



HAL
open science

Enumeration of hypermaps of a given genus

Alain Giorgetti, Timothy R.S. Walsh

► **To cite this version:**

Alain Giorgetti, Timothy R.S. Walsh. Enumeration of hypermaps of a given genus. *Ars Mathematica Contemporanea*, 2018, 15 (1), pp.225-266. 10.26493/1855-3974.1115.90f. hal-01948796

HAL Id: hal-01948796

<https://hal.science/hal-01948796>

Submitted on 8 Dec 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Enumeration of hypermaps of a given genus*

Alain Giorgetti †

*FEMTO-ST Institute, Univ. Bourgogne Franche-Comté, CNRS
16 route de Gray, 25030 Besançon cedex, France*

Timothy R. S. Walsh

*Department of Computer Science, University of Quebec in Montreal (UQAM)
P.O. Box 8888, Station A, Montreal, Quebec, Canada, HC3-3P8*

Received 23 May 2016, accepted 17 October 2017, published online 20 June 2018

Abstract

This paper addresses the enumeration of rooted and unrooted hypermaps of a given genus. For rooted hypermaps the enumeration method consists of considering the more general family of multirooted hypermaps, in which darts other than the root dart are distinguished. We give functional equations for the generating series counting multirooted hypermaps of a given genus by number of darts, vertices, edges, faces and the degrees of the vertices containing the distinguished darts. We solve these equations to get parametric expressions of the generating functions of rooted hypermaps of low genus. We also count unrooted hypermaps of given genus by number of darts, vertices, hyperedges and faces.

Keywords: Enumeration, surface, genus, rooted hypermap, unrooted hypermap.

Math. Subj. Class.: 05C30, 05A15

1 Introduction

A (*combinatorial*) hypermap is a triple (D, R, L) where D is a finite set of *darts* and R and L are permutations on D such that the group $\langle R, L \rangle$ generated by R and L acts transitively on D . A (*combinatorial ordinary*) map is a hypermap (D, R, L) whose permutation L is a fixed-point-free involution on D . For a hypermap (resp. map) the orbits of R , L and

*The authors wish to thank Alexander Mednykh, Roman Nedela and the referees for helpful suggestions to improve the presentation of this article.

†For this work Alain Giorgetti was supported by the French “Investissements d’Avenir” program, project ISITE-BFC (contract ANR-15-IDEX-03).

E-mail addresses: alain.giorgetti@femto-st.fr (Alain Giorgetti), walsh.timothy@uqam.ca (Timothy R. S. Walsh)

RL (L followed by R) are respectively called *vertices*, *hyperedges* (resp. *edges*) and *faces*. The *degree* of a vertex, edge, hyperedge or face is the number of darts it contains. The equivalence of combinatorial maps and topological maps having been established in [14], we use the word “map” to mean “combinatorial map” throughout this paper. The *genus* g of a map is given by the Euler-Poincaré formula [7]

$$v - e + f = 2(1 - g), \quad (1.1)$$

where v is the number of vertices, e is the number of edges and f is the number of faces. The genus of a hypermap with t darts, v vertices, e hyperedges and f faces was defined in [13] by the formula

$$v + e + f = t + 2(1 - g). \quad (1.2)$$

An *isomorphism* between two maps or hypermaps (D, R, L) and (D', R', L') is a bijection from D onto D' that takes R into R' and L into L' ; it corresponds to an orientation-preserving homeomorphism between two topological maps. A *sensed* hypermap (resp. map) is an isomorphism class of hypermaps (resp. maps). We admit the existence of a unique hypermap (resp. map) with an empty set of darts D , called the *empty* hypermap (resp. map). For both of these objects $v = f = 1$ and $g = e = 0$. A *rooted* hypermap (resp. map) is either the empty hypermap (resp. map) or a tuple (D, x, R, L) where (D, R, L) is a non-empty combinatorial hypermap (resp. map) and $x \in D$ is a distinguished dart, called the *root*.

The enumeration of maps and hypermaps has several non-trivial applications. One such application is based on the correspondence between hypermaps and algebraic curves established by the Belyi theorem [16]. For instance, the formula for the number of plane trees was used by A. Zvonkin in the computer generation of Shabat polynomials of bounded degree [16]. Another area where the map enumeration plays an important role is theoretical physics, in particular in 2-dimensional gravitation models. Roughly speaking, map enumeration is used to compute matrix integrals determining the properties of gravitational fields (see for instance the works of B. Eynard [9]). Some hypermaps have been shown to be related to contextuality in quantum physics [21]. Also, A. Mednykh and R. Nedela have applied the enumeration of rooted (resp. unrooted) hypermaps to the enumeration of subgroups (resp. conjugacy classes of subgroups) of the triangle group with three generators x, y, z and the relation $xyz = 1$ [20].

We enumerate rooted hypermaps of a given genus by number of darts, vertices, hyperedges and faces. To do so we consider more general families of rooted hypermaps and bipartite maps, in which other vertices or darts than the root dart are distinguished. We also use the genus-preserving bijection between hypermaps and 2-vertex-coloured bipartite maps presented in [23]. But since bipartite maps have all their **faces** of even degree and we're using the degrees of the **vertices** as parameters, we must instead study the face-vertex dual of a 2-coloured bipartite map, that is, a map whose faces are coloured in two colours (white and black) so that no two faces that share an edge have the same colour. All these maps are *Eulerian* – that is, all their vertices are of even degree – but not all Eulerian maps are 2-face-colourable. For example, the map on the torus with one vertex, one face and two edges is Eulerian because its only vertex is of degree 4, but its face cannot be coloured because it shares both edges with itself. Therefore we call the maps we are studying *face-bipartite*.

A *sequenced (rooted) map* is a rooted map with some vertices other than the *root vertex* (the vertex that contains the root) distinguished from each other and from all the other

vertices. The labels that distinguish these vertices can be taken to be $1, 2, \dots, k$, where k is the number of distinguished vertices. A *sequenced (rooted) hypermap* is defined similarly. We state (in Section 4) a bijective decomposition for the set $\mathcal{H}(g, t, f, e, n, D)$ of sequenced orientable hypermaps of genus g with t darts, f faces and e hyperedges, with the root vertex of degree n and with the sequence of degrees of the distinguished vertices equal to $D = (d_1, d_2, \dots, d_{|D|})$, where d_i is the degree of the distinguished vertex with label i . We obtain a bijective decomposition of the set $\mathcal{F}(g, e, w, b, n, D)$ of sequenced orientable face-bipartite maps of genus g with e edges, w white faces, b black faces, with the root face of degree $2n$ and with the sequence of half-degrees of the distinguished vertices equal to D . Then we apply face-vertex duality to obtain a bijective decomposition of the corresponding set of 2-coloured bipartite maps with distinguished faces. Next we use the bijection in [23] to obtain a bijective decomposition for hypermaps with distinguished faces, and finally we again apply face-vertex duality to obtain a bijective decomposition of $\mathcal{H}(g, t, f, e, n, D)$.

A *multirooted hypermap* is a hypermap in which a non-empty sequence of darts with pairwise distinct initial vertices is distinguished. We relate multirooted hypermaps to sequenced hypermaps and thus obtain a recurrence for the number of multirooted hypermaps and functional equations for the generating series counting multirooted hypermaps of a given genus by number of darts, vertices, edges, faces and the degrees of the initial vertices of the distinguished darts.

The paper is organized as follows. Section 2 fixes some notations, recalls a known decomposition for sequenced rooted maps and describes the bijection between hypermaps and bipartite maps presented in [23]. Sections 3 and 4 respectively enumerate sequenced face-bipartite maps and sequenced rooted hypermaps of a given genus. In Section 5 we consider multirooted hypermaps and we give equations for the generating functions that count these objects. In Section 6 we give functional equations relating the generating functions for rooted hypermaps with that for multirooted hypermaps. Then we show how to solve these equations. In Section 7 we obtain parametric expressions for the generating functions that count rooted hypermaps with a given small positive genus. Section 8 presents enumeration algorithms for sensed unrooted hypermaps counted by number of darts, vertices and hyperedges. Appendix A (resp. B) contains a table for numbers of rooted (resp. unrooted) hypermaps of genus g with d darts, v vertices and e hyperedges for $d \leq 14$.

2 Background

2.1 Notations

We first introduce the notations and conventions we use throughout the paper. Let D and D' be two lists of integers. The inclusion $D' \subseteq D$ means that D' is a sublist of D . In this case $D - D'$ is the complementary sublist of D' in D . For instance, the sublists of $D = [1, 1, 2]$ are the empty list $[], [1]$ (twice), $[2], [1, 1], [1, 2]$ (twice) and D itself. Their complementary sublists in the same order are $D, [1, 2]$ (twice), $[1, 1], [2], [1]$ (twice) and $[], [1, 2]$ (twice), $[1, 1], [2], [1]$ (twice) and $[], [1, 2]$ (twice). We denote by $D.D'$ the concatenation of the lists D and D' . If i is an integer and D is a list of integers, then $i.D$ is a shortcut for $[i].D$. For $1 \leq j \leq |D|$ we denote by d_j the j -th element of the list D of length $|D|$ and by $D - \{d_j\}$ the list obtained from D by removing its j -th element d_j . Let ρ be a positive integer. The abbreviation $D_{1.. \rho}$ denotes the list $[d_1, \dots, d_\rho]$. The abbreviation $v_{1.. \rho}^{D_{1.. \rho}}$ denotes $v_1^{d_1} \dots v_\rho^{d_\rho}$.

The sign $+$ (resp. \sum) denotes (resp. generalized) disjoint set union in the following decompositions and (resp. generalized) arithmetic sum in the following equations. By con-

vention, a disjoint set union (resp. sum) over an empty domain is equal to the empty set (resp. zero). For any logical formula φ the notation Δ_φ means the singleton set containing only the empty hypermap or map (depending on the context) and the empty set if φ is false. The notation δ_φ means 1 if φ is true and 0 if φ is false.

2.2 Bijective decomposition of the set of sequenced maps

In 1962 W. T. Tutte [22] presented a bijective decomposition of a planar map with all the vertices distinguished and a root in every vertex. In 1972 T. R. Walsh and A. B. Lehman [27] generalized this decomposition to maps of higher genus and used it to count rooted maps of a given genus by number of vertices and faces. In 1987 D. Arquès [3] used this latter decomposition to find a closed-form formula for the number of rooted maps of genus 1 by number of vertices and faces. In 1991 E. A. Bender and E. A. Canfield [4] presented a more efficient decomposition that roots only a single vertex and distinguishes only as many other vertices as necessary and used it to obtain explicit formulas for counting rooted maps of genus 2 and 3. In 1998 the first author [11] modified this decomposition and used it to obtain a bijective decomposition satisfied by the set $\mathcal{M}(g, e, f, n, D)$ of sequenced orientable maps of genus g with e edges and f faces, with the root vertex of degree n and with D the list of degrees of the distinguished vertices was obtained in [11]. Since this bijective decomposition contains an error, we present the correct bijective decomposition here, and we derive it to make the derivation more accessible than the contents of a Ph. D. thesis.

Theorem 2.1. *The set $\mathcal{M}(g, e, f, n, D)$ of sequenced orientable maps of genus g with e edges and f faces, with the root vertex of degree n and with the list D of degrees of the distinguished vertices is defined by the bijective decomposition*

$$\begin{aligned}
 \mathcal{M}(g, e, f, n, D) = & \sum_{\substack{g_1 + g_2 = g \\ e_1 + e_2 = e - 1 \\ f_1 + f_2 = f \\ n_1 + n_2 = n - 2 \\ D_1 \subseteq D}} \mathcal{M}(g_1, e_1, f_1, n_1, D_1) \times \mathcal{M}(g_2, e_2, f_2, n_2, D - D_1) \\
 & + \sum_{p=1}^{n-3} \mathcal{M}(g - 1, e - 1, f, n - 2 - p, p, D) \times \{1, \dots, p\} \tag{2.1} \\
 & + \sum_{p=n-1}^{p=2e-2} \mathcal{M}(g, e - 1, f, p, D) \\
 & + \sum_{j=1}^{|D|} \mathcal{M}(g, e - 1, f, d_j + n - 2, D - \{d_j\}) + \Delta_{(g,e,f,n,D)=(0,0,1,0,\{\})}.
 \end{aligned}$$

Proof. If a map m has at least one edge, we reduce by 1 the number of edges by the face-vertex dual of deleting the root edge. There are two cases of this operation, depending upon whether the root edge is a loop or a link, and each of these cases breaks down into two sub-cases.

Case 1: The root edge is a loop. We delete the root edge and split the root vertex into two parts, s_1 and s_2 . If r is the root, then s_1 consists of the darts $R(r), R^2(r), \dots, R^{-1}(L(r))$

and s_2 consists of the darts $R(L(r)), R^2(L(r)), \dots, R^{-1}(r)$. This case breaks down into two cases, depending upon whether or not this operation disconnects the map.

Case 1a: This operation disconnects the map into two maps, m_1 containing s_1 and m_2 containing s_2 . If m_1 has at least 1 edge, its root is $r_1 = R(r)$, and if m_2 has at least 1 edge, its root is $r_2 = R(L(r))$. Let g_1, e_1, f_1, n_1, D_1 and g_2, e_2, f_2, n_2, D_2 be the parameters of the maps m_1 and m_2 , respectively, corresponding to g, e, f, n, D . This operation reduces by 1 the total number of edges; so $e_1 + e_2 = e - 1$. It leaves unchanged the total number of faces because r and $L(r)$ simply get deleted from the cycle(s) of RL (L followed by R) containing them; so $f_1 + f_2 = f$. It increases by 1 the total number of vertices; so from Formula (1.1), which relates the genus of a map to the number of its vertices, faces and edges, it can easily be deduced that $g_1 + g_2 = g$. It decreases by 2 the total number of darts in s_1 and s_2 since r and $L(r)$, which belonged to the root vertex, get eliminated; so $n_1 + n_2 = n - 2$. Finally, D_1 can be any sublist of D and D_2 is just the complementary sublist, denoted by $D - D_1$. This operation is uniquely reversible; so the set of ordered pairs of sequenced maps obtained in this case is

$$\sum_{\substack{g_1 + g_2 = g \\ e_1 + e_2 = e - 1 \\ f_1 + f_2 = f \\ n_1 + n_2 = n - 2 \\ D_1 \subseteq D}} \mathcal{M}(g_1, e_1, f_1, n_1, D_1) \times \mathcal{M}(g_2, e_2, f_2, n_2, D - D_1), \quad (2.2)$$

where Σ means the union of disjoint sets.

Case 1b: This operation does not disconnect the map, but instead turns it into a new map m' with $e - 1$ edges and f faces and, since the number of vertices increases by 1, the genus of m' is $g - 1$, so that this case only occurs when $g \geq 1$. Neither s_1 nor s_2 can be of degree 0 (otherwise the map would be disconnected); so we can choose for m' the root $r_1 = R(r)$ belonging to s_1 . Let p be the degree of s_2 . Since the sum of the degrees of s_1 and s_2 is $n - 2$, the degree of s_1 , the root vertex, is $n - 2 - p$. We distinguish the vertex s_2 so that this operation can be reversed, and we put its degree p at the beginning of the list D , turning it into $p.D$. Now this operation is reversible in p distinct ways, since any of the p darts of s_2 can be chosen to be $R(L(r))$ when we merge the vertices s_1 and s_2 and replace the deleted root edge. Now p can be any integer from 1 up to $n - 3$ (so that $n - 2 - p \geq 1$). For both p and $n - 2 - p$ to be at least 1, n must be at least 4. The set of sequenced maps obtained in this case is

$$\sum_{p=1}^{n-3} \mathcal{M}(g - 1, e - 1, f, n - 2 - p, p.D) \times \{1, \dots, p\}. \quad (2.3)$$

Case 2: The root edge is a link. We contract the root edge, merging its two incident vertices s_1 containing the root r and s_2 containing $L(r)$ into a single vertex s with root $R(r)$. This operation decreases by 1 the number of edges and doesn't change the number of faces, since r and $L(r)$ simply get deleted from the cycle(s) containing them. Since the number of vertices is decreased by 1, the genus remains the same. This case breaks down into two sub-cases, depending upon whether or not s_2 is one of the distinguished vertices.

Case 2a: The vertex s_2 is not one of the distinguished vertices. Let p be the degree of the new vertex s . Then $p = n - 2 +$ the degree of s_2 , and since the degree of s_2 must be

at least 1, we have $p \geq n - 1$. Also, the new map has $2e - 2$ darts; so $p \leq 2e - 2$. This operation is uniquely reversible for each value of p ; so the set of maps so obtained is

$$\sum_{p=n-1}^{p=2e-2} \mathcal{M}(g, e - 1, f, p, D). \tag{2.4}$$

Case 2b: The vertex s_2 is one of the distinguished vertices. It can be any one of the $|D|$ distinguished vertices. If it is the j th distinguished vertex, then its degree is d_j . Then since it gets merged with s_1 into the new root vertex, d_j gets dropped from D . Finally, the degree of s is $d_j + n - 2$. This operation too is uniquely reversible; so the set of maps so obtained is

$$\sum_{j=1}^{|D|} \mathcal{M}(g, e - 1, f, d_j + n - 2, D - \{d_j\}). \tag{2.5}$$

Finally, suppose that m has no edges. It is of genus 0, has 1 face, its one vertex is of degree 0 and its list D is empty because it has no distinguished vertices; so it constitutes the singleton

$$\Delta_{(g,e,f,n,D)=(0,0,1,0,[])} \cdot \tag{2.6}$$

Then $\mathcal{M}(g, e, f, n, D)$ is the disjoint union of the sets given by (2.2)–(2.6). □

2.3 Bipartite maps and hypermaps

To motivate the transformation of (2.2)–(2.6) into the corresponding equations for sequenced hypermaps we briefly describe the bijection in [23] that takes a hypermap h into a 2-coloured bipartite map $m = I(h)$, its *incidence map*. The bijection I takes the darts, vertices and hyperedges of h into the edges, white vertices and black vertices of m . A root (distinguished dart) of h corresponds to a distinguished **edge** of m ; to make it correspond to a root of m we impose the condition that a root of m belongs to a white vertex. The permutation R in h corresponds to R in m acting on a dart in a white vertex and the permutation L in h corresponds to R in m acting on a dart in a black vertex. The permutation L in m doesn't correspond to any permutation in h ; rather, since it takes a dart belonging to a vertex of one colour into a dart belonging to a vertex of the opposite colour, it toggles R in m between R and L in h . A face (cycle of RL) in h corresponds to a face in m with twice the degree. To see this, we follow one application of RL in h starting with a dart d , which corresponds to an edge in m but we make it correspond to the dart d' in that edge that also belongs to a white vertex. Then the L in h takes d' first into $L(d')$, which belongs to a black vertex, and then into $RL(d')$ and the following R in h takes $RL(d')$ first into $LRL(d')$, which belongs to a white vertex, and then into $RLRL(d')$. Since the genus of a hypermap with t darts, v vertices, e hyperedges and f faces is defined by (1.2), m has the same genus as h .

Since the root of an incidence map of a rooted hypermap must belong to a white vertex, we impose the condition on a rooted 2-face-coloured face-bipartite map that the root belong to a white face and we transform (2.2)–(2.6) into the corresponding bijective decomposition for these maps.

3 Sequenced face-bipartite maps

Let $\mathcal{F}(g, e, w, b, n, D)$ be the set of sequenced orientable face-bipartite maps of genus g with e edges, w white faces, b black faces, with the root face of degree $2n$ and with the list of half-degrees of the distinguished vertices equal to D . For any dart d we denote by $f(d)$ the face containing d and we note that the face $f(R(d)) = f(L(d))$ must have the opposite colour from $f(d)$ because those two faces share the edge $\{d, L(d)\}$.

Theorem 3.1. *The set $\mathcal{F}(g, e, w, b, n, D)$ satisfies the bijective decomposition*

$$\begin{aligned}
 \mathcal{F}(g, e, w, b, n, D) = & \sum_{\substack{g_1 + g_2 = g \\ e_1 + e_2 = e - 1 \\ w_1 + b_2 = b \\ w_2 + b_1 = w \\ n_1 + n_2 = n - 1 \\ D_1 \subseteq D}} \mathcal{F}(g_1, e_1, w_1, b_1, n_1, D_1) \times \mathcal{F}(g_2, e_2, w_2, b_2, n_2, D - D_1) \\
 & + \sum_{p=1}^{n-2} \mathcal{F}(g - 1, e - 1, b, w, n - 1 - p, p, D) \times \{1, \dots, p\} \\
 & + \sum_{p=n}^{p=e-1} \mathcal{F}(g, e - 1, b, w, p, D) \\
 & + \sum_{j=1}^{|D|} \mathcal{F}(g, e - 1, b, w, d_j + n - 1, D - \{d_j\}) + \Delta_{(g,e,w,b,n,D)=(0,0,1,0,0,[])}.
 \end{aligned} \tag{3.1}$$

Proof. Case 1: The root edge is a loop. By definition, $f(r)$, where r is the root of the map m , is white, so that since $r_1 = R(r)$, $f(r_1)$ must be black. But when the loop is removed and the vertex s containing r is split, r_1 becomes a root; so $f(r_1)$ must change colour and so must all the faces of the new map m' (in case 1b) or the map m_1 containing r_1 (in case 1a). In case 1a, the other map m_2 has $r_2 = RL(r)$ as a root and $f(r_2)$ is white; so its faces stay the same colour. This implies that in case 1a $w_1 + b_2 = b$ and $w_2 + b_1 = w$, whereas in case 1b w and b switch in going from m to m' .

In case 1a, we have, as for general maps, $g_1 + g_2 = g$, $e_1 + e_2 = e - 1$ and D_1 is any subset of D , but instead of $n_1 + n_2 = n - 2$ we have $n_1 + n_2 = n - 1$ because the degrees satisfy the equation $2n_1 + 2n_2 = 2n - 2$. The analogue of formula (2.2) is thus

$$\sum_{\substack{g_1 + g_2 = g \\ e_1 + e_2 = e - 1 \\ w_1 + b_2 = b \\ w_2 + b_1 = w \\ n_1 + n_2 = n - 1 \\ D_1 \subseteq D}} \mathcal{F}(g_1, e_1, w_1, b_1, n_1, D_1) \times \mathcal{F}(g_2, e_2, w_2, b_2, n_2, D - D_1). \tag{3.2}$$

In case 1b, the reduced map m' is still of genus $g - 1$ and has $e - 1$ edges, but the degree of s_2 is now $2p$ instead of p and the degree of the new root vertex s_1 is $2(n - 1 - p)$; so the parameter $n - 2 - p$ in (2.3) changes to $n - 1 - p$. Also, $1 \leq 2p \leq 2n - 3$, but since $2p$ is even, we have $1 \leq p \leq n - 2$ instead of $1 \leq p \leq n - 3$, and the condition that $n \geq 4$

changes to $n \geq 3$. The analogue of formula (2.3) is thus

$$\sum_{p=1}^{n-2} \mathcal{F}(g-1, e-1, b, w, n-1-p, p, D) \times \{1, \dots, p\}. \tag{3.3}$$

Case 2: The root edge is a link. Since the new root $R(r)$ belongs to a black face, all the faces change colour; so b and w switch.

