Characteristics of a learning environment to support prospective secondary mathematics teachers' noticing of students' thinking related to the limit concept

C. Fernandez, S Llinares, M L Callejo, M. Moreno, Gloria Sánchez-Matamoros

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Fernández, C.[1], Llinares, S.[1], Callejo, M.L.[1], Moreno, M.[1] and Sánchez-Matamoros, G.[2]

[1]Universidad de Alicante, Spain, ceneida.fernandez@ua.es; sllinares@ua.es; luz.callejo@ua.es; mmoreno@ua.es

[2]Universidad de Sevilla (Spain), gsanchezmatamoros@us.es

The aim of this study is to describe changes in the way that prospective secondary school teachers notice students' mathematical thinking related to the limit concept in a learning environment designed ad hoc. The learning environment progressively nests the skills of attending to, interpreting and deciding as three interrelated skills of professional noticing. Results show characteristics of how prospective teachers gained expertise in the three skills since four out of five groups of prospective teachers interpreted students' mathematical reasoning attending to the mathematical elements of the dynamic conception of limit. The links between attending to and interpreting helped prospective teachers justify the teaching activities proposed to support the progression of students' mathematical reasoning: from a mathematical point of view or considering mathematical cognitive processes involved.

Keywords: Noticing, prospective teachers' learning, learning environment.

Introduction and theoretical background

Research has shown that noticing is an important component of teaching expertise (Mason, 2002). Teachers need to attend to students' mathematical reasoning and make sense of it in order to teach in ways that build on students' thinking (Choy, 2016; Sherin, Jacobs, & Philipp, 2011). Noticing has been conceptualised from different perspectives. One of them consists of two main processes: attending to particular teaching events and making sense of these events (Sherin et al., 2011). Jacobs, Lamb, and Philipp (2010) particularise the notion of noticing to children's mathematical thinking, conceptualising this notion as a set of three interrelated skills: attending to children's strategies, interpreting children's mathematical thinking, and deciding how to respond on the basis of children's mathematical thinking.

Previous research has focused on pre-service teachers' ability to interpret students' mathematical thinking (Bartell, Webel, Bowen, & Dyson, 2013; Callejo, & Zapatera, 2016; Fernández, Llinares, & Valls, 2012; Llinares, Fernández, & Sánchez-Matamoros, 2016; Magiera, van den Kieboom, & Moyer, 2013; Sánchez-Matamoros, Fernández, & Llinares, 2015) showing that the identification of the mathematical elements involved in the problem (mathematical content knowledge) plays a significant role in interpreting students' mathematical reasoning. Furthermore, previous research has shown that some contexts can help pre-service or prospective teachers develop the noticing skill: watching video clips (Coles, 2012; van Es, & Sherin, 2002), participating in online debates (Fernández et al., 2012) or participating in learning environments (interventions) designed considering specific mathematical topics. For example, Schack et al. (2013) in the area of early numeracy; Magiera et al. (2013) in algebra; Callejo and Zapatera (2016) in pattern generalization; Llinares et al. (2016) in classification of quadrilaterals; Sánchez-Matamoros et al. (2015) in the derivative concept; and Son (2013) in the concepts of ratio and proportion. These previous studies underline that the skill of deciding how to respond on the basis of children's mathematical thinking is the most difficult one to develop in teacher education programs. As Choy (2013) pointed *"the specificity of what teachers notice while necessary, is not sufficient for improved practices"* (p. 187). In other words, teachers can be very specific about what they notice without having a teaching decision in mind. So, the relation between how prospective teachers develop the skills of interpreting students' mathematical thinking and deciding how to respond on the basis of students' mathematical thinking deserves further research.

On the other hand, the concept of limit of a function is a difficult notion for high school students (16-18 years old) and is a key concept in the Spanish curriculum (Contreras, & García, 2011). Cottrill and colleagues (1996) indicated that the difficulty of students' understanding of the limit concept could be the result of a limited understanding of the dynamic conception. A way of overcoming this difficulty is by coordinating the processes of approaching in the domain and in the range in different modes of representation. Knowing these characteristics of students' understanding could provide prospective teachers with information to interpret students' mathematical thinking and to make instructional decisions based on students' reasoning.

Therefore, our study analyses changes in the way that prospective teachers notice students' mathematical thinking (attending to, interpreting and deciding) in relation to the limit concept when they participate in a learning environment designed ad hoc. The learning environment designed progressively nests the skills of attending to, interpreting and deciding and its relations. We hypothesise that the structure of the learning environment help prospective teachers to decide how to respond taking into account their previous interpretations of students' mathematical thinking.

