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Combining two theories in the design for learning about integrals

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We show how a combination of two theories, Abstraction in Context and Proceptual Thinking, served as basis for design decisions in the framework of a research study about learning the integral concept in high school via constructing knowledge about accumulation.

Keywords: Design of instruction, Abstraction in Context, proceptual thinking, integration.

Introduction

Designing learning units involves decisions about the transition from syllabus to curriculum (Dreyfus, Hershkowitz, & Bruckheimer, 1987); Mathematics Education offers home-grown theories supporting this transition; curriculum designers use these theories when design is a main goal.

Here, we present a case study of using theories for the design of a didactical tool – a learning unit intended for students' construction of a specific mathematical notion. This study is part of a larger research project (Kouropatov, 2016) offering a didactic approach supporting high school students in acquiring a conceptual understanding of the integral. For this purpose, we asked ourselves: a) What does "a conceptual understanding of the integral" mean? and b) How can we support students in acquiring such understanding? Answering question (b) required design. The design was preceded by a thorough didactical-mathematical analysis of approaching integration via the idea of accumulation (Thompson & Silverman, 2008). A tight interrelationship developed between the design process and the relevant theories, Abstraction in Context and Proceptual Thinking. In this paper, we exhibit this relationship via the need for and effect of these theories in the design process and hence the contribution of these theories to a basis for the transition from syllabus to curriculum.

Theoretical frameworks and their influence on the design

We follow Tabach, Hershkowitz, Arcavi, and Dreyfus (2008) in distinguishing

- a pre-design stage involving considerations before starting the actual development and research work with students;
- an initial design-research-redesign stage of sporadic and isolated activities, and the observation, data collection, and analysis of their implementation with a few students;
- a final stage that comprises further redesign, for the creation of a coherent, complete task-based curriculum, and its implementation (limited, in our case, to four pairs of students).

These stages indicate what to do but not how to do it. Design decisions require theories. For our study, we have adopted Abstraction in Context (AiC) because its roots in constructivism and in activity theory make it suitable to design for and analyse the construction of abstract knowledge during the learning process (Hershkowitz, Schwarz, & Dreyfus, 2001). We have also adopted the theory of Proceptual Thinking (PT) because of the proceptual nature of the mathematical notions in focus, in particular accumulation (Gray & Tall, 1994). The theories are compatible but different: PT deals with how students see mathematics; AiC deals with how students acquire knowledge.

Theoretical considerations at the pre-design stage

AiC as basis for design

AiC takes abstraction to be a learner's activity of vertical (in the sense of Freudenthal) reorganization of previous mathematical constructs in order to arrive at a new (to the learner) construct. Abstraction leads from an initial, vague first form, which may lack consistency, to an elaborated form (Davydov, 1990). The activity is interpreted in terms of epistemic actions performed by a learner, or a group of learners, for a specific purpose, in a particular context. The context includes the social setting as well as the learners' personal background, in particular the previous mathematical constructs resulting from previous processes of abstraction. 'Reorganization' includes establishing new connections between previous constructs, making mathematical generalizations, and discovering new strategies for solving problems. 'Vertical' implies building a new level of abstraction on top of a previous level. For the researcher, the question arises how to support and unveil the processes by which the students' new constructs may emerge as a vertical reorganization of previous constructs in the current context. AiC argues that for this purpose, it is crucial to carry out an a priori analysis. This theoretical content analysis aims at identifying the elements of knowledge (mathematical facts, notions, claims, strategies, representations, etc.) that together constitute 'learning a concept' and have a didactical perspective, namely can be constructed by learners in a suitable context using appropriate didactical tools. The results of the a priori analysis are *descriptions* of the elements of knowledge that belong to the world of mathematics but may be linked to the current context, and operational *definitions* that constitute descriptions of observable student behaviour: utterances or actions that provide criteria for assessing whether a student's constructing action corresponding to the said knowledge element has occurred.

AiC has originally been proposed in the framework of a curriculum development project, in which abstraction was a central concern, and Hershkowitz et al. (2001) have already then expressed the hope that it will be useful not only for analysing students' processes of abstraction but also for designing sequences of activities supporting students in such processes. At the pre-design stage, AiC requires a sequence of activities, each intended toward the construction of an appropriate element of knowledge, while the sequence is hierarchically structured as imposed by verticality. In other words, AiC helps us find a structure of the subject knowledge that is appropriate for implementation in the design. For example, based on the a priori analysis of the mathematical content, we decided that the unit should be designed as a four level vertical structure of the following conceptual components: *Approximation* in the context of given geometrical objects; *Accumulation Value* (the definite integral) in the context of given analytical objects or situations; *Accumulation Function* (the definite integral with varying upper bound); and *Integration-Differentiation interplay*, mainly the Fundamental Theorem of Calculus (FTC). Further analysis of each level provided the vertical structure of the elements of knowledge intended to be constructed by the students. We present two of the four levels in some detail.