In case 2a, we have $2n - 1 \leq 2p \leq 2e - 2$, but since $2p$ is even, we now have $n \leq p \leq e - 1$; so the analogue of (2.4) is

$$\sum_{p=n}^{p=e-1} \mathcal{F}(g, e-1, b, w, p, D). \tag{3.4}$$

In case 2b, the degree of the new root vertex is $2d_j + 2n - 2$; so the analogue of (2.5) is

$$\sum_{j=1}^{|D|} \mathcal{F}(g, e-1, b, w, d_j + n - 1, D - \{d_j\}). \tag{3.5}$$

Finally, the map with no edges has one white face and no black ones; so the analogue of (2.6) is

$$\Delta_{(g,e,w,b,n,D)=(0,0,1,0,0,[])} \tag{3.6}$$

□

After deriving this bijective decomposition, we became aware of the article [8], which presents a similar bijective decomposition but for multi-rooted face-bipartite maps, which are like sequenced face-bipartite maps except that every distinguished vertex has a root. However, we present our derivation here for several reasons: it makes our article self-contained, we obtained it independently of [8] and our main purpose is to count hypermaps rather than face-bipartite maps. Now [8] does present a construction that converts a hypermap into a face-bipartite map. However, that construction is not proved and it is far more complicated than the one in [23], which is not cited in [8]. We also recently became aware of the article [6], which generalizes the results of [15] by computing the generating functions for edge-labelled bipartite maps on an orientable surface of genus g with an unbounded number of faces and including the degrees of these faces as parameters.

4 Sequenced rooted hypermaps

Theorem 3.1 holds for rooted 2-coloured bipartite maps with distinguished faces, where e is the number of edges, w is the number of white vertices, b is the number of black vertices, n is half the degree of the root face and D is the list of half-degrees of the distinguished faces. By the bijection described in Section 2.3, it also holds for rooted hypermaps with distinguished faces, where e is the number of darts, w is the number of vertices, b is the number of hyperedges, n is the degree of the root face and D is the list of degrees of the distinguished faces. By duality, the theorem also holds for sequenced hypermaps, where e is the number of darts, w is the number of faces, b is the number of hyperedges, n is the degree of the root vertex and D is the list of degrees of the distinguished vertices. To make the letters correspond to the objects they represent, we change \mathcal{F} to \mathcal{H} , e to t , w to f and b to e . We thus obtain the following results.

Theorem 4.1 (Bijective decomposition for sequenced hypermaps). *Let $\mathcal{H}(g, t, f, e, n, D)$ be the set of sequenced orientable hypermaps of genus g with t darts, f faces and e hyperedges, with the root vertex of degree n and with the list of degrees of the distinguished vertices equal to $D = (d_1, d_2, \dots, d_{|D|})$, where d_i is the degree of the distinguished vertex with label i . The set $\mathcal{H}(g, t, f, e, n, D)$ satisfies the bijective decomposition*

$$\begin{aligned}
 \mathcal{H}(g, t, f, e, n, D) = & \sum_{\substack{g_1 + g_2 = g \\ t_1 + t_2 = t - 1 \\ f_1 + e_2 = e \\ f_2 + e_1 = f \\ n_1 + n_2 = n - 1 \\ D_1 \subseteq D}} \mathcal{H}(g_1, t_1, f_1, e_1, n_1, D_1) \times \mathcal{H}(g_2, t_2, f_2, e_2, n_2, D - D_1) \\
 & + \sum_{p=1}^{n-2} \mathcal{H}(g - 1, t - 1, e, f, n - 1 - p, p.D) \times \{1, \dots, p\} \\
 & + \sum_{p=n}^{p=t-1} \mathcal{H}(g, t - 1, e, f, p, D) \\
 & + \sum_{j=1}^{|D|} \mathcal{H}(g, t - 1, e, f, d_j + n - 1, D - \{d_j\}) + \Delta_{(g,t,f,e,n,D)=(0,0,1,0,0,[])} \cdot
 \end{aligned} \tag{4.1}$$

Corollary 4.2 (Recurrence between numbers of sequenced hypermaps). *Let $H(g, t, f, e, n, D)$ be the number of rooted sequenced hypermaps of genus g with t darts, f faces and e hyperedges such that the root vertex is of degree n and D is the list of degrees of the distinguished vertices. Then $H(0, 0, 1, 0, 0, []) = 1$ and if $t \geq 1$, then*

$$\begin{aligned}
 H(g, t, f, e, n, D) = & \sum_{\substack{g_1 + g_2 = g \\ t_1 + t_2 = t - 1 \\ f_1 + e_2 = e \\ f_2 + e_1 = f \\ n_1 + n_2 = n - 1 \\ D_1 \subseteq D}} H(g_1, t_1, f_1, e_1, n_1, D_1) H(g_2, t_2, f_2, e_2, n_2, D - D_1) \\
 & + \delta_{n \geq 3} \delta_{g \geq 1} \sum_{p=1}^{n-2} p H(g - 1, t - 1, e, f, n - 1 - p, p.D) \\
 & + \sum_{p=n}^{p=t-1} H(g, t - 1, e, f, p, D) \\
 & + \sum_{j=1}^{|D|} H(g, t - 1, e, f, d_j + n - 1, D - \{d_j\}).
 \end{aligned} \tag{4.2}$$

5 Multirooted hypermaps

For $\rho \geq 1$ a ρ -rooted hypermap is a hypermap in which a sequence of ρ darts with pairwise distinct initial vertices is distinguished. A multirooted hypermap is a ρ -rooted hypermap

for some $\rho \geq 1$. This section addresses the enumeration of multirooted hypermaps.

Theorem 5.1 (Recurrence between numbers of multirooted hypermaps). *Let $H_m(g, t, f, e, D)$ be the number of multirooted hypermaps of genus g with t darts, f faces and e hyperedges such that D is the list of degrees of the distinguished vertices. Then $H_m(0, 0, 1, 0, []) = 1$ and if $t \geq 1$, then*

$$\begin{aligned}
 H_m(g, t, f, e, n.D) = & \sum_{\substack{g_1 + g_2 = g \\ t_1 + t_2 = t - 1 \\ f_1 + e_2 = e \\ f_2 + e_1 = f \\ n_1 + n_2 = n - 1 \\ D_1 \subseteq D}} H_m(g_1, t_1, f_1, e_1, n_1.D_1) H_m(g_2, t_2, f_2, e_2, n_2.(D - D_1)) \\
 + \delta_{n \geq 3} \delta_{g \geq 1} \sum_{p=1}^{n-2} & H_m(g - 1, t - 1, e, f, (n - 1 - p).p.D) \\
 + \sum_{p=n}^{p=t-1} & H_m(g, t - 1, e, f, p.D) \\
 + \sum_{j=1}^{|D|} & d_j H_m(g, t - 1, e, f, (d_j + n - 1).(D - \{d_j\})).
 \end{aligned} \tag{5.1}$$

Proof. A multirooted hypermap is similar to a sequenced rooted hypermap except that for each distinguished non-root vertex a dart starting from it is distinguished. If the degree of the j th distinguished vertex is d_j , then there are d_j ways of distinguishing a dart of this vertex. It follows that for each sequenced rooted hypermap, there are $\prod_{j=1}^{|D|} d_j$ multirooted hypermaps. Let $H_m(g, t, f, e, D)$ be the number of multirooted hypermaps of genus g with t darts, f faces and e hyperedges such that D is the list of degrees of the initial vertex of the distinguished darts. Then

$$H_m(g, t, f, e, n.D) = H(g, t, f, e, n, D) \prod_{j=1}^{|D|} d_j. \tag{5.2}$$

Solving (5.2) for $H(g, t, f, e, n, D)$ and substituting into (4.2) proves the theorem. \square

For $\rho \geq 1$ let

$$H_g(v_1, \dots, v_\rho, x, y, u, z) = \sum_{\substack{t \geq 0, f \geq 1, e \geq 0 \\ d_1 \geq 1, \dots, d_\rho \geq 1 \\ v = t + 2(1 - g) - e - f}} H_m(g, t, f, e, D_{1.. \rho}) v_{1.. \rho}^{D_{1.. \rho}} x^f y^e u^v z^t \tag{5.3}$$

be the generating function that counts multirooted hypermaps of genus g with ρ distinguished darts if $g \geq 0$, and 0 otherwise. For $1 \leq i \leq \rho$, the exponent d_i of the variable v_i in this series is the degree of the initial vertex of the i -th distinguished dart. The exponent f of the variable x is the number of faces, the exponent e of the variable y is the number of hyperedges, the exponent t of the variable z is the number of darts and the exponent v of the variable u is the number of vertices (v is computable from the other parameters by Formula (1.2)).

Corollary 5.2 (Functional equations for multirooted hypermaps). *For $g \geq 0$ and $\rho \geq 1$ the generating functions H_g of multirooted hypermaps of genus g are defined by the following functional equations:*

$$\begin{aligned}
 H_g(v_1, W, x, y, u, z) = & \\
 & \frac{yv_1z}{xu} \sum_{j=0}^g \sum_{X \subseteq W} H_j(v_1, X, y, x, u, z) H_{g-j}(v_1, W - X, x, y, u, z) \\
 & + \frac{v_1z}{u} H_{g-1}(v_1, v_1, W, y, x, u, z) \tag{5.4} \\
 & + \frac{v_1uz}{v_1 - 1} (H_g(v_1, W, y, x, u, z) - H_g(1, W, y, x, u, z)) \\
 & + v_1uz \sum_{j=2}^{j=\rho} v_j \frac{\partial}{\partial v_j} \left(v_j \frac{H_g(v_j, W - \{v_j\}, y, x, u, z) - H_g(v_1, W - \{v_j\}, y, x, u, z)}{v_j - v_1} \right) \\
 & + xu\delta_{g=0}\delta_{\rho=1},
 \end{aligned}$$

where $W = v_2, \dots, v_\rho$.

Proof. By summation according to (5.3) of the recurrence between numbers of multirooted hypermaps from Theorem 5.1. □

By vertex-hyperedge duality, we have

$$H_g(v_1, W, y, x, u, z) = H_g(v_1, W, x, y, u, z) + \delta_{g=0}\delta_{\rho=1}(yu - xu) \tag{5.5}$$

and thus another functional equation without x, y swaps is:

$$\begin{aligned}
 H_g(v_1, W, x, y, u, z) = & \\
 & \frac{yv_1z}{xu} \sum_{j=0}^g \sum_{X \subseteq W} \left((H_j(v_1, X, x, y, u, z) + \delta_{j=0}\delta_{|X|=0}(yu - xu)) \right. \\
 & \qquad \qquad \qquad \left. H_{g-j}(v_1, W - X, x, y, u, z) \right) \\
 & + \frac{v_1z}{u} H_{g-1}(v_1, v_1, W, x, y, u, z) \tag{5.6} \\
 & + \frac{v_1uz}{v_1 - 1} (H_g(v_1, W, x, y, u, z) - H_g(1, W, x, y, u, z)) \\
 & + v_1uz \sum_{j=2}^{j=\rho} v_j \frac{\partial}{\partial v_j} \left(v_j \frac{H_g(v_j, W - \{v_j\}, x, y, u, z) - H_g(v_1, W - \{v_j\}, x, y, u, z)}{v_j - v_1} \right) \\
 & + xu\delta_{g=0}\delta_{\rho=1}.
 \end{aligned}$$

The former equation is given here for maximal generality. However, a consequence of the genus formula (1.2) is that three variables among the four variables x, y, u and z are sufficient. In the remainder of the paper we consider the generating functions

$$H_g(v_1, W, x, y, u) = H_g(v_1, W, x, y, u, 1)$$

with one fewer variable. They are defined by the following functional equations:

$$\begin{aligned}
 H_g(v_1, W, x, y, u) = & \\
 & \frac{yv_1}{xu} \sum_{j=0}^g \sum_{X \subseteq W} (H_j(v_1, X, x, y, u) + \delta_{j,0} \delta_{|X|,0} (yu - xu)) H_{g-j}(v_1, W - X, x, y, u) \\
 & + \frac{v_1}{u} H_{g-1}(v_1, v_1, W, x, y, u) \tag{5.7} \\
 & + \frac{v_1 u}{v_1 - 1} (H_g(v_1, W, x, y, u) - H_g(1, W, x, y, u)) \\
 & + v_1 u \sum_{j=2}^{j=\rho} v_j \frac{\partial}{\partial v_j} \left(\frac{v_j H_g(v_j, W - \{v_j\}, x, y, u) - H_g(v_1, W - \{v_j\}, x, y, u)}{v_j - v_1} \right) \\
 & + xu \delta_{g=0} \delta_{\rho=1}.
 \end{aligned}$$

For $g, \rho \neq 0, 1$, after grouping in the left-hand side the terms containing $H_g(v_1, W, x, y, u)$ in (5.7), one gets

$$\begin{aligned}
 \frac{A(v_1, x, y, u)}{v_1} H_g(v_1, W, x, y, u) = & \\
 & x(1 - v_1) \sum_{j=0}^g \sum_{\substack{X \subseteq W \\ (j, X) \neq (0, \{\}) \\ (j, X) \neq (g, W)}} H_j(v_1, X, x, y, u) H_{g-j}(v_1, W - X, x, y, u) \\
 & + \frac{1 - v_1}{u} H_{g-1}(v_1, v_1, W, x, y, u) + u H_g(1, W, x, y, u) \\
 & + u T_g(v_1, W, x, y, u) \tag{5.8}
 \end{aligned}$$

with

$$A(v, x, y, u) = vu + (1 - v)(1 - yv + xv - 2vH_0(v, x, y, u)/u) \tag{5.9}$$

and

$$\begin{aligned}
 T_g(v_1, W, x, y, u) = & \\
 & (1 - v_1) \sum_{j=2}^{j=\rho} v_j \frac{\partial}{\partial v_j} \left(\frac{v_j}{v_j - v_1} \left(H_g(v_j, W - \{v_j\}, x, y, u) - H_g(v_1, W - \{v_j\}, x, y, u) \right) \right). \tag{5.10}
 \end{aligned}$$

6 Rooted hypermap generating functions

Let $h_g(v, e, f)$ be the number of rooted genus- g hypermaps with v vertices, e hyperedges and f faces. Let

$$H_g(x, y, u) = \sum_{v, e, f \geq 1} h_g(v, e, f) x^v y^e u^f \tag{6.1}$$

be the ordinary generating function for counting rooted hypermaps on the orientable surface of genus $g \geq 0$, where the exponent of variable x is the number of vertices, the exponent of variable y is the number of hyperedges, and the exponent of variable u is the number of faces.

Rooted hypermaps being 1-rooted hypermaps,

$$H_g(x, y, u) = H_g(1, x, y, u), \tag{6.2}$$

where $H_g(v_1, \dots, v_\rho, x, y, u)$ is the generating function counting ρ -rooted genus- g hypermaps defined in Section 5 for $\rho \geq 1$.

We first recall in Section 6.1 a known parametric expression of the generating function that counts rooted planar hypermaps. Then we explain in Section 6.2 how to solve the functional equation of the generating functions $H_g(x, y, u)$ that count rooted hypermaps with a given positive genus g .

6.1 Rooted planar hypermaps

The following proposition is a reformulation of [1, Theorem 3], with the correspondence $s = x$, $f = u$ and $a = y$ for variables, $\lambda = p$, $\mu = q$ and $\nu = r$ for parameters, and $H_0 = sf(1 + J)$ for generating functions.

Proposition 6.1 ([1]). *The ordinary generating function $H_0(x, y, u)$ that counts rooted planar hypermaps by number of vertices (exponent of x), hyperedges (exponent of y) and faces (exponent of u) is the unique solution of the following parametric system:*

$$H_0(x, y, u) = 1 + pqr(1 - p - q - r) \tag{6.3}$$

with

$$\begin{cases} x = p(1 - q - r) \\ u = q(1 - p - r) \\ y = r(1 - p - q). \end{cases} \tag{6.4}$$

Proof. The generating function $H_0(v, x, y, u)$ that counts rooted planar hypermaps (genus 0) by number of vertices (exponent of x), hyperedges (exponent of y), faces (exponent of u) and degree of the root vertex (exponent of v) satisfies the functional equation

$$\begin{aligned} H_0(v, x, y, u) &= \frac{yv}{xu} (H_0(v, x, y, u) + yu - xu) H_0(v, x, y, u) \\ &\quad + \frac{vu}{v-1} (H_0(v, x, y, u) - H_0(1, x, y, u)) + xu \end{aligned} \tag{6.5}$$

obtained by instantiation of (5.7) with $g = 0$, $\rho = 1$ and $v_1 = v$.

This equation can be solved by the *quadratic method* [10, page 515]. The idea is to define auxiliary functions $A(v, x, y, u)$ and $B(v, x, y, u)$ by (5.9) and

$$B(v, x, y, u) = A(v, x, y, u)^2 \tag{6.6}$$

and look for a function $V(x, y, u)$ such that

$$A(V(x, y, u), x, y, u) = 0, \tag{6.7}$$

implying that $B(V(x, y, u), x, y, u) = 0$ and $\partial_v B(v, x, y, u)|_{v=V(x, y, u)} = 0$.

We get from (6.5), (5.9) and (6.6) that

$$\begin{aligned} B(v, x, y, u) &= \\ &1 - 2yv - 2xv - 2v^3y - 2v^3x - 2v^2u + v^4y^2 - 2v^3y^2 + y^2v^2 + v^4x^2 \\ &\quad - 2v^3x^2 + x^2v^2 + v^2u^2 + 4v^3yx - 2yv^2x - 2yv^2u + 2v^3yu - 2v^4yu \\ &\quad - 2v^3xu + 2xv^2u + 4v^2x + 4v^2y + 2vu + 4v^3H_0(1, x, y, u) \\ &\quad - 4v^2H_0(1, x, y, u) - 2v + v^2. \end{aligned} \tag{6.8}$$

The constraints $B(V(x, y, u), x, y, u) = 0$ and $\partial_v B(v, x, y, u)|_{v=V(x,y,u)} = 0$ respectively are

$$\begin{aligned}
 &1 - 2yV - 2xV - 2V^3y - 2V^3x - 2V^2u + V^4y^2 - 2V^3y^2 + y^2V^2 \\
 &\quad + V^4x^2 - 2V^3x^2 + x^2V^2 + V^2u^2 + 4V^3yx - 2yV^2x - 2yV^2u \\
 &\quad + 2V^3yux - 2V^4y - 2V^3xu + 2xV^2u + 4V^2x + 4V^2y + 2Vu \\
 &\quad + 4V^3H_0(1, x, y, u) - 4V^2H_0(1, x, y, u) - 2V + V^2 = 0 \quad (6.9)
 \end{aligned}$$

and

$$\begin{aligned}
 &-2 + 8yV + 8xV + 4V^3y^2 - 6y^2V^2 + 4V^3x^2 - 6x^2V^2 - 6V^2x \\
 &- 6V^2y - 4Vu + 2y^2V + 2x^2V + 2Vu^2 - 4yVu + 4xVu - 4yVx \\
 &\quad + 12yV^2x + 6yV^2u - 8V^3yx - 6xV^2u + 2V - 2x - 2y + 2u = 0. \quad (6.10)
 \end{aligned}$$

It can be checked that both equations are satisfied by

$$V = 1/(1 - q) \tag{6.11}$$

with x, u, y and $H_0(1, x, y, u)$ related to p, q and r by (6.4) and (6.3). □

6.2 Rooted hypermaps with positive genus

The following additional notations are used in this section. Let ρ be a positive integer. Let $H_j[n_1, \dots, n_\rho]$ denote the partial derivative of the function $H_j(v_1, \dots, v_\rho, x, y, u)$ with respect to the variables v_1, \dots, v_ρ to the respective orders n_1, \dots, n_ρ , computed at $v_1 = \dots = v_\rho = V$. The abbreviation $[\rho]$ denotes the list $[2, \dots, \rho]$ if $\rho \geq 2$ and the empty list $[\]$ if $\rho = 1$. The abbreviation $N_{[\rho]}$ denotes the list $[n_2, \dots, n_\rho]$. For any sublist $X \subseteq [\rho]$ of $[\rho]$, $[\rho] - X$ denotes the sublist of the elements of $[\rho]$ that are not in X , N_X denotes the list of those n_i in $N_{[\rho]}$ such that i is in X and N_j denotes the list $[n_2, \dots, n_{j-1}, n_{j+1}, \dots, n_\rho]$.

6.2.1 Equation for rooted hypermaps and recurrence relations

The special case of Formula (5.8) for $g \geq 1, \rho = 1$ and $v_1 = V$ is the following formula:

$$\begin{aligned}
 uH_g(1, x, y, u) = \\
 (V - 1) \left(x \sum_{j=1}^{g-1} H_j(V, x, y, u)H_{g-j}(V, x, y, u) + H_{g-1}(V, V, x, y, u)/u \right)
 \end{aligned}$$

i.e.

$$uH_g(1, x, y, u) = (V - 1) \left(x \sum_{j=1}^{g-1} H_j[0]H_{g-j}[0] + H_{g-1}[0, 0]/u \right). \tag{6.12}$$

In order to derive from (6.12) a value for $H_g(1, x, y, u)$, we are looking for a value for $H_j[0], H_{g-j}[0]$ and $H_{g-1}[0, 0]$. More generally, we will derive from the following proposition a closed form for the expressions $H_g[n_1, \dots, n_\rho]$.

