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# Enumeration of hypermaps of a given genus\*

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## Abstract

This paper addresses the enumeration of rooted and unrooted hypermaps of a given genus. For rooted hypermaps the enumeration method consists of considering the more general family of multirooted hypermaps, in which darts other than the root dart are distinguished. We give functional equations for the generating series counting multirooted hypermaps of a given genus by number of darts, vertices, edges, faces and the degrees of the vertices containing the distinguished darts. We solve these equations to get parametric expressions of the generating functions of rooted hypermaps of low genus. We also count unrooted hypermaps of given genus by number of darts, vertices, hyperedges and faces.

*Keywords:* Enumeration, surface, genus, rooted hypermap, unrooted hypermap.

*Math. Subj. Class.:* 05C30, 05A15

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## 1 Introduction

A (*combinatorial*) hypermap is a triple  $(D, R, L)$  where  $D$  is a finite set of *darts* and  $R$  and  $L$  are permutations on  $D$  such that the group  $\langle R, L \rangle$  generated by  $R$  and  $L$  acts transitively on  $D$ . A (*combinatorial ordinary*) map is a hypermap  $(D, R, L)$  whose permutation  $L$  is a fixed-point-free involution on  $D$ . For a hypermap (resp. map) the orbits of  $R$ ,  $L$  and

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$RL$  ( $L$  followed by  $R$ ) are respectively called *vertices*, *hyperedges* (resp. *edges*) and *faces*. The *degree* of a vertex, edge, hyperedge or face is the number of darts it contains. The equivalence of combinatorial maps and topological maps having been established in [14], we use the word “map” to mean “combinatorial map” throughout this paper. The *genus*  $g$  of a map is given by the Euler-Poincaré formula [7]

$$v - e + f = 2(1 - g), \quad (1.1)$$

where  $v$  is the number of vertices,  $e$  is the number of edges and  $f$  is the number of faces. The genus of a hypermap with  $t$  darts,  $v$  vertices,  $e$  hyperedges and  $f$  faces was defined in [13] by the formula

$$v + e + f = t + 2(1 - g). \quad (1.2)$$

An *isomorphism* between two maps or hypermaps  $(D, R, L)$  and  $(D', R', L')$  is a bijection from  $D$  onto  $D'$  that takes  $R$  into  $R'$  and  $L$  into  $L'$ ; it corresponds to an orientation-preserving homeomorphism between two topological maps. A *sensed* hypermap (resp. map) is an isomorphism class of hypermaps (resp. maps). We admit the existence of a unique hypermap (resp. map) with an empty set of darts  $D$ , called the *empty* hypermap (resp. map). For both of these objects  $v = f = 1$  and  $g = e = 0$ . A *rooted* hypermap (resp. map) is either the empty hypermap (resp. map) or a tuple  $(D, x, R, L)$  where  $(D, R, L)$  is a non-empty combinatorial hypermap (resp. map) and  $x \in D$  is a distinguished dart, called the *root*.

The enumeration of maps and hypermaps has several non-trivial applications. One such application is based on the correspondence between hypermaps and algebraic curves established by the Belyi theorem [16]. For instance, the formula for the number of plane trees was used by A. Zvonkin in the computer generation of Shabat polynomials of bounded degree [16]. Another area where the map enumeration plays an important role is theoretical physics, in particular in 2-dimensional gravitation models. Roughly speaking, map enumeration is used to compute matrix integrals determining the properties of gravitational fields (see for instance the works of B. Eynard [9]). Some hypermaps have been shown to be related to contextuality in quantum physics [21]. Also, A. Mednykh and R. Nedela have applied the enumeration of rooted (resp. unrooted) hypermaps to the enumeration of subgroups (resp. conjugacy classes of subgroups) of the triangle group with three generators  $x, y, z$  and the relation  $xyz = 1$  [20].

We enumerate rooted hypermaps of a given genus by number of darts, vertices, hyperedges and faces. To do so we consider more general families of rooted hypermaps and bipartite maps, in which other vertices or darts than the root dart are distinguished. We also use the genus-preserving bijection between hypermaps and 2-vertex-coloured bipartite maps presented in [23]. But since bipartite maps have all their **faces** of even degree and we're using the degrees of the **vertices** as parameters, we must instead study the face-vertex dual of a 2-coloured bipartite map, that is, a map whose faces are coloured in two colours (white and black) so that no two faces that share an edge have the same colour. All these maps are *Eulerian* – that is, all their vertices are of even degree – but not all Eulerian maps are 2-face-colourable. For example, the map on the torus with one vertex, one face and two edges is Eulerian because its only vertex is of degree 4, but its face cannot be coloured because it shares both edges with itself. Therefore we call the maps we are studying *face-bipartite*.

A *sequenced (rooted) map* is a rooted map with some vertices other than the *root vertex* (the vertex that contains the root) distinguished from each other and from all the other

vertices. The labels that distinguish these vertices can be taken to be  $1, 2, \dots, k$ , where  $k$  is the number of distinguished vertices. A *sequenced (rooted) hypermap* is defined similarly. We state (in Section 4) a bijective decomposition for the set  $\mathcal{H}(g, t, f, e, n, D)$  of sequenced orientable hypermaps of genus  $g$  with  $t$  darts,  $f$  faces and  $e$  hyperedges, with the root vertex of degree  $n$  and with the sequence of degrees of the distinguished vertices equal to  $D = (d_1, d_2, \dots, d_{|D|})$ , where  $d_i$  is the degree of the distinguished vertex with label  $i$ . We obtain a bijective decomposition of the set  $\mathcal{F}(g, e, w, b, n, D)$  of sequenced orientable face-bipartite maps of genus  $g$  with  $e$  edges,  $w$  white faces,  $b$  black faces, with the root face of degree  $2n$  and with the sequence of half-degrees of the distinguished vertices equal to  $D$ . Then we apply face-vertex duality to obtain a bijective decomposition of the corresponding set of 2-coloured bipartite maps with distinguished faces. Next we use the bijection in [23] to obtain a bijective decomposition for hypermaps with distinguished faces, and finally we again apply face-vertex duality to obtain a bijective decomposition of  $\mathcal{H}(g, t, f, e, n, D)$ .

A *multirooted hypermap* is a hypermap in which a non-empty sequence of darts with pairwise distinct initial vertices is distinguished. We relate multirooted hypermaps to sequenced hypermaps and thus obtain a recurrence for the number of multirooted hypermaps and functional equations for the generating series counting multirooted hypermaps of a given genus by number of darts, vertices, edges, faces and the degrees of the initial vertices of the distinguished darts.

The paper is organized as follows. Section 2 fixes some notations, recalls a known decomposition for sequenced rooted maps and describes the bijection between hypermaps and bipartite maps presented in [23]. Sections 3 and 4 respectively enumerate sequenced face-bipartite maps and sequenced rooted hypermaps of a given genus. In Section 5 we consider multirooted hypermaps and we give equations for the generating functions that count these objects. In Section 6 we give functional equations relating the generating functions for rooted hypermaps with that for multirooted hypermaps. Then we show how to solve these equations. In Section 7 we obtain parametric expressions for the generating functions that count rooted hypermaps with a given small positive genus. Section 8 presents enumeration algorithms for sensed unrooted hypermaps counted by number of darts, vertices and hyperedges. Appendix A (resp. B) contains a table for numbers of rooted (resp. unrooted) hypermaps of genus  $g$  with  $d$  darts,  $v$  vertices and  $e$  hyperedges for  $d \leq 14$ .

