the end of a lesson, where students come back into their respective groups. Here students are given the opportunity to consolidate their learning as a group, clarify any misunderstandings and learn from one another.

An example of inquiry based math center

Students were given the opportunity to solve a real life problem, using a task called, “how big is a trillion?”. In this problem, students were asked two open-ended questions: 1) Is it possible for trillion rice to fit into this room? Explain your solution 2) How much distance around the Earth can the rice cover if each rice is lined up in a line? Compare this with another non-metric unit. Students were asked to solve this problem in group of 4-5. Each group was given 6-7 classes (40 minutes each) to solve the problem in class. Students were allowed to do any background inquiry that they thought was necessary at home. The purpose of working on this problem in class was to have a common working place as a group, where they negotiated their learning and solution. At the end of the 7th class, groups were asked to submit their solution in form of poster which each group presented in the following class. Mathematically, this problem required students to learn about how to measure a unit in real life, length, metric versus non-metric units, volume, and capacity. Other than the mathematical knowledge, students had to discuss their ways of solving the problem, which means selecting appropriate tool to solve the problem in most efficient way, while self-regulating their learning and progress as a group. By the time this particular problem was given, students often became competitive. Teacher noticed that while students helped members of other groups with background mathematical knowledge, groups often tried to keep their solution a secret. This was because groups often wanted their solutions to be a unique solution.

Preparing the students for flipped classroom with math-center included teacher presenting and discussing math center code of conducts, where the purpose and the importance of self-regulation for one’s own learning was discussed in depth as a class. In order to utilize various centers, students were divided in groups of 4-5 students, these groups were often changed and redesigned either by the teacher or the students. Students were informed that they may seek help support from their teacher at any point, however they are encouraged to first discuss it with the members of their respective groups.

Flipped classroom

Flipped classroom approach for teaching mathematics is considered as an effective way for engaging students in active learning as well as in meaningful peer-to-peer and peer-to-teacher interactions during the in-class learning process (Forsey, Low, & Glance, 2013; Pluta, Richards, & Mutnick, 2013). Moreover, Bergmann and Sams (2012) indicated that flipped classrooms enable teachers to take individual students’ needs into account as well as to facilitate more interactions among peers and teachers in the classroom. The teaching and learning context of flipped classroom approach consists of two kinds of activities: in-class and out-class. In-class time is utilized for inquiry, application and assessment in order to better meet the needs of the individual learners. Technology-assisted out-of-class time involves personal instruction, where students acquire responsibility for their own learning, through studying course material on their own, using various sources (self-discovered and/or provided by teacher). The main goal in flipping a class is to cultivate deeper, richer, and active learning experiences for students where the instructor is present to coach and guide them. Further, emphasis is on higher-order thinking skills and application to complex problems, and which might include collaborative learning, case-based learning, peer instruction and problem set. In this sense the role of the teacher is to facilitate the learning process by helping students

individually and in groups. According to Bergmann & Sams (2012), there are many ways of implementing a flipped classroom approach. For this study, the participating teachers utilized various math-centers (discussed previously in the article) where students self-regulated their own learning in a math class.

Methodology

A qualitative research methodology was used to conduct this case study. According to Yin (2003) a case study design should be considered when: (a) the focus of the study is to answer “how” and “why” questions; (b) the behavior of those involved in the study cannot be manipulated; (c) you want to cover contextual conditions because you believe they are relevant to the phenomenon under study; or (d) the boundaries are not clear between the phenomenon and context. A case study was chosen because the study could not be considered without the context of flipped classroom approach, and more specifically the math centers classroom settings.

This case study involved three mathematics teachers at a Canadian middle school, who taught intermediate level (grade 6, 7 and 8) students. Two of the teachers were intermediate level mathematics teachers with their own classroom. One teacher was the resource teacher assigned by the district school board. The role of the resource teacher was to support the two classroom teachers by finding necessary resources needed to run the project. The two classroom teachers applied the flipped classroom approach in their mathematics classes. In preparation for the project, all three teachers sought opportunities to enhance their professional knowledge of using technology in mathematics teaching and learning. Further, each teacher read a number of monographs provided by the Ontario Ministry of Education in order to develop efficient knowledge of the Ontario elementary school mathematics curriculum, in particular knowledge of the mathematical processes such as problem solving, communicating, reasoning and proving.

Data was collected from:

- 1) Participating teachers’ observation of their teaching and learning experiences. Each teacher recorded field notes based on their own reflections as well as observation of their students in-class events related to math-centers (e.g., counting the number of students being engaged per center, how teachers guided the off track students to get back to work, etc.). In addition to these data was collected from teachers’ notes of their conferencing with students as individuals and in groups.
- 2) Transcriptions and field notes of group meetings (selectively audio-recorded). Each teacher’s observations were shared, discussed and reflected upon by teachers as a group for professional growth while focusing on their own professional growth and their students’ engagement in respective mathematics class.
- 3) Teachers’ artifacts such as lesson plans and assessment rubrics, and the teacher team’s final project report.

Findings and discussion

All transcribed data, field notes and teachers’ artifacts were analyzed by the university researcher and one teacher independently to identify major themes related to the guiding questions— How did the professional development project facilitate teachers’ understanding of the use of math centers in

a flipped classroom and student-centered approach for teaching and learning mathematics? How did the teachers negotiate constraints and possibilities as they engaged in their professional development project? The findings will be discussed in the following themes which emerged from the analysis: time for teachers to meet; awareness of initial resistance from parents and students; enhancement of student learning; and challenges. Findings from the study suggest that in order to sustain a collaborative professional development project teachers need time and need to meet. Teachers in this study were able to plan collaboratively and develop a practical, differentiated math program based on flipped classroom and student-centered pedagogical approach using math centers. The flipped classroom approach using math centers allowed students to engage in purposeful practice while freeing up the teacher to meet with individual and/or small groups of students for teaching and/or learning. However the two participating teachers taught different grades in different buildings of the same school, which became somewhat challenging to coordinate schedules for sit-down meetings. Often, the conversation between the three teachers would occur either in between classes (as the teachers would pass by each other's classroom) or through email (keeping each other informed on their status with the project). Although the teachers planned collaboratively, they had fewer time to compare what they had initially planned. This however, had an unexpected positive outcome which ended up by providing teachers with two different ideas of math-centers and student engagement. Both classroom teachers did utilize flipped classroom, student-centered pedagogical approach, however during their final group meeting, teachers discovered that they had each taken a different approach to the math center idea. This allowed for each teacher to talk about their thought process behind their choice for developing the math centers the way they did. Further, this provided both teachers to learn from one another's professional learning experience with their respective math centers. As different as each participating teachers' math centers were, the participating teachers observed that there were common themed categories to the math centers (these categories were presented earlier in this paper).