Proposition 6.2. For $g \geq 0$, $\rho \geq 1$ and $n_1, \dots, n_\rho \geq 0$ the function $H_g[n_1, \dots, n_\rho]$ is defined by

$$\begin{aligned} \frac{(n_1 + 1)A[1]}{V} H_g[n_1, N_{[\rho]}] = & \sum_{\substack{i+j+k=n_1+1 \\ i>0, k<n_1}} \binom{n_1 + 1}{i, j} \frac{(-1)^{j+1} j!}{V^{j+1}} A[i] H_g[k, N_{[\rho]}] \\ & + x \sum_{\substack{k+l+m=n_1+1 \\ 0 \leq j \leq g \\ X \subseteq [\rho] \\ (j, X) \neq (0, \{ \}) \\ (j, X) \neq (g, [\rho])}} \binom{n_1 + 1}{k, l} M[m] H_j[k, N_X] H_{g-j}[l, N_{[\rho]-X}] \\ & + \frac{1}{u} \sum_{i+j+k=n_1+1} \binom{n_1 + 1}{i, j} M[k] H_{g-1}[i, j, N_{[\rho]}] \\ & + u \sum_{j=2}^{\rho} \frac{(n_1 + 1)! n_j!}{(n_1 + n_j + 2)!} \left(n_j F_g[n_1 + n_j + 2, N_j] \right. \\ & \left. + \frac{V(n_j+1)}{n_1+n_j+3} F_g[n_1 + n_j + 3, N_j] \right), \end{aligned} \tag{6.13}$$

where

$$F_g(v_1, \dots, v_h, x, y, u) = L(v_1) H_g(v_1, \dots, v_h, x, y, u) \tag{6.14}$$

for $h \geq 1$, $M(v) = 1 - v$ and $L(v) = v(1 - v)$.

Proof. Equation (6.13) is obtained from Equation (5.8) as follows:

1. Partial derivation of (5.8) with respect to the variables v_1, v_2, \dots, v_ρ to the respective orders $n_1 + 1, n_2, \dots, n_\rho$.
2. Evaluation of this differential equation at $v_1 = \dots = v_\rho = V$. The function $H_g[n_1 + 1, \dots, n_\rho]$ is multiplied by $A[0]$ in the resulting equation, and $A[0]$ is known to be zero (6.7). The functions $T_g[\dots]$ are replaced by expressions with the functions $F_g[\dots]$ thanks to Lemma 6.3 below.
3. In the left-hand side of the resulting equation, isolation of the single term involving the function $H_g[n_1, \dots, n_\rho]$.

By inspection one can check that the right-hand side of (6.13) depends only on some functions $H_g[k, n_2, \dots, n_\rho]$ with $k < n_1$, some functions $H_g[n'_1, \dots, n'_{\rho'}]$ with $\rho' < \rho$ and some functions $H_j[\dots]$ for $j < g$. Thus, (6.13) is a recursive definition of the family of functions $H_g[n_1, \dots, n_\rho]$ for $g \geq 0$, $\rho \geq 1$ and $n_1, \dots, n_\rho \geq 0$. \square

The following lemma relates the partial derivatives of T_g at $v = V$ with the ones of F_g .

Lemma 6.3. For $\rho \geq 2$ and $g, n_1, \dots, n_\rho \geq 0$,

$$\begin{aligned} T_g[n_1 + 1, N_{[\rho]}] = & \sum_{j=2}^{j=\rho} \frac{(n_1 + 1)! n_j!}{(n_1 + n_j + 2)!} \left(n_j F_g[n_1 + n_j + 2, N_j] \right. \\ & \left. + \frac{V(n_j+1)}{n_1+n_j+3} F_g[n_1 + n_j + 3, N_j] \right). \end{aligned} \tag{6.15}$$

Proof. We can easily prove that

$$\frac{\partial}{\partial v_j} \left[\frac{(v_j - v_1)H_g(v_1, [\rho] - \{v_j\}, x, y, u)}{v_j - v_1} \right] = 0. \tag{6.16}$$

Then, $T_g(v_1, \dots, v_\rho, x, y, u)$ equals

$$\sum_{j=2}^{j=\rho} v_j \frac{\partial}{\partial v_j} \left((v_j - v_1)^{-1} \left(v_j(1 - v_1)H_g(v_j, [\rho] - \{v_j\}, x, y, u) - v_1(1 - v_1)H_g(v_1, [\rho] - \{v_j\}, x, y, u) \right) \right). \tag{6.17}$$

It also holds that

$$\frac{\partial^{n_1+1}}{\partial v_1^{n_1+1}} \left[\frac{v_j(v_j - v_1)H_g(v_j, [\rho] - \{v_j\}, x, y, u)}{v_j - v_1} \right] = 0, \tag{6.18}$$

so that $\frac{\partial^{n_1+1}}{\partial v_1^{n_1+1}} T_g(v_1, \dots, v_\rho, x, y, u)$ equals

$$\sum_{j=2}^{j=r} v_j \frac{\partial^{n_1+2}}{\partial v_1^{n_1+1} \partial v_j} \left((v_j - v_1)^{-1} \left(v_j(1 - v_j)H_g(v_j, [\rho] - \{v_j\}, x, y, u) - v_1(1 - v_1)H_g(v_1, [\rho] - \{v_j\}, x, y, u) \right) \right) \tag{6.19}$$

i.e.

$$\sum_{j=2}^{j=\rho} v_j \frac{\partial^{n_1+2}}{\partial v_1^{n_1+1} \partial v_j} \left(\frac{F_g(v_j, [\rho] - \{v_j\}, x, y, u) - F_g(v_1, [\rho] - \{v_j\}, x, y, u)}{v_j - v_1} \right). \tag{6.20}$$

Formula (6.15) is a consequence of

$$\frac{\partial^{n_1+n_2}}{\partial x_1^{n_1} \partial x_2^{n_2}} \left(\frac{\psi(x_1) - \psi(x_2)}{x_1 - x_2} \right)_{x_1=x_2=a} = \frac{n_1!n_2!}{(n_1 + n_2 + 1)!} \psi^{(n_1+n_2+1)}(a). \tag{6.21} \quad \square$$

The formula

$$F_g[n, N] = \sum_{k+l=n} \binom{n}{k} L[k]H_g[l, N] \tag{6.22}$$

is an easy consequence of (6.14). Thus the right-hand side of (6.13) only depends on some functions $H_g[k, \dots, n_\rho]$ with $k < n_1$, some functions $H_g[n'_1, \dots, n'_{\rho'}]$ with $\rho' < \rho$, some functions $H_j[\dots]$ for $j < g$ and some functions $A[i]$. A relation between $A[i]$ and some functions $H_0[j]$ is established in Section 6.2.2.

6.2.2 Case $g = 0$ and $\rho = 1$

The function $A[i]$ can be related to some functions $H_0[j]$ as follows: With $M(v) = 1 - v$ and $L(v) = v(1 - v)$, Equation (5.9) is

$$A(v, x, y, u) = vu + M(v) + L(v)(-y + x - 2xH_0(v, x, y, u)). \tag{6.23}$$

Its instantiation at $v = V$ gives

$$H_0[0] = \frac{1 - q}{1 - q - r}. \tag{6.24}$$

For $k \geq 1$, the k -th partial derivative of (6.23) in v is

$$\begin{aligned} \frac{\partial^k}{\partial v^k} A(v, x, y, u) &= \frac{\partial^k}{\partial v^k} (vu) + \frac{\partial^k}{\partial v^k} M(v) \\ &+ \frac{\partial^k}{\partial v^k} [L(v)(-y + x - 2xH_0(v, x, y, u))] \end{aligned} \tag{6.25}$$

and its instantiation in $v = V$ is

$$\begin{aligned} A[k] &= \frac{\partial^k}{\partial v^k} (vu)|_{v=V} + M[k] \\ &+ \sum_{i+j=k} \binom{k}{i} L[i] \left(\frac{\partial^j}{\partial v^j} (-y + x - 2xH_0(v, x, y, u))|_{v=V} \right). \end{aligned} \tag{6.26}$$

Solving (6.26) for $k = 1$ gives

$$H_0[1] = \frac{(1 - q)^2(A[1] + 1 - p - q - r)}{2pq(1 - q - r)}. \tag{6.27}$$

For $k \geq 2$, one gets

$$A[k] = -2x \sum_{i+j=k} \binom{k}{i} L[i] H_0[j]$$

since $M[k] = 0$, i.e.

$$A[k] = -2x \left(L[0]H_0[k] + kL[1]H_0[k - 1] + \frac{k(k - 1)}{2}L[2]H_0[k - 2] \right) \tag{6.28}$$

since $L[k] = 0$ if $k \geq 3$.

7 Explicit formulas for small genera

This section proposes explicit parametric expressions for the generating functions that count rooted hypermaps of small positive genus. In Section 7.1 we count by number of vertices, hyperedges and faces; the number of darts can be obtained from these parameters by Formula (1.2). In Section 7.2 we count by number of darts alone.

7.1 Rooted hypermap series enumerated with three parameters

For $g = 1, \dots, 5$ we have computed an explicit expression of $H_g(x, y, u)$ parameterized by p, q and r , with $x = p(1 - q - r)$, $u = q(1 - p - r)$ and $y = r(1 - p - q)$, by application of formulas in Section 6. For $g \geq 3$, the expressions are too large to be included in the present text, but a Maple file with all the generating functions up to genus 5 is available from the first author on request.

A parametric expression of $H_1(x, y, u)$ is

$$H_1(x, y, u) = \frac{pqr(1-p)(1-q)(1-r)}{[(1-p-q-r)^2 - 4pqr]^2}. \tag{7.1}$$

This expression can be derived from [2, Theorem 3], with the correspondence $s = x$, $f = u$, and $a = y$ between variables and the correspondence $H_1(x, y, u) = xuK_1(1, x, y, u)$ between generating functions.

A parametric expression of $H_2(x, y, u)$ is

$$H_2(x, y, u) = \frac{pqr(1-p)(1-q)(1-r)P_2(p, q, r)}{[(1-p-q-r)^2 - 4pqr]^7} \tag{7.2}$$

where

$$\begin{aligned} P_2(p, q, r) = & 76p^6q^2r^2 - 8p^4q^4r^2 - 8p^4q^2r^4 + 76p^2q^6r^2 - 8p^2q^4r^4 + 76p^2q^2r^6 \\ & + 40p^7qr - 76p^6q^2r - 76p^6qr^2 - 112p^5q^3r - 228p^5q^2r^2 - 112p^5qr^3 \\ & + 8p^4q^4r + 16p^4q^3r^2 + 16p^4q^2r^3 + 8p^4qr^4 - 112p^3q^5r + 16p^3q^4r^2 \\ & + 40p^3q^3r^3 + 16p^3q^2r^4 - 112p^3qr^5 - 76p^2q^6r - 228p^2q^5r^2 \\ & + 16p^2q^4r^3 + 16p^2q^3r^4 - 228p^2q^2r^5 - 76p^2qr^6 + 40pq^7r - 76pq^6r^2 \\ & - 112pq^5r^3 + 8pq^4r^4 - 112pq^3r^5 - 76pq^2r^6 + 40pqr^7 + p^8 - 20p^7q \\ & - 20p^7r - 35p^6q^2 - 64p^6qr - 35p^6r^2 + 56p^5q^3 + 396p^5q^2r + 396p^5qr^2 \\ & + 56p^5r^3 + 140p^4q^4 + 264p^4q^3r + 393p^4q^2r^2 + 264p^4qr^3 + 140p^4r^4 \\ & + 56p^3q^5 + 264p^3q^4r - 92p^3q^3r^2 - 92p^3q^2r^3 + 264p^3qr^4 + 56p^3r^5 \\ & - 35p^2q^6 + 396p^2q^5r + 393p^2q^4r^2 - 92p^2q^3r^3 + 393p^2q^2r^4 + 396p^2qr^5 \\ & - 35p^2r^6 - 20pq^7 - 64pq^6r + 396pq^5r^2 + 264pq^4r^3 + 264pq^3r^4 \\ & + 396pq^2r^5 - 64pqr^6 - 20pr^7 + q^8 - 20q^7r - 35q^6r^2 + 56q^5r^3 \\ & + 140q^4r^4 + 56q^3r^5 - 35q^2r^6 - 20qr^7 + r^8 + 6p^7 + 105p^6q + 105p^6r \\ & + 21p^5q^2 - 116p^5qr + 21p^5r^2 - 420p^4q^3 - 821p^4q^2r - 821p^4qr^2 \\ & - 420p^4r^3 - 420p^3q^4 - 648p^3q^3r - 316p^3q^2r^2 - 648p^3qr^3 - 420p^3r^4 \\ & + 21p^2q^5 - 821p^2q^4r - 316p^2q^3r^2 - 316p^2q^2r^3 - 821p^2qr^4 + 21p^2r^5 \\ & + 105pq^6 - 116pq^5r - 821pq^4r^2 - 648pq^3r^3 - 821pq^2r^4 - 116pqr^5 \\ & + 105pr^6 + 6q^7 + 105q^6r + 21q^5r^2 - 420q^4r^3 - 420q^3r^4 + 21q^2r^5 \\ & + 105qr^6 + 6r^7 - 49p^6 - 189p^5q - 189p^5r + 315p^4q^2 + 479p^4qr \\ & + 315p^4r^2 + 910p^3q^3 + 1162p^3q^2r + 1162p^3qr^2 + 910p^3r^3 + 315p^2q^4 \\ & + 1162p^2q^3r + 720p^2q^2r^2 + 1162p^2qr^3 + 315p^2r^4 - 189pq^5 + 479pq^4r \\ & + 1162pq^3r^2 + 1162pq^2r^3 + 479pqr^4 - 189pr^5 - 49q^6 - 189q^5r \\ & + 315q^4r^2 + 910q^3r^3 + 315q^2r^4 - 189qr^5 - 49r^6 + 112p^5 + 70p^4q \\ & + 70p^4r - 770p^3q^2 - 876p^3qr - 770p^3r^2 - 770p^2q^3 - 1380p^2q^2r \\ & - 1380p^2qr^2 - 770p^2r^3 + 70pq^4 - 876pq^3r - 1380pq^2r^2 - 876pqr^3 \\ & + 70pr^4 + 112q^5 + 70q^4r - 770q^3r^2 - 770q^2r^3 + 70qr^4 + 112r^5 \end{aligned}$$

$$\begin{aligned}
 & - 105p^4 + 210p^3q + 210p^3r + 735p^2q^2 + 1034p^2qr + 735p^2r^2 + 210pq^3 \\
 & + 1034pq^2r + 1034pqr^2 + 210pr^3 - 105q^4 + 210q^3r + 735q^2r^2 + 210qr^3 \\
 & - 105r^4 + 14p^3 - 315p^2q - 315p^2r - 315pq^2 - 672pqr - 315pr^2 + 14q^3 \\
 & - 315q^2r - 315qr^2 + 14r^3 + 49p^2 + 175pq + 175pr + 49q^2 + 175qr \\
 & + 49r^2 - 36p - 36q - 36r + 8.
 \end{aligned}$$

Remark: For $g = 0$, the formula

$$H_0(x, y, u) = pqr(1 - p - q - r) \tag{7.3}$$

can be derived from [1], with the correspondence $s = x$, $f = u$, and $a = y$ between variables and the correspondence $H_0(x, y, u) = xuK_0(1, x, y, u)$ between generating functions.

7.2 Rooted hypermap series enumerated by number of darts

Let $H_g(z)$ be the ordinary generating function of rooted hypermaps on the orientable surface of genus $g \geq 0$, where the exponent of variable z is the number d of darts.

7.2.1 Generating functions

For g from 0 to 6, a parametric expression of $H_g(z)$, where $z = \tau(1 - 2\tau)$ and $\tau = 0$ when $z = 0$, is

$$H_0(z) = \frac{\tau^3 (1 - 3\tau)}{z^2}, \tag{7.4}$$

$$H_1(z) = \frac{\tau^3}{(1 - \tau) (1 - 4\tau)^2}, \tag{7.5}$$

$$H_2(z) = \frac{4 z^2 \tau^3 (51 \tau^4 - 77 \tau^3 + 48 \tau^2 - 15 \tau + 2)}{(1 - \tau)^5 (1 - 4\tau)^7}, \tag{7.6}$$

$$H_3(z) = \frac{4 z^4 \tau^3 P_3(z)}{(1 - \tau)^9 (1 - 4\tau)^{12}}, \tag{7.7}$$

$$H_4(z) = \frac{4 z^6 \tau^3 P_4(z)}{(1 - \tau)^{13} (1 - 4\tau)^{17}}, \tag{7.8}$$

$$H_5(z) = \frac{4 z^8 \tau^3 P_5(z)}{(1 - \tau)^{17} (1 - 4\tau)^{22}}, \tag{7.9}$$

$$H_6(z) = \frac{4 z^{10} \tau^3 P_6(z)}{(1 - \tau)^{21} (1 - 4\tau)^{27}}, \tag{7.10}$$

with

$$\begin{aligned}
 P_3(z) = & 28496 \tau^9 - 36888 \tau^8 - 13164 \tau^7 + 61676 \tau^6 - 61872 \tau^5 + 35172 \tau^4 \\
 & - 13168 \tau^3 + 3360 \tau^2 - 552 \tau + 45,
 \end{aligned}$$

$$\begin{aligned}
 P_4(z) = & 32375616 \tau^{14} + 15509760 \tau^{13} - 243313744 \tau^{12} + 442844592 \tau^{11} \\
 & - 389268768 \tau^{10} + 170357328 \tau^9 + 1281984 \tau^8 - 53553072 \tau^7 \\
 & + 39814032 \tau^6 - 17597520 \tau^5 + 5541192 \tau^4 - 1320920 \tau^3 + 239697 \tau^2 \\
 & - 30456 \tau + 2016,
 \end{aligned}$$

$$\begin{aligned}
 P_5(z) = & 61742404608 \tau^{19} + 239043447552 \tau^{18} - 1163002515456 \tau^{17} \\
 & + 1403096348736 \tau^{16} + 338393916800 \tau^{15} - 2962590413376 \tau^{14} \\
 & + 4243997599488 \tau^{13} - 3552865706240 \tau^{12} + 2000782619136 \tau^{11} \\
 & - 761565230016 \tau^{10} + 165542511744 \tau^9 + 7568059872 \tau^8 \\
 & - 23295865824 \tau^7 + 11016156244 \tau^6 - 3336459144 \tau^5 + 761835465 \tau^4 \\
 & - 141393220 \tau^3 + 21738240 \tau^2 - 2490480 \tau + 151200
 \end{aligned}$$

and

$$\begin{aligned}
 P_6(z) = & 178054771302400 \tau^{24} + 1584534210564096 \tau^{23} - 4933663711730688 \tau^{22} \\
 & - 2073822560019456 \tau^{21} + 28025505345377280 \tau^{20} \\
 & - 55010184951564288 \tau^{19} + 54283457920223232 \tau^{18} \\
 & - 22997164994372352 \tau^{17} - 13439214645718272 \tau^{16} \\
 & + 31734000656779264 \tau^{15} - 29719458122609664 \tau^{14} \\
 & + 18704646148809216 \tau^{13} - 8736443315384448 \tau^{12} \\
 & + 3098312828500416 \tau^{11} - 813298324826016 \tau^{10} + 138473163256176 \tau^9 \\
 & - 4043551301232 \tau^8 - 6580517850696 \tau^7 + 2630924485729 \tau^6 \\
 & - 626336383104 \tau^5 + 112079088144 \tau^4 - 17314508592 \tau^3 + 2485496880 \tau^2 \\
 & - 284717376 \tau + 17107200.
 \end{aligned}$$

We have also computed the generating functions for $7 \leq g \leq 11$. Their expressions are too large to be included in the present text, but a Maple file is available from the first author on request.

A. Mednykh and R. Nedela used our formulas (7.4) to (7.7) to find explicit formulas for the number of rooted hypermaps for genus $g = 0, 1, 2$ and 3 [19].

7.3 Other parameterization

In a private communication to the second author, P. Zograf suggests the parameterization

$$z = \frac{t}{(1 + 2t)^2}. \tag{7.11}$$

After adding the condition that $t = 0$ when $z = 0$, it corresponds to

$$t = \frac{1 - 4z - \sqrt{1 - 8z}}{8z}. \tag{7.12}$$

These two parameterizations are equivalent. The one can be transformed into the other by means of the following substitutions:

$$\tau = \frac{t}{1 + 2t} \tag{7.13}$$

and

$$t = \frac{\tau}{1 - 2\tau}. \tag{7.14}$$

By means of these substitutions, the following parametric expressions in t can be obtained from the parametric expressions (7.4)–(7.10) for $H_g(t)$ in τ :

$$\begin{aligned} H_0(z) &= t(1-t), \\ H_1(z) &= \frac{t^3}{(1+t)(1-2t)^2}, \\ H_2(z) &= \frac{4t^5(1+2t)(t^4-t^3+6t^2+t+2)}{(1+t)^5(1-2t)^7}, \\ H_3(z) &= 4t^7(1+2t)(1+t)^{-9}(1-2t)^{-12}(80t^9-120t^8+1500t^7+1036t^6 \\ &\quad + 3768t^5+2820t^4+2288t^3+1008t^2+258t+45), \\ H_4(z) &= 4t^9(1+2t)(1+t)^{-13}(1-2t)^{-17}(16768t^{14}-33536t^{13} \\ &\quad + 653776t^{12}+786480t^{11}+4358016t^{10}+6151056t^9+10059552t^8 \\ &\quad + 10217040t^7+8418240t^6+5227024t^5+2365888t^4+800128t^3 \\ &\quad + 181665t^2+25992t+2016), \\ H_5(z) &= 4t^{11}(1+2t)(1+t)^{-17}(1-2t)^{-22}(6732800t^{19}-16832000t^{18} \\ &\quad + 450011520t^{17}+773106240t^{16}+5764983552t^{15}+11910647232t^{14} \\ &\quad + 29130502912t^{13}+46090300928t^{12}+63452543616t^{11} \\ &\quad + 68713116608t^{10}+60654218080t^9+43591208976t^8 \\ &\quad + 25142796864t^7+11637842232t^6+4232899206t^5+1181820745t^4 \\ &\quad + 245635580t^3+35501760t^2+3255120t+151200), \\ H_6(z) &= 4t^{13}(1+2t)(1+t)^{-21}(1-2t)^{-27}(4424052736t^{24}-13272158208t^{23} \\ &\quad + 452750478336t^{22}+1012254206976t^{21}+9488911137792t^{20} \\ &\quad + 25803592571904t^{19}+83891900050944t^{18}+180120643165440t^{17} \\ &\quad + 346626234587904t^{16}+535272874975232t^{15}+701152993531392t^{14} \\ &\quad + 771688966862592t^{13}+716686355273472t^{12}+563018634260736t^{11} \\ &\quad + 372549313187520t^{10}+207088794784752t^9+96021082581732t^8 \\ &\quad + 36765061031004t^7+11475757049569t^6+2863185376896t^5 \\ &\quad + 556090776432t^4+80913152016t^3+8274846384t^2+536428224t \\ &\quad + 17107200). \end{aligned}$$

For $0 \leq g \leq 3$, these expressions correspond to $F_g(t)$ in Zograf’s communication. Moreover, they reveal an extra factorization by $4(1+2t)$ for $g \geq 2$.