## 2 Background

### 2.1 Notations

We first introduce the notations and conventions we use throughout the paper. Let  $D$  and  $D'$  be two lists of integers. The inclusion  $D' \subseteq D$  means that  $D'$  is a sublist of  $D$ . In this case  $D - D'$  is the complementary sublist of  $D'$  in  $D$ . For instance, the sublists of  $D = [1, 1, 2]$  are the empty list  $[], [1]$  (twice),  $[2], [1, 1], [1, 2]$  (twice) and  $D$  itself. Their complementary sublists in the same order are  $D, [1, 2]$  (twice),  $[1, 1], [2], [1]$  (twice) and  $[], [1, 2]$  (twice),  $[1, 1], [2], [1]$  (twice) and  $[], [1, 2]$  (twice). We denote by  $D.D'$  the concatenation of the lists  $D$  and  $D'$ . If  $i$  is an integer and  $D$  is a list of integers, then  $i.D$  is a shortcut for  $[i].D$ . For  $1 \leq j \leq |D|$  we denote by  $d_j$  the  $j$ -th element of the list  $D$  of length  $|D|$  and by  $D - \{d_j\}$  the list obtained from  $D$  by removing its  $j$ -th element  $d_j$ . Let  $\rho$  be a positive integer. The abbreviation  $D_{1.. \rho}$  denotes the list  $[d_1, \dots, d_\rho]$ . The abbreviation  $v_{1.. \rho}^{D_{1.. \rho}}$  denotes  $v_1^{d_1} \dots v_\rho^{d_\rho}$ .

The sign  $+$  (resp.  $\sum$ ) denotes (resp. generalized) disjoint set union in the following decompositions and (resp. generalized) arithmetic sum in the following equations. By con-

vention, a disjoint set union (resp. sum) over an empty domain is equal to the empty set (resp. zero). For any logical formula  $\varphi$  the notation  $\Delta_\varphi$  means the singleton set containing only the empty hypermap or map (depending on the context) and the empty set if  $\varphi$  is false. The notation  $\delta_\varphi$  means 1 if  $\varphi$  is true and 0 if  $\varphi$  is false.

### 2.2 Bijective decomposition of the set of sequenced maps

In 1962 W. T. Tutte [22] presented a bijective decomposition of a planar map with all the vertices distinguished and a root in every vertex. In 1972 T. R. Walsh and A. B. Lehman [27] generalized this decomposition to maps of higher genus and used it to count rooted maps of a given genus by number of vertices and faces. In 1987 D. Arquès [3] used this latter decomposition to find a closed-form formula for the number of rooted maps of genus 1 by number of vertices and faces. In 1991 E. A. Bender and E. A. Canfield [4] presented a more efficient decomposition that roots only a single vertex and distinguishes only as many other vertices as necessary and used it to obtain explicit formulas for counting rooted maps of genus 2 and 3. In 1998 the first author [11] modified this decomposition and used it to obtain a bijective decomposition satisfied by the set  $\mathcal{M}(g, e, f, n, D)$  of sequenced orientable maps of genus  $g$  with  $e$  edges and  $f$  faces, with the root vertex of degree  $n$  and with  $D$  the list of degrees of the distinguished vertices was obtained in [11]. Since this bijective decomposition contains an error, we present the correct bijective decomposition here, and we derive it to make the derivation more accessible than the contents of a Ph. D. thesis.

**Theorem 2.1.** *The set  $\mathcal{M}(g, e, f, n, D)$  of sequenced orientable maps of genus  $g$  with  $e$  edges and  $f$  faces, with the root vertex of degree  $n$  and with the list  $D$  of degrees of the distinguished vertices is defined by the bijective decomposition*

$$\begin{aligned}
 \mathcal{M}(g, e, f, n, D) = & \sum_{\substack{g_1 + g_2 = g \\ e_1 + e_2 = e - 1 \\ f_1 + f_2 = f \\ n_1 + n_2 = n - 2 \\ D_1 \subseteq D}} \mathcal{M}(g_1, e_1, f_1, n_1, D_1) \times \mathcal{M}(g_2, e_2, f_2, n_2, D - D_1) \\
 & + \sum_{p=1}^{n-3} \mathcal{M}(g - 1, e - 1, f, n - 2 - p, p, D) \times \{1, \dots, p\} \tag{2.1} \\
 & + \sum_{p=n-1}^{p=2e-2} \mathcal{M}(g, e - 1, f, p, D) \\
 & + \sum_{j=1}^{|D|} \mathcal{M}(g, e - 1, f, d_j + n - 2, D - \{d_j\}) + \Delta_{(g,e,f,n,D)=(0,0,1,0,\{\})}.
 \end{aligned}$$

*Proof.* If a map  $m$  has at least one edge, we reduce by 1 the number of edges by the face-vertex dual of deleting the root edge. There are two cases of this operation, depending upon whether the root edge is a loop or a link, and each of these cases breaks down into two sub-cases.

**Case 1: The root edge is a loop.** We delete the root edge and split the root vertex into two parts,  $s_1$  and  $s_2$ . If  $r$  is the root, then  $s_1$  consists of the darts  $R(r), R^2(r), \dots, R^{-1}(L(r))$

and  $s_2$  consists of the darts  $R(L(r)), R^2(L(r)), \dots, R^{-1}(r)$ . This case breaks down into two cases, depending upon whether or not this operation disconnects the map.

**Case 1a: This operation disconnects the map into two maps**,  $m_1$  containing  $s_1$  and  $m_2$  containing  $s_2$ . If  $m_1$  has at least 1 edge, its root is  $r_1 = R(r)$ , and if  $m_2$  has at least 1 edge, its root is  $r_2 = R(L(r))$ . Let  $g_1, e_1, f_1, n_1, D_1$  and  $g_2, e_2, f_2, n_2, D_2$  be the parameters of the maps  $m_1$  and  $m_2$ , respectively, corresponding to  $g, e, f, n, D$ . This operation reduces by 1 the total number of edges; so  $e_1 + e_2 = e - 1$ . It leaves unchanged the total number of faces because  $r$  and  $L(r)$  simply get deleted from the cycle(s) of  $RL$  ( $L$  followed by  $R$ ) containing them; so  $f_1 + f_2 = f$ . It increases by 1 the total number of vertices; so from Formula (1.1), which relates the genus of a map to the number of its vertices, faces and edges, it can easily be deduced that  $g_1 + g_2 = g$ . It decreases by 2 the total number of darts in  $s_1$  and  $s_2$  since  $r$  and  $L(r)$ , which belonged to the root vertex, get eliminated; so  $n_1 + n_2 = n - 2$ . Finally,  $D_1$  can be any sublist of  $D$  and  $D_2$  is just the complementary sublist, denoted by  $D - D_1$ . This operation is uniquely reversible; so the set of ordered pairs of sequenced maps obtained in this case is

$$\sum_{\substack{g_1 + g_2 = g \\ e_1 + e_2 = e - 1 \\ f_1 + f_2 = f \\ n_1 + n_2 = n - 2 \\ D_1 \subseteq D}} \mathcal{M}(g_1, e_1, f_1, n_1, D_1) \times \mathcal{M}(g_2, e_2, f_2, n_2, D - D_1), \quad (2.2)$$

where  $\Sigma$  means the union of disjoint sets.

**Case 1b: This operation does not disconnect the map**, but instead turns it into a new map  $m'$  with  $e - 1$  edges and  $f$  faces and, since the number of vertices increases by 1, the genus of  $m'$  is  $g - 1$ , so that this case only occurs when  $g \geq 1$ . Neither  $s_1$  nor  $s_2$  can be of degree 0 (otherwise the map would be disconnected); so we can choose for  $m'$  the root  $r_1 = R(r)$  belonging to  $s_1$ . Let  $p$  be the degree of  $s_2$ . Since the sum of the degrees of  $s_1$  and  $s_2$  is  $n - 2$ , the degree of  $s_1$ , the root vertex, is  $n - 2 - p$ . We distinguish the vertex  $s_2$  so that this operation can be reversed, and we put its degree  $p$  at the beginning of the list  $D$ , turning it into  $p.D$ . Now this operation is reversible in  $p$  distinct ways, since any of the  $p$  darts of  $s_2$  can be chosen to be  $R(L(r))$  when we merge the vertices  $s_1$  and  $s_2$  and replace the deleted root edge. Now  $p$  can be any integer from 1 up to  $n - 3$  (so that  $n - 2 - p \geq 1$ ). For both  $p$  and  $n - 2 - p$  to be at least 1,  $n$  must be at least 4. The set of sequenced maps obtained in this case is

$$\sum_{p=1}^{n-3} \mathcal{M}(g - 1, e - 1, f, n - 2 - p, p.D) \times \{1, \dots, p\}. \quad (2.3)$$

**Case 2: The root edge is a link.** We contract the root edge, merging its two incident vertices  $s_1$  containing the root  $r$  and  $s_2$  containing  $L(r)$  into a single vertex  $s$  with root  $R(r)$ . This operation decreases by 1 the number of edges and doesn't change the number of faces, since  $r$  and  $L(r)$  simply get deleted from the cycle(s) containing them. Since the number of vertices is decreased by 1, the genus remains the same. This case breaks down into two sub-cases, depending upon whether or not  $s_2$  is one of the distinguished vertices.