For didactical reasons and based on verticality, *Approximation* was interpreted as Geometrical Shapes Approximation (GSA) with the following three knowledge elements:

AP_G "General approximation": The size of a given object can be approximated by replacing the given object with known objects;

AP_R "Refined approximation": The approximation can be made more precise by decreasing the size of the replacing objects and increasing their number;

AP_L "Approximation limit": The size of a given object can be determined as precisely as one wants by continued refinement.

The corresponding operational definitions are that we will say students have constructed

AP_G if they explicitly (verbally and/or graphically) replace a given object with known objects;

AP_R if they explicitly (verbally and/or graphically) refine the approximation by decreasing the size of the replacing objects and increasing their number;

AP_L if they explicitly (verbally and/or graphically) identify a value as the exact size of a given object by continued refinement.

Knowledge elements for the other three conceptual components were similarly described and defined (Kouropatov, 2016; Kouropatov & Dreyfus, 2014). These descriptions and definitions constitute the framework for the design of activities in the learning unit. In other words, by means of the description of the elements of knowledge, AiC informs the decisions of what should be designed and in what hierarchy it should be organized. AiC does not inform how to design for each notion by means of micro-tasks. For that purpose we used PT.

PT as basis for content design

Gray and Tall (1994) defined the notion of procept as an aggregate of three things: process, concept (or object), and symbol. For example, the symbol $\int_0^x f(t)dt$ is meant to evoke both the process of

accumulation (integration) and the concept of accumulation function (integral), with the cognitive combination of all three, process, concept, and symbol, being called a procept. This stance has crucial didactical implications: students might first meet a process; later, a symbol is introduced for that process and/or its product, and this symbol takes on the dual meaning of the process and the object created by the process. Proceptual Thinking is then defined as the ability to switch one's focus between these dual roles of the symbols as is useful and efficient in the current context, for example solving a problem. Someone who has the ability to think in this way may be described as versatile (Tall & Thomas, 1991). Versatility includes a global picture of a concept as well as the ability to break it down into a process, seeing each stage as part of the whole concept. According to Hong and Thomas (1998) versatility is critical for comprehension of the integral concept.

We see the integral as a multilevel, hierarchic procept, which is composed of (in the sense of AiC, and hence intended to be constructed by students from) other procepts including function, graph, approximation, sum, and accumulation; hence, we continue the pre-design stage by

- using the result of the above a priori analysis in order to identify and describe the main sub-procepts of the integral procept;
- identifying the hierarchic structure of the integral as an aggregate of procepts.

The main didactical flow of ideas was derived directly from the procept hierarchy of the mathematical notion of the concept of the integral. In particular, the didactical goals are: to create an opportunity for the learners *to carry out a process* that is meaningful for them (e.g., to

approximate an unknown area of some shape by accumulating the known areas of small parts of this shape); to give the learner the possibility *to internalize this process as a concept* (e.g., by quantifying the process, by discussing the characteristics of this process); *to introduce* the learner to the common mathematical *symbol* as encapsulation of the completed process and the internalized object; these considerations became the leading considerations of the initial stage of design. In other words, PT allows us to answer the question of how the learning activities should be designed.

We present two examples from the learning unit that show how we took into account the proceptual nature of the intended elements. The first one, is the initial activity for introducing *Approximation* via GSA. And the second one is from the middle of the unit, and is intended to lead students to constructing the concept of *Accumulation Function*. As mentioned above, these two concepts, together with the *Accumulation Value* and the *FTC*, are the four components of the vertical structure of the suggested design of the unit.

Regarding GSA, students *carried out the process* of approximating the length of a given (sketched) curve (interval, semicircle, non-standard curves) using a ruler, compass, protractor, square paper (with two different mesh sizes) and calculator. Then, students discussed the "quality" of the resulting approximation and were asked to refine it (for example by using more sophisticated measurements) with the intention to lead to *internalization of this process as a concept*. Finally, students were asked to *find the length* as precisely as possible (the existence of such a value was taken as intuitively obvious). This "process, concept, existence" triad constitutes the GSA procept according to the above analysis.