8 Efficient enumeration of rooted and sensed unrooted hypermaps by number of darts, vertices and hyperedges

We recall that a sensed map or hypermap is an equivalence class of (unrooted) maps or hypermaps under orientation-preserving isomorphism.

Before enumerating sensed hypermaps we first need to enumerate rooted hypermaps. We use an efficient method of counting rooted hypermaps by number of darts, faces, ver-

tices and hyperedges or, equivalently [23], 2-coloured bipartite maps rooted at a white vertex by number of edges, faces, white vertices and black vertices, presented by Kazarian and Zograf [15], and then count sensed 2-coloured bipartite maps and hypermaps with the same parameters using the same method we used [26, 12] to count sensed maps by number of edges, faces and vertices. The recurrence (formula (11) in [15]), with f changed to H , is as follows. Define $H_{g,d}$ to be a homogeneous polynomial in the three variables t, u , and v . The coefficient of $t^f u^b v^w$ in $H_{g,d}$ is the number of 2-coloured bipartite maps of genus g with d edges, f faces, b black vertices and w white vertices rooted at a white vertex or, equivalently, the number of rooted hypermaps of genus g with d darts, f faces, b hyperedges and w vertices. Then $H_{0,1} = tuv$ and

$$\begin{aligned}
 (d + 1)H_{g,d} &= \\
 &(2d - 1)(t + u + v)H_{g,d-1} \\
 &+ (d - 2) (2(tu + tv + uv) - (t^2 + u^2 + v^2)) H_{g,d-2} \tag{8.1} \\
 &+ (d - 1)^2(d - 2)H_{g-1,d-2} + \sum_{i=0}^g \sum_{j=1}^{d-3} (4 + 6j)(d - 2 - j)H_{i,j}H_{g-i,d-2-j}.
 \end{aligned}$$

In [26] we collaborated with Mednykh to enumerate rooted and sensed maps. Mednykh enumerated maps of genus up to 11 by number of edges alone, while we enumerated maps of genus up to 10 by number of edges and vertices. The method we used to enumerate rooted maps is presented in [25]. The method we used to enumerate sensed maps is based on Liskovets’ refinement [17] of the method Mednykh and Nedela used to enumerate sensed map of genus up to 3 by number of edges [18]. Later we used a more efficient method of enumerating rooted maps, presented in [5], to enumerate rooted and sensed maps of genus up to 50 [12].

To describe here the modifications we made to pass from maps to 2-coloured bipartite maps we need to briefly discuss a few of the concepts described in more detail in [26]. All the automorphisms of a map on an orientable surface are periodic. If the period is $L > 1$, then the automorphism divides the map into L isomorphic copies of a smaller map, called the *quotient map*. Most of the *cells* (vertices, edges and faces) are in orbits of length L under the automorphism; those that aren’t are called *branch points*. For example, if a map is drawn on the surface of a sphere which undergoes a rotation through $360/L$ degrees, the two cells through which the axis of rotation pass are fixed; so they are each in an orbit of length 1 for any L . For maps of higher genus, not all the branch points are on orbits of length 1. For example, if a torus is represented as a square with opposite edges identified in pairs, and is rotated by 90 degrees (period 4), then the centre of the square is a branch point of orbit length 1 and so is the point represented by all four corners of the square, but the middle of the sides of the square are two branch points of orbit length 2: the point represented by the middle of both vertical sides of the square is taken by the rotation onto the point represented by the middle of both horizontal sides, and vice versa; so it takes two rotations to take either of these points back onto itself. Also, if the middle of an edge is a branch point, then the quotient map contains half of that edge – a *dangling semi-edge*.

An automorphism of a map M of genus G is characterized by the following parameters: the period L , the genus g of its quotient map and the number of branch points of each orbit length. If each orbit length is replaced by its *branch index* (L divided by the orbit length), we obtain what is called an *orbifold signature* in [18]. In [18] a method is presented for determining which orbifold signatures could characterize an automorphism

of a map of genus G (a G -admissible orbifold) and how many such automorphisms could be characterized by that orbifold signature; a variant of that method is presented in [17], and this is the one we use except that we deal with orbit lengths instead of branch indices. The method used in [18] to enumerate sensed maps of genus G with E edges by number of edges can be roughly described as follows. For each G -admissible orbifold O , let g be the genus of the quotient map, L be the period and q_i be the number of branch points with branch index i . Then the number $\nu_O(d)$ of rooted maps with d darts that could serve as a quotient map for an automorphism with that signature once the branch points are pasted onto the map in all possible ways is given by

$$\nu_O(d) = \sum_{s=0}^{q_2} \binom{d}{s} \binom{(d-s)/2+2-2g}{q_2-s, q_3, \dots, q_L} N_g((d-s)/2), \tag{8.2}$$

where $N_g(n)$ is the number of rooted maps of genus g with n edges (0 if n is not an integer). Here s is the number of dangling semi-edges in the quotient map m , all of which must be in orbits of length $L/2$ so that they represent normal edges in the original map M . The binomial coefficient is the number of ways of inserting dangling semi-edges into the rooted map multiplied by $d/(d-s)$ because there are d ways to root the map once the dangling edges have been inserted and only $d-s$ ways to root it without the dangling edges. The multinomial coefficient is the number of ways to distribute the branch points with the various branch indices among the non-edges of the quotient map; the number at the top of the multinomial coefficient is the number of non-edges and is given by the Euler-Poincaré formula (1.1). Then the number of sensed maps of genus G with E edges is

$$\frac{1}{2E} \sum_{L|E} \sum_O \text{Epi}_0(\pi_1(O), Z_L) \nu_O(2E/L), \tag{8.3}$$

where O runs over all the G -admissible orbifolds with period L and $\text{Epi}_0(\pi_1(O), Z_L)$ is the number of automorphisms that have the orbifold signature of O .

In [26] we distributed the branch points that aren't on dangling semi-edges among the vertices and faces separately. The quotient map of a bipartite map can't contain any dangling semi-edges; otherwise the lifted map would have an edge joining two vertices of the same colour. Here we distribute the branch points among the white vertices, black vertices and faces, and, like in [26], we don't use a formula like (8.3); instead we compute the contribution of each orbifold signature to the number of sensed 2-coloured bipartite maps whose number of white vertices, black vertices, faces and edges are allowed to vary within a user-defined upper bound on the number of edges.

Suppose that the quotient map is of genus g and has w white vertices, b black vertices and f faces. Then the number e of edges can be calculated from the formula

$$f - e + w + b = 2(1 - g) \tag{8.4}$$

and the number d of darts is $2e$. Suppose also that among the branch points of orbit length i , w_i are on a white vertex, b_i are on a black vertex and f_i are in a face. We denote by w_L , b_L and f_L the number of white vertices, black vertices and faces, respectively, that do not contain a branch point. The original map will have W white vertices, B black vertices and F faces, where

$$W = \sum_{i=1}^L iw_i, \quad B = \sum_{i=1}^L ib_i \quad \text{and} \quad F = \sum_{i=1}^L if_i, \tag{8.5}$$

and the total number E of edges is equal to $Le = F + W + B - 2(1 - g)$.

The binomial coefficient in (8.2) disappears because the quotient map can't contain any dangling semi-edges. The multinomial coefficient must be replaced by the number of ways to distribute the branch points among the white vertices, black vertices and faces. Then (8.2) becomes

$$\nu_O(d, w, b, f) = \binom{w}{w_1, w_2, \dots, w_L} \binom{b}{b_1, b_2, \dots, b_L} \binom{f}{f_1, f_2, \dots, f_L} N_g(d, w, b, f), \quad (8.6)$$

where d is the number of edges in the quotient maps on both sides of the formula (or the number of darts in the corresponding hypermaps) and $N_g(d, w, b, f)$ is the number of 2-coloured bipartite maps with d edges with w white vertices, b black vertices and f faces, rooted at a white vertex. For this number to be positive, the sum of all the w_i cannot exceed w with a similar bound on the sum of all the b_i and the sum of all the f_i ; so w , b and f each starts at its respective sum and increases by 1 until the number E of edges in the original map exceeds a user-defined maximum. With each increase of w , b or f , one of the multinomial coefficients in (8.6) gets updated using a single multiplication and division. The product of these three multinomial coefficients must be computed for all sets of non-negative integers such that for each i , $w_i + b_i + f_i$ is equal to the total number of branch points of orbit length i .

Once (8.6) is multiplied by the number of automorphisms with the current orbifold signature, we get the contribution of that signature and the numbers w_i , b_i and f_i to E times the number of sensed 2-coloured bipartite maps of genus G with E edges, F faces, B black vertices and W white vertices. This contribution is added to the appropriate element of an array, initially 0, and when all the contributions have been tallied, for each E , F , W and B the corresponding array element is divided by E (not $2E$ because the root must be incident to a white vertex) to give the number of sensed 2-coloured bipartite maps of genus G with E edges, F faces, B black vertices and W white vertices or, equivalently, the number of sensed hypermaps of genus G with E darts, F faces, B hyperedges and W vertices.

This enumeration was done with a program written in C++ using CLN to treat big integers. It enumerated rooted and sensed hypermaps of genus up to 24 with up to 50 darts as fast as it could display the numbers on the screen. The numbers coincide with those obtained by generating the hypermaps [24]. The source code is available from the second author on request.

References

- [1] D. Arquès, Relations fonctionnelles et dénombrement des hypercartes planaires pointées, in: G. Labelle and P. Leroux (eds.), *Combinatoire énumérative*, Springer, Berlin, volume 1234 of *Lecture Notes in Mathematics*, pp. 5–26, 1986, doi:10.1007/bfb0072505.
- [2] D. Arquès, Hypercartes pointées sur le tore: Décompositions et dénombrements, *J. Comb. Theory Ser. B* **43** (1987), 275–286, doi:10.1016/0095-8956(87)90003-7.
- [3] D. Arquès, Relations fonctionnelles et dénombrement des cartes pointées sur le tore, *J. Comb. Theory Ser. B* **43** (1987), 253–274, doi:10.1016/0095-8956(87)90002-5.
- [4] E. A. Bender and E. R. Canfield, The number of rooted maps on an orientable surface, *J. Comb. Theory Ser. B* **53** (1991), 293–299, doi:10.1016/0095-8956(91)90079-y.

- [5] S. R. Carrell and G. Chapuy, Simple recurrence formulas to count maps on orientable surfaces, *J. Comb. Theory Ser. A* **133** (2015), 58–75, doi:10.1016/j.jcta.2015.01.005.
- [6] G. Chapuy and W. Fang, Generating functions of bipartite maps on orientable surfaces, *Electron. J. Comb.* **23** (2016), #P3.31, <http://www.combinatorics.org/ojs/index.php/eljc/article/view/v23i3p31>.
- [7] H. S. M. Coxeter, *Regular Polytopes*, Dover, New York, 3rd edition, 1973.
- [8] P. Dunin-Barkowski, N. Orantin, A. Popolitov and S. Shadrin, Combinatorics of loop equations for branched covers of sphere, *Int. Math. Res. Notices* (2017), rnx047, doi:10.1093/imrn/rnx047.
- [9] B. Eynard, Formal matrix integrals and combinatorics of maps, in: J. Harnad (ed.), *Random Matrices, Random Processes and Integrable Systems*, Springer, New York, pp. 415–442, 2011, doi:10.1007/978-1-4419-9514-8.6.
- [10] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, New York, NY, USA, 1st edition, 2009, <http://ac.cs.princeton.edu/home/>.
- [11] A. Giorgetti, *Combinatoire bijective et énumérative des cartes pointées sur une surface*, Ph.D. thesis, Université de Marne-la-Vallée, Institut Gaspard Monge, 1998.
- [12] A. Giorgetti and T. R. S. Walsh, Constructing large tables of numbers of maps by orientable genus, 2014, [arXiv:1405.0615](https://arxiv.org/abs/1405.0615) [math.CO].
- [13] A. Jacques, Sur le genre d’une paire de substitutions, *C. R. Acad. Sci. Paris Ser. A* **267** (1968), 625–627.
- [14] G. A. Jones and D. Singerman, Theory of maps on orientable surfaces, *Proc. London Math. Soc.* **37** (1978), 273–307, doi:10.1112/plms/s3-37.2.273.
- [15] M. Kazarian and P. Zograf, Virasoro constraints and topological recursion for Grothendieck’s dessin counting, *Lett. Math. Phys.* **105** (2015), 1057–1084, doi:10.1007/s11005-015-0771-0.
- [16] S. K. Lando and A. K. Zvonkin, *Graphs on Surfaces and Their Applications*, volume 141 of *Encyclopaedia of Mathematical Sciences*, Springer-Verlag, Berlin, 2004, doi:10.1007/978-3-540-38361-1, see especially chapter 2, “Dessins d’Enfants”, pp. 79–153.
- [17] V. A. Liskovets, A multivariate arithmetic function of combinatorial and topological significance, *Integers* **10** (2010), 155–177, doi:10.1515/integ.2010.012.
- [18] A. Mednykh and R. Nedela, Enumeration of unrooted maps of a given genus, *J. Comb. Theory Ser. B* **96** (2006), 706–729, doi:10.1016/j.jctb.2006.01.005.
- [19] A. Mednykh and R. Nedela, Counting hypermaps by Egorychev’s method, *Anal. Math. Phys.* **6** (2016), 301–314, doi:10.1007/s13324-015-0119-z.
- [20] A. Mednykh and R. Nedela, Recent progress in enumeration of hypermaps, *J. Math. Sci.* **226** (2017), 635–654, doi:10.1007/s10958-017-3555-5.
- [21] M. Planat, A. Giorgetti, F. Holweck and M. Saniga, Quantum contextual finite geometries from dessins d’enfants, *Int. J. Geom. Methods Mod. Phys.* **12** (2015), 1550067, doi:10.1142/s021988781550067x.
- [22] W. T. Tutte, A census of slicings, *Canad. J. Math.* **14** (1962), 708–722, doi:10.4153/cjm-1962-061-1.
- [23] T. R. S. Walsh, Hypermaps versus bipartite maps, *J. Comb. Theory Ser. B* **18** (1975), 155–163, doi:10.1016/0095-8956(75)90042-8.
- [24] T. R. S. Walsh, Space-efficient generation of nonisomorphic maps and hypermaps, *J. Integer Seq.* **18** (2015), Article 15.4.3, <https://cs.uwaterloo.ca/journals/JIS/VOL18/Walsh/walsh3.html>.

- [25] T. R. S. Walsh and A. Giorgetti, Efficient enumeration of rooted maps of a given orientable genus by number of faces and vertices, *Ars Math. Contemp.* **7** (2014), 263–280, <http://amc-journal.eu/index.php/amc/article/view/190>.
- [26] T. R. S. Walsh, A. Giorgetti and A. Mednykh, Enumeration of unrooted orientable maps of arbitrary genus by number of edges and vertices, *Discrete Math.* **312** (2012), 2660–2671, doi:10.1016/j.disc.2011.11.027.
- [27] T. R. S. Walsh and A. B. Lehman, Counting rooted maps by genus I, *J. Comb. Theory Ser. B* **13** (1972), 192–218, doi:10.1016/0095-8956(75)90050-7.

A First numbers of rooted hypermaps

The following sections show the numbers h of rooted hypermaps of genus g with d darts, v vertices, e edges and $d - v - e + 2(1 - g)$ faces, for $g \leq 6$ and $d \leq 14$.

A.1 Genus 0

| d | v | e | f | h | | | | | | | | | | | | | | | |
|---|---|---|---|------|---|--|-----|------|--|---|---|---|---|-------|-------|--|--|--|--|
| 6 | 1 | 5 | 2 | 15 | | | | | | 8 | 2 | 5 | 3 | 2436 | | | | | |
| 6 | 2 | 4 | 2 | 135 | | | | | | 8 | 3 | 4 | 3 | 7500 | | | | | |
| 6 | 3 | 3 | 2 | 262 | | | | | | 8 | 4 | 3 | 3 | 7500 | | | | | |
| 6 | 4 | 2 | 2 | 135 | | | | | | 8 | 5 | 2 | 3 | 2436 | | | | | |
| 6 | 5 | 1 | 2 | 15 | | | | | | 8 | 6 | 1 | 3 | 196 | | | | | |
| 6 | 1 | 6 | 1 | 1 | | | | | | 8 | 1 | 7 | 2 | 28 | | | | | |
| 6 | 2 | 5 | 1 | 15 | | | | | | 8 | 2 | 6 | 2 | 518 | | | | | |
| 6 | 3 | 4 | 1 | 50 | | | | | | 8 | 3 | 5 | 2 | 2436 | | | | | |
| 6 | 4 | 3 | 1 | 50 | | | | | | 8 | 4 | 4 | 2 | 3985 | | | | | |
| 6 | 5 | 2 | 1 | 15 | | | | | | 8 | 5 | 3 | 2 | 2436 | | | | | |
| 6 | 6 | 1 | 1 | 1 | | | | | | 8 | 6 | 2 | 2 | 518 | | | | | |
| 6 | | | | | | | | | | 8 | 7 | 1 | 2 | 28 | | | | | |
| 6 | | | | | 6 | | sum | 1584 | | 8 | 1 | 8 | 1 | 1 | | | | | |
| 7 | 1 | 1 | 7 | 1 | | | | | | 8 | 2 | 7 | 1 | 28 | | | | | |
| 7 | 1 | 2 | 6 | 21 | | | | | | 8 | 3 | 6 | 1 | 196 | | | | | |
| 7 | 1 | 2 | 6 | 21 | | | | | | 8 | 4 | 5 | 1 | 490 | | | | | |
| 7 | 2 | 1 | 6 | 21 | | | | | | 8 | 5 | 4 | 1 | 490 | | | | | |
| 7 | 1 | 3 | 5 | 105 | | | | | | 8 | 6 | 3 | 1 | 196 | | | | | |
| 7 | 2 | 2 | 5 | 280 | | | | | | 8 | 7 | 2 | 1 | 28 | | | | | |
| 7 | 3 | 1 | 5 | 105 | | | | | | 8 | 8 | 1 | 1 | 1 | | | | | |
| 7 | 1 | 4 | 4 | 175 | | | | | | | | | | | | | | | |
| 7 | 2 | 3 | 4 | 889 | | | | | | 8 | | | | sum | 54912 | | | | |
| 7 | 3 | 2 | 4 | 889 | | | | | | | | | | | | | | | |
| 7 | 4 | 1 | 4 | 175 | | | | | | 9 | 1 | 1 | 9 | 1 | | | | | |
| 7 | 1 | 5 | 3 | 105 | | | | | | 9 | 1 | 2 | 8 | 36 | | | | | |
| 7 | 2 | 4 | 3 | 889 | | | | | | 9 | 2 | 1 | 8 | 36 | | | | | |
| 7 | 3 | 3 | 3 | 1694 | | | | | | 9 | 1 | 3 | 7 | 336 | | | | | |
| 7 | 4 | 2 | 3 | 889 | | | | | | 9 | 2 | 2 | 7 | 882 | | | | | |
| 7 | 5 | 1 | 3 | 105 | | | | | | 9 | 3 | 1 | 7 | 336 | | | | | |
| 7 | 1 | 6 | 2 | 21 | | | | | | 9 | 1 | 4 | 6 | 1176 | | | | | |
| 7 | 2 | 5 | 2 | 280 | | | | | | 9 | 2 | 3 | 6 | 5754 | | | | | |
| 7 | 3 | 4 | 2 | 889 | | | | | | 9 | 3 | 2 | 6 | 5754 | | | | | |
| 7 | 4 | 3 | 2 | 889 | | | | | | 9 | 4 | 1 | 6 | 1176 | | | | | |
| 7 | 5 | 2 | 2 | 280 | | | | | | 9 | 1 | 5 | 5 | 1764 | | | | | |
| 7 | 6 | 1 | 2 | 21 | | | | | | 9 | 2 | 4 | 5 | 13941 | | | | | |
| 7 | 1 | 7 | 1 | 1 | | | | | | 9 | 3 | 3 | 5 | 26004 | | | | | |
| 7 | 2 | 6 | 1 | 21 | | | | | | 9 | 4 | 2 | 5 | 13941 | | | | | |
| 7 | 3 | 5 | 1 | 105 | | | | | | 9 | 5 | 1 | 5 | 1764 | | | | | |
| 7 | 4 | 4 | 1 | 175 | | | | | | 9 | 1 | 6 | 4 | 1176 | | | | | |
| 7 | 5 | 3 | 1 | 105 | | | | | | 9 | 2 | 5 | 4 | 13941 | | | | | |
| 7 | 6 | 2 | 1 | 21 | | | | | | 9 | 3 | 4 | 4 | 42015 | | | | | |
| 7 | 7 | 1 | 1 | 1 | | | | | | 9 | 4 | 3 | 4 | 42015 | | | | | |
| 7 | | | | | | | | | | 9 | 5 | 2 | 4 | 13941 | | | | | |
| 7 | | | | | 7 | | sum | 9152 | | 9 | 6 | 1 | 4 | 1176 | | | | | |
| 8 | 1 | 1 | 8 | 1 | | | | | | 9 | 1 | 7 | 3 | 336 | | | | | |
| 8 | 1 | 2 | 7 | 28 | | | | | | 9 | 2 | 6 | 3 | 5754 | | | | | |
| 8 | 2 | 1 | 7 | 28 | | | | | | 9 | 3 | 5 | 3 | 26004 | | | | | |
| 8 | 1 | 3 | 6 | 196 | | | | | | 9 | 4 | 4 | 3 | 42015 | | | | | |
| 8 | 2 | 2 | 6 | 518 | | | | | | 9 | 5 | 3 | 3 | 26004 | | | | | |
| 8 | 3 | 1 | 6 | 196 | | | | | | 9 | 6 | 2 | 3 | 5754 | | | | | |
| 8 | 1 | 4 | 5 | 490 | | | | | | 9 | 7 | 1 | 3 | 336 | | | | | |
| 8 | 2 | 3 | 5 | 2436 | | | | | | 9 | 1 | 8 | 2 | 36 | | | | | |
| 8 | 3 | 2 | 5 | 2436 | | | | | | 9 | 2 | 7 | 2 | 882 | | | | | |
| 8 | 4 | 1 | 5 | 490 | | | | | | 9 | 3 | 6 | 2 | 5754 | | | | | |
| 8 | 1 | 5 | 4 | 490 | | | | | | 9 | 4 | 5 | 2 | 13941 | | | | | |
| 8 | 2 | 4 | 4 | 3985 | | | | | | 9 | 5 | 4 | 2 | 13941 | | | | | |
| 8 | 3 | 3 | 4 | 7500 | | | | | | 9 | 6 | 3 | 2 | 5754 | | | | | |
| 8 | 4 | 2 | 4 | 3985 | | | | | | 9 | 7 | 2 | 2 | 882 | | | | | |
| 8 | 5 | 1 | 4 | 490 | | | | | | 9 | 8 | 1 | 2 | 36 | | | | | |
| 8 | 1 | 6 | 3 | 196 | | | | | | 9 | 1 | 9 | 1 | 1 | | | | | |
| 8 | | | | | | | | | | 9 | 2 | 8 | 1 | 36 | | | | | |