**Case 2a: The vertex  $s_2$  is not one of the distinguished vertices.** Let  $p$  be the degree of the new vertex  $s$ . Then  $p = n - 2 +$  the degree of  $s_2$ , and since the degree of  $s_2$  must be

at least 1, we have  $p \geq n - 1$ . Also, the new map has  $2e - 2$  darts; so  $p \leq 2e - 2$ . This operation is uniquely reversible for each value of  $p$ ; so the set of maps so obtained is

$$\sum_{p=n-1}^{p=2e-2} \mathcal{M}(g, e - 1, f, p, D). \tag{2.4}$$

**Case 2b: The vertex  $s_2$  is one of the distinguished vertices.** It can be any one of the  $|D|$  distinguished vertices. If it is the  $j$ th distinguished vertex, then its degree is  $d_j$ . Then since it gets merged with  $s_1$  into the new root vertex,  $d_j$  gets dropped from  $D$ . Finally, the degree of  $s$  is  $d_j + n - 2$ . This operation too is uniquely reversible; so the set of maps so obtained is

$$\sum_{j=1}^{|D|} \mathcal{M}(g, e - 1, f, d_j + n - 2, D - \{d_j\}). \tag{2.5}$$

Finally, suppose that  $m$  has no edges. It is of genus 0, has 1 face, its one vertex is of degree 0 and its list  $D$  is empty because it has no distinguished vertices; so it constitutes the singleton

$$\Delta_{(g,e,f,n,D)=(0,0,1,0,[])} \cdot \tag{2.6}$$

Then  $\mathcal{M}(g, e, f, n, D)$  is the disjoint union of the sets given by (2.2)–(2.6). □

### 2.3 Bipartite maps and hypermaps

To motivate the transformation of (2.2)–(2.6) into the corresponding equations for sequenced hypermaps we briefly describe the bijection in [23] that takes a hypermap  $h$  into a 2-coloured bipartite map  $m = I(h)$ , its *incidence map*. The bijection  $I$  takes the darts, vertices and hyperedges of  $h$  into the edges, white vertices and black vertices of  $m$ . A root (distinguished dart) of  $h$  corresponds to a distinguished **edge** of  $m$ ; to make it correspond to a root of  $m$  we impose the condition that a root of  $m$  belongs to a white vertex. The permutation  $R$  in  $h$  corresponds to  $R$  in  $m$  acting on a dart in a white vertex and the permutation  $L$  in  $h$  corresponds to  $R$  in  $m$  acting on a dart in a black vertex. The permutation  $L$  in  $m$  doesn't correspond to any permutation in  $h$ ; rather, since it takes a dart belonging to a vertex of one colour into a dart belonging to a vertex of the opposite colour, it toggles  $R$  in  $m$  between  $R$  and  $L$  in  $h$ . A face (cycle of  $RL$ ) in  $h$  corresponds to a face in  $m$  with twice the degree. To see this, we follow one application of  $RL$  in  $h$  starting with a dart  $d$ , which corresponds to an edge in  $m$  but we make it correspond to the dart  $d'$  in that edge that also belongs to a white vertex. Then the  $L$  in  $h$  takes  $d'$  first into  $L(d')$ , which belongs to a black vertex, and then into  $RL(d')$  and the following  $R$  in  $h$  takes  $RL(d')$  first into  $LRL(d')$ , which belongs to a white vertex, and then into  $RLRL(d')$ . Since the genus of a hypermap with  $t$  darts,  $v$  vertices,  $e$  hyperedges and  $f$  faces is defined by (1.2),  $m$  has the same genus as  $h$ .

Since the root of an incidence map of a rooted hypermap must belong to a white vertex, we impose the condition on a rooted 2-face-coloured face-bipartite map that the root belong to a white face and we transform (2.2)–(2.6) into the corresponding bijective decomposition for these maps.

### 3 Sequenced face-bipartite maps

Let  $\mathcal{F}(g, e, w, b, n, D)$  be the set of sequenced orientable face-bipartite maps of genus  $g$  with  $e$  edges,  $w$  white faces,  $b$  black faces, with the root face of degree  $2n$  and with the list of half-degrees of the distinguished vertices equal to  $D$ . For any dart  $d$  we denote by  $f(d)$  the face containing  $d$  and we note that the face  $f(R(d)) = f(L(d))$  must have the opposite colour from  $f(d)$  because those two faces share the edge  $\{d, L(d)\}$ .

**Theorem 3.1.** *The set  $\mathcal{F}(g, e, w, b, n, D)$  satisfies the bijective decomposition*

$$\begin{aligned} \mathcal{F}(g, e, w, b, n, D) = & \sum_{\substack{g_1 + g_2 = g \\ e_1 + e_2 = e - 1 \\ w_1 + b_2 = b \\ w_2 + b_1 = w \\ n_1 + n_2 = n - 1 \\ D_1 \subseteq D}} \mathcal{F}(g_1, e_1, w_1, b_1, n_1, D_1) \times \mathcal{F}(g_2, e_2, w_2, b_2, n_2, D - D_1) \\ & + \sum_{p=1}^{n-2} \mathcal{F}(g - 1, e - 1, b, w, n - 1 - p, p, D) \times \{1, \dots, p\} \\ & + \sum_{p=n}^{p=e-1} \mathcal{F}(g, e - 1, b, w, p, D) \\ & + \sum_{j=1}^{|D|} \mathcal{F}(g, e - 1, b, w, d_j + n - 1, D - \{d_j\}) + \Delta_{(g,e,w,b,n,D)=(0,0,1,0,0,[])}. \end{aligned} \tag{3.1}$$

*Proof. Case 1: The root edge is a loop.* By definition,  $f(r)$ , where  $r$  is the root of the map  $m$ , is white, so that since  $r_1 = R(r)$ ,  $f(r_1)$  must be black. But when the loop is removed and the vertex  $s$  containing  $r$  is split,  $r_1$  becomes a root; so  $f(r_1)$  must change colour and so must all the faces of the new map  $m'$  (in case 1b) or the map  $m_1$  containing  $r_1$  (in case 1a). In case 1a, the other map  $m_2$  has  $r_2 = RL(r)$  as a root and  $f(r_2)$  is white; so its faces stay the same colour. This implies that in case 1a  $w_1 + b_2 = b$  and  $w_2 + b_1 = w$ , whereas in case 1b  $w$  and  $b$  switch in going from  $m$  to  $m'$ .

In case 1a, we have, as for general maps,  $g_1 + g_2 = g$ ,  $e_1 + e_2 = e - 1$  and  $D_1$  is any subset of  $D$ , but instead of  $n_1 + n_2 = n - 2$  we have  $n_1 + n_2 = n - 1$  because the degrees satisfy the equation  $2n_1 + 2n_2 = 2n - 2$ . The analogue of formula (2.2) is thus

$$\sum_{\substack{g_1 + g_2 = g \\ e_1 + e_2 = e - 1 \\ w_1 + b_2 = b \\ w_2 + b_1 = w \\ n_1 + n_2 = n - 1 \\ D_1 \subseteq D}} \mathcal{F}(g_1, e_1, w_1, b_1, n_1, D_1) \times \mathcal{F}(g_2, e_2, w_2, b_2, n_2, D - D_1). \tag{3.2}$$

In case 1b, the reduced map  $m'$  is still of genus  $g - 1$  and has  $e - 1$  edges, but the degree of  $s_2$  is now  $2p$  instead of  $p$  and the degree of the new root vertex  $s_1$  is  $2(n - 1 - p)$ ; so the parameter  $n - 2 - p$  in (2.3) changes to  $n - 1 - p$ . Also,  $1 \leq 2p \leq 2n - 3$ , but since  $2p$  is even, we have  $1 \leq p \leq n - 2$  instead of  $1 \leq p \leq n - 3$ , and the condition that  $n \geq 4$



changes to  $n \geq 3$ . The analogue of formula (2.3) is thus

$$\sum_{p=1}^{n-2} \mathcal{F}(g-1, e-1, b, w, n-1-p, p, D) \times \{1, \dots, p\}. \tag{3.3}$$

**Case 2: The root edge is a link.** Since the new root  $R(r)$  belongs to a black face, all the faces change colour; so  $b$  and  $w$  switch.

