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► **To cite this version:**

Alain Kuzniak, Assia Nechache. On dialectic and dynamic links between the Mathematical Working Space model and practice in the teaching and learning of mathematics. CERME 10, Feb 2017, Dublin, Ireland. hal-01948867

HAL Id: hal-01948867

<https://hal.science/hal-01948867>

Submitted on 8 Dec 2018

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On dialectic and dynamic links between the Mathematical Working Space model and practice in the teaching and learning of mathematics

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In this communication, we address the specific relationships between the Mathematical Working Space model (MWS model) and practice in the teaching and learning of mathematics. The strong and positive interactions existing between these two aspects are illustrated with two examples from geometry and probability teaching. They show how some theoretical constructs as MWS diagram can enlighten practice and, conversely, how studies on practice nourish the model with new tools such as “comics”, “complete mathematical work” or “emblematic tasks”.

Keywords: Mathematical work, MWS model, emblematic task, mathematics teaching.

A decade ago, the Mathematical Working Space (MWS) model has been introduced as a theoretical and methodological framework dedicated to identify and shape the mathematical work in schooling. Developed by researchers working collaboratively in various countries with, sometimes, very different educational approaches in Europe (France, Spain, Cyprus), Latin America (Chile, Mexico...) and North America (Canada), the model is deeply rooted in the teaching of mathematics in real classrooms. This communication aims at showing dynamics and dialectics between the MWS model and relevant questions to education practice. After a short presentation of the MWS model, we show first how it can be used to deal with the question of planning series of tasks in the teaching and learning of geometry. That leads us to introduce two new constructs: the methodological tool of “comics” used to describe the evolution and circulation of the mathematical work and the more theoretical idea of “complete mathematical work” which allows qualifying the final nature of this circulation within the diagram. Then, these two new tools are used for the study of the teaching of probability and statistics. In France, this teaching is relatively new and it is now initiated on modeling tasks with use of technological tools. This leads us to check the relevance of the above constructs and identify two kinds of incompleteness of mathematical work and, in addition, to draw out certain specific tasks, named “emblematic tasks”. Designed on these former results, our present research aims at tracking transformations made by teachers when they adapt these “emblematic tasks” to their classrooms.

A short insight in the MWS model

Extending the research work developed by Houdement and Kuzniak (2006) in didactics of geometry, the MWS model emerged during the last decade. The model, especially in geometry, had already been presented during former CERME meetings (Kuzniak and Nechache, 2015). A recent issue of ZDM-Mathematics Education (48-6, 2016) is devoted to this model and we refer the reader to this issue for further details and discussions about the MWS model. Some elements of the introduction (Kuzniak, Tanguay and Elia, 2016) to this issue are used to present the framework.