| | | | | | | | | | | | | | | | |
|----|----|----|-----|---------|----|----|----|----|---------|----------|----|----|----|---------|-------|
| 9 | 3 | 7 | 1 | 336 | 11 | 2 | 1 | 10 | 55 | 12 | 1 | 3 | 10 | 1210 | |
| 9 | 4 | 6 | 1 | 1176 | 11 | 1 | 3 | 9 | 825 | 12 | 2 | 2 | 10 | 3135 | |
| 9 | 5 | 5 | 1 | 1764 | 11 | 2 | 2 | 9 | 2145 | 12 | 3 | 1 | 10 | 1210 | |
| 9 | 6 | 4 | 1 | 1176 | 11 | 3 | 1 | 9 | 825 | 12 | 1 | 4 | 9 | 9075 | |
| 9 | 7 | 3 | 1 | 336 | 11 | 1 | 4 | 8 | 4950 | 12 | 2 | 3 | 9 | 43098 | |
| 9 | 8 | 2 | 1 | 36 | 11 | 2 | 3 | 8 | 23694 | 12 | 3 | 2 | 9 | 43098 | |
| 9 | 9 | 1 | 1 | 1 | 11 | 3 | 2 | 8 | 23694 | 12 | 4 | 1 | 9 | 9075 | |
| | | | | | 11 | 4 | 1 | 8 | 4950 | 12 | 1 | 5 | 8 | 32670 | |
| 9 | | | sum | 339456 | 11 | 1 | 5 | 7 | 13860 | 12 | 2 | 4 | 8 | 245223 | |
| | | | | | 11 | 2 | 4 | 7 | 105435 | 12 | 3 | 3 | 8 | 449988 | |
| 10 | 1 | 1 | 10 | 1 | 11 | 3 | 3 | 7 | 194304 | 12 | 4 | 2 | 8 | 245223 | |
| 10 | 1 | 2 | 9 | 45 | 11 | 4 | 2 | 7 | 105435 | 12 | 5 | 1 | 8 | 32670 | |
| 10 | 2 | 1 | 9 | 45 | 11 | 5 | 1 | 7 | 13860 | 12 | 1 | 6 | 7 | 60984 | |
| 10 | 1 | 3 | 8 | 540 | 11 | 1 | 6 | 6 | 19404 | 12 | 2 | 5 | 7 | 666996 | |
| 10 | 2 | 2 | 8 | 1410 | 11 | 2 | 5 | 6 | 216601 | 12 | 3 | 4 | 7 | 1936308 | |
| 10 | 3 | 1 | 8 | 540 | 11 | 3 | 4 | 6 | 634865 | 12 | 4 | 3 | 7 | 1936308 | |
| 10 | 1 | 4 | 7 | 2520 | 11 | 4 | 3 | 6 | 634865 | 12 | 5 | 2 | 7 | 666996 | |
| 10 | 2 | 3 | 7 | 12180 | 11 | 5 | 2 | 6 | 216601 | 12 | 6 | 1 | 7 | 60984 | |
| 10 | 3 | 2 | 7 | 12180 | 11 | 6 | 1 | 6 | 19404 | 12 | 1 | 7 | 6 | 60984 | |
| 10 | 4 | 1 | 7 | 2520 | 11 | 1 | 7 | 5 | 13860 | 12 | 2 | 6 | 6 | 925190 | |
| 10 | 1 | 5 | 6 | 5292 | 11 | 2 | 6 | 5 | 216601 | 12 | 3 | 5 | 6 | 3915576 | |
| 10 | 2 | 4 | 6 | 40935 | 11 | 3 | 5 | 5 | 931854 | 12 | 4 | 4 | 6 | 6195560 | |
| 10 | 3 | 3 | 6 | 75840 | 11 | 4 | 4 | 5 | 1482250 | 12 | 5 | 3 | 6 | 3915576 | |
| 10 | 4 | 2 | 6 | 40935 | 11 | 5 | 3 | 5 | 931854 | 12 | 6 | 2 | 6 | 925190 | |
| 10 | 5 | 1 | 6 | 5292 | 11 | 6 | 2 | 5 | 216601 | 12 | 7 | 1 | 6 | 60984 | |
| 10 | 1 | 6 | 5 | 5292 | 11 | 7 | 1 | 5 | 13860 | 12 | 1 | 8 | 5 | 32670 | |
| 10 | 2 | 5 | 5 | 60626 | 11 | 1 | 8 | 4 | 4950 | 12 | 2 | 7 | 5 | 666996 | |
| 10 | 3 | 4 | 5 | 179860 | 11 | 2 | 7 | 4 | 105435 | 12 | 3 | 6 | 5 | 3915576 | |
| 10 | 4 | 3 | 5 | 179860 | 11 | 3 | 6 | 4 | 634865 | 12 | 4 | 5 | 5 | 9032898 | |
| 10 | 5 | 2 | 5 | 60626 | 11 | 4 | 5 | 4 | 1482250 | 12 | 5 | 4 | 5 | 9032898 | |
| 10 | 6 | 1 | 5 | 5292 | 11 | 5 | 4 | 4 | 1482250 | 12 | 6 | 3 | 5 | 3915576 | |
| 10 | 1 | 7 | 4 | 2520 | 11 | 6 | 3 | 4 | 634865 | 12 | 7 | 2 | 5 | 666996 | |
| 10 | 2 | 6 | 4 | 40935 | 11 | 7 | 2 | 4 | 105435 | 12 | 8 | 1 | 5 | 32670 | |
| 10 | 3 | 5 | 4 | 179860 | 11 | 8 | 1 | 4 | 4950 | 12 | 1 | 9 | 4 | 9075 | |
| 10 | 4 | 4 | 4 | 288025 | 11 | 1 | 9 | 3 | 825 | 12 | 2 | 8 | 4 | 245223 | |
| 10 | 5 | 3 | 4 | 179860 | 11 | 2 | 8 | 3 | 23694 | 12 | 3 | 7 | 4 | 1936308 | |
| 10 | 6 | 2 | 4 | 40935 | 11 | 3 | 7 | 3 | 194304 | 12 | 4 | 6 | 4 | 6195560 | |
| 10 | 7 | 1 | 4 | 2520 | 11 | 4 | 6 | 3 | 634865 | 12 | 5 | 5 | 4 | 9032898 | |
| 10 | 1 | 8 | 3 | 540 | 11 | 5 | 5 | 3 | 931854 | 12 | 6 | 4 | 4 | 6195560 | |
| 10 | 2 | 7 | 3 | 12180 | 11 | 6 | 4 | 3 | 634865 | 12 | 7 | 3 | 4 | 1936308 | |
| 10 | 3 | 6 | 3 | 75840 | 11 | 7 | 3 | 3 | 194304 | 12 | 8 | 2 | 4 | 245223 | |
| 10 | 4 | 5 | 3 | 179860 | 11 | 8 | 2 | 3 | 23694 | 12 | 9 | 1 | 4 | 9075 | |
| 10 | 5 | 4 | 3 | 179860 | 11 | 9 | 1 | 3 | 825 | 12 | 1 | 10 | 3 | 1210 | |
| 10 | 6 | 3 | 3 | 75840 | 11 | 1 | 10 | 2 | 55 | 12 | 2 | 9 | 3 | 43098 | |
| 10 | 7 | 2 | 3 | 12180 | 11 | 2 | 9 | 2 | 2145 | 12 | 3 | 8 | 3 | 449988 | |
| 10 | 8 | 1 | 3 | 540 | 11 | 3 | 8 | 2 | 23694 | 12 | 4 | 7 | 3 | 1936308 | |
| 10 | 1 | 9 | 2 | 45 | 11 | 4 | 7 | 2 | 105435 | 12 | 5 | 6 | 3 | 3915576 | |
| 10 | 2 | 8 | 2 | 1410 | 11 | 5 | 6 | 2 | 216601 | 12 | 6 | 5 | 3 | 3915576 | |
| 10 | 3 | 7 | 2 | 12180 | 11 | 6 | 5 | 2 | 216601 | 12 | 7 | 4 | 3 | 1936308 | |
| 10 | 4 | 6 | 2 | 40935 | 11 | 7 | 4 | 2 | 105435 | 12 | 8 | 3 | 3 | 449988 | |
| 10 | 5 | 5 | 2 | 60626 | 11 | 8 | 3 | 2 | 23694 | 12 | 9 | 2 | 3 | 43098 | |
| 10 | 6 | 4 | 2 | 40935 | 11 | 9 | 2 | 2 | 2145 | 12 | 10 | 1 | 3 | 1210 | |
| 10 | 7 | 3 | 2 | 12180 | 11 | 10 | 1 | 2 | 55 | 12 | 1 | 11 | 2 | 66 | |
| 10 | 8 | 2 | 2 | 1410 | 11 | 1 | 11 | 1 | 1 | 12 | 2 | 10 | 2 | 3135 | |
| 10 | 9 | 1 | 2 | 45 | 11 | 2 | 10 | 1 | 55 | 12 | 3 | 9 | 2 | 43098 | |
| 10 | 1 | 10 | 1 | 1 | 11 | 3 | 9 | 1 | 825 | 12 | 4 | 8 | 2 | 245223 | |
| 10 | 2 | 9 | 1 | 45 | 11 | 4 | 8 | 1 | 4950 | 12 | 5 | 7 | 2 | 666996 | |
| 10 | 3 | 8 | 1 | 540 | 11 | 5 | 7 | 1 | 13860 | 12 | 6 | 6 | 2 | 925190 | |
| 10 | 4 | 7 | 1 | 2520 | 11 | 6 | 6 | 1 | 19404 | 12 | 7 | 5 | 2 | 666996 | |
| 10 | 5 | 6 | 1 | 5292 | 11 | 7 | 5 | 1 | 13860 | 12 | 8 | 4 | 2 | 245223 | |
| 10 | 6 | 5 | 1 | 5292 | 11 | 8 | 4 | 1 | 4950 | 12 | 9 | 3 | 2 | 43098 | |
| 10 | 7 | 4 | 1 | 2520 | 11 | 9 | 3 | 1 | 825 | 12 | 10 | 2 | 2 | 3135 | |
| 10 | 8 | 3 | 1 | 540 | 11 | 10 | 2 | 1 | 55 | 12 | 11 | 1 | 2 | 66 | |
| 10 | 9 | 2 | 1 | 45 | 11 | 11 | 1 | 1 | 1 | 12 | 1 | 12 | 1 | 1 | |
| 10 | 10 | 1 | 1 | 1 | | | | | | 12 | 2 | 11 | 1 | 66 | |
| | | | | | 11 | | | | sum | 13891584 | 12 | 3 | 10 | 1 | 1210 |
| 10 | | | sum | 2149888 | | | | | | | 12 | 4 | 9 | 1 | 9075 |
| | | | | | 12 | 1 | 1 | 12 | 1 | | 12 | 5 | 8 | 1 | 32670 |
| 11 | 1 | 1 | 11 | 1 | 12 | 1 | 2 | 11 | 66 | | 12 | 6 | 7 | 1 | 60984 |
| 11 | 1 | 2 | 10 | 55 | 12 | 2 | 1 | 11 | 66 | | 12 | 7 | 6 | 1 | 60984 |

| | | | | | | | | | | | | | | |
|----|----|-----|----|----------|----|----|-----|----|-----------|----|----|-----|---|------------|
| 12 | 8 | 5 | 1 | 32670 | 13 | 8 | 4 | 3 | 5264545 | 14 | 3 | 7 | 6 | 80231508 |
| 12 | 9 | 4 | 1 | 9075 | 13 | 9 | 3 | 3 | 960960 | 14 | 4 | 6 | 6 | 249321114 |
| 12 | 10 | 3 | 1 | 1210 | 13 | 10 | 2 | 3 | 74217 | 14 | 5 | 5 | 6 | 360078558 |
| 12 | 11 | 2 | 1 | 66 | 13 | 11 | 1 | 3 | 1716 | 14 | 6 | 4 | 6 | 249321114 |
| 12 | 12 | 1 | 1 | 1 | 13 | 1 | 12 | 2 | 78 | 14 | 7 | 3 | 6 | 80231508 |
| | | | | | 13 | 2 | 11 | 2 | 4433 | 14 | 8 | 2 | 6 | 10701873 |
| 12 | | sum | | 91287552 | 13 | 3 | 10 | 2 | 74217 | 14 | 9 | 1 | 6 | 429429 |
| | | | | | 13 | 4 | 9 | 2 | 525525 | 14 | 1 | 10 | 5 | 143143 |
| 13 | 1 | 1 | 13 | 1 | 13 | 5 | 8 | 2 | 1827683 | 14 | 2 | 9 | 5 | 4557553 |
| 13 | 1 | 2 | 12 | 78 | 13 | 6 | 7 | 2 | 3356522 | 14 | 3 | 8 | 5 | 44221632 |
| 13 | 2 | 1 | 12 | 78 | 13 | 7 | 6 | 2 | 3356522 | 14 | 4 | 7 | 5 | 181925268 |
| 13 | 1 | 3 | 11 | 1716 | 13 | 8 | 5 | 2 | 1827683 | 14 | 5 | 6 | 5 | 360078558 |
| 13 | 2 | 2 | 11 | 4433 | 13 | 9 | 4 | 2 | 525525 | 14 | 6 | 5 | 5 | 360078558 |
| 13 | 3 | 1 | 11 | 1716 | 13 | 10 | 3 | 2 | 74217 | 14 | 7 | 4 | 5 | 181925268 |
| 13 | 1 | 4 | 10 | 15730 | 13 | 11 | 2 | 2 | 4433 | 14 | 8 | 3 | 5 | 44221632 |
| 13 | 2 | 3 | 10 | 74217 | 13 | 12 | 1 | 2 | 78 | 14 | 9 | 2 | 5 | 4557553 |
| 13 | 3 | 2 | 10 | 74217 | 13 | 1 | 13 | 1 | 1 | 14 | 10 | 1 | 5 | 143143 |
| 13 | 4 | 1 | 10 | 15730 | 13 | 2 | 12 | 1 | 78 | 14 | 1 | 11 | 4 | 26026 |
| 13 | 1 | 5 | 9 | 70785 | 13 | 3 | 11 | 1 | 1716 | 14 | 2 | 10 | 4 | 1053052 |
| 13 | 2 | 4 | 9 | 525525 | 13 | 4 | 10 | 1 | 15730 | 14 | 3 | 9 | 4 | 13043030 |
| 13 | 3 | 3 | 9 | 960960 | 13 | 5 | 9 | 1 | 70785 | 14 | 4 | 8 | 4 | 69432090 |
| 13 | 4 | 2 | 9 | 525525 | 13 | 6 | 8 | 1 | 169884 | 14 | 5 | 7 | 4 | 181925268 |
| 13 | 5 | 1 | 9 | 70785 | 13 | 7 | 7 | 1 | 226512 | 14 | 6 | 6 | 4 | 249321114 |
| 13 | 1 | 6 | 8 | 169884 | 13 | 8 | 6 | 1 | 169884 | 14 | 7 | 5 | 4 | 181925268 |
| 13 | 2 | 5 | 8 | 1827683 | 13 | 9 | 5 | 1 | 70785 | 14 | 8 | 4 | 4 | 69432090 |
| 13 | 3 | 4 | 8 | 5264545 | 13 | 10 | 4 | 1 | 15730 | 14 | 9 | 3 | 4 | 13043030 |
| 13 | 4 | 3 | 8 | 5264545 | 13 | 11 | 3 | 1 | 1716 | 14 | 10 | 2 | 4 | 1053052 |
| 13 | 5 | 2 | 8 | 1827683 | 13 | 12 | 2 | 1 | 78 | 14 | 11 | 1 | 4 | 26026 |
| 13 | 6 | 1 | 8 | 169884 | 13 | 13 | 1 | 1 | 1 | 14 | 1 | 12 | 3 | 2366 |
| 13 | 1 | 7 | 7 | 226512 | | | | | | 14 | 2 | 11 | 3 | 122122 |
| 13 | 2 | 6 | 7 | 3356522 | 13 | | sum | | 608583680 | 14 | 3 | 10 | 3 | 1919918 |
| 13 | 3 | 5 | 7 | 14019928 | | | | | | 14 | 4 | 9 | 3 | 13043030 |
| 13 | 4 | 4 | 7 | 22089600 | 14 | 1 | 1 | 14 | 1 | 14 | 5 | 8 | 3 | 44221632 |
| 13 | 5 | 3 | 7 | 14019928 | 14 | 1 | 2 | 13 | 91 | 14 | 6 | 7 | 3 | 80231508 |
| 13 | 6 | 2 | 7 | 3356522 | 14 | 2 | 1 | 13 | 91 | 14 | 7 | 6 | 3 | 80231508 |
| 13 | 7 | 1 | 7 | 226512 | 14 | 1 | 3 | 12 | 2366 | 14 | 8 | 5 | 3 | 44221632 |
| 13 | 1 | 8 | 6 | 169884 | 14 | 2 | 2 | 12 | 6097 | 14 | 9 | 4 | 3 | 13043030 |
| 13 | 2 | 7 | 6 | 3356522 | 14 | 3 | 1 | 12 | 2366 | 14 | 10 | 3 | 3 | 1919918 |
| 13 | 3 | 6 | 6 | 19315114 | 14 | 1 | 4 | 11 | 26026 | 14 | 11 | 2 | 3 | 122122 |
| 13 | 4 | 5 | 6 | 44136820 | 14 | 2 | 3 | 11 | 122122 | 14 | 12 | 1 | 3 | 2366 |
| 13 | 5 | 4 | 6 | 44136820 | 14 | 3 | 2 | 11 | 122122 | 14 | 1 | 13 | 2 | 91 |
| 13 | 6 | 3 | 6 | 19315114 | 14 | 4 | 1 | 11 | 26026 | 14 | 2 | 12 | 2 | 6097 |
| 13 | 7 | 2 | 6 | 3356522 | 14 | 1 | 5 | 10 | 143143 | 14 | 3 | 11 | 2 | 122122 |
| 13 | 8 | 1 | 6 | 169884 | 14 | 2 | 4 | 10 | 1053052 | 14 | 4 | 10 | 2 | 1053052 |
| 13 | 1 | 9 | 5 | 70785 | 14 | 3 | 3 | 10 | 1919918 | 14 | 5 | 9 | 2 | 4557553 |
| 13 | 2 | 8 | 5 | 1827683 | 14 | 4 | 2 | 10 | 1053052 | 14 | 6 | 8 | 2 | 10701873 |
| 13 | 3 | 7 | 5 | 14019928 | 14 | 5 | 1 | 10 | 143143 | 14 | 7 | 7 | 2 | 14168988 |
| 13 | 4 | 6 | 5 | 44136820 | 14 | 1 | 6 | 9 | 429429 | 14 | 8 | 6 | 2 | 10701873 |
| 13 | 5 | 5 | 5 | 64013222 | 14 | 2 | 5 | 9 | 4557553 | 14 | 9 | 5 | 2 | 4557553 |
| 13 | 6 | 4 | 5 | 44136820 | 14 | 3 | 4 | 9 | 13043030 | 14 | 10 | 4 | 2 | 1053052 |
| 13 | 7 | 3 | 5 | 14019928 | 14 | 4 | 3 | 9 | 13043030 | 14 | 11 | 3 | 2 | 122122 |
| 13 | 8 | 2 | 5 | 1827683 | 14 | 5 | 2 | 9 | 4557553 | 14 | 12 | 2 | 2 | 6097 |
| 13 | 9 | 1 | 5 | 70785 | 14 | 6 | 1 | 9 | 429429 | 14 | 13 | 1 | 2 | 91 |
| 13 | 1 | 10 | 4 | 15730 | 14 | 1 | 7 | 8 | 736164 | 14 | 1 | 14 | 1 | 1 |
| 13 | 2 | 9 | 4 | 525525 | 14 | 2 | 6 | 8 | 10701873 | 14 | 2 | 13 | 1 | 91 |
| 13 | 3 | 8 | 4 | 5264545 | 14 | 3 | 5 | 8 | 44221632 | 14 | 3 | 12 | 1 | 2366 |
| 13 | 4 | 7 | 4 | 22089600 | 14 | 4 | 4 | 8 | 69432090 | 14 | 4 | 11 | 1 | 26026 |
| 13 | 5 | 6 | 4 | 44136820 | 14 | 5 | 3 | 8 | 44221632 | 14 | 5 | 10 | 1 | 143143 |
| 13 | 6 | 5 | 4 | 44136820 | 14 | 6 | 2 | 8 | 10701873 | 14 | 6 | 9 | 1 | 429429 |
| 13 | 7 | 4 | 4 | 22089600 | 14 | 7 | 1 | 8 | 736164 | 14 | 7 | 8 | 1 | 736164 |
| 13 | 8 | 3 | 4 | 5264545 | 14 | 1 | 8 | 7 | 736164 | 14 | 8 | 7 | 1 | 736164 |
| 13 | 9 | 2 | 4 | 525525 | 14 | 2 | 7 | 7 | 14168988 | 14 | 9 | 6 | 1 | 429429 |
| 13 | 10 | 1 | 4 | 15730 | 14 | 3 | 6 | 7 | 80231508 | 14 | 10 | 5 | 1 | 143143 |
| 13 | 1 | 11 | 3 | 1716 | 14 | 4 | 5 | 7 | 181925268 | 14 | 11 | 4 | 1 | 26026 |
| 13 | 2 | 10 | 3 | 74217 | 14 | 5 | 4 | 7 | 181925268 | 14 | 12 | 3 | 1 | 2366 |
| 13 | 3 | 9 | 3 | 960960 | 14 | 6 | 3 | 7 | 80231508 | 14 | 13 | 2 | 1 | 91 |
| 13 | 4 | 8 | 3 | 5264545 | 14 | 7 | 2 | 7 | 14168988 | 14 | 14 | 1 | 1 | 1 |
| 13 | 5 | 7 | 3 | 14019928 | 14 | 8 | 1 | 7 | 736164 | | | | | |
| 13 | 6 | 6 | 3 | 19315114 | 14 | 1 | 9 | 6 | 429429 | 14 | | sum | | 4107939840 |
| 13 | 7 | 5 | 3 | 14019928 | 14 | 2 | 8 | 6 | 10701873 | | | | | |