In case 2a, we have  $2n-1 \leq 2p \leq 2e-2$ , but since  $2p$  is even, we now have  $n \leq p \leq e-1$ ; so the analogue of (2.4) is

$$\sum_{p=n}^{p=e-1} \mathcal{F}(g, e-1, b, w, p, D). \tag{3.4}$$

In case 2b, the degree of the new root vertex is  $2d_j + 2n - 2$ ; so the analogue of (2.5) is

$$\sum_{j=1}^{|D|} \mathcal{F}(g, e-1, b, w, d_j + n - 1, D - \{d_j\}). \tag{3.5}$$

Finally, the map with no edges has one white face and no black ones; so the analogue of (2.6) is

$$\Delta_{(g,e,w,b,n,D)=(0,0,1,0,0,[])} \tag{3.6}$$

□

After deriving this bijective decomposition, we became aware of the article [8], which presents a similar bijective decomposition but for multi-rooted face-bipartite maps, which are like sequenced face-bipartite maps except that every distinguished vertex has a root. However, we present our derivation here for several reasons: it makes our article self-contained, we obtained it independently of [8] and our main purpose is to count hypermaps rather than face-bipartite maps. Now [8] does present a construction that converts a hypermap into a face-bipartite map. However, that construction is not proved and it is far more complicated than the one in [23], which is not cited in [8]. We also recently became aware of the article [6], which generalizes the results of [15] by computing the generating functions for edge-labelled bipartite maps on an orientable surface of genus  $g$  with an unbounded number of faces and including the degrees of these faces as parameters.

### 4 Sequenced rooted hypermaps

Theorem 3.1 holds for rooted 2-coloured bipartite maps with distinguished faces, where  $e$  is the number of edges,  $w$  is the number of white vertices,  $b$  is the number of black vertices,  $n$  is half the degree of the root face and  $D$  is the list of half-degrees of the distinguished faces. By the bijection described in Section 2.3, it also holds for rooted hypermaps with distinguished faces, where  $e$  is the number of darts,  $w$  is the number of vertices,  $b$  is the number of hyperedges,  $n$  is the degree of the root face and  $D$  is the list of degrees of the distinguished faces. By duality, the theorem also holds for sequenced hypermaps, where  $e$  is the number of darts,  $w$  is the number of faces,  $b$  is the number of hyperedges,  $n$  is the degree of the root vertex and  $D$  is the list of degrees of the distinguished vertices. To make the letters correspond to the objects they represent, we change  $\mathcal{F}$  to  $\mathcal{H}$ ,  $e$  to  $t$ ,  $w$  to  $f$  and  $b$  to  $e$ . We thus obtain the following results.

**Theorem 4.1** (Bijective decomposition for sequenced hypermaps). *Let  $\mathcal{H}(g, t, f, e, n, D)$  be the set of sequenced orientable hypermaps of genus  $g$  with  $t$  darts,  $f$  faces and  $e$  hyperedges, with the root vertex of degree  $n$  and with the list of degrees of the distinguished vertices equal to  $D = (d_1, d_2, \dots, d_{|D|})$ , where  $d_i$  is the degree of the distinguished vertex with label  $i$ . The set  $\mathcal{H}(g, t, f, e, n, D)$  satisfies the bijective decomposition*

$$\begin{aligned} \mathcal{H}(g, t, f, e, n, D) = & \sum_{\substack{g_1 + g_2 = g \\ t_1 + t_2 = t - 1 \\ f_1 + e_2 = e \\ f_2 + e_1 = f \\ n_1 + n_2 = n - 1 \\ D_1 \subseteq D}} \mathcal{H}(g_1, t_1, f_1, e_1, n_1, D_1) \times \mathcal{H}(g_2, t_2, f_2, e_2, n_2, D - D_1) \\ & + \sum_{p=1}^{n-2} \mathcal{H}(g - 1, t - 1, e, f, n - 1 - p, p, D) \times \{1, \dots, p\} \\ & + \sum_{p=n}^{p=t-1} \mathcal{H}(g, t - 1, e, f, p, D) \\ & + \sum_{j=1}^{|D|} \mathcal{H}(g, t - 1, e, f, d_j + n - 1, D - \{d_j\}) + \Delta_{(g,t,f,e,n,D)=(0,0,1,0,0,[])} \end{aligned} \tag{4.1}$$

**Corollary 4.2** (Recurrence between numbers of sequenced hypermaps). *Let  $H(g, t, f, e, n, D)$  be the number of rooted sequenced hypermaps of genus  $g$  with  $t$  darts,  $f$  faces and  $e$  hyperedges such that the root vertex is of degree  $n$  and  $D$  is the list of degrees of the distinguished vertices. Then  $H(0, 0, 1, 0, 0, []) = 1$  and if  $t \geq 1$ , then*

$$\begin{aligned} H(g, t, f, e, n, D) = & \sum_{\substack{g_1 + g_2 = g \\ t_1 + t_2 = t - 1 \\ f_1 + e_2 = e \\ f_2 + e_1 = f \\ n_1 + n_2 = n - 1 \\ D_1 \subseteq D}} H(g_1, t_1, f_1, e_1, n_1, D_1) H(g_2, t_2, f_2, e_2, n_2, D - D_1) \\ & + \delta_{n \geq 3} \delta_{g \geq 1} \sum_{p=1}^{n-2} p H(g - 1, t - 1, e, f, n - 1 - p, p, D) \\ & + \sum_{p=n}^{p=t-1} H(g, t - 1, e, f, p, D) \\ & + \sum_{j=1}^{|D|} H(g, t - 1, e, f, d_j + n - 1, D - \{d_j\}). \end{aligned} \tag{4.2}$$

### 5 Multirooted hypermaps

For  $\rho \geq 1$  a  $\rho$ -rooted hypermap is a hypermap in which a sequence of  $\rho$  darts with pairwise distinct initial vertices is distinguished. A multirooted hypermap is a  $\rho$ -rooted hypermap

for some  $\rho \geq 1$ . This section addresses the enumeration of multirooted hypermaps.

**Theorem 5.1** (Recurrence between numbers of multirooted hypermaps). *Let  $H_m(g, t, f, e, D)$  be the number of multirooted hypermaps of genus  $g$  with  $t$  darts,  $f$  faces and  $e$  hyperedges such that  $D$  is the list of degrees of the distinguished vertices. Then  $H_m(0, 0, 1, 0, []) = 1$  and if  $t \geq 1$ , then*

$$\begin{aligned}
 H_m(g, t, f, e, n.D) = & \sum_{\substack{g_1 + g_2 = g \\ t_1 + t_2 = t - 1 \\ f_1 + e_2 = e \\ f_2 + e_1 = f \\ n_1 + n_2 = n - 1 \\ D_1 \subseteq D}} H_m(g_1, t_1, f_1, e_1, n_1.D_1) H_m(g_2, t_2, f_2, e_2, n_2.(D - D_1)) \\
 & + \delta_{n \geq 3} \delta_{g \geq 1} \sum_{p=1}^{n-2} H_m(g - 1, t - 1, e, f, (n - 1 - p).p.D) \\
 & + \sum_{p=n}^{p=t-1} H_m(g, t - 1, e, f, p.D) \\
 & + \sum_{j=1}^{|D|} d_j H_m(g, t - 1, e, f, (d_j + n - 1).(D - \{d_j\})).
 \end{aligned} \tag{5.1}$$

*Proof.* A multirooted hypermap is similar to a sequenced rooted hypermap except that for each distinguished non-root vertex a dart starting from it is distinguished. If the degree of the  $j$ th distinguished vertex is  $d_j$ , then there are  $d_j$  ways of distinguishing a dart of this vertex. It follows that for each sequenced rooted hypermap, there are  $\prod_{j=1}^{|D|} d_j$  multirooted hypermaps. Let  $H_m(g, t, f, e, D)$  be the number of multirooted hypermaps of genus  $g$  with  $t$  darts,  $f$  faces and  $e$  hyperedges such that  $D$  is the list of degrees of the initial vertex of the distinguished darts. Then

$$H_m(g, t, f, e, n.D) = H(g, t, f, e, n, D) \prod_{j=1}^{|D|} d_j. \tag{5.2}$$

Solving (5.2) for  $H(g, t, f, e, n, D)$  and substituting into (4.2) proves the theorem.  $\square$

