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► **To cite this version:**

Martin A Simon. Learning Through Activity: A developing integrated theory of mathematics learning and teaching. CERME 10, Feb 2017, Dublin, Ireland. hal-01948860

HAL Id: hal-01948860

<https://hal.science/hal-01948860>

Submitted on 8 Dec 2018

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Learning Through Activity: A developing integrated theory of mathematics learning and teaching

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Sociocultural theory (Vygotsky, 1978) has left the process of internalization relatively unexplored. In the Learning Through Activity (LTA) research program, we use basic constructs of constructivism to address this issue. The goal of our empirical and theoretical work has been to elaborate an integrated theory of mathematics learning and teaching. Towards this end and grounded in our empirical research, we have engaged in explicating reflective abstraction for mathematical concepts and developing a design approach for fostering reflective abstraction. The LTA approach is complimentary to a problem-solving approach; learning engendered by the LTA approach is not dependent on the uncertainty inherent in solving authentic problems.

Keywords: Constructivism, learning theories, task design, mathematical concepts, learning trajectories.

In Simon (2009), I argued that different theories of learning can be thought of as different tools for research affording different kinds of work. It is the job of mathematics education researchers/theorists to not only develop and articulate theories, but to specify the work for which they are designed. Such specification allows fruitful discussion about the relationship of different theories and the possibilities of particular multi-theoretical approaches. In this paper, I describe a developing theory and the work for which it is intended.

Background

In the Learning Through Activity (LTA) research program, we use multiple theories. In particular, we use sociocultural theories to think about cultural factors and the role of artifacts, social theories to think about the norms negotiated in situations of learning and teaching, and cognitive (constructivist) theories to think about the development of particular concepts. The latter has been our primary theoretical tool and the theoretical area to which we have been contributing.

Sociocultural theory views all knowledge as socially constructed. Knowledge development proceeds from a social level to an individual level through a process termed “internalization” (Vygotsky, 1978). Bereiter (1985) wrote, “How does internalization take place? It is evident from Luria’s first-hand account (1979) of Vygotsky and his group that they recognized this as a problem yet to be solved (p. 206). My colleagues and I see constructivism as a theory that has the potential to explicate internalization.

Constructivism, particularly the work of Piaget (1985), is a major theory of learning and has been the basis of important research on the learning of mathematics (Steffe & Kieren, 1994). However, Piaget’s work has not had a comparable effect on mathematics pedagogy.¹ DiSessa and Cobb (2004) observed, “Piaget’s theory is powerful and continues to be an important source of insight. However, it was not developed with the intention of informing design and is inadequate, by itself, to

¹ I use “pedagogy” to refer to all contributions to instruction, instructional design, instructional planning, and teaching.

do so deeply and effectively” (p. 81). We believe that it is Piaget’s later work on reflective abstraction, rather than his earlier work on equilibration, that has the potential to be the basis for pedagogical theory development.

The goal of our empirical and theoretical work is to elaborate an integrated theory of mathematics learning and teaching. This involves articulating a theory of conceptual learning that is useful for orienting mathematics pedagogy and building on that theory to explicate mathematics instructional design and teaching. Towards this end and grounded in our empirical research, we have engaged in explicating reflective abstraction for mathematical concepts and developing a design approach for fostering reflective abstraction. This empirical and theoretical work has focused on the learning and teaching of mathematical concepts (as opposed to problem solving or other areas of mathematics learning), and on the generation of hypothetical learning trajectories (Simon, 1995), including the design of task sequences.

The term “mathematical concept” is an underspecified construct. Because it is central to our empirical and theoretical work, I have characterized the construct for research and development purposes (Simon, 2017). I refer to one aspect of that characterization here:

A mathematical concept is a researcher’s articulation of intended or inferred student knowledge of the logical necessity involved in a particular mathematical relationship.

Elaborating reflective abstraction

Outline of the theory²

One challenge that we accepted in explaining the development of new mathematical concepts was that the explanation must account for building more advanced concepts from prior concepts. Thus, we endeavored to describe a recursive structure in which the result of conceptual development at one level serves as a building block of a concept at the next level. Piaget (1980, p. 90) described reflective abstraction as a “coordination of actions.” We built on this idea in the following way.