A.2 Genus 1

| | | | | | | | | | | | | | | |
|---|---|---|-----|-------|----|---|---|-----|---------|----|---|---|-----|----------|
| d | v | e | f | h | 8 | 2 | 5 | 1 | 1470 | 10 | 1 | 8 | 1 | 330 |
| 3 | 1 | 1 | 1 | 1 | 8 | 3 | 4 | 1 | 4410 | 10 | 2 | 7 | 1 | 6930 |
| | | | | | 8 | 4 | 3 | 1 | 4410 | 10 | 3 | 6 | 1 | 41580 |
| | | | sum | 1 | 8 | 5 | 2 | 1 | 1470 | 10 | 4 | 5 | 1 | 97020 |
| | | | | | 8 | 6 | 1 | 1 | 126 | 10 | 5 | 4 | 1 | 97020 |
| 4 | 1 | 1 | 2 | 5 | | | | | | 10 | 6 | 3 | 1 | 41580 |
| 4 | 1 | 2 | 1 | 5 | 8 | | | sum | 131307 | 10 | 7 | 2 | 1 | 6930 |
| 4 | 2 | 1 | 1 | 5 | | | | | | 10 | 8 | 1 | 1 | 330 |
| | | | | | 9 | 1 | 1 | 7 | 210 | | | | | |
| | | | sum | 15 | 9 | 1 | 2 | 6 | 3360 | 10 | | | sum | 9713835 |
| | | | | | 9 | 2 | 1 | 6 | 3360 | | | | | |
| 5 | 1 | 1 | 3 | 15 | 9 | 1 | 3 | 5 | 14700 | 11 | 1 | 1 | 9 | 495 |
| 5 | 1 | 2 | 2 | 40 | 9 | 2 | 2 | 5 | 37035 | 11 | 1 | 2 | 8 | 13200 |
| 5 | 2 | 1 | 2 | 40 | 9 | 3 | 1 | 5 | 14700 | 11 | 2 | 1 | 8 | 13200 |
| 5 | 1 | 3 | 1 | 15 | 9 | 1 | 4 | 4 | 23520 | 11 | 1 | 3 | 7 | 103950 |
| 5 | 2 | 2 | 1 | 40 | 9 | 2 | 3 | 4 | 108285 | 11 | 2 | 2 | 7 | 259017 |
| 5 | 3 | 1 | 1 | 15 | 9 | 3 | 2 | 4 | 108285 | 11 | 3 | 1 | 7 | 103950 |
| | | | sum | 165 | 9 | 4 | 1 | 4 | 23520 | 11 | 1 | 4 | 6 | 332640 |
| | | | | | 9 | 1 | 5 | 3 | 14700 | 11 | 2 | 3 | 6 | 1493525 |
| | | | | | 9 | 2 | 4 | 3 | 108285 | 11 | 3 | 2 | 6 | 1493525 |
| 6 | 1 | 1 | 4 | 35 | 9 | 3 | 3 | 3 | 197896 | 11 | 4 | 1 | 6 | 332640 |
| 6 | 1 | 2 | 3 | 175 | 9 | 4 | 2 | 3 | 108285 | 11 | 1 | 5 | 5 | 485100 |
| 6 | 2 | 1 | 3 | 175 | 9 | 5 | 1 | 3 | 14700 | 11 | 2 | 4 | 5 | 3420835 |
| 6 | 1 | 3 | 2 | 175 | 9 | 1 | 6 | 2 | 3360 | 11 | 3 | 3 | 5 | 6165478 |
| 6 | 2 | 2 | 2 | 456 | 9 | 2 | 5 | 2 | 37035 | 11 | 4 | 2 | 5 | 3420835 |
| 6 | 3 | 1 | 2 | 175 | 9 | 3 | 4 | 2 | 108285 | 11 | 5 | 1 | 5 | 485100 |
| 6 | 1 | 4 | 1 | 35 | 9 | 4 | 3 | 2 | 108285 | 11 | 1 | 6 | 4 | 332640 |
| 6 | 2 | 3 | 1 | 175 | 9 | 5 | 2 | 2 | 37035 | 11 | 2 | 5 | 4 | 3420835 |
| 6 | 3 | 2 | 1 | 175 | 9 | 6 | 1 | 2 | 3360 | 11 | 3 | 4 | 4 | 9684433 |
| 6 | 4 | 1 | 1 | 35 | 9 | 1 | 7 | 1 | 210 | 11 | 4 | 3 | 4 | 9684433 |
| | | | sum | 1611 | 9 | 2 | 6 | 1 | 3360 | 11 | 5 | 2 | 4 | 3420835 |
| | | | | | 9 | 3 | 5 | 1 | 14700 | 11 | 6 | 1 | 4 | 332640 |
| | | | | | 9 | 4 | 4 | 1 | 23520 | 11 | 1 | 7 | 3 | 103950 |
| 7 | 1 | 1 | 5 | 70 | 9 | 5 | 3 | 1 | 14700 | 11 | 2 | 6 | 3 | 1493525 |
| 7 | 1 | 2 | 4 | 560 | 9 | 6 | 2 | 1 | 3360 | 11 | 3 | 5 | 3 | 6165478 |
| 7 | 2 | 1 | 4 | 560 | 9 | 7 | 1 | 1 | 210 | 11 | 4 | 4 | 3 | 9684433 |
| 7 | 1 | 3 | 3 | 1050 | | | | | | 11 | 5 | 3 | 3 | 6165478 |
| 7 | 2 | 2 | 3 | 2695 | 9 | | | sum | 1138261 | 11 | 6 | 2 | 3 | 1493525 |
| 7 | 3 | 1 | 3 | 1050 | | | | | | 11 | 7 | 1 | 3 | 103950 |
| 7 | 1 | 4 | 2 | 560 | 10 | 1 | 1 | 8 | 330 | 11 | 1 | 8 | 2 | 13200 |
| 7 | 2 | 3 | 2 | 2695 | 10 | 1 | 2 | 7 | 6930 | 11 | 2 | 7 | 2 | 259017 |
| 7 | 3 | 2 | 2 | 2695 | 10 | 2 | 1 | 7 | 6930 | 11 | 3 | 6 | 2 | 1493525 |
| 7 | 4 | 1 | 2 | 560 | 10 | 1 | 3 | 6 | 41580 | 11 | 4 | 5 | 2 | 3420835 |
| 7 | 1 | 5 | 1 | 70 | 10 | 2 | 2 | 6 | 104115 | 11 | 5 | 4 | 2 | 3420835 |
| 7 | 2 | 4 | 1 | 560 | 10 | 3 | 1 | 6 | 41580 | 11 | 6 | 3 | 2 | 1493525 |
| 7 | 3 | 3 | 1 | 1050 | 10 | 1 | 4 | 5 | 97020 | 11 | 7 | 2 | 2 | 259017 |
| 7 | 4 | 2 | 1 | 560 | 10 | 2 | 3 | 5 | 440440 | 11 | 8 | 1 | 2 | 13200 |
| 7 | 5 | 1 | 1 | 70 | 10 | 3 | 2 | 5 | 440440 | 11 | 1 | 9 | 1 | 495 |
| | | | sum | 14805 | 10 | 4 | 1 | 5 | 97020 | 11 | 2 | 8 | 1 | 13200 |
| | | | | | 10 | 1 | 5 | 4 | 97020 | 11 | 3 | 7 | 1 | 103950 |
| | | | | | 10 | 2 | 4 | 4 | 697250 | 11 | 4 | 6 | 1 | 332640 |
| 8 | 1 | 1 | 6 | 126 | 10 | 3 | 3 | 4 | 1264310 | 11 | 5 | 5 | 1 | 485100 |
| 8 | 1 | 2 | 5 | 1470 | 10 | 4 | 2 | 4 | 697250 | 11 | 6 | 4 | 1 | 332640 |
| 8 | 2 | 1 | 5 | 1470 | 10 | 5 | 1 | 4 | 97020 | 11 | 7 | 3 | 1 | 103950 |
| 8 | 1 | 3 | 4 | 4410 | 10 | 1 | 6 | 3 | 41580 | 11 | 8 | 2 | 1 | 13200 |
| 8 | 2 | 2 | 4 | 11199 | 10 | 2 | 5 | 3 | 440440 | 11 | 9 | 1 | 1 | 495 |
| 8 | 3 | 1 | 4 | 4410 | 10 | 3 | 4 | 3 | 1264310 | | | | | |
| 8 | 1 | 4 | 3 | 4410 | 10 | 4 | 3 | 3 | 1264310 | 11 | | | sum | 81968469 |
| 8 | 2 | 3 | 3 | 20684 | 10 | 5 | 2 | 3 | 440440 | | | | | |
| 8 | 3 | 2 | 3 | 20684 | 10 | 6 | 1 | 3 | 41580 | 12 | 1 | 1 | 10 | 715 |
| 8 | 4 | 1 | 3 | 4410 | 10 | 1 | 7 | 2 | 6930 | 12 | 1 | 2 | 9 | 23595 |
| 8 | 1 | 5 | 2 | 1470 | 10 | 2 | 6 | 2 | 104115 | 12 | 2 | 1 | 9 | 23595 |
| 8 | 2 | 4 | 2 | 11199 | 10 | 3 | 5 | 2 | 440440 | 12 | 1 | 3 | 8 | 235950 |
| 8 | 3 | 3 | 2 | 20684 | 10 | 4 | 4 | 2 | 697250 | 12 | 2 | 2 | 8 | 585585 |
| 8 | 4 | 2 | 2 | 11199 | 10 | 5 | 3 | 2 | 440440 | 12 | 3 | 1 | 8 | 235950 |
| 8 | 5 | 1 | 2 | 1470 | 10 | 6 | 2 | 2 | 104115 | 12 | 1 | 4 | 7 | 990990 |
| 8 | 1 | 6 | 1 | 126 | 10 | 7 | 1 | 2 | 6930 | 12 | 2 | 3 | 7 | 4410120 |

A.3 Genus 2

| | | | | |
|------------------|----------|---------------|-----------|-----------------|
| d v e f h | 10 2 5 1 | 167013 | 12 1 8 1 | 88803 |
| 5 1 1 1 8 | 10 3 4 1 | 471240 | 12 2 7 1 | 1585584 |
| | 10 4 3 1 | 471240 | 12 3 6 1 | 8654646 |
| 5 | 10 5 2 1 | 167013 | 12 4 5 1 | 19324305 |
| sum 8 | 10 6 1 1 | 16401 | 12 5 4 1 | 19324305 |
| | | | 12 6 3 1 | 8654646 |
| 6 1 1 2 84 | | | 12 7 2 1 | 1585584 |
| 6 1 2 1 84 | 10 | sum 13545216 | 12 8 1 1 | 88803 |
| 6 2 1 1 84 | | | | |
| | 11 1 1 7 | 39963 | | |
| 6 | 11 1 2 6 | 550011 | 12 | sum 1805010948 |
| sum 252 | 11 2 1 6 | 550011 | | |
| | 11 1 3 5 | 2221065 | 13 1 1 9 | 183183 |
| 7 1 1 3 469 | 11 2 2 5 | 5409019 | 13 1 2 8 | 4114110 |
| 7 1 2 2 1183 | 11 3 1 5 | 2221065 | 13 2 1 8 | 4114110 |
| 7 2 1 2 1183 | 11 1 4 4 | 3465000 | 13 1 3 7 | 29135106 |
| 7 1 3 1 469 | 11 2 3 4 | 15014846 | 13 2 2 7 | 70367479 |
| 7 2 2 1 1183 | 11 3 2 4 | 15014846 | 13 3 1 7 | 29135106 |
| 7 3 1 1 469 | 11 4 1 4 | 3465000 | 13 1 4 6 | 87933846 |
| | 11 1 5 3 | 2221065 | 13 2 3 6 | 374127663 |
| 7 | 11 2 4 3 | 15014846 | 13 3 2 6 | 374127663 |
| sum 4956 | 11 3 3 3 | 26717482 | 13 4 1 6 | 87933846 |
| | 11 4 2 3 | 15014846 | 13 1 5 5 | 125855730 |
| 8 1 1 4 1869 | 11 5 1 3 | 2221065 | 13 2 4 5 | 824962502 |
| 8 1 2 3 8526 | 11 1 6 2 | 550011 | 13 3 3 5 | 1453414846 |
| 8 2 1 3 8526 | 11 2 5 2 | 5409019 | 13 4 2 5 | 824962502 |
| 8 1 3 2 8526 | 11 3 4 2 | 15014846 | 13 5 1 5 | 125855730 |
| 8 2 2 2 21229 | 11 4 3 2 | 15014846 | 13 1 6 4 | 87933846 |
| 8 3 1 2 8526 | 11 5 2 2 | 5409019 | 13 2 5 4 | 824962502 |
| 8 1 4 1 1869 | 11 6 1 2 | 550011 | 13 3 4 4 | 2239280420 |
| 8 2 3 1 8526 | 11 6 1 2 | 550011 | 13 4 3 4 | 2239280420 |
| 8 3 2 1 8526 | 11 7 1 1 | 39963 | 13 5 2 4 | 824962502 |
| 8 4 1 1 1869 | 11 2 6 1 | 550011 | 13 6 1 4 | 87933846 |
| | 11 3 5 1 | 2221065 | 13 7 3 3 | 29135106 |
| 8 | 11 4 4 1 | 3465000 | 13 2 6 3 | 374127663 |
| sum 77992 | 11 5 3 1 | 2221065 | 13 3 5 3 | 1453414846 |
| | 11 6 2 1 | 550011 | 13 4 4 3 | 2239280420 |
| 9 1 1 5 5985 | 11 7 1 1 | 39963 | 13 5 3 3 | 1453414846 |
| 9 1 2 4 42588 | | | 13 6 2 3 | 374127663 |
| 9 2 1 4 42588 | 11 | sum 160174960 | 13 7 1 3 | 29135106 |
| 9 1 3 3 77028 | | | 13 1 8 2 | 4114110 |
| 9 2 2 3 189999 | 12 1 1 8 | 88803 | 13 2 7 2 | 70367479 |
| 9 3 1 3 77028 | 12 1 2 7 | 1585584 | 13 3 6 2 | 374127663 |
| 9 1 4 2 42588 | 12 2 1 7 | 1585584 | 13 4 5 2 | 824962502 |
| 9 2 3 2 189999 | 12 1 3 6 | 8654646 | 13 5 4 2 | 824962502 |
| 9 3 2 2 189999 | 12 2 2 6 | 20981337 | 13 6 3 2 | 374127663 |
| 9 4 1 2 42588 | 12 3 1 6 | 8654646 | 13 7 2 2 | 70367479 |
| 9 1 5 1 5985 | 12 1 4 5 | 19324305 | 13 8 1 2 | 4114110 |
| 9 2 4 1 42588 | 12 2 3 5 | 82897296 | 13 9 1 1 | 183183 |
| 9 3 3 1 77028 | 12 3 2 5 | 82897296 | 13 2 8 1 | 4114110 |
| 9 4 2 1 42588 | 12 4 1 5 | 19324305 | 13 3 7 1 | 29135106 |
| 9 5 1 1 5985 | 12 2 4 4 | 128420004 | 13 4 6 1 | 87933846 |
| | 12 3 3 4 | 227256510 | 13 5 5 1 | 125855730 |
| 9 | 12 4 2 4 | 128420004 | 13 6 4 1 | 87933846 |
| sum 1074564 | 12 5 1 4 | 19324305 | 13 7 3 1 | 29135106 |
| | 12 1 6 3 | 8654646 | 13 8 2 1 | 4114110 |
| 10 1 1 6 16401 | 12 2 5 3 | 82897296 | 13 9 1 1 | 183183 |
| 10 1 2 5 167013 | 12 3 4 3 | 227256510 | | |
| 10 2 1 5 167013 | 12 4 3 3 | 227256510 | 13 | sum 19588944336 |
| 10 1 3 4 471240 | 12 5 2 3 | 82897296 | | |
| 10 2 2 4 1154095 | 12 6 1 3 | 8654646 | 14 1 1 10 | 355355 |
| 10 3 1 4 471240 | 12 1 7 2 | 1585584 | 14 1 2 9 | 9798789 |
| 10 1 4 3 471240 | 12 2 6 2 | 20981337 | 14 2 1 9 | 9798789 |
| 10 2 3 3 2068070 | 12 3 5 2 | 82897296 | 14 1 3 8 | 87291204 |
| 10 3 2 3 2068070 | 12 4 4 2 | 128420004 | 14 2 2 8 | 210164227 |
| 10 4 1 3 471240 | 12 5 3 2 | 82897296 | 14 3 1 8 | 87291204 |
| 10 1 5 2 167013 | 12 6 2 2 | 20981337 | 14 1 4 7 | 341825484 |
| 10 2 4 2 1154095 | 12 7 1 2 | 1585584 | 14 2 3 7 | 1444432612 |
| 10 3 3 2 2068070 | | | | |
| 10 4 2 2 1154095 | | | | |
| 10 5 1 2 167013 | | | | |
| 10 1 6 1 16401 | | | | |

B First numbers of unrooted hypermaps

The following sections show the numbers H of unrooted hypermaps of genus g with d darts, v vertices, e edges and $d - v - e + 2(1 - g)$ faces, for $g \leq 6$ and $d \leq 14$.