For  $\rho \geq 1$  let

$$\begin{aligned}
 H_g(v_1, \dots, v_\rho, x, y, u, z) = & \sum_{\substack{t \geq 0, f \geq 1, e \geq 0 \\ d_1 \geq 1, \dots, d_\rho \geq 1 \\ v = t + 2(1 - g) - e - f}} H_m(g, t, f, e, D_{1.. \rho}) v_{1.. \rho}^{D_{1.. \rho}} x^f y^e u^v z^t \tag{5.3}
 \end{aligned}$$

be the generating function that counts multirooted hypermaps of genus  $g$  with  $\rho$  distinguished darts if  $g \geq 0$ , and 0 otherwise. For  $1 \leq i \leq \rho$ , the exponent  $d_i$  of the variable  $v_i$  in this series is the degree of the initial vertex of the  $i$ -th distinguished dart. The exponent  $f$  of the variable  $x$  is the number of faces, the exponent  $e$  of the variable  $y$  is the number of hyperedges, the exponent  $t$  of the variable  $z$  is the number of darts and the exponent  $v$  of the variable  $u$  is the number of vertices ( $v$  is computable from the other parameters by Formula (1.2)).

**Corollary 5.2** (Functional equations for multirooted hypermaps). *For  $g \geq 0$  and  $\rho \geq 1$  the generating functions  $H_g$  of multirooted hypermaps of genus  $g$  are defined by the following functional equations:*

$$\begin{aligned}
 H_g(v_1, W, x, y, u, z) = & \\
 & \frac{yv_1z}{xu} \sum_{j=0}^g \sum_{X \subseteq W} H_j(v_1, X, y, x, u, z) H_{g-j}(v_1, W - X, x, y, u, z) \\
 & + \frac{v_1z}{u} H_{g-1}(v_1, v_1, W, y, x, u, z) \tag{5.4} \\
 & + \frac{v_1uz}{v_1 - 1} (H_g(v_1, W, y, x, u, z) - H_g(1, W, y, x, u, z)) \\
 & + v_1uz \sum_{j=2}^{j=\rho} v_j \frac{\partial}{\partial v_j} \left( v_j \frac{H_g(v_j, W - \{v_j\}, y, x, u, z) - H_g(v_1, W - \{v_j\}, y, x, u, z)}{v_j - v_1} \right) \\
 & + xu\delta_{g=0}\delta_{\rho=1},
 \end{aligned}$$

where  $W = v_2, \dots, v_\rho$ .

*Proof.* By summation according to (5.3) of the recurrence between numbers of multirooted hypermaps from Theorem 5.1. □

By vertex-hyperedge duality, we have

$$H_g(v_1, W, y, x, u, z) = H_g(v_1, W, x, y, u, z) + \delta_{g=0}\delta_{\rho=1}(yu - xu) \tag{5.5}$$

and thus another functional equation without  $x, y$  swaps is:

$$\begin{aligned}
 H_g(v_1, W, x, y, u, z) = & \\
 & \frac{yv_1z}{xu} \sum_{j=0}^g \sum_{X \subseteq W} \left( (H_j(v_1, X, x, y, u, z) + \delta_{j=0}\delta_{|X|=0}(yu - xu)) \right. \\
 & \qquad \qquad \qquad \left. H_{g-j}(v_1, W - X, x, y, u, z) \right) \\
 & + \frac{v_1z}{u} H_{g-1}(v_1, v_1, W, x, y, u, z) \tag{5.6} \\
 & + \frac{v_1uz}{v_1 - 1} (H_g(v_1, W, x, y, u, z) - H_g(1, W, x, y, u, z)) \\
 & + v_1uz \sum_{j=2}^{j=\rho} v_j \frac{\partial}{\partial v_j} \left( v_j \frac{H_g(v_j, W - \{v_j\}, x, y, u, z) - H_g(v_1, W - \{v_j\}, x, y, u, z)}{v_j - v_1} \right) \\
 & + xu\delta_{g=0}\delta_{\rho=1}.
 \end{aligned}$$

The former equation is given here for maximal generality. However, a consequence of the genus formula (1.2) is that three variables among the four variables  $x, y, u$  and  $z$  are sufficient. In the remainder of the paper we consider the generating functions

$$H_g(v_1, W, x, y, u) = H_g(v_1, W, x, y, u, 1)$$

with one fewer variable. They are defined by the following functional equations:

$$\begin{aligned}
 H_g(v_1, W, x, y, u) = & \\
 & \frac{yv_1}{xu} \sum_{j=0}^g \sum_{X \subseteq W} (H_j(v_1, X, x, y, u) + \delta_{j,0} \delta_{|X|,0} (yu - xu)) H_{g-j}(v_1, W - X, x, y, u) \\
 & + \frac{v_1}{u} H_{g-1}(v_1, v_1, W, x, y, u) \tag{5.7} \\
 & + \frac{v_1 u}{v_1 - 1} (H_g(v_1, W, x, y, u) - H_g(1, W, x, y, u)) \\
 & + v_1 u \sum_{j=2}^{j=\rho} v_j \frac{\partial}{\partial v_j} \left( v_j \frac{H_g(v_j, W - \{v_j\}, x, y, u) - H_g(v_1, W - \{v_j\}, x, y, u)}{v_j - v_1} \right) \\
 & + xu \delta_{g=0} \delta_{\rho=1}.
 \end{aligned}$$

For  $g, \rho \neq 0, 1$ , after grouping in the left-hand side the terms containing  $H_g(v_1, W, x, y, u)$  in (5.7), one gets

$$\begin{aligned}
 \frac{A(v_1, x, y, u)}{v_1} H_g(v_1, W, x, y, u) = & \\
 & x(1 - v_1) \sum_{j=0}^g \sum_{\substack{X \subseteq W \\ (j, X) \neq (0, \{\}) \\ (j, X) \neq (g, W)}} H_j(v_1, X, x, y, u) H_{g-j}(v_1, W - X, x, y, u) \\
 & + \frac{1 - v_1}{u} H_{g-1}(v_1, v_1, W, x, y, u) + u H_g(1, W, x, y, u) \\
 & + u T_g(v_1, W, x, y, u) \tag{5.8}
 \end{aligned}$$

with

$$A(v, x, y, u) = vu + (1 - v)(1 - yv + xv - 2vH_0(v, x, y, u)/u) \tag{5.9}$$

and

$$\begin{aligned}
 T_g(v_1, W, x, y, u) = & \\
 & (1 - v_1) \sum_{j=2}^{j=\rho} v_j \frac{\partial}{\partial v_j} \left( \frac{v_j}{v_j - v_1} \left( H_g(v_j, W - \{v_j\}, x, y, u) \right. \right. \\
 & \left. \left. - H_g(v_1, W - \{v_j\}, x, y, u) \right) \right). \tag{5.10}
 \end{aligned}$$

### 6 Rooted hypermap generating functions

Let  $h_g(v, e, f)$  be the number of rooted genus- $g$  hypermaps with  $v$  vertices,  $e$  hyperedges and  $f$  faces. Let

$$H_g(x, y, u) = \sum_{v, e, f \geq 1} h_g(v, e, f) x^v y^e u^f \tag{6.1}$$

be the ordinary generating function for counting rooted hypermaps on the orientable surface of genus  $g \geq 0$ , where the exponent of variable  $x$  is the number of vertices, the exponent of variable  $y$  is the number of hyperedges, and the exponent of variable  $u$  is the number of faces.

Rooted hypermaps being 1-rooted hypermaps,

$$H_g(x, y, u) = H_g(1, x, y, u), \tag{6.2}$$

where  $H_g(v_1, \dots, v_\rho, x, y, u)$  is the generating function counting  $\rho$ -rooted genus- $g$  hypermaps defined in Section 5 for  $\rho \geq 1$ .

