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# Using a fraction learning trajectory as a tool to develop pre-service primary teachers' noticing of students' fractional reasoning

Pere Ivars, Ceneida Fernández and Salvador Llinares

Universidad de Alicante, Spain

[pere.ivars@ua.es](mailto:pere.ivars@ua.es); [ceneida.fernandez@ua.es](mailto:ceneida.fernandez@ua.es); [sllinares@ua.es](mailto:sllinares@ua.es)

*Since noticing has been identified as a critical skill that teachers must develop, research on how pre-service teachers develop this skill in teacher education programs has emerged. In this study, we focus on how pre-service teachers notice students' fractional reasoning through a task designed taking into account a students' Learning Trajectory of fractional reasoning. Our results show that pre-service teachers' learning of the Learning Trajectory helped them to notice students' fractional reasoning in a structured way: identifying important mathematical elements of the problems and, establishing relationships between the mathematical elements and students' fractional reasoning levels of the Learning Trajectory to help students progress in their fractional reasoning.*

*Keywords: Noticing, fractional reasoning, learning trajectories.*

## Noticing and learning trajectories

Noticing has been shown as an important skill for teachers. This skill has been conceptualised from different perspectives (Jacobs, Lamb, & Philipp, 2010; Mason, 2002, 2011; Sherin, Jacobs, & Philipp, 2011) but all of them emphasise the importance of identifying the relevant aspects in teaching and learning situations and interpreting them to make teaching decisions. Mason stated that “noticing is a movement or shift of attention” (Mason, 2011, p. 45) and identified different ways in which people can attend (p.47):

*Holding wholes* is attending by gazing at something without particularly discerning details.

*Discerning details* is picking out bits, discriminating this from that, decomposing or subdividing and so distinguish and, hence, creating things.

*Recognizing relationships* is becoming aware of sameness and difference or other relationships among the discerned details in the situation.

*Perceiving properties* is becoming aware of particular relationships as instances of properties that could hold in other situations.

*Reasoning on the basis of agreed properties* is going beyond the assembling of things you think you know, intuit, or induce must be true in order to use previously justified properties as the basis for convincing yourself and others, leading to reasoning from definitions and axioms.

This perspective emphasises the importance of identifying the relevant aspects of the teaching-learning situations (discerning details) and interpreting them (recognising relationships) to support instructional decisions (perceiving properties).

On the other hand, research has shown that when pre-service teachers attend to students learning progressions in a particular mathematical domain, they are better able to make decisions about next

instructional steps (Wilson, Mojica, & Confrey, 2013). In this context, students' learning trajectories (Battista, 2012) can assist pre-service teachers in identifying learning goals for their students, in anticipating and interpreting students' mathematical reasoning and in responding with appropriate instruction (Sztajn, Confrey, Wilson, & Edgington, 2012). Our study is embedded in this line of research and analyse how pre-service teachers' learning of a fractional reasoning Learning Trajectory supports their development of noticing students' fractional reasoning. Our research question is: how do pre-service teachers interpret student' fractional reasoning and respond with instructional actions using a learning trajectory of fractional reasoning?

### **A learning trajectory of fractional reasoning**

A Learning Trajectory consists of three components: a learning goal, learning activities, and a hypothetical learning process (Battista, 2011). A Learning Trajectory includes descriptions of learning activities that are designed to support students in the transition through intermediate stages to a more sophisticated level of reasoning.

The learning goal of the fractional reasoning Learning Trajectory used in this study is derived from the Spanish Primary Education's curriculum: the meaning of fraction and its different representations and, the meaning of fractions operations. This learning goal highlights two key aspects: a) the transition from an intuitive meaning of splitting into equal parts to the idea of fraction as part-whole taking into account different representations, and b) the construction of the meaning of operations with fractions.

The student's learning process takes into account how the student reasoning about fractions develops over time (Battista, 2012; Steffe, 2004; Steffe, & Olive, 2009). We have considered six different levels of students' fractional reasoning (learning trajectory proficiency levels): **at level 1**, students have difficulties in recognising that the parts of the whole must be congruent; **at level 2**, students recognise that the parts could be different in form but congruent in relation to the whole. This allows them to identify and represent fractions in a continuous context but they have difficulties with discrete contexts. They also begin to use unit fractions as an iterative unit (i) to represent proper fractions (although they have difficulties with improper fractions) and (ii) to solve some fraction addition problems with the same denominator; **at level 3**, students identify and represent fractions in discrete contexts recognising that the groups must be equal. They also recognise that a part could be divided into other parts. When comparing fractions, they recognise that the size of a part decreases when the number of parts increases. They can use a part (not necessarily the unit fraction) as an iterative unit to represent proper ( $f < 1$ ) and improper ( $f > 1$ ) fractions. They can also reconstruct the whole using any fraction as an iterative unit (continuous and discrete contexts). In addition, they use intuitive graphical representations to add/ subtract fractions with different denominators; **at level 4**, students can solve simple arithmetic problems with the help of a guide or support. They can do equivalent fractions so that operations can be graphically represented. When they add or subtract fractions with different denominators, they understand that the parts must be congruent to join/separate although they need a guide that allows them to choose the unit correctly. When they multiply, they understand the fraction as an operator "a/b of c/d" and when they divide, they develop two types of reasoning; (i) division as a measure and (ii) division as a partition; **at level 5**, students can operate and solve arithmetic problems symbolically, identifying patterns. They can graphically justify what they do but only in simple situations. At this level, they