B.1 Genus 0

| | | | | | | | | | | | | | | |
|---|---|---|-----|----|---|---|---|-----|------|---|---|---|-----|------|
| d | v | e | f | H | 6 | 1 | 5 | 2 | 3 | 8 | 2 | 5 | 3 | 309 |
| 1 | 1 | 1 | 1 | 1 | 6 | 2 | 4 | 2 | 24 | 8 | 3 | 4 | 3 | 946 |
| 1 | | | sum | 1 | 6 | 3 | 3 | 2 | 46 | 8 | 4 | 3 | 3 | 946 |
| | | | | | 6 | 4 | 2 | 2 | 24 | 8 | 5 | 2 | 3 | 309 |
| | | | | | 6 | 5 | 1 | 2 | 3 | 8 | 6 | 1 | 3 | 26 |
| 2 | 1 | 1 | 2 | 1 | 6 | 1 | 6 | 1 | 1 | 8 | 1 | 7 | 2 | 4 |
| 2 | 1 | 2 | 1 | 1 | 6 | 2 | 5 | 1 | 3 | 8 | 2 | 6 | 2 | 67 |
| 2 | 2 | 1 | 1 | 1 | 6 | 3 | 4 | 1 | 10 | 8 | 3 | 5 | 2 | 309 |
| | | | sum | 3 | 6 | 4 | 3 | 1 | 10 | 8 | 4 | 4 | 2 | 505 |
| | | | | | 6 | 5 | 2 | 1 | 3 | 8 | 5 | 3 | 2 | 309 |
| | | | | | 6 | 6 | 1 | 1 | 1 | 8 | 6 | 2 | 2 | 67 |
| 3 | 1 | 1 | 3 | 1 | | | | | | 8 | 7 | 1 | 2 | 4 |
| 3 | 1 | 2 | 2 | 1 | 6 | | | sum | 291 | 8 | 1 | 8 | 1 | 1 |
| 3 | 2 | 1 | 2 | 1 | | | | | | 8 | 2 | 7 | 1 | 4 |
| 3 | 1 | 3 | 1 | 1 | 7 | 1 | 1 | 7 | 1 | 8 | 3 | 6 | 1 | 26 |
| 3 | 2 | 2 | 1 | 1 | 7 | 1 | 2 | 6 | 3 | 8 | 4 | 5 | 1 | 64 |
| 3 | 3 | 1 | 1 | 1 | 7 | 2 | 1 | 6 | 3 | 8 | 5 | 4 | 1 | 64 |
| | | | sum | 6 | 7 | 1 | 3 | 5 | 15 | 8 | 6 | 3 | 1 | 26 |
| | | | | | 7 | 2 | 2 | 5 | 40 | 8 | 7 | 2 | 1 | 4 |
| | | | | | 7 | 3 | 1 | 5 | 15 | 8 | 8 | 1 | 1 | 1 |
| 4 | 1 | 1 | 4 | 1 | 7 | 1 | 4 | 4 | 25 | | | | | |
| 4 | 1 | 2 | 3 | 2 | 7 | 2 | 3 | 4 | 127 | 8 | | | sum | 6975 |
| 4 | 2 | 1 | 3 | 2 | 7 | 3 | 2 | 4 | 127 | | | | | |
| 4 | 1 | 3 | 2 | 2 | 7 | 4 | 1 | 4 | 25 | 9 | 1 | 1 | 9 | 1 |
| 4 | 2 | 2 | 2 | 5 | 7 | 1 | 5 | 3 | 15 | 9 | 1 | 2 | 8 | 4 |
| 4 | 3 | 1 | 2 | 2 | 7 | 2 | 4 | 3 | 127 | 9 | 2 | 1 | 8 | 4 |
| 4 | 1 | 4 | 1 | 1 | 7 | 3 | 3 | 3 | 242 | 9 | 1 | 3 | 7 | 38 |
| 4 | 2 | 3 | 1 | 2 | 7 | 4 | 2 | 3 | 127 | 9 | 2 | 2 | 7 | 98 |
| 4 | 3 | 2 | 1 | 2 | 7 | 5 | 1 | 3 | 15 | 9 | 3 | 1 | 7 | 38 |
| 4 | 4 | 1 | 1 | 1 | 7 | 1 | 6 | 2 | 3 | 9 | 1 | 4 | 6 | 132 |
| | | | sum | 20 | 7 | 2 | 5 | 2 | 40 | 9 | 2 | 3 | 6 | 640 |
| | | | | | 7 | 3 | 4 | 2 | 127 | 9 | 3 | 2 | 6 | 640 |
| | | | | | 7 | 4 | 3 | 2 | 127 | 9 | 4 | 1 | 6 | 132 |
| 5 | 1 | 1 | 5 | 1 | 7 | 5 | 2 | 2 | 40 | 9 | 1 | 5 | 5 | 196 |
| 5 | 1 | 2 | 4 | 2 | 7 | 6 | 1 | 2 | 3 | 9 | 2 | 4 | 5 | 1549 |
| 5 | 2 | 1 | 4 | 2 | 7 | 1 | 7 | 1 | 1 | 9 | 3 | 3 | 5 | 2890 |
| 5 | 1 | 3 | 3 | 4 | 7 | 2 | 6 | 1 | 3 | 9 | 4 | 2 | 5 | 1549 |
| 5 | 2 | 2 | 3 | 11 | 7 | 3 | 5 | 1 | 15 | 9 | 5 | 1 | 5 | 196 |
| 5 | 3 | 1 | 3 | 4 | 7 | 4 | 4 | 1 | 25 | 9 | 1 | 6 | 4 | 132 |
| 5 | 1 | 4 | 2 | 2 | 7 | 5 | 3 | 1 | 15 | 9 | 2 | 5 | 4 | 1549 |
| 5 | 2 | 3 | 2 | 11 | 7 | 6 | 2 | 1 | 3 | 9 | 3 | 4 | 4 | 4671 |
| 5 | 3 | 2 | 2 | 11 | 7 | 7 | 1 | 1 | 1 | 9 | 4 | 3 | 4 | 4671 |
| 5 | 4 | 1 | 2 | 2 | | | | | | 9 | 5 | 2 | 4 | 1549 |
| 5 | 1 | 5 | 1 | 1 | 7 | | | sum | 1310 | 9 | 6 | 1 | 4 | 132 |
| 5 | 2 | 4 | 1 | 2 | | | | | | 9 | 1 | 7 | 3 | 38 |
| 5 | 3 | 3 | 1 | 4 | 8 | 1 | 1 | 8 | 1 | 9 | 2 | 6 | 3 | 640 |
| 5 | 4 | 2 | 1 | 2 | 8 | 1 | 2 | 7 | 4 | 9 | 3 | 5 | 3 | 2890 |
| 5 | 5 | 1 | 1 | 1 | 8 | 2 | 1 | 7 | 4 | 9 | 4 | 4 | 3 | 4671 |
| | | | sum | 60 | 8 | 1 | 3 | 6 | 26 | 9 | 5 | 3 | 3 | 2890 |
| | | | | | 8 | 2 | 2 | 6 | 67 | 9 | 6 | 2 | 3 | 640 |
| | | | | | 8 | 3 | 1 | 6 | 26 | 9 | 7 | 1 | 3 | 38 |
| 6 | 1 | 1 | 6 | 1 | 8 | 1 | 4 | 5 | 64 | 9 | 1 | 8 | 2 | 4 |
| 6 | 1 | 2 | 5 | 3 | 8 | 2 | 3 | 5 | 309 | 9 | 2 | 7 | 2 | 98 |
| 6 | 2 | 1 | 5 | 3 | 8 | 3 | 2 | 5 | 309 | 9 | 3 | 6 | 2 | 640 |
| 6 | 1 | 3 | 4 | 10 | 8 | 4 | 1 | 5 | 64 | 9 | 4 | 5 | 2 | 1549 |
| 6 | 2 | 2 | 4 | 24 | 8 | 1 | 5 | 4 | 64 | 9 | 5 | 4 | 2 | 1549 |
| 6 | 3 | 1 | 4 | 10 | 8 | 2 | 4 | 4 | 505 | 9 | 6 | 3 | 2 | 640 |
| 6 | 1 | 4 | 3 | 10 | 8 | 3 | 3 | 4 | 946 | 9 | 7 | 2 | 2 | 98 |
| 6 | 2 | 3 | 3 | 46 | 8 | 4 | 2 | 4 | 505 | 9 | 8 | 1 | 2 | 4 |
| 6 | 3 | 2 | 3 | 46 | 8 | 5 | 1 | 4 | 64 | 9 | 1 | 9 | 1 | 1 |
| 6 | 4 | 1 | 3 | 10 | 8 | 1 | 6 | 3 | 26 | 9 | 2 | 8 | 1 | 4 |

| | | | | | | | | | | | | | | |
|----|----|-----|----|--------|----|----|-----|----|---------|----|----|----|----|--------|
| 9 | 3 | 7 | 1 | 38 | 11 | 2 | 1 | 10 | 5 | 12 | 1 | 3 | 10 | 104 |
| 9 | 4 | 6 | 1 | 132 | 11 | 1 | 3 | 9 | 75 | 12 | 2 | 2 | 10 | 265 |
| 9 | 5 | 5 | 1 | 196 | 11 | 2 | 2 | 9 | 195 | 12 | 3 | 1 | 10 | 104 |
| 9 | 6 | 4 | 1 | 132 | 11 | 3 | 1 | 9 | 75 | 12 | 1 | 4 | 9 | 765 |
| 9 | 7 | 3 | 1 | 38 | 11 | 1 | 4 | 8 | 450 | 12 | 2 | 3 | 9 | 3605 |
| 9 | 8 | 2 | 1 | 4 | 11 | 2 | 3 | 8 | 2154 | 12 | 3 | 2 | 9 | 3605 |
| 9 | 9 | 1 | 1 | 1 | 11 | 3 | 2 | 8 | 2154 | 12 | 4 | 1 | 9 | 765 |
| 9 | | | | | 11 | 4 | 1 | 8 | 450 | 12 | 1 | 5 | 8 | 2736 |
| 9 | | sum | | 37746 | 11 | 1 | 5 | 7 | 1260 | 12 | 2 | 4 | 8 | 20472 |
| 10 | 1 | 1 | 10 | 1 | 11 | 2 | 4 | 7 | 9585 | 12 | 3 | 3 | 8 | 37545 |
| 10 | 1 | 2 | 9 | 5 | 11 | 3 | 3 | 7 | 17664 | 12 | 4 | 2 | 8 | 20472 |
| 10 | 2 | 1 | 9 | 5 | 11 | 4 | 2 | 7 | 9585 | 12 | 5 | 1 | 8 | 2736 |
| 10 | 1 | 3 | 8 | 56 | 11 | 5 | 1 | 7 | 1260 | 12 | 1 | 6 | 7 | 5102 |
| 10 | 2 | 2 | 8 | 144 | 11 | 1 | 6 | 6 | 1764 | 12 | 2 | 5 | 7 | 55633 |
| 10 | 3 | 1 | 8 | 56 | 11 | 2 | 5 | 6 | 19691 | 12 | 3 | 4 | 7 | 161455 |
| 10 | 1 | 4 | 7 | 256 | 11 | 3 | 4 | 6 | 57715 | 12 | 4 | 3 | 7 | 161455 |
| 10 | 2 | 3 | 7 | 1226 | 11 | 4 | 3 | 6 | 57715 | 12 | 5 | 2 | 7 | 55633 |
| 10 | 3 | 2 | 7 | 1226 | 11 | 5 | 2 | 6 | 19691 | 12 | 6 | 1 | 7 | 5102 |
| 10 | 4 | 1 | 7 | 256 | 11 | 6 | 1 | 6 | 1764 | 12 | 1 | 7 | 6 | 5102 |
| 10 | 1 | 5 | 6 | 536 | 11 | 1 | 7 | 5 | 1260 | 12 | 2 | 6 | 6 | 77174 |
| 10 | 2 | 4 | 6 | 4111 | 11 | 2 | 6 | 5 | 19691 | 12 | 3 | 5 | 6 | 326432 |
| 10 | 3 | 3 | 6 | 7606 | 11 | 3 | 5 | 5 | 84714 | 12 | 4 | 4 | 6 | 516507 |
| 10 | 4 | 2 | 6 | 4111 | 11 | 4 | 4 | 5 | 134750 | 12 | 5 | 3 | 6 | 326432 |
| 10 | 5 | 1 | 6 | 536 | 11 | 5 | 3 | 5 | 84714 | 12 | 6 | 2 | 6 | 77174 |
| 10 | 1 | 6 | 5 | 536 | 11 | 6 | 2 | 5 | 19691 | 12 | 7 | 1 | 6 | 5102 |
| 10 | 2 | 5 | 5 | 6081 | 11 | 7 | 1 | 5 | 1260 | 12 | 1 | 8 | 5 | 2736 |
| 10 | 3 | 4 | 5 | 18019 | 11 | 1 | 8 | 4 | 450 | 12 | 2 | 7 | 5 | 55633 |
| 10 | 4 | 3 | 5 | 18019 | 11 | 2 | 7 | 4 | 9585 | 12 | 3 | 6 | 5 | 326432 |
| 10 | 5 | 2 | 5 | 6081 | 11 | 3 | 6 | 4 | 57715 | 12 | 4 | 5 | 5 | 752940 |
| 10 | 6 | 1 | 5 | 536 | 11 | 4 | 5 | 4 | 134750 | 12 | 5 | 4 | 5 | 752940 |
| 10 | 1 | 7 | 4 | 256 | 11 | 5 | 4 | 4 | 134750 | 12 | 6 | 3 | 5 | 326432 |
| 10 | 2 | 6 | 4 | 4111 | 11 | 6 | 3 | 4 | 57715 | 12 | 7 | 2 | 5 | 55633 |
| 10 | 3 | 5 | 4 | 18019 | 11 | 7 | 2 | 4 | 9585 | 12 | 8 | 1 | 5 | 2736 |
| 10 | 4 | 4 | 4 | 28852 | 11 | 8 | 1 | 4 | 450 | 12 | 1 | 9 | 4 | 765 |
| 10 | 5 | 3 | 4 | 18019 | 11 | 1 | 9 | 3 | 75 | 12 | 2 | 8 | 4 | 20472 |
| 10 | 6 | 2 | 4 | 4111 | 11 | 2 | 8 | 3 | 2154 | 12 | 3 | 7 | 4 | 161455 |
| 10 | 7 | 1 | 4 | 256 | 11 | 3 | 7 | 3 | 17664 | 12 | 4 | 6 | 4 | 516507 |
| 10 | 1 | 8 | 3 | 56 | 11 | 4 | 6 | 3 | 57715 | 12 | 5 | 5 | 4 | 752940 |
| 10 | 2 | 7 | 3 | 1226 | 11 | 5 | 5 | 3 | 84714 | 12 | 6 | 4 | 4 | 516507 |
| 10 | 3 | 6 | 3 | 7606 | 11 | 6 | 4 | 3 | 57715 | 12 | 7 | 3 | 4 | 161455 |
| 10 | 4 | 5 | 3 | 18019 | 11 | 7 | 3 | 3 | 17664 | 12 | 8 | 2 | 4 | 20472 |
| 10 | 5 | 4 | 3 | 18019 | 11 | 8 | 2 | 3 | 2154 | 12 | 9 | 1 | 4 | 765 |
| 10 | 6 | 3 | 3 | 7606 | 11 | 9 | 1 | 3 | 75 | 12 | 1 | 10 | 3 | 104 |
| 10 | 7 | 2 | 3 | 1226 | 11 | 1 | 10 | 2 | 5 | 12 | 2 | 9 | 3 | 3605 |
| 10 | 8 | 1 | 3 | 56 | 11 | 2 | 9 | 2 | 195 | 12 | 3 | 8 | 3 | 37545 |
| 10 | 1 | 9 | 2 | 5 | 11 | 3 | 8 | 2 | 2154 | 12 | 4 | 7 | 3 | 161455 |
| 10 | 2 | 8 | 2 | 144 | 11 | 4 | 7 | 2 | 9585 | 12 | 5 | 6 | 3 | 326432 |
| 10 | 3 | 7 | 2 | 1226 | 11 | 5 | 6 | 2 | 19691 | 12 | 6 | 5 | 3 | 326432 |
| 10 | 4 | 6 | 2 | 4111 | 11 | 6 | 5 | 2 | 19691 | 12 | 7 | 4 | 3 | 161455 |
| 10 | 5 | 5 | 2 | 6081 | 11 | 7 | 4 | 2 | 9585 | 12 | 8 | 3 | 3 | 37545 |
| 10 | 6 | 4 | 2 | 4111 | 11 | 8 | 3 | 2 | 2154 | 12 | 9 | 2 | 3 | 3605 |
| 10 | 7 | 3 | 2 | 1226 | 11 | 9 | 2 | 2 | 195 | 12 | 10 | 1 | 3 | 104 |
| 10 | 8 | 2 | 2 | 144 | 11 | 10 | 1 | 2 | 5 | 12 | 1 | 11 | 2 | 6 |
| 10 | 9 | 1 | 2 | 5 | 11 | 1 | 11 | 1 | 1 | 12 | 2 | 10 | 2 | 265 |
| 10 | 1 | 10 | 1 | 1 | 11 | 2 | 10 | 1 | 5 | 12 | 3 | 9 | 2 | 3605 |
| 10 | 2 | 9 | 1 | 5 | 11 | 3 | 9 | 1 | 75 | 12 | 4 | 8 | 2 | 20472 |
| 10 | 3 | 8 | 1 | 56 | 11 | 4 | 8 | 1 | 450 | 12 | 5 | 7 | 2 | 55633 |
| 10 | 4 | 7 | 1 | 256 | 11 | 5 | 7 | 1 | 1260 | 12 | 6 | 6 | 2 | 77174 |
| 10 | 5 | 6 | 1 | 536 | 11 | 6 | 6 | 1 | 1764 | 12 | 7 | 5 | 2 | 55633 |
| 10 | 6 | 5 | 1 | 536 | 11 | 7 | 5 | 1 | 1260 | 12 | 8 | 4 | 2 | 20472 |
| 10 | 7 | 4 | 1 | 256 | 11 | 8 | 4 | 1 | 450 | 12 | 9 | 3 | 2 | 3605 |
| 10 | 8 | 3 | 1 | 56 | 11 | 9 | 3 | 1 | 75 | 12 | 10 | 2 | 2 | 265 |
| 10 | 9 | 2 | 1 | 5 | 11 | 10 | 2 | 1 | 5 | 12 | 11 | 1 | 2 | 6 |
| 10 | 10 | 1 | 1 | 1 | 11 | 11 | 1 | 1 | 1 | 12 | 1 | 12 | 1 | 1 |
| 10 | | | | | 11 | | | | | 12 | 2 | 11 | 1 | 6 |
| 10 | | sum | | 215602 | 11 | | sum | | 1262874 | 12 | 3 | 10 | 1 | 104 |
| 11 | 1 | 1 | 11 | 1 | 12 | 1 | 1 | 12 | 1 | 12 | 4 | 9 | 1 | 765 |
| 11 | 1 | 2 | 10 | 5 | 12 | 1 | 2 | 11 | 6 | 12 | 5 | 8 | 1 | 2736 |
| | | | | | 12 | 2 | 1 | 11 | 6 | 12 | 6 | 7 | 1 | 5102 |
| | | | | | | | | | | 12 | 7 | 6 | 1 | 5102 |

B.2 Genus 1

| | | | | | | | | | | | | | | |
|---|---|---|-----|------|----|---|---|-----|--------|----|---|-----|----|---------|
| d | v | e | f | H | 8 | 2 | 5 | 1 | 187 | 10 | 1 | 8 | 1 | 34 |
| 3 | 1 | 1 | 1 | 1 | 8 | 3 | 4 | 1 | 557 | 10 | 2 | 7 | 1 | 698 |
| | | | | | 8 | 4 | 3 | 1 | 557 | 10 | 3 | 6 | 1 | 4172 |
| | | | sum | 1 | 8 | 5 | 2 | 1 | 187 | 10 | 4 | 5 | 1 | 9724 |
| | | | | | 8 | 6 | 1 | 1 | 17 | 10 | 5 | 4 | 1 | 9724 |
| 4 | 1 | 1 | 2 | 2 | | | | | | 10 | 6 | 3 | 1 | 4172 |
| 4 | 1 | 2 | 1 | 2 | 8 | | | sum | 16533 | 10 | 7 | 2 | 1 | 698 |
| 4 | 2 | 1 | 1 | 2 | | | | | | 10 | 8 | 1 | 1 | 34 |
| | | | sum | 6 | 9 | 1 | 1 | 7 | 24 | | | | | |
| | | | | | 9 | 1 | 2 | 6 | 374 | 10 | | sum | | 972441 |
| | | | | | 9 | 2 | 1 | 6 | 374 | | | | | |
| 5 | 1 | 1 | 3 | 3 | 9 | 1 | 3 | 5 | 1634 | 11 | 1 | 1 | 9 | 45 |
| 5 | 1 | 2 | 2 | 8 | 9 | 2 | 2 | 5 | 4115 | 11 | 1 | 2 | 8 | 1200 |
| 5 | 2 | 1 | 2 | 8 | 9 | 3 | 1 | 5 | 1634 | 11 | 2 | 1 | 8 | 1200 |
| 5 | 1 | 3 | 1 | 3 | 9 | 1 | 4 | 4 | 2616 | 11 | 1 | 3 | 7 | 9450 |
| 5 | 2 | 2 | 1 | 8 | 9 | 2 | 3 | 4 | 12033 | 11 | 2 | 2 | 7 | 23547 |
| 5 | 3 | 1 | 1 | 3 | 9 | 3 | 2 | 4 | 12033 | 11 | 3 | 1 | 7 | 9450 |
| | | | sum | 33 | 9 | 4 | 1 | 4 | 2616 | 11 | 1 | 4 | 6 | 30240 |
| | | | | | 9 | 1 | 5 | 3 | 1634 | 11 | 2 | 3 | 6 | 135775 |
| | | | | | 9 | 2 | 4 | 3 | 12033 | 11 | 3 | 2 | 6 | 135775 |
| 6 | 1 | 1 | 4 | 7 | 9 | 3 | 3 | 3 | 21990 | 11 | 4 | 1 | 6 | 30240 |
| 6 | 1 | 2 | 3 | 31 | 9 | 4 | 2 | 3 | 12033 | 11 | 1 | 5 | 5 | 44100 |
| 6 | 2 | 1 | 3 | 31 | 9 | 5 | 1 | 3 | 1634 | 11 | 2 | 4 | 5 | 310985 |
| 6 | 1 | 3 | 2 | 31 | 9 | 1 | 6 | 2 | 374 | 11 | 3 | 3 | 5 | 560498 |
| 6 | 2 | 2 | 2 | 78 | 9 | 2 | 5 | 2 | 4115 | 11 | 4 | 2 | 5 | 310985 |
| 6 | 3 | 1 | 2 | 31 | 9 | 3 | 4 | 2 | 12033 | 11 | 5 | 1 | 5 | 44100 |
| 6 | 1 | 4 | 1 | 7 | 9 | 4 | 3 | 2 | 12033 | 11 | 1 | 6 | 4 | 30240 |
| 6 | 2 | 3 | 1 | 31 | 9 | 5 | 2 | 2 | 4115 | 11 | 2 | 5 | 4 | 310985 |
| 6 | 3 | 2 | 1 | 31 | 9 | 6 | 1 | 2 | 374 | 11 | 3 | 4 | 4 | 880403 |
| 6 | 4 | 1 | 1 | 7 | 9 | 1 | 7 | 1 | 24 | 11 | 4 | 3 | 4 | 880403 |
| | | | sum | 285 | 9 | 2 | 6 | 1 | 374 | 11 | 5 | 2 | 4 | 310985 |
| | | | | | 9 | 3 | 5 | 1 | 1634 | 11 | 6 | 1 | 4 | 30240 |
| | | | | | 9 | 4 | 4 | 1 | 2616 | 11 | 1 | 7 | 3 | 9450 |
| 7 | 1 | 1 | 5 | 10 | 9 | 5 | 3 | 1 | 1634 | 11 | 2 | 6 | 3 | 135775 |
| 7 | 1 | 2 | 4 | 80 | 9 | 6 | 2 | 1 | 374 | 11 | 3 | 5 | 3 | 560498 |
| 7 | 2 | 1 | 4 | 80 | 9 | 7 | 1 | 1 | 24 | 11 | 4 | 4 | 3 | 880403 |
| 7 | 1 | 3 | 3 | 150 | | | | | | 11 | 5 | 3 | 3 | 560498 |
| 7 | 2 | 2 | 3 | 385 | 9 | | | sum | 126501 | 11 | 6 | 2 | 3 | 135775 |
| 7 | 3 | 1 | 3 | 150 | | | | | | 11 | 7 | 1 | 3 | 9450 |
| 7 | 1 | 4 | 2 | 80 | 10 | 1 | 1 | 8 | 34 | 11 | 1 | 8 | 2 | 1200 |
| 7 | 2 | 3 | 2 | 385 | 10 | 1 | 2 | 7 | 698 | 11 | 2 | 7 | 2 | 23547 |
| 7 | 3 | 2 | 2 | 385 | 10 | 2 | 1 | 7 | 698 | 11 | 3 | 6 | 2 | 135775 |
| 7 | 4 | 1 | 2 | 80 | 10 | 1 | 3 | 6 | 4172 | 11 | 4 | 5 | 2 | 310985 |
| 7 | 1 | 5 | 1 | 10 | 10 | 2 | 2 | 6 | 10434 | 11 | 5 | 4 | 2 | 310985 |
| 7 | 2 | 4 | 1 | 80 | 10 | 3 | 1 | 6 | 4172 | 11 | 6 | 3 | 2 | 135775 |
| 7 | 3 | 3 | 1 | 150 | 10 | 1 | 4 | 5 | 9724 | 11 | 7 | 2 | 2 | 23547 |
| 7 | 4 | 2 | 1 | 80 | 10 | 2 | 3 | 5 | 44091 | 11 | 8 | 1 | 2 | 1200 |
| 7 | 5 | 1 | 1 | 10 | 10 | 3 | 2 | 5 | 44091 | 11 | 1 | 9 | 1 | 45 |
| | | | sum | 2115 | 10 | 4 | 1 | 5 | 9724 | 11 | 2 | 8 | 1 | 1200 |
| | | | | | 10 | 1 | 5 | 4 | 9724 | 11 | 3 | 7 | 1 | 9450 |
| | | | | | 10 | 2 | 4 | 4 | 69790 | 11 | 4 | 6 | 1 | 30240 |
| 8 | 1 | 1 | 6 | 17 | 10 | 3 | 3 | 4 | 126519 | 11 | 5 | 5 | 1 | 44100 |
| 8 | 1 | 2 | 5 | 187 | 10 | 4 | 2 | 4 | 69790 | 11 | 6 | 4 | 1 | 30240 |
| 8 | 2 | 1 | 5 | 187 | 10 | 5 | 1 | 4 | 9724 | 11 | 7 | 3 | 1 | 9450 |
| 8 | 1 | 3 | 4 | 557 | 10 | 1 | 6 | 3 | 4172 | 11 | 8 | 2 | 1 | 1200 |
| 8 | 2 | 2 | 4 | 1409 | 10 | 2 | 5 | 3 | 44091 | 11 | 9 | 1 | 1 | 45 |
| 8 | 3 | 1 | 4 | 557 | 10 | 3 | 4 | 3 | 126519 | | | | | |
| 8 | 1 | 4 | 3 | 557 | 10 | 4 | 3 | 3 | 126519 | 11 | | sum | | 7451679 |
| 8 | 2 | 3 | 3 | 2597 | 10 | 5 | 2 | 3 | 44091 | | | | | |
| 8 | 3 | 2 | 3 | 2597 | 10 | 6 | 1 | 3 | 4172 | 12 | 1 | 1 | 10 | 62 |
| 8 | 4 | 1 | 3 | 557 | 10 | 1 | 7 | 2 | 698 | 12 | 1 | 2 | 9 | 1976 |
| 8 | 1 | 5 | 2 | 187 | 10 | 2 | 6 | 2 | 10434 | 12 | 2 | 1 | 9 | 1976 |
| 8 | 2 | 4 | 2 | 1409 | 10 | 3 | 5 | 2 | 44091 | 12 | 1 | 3 | 8 | 19694 |
| 8 | 3 | 3 | 2 | 2597 | 10 | 4 | 4 | 2 | 69790 | 12 | 2 | 2 | 8 | 48846 |
| 8 | 4 | 2 | 2 | 1409 | 10 | 5 | 3 | 2 | 44091 | 12 | 3 | 1 | 8 | 19694 |
| 8 | 5 | 1 | 2 | 187 | 10 | 6 | 2 | 2 | 10434 | 12 | 1 | 4 | 7 | 82652 |
| 8 | 1 | 6 | 1 | 17 | 10 | 7 | 1 | 2 | 698 | 12 | 2 | 3 | 7 | 367645 |