We first recall in Section 6.1 a known parametric expression of the generating function that counts rooted planar hypermaps. Then we explain in Section 6.2 how to solve the functional equation of the generating functions  $H_g(x, y, u)$  that count rooted hypermaps with a given positive genus  $g$ .

### 6.1 Rooted planar hypermaps

The following proposition is a reformulation of [1, Theorem 3], with the correspondence  $s = x$ ,  $f = u$  and  $a = y$  for variables,  $\lambda = p$ ,  $\mu = q$  and  $\nu = r$  for parameters, and  $H_0 = sf(1 + J)$  for generating functions.

**Proposition 6.1 ([1]).** *The ordinary generating function  $H_0(x, y, u)$  that counts rooted planar hypermaps by number of vertices (exponent of  $x$ ), hyperedges (exponent of  $y$ ) and faces (exponent of  $u$ ) is the unique solution of the following parametric system:*

$$H_0(x, y, u) = 1 + pqr(1 - p - q - r) \tag{6.3}$$

with

$$\begin{cases} x = p(1 - q - r) \\ u = q(1 - p - r) \\ y = r(1 - p - q). \end{cases} \tag{6.4}$$

*Proof.* The generating function  $H_0(v, x, y, u)$  that counts rooted planar hypermaps (genus 0) by number of vertices (exponent of  $x$ ), hyperedges (exponent of  $y$ ), faces (exponent of  $u$ ) and degree of the root vertex (exponent of  $v$ ) satisfies the functional equation

$$\begin{aligned} H_0(v, x, y, u) &= \frac{yv}{xu} (H_0(v, x, y, u) + yu - xu) H_0(v, x, y, u) \\ &\quad + \frac{vu}{v-1} (H_0(v, x, y, u) - H_0(1, x, y, u)) + xu \end{aligned} \tag{6.5}$$

obtained by instantiation of (5.7) with  $g = 0$ ,  $\rho = 1$  and  $v_1 = v$ .

This equation can be solved by the *quadratic method* [10, page 515]. The idea is to define auxiliary functions  $A(v, x, y, u)$  and  $B(v, x, y, u)$  by (5.9) and

$$B(v, x, y, u) = A(v, x, y, u)^2 \tag{6.6}$$

and look for a function  $V(x, y, u)$  such that

$$A(V(x, y, u), x, y, u) = 0, \tag{6.7}$$

implying that  $B(V(x, y, u), x, y, u) = 0$  and  $\partial_v B(v, x, y, u)|_{v=V(x, y, u)} = 0$ .

We get from (6.5), (5.9) and (6.6) that

$$\begin{aligned} B(v, x, y, u) &= \\ &1 - 2yv - 2xv - 2v^3y - 2v^3x - 2v^2u + v^4y^2 - 2v^3y^2 + y^2v^2 + v^4x^2 \\ &\quad - 2v^3x^2 + x^2v^2 + v^2u^2 + 4v^3yx - 2yv^2x - 2yv^2u + 2v^3yu - 2v^4yu \\ &\quad - 2v^3xu + 2xv^2u + 4v^2x + 4v^2y + 2vu + 4v^3H_0(1, x, y, u) \\ &\quad - 4v^2H_0(1, x, y, u) - 2v + v^2. \end{aligned} \tag{6.8}$$

The constraints  $B(V(x, y, u), x, y, u) = 0$  and  $\partial_v B(v, x, y, u)|_{v=V(x,y,u)} = 0$  respectively are

$$\begin{aligned}
 &1 - 2yV - 2xV - 2V^3y - 2V^3x - 2V^2u + V^4y^2 - 2V^3y^2 + y^2V^2 \\
 &\quad + V^4x^2 - 2V^3x^2 + x^2V^2 + V^2u^2 + 4V^3yx - 2yV^2x - 2yV^2u \\
 &\quad + 2V^3yux - 2V^4y - 2V^3xu + 2xV^2u + 4V^2x + 4V^2y + 2Vu \\
 &\quad + 4V^3H_0(1, x, y, u) - 4V^2H_0(1, x, y, u) - 2V + V^2 = 0 \quad (6.9)
 \end{aligned}$$

and

$$\begin{aligned}
 &-2 + 8yV + 8xV + 4V^3y^2 - 6y^2V^2 + 4V^3x^2 - 6x^2V^2 - 6V^2x \\
 &- 6V^2y - 4Vu + 2y^2V + 2x^2V + 2Vu^2 - 4yVu + 4xVu - 4yVx \\
 &\quad + 12yV^2x + 6yV^2u - 8V^3yx - 6xV^2u + 2V - 2x - 2y + 2u = 0. \quad (6.10)
 \end{aligned}$$

It can be checked that both equations are satisfied by

$$V = 1/(1 - q) \tag{6.11}$$

with  $x, u, y$  and  $H_0(1, x, y, u)$  related to  $p, q$  and  $r$  by (6.4) and (6.3). □

### 6.2 Rooted hypermaps with positive genus

The following additional notations are used in this section. Let  $\rho$  be a positive integer. Let  $H_j[n_1, \dots, n_\rho]$  denote the partial derivative of the function  $H_j(v_1, \dots, v_\rho, x, y, u)$  with respect to the variables  $v_1, \dots, v_\rho$  to the respective orders  $n_1, \dots, n_\rho$ , computed at  $v_1 = \dots = v_\rho = V$ . The abbreviation  $[\rho]$  denotes the list  $[2, \dots, \rho]$  if  $\rho \geq 2$  and the empty list  $[\ ]$  if  $\rho = 1$ . The abbreviation  $N_{[\rho]}$  denotes the list  $[n_2, \dots, n_\rho]$ . For any sublist  $X \subseteq [\rho]$  of  $[\rho]$ ,  $[\rho] - X$  denotes the sublist of the elements of  $[\rho]$  that are not in  $X$ ,  $N_X$  denotes the list of those  $n_i$  in  $N_{[\rho]}$  such that  $i$  is in  $X$  and  $N_j$  denotes the list  $[n_2, \dots, n_{j-1}, n_{j+1}, \dots, n_\rho]$ .

#### 6.2.1 Equation for rooted hypermaps and recurrence relations

The special case of Formula (5.8) for  $g \geq 1, \rho = 1$  and  $v_1 = V$  is the following formula:

$$\begin{aligned}
 uH_g(1, x, y, u) = \\
 (V - 1) \left( x \sum_{j=1}^{g-1} H_j(V, x, y, u)H_{g-j}(V, x, y, u) + H_{g-1}(V, V, x, y, u)/u \right)
 \end{aligned}$$

i.e.

$$uH_g(1, x, y, u) = (V - 1) \left( x \sum_{j=1}^{g-1} H_j[0]H_{g-j}[0] + H_{g-1}[0, 0]/u \right). \tag{6.12}$$

In order to derive from (6.12) a value for  $H_g(1, x, y, u)$ , we are looking for a value for  $H_j[0], H_{g-j}[0]$  and  $H_{g-1}[0, 0]$ . More generally, we will derive from the following proposition a closed form for the expressions  $H_g[n_1, \dots, n_\rho]$ .

**Proposition 6.2.** For  $g \geq 0$ ,  $\rho \geq 1$  and  $n_1, \dots, n_\rho \geq 0$  the function  $H_g[n_1, \dots, n_\rho]$  is defined by

$$\begin{aligned} & \frac{(n_1 + 1)A[1]}{V} H_g[n_1, N_{[\rho]}] = \\ & \sum_{\substack{i+j+k=n_1+1 \\ i>0, k<n_1}} \binom{n_1 + 1}{i, j} \frac{(-1)^{j+1} j!}{V^{j+1}} A[i] H_g[k, N_{[\rho]}] \\ & + x \sum_{\substack{k+l+m=n_1+1 \\ 0 \leq j \leq g \\ X \subseteq [\rho] \\ (j, X) \neq (0, \{1\}) \\ (j, X) \neq (g, [\rho])}} \binom{n_1 + 1}{k, l} M[m] H_j[k, N_X] H_{g-j}[l, N_{[\rho]-X}] \\ & + \frac{1}{u} \sum_{i+j+k=n_1+1} \binom{n_1 + 1}{i, j} M[k] H_{g-1}[i, j, N_{[\rho]}] \\ & + u \sum_{j=2}^{\rho} \frac{(n_1 + 1)! n_j!}{(n_1 + n_j + 2)!} \left( n_j F_g[n_1 + n_j + 2, N_j] \right. \\ & \quad \left. + \frac{V(n_j+1)}{n_1+n_j+3} F_g[n_1 + n_j + 3, N_j] \right), \end{aligned} \tag{6.13}$$

where

$$F_g(v_1, \dots, v_h, x, y, u) = L(v_1) H_g(v_1, \dots, v_h, x, y, u) \tag{6.14}$$

for  $h \geq 1$ ,  $M(v) = 1 - v$  and  $L(v) = v(1 - v)$ .