Method

Participants and the learning environment

The participants were 25 prospective secondary school teachers (mathematics, physics and engineering) who were enrolled in an initial secondary mathematics teacher training program. One of the subjects of this program is focused on developing the skill of noticing students' mathematical thinking in different mathematical topics and on planning the instruction attending to students' mathematical thinking. One of the mathematical topics considered was the limit concept.

The learning environment consisted of 5 sessions of two hours each and was designed taking into account the nested nature of the skills of attending to, interpreting and deciding (Jacobs et al., 2010). Prospective teachers were divided into five groups of 5 persons to perform the tasks of the learning environment. Firstly, prospective teachers solved three problems related to the limit concept selected from high school textbooks (Figure 1) in order to unpack the important mathematical elements of the limit concept (session 1). Then, prospective teachers had to anticipate hypothetical students' answers to these problems reflecting different characteristics of conceptual development (session 2). That is, they had to anticipate what students are likely to do. Prospective teachers had a document with the definition of the dynamic conception of limit and its mathematical elements (Pons, 2014): (i) approaches from the right and from the left (in the domain and in the range), and

(ii) coordination of the processes of approaching in the domain and in the range considering different modes of representation (graphical, algebraic and numerical).

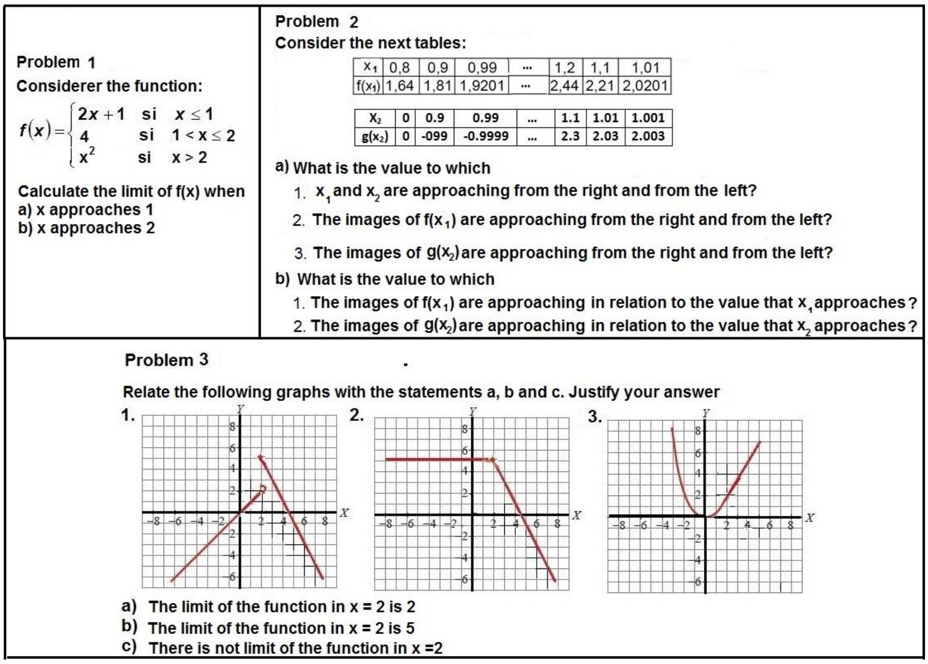


Figure 1: The three problems related to the limit concept

The aim of the tasks of identifying the mathematical elements in the resolution of the problems and anticipating hypothetical students' answers was to help prospective teachers focus their attention on the relationship between the specific mathematical content and students' mathematical thinking. We conjecture that focusing on this relationship is needed to develop the skill of noticing in a first step. Next, prospective teachers analysed a set of four high school students' answers (Pablo, Rebecca, Luiggi and Jorge) to the same problems. Prospective teachers had to attend to students' strategies, interpret students' mathematical thinking and propose new activities (or modify them) to help students progress in their conceptual reasoning (according to their previous interpretations of students' mathematical thinking) (session 3 and 4). The high school students' answers, provided to prospective teachers, reflected different levels of high school students' reasoning of the limit concept (Table 1; Pons, 2014). We also provided prospective teachers with theoretical information that summarise the characteristics of high school students' reasoning of the limit concept from previous research to solve the task (Cornu, 1991; Cottrill et al., 1996; Swinyard, & Larsen, 2012). In figure 2, the answers of Pablo to the three problems are given.

Prospective teachers had to answer the next three questions: (i) which mathematical elements has the student used in each problem? Indicate if he/she has had difficulties with them; (ii) identify some characteristics of how the student understands the limit of a function. Explain your answer using the mathematical elements identified before; (iii) considering the student reasoning, propose an activity that helps the student progress in their conceptual reasoning of the limit concept. Therefore, the objective of sessions 3 and 4 was that prospective teachers focus their attention on the relation between identifying-interpreting and between interpreting-deciding. We conjecture that these relationships are necessary to develop the skill of noticing. Finally, in the session 5, prospective teachers had to answer a similar task individually.