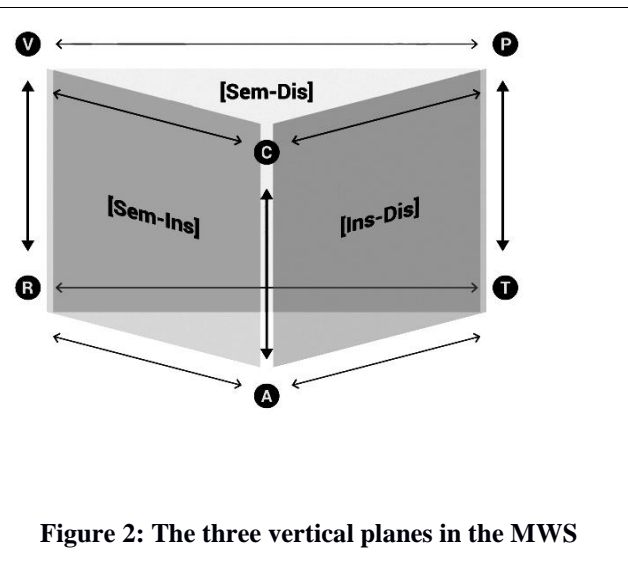
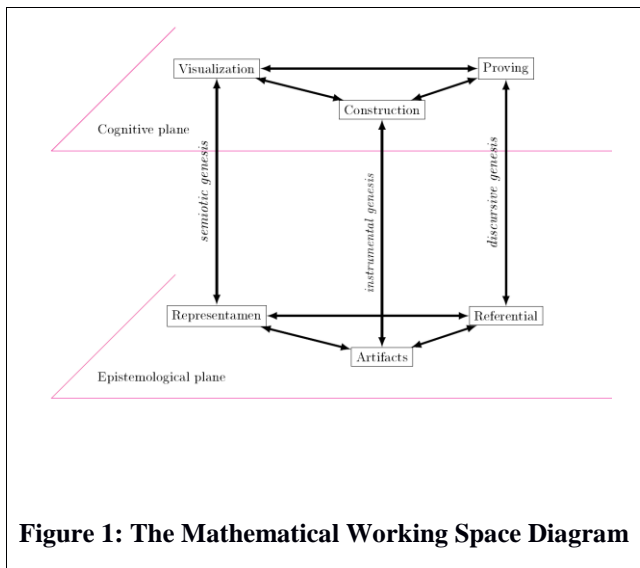
The theoretical model of Mathematical Working Space (MWS) provides a tool for the specific study of the mathematical work in which students and teachers are effectively engaged during mathematics sessions. The abstract space thus conceived refers to a structure organized in a way that allows the mathematical activity of individuals who are facing mathematical problems. It establishes the reference to the complex setting in which the problem solver acts. In this approach, the crucial function of educational institutions and teachers is to develop a rich environment which enables students to properly solve mathematics problems. To describe the specific activity of students solving problems in mathematics, the idea of organizing the MWS into two articulated planes is retained: one of an epistemological nature in close relation to the mathematical content in the field being studied; the other of a cognitive nature, related to the thinking of the individual solving problems. Three components in interaction are characterized for the purpose of describing the work in its epistemological dimension, organized according to purely mathematical criteria: a set of concrete and tangible objects, the term *sign* or *representamen*¹ is used to summarize this component; a set of *artifacts* such as drawing instruments or software; a *theoretical system of reference* based on definitions, properties and theorems.

The second level of the MWS model is centered on the subject, considered as a cognitive subject. In close relation to the components of the epistemological level, three cognitive components are introduced as follows: *visualization* related to deciphering and interpreting signs, and to internally building (psychological) representation of the involved objects and relations; *construction* depending on the used artifacts and the associated techniques; *proving* conveyed through processes producing validations, and based on the theoretical frame of reference. Furthermore, the development by communities or an individual, whether generic or not, of appropriate mathematical work is a gradual process by which a suitable MWS is settled through a progressive approach and fine tuning. Therefore, analyzing mathematical work through the lens of MWSs allows tracking down how meaning is progressively constructed, as a process of bridging the epistemological plane and the cognitive plane, in accordance with different specific yet intertwined genetic developments, each being identified as a genesis related to a specific dimension in the model: semiotic, instrumental and discursive geneses. This set of relationships can be described proceeding from the elements of the following diagram (Figure 1) which, in addition, shows the interactions between the two levels with three different dimensions or geneses: semiotic, instrumental, and discursive:

- The Semiotic genesis is the process associated with representamen (or signifiers), and accounts for the dialectical relationship between the syntactic and the semantic perspectives on mathematical objects, displayed and organized through semiotic systems of representation.
- The Instrumental genesis enables making artifacts operational in the construction processes contributing to the achievement of mathematical work.
- The Discursive genesis of proof is the process by which the properties and results organized in the theoretical reference system are being actuated in order to be available for mathematical reasoning and discursive validations.