Regarding the notion of *accumulation function*, the activities offered students opportunities to *carry out the process* of co-variational change of the accumulation value according to the value of the right end-point of a certain sub-interval (using approximation or algebraic considerations); students dealt with a table and/or graph and/or verbal representation of this change with the intention to lead to *internalization of this process as a concept*; finally, the symbol $A(x) = \int_a^x f(t)dt$ was introduced.

This "process, concept, symbol" triad constitutes the accumulation function procept.

Theoretical considerations at the initial stage of design

The influence of AiC and PT on the design of the unit could, in principle, best be demonstrated by the design of activities about the procept of accumulation, the central notion of the learning unit. Because of space limitations, we concentrate instead on a small part of this: When describing a process of accumulation, one should know "how to start accumulating" - in other words, how to calculate an initial quantity. Then, one should know how to calculate further pieces of the accumulating quantity. The general answer to this problem is *approximation*. Here, the proceptual nature of approximation is particularly important: We see approximation as a process, and the result of this process, of calculating (as accurately as required) some unknown value (length, area or volume) by using known values.

While approximation refers to many kinds of quantity, verticality suggests a sequence of activities that starts with concrete geometrical shapes (lines, 2-dimensional and 3-dimensional shapes) followed by geometrical shapes that are given analytically (using elementary functions) in a coordinate plane (space); only then, more general quantities, given analytically, are considered.

Such a sequence allows students to construct their knowledge, starting in a concrete context of geometrical drawings and bodies that is intuitively clear to them, and where all quantities (i.e. length, area, volume) have positive values. This context requires relatively little previous knowledge and allows for rather linear and smooth vertical reorganization. Next, follows a more formal context of analytically given objects or situations (all quantities still having positive values). And finally, students are asked to deal with general quantities. Practically, approximation may be made by measurement, by geometrical consideration (with known formulas), or by algebraic considerations (analysing some algebraic term). In light of these considerations we have designed a sequence of activities, which we now, following an a priori analysis, interpret as focusing on (i) GSA with its three elements of knowledge ‘General Approximation’ (replacing the given object by known objects), ‘Refined Approximation’, and ‘Approximation Limit’; and (ii) parallel elements of knowledge for ‘Analytical Shape Approximation’.

Similar considerations apply to the concept of *accumulation function*. We see the accumulation function as a process of change (e.g. the change of accumulating area beneath the graph of the function while "the upper bound is moving") and the product corresponding to this process (e.g., a graph of this process demonstrating the ability to characterize it). This approach to the accumulation function immediately leads to the following conclusions: for constructing the accumulation function element of knowledge, students should know (even if only intuitively) that if we change the upper end-point of some sub-interval of the function domain, the appropriate value of the given function and the accumulation value of the given function will also change; and they should know how to characterize the process of the changing of the accumulation value. In light of these considerations, we have designed a sequence of activities, focusing on the *Accumulation Function* element of knowledge via its component elements (not specified in this paper).

On the basis of the above we claim that the combination of AiC and PT allows us to make decisions regarding the design of activities for the learning unit.

Theoretical considerations at the final stage of design

As a result of the two previous stages (pre-design and initial design) we developed a sequence of activities that were organized according to the above four component vertical structure. Each of its four components constitutes a hierarchical procept that is vertically composed of sub-procepts. We interpret this whole structure as the procept of Integral.

At the final stage of the design we analysed the developed activities with the purpose of avoiding inconsistent usage of terms, symbols, and visual representations. Another important issue we took into account at this stage was adaptation of the unit to students' previous knowledge. Thus, for example, at the previous stages we had used the number e for some of tasks. We recognized that this notion is not familiar to the students, so at the final stage of the design certain activities have been changed (e.g., by using π instead of e).

The implementation of the unit was organized in the form of learning sessions of pairs of students with in the presence of a researcher. The time interval between the sessions was typically between one and two weeks. We considered that for every part of the unit, students need some introductory and some summarizing activities. These activities aimed to provide a smooth flow of the learning

process and were developed (in the form of a short discussion that was led by researcher) at the final stage of design.

As a result of the final stage, we created a task-based curriculum unit introducing the concept of integral via the idea of accumulation with a fair measure of internal coherence. This unit has been implemented with four pairs of students.

AiC as a tool for design evaluation

An essential component of AiC is the nested epistemic actions model for describing and analysing, at the micro-level, processes of abstraction by which learners construct new knowledge. The model uses the three epistemic actions of Recognizing (previous constructs as relevant in the present situation, R), Building-with (the recognized constructs to achieve a local goal, B), and Constructing (assemble and integrate previous constructs so that a new construct emerges by vertical mathematization, C). In processes of abstraction, R-actions are nested in B-actions, and R and B-actions are nested in C-actions.