Another theme that emerged from the case study is that the professional development project provided opportunity for participating teachers to be aware of and understand about the initial resistance from the students and their parents toward flipped classroom and student-center pedagogical approach. Teachers developed awareness of the fact that both the students and their parents perceived mathematics teaching and learning in a traditional manner. And that for both the students and their parents, mathematics was a subject where the teacher taught a lesson, the students completed assigned tasks like doing practice questions from a mathematics textbook related to the lesson, followed by an assessment in form of a test. The students in participating teachers' classes and their parents' perception of how mathematics should be taught presented with complex challenges. This resulted in the professional development project allowing teachers to learn about strategies for alleviating these challenges including having to do a lot of community building exercise in class, while also having conversation with parents through emails, phone calls and/or one-on-one meeting, about the importance of math centers for their child's learning. These conversations with parents often revolved around the topic of how math centers not only helped students to become more engaged with mathematics but also helped to develop importance skill set of becoming more self-regulated toward their own learning.

Another theme that emerged is how the professional development provided opportunity for teachers to enhance students' learning and development through using math centers. Teachers noticed that

after the initial resistance, the students began to be engaged with math centers and by the end of school year, they began to self-regulate their learning. One teacher noted about a grade 8 student who reflected on his journey with math centers and stated that it helped him to become more resilient to mathematics learning.

First, I did not know why we were doing math centers. I felt that the teacher did not want to teach anything.... but now, when I go through different centers in class I know that I am able to do things on my own...I feel happy.... I have done these many [math centers] today...which means I can do math...I just have to take my time with each center and not worry about how much time my group members are taking with centers.

In relation to the same theme, another teacher expressed how her grade 8 student commented,

I used to think that my teacher should know everything...you know, like all the answers...but now I know that I can find all the answers...and if I am stuck, I can take help from my friends...which is okay, because we are learning together

Both of the participating teachers experienced a sense of fulfillment in terms of their professional development experiences in relation to their impact on growth in their students in terms of both mathematical understanding and self-regulation toward their own learning. Teachers noted that many of their students grew stronger in their ability to self-regulate, as they had to make choices toward their learning in terms of what to work on, how long to work for and with whom. Teachers expressed satisfaction on how the one-on-one time with the teacher allowed the struggling learners to take risk and seek clarification without feeling restricted by the classroom environment.

Given the many positive outcomes of the professional development project that focused on flipped classroom and student-centered pedagogical approach through math centers, there were some challenges. These were mainly due to the fact that teachers became aware that some students needed more time with this approach, which was not possible given that there was a limited time that these students were with their mathematics teachers and that the teachers were expected to cover the curriculum expectations. Also teachers in the professional development project realized that for a small number of students, it was extremely difficult to adapt to this approach, even if they loved mathematics. This was because these students had only experienced learning only from a textbook teaching approach in mathematics, and flipped classroom and math centers approaches were a significant departure from their past mathematics learning experiences.

Implications

This project utilized a case study research design and was conducted at one Canadian middle school. Hence, the findings of the study should not be read in terms of generalizability, but of transferability to other cases (Creswell, 2008). Recently, mathematics educators have realized the potential for a flipped classroom and student centered pedagogical approach for enhancing student engagement and learning. However, very little is known in terms of the implementation of this approach in elementary schools. This study explored the mathematics professional learning experiences of elementary school teachers as they implement the flipped classroom and student centered pedagogical approach. The professional development project provided opportunity for teachers to enhance their understanding of flipped classroom and its impact on students' learning. Teachers noted that their students became engaged with mathematics and self-regulated toward their own

math learning. The findings suggest that given opportunity to learn in a professional development setting that ensures autonomy, teachers learn and are capable of teaching through flipped classrooms and student centered pedagogical approach. The study also suggests that professional development project provides opportunity for teachers to be aware of the need for communication and collaboration among teachers, parents and students regarding the benefits and implementation of flipped classroom. As a result, further research is needed on how professional development can facilitate teachers' learning about how to communicate and collaborate with parents and students in flipped classrooms.

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Noticing aspects of example use in the classroom: Analysis of a case

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This paper investigates promoting knowledge of example use in mathematics education by way of analyzing a case using theoretical tools. Participants were both prospective and practicing teachers attending a university course. An event taken from a tenth grade geometry class was analyzed in terms of example use, and then discussed. Participants related to the type of example given, the timing of the example, agency, what the example was an example of, and the aim of giving the example.

Keywords: Examples, case analysis, teacher education, theoretical tools.

Introduction

Examples and non-examples are an integral part of learning and teaching mathematics. They are used in concept formation, when seeking relations between elements, abstraction, and generalization (e.g., Smith & Medin, 1981; Watson & Chick, 2011). Acknowledging the importance of example use in mathematics education, Shulman (1986) included knowledge of examples within the category of pedagogical content knowledge (PCK). This knowledge, he claimed, is essential for representing the subject, so that it will be comprehensible to others. Rowland, Huckstep, and Thwaites (2005) also included teachers' use of examples in their description of 'the knowledge quartet', a framework for thinking about the ways subject matter knowledge comes into play in the classroom. In their framework, the ways teachers use examples and the types of examples they use, are manifestations of teachers' own content knowledge, their meanings and descriptions, being transformed and presented in order for students to learn the mathematics. Ball, Thames, and Phelps (2008) also noted the importance of teachers' knowing how to sequence examples.

Although example use is a complex matter and promoting teachers' knowledge of example use could be quite challenging, few studies specifically investigated this aspect of teacher education. This study proposes fostering prospective and practicing teachers' knowledge and awareness of example use by applying research and theoretical tools to analyze cases, helping to bridge theory and practice for mathematics teachers. The material, which is the focus of this paper, is a case based on a classroom event, and used as an exemplar. The case material consisted of a classroom transcript, along with guiding questions to be answered by each participant. This was followed by a group discussion with the teacher educator. Specifically, we ask: What aspects of example use do individual participants notice when studying an authentic case? Can those aspects be traced back to theories learned during the course? What additional aspects arise during group discussion?