The epistemological and cognitive planes structure the MWS into two levels and help to understand the circulation of knowledge within mathematical work. How then, proceeding from here, may one

articulate efficiently the epistemological and cognitive levels in order to make possible the expected mathematical work? How may one organize and describe interrelationships existing between our former three geneses? In order to understand this complex process, the interactions that are specific to the execution of given mathematical tasks will be associated to the three vertical planes, naturally occurring in the diagram of Figure 1: the [Sem-Dis] plane, conjoining the semiotic genesis and the discursive genesis of proof, the [Ins-Dis] plane, conjoining the instrumental genesis and the discursive genesis of proof, the [Sem-Ins] plane, conjoining the semiotic genesis and the instrumental genesis (Figure 2). The three planes are valuable tools for describing the interrelationships between the different geneses, for identifying and characterizing phases in the solving processes, for analyzing the shifts occurring in the course of these processes when specific aspects are, unexpectedly or gradually, either left aside or given more prominence.



The exact definition and precise description of the nature and dynamics between these planes during the solving of a series of mathematical problems remains a central concern for a deeper understanding of the MWS model. They vary with the mathematical field at issue, with the tasks, with the schooling level, with the type of work promoted or expected, etc.

Planning of a teaching sequence in geometry at primary school

In France, at primary school level, numerous and interesting tasks in geometry are available and relatively easy to access. By contrast, few resources are available to help teachers to plan a series of geometrical tasks and activities for elementary schools students. To move forward on this issue, the MWS model (Kuzniak & Nechache, 2015) was used to identify some key points in organizing a long teaching sequence on a specific topic. Designed by two well-known French researchers in the domain (Fenichel & Taveau, 2009), the selected sequence “Le cercle sans tourner en rond” is dedicated to Grade 4-6 students. The sequence includes eight sessions from half an hour to one hour. Its main objectives are the introduction of the global notion of circle as the set of all points equidistant from a given point, named the center; to use this property for solving distance problems and make constructions with compass used also to transfer distances. The MWS diagram was used to analyse each of the sessions and to observe various circulations of the geometrical work through the different planes of the MWS diagram (Figure 2). For example, in session 1, the objective is to identify the circle as the set of all points equidistant from a given point, the centre. Students are

asked to place a point A on a white sheet and then a point B (semiotic dimension). After that, they have to place 15 points “situated at a distance from A which is the same as the distance of B from A” (semiotic dimension). They may use various artifacts: blank and tracing paper, twine, square set, compass (instrumental dimension). The geometric work starts in the [Sem-Ins] plane. Then, during a formulation phase, some students’ productions are displayed on the blackboard and discussed. The strategies used by the students to carry out the task are clarified and formulated. The notion of equidistance from a given point is expected to emerge. Some geometric terms are institutionalized and the characteristic property of the circle is given by the teacher and enriches the theoretical referential (discursive dimension) in the MWS. In summary, the geometric work starts in the [Sem-Ins] plane and is concluded in the discursive dimension (Dis). The same analysis has been made on five sessions and allows describing the dynamic evolution of geometric work. This evolution is visualised with the following “comics” which highlight the key-points of the sequence.