are able to interpret the remainder of a division of fractions; **at level 6**, students can explain operations graphically. They do not need a guide to represent fraction operations.

## Method

### Participants and context

Participants were 31 pre-service primary school teachers (PT) enrolled in a degree to become primary school teachers. They were enrolled in a subject of 150 hours (60/90 attendance/nonattendance) related to teaching and learning of mathematics in primary school. In previous courses, these pre-service teachers had participated in a subject related to Numerical Sense and in a subject related to Geometrical Sense.

### Instrument: The task

The task consists of three pairs of primary school students answers, with different learning trajectory proficiency levels of fractional reasoning, to a problem that implies the identification of a fraction (adapted from Battista, 2012) (Figure 1). These answers reflect characteristics of the first three levels of the Learning Trajectory. The answers of Xavi and Victor show characteristics of the level 1 since they are not able to identify that the parts of a whole must be congruent. The answers of Joan and Tere reflect characteristics of the second level since they are able to identify that the parts of a whole must be congruent in continuous contexts but they still do not recognise that a part can be divided into other parts. This last characteristic is evidenced when they say that Figure E is not three quarters because it is divided into 24 equal parts and there are 18 shaded. Finally, the answers of Álvaro and Félix show that not only they are able to recognise that the whole must be divided into congruent parts but also they acknowledge that a part could be divided into other parts.

Pre-service teachers had to answer the next four questions. To answer them, we provided pre-service teachers with theoretical information about the mathematical elements of the fraction concept and about the Learning Trajectory of fractional reasoning used in this study.

**Q1-** Describe the problem taking into account the learning objective: what are the mathematical elements that the student needs to know to solve it?

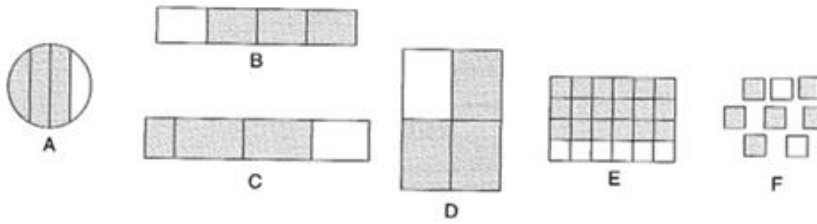
**Q2-** Describe how each pair of students has solved the problem identifying how they have used the mathematical elements involved and the difficulties they have had with them.

**Q3-** What are the characteristics of students' reasoning (Learning Trajectory) that can be inferred from their responses? Explain your answer.

**Q4-** How could you respond to these students? Propose a learning objective and a new activity to help students progress in their fractional reasoning.

These questions and the theoretical information given (Learning Trajectory of fractional reasoning) focus pre-service teachers' attention on relevant aspects of students' answers (discerning details) identifying the relevant mathematical elements; on interpreting these answers (recognising relationships between the mathematical elements and students' reasoning) and on supporting instructional decisions (attending students' mathematical reasoning).

1. Choose the figures below that show  $\frac{3}{4}$ . Explain your answers.



### Xavi and Víctor's answers

Víctor: Mmmm, well we think Figures A, B, C and D represent three-quarters.

Teacher: Xavi, do you agree with Víctor?

Xavi: Yes, A, B, C and D are divided in 4 parts, and 3 are shaded.

### Joan and Tere's answers

Tere: We believe that Figures B and D are three quarters because they are divided into four equal parts and three are shaded. Figures A and C have 3 parts of 4 shaded, but the parts are not equal...

Teacher: And Figure E? What do you think about Figure E?

Joan: Figure E is not three quarters because it is divided into 24 equal parts and there are 18 shaded.

Tere: Sure, it is not three-quarters.

Teacher: And the F?

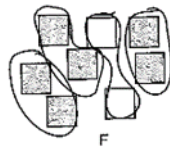
Both: It is not a fraction. In figure F, there are only 6 shaded squares.