*Proof.* Equation (6.13) is obtained from Equation (5.8) as follows:

1. Partial derivation of (5.8) with respect to the variables  $v_1, v_2, \dots, v_\rho$  to the respective orders  $n_1 + 1, n_2, \dots, n_\rho$ .
2. Evaluation of this differential equation at  $v_1 = \dots = v_\rho = V$ . The function  $H_g[n_1 + 1, \dots, n_\rho]$  is multiplied by  $A[0]$  in the resulting equation, and  $A[0]$  is known to be zero (6.7). The functions  $T_g[\dots]$  are replaced by expressions with the functions  $F_g[\dots]$  thanks to Lemma 6.3 below.
3. In the left-hand side of the resulting equation, isolation of the single term involving the function  $H_g[n_1, \dots, n_\rho]$ .

By inspection one can check that the right-hand side of (6.13) depends only on some functions  $H_g[k, n_2, \dots, n_\rho]$  with  $k < n_1$ , some functions  $H_g[n'_1, \dots, n'_{\rho'}]$  with  $\rho' < \rho$  and some functions  $H_j[\dots]$  for  $j < g$ . Thus, (6.13) is a recursive definition of the family of functions  $H_g[n_1, \dots, n_\rho]$  for  $g \geq 0$ ,  $\rho \geq 1$  and  $n_1, \dots, n_\rho \geq 0$ .  $\square$

The following lemma relates the partial derivatives of  $T_g$  at  $v = V$  with the ones of  $F_g$ .

**Lemma 6.3.** For  $\rho \geq 2$  and  $g, n_1, \dots, n_\rho \geq 0$ ,

$$\begin{aligned} T_g[n_1 + 1, N_{[\rho]}] = \\ \sum_{j=2}^{j=\rho} \frac{(n_1 + 1)! n_j!}{(n_1 + n_j + 2)!} \left( n_j F_g[n_1 + n_j + 2, N_j] \right. \\ \quad \left. + \frac{V(n_j+1)}{n_1+n_j+3} F_g[n_1 + n_j + 3, N_j] \right). \end{aligned} \tag{6.15}$$



*Proof.* We can easily prove that

$$\frac{\partial}{\partial v_j} \left[ \frac{(v_j - v_1)H_g(v_1, [\rho] - \{v_j\}, x, y, u)}{v_j - v_1} \right] = 0. \tag{6.16}$$

Then,  $T_g(v_1, \dots, v_\rho, x, y, u)$  equals

$$\sum_{j=2}^{j=\rho} v_j \frac{\partial}{\partial v_j} \left( (v_j - v_1)^{-1} \left( v_j(1 - v_1)H_g(v_j, [\rho] - \{v_j\}, x, y, u) - v_1(1 - v_1)H_g(v_1, [\rho] - \{v_j\}, x, y, u) \right) \right). \tag{6.17}$$

It also holds that

$$\frac{\partial^{n_1+1}}{\partial v_1^{n_1+1}} \left[ \frac{v_j(v_j - v_1)H_g(v_j, [\rho] - \{v_j\}, x, y, u)}{v_j - v_1} \right] = 0, \tag{6.18}$$

so that  $\frac{\partial^{n_1+1}}{\partial v_1^{n_1+1}} T_g(v_1, \dots, v_\rho, x, y, u)$  equals

$$\sum_{j=2}^{j=r} v_j \frac{\partial^{n_1+2}}{\partial v_1^{n_1+1} \partial v_j} \left( (v_j - v_1)^{-1} \left( v_j(1 - v_j)H_g(v_j, [\rho] - \{v_j\}, x, y, u) - v_1(1 - v_1)H_g(v_1, [\rho] - \{v_j\}, x, y, u) \right) \right) \tag{6.19}$$

i.e.

$$\sum_{j=2}^{j=\rho} v_j \frac{\partial^{n_1+2}}{\partial v_1^{n_1+1} \partial v_j} \left( \frac{F_g(v_j, [\rho] - \{v_j\}, x, y, u) - F_g(v_1, [\rho] - \{v_j\}, x, y, u)}{v_j - v_1} \right). \tag{6.20}$$

Formula (6.15) is a consequence of

$$\frac{\partial^{n_1+n_2}}{\partial x_1^{n_1} \partial x_2^{n_2}} \left( \frac{\psi(x_1) - \psi(x_2)}{x_1 - x_2} \right)_{x_1=x_2=a} = \frac{n_1!n_2!}{(n_1 + n_2 + 1)!} \psi^{(n_1+n_2+1)}(a). \tag{6.21} \quad \square$$

The formula

$$F_g[n, N] = \sum_{k+l=n} \binom{n}{k} L[k]H_g[l, N] \tag{6.22}$$

is an easy consequence of (6.14). Thus the right-hand side of (6.13) only depends on some functions  $H_g[k, \dots, n_\rho]$  with  $k < n_1$ , some functions  $H_g[n'_1, \dots, n'_{\rho'}]$  with  $\rho' < \rho$ , some functions  $H_j[\dots]$  for  $j < g$  and some functions  $A[i]$ . A relation between  $A[i]$  and some functions  $H_0[j]$  is established in Section 6.2.2.

**6.2.2 Case  $g = 0$  and  $\rho = 1$**

The function  $A[i]$  can be related to some functions  $H_0[j]$  as follows: With  $M(v) = 1 - v$  and  $L(v) = v(1 - v)$ , Equation (5.9) is

$$A(v, x, y, u) = vu + M(v) + L(v)(-y + x - 2xH_0(v, x, y, u)). \tag{6.23}$$

Its instantiation at  $v = V$  gives

$$H_0[0] = \frac{1 - q}{1 - q - r}. \tag{6.24}$$

For  $k \geq 1$ , the  $k$ -th partial derivative of (6.23) in  $v$  is

$$\begin{aligned} \frac{\partial^k}{\partial v^k} A(v, x, y, u) &= \frac{\partial^k}{\partial v^k} (vu) + \frac{\partial^k}{\partial v^k} M(v) \\ &+ \frac{\partial^k}{\partial v^k} [L(v)(-y + x - 2xH_0(v, x, y, u))] \end{aligned} \tag{6.25}$$

and its instantiation in  $v = V$  is

$$\begin{aligned} A[k] &= \frac{\partial^k}{\partial v^k} (vu)|_{v=V} + M[k] \\ &+ \sum_{i+j=k} \binom{k}{i} L[i] \left( \frac{\partial^j}{\partial v^j} (-y + x - 2xH_0(v, x, y, u))|_{v=V} \right). \end{aligned} \tag{6.26}$$

Solving (6.26) for  $k = 1$  gives

$$H_0[1] = \frac{(1 - q)^2(A[1] + 1 - p - q - r)}{2pq(1 - q - r)}. \tag{6.27}$$

For  $k \geq 2$ , one gets

$$A[k] = -2x \sum_{i+j=k} \binom{k}{i} L[i] H_0[j]$$

since  $M[k] = 0$ , i.e.

$$A[k] = -2x \left( L[0]H_0[k] + kL[1]H_0[k - 1] + \frac{k(k - 1)}{2}L[2]H_0[k - 2] \right) \tag{6.28}$$

since  $L[k] = 0$  if  $k \geq 3$ .

## 7 Explicit formulas for small genera

This section proposes explicit parametric expressions for the generating functions that count rooted hypermaps of small positive genus. In Section 7.1 we count by number of vertices, hyperedges and faces; the number of darts can be obtained from these parameters by Formula (1.2). In Section 7.2 we count by number of darts alone.