1. We specify a concept as a complex of a goal and an action (represented as G_n-A_n) constructed through reflective abstraction. We represent the prior concepts of the learner as G_0-A_0 and the concept whose development we are attempting to explain as G_1-A_1 .
2. The learning process begins with the learner setting a new goal (generally in response to a mathematical task) and calling on a sequence of *available* actions to achieve that goal. This sequence of available actions is what we call an activity, represented as $(A_{0a} \rightarrow A_{0b} \rightarrow A_{0c})$. An activity is the precursor to a new concept.
3. The actions that are part of the activity do not exist in isolation. Each is part of an existing concept (e.g., $G_{0a}-A_{0a}$), and each is called upon, because the goal of that existing concept (e.g., G_{0a}) is a subgoal of the activity used to solve the task. Thus, the set of actions that make up the activity are part of a set of concepts that are activated to achieve the goal (solution of the task). Thus, whereas Piaget defined reflective abstraction as a coordination

² This description has been abbreviated. In particular, there is no discussion of the stages of concept development. For a full treatment of reflective abstraction and discussion of the stages, see Simon, Placa and Avitzur (2016).

of actions, we assert that this coordination takes place in the context of a coordination of concepts. This point is important, because it allows us to explain how concepts build on concepts recursively.

4. The coordination of actions results in a new, higher-level action linked to a goal (G_1-A_1). Reflective abstraction results in a learned anticipation. That is, the learner can now solve the task without going through the sequence of actions that was originally necessary, but rather by enacting the higher-level action. (This will be demonstrated in the example.)

Example from data

The following example is taken from Simon, Placa and Avitzur (2016). The data derived from a single-subject teaching experiment focusing on learning fraction concepts. Kylie was 10 years old at the time of the study. “R” refers to the researcher, Simon, and “K” to Kylie.

In this example, Kylie is developing an abstraction of recursive partitioning (i.e., a unit fraction of a unit fraction). Hackenberg and Tillema (2009) defined recursive partitioning as “partitioning a partition in service of a non-partitioning goal, such as determining the size of $1/3$ of $1/5$ of one yard in relation to the whole yard” (p. 2).

• **Task 4.1:** [Using JavaBars, R draws a bar on the screen.] *This is one third of a unit. Make a bar that is one sixth of a unit.* Kylie made it clear that she did not know how to just “cut up” the bar on the screen. She made the whole by iterating the third three times and then cut the first third in half. She indicated immediately that one of the small pieces is one sixth.

• **Task 4.2:** *This is one fifth of a unit. Make one tenth of a unit.* Kylie used the same process. She iterated the one fifth 5 times to make the whole and then partitioned the first fifth into two subparts. She reported, “Here, you have one tenth of a unit.”

• **Task 4.3:** *This is one third of a unit. Make one ninth of a unit.* This time Kylie immediately divided the one-third bar into three pieces (without iterating to make the whole).

K: One of those is one ninth.

R: How do you know?

K: Because, um. How many times does three go into nine? . . . Three times. And it’s one third! So. Three times three is nine, and one of—if you cut it up into thirds again. That is, um. . . . And you take one, it would be . . . one third. . . . But that’s really one ninth of a unit.

Kylie seemed to indicate that she thought about what number of parts would iterate three times to the whole. She therefore knew that one third of the one third would iterate nine times in the whole.

• **Task 4.4:** *This is one fifth of a unit. Make one twentieth of a unit.* She immediately cut the fifth into four pieces. She went on to complete two more tasks in this way. (pp. 77-78)

In this example, Kylie developed an abstraction that taking $1/m$ of $1/n$ creates $1/mn$, that is, a part that when iterated mn times recreates the unit. The example illustrates several aspects of the theory discussed above. Initially, Kylie had no way to think about making $1/mn$ by simply partitioning $1/n$. However, she did have knowledge that allowed her to make $1/mn$. That is, she had concepts that she

was able to call on producing a sequence of actions (an activity) to achieve her goal. She conceptualized $1/n$ of a unit as a part that can be iterated n times to make the unit. She also knew that she could partition the unit to make any unit fraction. In Task 4.1, she called on these two concepts. She sequentially iterated the original part, $1/3$, three times to make the unit and then partitioned the unit to make $1/6$ of a unit. However, because the unit bar that she created was already partitioned into three parts, she called on her concept of partitive division (6 divided by 3) to determine how many times to partition each of the three parts. Thus, Kylie created an activity made up of three actions involving three extant concepts: iterating the part to make the unit, using partitive division to determine the number of subparts per part, partitioning a subpart.