B.3 Genus 2

| | | | | | | | | | | | | | | |
|----|---|---|-----|--------|----|---|---|-----|----------|----|---|-----|----|------------|
| d | v | e | f | H | 10 | 2 | 5 | 1 | 16725 | 12 | 1 | 8 | 1 | 7417 |
| 5 | 1 | 1 | 1 | 4 | 10 | 3 | 4 | 1 | 47164 | 12 | 2 | 7 | 1 | 132202 |
| | | | | | 10 | 4 | 3 | 1 | 47164 | 12 | 3 | 6 | 1 | 721382 |
| 5 | | | sum | 4 | 10 | 5 | 2 | 1 | 16725 | 12 | 4 | 5 | 1 | 1610617 |
| | | | | | 10 | 6 | 1 | 1 | 1649 | 12 | 5 | 4 | 1 | 1610617 |
| 6 | 1 | 1 | 2 | 16 | | | | | | 12 | 6 | 3 | 1 | 721382 |
| 6 | 1 | 2 | 1 | 16 | 10 | | | sum | 1355400 | 12 | 7 | 2 | 1 | 132202 |
| 6 | 2 | 1 | 1 | 16 | | | | | | 12 | 8 | 1 | 1 | 7417 |
| | | | | | 11 | 1 | 1 | 7 | 3633 | | | | | |
| 6 | | | sum | 48 | 11 | 1 | 2 | 6 | 50001 | 12 | | sum | | 150429819 |
| | | | | | 11 | 2 | 1 | 6 | 50001 | | | | | |
| 7 | 1 | 1 | 3 | 67 | 11 | 1 | 3 | 5 | 201915 | 13 | 1 | 1 | 9 | 14091 |
| 7 | 1 | 2 | 2 | 169 | 11 | 2 | 2 | 5 | 491729 | 13 | 1 | 2 | 8 | 316470 |
| 7 | 2 | 1 | 2 | 169 | 11 | 3 | 1 | 5 | 201915 | 13 | 2 | 1 | 8 | 316470 |
| 7 | 1 | 3 | 1 | 67 | 11 | 1 | 4 | 4 | 315000 | 13 | 1 | 3 | 7 | 2241162 |
| 7 | 2 | 2 | 1 | 169 | 11 | 2 | 3 | 4 | 1364986 | 13 | 2 | 2 | 7 | 5412883 |
| 7 | 3 | 1 | 1 | 67 | 11 | 3 | 2 | 4 | 1364986 | 13 | 3 | 1 | 7 | 2241162 |
| | | | | | 11 | 4 | 1 | 4 | 315000 | 13 | 1 | 4 | 6 | 6764142 |
| 7 | | | sum | 708 | 11 | 1 | 5 | 3 | 201915 | 13 | 2 | 3 | 6 | 28779051 |
| | | | | | 11 | 2 | 4 | 3 | 1364986 | 13 | 3 | 2 | 6 | 28779051 |
| 8 | 1 | 1 | 4 | 237 | 11 | 3 | 3 | 3 | 2428862 | 13 | 4 | 1 | 6 | 6764142 |
| 8 | 1 | 2 | 3 | 1072 | 11 | 4 | 2 | 3 | 1364986 | 13 | 1 | 5 | 5 | 9681210 |
| 8 | 2 | 1 | 3 | 1072 | 11 | 5 | 1 | 3 | 201915 | 13 | 2 | 4 | 5 | 63458654 |
| 8 | 1 | 3 | 2 | 1072 | 11 | 1 | 6 | 2 | 50001 | 13 | 3 | 3 | 5 | 111801142 |
| 8 | 2 | 2 | 2 | 2664 | 11 | 2 | 5 | 2 | 491729 | 13 | 4 | 2 | 5 | 63458654 |
| 8 | 3 | 1 | 2 | 1072 | 11 | 3 | 4 | 2 | 1364986 | 13 | 5 | 1 | 5 | 9681210 |
| 8 | 1 | 4 | 1 | 237 | 11 | 4 | 3 | 2 | 1364986 | 13 | 1 | 6 | 4 | 6764142 |
| 8 | 2 | 3 | 1 | 1072 | 11 | 5 | 2 | 2 | 491729 | 13 | 2 | 5 | 4 | 63458654 |
| 8 | 3 | 2 | 1 | 1072 | 11 | 6 | 1 | 2 | 50001 | 13 | 3 | 4 | 4 | 172252340 |
| 8 | 4 | 1 | 1 | 237 | 11 | 1 | 7 | 1 | 3633 | 13 | 4 | 3 | 4 | 172252340 |
| | | | | | 11 | 2 | 6 | 1 | 50001 | 13 | 5 | 2 | 4 | 63458654 |
| 8 | | | sum | 9807 | 11 | 3 | 5 | 1 | 201915 | 13 | 6 | 1 | 4 | 6764142 |
| | | | | | 11 | 4 | 4 | 1 | 315000 | 13 | 1 | 7 | 3 | 2241162 |
| 9 | 1 | 1 | 5 | 667 | 11 | 5 | 3 | 1 | 201915 | 13 | 2 | 6 | 3 | 28779051 |
| 9 | 1 | 2 | 4 | 4736 | 11 | 6 | 2 | 1 | 50001 | 13 | 3 | 5 | 3 | 111801142 |
| 9 | 2 | 1 | 4 | 4736 | 11 | 7 | 1 | 1 | 3633 | 13 | 4 | 4 | 3 | 172252340 |
| 9 | 1 | 3 | 3 | 8560 | | | | | | 13 | 5 | 3 | 3 | 111801142 |
| 9 | 2 | 2 | 3 | 21113 | 11 | | | sum | 14561360 | 13 | 6 | 2 | 3 | 28779051 |
| 9 | 3 | 1 | 3 | 8560 | | | | | | 13 | 7 | 1 | 3 | 2241162 |
| 9 | 1 | 4 | 2 | 4736 | 12 | 1 | 1 | 8 | 7417 | 13 | 1 | 8 | 2 | 316470 |
| 9 | 2 | 3 | 2 | 21113 | 12 | 1 | 2 | 7 | 132202 | 13 | 2 | 7 | 2 | 5412883 |
| 9 | 3 | 2 | 2 | 21113 | 12 | 2 | 1 | 7 | 132202 | 13 | 3 | 6 | 2 | 28779051 |
| 9 | 4 | 1 | 2 | 4736 | 12 | 1 | 3 | 6 | 721382 | 13 | 4 | 5 | 2 | 63458654 |
| 9 | 1 | 5 | 1 | 667 | 12 | 2 | 2 | 6 | 1748723 | 13 | 5 | 4 | 2 | 63458654 |
| 9 | 2 | 4 | 1 | 4736 | 12 | 3 | 1 | 6 | 721382 | 13 | 6 | 3 | 2 | 28779051 |
| 9 | 3 | 3 | 1 | 8560 | 12 | 1 | 4 | 5 | 1610617 | 13 | 7 | 2 | 2 | 5412883 |
| 9 | 4 | 2 | 1 | 4736 | 12 | 2 | 3 | 5 | 6908644 | 13 | 8 | 1 | 2 | 316470 |
| 9 | 5 | 1 | 1 | 667 | 12 | 3 | 2 | 5 | 6908644 | 13 | 1 | 9 | 1 | 14091 |
| | | | | | 12 | 4 | 1 | 5 | 1610617 | 13 | 2 | 8 | 1 | 316470 |
| 9 | | | sum | 119436 | 12 | 1 | 5 | 4 | 1610617 | 13 | 3 | 7 | 1 | 2241162 |
| | | | | | 12 | 2 | 4 | 4 | 10702449 | 13 | 4 | 6 | 1 | 6764142 |
| 10 | 1 | 1 | 6 | 1649 | 12 | 3 | 3 | 4 | 18938994 | 13 | 5 | 5 | 1 | 9681210 |
| 10 | 1 | 2 | 5 | 16725 | 12 | 4 | 2 | 4 | 10702449 | 13 | 6 | 4 | 1 | 6764142 |
| 10 | 2 | 1 | 5 | 16725 | 12 | 5 | 1 | 4 | 1610617 | 13 | 7 | 3 | 1 | 2241162 |
| 10 | 1 | 3 | 4 | 47164 | 12 | 1 | 6 | 3 | 721382 | 13 | 8 | 2 | 1 | 316470 |
| 10 | 2 | 2 | 4 | 115478 | 12 | 2 | 5 | 3 | 6908644 | 13 | 9 | 1 | 1 | 14091 |
| 10 | 3 | 1 | 4 | 47164 | 12 | 3 | 4 | 3 | 18938994 | | | | | |
| 10 | 1 | 4 | 3 | 47164 | 12 | 4 | 3 | 3 | 18938994 | 13 | | sum | | 1506841872 |
| 10 | 2 | 3 | 3 | 206895 | 12 | 5 | 2 | 3 | 6908644 | | | | | |
| 10 | 3 | 2 | 3 | 206895 | 12 | 6 | 1 | 3 | 721382 | 14 | 1 | 1 | 10 | 25405 |
| 10 | 4 | 1 | 3 | 47164 | 12 | 1 | 7 | 2 | 132202 | 14 | 1 | 2 | 9 | 700045 |
| 10 | 1 | 5 | 2 | 16725 | 12 | 2 | 6 | 2 | 1748723 | 14 | 2 | 1 | 9 | 700045 |
| 10 | 2 | 4 | 2 | 115478 | 12 | 3 | 5 | 2 | 6908644 | 14 | 1 | 3 | 8 | 6235526 |
| 10 | 3 | 3 | 2 | 206895 | 12 | 4 | 4 | 2 | 10702449 | 14 | 2 | 2 | 8 | 15012496 |
| 10 | 4 | 2 | 2 | 115478 | 12 | 5 | 3 | 2 | 6908644 | 14 | 3 | 1 | 8 | 6235526 |
| 10 | 5 | 1 | 2 | 16725 | 12 | 6 | 2 | 2 | 1748723 | 14 | 1 | 4 | 7 | 24417030 |
| 10 | 1 | 6 | 1 | 1649 | 12 | 7 | 1 | 2 | 132202 | 14 | 2 | 3 | 7 | 103175785 |

| | | | | | | | | | | | | | | |
|----|---|---|---|------------|----|---|---|---|------------|----|----|----|-----|-------------|
| 14 | 3 | 2 | 7 | 103175785 | 14 | 5 | 3 | 4 | 1173398706 | 14 | 7 | 3 | 2 | 103175785 |
| 14 | 4 | 1 | 7 | 24417030 | 14 | 6 | 2 | 4 | 306159286 | 14 | 8 | 2 | 2 | 15012496 |
| 14 | 1 | 5 | 6 | 47238510 | 14 | 7 | 1 | 4 | 24417030 | 14 | 9 | 1 | 2 | 700045 |
| 14 | 2 | 4 | 6 | 306159286 | 14 | 1 | 8 | 3 | 6235526 | 14 | 1 | 10 | 1 | 25405 |
| 14 | 3 | 3 | 6 | 537417269 | 14 | 2 | 7 | 3 | 103175785 | 14 | 2 | 9 | 1 | 700045 |
| 14 | 4 | 2 | 6 | 306159286 | 14 | 3 | 6 | 3 | 537417269 | 14 | 3 | 8 | 1 | 6235526 |
| 14 | 5 | 1 | 6 | 47238510 | 14 | 4 | 5 | 3 | 1173398706 | 14 | 4 | 7 | 1 | 24417030 |
| 14 | 1 | 6 | 5 | 47238510 | 14 | 5 | 4 | 3 | 1173398706 | 14 | 5 | 6 | 1 | 47238510 |
| 14 | 2 | 5 | 5 | 435785878 | 14 | 6 | 3 | 3 | 537417269 | 14 | 6 | 5 | 1 | 47238510 |
| 14 | 3 | 4 | 5 | 1173398706 | 14 | 7 | 2 | 3 | 103175785 | 14 | 7 | 4 | 1 | 24417030 |
| 14 | 4 | 3 | 5 | 1173398706 | 14 | 8 | 1 | 3 | 6235526 | 14 | 8 | 3 | 1 | 6235526 |
| 14 | 5 | 2 | 5 | 435785878 | 14 | 1 | 9 | 2 | 700045 | 14 | 9 | 2 | 1 | 700045 |
| 14 | 6 | 1 | 5 | 47238510 | 14 | 2 | 8 | 2 | 15012496 | 14 | 10 | 1 | 1 | 25405 |
| 14 | 1 | 7 | 4 | 24417030 | 14 | 3 | 7 | 2 | 103175785 | | | | | |
| 14 | 2 | 6 | 4 | 306159286 | 14 | 4 | 6 | 2 | 306159286 | 14 | | | sum | 14732613116 |
| 14 | 3 | 5 | 4 | 1173398706 | 14 | 5 | 5 | 2 | 435785878 | | | | | |
| 14 | 4 | 4 | 4 | 1799940644 | 14 | 6 | 4 | 2 | 306159286 | | | | | |

B.4 Genus 3

| | | | | | | | | | | | | | | |
|----|---|---|-----|---------|----|-----|----------|-----------|-----------|----|---|---|-----|-------------|
| d | v | e | f | H | | | | | | 13 | 1 | 7 | 1 | 668591 |
| 7 | 1 | 1 | 1 | 30 | 11 | sum | 10260804 | | | 13 | 2 | 6 | 1 | 8342532 |
| | | | sum | 30 | | | | | | 13 | 3 | 5 | 1 | 31916775 |
| 7 | | | | | 12 | 1 | 1 | 6 | 220244 | 13 | 4 | 4 | 1 | 48937240 |
| | | | | | 12 | 1 | 2 | 5 | 2054974 | 13 | 5 | 3 | 1 | 31916775 |
| 8 | 1 | 1 | 2 | 385 | 12 | 2 | 1 | 5 | 2054974 | 13 | 6 | 2 | 1 | 8342532 |
| 8 | 1 | 2 | 1 | 385 | 12 | 1 | 3 | 4 | 5567550 | 13 | 7 | 1 | 1 | 668591 |
| 8 | 2 | 1 | 1 | 385 | 12 | 2 | 2 | 4 | 13314231 | | | | | |
| | | | sum | 1155 | 12 | 3 | 1 | 4 | 5567550 | 13 | | | sum | 2195431068 |
| 8 | | | | | 12 | 1 | 4 | 3 | 5567550 | | | | | |
| | | | | | 12 | 2 | 3 | 3 | 23377106 | 14 | 1 | 1 | 8 | 1824323 |
| 9 | 1 | 1 | 3 | 2900 | 12 | 3 | 2 | 3 | 23377106 | 14 | 1 | 2 | 7 | 29267487 |
| 9 | 1 | 2 | 2 | 7070 | 12 | 4 | 1 | 3 | 5567550 | 14 | 2 | 1 | 7 | 29267487 |
| 9 | 2 | 1 | 2 | 7070 | 12 | 1 | 5 | 2 | 2054974 | 14 | 1 | 3 | 6 | 149721473 |
| 9 | 1 | 3 | 1 | 2900 | 12 | 2 | 4 | 2 | 13314231 | 14 | 2 | 2 | 6 | 355267058 |
| 9 | 2 | 2 | 1 | 7070 | 12 | 3 | 3 | 2 | 23377106 | 14 | 3 | 1 | 6 | 149721473 |
| 9 | 3 | 1 | 1 | 2900 | 12 | 4 | 2 | 2 | 13314231 | 14 | 1 | 4 | 5 | 324171185 |
| | | | sum | 29910 | 12 | 5 | 1 | 2 | 2054974 | 14 | 2 | 3 | 5 | 1338142324 |
| 9 | | | | | 12 | 1 | 6 | 1 | 220244 | 14 | 3 | 2 | 5 | 1338142324 |
| | | | | | 12 | 2 | 5 | 1 | 2054974 | 14 | 4 | 1 | 5 | 324171185 |
| 10 | 1 | 1 | 4 | 15308 | 12 | 3 | 4 | 1 | 5567550 | 14 | 1 | 5 | 4 | 324171185 |
| 10 | 1 | 2 | 3 | 65972 | 12 | 4 | 3 | 1 | 5567550 | 14 | 2 | 4 | 4 | 2041388556 |
| 10 | 2 | 1 | 3 | 65972 | 12 | 5 | 2 | 1 | 2054974 | 14 | 3 | 3 | 4 | 3551405485 |
| 10 | 1 | 3 | 2 | 65972 | 12 | 6 | 1 | 1 | 220244 | 14 | 4 | 2 | 4 | 2041388556 |
| 10 | 2 | 2 | 2 | 159608 | | | | | | 14 | 5 | 1 | 4 | 324171185 |
| 10 | 3 | 1 | 2 | 65972 | 12 | | | | | 14 | 1 | 6 | 3 | 149721473 |
| 10 | 1 | 4 | 1 | 15308 | 12 | | sum | 156469887 | | 14 | 2 | 5 | 3 | 1338142324 |
| 10 | 2 | 3 | 1 | 65972 | 13 | 1 | 1 | 7 | 668591 | 14 | 3 | 4 | 3 | 3551405485 |
| 10 | 3 | 2 | 1 | 65972 | 13 | 1 | 2 | 6 | 8342532 | 14 | 4 | 3 | 3 | 3551405485 |
| 10 | 4 | 1 | 1 | 15308 | 13 | 2 | 1 | 6 | 8342532 | 14 | 5 | 2 | 3 | 1338142324 |
| | | | sum | 601364 | 13 | 1 | 3 | 5 | 31916775 | 14 | 6 | 1 | 3 | 149721473 |
| 10 | | | | | 13 | 2 | 2 | 5 | 76003367 | 14 | 1 | 7 | 2 | 29267487 |
| 11 | 1 | 1 | 5 | 63355 | 13 | 3 | 1 | 5 | 31916775 | 14 | 2 | 6 | 2 | 355267058 |
| 11 | 1 | 2 | 4 | 418810 | 13 | 1 | 4 | 4 | 48937240 | 14 | 3 | 5 | 2 | 1338142324 |
| 11 | 2 | 1 | 4 | 418810 | 13 | 2 | 3 | 4 | 203571133 | 14 | 4 | 4 | 2 | 2041388556 |
| 11 | 1 | 3 | 3 | 740100 | 13 | 3 | 2 | 4 | 203571133 | 14 | 5 | 3 | 2 | 1338142324 |
| 11 | 2 | 2 | 3 | 1779193 | 13 | 4 | 1 | 4 | 48937240 | 14 | 6 | 2 | 2 | 355267058 |
| 11 | 3 | 1 | 3 | 740100 | 13 | 1 | 5 | 3 | 31916775 | 14 | 7 | 1 | 2 | 29267487 |
| 11 | 1 | 4 | 2 | 418810 | 13 | 2 | 4 | 3 | 203571133 | 14 | 1 | 8 | 1 | 1824323 |
| 11 | 2 | 3 | 2 | 1779193 | 13 | 3 | 3 | 3 | 355260834 | 14 | 2 | 7 | 1 | 29267487 |
| 11 | 3 | 2 | 2 | 1779193 | 13 | 4 | 2 | 3 | 203571133 | 14 | 3 | 6 | 1 | 149721473 |
| 11 | 4 | 1 | 2 | 418810 | 13 | 5 | 1 | 3 | 31916775 | 14 | 4 | 5 | 1 | 324171185 |
| 11 | 1 | 5 | 1 | 63355 | 13 | 1 | 6 | 2 | 8342532 | 14 | 5 | 4 | 1 | 324171185 |
| 11 | 2 | 4 | 1 | 418810 | 13 | 2 | 5 | 2 | 76003367 | 14 | 6 | 3 | 1 | 149721473 |
| 11 | 3 | 3 | 1 | 740100 | 13 | 3 | 4 | 2 | 203571133 | 14 | 7 | 2 | 1 | 29267487 |
| 11 | 4 | 2 | 1 | 418810 | 13 | 4 | 3 | 2 | 203571133 | 14 | 8 | 1 | 1 | 1824323 |
| 11 | 5 | 1 | 1 | 63355 | 13 | 5 | 2 | 2 | 76003367 | | | | | |
| | | | | | 13 | 6 | 1 | 2 | 8342532 | 14 | | | sum | 28897471080 |