Example use in mathematics education

There are many aspects of example use which have been investigated. To begin with, there are different types of examples. Arcavi (2003) discussed visual representations, while Tabach et al.

(2010) focused on numerical examples. In addition to the form of the example, studies have categorized examples by how they are identified. For instance, there are intuitive and non-intuitive examples and non-examples (Tsamir, Tirosh, & Levenson, 2008). Intuitive examples are those examples which students immediately identify as such and are often derived from practical experience. Likewise, intuitive non-examples may encourage visual, rather than analytical thinking. On the other hand, it seems that non-intuitive examples and non-examples can encourage students to use reasoning based on critical attributes. Tsamir, Tirosh, and Levenson (2008) also discussed the sequencing of examples and non-examples and its effects on students' learning.

The difference between examples and non-examples is dependent on the mathematical lens one is looking through. When looking for polygons, a triangle is an example; when looking for quadrilaterals, a triangle is a non-example. In other words, another aspect of example use to consider is the way an example is being used. Zazkis and Chernoff (2008) introduced the idea of pivotal examples which can cause the learner to change his or her cognitive perception or way of approaching a problem. Rowland (2008) differentiated between two uses of examples in teaching. The first is when examples are used to motivate generality. In this case, the examples are examples *of* something where the aim is to teach a general procedure or to support abstraction and concept formation. The second type of example use is for students to practice what was taught. That is, students are given many examples to practice some procedure. This type of example use allows students to experience variation and can lead to additional awareness and understanding. Watson and Chick (2011) found that examples can be used as templates for dealing with other class members, indicating a relation between classes or to express equivalence.

If we take into account that the way an example (or non-example) is used by a learner is determined by the focus of that learner, then it becomes the teacher's task to choose examples and set them up in such a way that students will view those examples through the intended lens and focus on the intended pedagogical aim. Zodik and Zaslavsky (2008) identified six types of considerations employed by teachers when selecting or generating examples: starting with a simple or familiar case, attending to students' errors, drawing attention to relevant features, conveying generality, including uncommon cases, and keeping unnecessary work to a minimum. They found that the most frequent consideration was choosing to begin with the simple or familiar example. They also found that on the spot, teachers often choose an example that will attend to an error which arose in class, whereas pre-planned examples tend more to consider uncommon cases.

To notice something means to make a distinction, to stress some perceived feature and ignore others (Mason, 1991). As shown above, various aspects of example use can be considered when analyzing a classroom event. In this study we investigate which of these aspects, and perhaps other aspects of example use, practicing and prospective teachers notice when analyzing a case.

Methodology

The participants in this study were 13 practicing teachers (denoted as T) with between 1 and 11 years of experience (mean years of experience was four) and 10 prospective teachers (PTs) who had completed their first degree in mathematics or a mathematically rich field of study, such as engineering. The specific course which is the context of this study, aimed to promote participants' knowledge of explanations and examples, and the relationship between them in mathematics

education. The course consisted of a total of 13 lessons which took place once a week for a period of 90 minutes each. The first six lessons were devoted to explanations in mathematics education. Different types of explanations were reviewed, such as conceptual, procedural, mathematically-based and practically-based explanations (e.g., Levenson, Tsamir, & Tirosh, 2010), and theories concerning the roles of explanations in mathematics education were discussed (Levenson & Barkai, 2008). The last seven lessons were devoted to examples in mathematics education. Theoretical perspectives of examples, like those described in the background of this paper, were read and discussed.

During the fourth course lesson on examples (lesson #10), the teacher educator handed out a transcript of a geometry lesson which took place in a tenth grade classroom. The geometry lesson had taken place within a few weeks of the course lesson and was observed by three of the prospective teachers attending the course, lending the case authenticity and relatedness. The overall aim of the geometry lesson was to introduce students to Thales' theorem and show its connection to similar triangles. The students had previously learned about similar triangles in ninth grade. The case transcript began with the classroom teacher stating Thales' theorem. This was accompanied by a drawing on the board, made by the teacher, of two similar triangles, under the headline: Geometry – Thales' theorem. Next to the drawing it says that if the given is $BC \parallel DE$ then from Thales' theorem we conclude proportional line segments (see Figure 1).

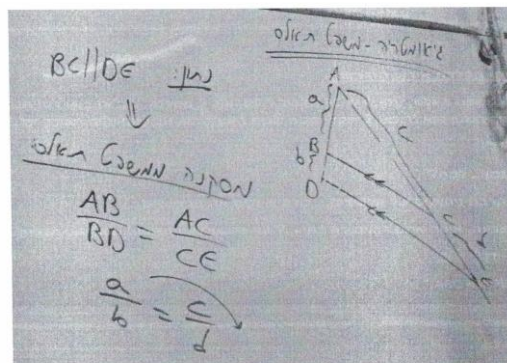


Figure 1: Presenting Thales' theorem on the board

The case transcript was handed out in the beginning of the course lesson with instructions to read it from beginning to end, without interruption, in order to understand the context and get a feel for the classroom. After reading through the transcript, participants were asked to reread the transcript and fill out a worksheet with the following questions: What did you learn about the use of examples during mathematics instruction from the examples given in the case presented? What would you do the same as the teacher did with regard to examples? What would you do differently from the teacher with regard to examples? After the participants wrote their answers and handed them in, a discussion followed. This discussion was audio-recorded and transcribed.

Data analysis

When analyzing participants' responses to the worksheet, our guiding question was, what aspects of using examples did the participants relate to when studying the case. We then used the literature background to help form a categorization scheme of those aspects. For example, we examined participant's responses for comments related to various types of examples that were discussed in class and were present in the case, such as intuitive examples, non-examples, familiar examples, and uncommon examples. Participants also commented on didactical aims of giving examples, such

as responding to a student's error. Because participants wrote freely, it happened that one sentence could encompass more than one aspect of example use. For instance, one teacher (T12) wrote "the teacher gave a numerical example, which in my opinion served the purpose of making it easier for the students to understand and generalize the idea." T12 refers to the type of example (numerical) as well as the purpose of the example (to help students generalize). Participants' comments that were thought not to be related to the giving and use of examples were categorized as 'unrelated'. Table 1 lists the categories along with examples from the data of each category.