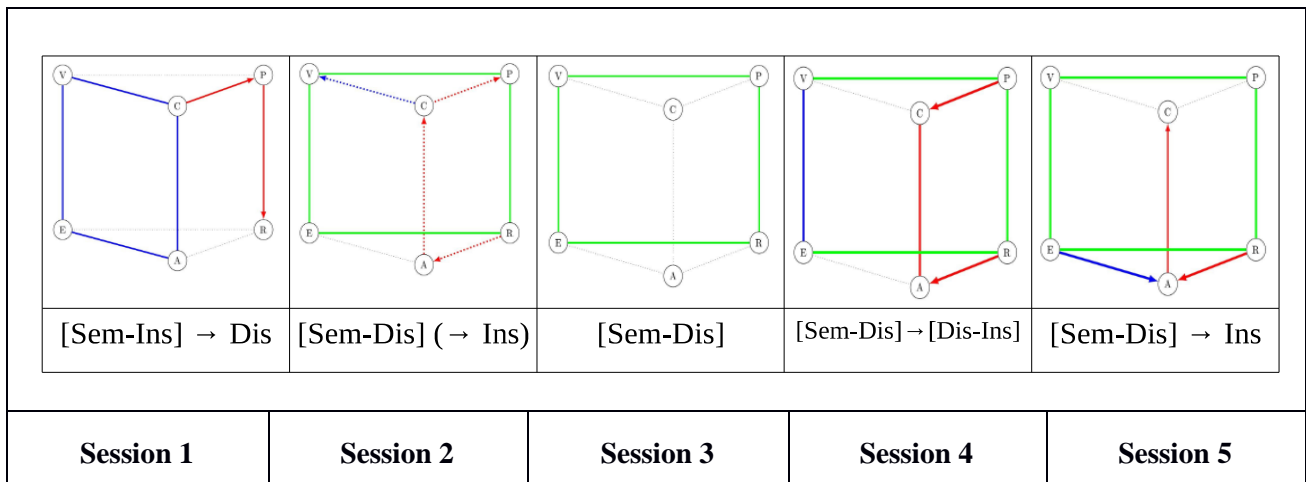


Figure 3: The dynamic evolution of the mathematical work during the five sessions

The analysis, supported on “comics”, demonstrates a comprehensive circulation through the three vertical planes of the MWS model (figure 2) leading to what we identify as a “complete geometric work”. More generally, a mathematical work is considered “complete” when both conditions (A) and (B) are satisfied:

(A) A genuine relationship between the epistemological and cognitive planes. This aspect means that students, be they generic or not, are able to select the useful tools to deal with a problem and then to use them appropriately as instruments to solve the given task.

(B) An articulation of a rich diversity between the different geneses and vertical planes of the model. This aspect means that various dimensions of the work related to tools, techniques and properties are taken into account.

Identifying blockages and misunderstandings and checking if the mathematical work is complete and coherent

Identifying blockages and misunderstandings requires observing how teachers implement tasks in their classroom. That allows us to describe what we call *suitable* MWS which depends on the institution involved, and is defined according to the way the knowledge must be taught, in relation to its specific place and function within the institutional curriculum.

Identifying blockages and misunderstandings through the study of circulation within the MWS diagram

Our analysis is based on a classroom session at Grade 10 (age 15) (Kuzniak, Nechache & Drouhard, 2016) in which a task is given to the students with two questions on the probability values of an event. The statement of the exercise is written as follows in the textbook used by the teacher:

Two identical wallets are at disposal. The first contains 3 banknotes of 10 euro and 5 banknotes of 20 euro. The second contains 2 banknotes of 10 euro and 4 banknotes of 20 euro. One wallet is chosen randomly and a banknote is drawn “blindly” from this wallet. What is the probability of choosing one banknote of 10 euro? One banknote of 20 euro?

The underlying probabilistic model is that of equal probability. This model is not explicit, but the text makes reference to it with the following terms: identical, randomly, blindly. Moreover, this exercise involves a random experiment with two successive and not independent draws. The use of a weighted tree to solve the problem would be the most effective way to solve the problem. But, this particular type of tree only appears officially in Grade 12, the introduction of this kind of tree is something that is left for teachers to do. In the textbook, weighted trees are introduced before the exercise which is not the case in the observed class.

After some time left to search for a solution, a student is invited by the teacher to write his answer on the blackboard. He draws a non-weighted tree semiotic dimension to represent the situation and then gives his answer in the form of a fraction (Figure 4). The student gives numerical results without any justification and the tree is not only used for representing the situation but also as an implicit support for calculation instrumental dimension. His mathematical work starts in the semiotic dimension, which allows him to convert the problem into the form of a tree, the latter being then used to get the solution of the given problem. The student has performed his work in the [Sem-Ins] plane.