The activity Kylie employed for Tasks 4.1 and 4.2 led to the abstraction that was apparent in Task 4.3 and beyond. In Task 4.3, Kylie no longer needed to go through the sequence of actions used in the preceding tasks. The actions that made up the activity were now coordinated into a single higher-level action. She knew immediately in Task 4.4 that cutting $1/5$ into 4 subparts allows a subpart to iterate 5×4 times to the whole.³ That is, she had developed an anticipation that the denominator of the part has a multiplicative effect on the number of times the subpart iterates to the unit.

Building a pedagogical theory: the LTA instructional approach

As stated our goal was to generate a theory of mathematics concept learning that can serve as a basis for mathematics pedagogy. In this section, I describe how we have built an instructional design approach on the basis of the explication of reflective abstraction, discussed above.

The first two steps in our design of instructional sequences are part of various design approaches. In Step 1, we specify the prior knowledge needed to engage with the sequence. This is particularly important in our design approach, because it identifies the concepts that students can call upon as components of their activity. In Step 2, we identify specific learning goals for the students, that is, we articulate the particular abstractions we intend to promote.

Step 3 makes direct use of our explication of reflective abstraction. In this step, we specify an activity (sequence of concepts/actions) available to the (actual or hypothetical) students that could serve as the raw material for the intended abstraction. There are two requirements here. First, the students must be able to call on the activity. Second, the researchers/designers must be able to describe how the students could come to the abstraction as a result of engaging in the activity. In our example above, the activity would be iterating the part to make the unit and then partitioning the unit by subdividing each part – the number of partitions determined through partitive division.

Step 4 involves generating a sequence of tasks designed to elicit the activity specified in Step 3 (in our example Tasks 4.1-4.4) and promote reflective abstraction. Sometimes the tasks that promote the activity are sufficient as in the example presented (by the third task, Kylie had made the abstraction). In some cases, subsequent tasks are created that restrict the student's ability to carry out the sequence of actions in the activity – prompting the students to use developing anticipations

³ Although Kylie's justification was given for Task 4.3, I refer to the numbers from Task 4.4, because in Task 4.3, the use of $1/3$ as both the fraction of the unit and the fraction of the part makes articulation of the ideas confusing.

if available. For example, in our work on promoting a reinvention/abstraction of the multiplication of fractions algorithm, Kylie had developed a reasoned activity beginning with thinking through the effect of the denominator of the multiplier on the denominator of the multiplicand. Her reasoning then included the numerator of the multiplicand and finally the numerator of the multiplier. Each step was dependent on the prior one. To promote and elicit use of a developing anticipation, we gave her tasks with the denominators hidden and asked for the numerator of the product. She was only able to do these tasks when she had developed sufficient anticipation of the effect of the numerator of the multiplicand in the context of her activity. In other cases, particular tasks are sequenced to increase the probability that students will attend to the commonality in their activity. In Simon et al (2010), Erin was reinventing/abstracting a common-denominator algorithm for multiplication of fractions. She had developed diagram solutions to division tasks whose dividend and divisor had common denominators. She had also developed the ability to talk through a diagram solution (without drawing). For example, she was able to talk through $37/31 \div 17/31$. However, she also made it clear that without talking through the solution, she could not come up with the quotient. At this point, I gave her consecutively two tasks with the same pair of numerators, but different common denominators (e.g., $7/167 \div 2/167$ and $7/103 \div 2/103$). Although she needed to talk through the first, she was able to give the answer immediately to the second. Not only that, she was able to elegantly explain the abstraction she had made and do subsequent tasks (involving common denominators) simply by dividing the numerators.

I have highlighted the first four steps. However, as in other approaches to instruction, these steps might be followed by symbolization, introduction of vocabulary, and institutionalization of ideas.⁴

A couple of clarifications are in order. First, when we refer to a *task*, it includes the resources available to the students for solving it. Second, the goal of our research is to specify a sequence of tasks that can promote a particular abstraction. Thus, the sequence should work without the instructor asking leading questions, telling or showing solutions, or giving hints or suggestions. Also, the sequence should allow students to make the abstraction without needing to hear the solutions of others. This does not mean that there is not a role for teachers. The teacher is important in developing norms for the mathematical work, promoting justification at appropriate times, introducing symbols and vocabulary, and leading discussions that institutionalize the learned abstraction. Also, teachers should be able to monitor student progress and modify task sequences in response to student progress.