### Félix and Alvaro's answers

Félix: Well ... yes. We agree with Joan and Tere answers related to figures A, B, C, and D but we think differently about figure E...

Teacher: What do you think? Could you explain your answer?

Álvaro: Well ... mmm sure. If you look each line of Figure E, each line has 6 squares, and as there are 3 lines shaded of the 4 total lines then it is three quarters. In addition, Figure F also represents three quarters because if you group the squares in groups of 2, you get 4 groups of 2, and there are three groups shaded.



Álvaro and Félix answer to Figure F

**Figure 1: Task to support pre-service teachers' learning of a fractional reasoning Learning Trajectory to notice students' mathematical reasoning**

### Analysis

Taking into account Mason's work and the Learning Trajectory of fractional reasoning, we analysed pre-service teachers' answers according to if they had (i) identified relevant elements of fractional reasoning in the student's answers (discerning details); (ii) interpreted the student's reasoning considering the characteristics of students' fractional reasoning from the Learning Trajectory (recognising relationships between the elements identified and the different levels of students' learning progress of fractional reasoning); (iii) made instructional decisions (reasoning about next steps providing different activities that promote students' progression in the Learning Trajectory).

To carry out the analysis, initially a subset of pre-service teachers' answers was analysed by three researchers independently considering the points mentioned above. Then, we put together our respective analyses and compared and discussed our discrepancies until reaching an agreement. Afterwards, new data samples were added to review our allocation.

## Results

From the analysis, we have identified three groups of pre-service primary school teachers according to the way that they used the Learning Trajectory to interpret students' fractional reasoning and make teaching decisions. These results show that 20 pre-service teachers were able to use the Learning Trajectory to interpret students' fractional reasoning, while the other pre-service teachers (group 1) had difficulties in using the Learning Trajectory to interpret students' answers. The characteristics of the different groups of pre-service teachers are:

- Group 1. Pre-service teachers who used some mathematical elements of the Learning Trajectory but in rhetoric way or without sense (11 PT).
- Group 2. Pre-service teachers who used the mathematical elements of the Learning Trajectory to recognise different levels of students' fractional reasoning, but they were not able to propose new activities considering the learning trajectory proficiency levels (11 PT)
- Group 3. Pre-service teachers who used the mathematical elements of the Learning Trajectory to recognise different levels of students' fractional reasoning, and proposed new activities to help students progress in their fractional reasoning taking into account the learning trajectory proficiency levels (9 PT)

### **Group 1: Pre-service teachers who used some mathematical elements of the Learning Trajectory but in rhetoric way or without sense**

Pre-service teachers of this group used the mathematical elements implied in the problem (the parts of the whole must be congruent and a part can be divided in other parts) in a rhetoric way when they described students' answers but they did not recognise characteristics of the different Learning Trajectory proficiency levels in students' answers. For instance, the pre-service teacher E27 answered question 3 of the task, pointing out (emphasis has been added underlying the mathematical elements):

Víctor and Xavi: They are at Level 1 of the Learning Trajectory because they do not know the concept of congruence and they do not know that a part could be divided in other parts

Joan and Tere: They are at Level 1 because they have difficulties in recognising that the part must be congruent and they do not recognise that a part could be divided in other parts.

Félix and Álvaro: They are at Level 1 because, related to congruence they know the same that Joan and Tere, although they recognise that a part could be divided in other parts in continuous and discrete contexts.

This pre-service teacher did not recognise differences between students' fractional reasoning saying that all pairs of students have difficulties with the mathematical element *the parts of the whole must be congruent* although he used the mathematical elements to describe students' answers.

**Group 2: Pre-service teachers who used the mathematical elements of the Learning Trajectory to recognise different levels of students' fractional reasoning, but they were not able to propose new activities considering the Learning Trajectory proficiency levels**

Pre-service teachers of this group used the mathematical elements of the Learning Trajectory that correspond with the problem (the parts of the whole must be congruent and a part can be divided in other parts) to recognise the different levels of students' fractional reasoning. However, these pre-service teachers did not justify a new activity taking into account the students' fractional reasoning. For instance, the pre-service teacher E09 answered to question 2 and 3 for each pair of students (emphasis has been added underlying the mathematical elements):

Víctor and Xavi have difficulties in recognising that the parts must be congruent as they identify as a  $\frac{3}{4}$  figures A and C whose parts are not equal. Another characteristic that we can identify is that they have difficulties in recognising that a part could be divided in other parts. They do not notice that figures E and F are divided in 4 parts, maybe they notice that E has 24 squares and F has 8 squares. Thus they do not realise that both are equivalents. So, these students are at Level 1.