High school students	Level	Levels of students' reasoning about the dynamical conception of the limit concept
Pablo and Luiggi	High	Pablo and Luiggi coordinate the processes of approaching in the domain and in the range in the three modes of representation
Rebecca	Low	Rebecca coordinates the processes of approaching in the domain and in the range in the graphical mode of representation when limits coincide
Jorge	Intermediate	Jorge coordinates the processes of approaching in the domain and in the range in the algebraic and graphical mode of representation (when limits coincide in this last mode of representation)

Table 1: Characteristics of high school students' answers

Answer to Problem 1	Answer to Problem 2	Answer to Problem 3
$\lim_{x\to 1} \to \lim_{x\to 1^-} 2x+1 = 3$ $\lim_{x\to 1} \to \lim_{x\to 1^+} 4 = 4$ $\lim_{x\to 2} \to \lim_{x\to 2^-} 4 = 4$ $\lim_{x\to 2} \to \lim_{x\to 2^+} x^2 = 4$	a*1) x_1 is approaching to 1 from the left and from the right and x_2 is approaching to 1 from the left and from the right* *a2) The images of $f(x_1)$ are approaching to 2 from the left and from the right* *a3) The images of $g(x_2)$ are approaching to 2 from the right and is approaching to -1 from the left* *b1) when x_1 is approaching to 1, images of $f(x_1)$ tend to 2* *b2) when x_2 is approaching to 1, the images of $g(x_2)$ tend to -1 from the left and tend to 2 from the right.*	*a) Graph 3 because the limit of the function in x=2 from the right and from the left is 2.* *b) Graph 2 because the limit of the function in x=2 from the left and from the right is 5.* *c) Graph 1 because the limit of the function in x=2 is not the same from the right and from the left.*

Figure 2: Pablo's answers to the three problems

Data and analysis

Data of this study are prospective teachers' answers to the tasks of session 2 (anticipation) and sessions 3 and 4 (interpretation). Through an inductive analysis (Strauss & Corbin, 1994), we generated similarities and differences about how prospective teachers conceived high school students' reasoning of the limit concept and the type of activities they provided to help students progress in their conceptual reasoning. To carry out this analysis, five researchers analysed individually prospective teachers' answers to the anticipation and interpretation tasks and then, the agreements and disagreements were discussed to reach a consensus on these issues.

This analysis let us identify two ways of how prospective teachers conceived high school students' reasoning: as dichotomous (right or wrong) and as a progression (identifying different levels of students' reasoning). The type of activities that prospective teachers provided were categorised in three categories: general decisions, decision based on curricula contents and decisions based on cognitive processes. Examples of these categories are presented in the results section.

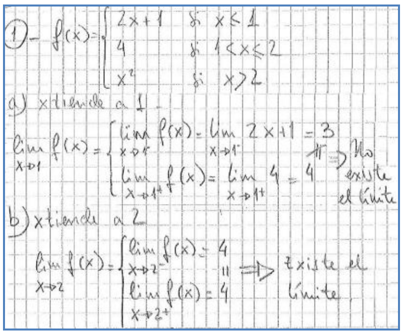
Finally, we compared categories obtained in the anticipation task with the categories obtained in the interpretation task to identify changes in the way of how prospective teachers conceived high school students' reasoning and proposed activities to help students progress in their conceptual reasoning.

Results

Our results show that prospective teachers changed the way that they conceived students' reasoning from a dichotomous to a progression way and this shift influenced the type of activities that they proposed to help students progress in their conceptual reasoning.

Changes in the way that prospective teachers conceived students' reasoning: From a dichotomous to a progression

In the anticipation task, three out of five groups conceived students' reasoning as dichotomous (right or wrong). For example, the group of prospective teachers G2 anticipated that a high school student with high level of reasoning of the limit concept (Maria) would coordinate in all modes of representation. For example, this group of prospective teachers anticipated the next answer for the algebraic representation:



Maria understands the limit concept. The idea of approximation in the domain corresponds to the fact that she properly selects the branch of the function and uses the notion of approximation in the range adequately. It is demonstrated when she replaces on the limit the approach of the independent variable. This student also coordinates the approximations to establish the value of the limit according to the branch.