Affordances of the LTA instructional approach

To provide a context for discussing the affordances of the LTA instructional approach, I first discuss a commonly used and important approach to instruction, a problem-solving approach. I will then highlight some of the contrasts and complementarities between the LTA approach and a problem-solving approach.⁵

⁴ See also Simon (2016).

⁵ Discussion of contrast with Harel's DNR can be found in Simon (2013).

Although there is no single problem-solving approach, I will discuss some typical features. One of the main strengths of a problem-solving approach is the engagement of students in the critical activity of mathematical problem solving, attacking a problem for which the student has no solution at the outset. I will not highlight here the abilities and dispositions that can be developed through regular engagement in problem solving; these have been well documented. Rather, I will focus on one feature of this approach that provides a contrast with the LTA approach. Problem solving is by definition uncertain. There is no assurance that those who engage in solving an authentic problem will succeed in solving it. When a diverse set of students in a mathematics class attempt to solve a problem, it is likely that only those who are the stronger problem solvers and who have the more powerful mathematical concepts will succeed in solving the problem. The other students must try to *follow* the reasoning (in small groups or whole class discussions) of their more able peers.

The LTA approach is intended to provide a complimentary approach, one in which learning of a concept is not dependent on the uncertainty inherent in attempting to solve authentic problems. If an LTA sequence is designed effectively, students should be successful in solving every task in the sequence.⁶ In the LTA approach the learning (the new abstraction) is not the ability to solve the task. Rather it is the insight that is gleaned through the students' solutions to tasks using available activity. In our example, Kylie was successful in solving each of the four tasks. She was not trying to learn anything – just to solve the tasks presented. However, by the third task, she understood something that she had not understood at the beginning of the instructional sequence.

We conceive of the LTA instructional approach as a technology for engendering the construction of particular mathematical concepts on the basis of particular prior knowledge. I call attention to two potential contributions of this approach:

1. For concepts that tend to be difficult to teach and learn, the LTA approach provides a technology for building up those understandings (promoting particular abstractions).
2. For students who have previously been unsuccessful in learning mathematical concepts, it provides a specific methodology for building up their conceptual foundation.

Affordances of the LTA theory for research and development

The elaboration of reflective abstraction discussed above provides a lens for looking at conceptual learning in different situations, not just in situations designed using the LTA instructional approach. For example, the LTA elaboration of reflective abstraction could be used to understand conceptual learning in the context of a problem-solving approach to instruction. How do we explain the success or failure of a lesson for particular students? Of course, the students' prior knowledge and problem solving skills are important. But how can we consider the usefulness of the problem or problems? The LTA elaboration of reflective abstraction allows for analysis of the students' activity and its relationship to the abstraction that they make.

⁶ Of course, there is no curriculum that works flawlessly for all students. The issue is not whether we can create a sequence in which every student can solve every task. Rather the issue is that we intentionally create tasks that we predict students will be able to solve. This is in contrast to putting them in a problem-solving situation.

In Simon (1995), I postulated the construct of a *hypothetical learning trajectory (HLT)*. An HLT can describe a hypothesized trajectory for a single lesson or for a sequence of concepts in a conceptual area (also referred to as a “learning progression”). Learning trajectories has become a hot area for mathematics education researchers. HLTs are not just a series of conceptual steps through which learners progress, they involve articulation of sequences of learning situations and hypotheses of how these situations will be used by the students to learn the target concepts.

The LTA integrated theory of teaching and learning can provide the framework for learning trajectories in various conceptual areas. As a framework, it provides a basis for selecting and sequencing tasks and for hypothesizing how the students will abstract from their activity in working with those tasks. In our current project, we designed, enacted, and modified in teaching experiments nine trajectories for different concepts involving fractions.⁷ This work has been grounded in and contributed to the LTA theory of teaching and learning. Also essential to the design and modification of the trajectories has been (but beyond the scope of this short paper) our work on reversibility (Simon, Kara, Placa, & Sandir, 2016) and on the stages of concept development (Simon, Placa, & Avitzur, 2016). Both build on the theory described in this paper.

Acknowledgment

This research was supported by the National Science Foundation (DRL-1020154). The opinions expressed do not necessarily reflect the views of the foundation.

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⁷ I am currently planning the subsequent project in which we will use these trajectories, which were developed primarily in single subject teaching experiments, as the bases for creating and studying curricular interventions for whole-class lessons.

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