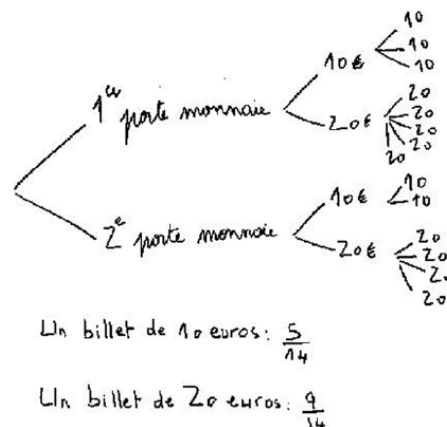


Figure 4: The student's tree on blackboard

Unsatisfied with the student's solution, the teacher asks him to explain his answer, and in particular, to explain the two results written on the blackboard (namely $\frac{5}{14}$ and $\frac{9}{14}$). Arguments given by the student are uniquely grounded on the semiotic dimension and the teacher is expecting one based on the discursive dimension, using properties. Then, asking various questions to the whole classroom, he attempts to shift the mathematical work to discursive dimension in order to develop a discursive proof of the results. The teacher emphasises strongly the importance of justification based on tools coming from the theoretical system of reference and this focus prevents him to notice the non-validity of the results provided by the student (the right results are $\frac{17}{48}$ and $\frac{31}{48}$). In fact, the mistake is linked to the student's insufficient knowledge about the nature and use of the tool “tree”. The student draws a choice tree which allows counting the outcomes, but which is not a

weighted tree. At this Grade, the teacher avoids the use of probability trees which are spontaneously used by his students. The mathematical work done by students remains in the [Sem-Ins] plane while the teacher confines it in the discursive dimension to promote a discursive proof. Thus, this leads to a misunderstanding and blockages among some students which can be related to the two different forms of mathematical work expected to solve the task.

Mathematical work: Completeness and mathematical coherency

The following example is based on the analysis of a class session at Grade 9 (Kuzniak, Nechache & Drouhard, 2016) in which students are asked to solve the following task taken from Education Ministry resources:

On a segment S, two points A and B are taken randomly. The following outcome is considered “The length of segment [AB] is strictly superior to half the length of segment S”. What is the probability of this event?

The event “The length of segment [AB] is strictly superior to half the length of segment S” is labelled D. The solution suggested into the resource document is divided in two parts. In the first part, the reasoning work starts with an visual exploration on the segment (semiotic dimension) which is closely related to the use of an artefact (here a spreadsheet) for calculating numbers randomly with the *random* function (instrumental dimension). So, the mathematical work begins in the plane [Sem-Ins]. Then, based on the results given by the artefacts, an estimated value, closed to 0.25, is given and the estimation process is justified with the law of large numbers. The work done in this phase ends in the plane [Ins-Dis].

In the second part, the exact value (0.25) is justified with a discursive proof. It is first suggested to find all the couples (X;Y) such that $|X-Y| > 1/2$, where X and Y are two random variables with a continuous uniform distribution on the interval [0 ;1] (use of the theoretical referential). The inequation is solved graphically (Figure 5) on the square $[0 ;1] \times [0 ;1]$ (semiotic use of the square). Thus, the suitable couples (X ; Y) belong to the gray zone (Figure 5), hence the probability of the event D is equal to $1/4$ (based on visualisation). The mathematical work, really implemented, is placed in the [Sem-Dis] plane.

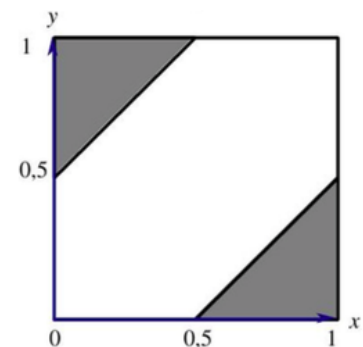


Figure 5: Geometric solution

In summary, the analysis, with the MWS model of the solution given by authors, of the resource document, serves to identify a circulation of the mathematical work through the three vertical planes of the diagram (Figure 2). Thus, a priori, the mathematical work can be regarded as potentially complete and mathematically coherent.