Aspect of example use	Sample data
Type: What type of example is given?	The example uses simple numbers; it is a visual example of Thales' theorem.
Agency: Who is giving the example?	Only the teacher gives examples.
Aim: what seems to be the aim of giving the example?	The example is given to explain it again; the example shows the students how the theorem works.
Timing: When is the example given?	The example is good for the beginning of a lesson.
Example of: What is the example an example of?	The teacher gives examples of proportions.
Unrelated	The teacher presents a dry definition of Thales' theorem.

Table 1: Categorizing participants' comments

Findings

This section analyzes participants' comments from the worksheet on two segments of the case along with excerpts from the discussion which focused on those case segments. Thus, we review what participants noticed individually and what they discussed collectively. Analysis of participants' comments is carried out according to the aspects of examples listed in Table 1.

Segment one- Introducing Thales' theorem

The case transcript began a picture of the teacher at the board presenting to his students an example (see Figure 1). Accompanying the picture were the following lines from the case transcript:

- 1 T: In Thales' theorem, it is given that BC is parallel to DE. The conclusion is... wait, guess. If the length here (pointing to AB) is 6 and here (pointing to BD) is 2. And let's say that the length here (pointing to AC) is 12, what is the length of CE?
- 2 S: 4!
- 3 T: Right!

There are several ways to look at the examples in the above segment. One way is that there are two explicit examples. First, there is the drawing on the board (see Figure 1). Second, there is the numerical example given by the teacher in Line 1. However, the drawing on the board can be an example of Thales' theorem or an example of similar triangles, and indeed, that is what the teacher is trying to convey. In addition, depending on one's point of view, the example on the board and the

numerical example may be considered one complete example, with the example on the board written in a general matter, using parameters (a, b, c, and d), and the oral example, a specific example given with numbers.

The most frequent aspect of examples mentioned by participants was example *type*. Participants used such descriptions as: a simple example (PT2), an unfinished example (T8), a visual example (T12), a numerical example (T12, T13), and an intuitive example (PT 22). Although most participants wrote that the teacher gave the example, we do not take this as commenting on the *agency*, but rather as a description of what is going on. On the other hand, for a different segment, one participant wrote, “Only the teacher gives examples... but he should have requested examples from the students.” This participant is not merely describing the situation, but commenting on who is and who should be giving the examples. The *aim* of giving the examples was mentioned by T8 who wrote, “In Line 1, the teacher gives an example that students have to finish. He is checking to see if the students are listening and if they understand.” T13 wrote, “The teacher gave a numerical example so that the students could understand the example and draw on their previous knowledge of similarity and proportional triangle sides.” *Timing* of the example, that the example was appropriate for the beginning of the lesson, was noted by PT2. Two participants commented on what the example was *an example of* – T12 wrote that it was an example of the theorem, meaning that the example showed how the theorem could be applied. PT22, referring to the numerical example in Line 1, wrote that it was an example of equivalent ratios.

Segment two – Proportional segments

The following case segment is a direct continuation of the first one:

- 4-6. (The teacher and students review the concept of similar triangles.)
7. T: So, what is the ratio of their corresponding sides (referring to the example given in Line 1)?
- 8-10. Students: 1 to 3. 3 to 1. It depends on how you look at it.
11. T: The ratio is ... 3 to 4 because BD and CE are not sides of the triangle. So 6 is to 8 like 12 is to 16. Now, ..., what came first in Euclidian geometry? First, there was Thales’ theorem and only after that came the similar triangles theorems. So, let’s say we are in ancient Greece and we don’t know yet about similar triangles, but we do know Thales’ theorem. With that theorem, we can prove proportional sides in similar triangles.

In lines 4-11, no new examples are given. Instead, the teacher and students still refer to the first examples given on the board. Like the comments on the first segment, here too, most comments related to the *types* of examples being given: simple numbers (PT7), a non-intuitive example (T11), a general example with parameters (T17), and a numerical example with familiar numbers (T19). None of the participants mentioned aspects of *agency*, *aim*, or the *timing* of examples. T11 noted that the example was an *example of* ratios. Two PTs wrote remarks connected with the story of ancient Greece. PT23 wrote, “There is an example from real life that I like – in ancient Greece.” PT16 wrote, “In Line 11, the teacher made the material come alive when she told the story about Thales’ theorem from ancient Greece ... This example raises the question of why the students first learn about similarity and then about Thales.” T23 calls it an example from real life. Yet, it is not a

mathematical problem related to real life. The term ‘example’ when describing the ancient Greece story is not in line with the notion of examples discussed in the course.

Discussing the case

After the worksheets were handed in, the teacher educator (TE) opened up the discussion by asking who wished to comment on the case. After nine minutes of discussing general ideas, the discussion turned to the example given in Line 1 of the case transcript. Note that PT21 was present during the geometry lesson as part of their field work.

TE: Let’s look at Line 1.

PT21: But it’s not complete... He wanted to give an example of a ratio, but... Instead, he asked the students... He told the students to guess. And they did. He didn’t really give an example, in my opinion. But, in the next example...

TE: Where is the next example, in your opinion?

PT21: Line 11... In my opinion, it’s an explanation with a few numbers and that’s so you can see... I can’t decide. It’s, like, a numerical example. Here (in Line 11) is what was missing beforehand (in Line 1).

TE: Is it an explanation or an example? Can an example also be an explanation?

PT21: No. An explanation can be accompanied by an example. There can be an example and then the explanation generalizes it.

T8: I felt that way about Line 1. It feels like an explanation, and also like an example. On the one hand, there are numbers. On the other hand, it’s not complete.

The above excerpt gives us a glimpse into PT21’s and T8’s concept image of an example. Both infer that an example has numbers, but it must be complete, without any missing parts. There is also the question of the different roles an example may play in the classroom. Can an example be an explanation? Must all examples be specific and only the explanation can generalize it? These are some of the questions that the participants are grappling with. In the following excerpt, PT22, who was also present as an observer of the geometry lesson, tells what he observed.