Joan and Tere are able to identify and represent fractions in a continuous context recognising that the parts must be congruent as they recognise that, although figures A and C are divided in 4 parts and 3 are shaded they do not represent  $\frac{3}{4}$  because the parts are not congruent. They also identify that B and D are  $\frac{3}{4}$ . They are not able to recognise that a part could be divided in other parts/consider a group of parts as a part since they do not identify that even though E and F are divided in more parts, they represent  $\frac{3}{4}$ . So, these students are at Level 2.

Félix and Álvaro agree with Joan and Tere about figures A, B, C, and D, thus they recognise that the parts must be congruent. Furthermore they recognise that a part could be divided in other parts and they identify fractions in discrete contexts since for figure E they say that, although it is divided in 24 squares, it represents  $\frac{3}{4}$  because there are 4 lines with 6 squares each and 3 of those 4 are shaded (they recognise the equivalence  $18/24=3/4$ ). Besides of that, in figure F they group in pairs the eight squares of the whole to represent the  $\frac{3}{4}$ . So, these students are at Level 3.

Nevertheless, this pre-service teacher was not able to propose a specific activity considering the Learning Trajectory in order to help students progress in their conceptual reasoning. For instance, this pre-service teacher proposed for the first pair of students: *“With Víctor and Xavi we would work with the recognition that the parts must be congruent. To do that, we could propose the same task but with other figures and they (students) could represent  $4/6$ ”*.

The answers of this group of pre-service teachers indicated the difficulty of making instructional decisions considering the Learning Trajectory proficiency levels.

**Group 3: Pre-service teachers who used the mathematical elements of the Learning Trajectory to recognise different levels of students' fractional reasoning, and proposed new activities to help students progress in their fractional reasoning taking into account the learning trajectory proficiency levels**

Pre-service teachers of this group, after using the mathematical elements (the parts of the whole must be congruent and a part can be divided in other parts) to recognise different levels of students' fractional reasoning, proposed new activities focused on helping students progress in their fractional

reasoning according to the learning trajectory proficiency levels. For example the pre-service teacher E25 proposed the next objective and activity to help Victor and Xavi progress in their fractional reasoning:

Objective: In order to progress from Level 1 to Level 2, students have to recognise that the parts of a whole must be congruent (although they could be different in form).

Activity: Represent in the following figure (square)  $\frac{2}{4}$  in three different ways

This group of pre-service teachers used their knowledge of the Learning Trajectory to interpret students' fractional reasoning, and proposed new activities to help students develop their fractional reasoning.

## Discussion and conclusions

The aim of this research was to analyse how pre-service teachers' learning of a Learning Trajectory of fractional reasoning supports their development of noticing students' fractional reasoning. We focus on how pre-service teachers interpret student' fractional reasoning and respond with instructional actions using a learning trajectory of fractional reasoning.

Twenty out of thirty-one pre-service teachers who participated in the task were able to use the mathematical elements to interpret students' fractional reasoning considering the characteristics of the students learning progression of fractional reasoning and identifying different levels of students reasoning. This result indicates that the information about a Learning Trajectory of a particular mathematic topic can be used by pre-service teachers to begin to notice features of students' mathematical thinking in a particular domain and therefore, to develop the skill of noticing. The Learning Trajectory can be seen as a powerful tool that help pre-service teachers focus their attention on important mathematical aspects of the problem, on the students' mathematical reasoning and on making instructional decisions on the basis of students' mathematical reasoning. The other eleven pre-service teachers had difficulties in using the Learning Trajectory to interpret students' answers. This result is in line with other studies that have shown that interpreting students' mathematical reasoning is a challenging task for some pre-service teachers (Llinares, Fernández, & Sánchez-Matamoros, 2016; Sánchez-Matamoros, Fernández, & Llinares, 2015).

However, only nine out of these twenty pre-service teachers could use their interpretations of students' fractional reasoning to propose new activities according to the Learning Trajectory in order to help students progress in their fractional reasoning. Previous research has pointed out that the skill of making instructional decisions is the most difficult one to develop in teacher education programs (Callejo & Zapatera, 2016; Ivars & Fernández, 2016; Llinares, Fernández, & Sánchez-Matamoros, 2016; Sánchez-Matamoros, Fernández, & Llinares, 2015). Nevertheless, approximately one third of the participants, in our task, were able to design an activity to promote students' progressions of fractional reasoning according to the Learning Trajectory. Therefore, we think that the task of our study, designed according to a Learning Trajectory, seems to have a relevant paper in the development of the skill of providing activities that could help students progress in their learning. The Learning Trajectory could be seen as a referent or guide for pre-service teachers that could help them to link the mathematical domain (mathematical elements), the student's reasoning and the instruction that considers students' learning progressions.



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