Furthermore, these prospective teachers (G2) anticipated that a high school student with a not suitable level of reasoning of the limit concept (Pedro) would not coordinate in any mode of representation pointing out: *"Pedro only approximates (from the left or from the right) when the function is defined".*

Then (in the interpretation task), four out of five groups of prospective teachers were able to interpret students' mathematical reasoning. They linked students' reasoning with the mathematical elements of the dynamic conception of limit: the approaches from the right and from the left (in the domain and in the range), and the coordination of the approaches in the domain and in the range considering different modes of representation (graphical, algebraic and numerical). For instance, the group of prospective teachers (G2) interpreted the student's answer of problem 1 (Figure 2) as:

> The resolution of the student is correct (Pablo). We can notice that the student has identified the kind of function (piecewise function) since he (the student) has approximated in the range (he has calculated the approximation to x=1 from the left and from the right and the approximation to x=2 from the left and from the right) and in the domain (taking the correct definition of function in each interval). Furthermore, he has coordinated the processes of approaching in the domain and in the range since he has written, for example, that when x tends to 1 from the left, the image of the function tends to 3 (using the function 2x+1).

This group gave similar comments for the student's answers to the other two problems linking students' reasoning with the important mathematical elements in the other two modes of representations (problems 2 and 3). Afterwards, they wrote a summary about this student level of reasoning:

> This student understands the limit concept since he approaches from the right and from the left (in the domain and in the range), and coordinates the processes of approaching in the domain and in the range in the three modes of representation (graphical, algebraic and numerical). This student would be in the high level of reasoning.

These prospective teachers were able to identify different levels of students' reasoning. Therefore, they conceived students' reasoning as gradual.

Changes in the type of activities they proposed to help students progress in their conceptual reasoning

Prospective teachers who conceived students' reasoning as dichotomous did not propose specific activities to help students progress in their reasoning. These prospective teachers gave general comments about teaching as instructional actions. For instance, the group of prospective teachers G2 proposed to Maria (in the anticipation task) the representation of the graph of the function of problem 1. This decision was not based on the conceptual progression of the student.

When prospective teachers interpreted students' mathematical thinking identifying different levels of students' reasoning (linking students' mathematical reasoning with the important mathematical elements), they were able to provide specific activities to help students progress in their conceptual reasoning. For the students who only coordinate the approaches in the domain and in the range in one mode of representation, they proposed new activities to integrate these mathematical elements gradually in the different modes of representation. The proposed activities required a coordination of approaches in the domain and in the range in the different modes of representation. For the students who coordinate the approximations in the domain and in the rage in all modes of representation (such as the student of Figure 2), they also provided activities to help students progress in their reasoning.

We have identified two ways in which they justified their new activities: some justifications were based on the mathematical elements and others on the cognitive processes involved. In the first case, prospective teachers focused their attention on introducing a new mathematic content. In the following example, they introduced a new type of discontinuity – an avoidable discontinuity. The justification of this type of activities was based on the use of new mathematical elements (in this case, introducing other type of functions).

<u>The activity</u>: We would modify the function of problem 1 and we would use:

$$f(x) = \begin{cases} 2x & si \quad x > 2 \\ 5 & si \quad x = 2 \\ 2x & si \quad x < 2 \end{cases}$$

<u>Our justification</u>: The student (Figure 2) seems to understand the limit concept in the three modes of representation, so we would provide him a more difficult function with an avoidable discontinuity.

In the second case, prospective teachers focused their attention on the cognitive processes involved to understand the limit concept. In the next example, prospective teachers focused on the reversal as a cognitive mechanism that leads students to a new reasoning level. That is to say, prospective teachers justified the proposed activity by the need of generating learning opportunities to develop the reverse mechanism that allows the construction of cognitive objects.

<u>The activity</u>: Represent a graph of a function which limit in x=-1 is 4 and that there is not limit in x=1.

<u>Our justification:</u> with this activity students need to do the inverse process that is, they need to use all the important mathematical elements to build that function.

Discussion and conclusions

Results show that after the participation in a learning environment that progressively nests the three interrelated skills of professional noticing (attending to, interpreting and deciding), prospective secondary school teachers gained expertise in noticing. Four out of five groups of prospective teachers were able to interpret students' mathematical thinking linking students' reasoning with the important mathematical elements of the dynamic conception of limit. These findings support the claim that some characteristics of the learning environment such as considering the nested nature of the skills help prospective teachers develop the skill of noticing (Sánchez-Matamoros et al, 2015; Schack et al., 2013).

Furthermore, prospective teachers were able to provide specific activities to help students progress in their conceptual reasoning. Therefore, the characteristics of the learning environment in which prospective teachers were engaged in the analysis of mathematical elements of limit problems, in the analysis of students' reasoning and in proposing new activities to support students' conceptual development enable them to gain more accurate understanding of the relation between the mathematical content and students' mathematical thinking. This new understanding provides prospective teachers with the needed knowledge to give their teaching decisions based on the progression of students' reasoning: from a mathematical point of view or considering the cognitive processes involved.

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