PT22: He was trying to show how Thales’ theorem is really intuitive. That is, he gave an intuitive example.

TE: An intuitive example of what?

Many voices: Equivalent fractions.

PT22: And most answered correctly. The teacher was trying to show how easy it is.

TE: But what was it an intuitive example of?

T12: Of proportions. I know that it’s proportional because I know, I recognize it. It doesn’t really have anything to do with Thales’ theorem. Simply, 6 is to 2 as 12 is to 4. That’s it. That’s the example.

As stated in the background, what an example is an example of, depends on your focus and point of view. PT22 claims that the examples in Line 1 and Line 11 are examples of Thales’ theorem and

that the teacher used intuitive examples to simplify the concept for his students. However, other participants, those not necessarily present during the geometry lesson, see those examples as intuitive examples of equivalent fractions or of a proportion.

Discussion

During teacher preparation and professional development, participants are introduced to various theories. While field work is important, theories can help prospective and practicing teachers make the most of their field work by focusing their attention on different elements of practice. Findings showed that analyzing the case gave participants a chance to apply their knowledge of example theory when examining a classroom situation. Findings also showed that participants did not necessarily draw on the same theories when analyzing the same event. The example given in Line 1 of the case transcript was described alternatively as simple, numerical, visual, and intuitive. Each of these types can be traced back to different theories discussed in the course, but they focus on different issues. An intuitive example may also be numerical, but if the prospective teacher specifically comments on its intuitiveness, then that participant is remarking that a student will easily recognize it as an example (Tsamir, Tirosh, & Levenson, 2008). In other words, that participant is integrating knowledge of students (Ball, Thames, & Phelps, 2008) when analyzing example use and has appropriated a specific theory to accompany this integration. As teacher educators, we wish to encourage such integration of knowledge.

During the discussion, additional aspects of example use arose. Participants grappled with the nature of examples, and whether or not an example can be an explanation. This link between examples and explanations could have stemmed from the first half of the course which dealt with the topic of explanations in mathematics education. In any event, this question can help teachers and prospective teachers focus on the roles examples may play in the greater picture of teaching and learning mathematics. Finally, we also note how the integration of an authentic case, one that at least some participants actually observed, can help bridge theory and practice. In the discussion, participants held different views regarding what the example in Line 11 was an example of. Those who had actually observed the lesson had a chance to review the lesson again, focusing now on example use. They were also able to share with others some background of the lesson, perhaps adding to everyone's sense of 'being there'. This excerpt illustrates how fieldwork may be integrated into course work. It also reminds us, as teachers and as teacher educators, that it is not enough to offer examples. As Goldenberg and Mason (2008) said, exemplification is dependent on one's point of view. Analyzing a case using theories, and then discussing these analyses with participants, can raise awareness of how students may view examples and encourage planning example use in mathematics classrooms.

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Researching the sustainable impact of professional development programmes

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Impact analyses and evaluations of professional development programmes are mainly scheduled during or at the end of a particular programme. They aim at and provide results regarding immediate and short-term effects. However, apart from and beyond that, an analysis of sustainable effects is crucial. To address this issue, this contribution deals with the central question: What is the sustainable impact of professional development programmes? Theoretical models and empirical findings are introduced. In particular, this contribution provides two case studies' results regarding Austrian professional development programmes' impact. Here, the factors which foster or hinder the sustainability of impact are in the focus. Finally, implications for professional development programmes' implementation and research are discussed.

Keywords: Impact, sustainability, professional development programme, case study.

Introduction

The question of how to promote mathematics teachers' professional development has been discussed in various papers (e.g. Krainer & Zehetmeier, 2013; Loucks-Horsley et al., 1996; Sowder, 2007; Zehetmeier, 2010, 2014a, b; Zehetmeier & Krainer, 2011). In this context, the question of impact is of particular relevance. Evaluations and impact analyses of professional development programmes are mostly conducted during or at the end of a project and exclusively provide results regarding short-term effects. These findings are highly relevant for critical reflection of the terminated project and necessary for the conception of similar projects in the future (Fullan, 2006). However, apart from and beyond that, an analysis of sustainable effects is crucial (Loucks-Horsley et al., 1996). Despite its central importance for both teachers and teacher educators, research on sustainable impact is generally lacking within teacher education disciplines (Datnow, 2006; Rogers, 2003). This kind of sustainability analysis is often missing because of a lack of material, financial and personal resources (McLaughlin & Mitra, 2001; Hargreaves, 2002).

Theoretical framework

The expected impacts of professional development programmes are not only focused on short-term effects that occur during or at the end of the project, but also on long-term effects that emerge (even after some years) after the project's termination. Effects that are both short-term and long-term can be considered to be sustainable. Sustainability may refer to both system and/or individual level. Sustainability can be defined as the lasting continuation of achieved benefits and effects of a project or initiative beyond its termination (DEZA, 2002).

Empirical evidence concerning the impact of professional development programmes points to the finding that "prior large-scale improvement efforts (...) have rarely produced lasting changes in either teachers' instructional practices or the organization of schools" (Cobb & Smith, 2008, p.

232). Thus, it seems reasonable to focus on factors which might foster the broad effects and scale-up of professional development programmes' innovations. Cobb and Smith (2008) highlight networks, shared vision and mutual accountability as key factors.

Teacher networks are described, for example, as groups of colleagues who provide social support in developing demanding instructional practices; this affords time built into the school schedule for collaboration among teachers and access to colleagues who have already developed relatively accomplished instructional practices.

Moreover, a shared vision of high quality instruction fosters the scale-up of impact: this includes a shared vision concerning the question of instructional goals (what pupils should know and be able to do) and the question of how pupils' development of these forms of knowing can be supported.

Another key factor which fosters the scale-up of innovations and impact in teacher education is mutual accountability. This means, for example, that if school leaders hold teachers accountable for developing high-quality instructional practices, then – in turn – school leaders are mutually accountable to teachers for supporting teachers' learning.

Examples of Austrian professional development programmes

This paper deals with the analysis of sustainable impact of professional development programmes. In particular, two Austrian professional development programmes are in the focus: the IMST project and the PFL courses:

IMST project

In Austria, a national initiative with the aim to foster mathematics and science education was launched in 1998: the IMST project. Since then, this initiative has undergone several adaptations and is still running.