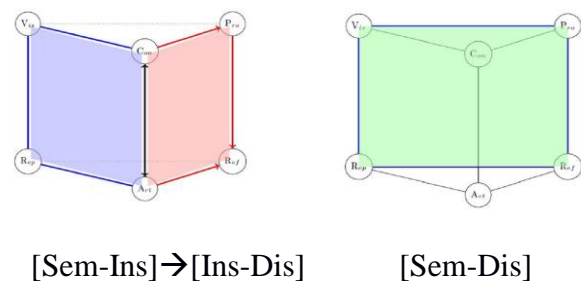


Figure 6: The evolution of the mathematical work

In the session we observed, the *suitable* MWS implemented by the teacher, and, thus, the resulting mathematical work, is really different from the potential one described above. The teacher asks the students to realize the random experiment first. They have to draw a segment with a given length, place two points randomly on this segment and, measure the distance between this two points and, compare the measure to half the length of segment S . Then, the teacher engages students to use a discrete model of the experience with throws of two six-sided dice to get an experimental value of the probability of D and, they get 0,3. Finally, the teacher gives the students a table (6×6) with 36 cells to complete and asks them to calculate the probability of D , which is equal to $1/3$.

In figurative terms, we can say that each phase favors one of the MWS vertical planes (Figure 2) moving from the [Sem-Ins] plane to help the understanding of the random experiment, to the [Ins-Dis] plane to obtain an experimental value of the probability, and finally to the [Sem-Dis] plane to give a theoretical validation based on counting numbers. In summary, the mathematical work proposed by the teacher provides an articulation between the various working contexts and can be considered complete. But, the probability of D in the model chosen by the teacher is $1/3$ and is different from that expected in the official resource, which is $1/4$. This difference is due to the fact that the teacher wants to adapt the task to his classroom and changes the initial task by using a discrete model instead of a continuous model. This difference highlights the contradiction between the reference MWS expected by the authors of the resource document and the suitable MWS developed by the teacher. The consequence is that the mathematical work is not mathematically coherent according the expectations of the reference MWS, at this level, even if the mathematical work can be considered complete.

On mutual influence of theory and practice on the MWS development

In this paper, we intend to show how analysis of tasks and teaching-learning sessions can benefit from and participate in the development of the MWS model. Is it possible to generalize our results to other theoretical approaches? We cannot assert, because the MWS model is still an emerging and growing model that is difficult to compare with mature theories. As Artigue (2016) underlines, one of the current characteristics of the model is precisely its plasticity and adaptability that, according her, big and mature French theories do not have. Moreover, conceived to describe and ensure the dynamics of mathematical work, the MWS model cannot be improved without a close and dialectic link with researches on tasks and activities favoring the tuning of the mathematical work.

Research perspective: Teaching trajectory and mathematical work

In the previous section, we have shown how, in some cases, teachers have transformed tasks in such a way that students have been blocked or engaged in mathematical work far from the intended one. In our present research and using the MWS model, we address the following questions: When do some blockages arise in the mathematical work? How can they be characterized? What is their origin? Which kind of teachers' adaptations and changes allows keeping (or not) a complete and mathematically coherent mathematical work? The research objective is to identify misunderstandings or resistance points or, instead, favorable rebounds which allow that an activity goes on nicely in the classroom. It is also possible to focus on tasks transformations leading to

denaturing when the intended mathematical objective is lost and questions of reproducibility and didactic obsolescence can be addressed.

To do this, some specific tasks, named “emblematic tasks” and verifying several conditions, are chosen. They must benefit first from an institutional recognition which ensures their compatibility with the intended mathematical work. Then, they are already provided by textbooks and, above all, implemented in some regular classrooms. Lastly, they may convey a complete mathematical work as defined above. We make the assumption that adequate and solid learning can result from the implementation of these tasks in classrooms if they are not too distorted through the teaching process. To study this assumption, these emblematic tasks are first implemented in pre-service teachers training by experienced teachers trainers and their transformations by the pre-service teachers are studied. The teachers training framework helps us to monitor the development and implementation of tasks in classrooms and makes easier the study of teaching trajectories according our research objectives. Moreover, two other specific objectives related to teacher training can be added to our research program: the use of “emblematic tasks” may initiate students to new and interesting forms of mathematical work for those who are not familiar with; the assessment of the impact of this approach on students' belief by analyzing the different transformations and adaptations of the tasks. In a way, emblematic tasks can help to understand the link between teaching and learning.

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