Following Dreyfus, Hershkowitz and Schwarz (2015) the core of the method is the analysis, utterance by utterance, of transcripts to identify R, B and C epistemic actions as building blocks of abstraction. The RBC methodology helps making processes of knowledge constructing observable. This claim is based on empirical results regarding many content areas including integration (Kouropatov & Dreyfus, 2014).

RBC analysis of the learning sessions has been successfully used for evaluating the design of the activities by identifying problems with the implementation; this evaluation has uncovered instances where the design (or micro-design) of activities or their sequencing needed to be improved. We present two examples.

The first example concerns the concept of approximation limit (AP_L) referred to above. The RBC-analysis of the performance of one pair of students (A and B) supplied empirical evidence about students' constructing processes of AP_G and of AP_R but not of AP_L , which is a crucial component of approximation. Therefore, the design of the activity for the following groups of students has been refined in a way that supports the constructing process of AP_L . The elaboration consisted of adding questions leading the students to intuitively distinguish between overestimates (decreasing to the exact value) and underestimates (increasing to the exact value) of the approximated value. The revision was successful in the sense that all following student pairs succeeded in constructing AP_L .

The second example relates to the issue of lacking previous constructs assumed by the design. For example, when constructing the concept of approximation via GSA, students M and N demonstrated a lack of previous constructs such as identifying coordinates of points on a graph, or calculating lengths of segments. For example, in the activity of finding the length of a quarter-circle, the students quickly recognized the relevance of approximation. They replaced the curve with a set of chords but then got stuck because they didn't know how to calculate the chord-lengths. The idea of choosing the segment endpoints according to some division of the given interval was new to them and outside their current grasp. The teacher's intervention was needed and was locally helpful. So, we can argue that there was a need for an additional element of knowledge that our design did not take into account: division of the given interval creating an appropriate division of the graph.

Conclusions and further questions

We presented a case study of using theories for design decisions; this case dealt with learning the integral concept in high school via constructing knowledge about accumulation. The theories were most significant but not the only resource for decision making. The decisions were inspired by theory (e.g., in the case of verticality of the structure of the elements of the intended knowledge), by practical experience (e.g., in the case of assumption regarding intuitive accessibility of some elements of knowledge for students), or by both (e.g., in the case of building the system of sub-procepts of the procept of the integral or in the case of designing the sequence of learning activities). However, we argue that in the process of designing the learning unit on integrals for high school students, the theories have been interwoven and have played crucial roles in the process of development and implementation of the unit at all stages: the pre-design, the initial, and the final stages, as well as for fine-tuning the design after its evaluation.

The theories that we adopted for the purpose of the design are Abstraction in Context (AiC) and Proceptual Thinking (PT). These theories were adopted on two levels: AiC on a cognitive-epistemological level with the purpose of coming to design decisions regarding the nature and the structure of knowledge to be learned (at a macro-level, which seems to be efficient in a more general context); PT on a didactic-implementation level with the purpose of coming to design decisions regarding how to help learners to achieve this knowledge (at a micro-level, which seems to be efficient in the context of mathematics procepts). The role of the theories differed from stage to stage: AiC was more essential at the pre-design stage while PT was more fruitful at the initial stage. However, the synergy of the theories was more influential than their diversity: Our design aims at supporting students in constructing proceptual knowledge of the Integral that we interpret as a hierarchical procept that is vertically composed of sub-procepts. Our research allows us to follow how students acquire a proceptual view while they construct their knowledge. We speculate that students' behaviour that is coherent with the suggested operational definitions (in terms of AiC) can be interpreted as evidence for the acquisition of a proceptual view (in terms of PT).

Following the research, we find ourselves in a better position to pose two relevant yet unsolved problems, a practical and a theoretical one. The practical problem is how to optimally profit from theory when designing instruction. The findings of the research show that the AiC and PT frameworks can be used for development and evaluation of an important instructional instrument – a learning unit. What about other instructional instruments, such as homework assignments, tests, and so on? Could we also use these theories for the design of such instruments? Additional research and experiments are needed in order to suggest the adaptation of the discussed theories for these types of instrument. The theoretical problem concerns the consistency of theories: Could we have used other theoretical frameworks instead or in addition to AiC and PT, what consistency issues would have arisen, and how different a design would have resulted?

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