IMST was implemented in three steps:

1. The task of the IMST research project (1998–1999) was to analyse the situation of upper secondary mathematics and science teaching in Austria and to work out suggestions for its further development. This research identified a complex picture of diverse problematic influences on the status and quality of mathematics and science teaching: For example, mathematics education and related research was seen as poorly anchored at Austrian teacher education institutions. Subject experts dominated university teacher education, while other teacher education institutions showed a lack of research in mathematics education. Also, the overall structure showed a fragmented educational system consisting of lone fighters with a high level of (individual) autonomy and action, but little evidence of reflection and networking (Krainer, 2003; see summarized in Pegg & Krainer, 2008).
2. The IMST² development project (2000–2004) focused on the upper secondary level in response to the problems and findings described. The two major tasks of IMST² were (a) the initiation, promotion, dissemination, networking and analysis of innovations in schools (and to some extent also in teacher education at university); and (b) recommendations for a support system for the quality development of mathematics, science and technology teaching. In order to take systemic steps to overcome the “fragmented educational system”, a “learning system” (Krainer, 2005) approach was taken. It adopted enhanced reflection and networking as the basic intervention

strategy to initiate and promote innovations at schools. Besides stressing the dimensions of reflection and networking, “innovation” and “working with teams” were two additional features. Teachers and schools defined their own starting point for innovations and were individually supported by researchers and project facilitators.

3. The IMST3 support system started to continuously broadening the focus to all school levels and to the kindergarten, and also to the subject German language (due to the poor results in PISA). The overall goal of IMST is to establish a culture of innovation and thus to strengthen the teaching of mathematics, information technology, natural sciences, technology, and related subjects in Austrian schools (see e.g. Krainer et al., 2009). Here, culture of innovation means starting from teachers’ strengths, understanding teachers and schools as owners of their innovations, and regarding innovations as continuous processes that lead to a natural further development of practice, as opposed to singular events that replace an ineffective practice (for more details see e.g. Krainer, 2003).

For the future, the ministry expressed its intention to continue IMST. The overall goal is setting up and strengthening a culture of innovations in schools and classrooms, and anchoring this culture within the Austrian educational system.

PFL courses

In Austria, in-service professional development courses (PFL - Pedagogy and Subject-specific Methodology for Teachers) support teachers in developing their teaching skills and updating their knowledge of the subject they teach. The participants systematically reflect their professional work. PFL started in 1982, has undergone several adaptations, and is still running (for more detail, see Rauch et al., 2014). The programme is designed for teachers from all types of schools across the nation, including all age groups of pupils. The overall focus of PFL is on the professional development of teachers in the fields of content, didactics and pedagogy. School development plays a central role without losing sight of classroom instruction. The PFL concept is based on the implicit knowledge, which teachers possess concerning their work in class, their experience and their individual strengths. The course is intended to contribute to the further development of the teachers as professionals. Teachers are introduced to the methods of action research (Altrichter & Posch, 2009). They investigate different aspects of their teaching by defining research questions of relevance to their work, by collecting data, interpreting and drawing conclusions and writing down their findings in reflective papers.

The major goals of the teaching process should be primarily achieved through – and not detached from – the subject-related design of teaching and learning. PFL takes two years and focuses on the individual teachers’ own reflective practice using action research methods (Altrichter & Posch, 2009). By the end of the course, each participant is obliged to write a reflective paper using the data he/she has gathered throughout the process using qualitative and quantitative research methods. Participants are part of a community of practice (Wenger, 1998), since their work is embedded in a structure of mutual assistance and external support.

Case studies

Within both professional development programmes (IMST and PFL) several case studies were conducted, with the aim to research the sustainability of the programmes’ impact. The case studies

presented here were based on data from various sources and time periods to gain validity by “convergence of evidence” (Yin, 2003, p. 100): data collection contained documents (e.g. teachers’ project reports, which were written during and at the end of teachers’ participation in the project) and archival records (e.g. author’s field-notes, which originate from author’s activities as teacher educator in the project). Moreover, interviews were conducted from an ex-post perspective with former participating teachers, teachers’ colleagues, principals, and project facilitators and teacher educators. Data analysis included both inductive and deductive elements (Altrichter & Posch, 2009) to analyse both the impact and the respective fostering (or hindering) factors. For example: document analysis aimed at gathering information concerning short-term impact which (a) occurred during and/or at the end of the teachers’ participation and (b) might hold the possibility of sustainability and scale-up. Subsequently, this document analysis formed the basis for the interviews series. The interviews were semi-structured, since they were based on the analysis of existing data (document analysis), which identified various levels of short-term impact which occurred during and/or at the end of teachers’ participation. The interviews were designed accordingly (a) to gather data concerning the sustainability and scale-up of impact and (b) to reveal other types of impact which were not already coded. Data was analysed by qualitative content analysis (Mayring, 2003) in order to identify common topics, elaborate emerging categories, and gain deeper insight into teachers’ professional growth over time. The case studies’ results were validated by means of member checking.

Exemplary case study from IMST

The case of Barbara, a former participant of the IMST² project, provides exemplary results concerning the issue of sustainable impact:

Barbara’s beliefs regarding inquiry based learning (IBL) and open learning environments were changed during her participation in IMST²: Due to her participation in the project, she regularly used IBL settings (which she did not before her participation) and experienced positive effects on students’ content knowledge, as well as on their self-confidence. In particular, she stated that there were positive changes regarding low-performing students’ self-esteem, as well as the further development of high-performing students’ competencies. This change was evidenced by data (both document analysis and interviews). In the interview, Barbara highlighted that this impact was sustained and enabled her to create and implement innovative teaching methods in a long-term way.

Besides this impact, she also developed (due to her participation in the IMST²) an inquiry stance towards the content and the method of her teaching. This inquiry stance was mirrored by her new belief about the value of feedback: due to the teacher education programme, topics such as classroom atmosphere and teaching quality were discussed with and evaluated by her students on a regular basis. This impact on her own IBL was sustained: Barbara stated in the interview, that she was convinced of the importance of critically evaluating her teaching. Even after the programme’s termination, she continued to actively facilitate her students’ communication and discussion about her teaching practices.

One of the central factors that fostered the sustainability of impacts was the engagement of the school’s principal. The school had an efficiently organized management and school development structure, which represented another fostering factor. Additionally, Barbara experienced personal

benefit, which also helped the impact persist after the programme's termination. Both the teacher and principal highlighted (in the interviews) the role of the IMST² project facilitator as a fostering factor. Yet another fostering factor was represented by the IMST² workshops and seminars, where Barbara (according the interview data) got support and opportunities to share her experiences and to make her success and remaining challenges visible.

Exemplary case study from PFL

Eve participated in the PFL course and had the goal to promote open learning settings by implementing new teaching approaches in her mathematics classes. Document analysis showed that she aimed at enhancing pupils' inquiry-based learning opportunities. During her participation in PFL, Eve changed her teaching practices and implemented innovative teaching approaches to enhance her pupils' self-directed and independent learning. Interview data clearly shows that this impact was sustained: the changes in Eve's teaching practices stayed effective even after the termination of PFL. Core fostering factors were the school principal's support and a high level of mutual appreciation within the school staff, and pupils' benefit. In particular, Eve highlighted in the interview that the pupils' joy and success are core reasons for her to keep this impact sustained.

Document analysis further showed that Eve conducted various self-evaluations during her participation in PFL and gained new knowledge concerning action research methods. In the interview, she stated that she continued to reflect on her teaching practices, even after the end of her participation. This impact was sustainable, due to Eve's direct advantage (by getting information on her classroom performance) and the support of the school's principal (who was convinced that reflections and self-evaluations are important steps on the journey to school quality). This impact was also fostered by Eve's colleagues' joint reflection and communication. Interview data shows that teachers cooperated beyond school subjects and held similar values and standards concerning pedagogical or subject-related issues. The school's principal showed great interest in, and provided support for, the teachers' activities. He participated in the school's mathematics study group and shared his perspective with the teachers.

Discussion

The factors that fostered the sustainability of the case studies' impacts are mirrored by the theoretical framework (see above):

IMST and PFL enabled networking (Cobb & Smith, 2008) by community building, mutual appreciation and joint reflection. A particular factor was the principals' content knowledge (Cobb & Smith, 2008). Teachers' colleagues provided communication and social support in developing and reflecting instructional practices (Cobb & Smith, 2008). Moreover, a shared vision (Cobb & Smith, 2008) of values and standards regarding high quality mathematics instruction was established. In particular, the case studies' results highlight that the promotion of reflection and networking as key interventions (Krainer, 1998) turned out to be supportive for the sustainability of the professional development programmes' impacts.

NOTE

Parts of this paper are based on Krainer & Zehetmeier (2013), Zehetmeier (2015) and Zehetmeier, Erlacher, Andreitz, and Rauch (2015).

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The impact of professional development on teachers' autonomy-supportive teaching practices

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Keywords: Professional development, autonomy-supportive practices, teacher beliefs.

Description of the research topic

The benefits to student academic and dispositional outcomes when exposed to autonomy supportive learning environments have been acknowledged for more than a decade (Assor, 2012; Reeve, 2009). Autonomy supportive teaching practices nurture students' internal motivations to learn, resulting in learning that is self-directed and both cognitively and emotionally engaging (Wolters & Taylor, 2012). While research affirms the benefits of instruction incorporating autonomy-supportive practices (e.g., Assor, 2012) it also shows that the mathematical beliefs of teachers can be an impediment to their commitment and enactment of such practices (Bobis, Way, Anderson, & Martin, 2016). With this in mind, an intervention study was conducted with mathematics teachers (grades 5-7) that aimed to enhance their use of engagement supportive teaching strategies in their mathematics classrooms. The intervention was a year-long professional development program that focused on shifting teachers' beliefs about student engagement and building knowledge of instructional strategies for promoting student autonomy in the mathematics classroom. The specific research question addressed was: What impact did the professional development program have on teachers' beliefs and practices that promote learner autonomy in mathematics?

Theoretical framework and methodology

Self-determination theorists (SDT, Deci & Ryan, 1985) advocate that autonomous motivation will improve students' academic and dispositional outcomes because activities undertaken for autonomous reasons are likely to increase students' willingness to apply effort when learning. According to SDT, students will be more intrinsically motivated to learn when teachers adopt autonomy-supportive pedagogy rather than controlling pedagogical approaches. SDT was used to guide our examination of self-reported data regarding mathematics teachers' instructional beliefs and practices as a result of their involvement in the professional development program.

Pre- and post-intervention data were collected from 32 grade 5 to 7 teachers of mathematics from four secondary and ten elementary schools located in Sydney, Australia. Participants included five male and 27 female teachers. Data were collected via focus groups and a 20 Likert-type item questionnaire that measured the extent to which teachers were committed to instructional beliefs and practices considered supportive of student engagement, including learner autonomy.

Findings and conclusion

Dependent T-tests were used to determine whether there were significant differences between teachers' pre- and post-intervention responses on each dimension of the questionnaire. Results for two dimensions pertaining to teachers' autonomy supportive beliefs and practices—discovery (the construction of ideas through student discovery) and teacher's role (co-learner and constructor of a learning community) are presented on the poster. During the pre-intervention focus groups, most teachers described their roles as a 'giver' of knowledge to students. However, in the post-intervention focus groups, teachers reported how they now tried to develop more autonomous learning strategies in their students and to encourage them to take greater responsibility for their own learning. The results indicate that teachers expressed beliefs and practices that were more supportive of student autonomy at the end of the intervention than prior to undertaking the professional development program.

Presentation of the poster

The poster is structured in four major sections: Section one provides a succinct introduction to the literature, providing a justification for the study and presents the research question. Section two presents a visual representation of the theoretical framework (SDT) underpinning the study and our analysis of results. Section three presents results from the questionnaire and focus groups. The final section presents implications of the findings and argues that such shifts in teachers' beliefs/practices can have practical consequences in terms of improving students' autonomy for learning mathematics.

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Supporting teachers' practices: A teacher educator-embedded professional development model

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The poster describes preliminary results from ongoing professional development with two U.S. mathematics teachers, one Algebra 1 and one eighth grade mathematics teacher, designed to increase and enhance teachers' content knowledge and transform their classroom instruction by embedding the author (i.e., researcher) in teachers' practices. The poster also articulates the embedded PD model. Preliminary results show participating teachers are engaging their students in more rigorous mathematics, teachers are demonstrating increased self-efficacy and are more frequently engaging students in mathematical sense making, reasoning, modeling, generalizing, and communicating.

Keywords: Mathematics instruction, mathematics teachers, theory practice relationship.

Description of research topic

The poster describes ongoing professional development (PD) in the United States with one eighth grade math teacher (students 13-14 years) and one ninth grade Algebra 1 teacher (students 14-15 years) designed to increase and enhance teachers' content knowledge and transform their classroom instruction by embedding the author (i.e., researcher) in teachers' practices. The poster focuses on the following research questions: How does embedding a mathematics teacher educator in a mathematics classroom (embedded PD) impact (1) participating teachers' content knowledge, (2) participating teachers' instruction, (3) students' self-efficacy, and (4) student achievement?

Theoretical framework

The embedded PD model is grounded in a constructivist approach to learning mathematics and aligns with Simon's (1995) Mathematics Teaching Cycle and Thompson's (2013) interpretive framework for the development of powerful Mathematical Meanings for Teaching (MMT).

Method

Throughout embedded PD, the author (i.e., researcher) and participating teachers relied on: participating teachers' prior assignments, assessments and notes; textbooks and district generated documents (e.g., curriculum maps); state-level standards and documents; and a variety of Internet resources (e.g., GeoGebra, NRIC Project). Throughout the planning of a lesson, which may take multiple in-person or online meetings (or both), the researcher attempts to motivate the teacher to make explicit (and objects thought, discussion and subsequent reflection) her (i.e., teacher's) understandings of: the mathematics inherent in the lesson, hypotheses of their students' knowledge, theories of mathematics learning and teaching, activities and assessments (Simon, 1995; Thompson, 2013). As such, notions of meanings, ways of thinking and the need for the teacher and her students to articulate their meanings, thinking, and reasoning are consistently addressed (Thompson, 2013).

Participants

Participating teachers reported on the poster involve Tami (eighth grade math) and Jeremy (Algebra 1). Tami's eighth grade class contained 15 students of both genders and multiple races. Five (of the

15) students were on individual education plans. Jeremy's Algebra 1 class contained 25 students of both genders and multiple races.

Data and analysis

Data consisted of: (1) video- and audio-recordings of and physical documents related to lesson co-planning sessions; (2) video- and audio-recordings of lesson implementations (i.e., co-teaching); and, (3) video- and audio-recordings of and physical documents related to teachers' reflection on student work and classroom instruction. Two embedded co-teaching descriptions will be described on the poster, one involving a co-planning session with Tami, the other involving co-teaching with Jeremy. Analysis will serve to characterize some of the differences exhibited in tasks, activities, and classroom interactions highlighted as a result of the embedded model.

Preliminary results

Teachers engaged in embedded PD have indicated their participation has provided them the support to do what they believe is best for their students and their practice while not feeling constrained by district and state demands. Rather than feeling the need to rush through content and focus on skills and procedures, embedded co-teaching has allowed participating teachers to focus on understanding, coherence, and discourse. Preliminary results show participating teachers are engaging their students in more rigorous mathematics and both students and teachers are demonstrating increased self-efficacy and are more frequently engaging in mathematical sense making, reasoning, modeling, generalizing, and communicating.

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Teachers learning through participatory action research – developing instructional tools in mathematics primary classrooms

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Keywords: Mathematics education, action research, teacher professional development, multilingual classrooms, instructional tools.

Introduction

In many classrooms, from first grade through the whole school system, there are many students whose mother tongue is not the same as the teachers' language used for instruction (Khisty, 2001). In Sweden, newly arrived immigrant children with limited Swedish language knowledge are learning Mathematics together with children who have spent their entire life in a Swedish context. Considering the large number of students with limited knowledge in the language of instruction one of the most important tasks for teachers in Swedish primary education is to create conditions to support the development of mathematical knowledge in these students.

Lately, the role of language in mathematics education has received a profound interest in educational research. Researchers have emphasized the importance of teachers using specific strategies to facilitate the classroom communication to and support students' mathematical thinking (O'Connor & Michaels, 1993).

Specifically, in the syllabuses (Skolverket, 2011) it is particularly prominent that mathematics is dominated by discourse-intensive approaches, and the use of instructional tools such as talk moves, give ample opportunities for student learning (Chapin & O'Connor, 2007). Similar strategies for supporting students' learning in mathematics have received attention among effective teachers of second language learners in mathematics (Khisty, 2001).

This study uses action research which is characterized by ongoing processes of self-reflection, which can be thought of as a spiral of self-reflective cycles on planning a change, followed by acting and observing the process and reflecting on the process and then re-planning and so forth (Kemmis & Wilkinsson, 1998). Using PAR gives an attempt "to help people investigate and change their social and educational realities by changing some of the practices which constitute their lived realities" (Kemmis & Wilkinsson, 1998, p.22).

Method

The poster gives a brief presentation of a one-year research project where four primary teachers at the same school (year 2, 4 and 5) have been working together with a researcher, using participatory action research (PAR) (Kemmis & Wilkinsson, 1998) to develop their instructional tools in order to support students' mathematical development in multilingual classrooms. Data collection has continued throughout the whole action research process during the academic year. The empirical data includes teachers' logs, teacher questionnaires with open answers, researcher's notes, audio-

recorded discussions from the meetings twice a month in the project group and 3-4 video-taped mathematics lessons in each classes, 14 lessons altogether

Results

Although the focus in this project has been on instructional tools for supporting students' talk in order to enhance their development in communicating and reasoning mathematically, it is noteworthy that the teachers express their development, not only in terms of (1) instructional tools but also regarding other aspects such as (2) classroom organization and (3) focus on mathematical content.

Methods structured in these three themes above constitute a teacher *tool kit* to support students' learning mathematics in multicultural classrooms.

Conclusions

By using PAR, the teachers had the opportunity to reflect critically, analyze and act as coparticipants in the challenge to change the practices in which they interact, which also challenged their approach to teaching.

When teachers act and reflect on their use of specific strategies of classroom talk they also start reflecting and acting on other aspects of teaching, such as classroom organization and how to keep attention to the taught content. Thereby, the change in practice became more than just temporary changes.

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