### 7.1 Rooted hypermap series enumerated with three parameters

For  $g = 1, \dots, 5$  we have computed an explicit expression of  $H_g(x, y, u)$  parameterized by  $p, q$  and  $r$ , with  $x = p(1 - q - r)$ ,  $u = q(1 - p - r)$  and  $y = r(1 - p - q)$ , by application of formulas in Section 6. For  $g \geq 3$ , the expressions are too large to be included in the present text, but a Maple file with all the generating functions up to genus 5 is available from the first author on request.

A parametric expression of  $H_1(x, y, u)$  is

$$H_1(x, y, u) = \frac{pqr(1-p)(1-q)(1-r)}{[(1-p-q-r)^2 - 4pqr]^2}. \tag{7.1}$$

This expression can be derived from [2, Theorem 3], with the correspondence  $s = x$ ,  $f = u$ , and  $a = y$  between variables and the correspondence  $H_1(x, y, u) = xuK_1(1, x, y, u)$  between generating functions.

A parametric expression of  $H_2(x, y, u)$  is

$$H_2(x, y, u) = \frac{pqr(1-p)(1-q)(1-r)P_2(p, q, r)}{[(1-p-q-r)^2 - 4pqr]^7} \tag{7.2}$$

where

$$\begin{aligned} P_2(p, q, r) = & 76p^6q^2r^2 - 8p^4q^4r^2 - 8p^4q^2r^4 + 76p^2q^6r^2 - 8p^2q^4r^4 + 76p^2q^2r^6 \\ & + 40p^7qr - 76p^6q^2r - 76p^6qr^2 - 112p^5q^3r - 228p^5q^2r^2 - 112p^5qr^3 \\ & + 8p^4q^4r + 16p^4q^3r^2 + 16p^4q^2r^3 + 8p^4qr^4 - 112p^3q^5r + 16p^3q^4r^2 \\ & + 40p^3q^3r^3 + 16p^3q^2r^4 - 112p^3qr^5 - 76p^2q^6r - 228p^2q^5r^2 \\ & + 16p^2q^4r^3 + 16p^2q^3r^4 - 228p^2q^2r^5 - 76p^2qr^6 + 40pq^7r - 76pq^6r^2 \\ & - 112pq^5r^3 + 8pq^4r^4 - 112pq^3r^5 - 76pq^2r^6 + 40pqr^7 + p^8 - 20p^7q \\ & - 20p^7r - 35p^6q^2 - 64p^6qr - 35p^6r^2 + 56p^5q^3 + 396p^5q^2r + 396p^5qr^2 \\ & + 56p^5r^3 + 140p^4q^4 + 264p^4q^3r + 393p^4q^2r^2 + 264p^4qr^3 + 140p^4r^4 \\ & + 56p^3q^5 + 264p^3q^4r - 92p^3q^3r^2 - 92p^3q^2r^3 + 264p^3qr^4 + 56p^3r^5 \\ & - 35p^2q^6 + 396p^2q^5r + 393p^2q^4r^2 - 92p^2q^3r^3 + 393p^2q^2r^4 + 396p^2qr^5 \\ & - 35p^2r^6 - 20pq^7 - 64pq^6r + 396pq^5r^2 + 264pq^4r^3 + 264pq^3r^4 \\ & + 396pq^2r^5 - 64pqr^6 - 20pr^7 + q^8 - 20q^7r - 35q^6r^2 + 56q^5r^3 \\ & + 140q^4r^4 + 56q^3r^5 - 35q^2r^6 - 20qr^7 + r^8 + 6p^7 + 105p^6q + 105p^6r \\ & + 21p^5q^2 - 116p^5qr + 21p^5r^2 - 420p^4q^3 - 821p^4q^2r - 821p^4qr^2 \\ & - 420p^4r^3 - 420p^3q^4 - 648p^3q^3r - 316p^3q^2r^2 - 648p^3qr^3 - 420p^3r^4 \\ & + 21p^2q^5 - 821p^2q^4r - 316p^2q^3r^2 - 316p^2q^2r^3 - 821p^2qr^4 + 21p^2r^5 \\ & + 105pq^6 - 116pq^5r - 821pq^4r^2 - 648pq^3r^3 - 821pq^2r^4 - 116pqr^5 \\ & + 105pr^6 + 6q^7 + 105q^6r + 21q^5r^2 - 420q^4r^3 - 420q^3r^4 + 21q^2r^5 \\ & + 105qr^6 + 6r^7 - 49p^6 - 189p^5q - 189p^5r + 315p^4q^2 + 479p^4qr \\ & + 315p^4r^2 + 910p^3q^3 + 1162p^3q^2r + 1162p^3qr^2 + 910p^3r^3 + 315p^2q^4 \\ & + 1162p^2q^3r + 720p^2q^2r^2 + 1162p^2qr^3 + 315p^2r^4 - 189pq^5 + 479pq^4r \\ & + 1162pq^3r^2 + 1162pq^2r^3 + 479pqr^4 - 189pr^5 - 49q^6 - 189q^5r \\ & + 315q^4r^2 + 910q^3r^3 + 315q^2r^4 - 189qr^5 - 49r^6 + 112p^5 + 70p^4q \\ & + 70p^4r - 770p^3q^2 - 876p^3qr - 770p^3r^2 - 770p^2q^3 - 1380p^2q^2r \\ & - 1380p^2qr^2 - 770p^2r^3 + 70pq^4 - 876pq^3r - 1380pq^2r^2 - 876pqr^3 \\ & + 70pr^4 + 112q^5 + 70q^4r - 770q^3r^2 - 770q^2r^3 + 70qr^4 + 112r^5 \end{aligned}$$

$$\begin{aligned}
 & - 105p^4 + 210p^3q + 210p^3r + 735p^2q^2 + 1034p^2qr + 735p^2r^2 + 210pq^3 \\
 & + 1034pq^2r + 1034pqr^2 + 210pr^3 - 105q^4 + 210q^3r + 735q^2r^2 + 210qr^3 \\
 & - 105r^4 + 14p^3 - 315p^2q - 315p^2r - 315pq^2 - 672pqr - 315pr^2 + 14q^3 \\
 & - 315q^2r - 315qr^2 + 14r^3 + 49p^2 + 175pq + 175pr + 49q^2 + 175qr \\
 & + 49r^2 - 36p - 36q - 36r + 8.
 \end{aligned}$$

Remark: For  $g = 0$ , the formula

$$H_0(x, y, u) = pqr(1 - p - q - r) \tag{7.3}$$

can be derived from [1], with the correspondence  $s = x$ ,  $f = u$ , and  $a = y$  between variables and the correspondence  $H_0(x, y, u) = xuK_0(1, x, y, u)$  between generating functions.

## 7.2 Rooted hypermap series enumerated by number of darts

Let  $H_g(z)$  be the ordinary generating function of rooted hypermaps on the orientable surface of genus  $g \geq 0$ , where the exponent of variable  $z$  is the number  $d$  of darts.

### 7.2.1 Generating functions

For  $g$  from 0 to 6, a parametric expression of  $H_g(z)$ , where  $z = \tau(1 - 2\tau)$  and  $\tau = 0$  when  $z = 0$ , is

$$H_0(z) = \frac{\tau^3 (1 - 3\tau)}{z^2}, \tag{7.4}$$

$$H_1(z) = \frac{\tau^3}{(1 - \tau) (1 - 4\tau)^2}, \tag{7.5}$$

$$H_2(z) = \frac{4 z^2 \tau^3 (51 \tau^4 - 77 \tau^3 + 48 \tau^2 - 15 \tau + 2)}{(1 - \tau)^5 (1 - 4\tau)^7}, \tag{7.6}$$

$$H_3(z) = \frac{4 z^4 \tau^3 P_3(z)}{(1 - \tau)^9 (1 - 4\tau)^{12}}, \tag{7.7}$$

$$H_4(z) = \frac{4 z^6 \tau^3 P_4(z)}{(1 - \tau)^{13} (1 - 4\tau)^{17}}, \tag{7.8}$$

$$H_5(z) = \frac{4 z^8 \tau^3 P_5(z)}{(1 - \tau)^{17} (1 - 4\tau)^{22}}, \tag{7.9}$$

$$H_6(z) = \frac{4 z^{10} \tau^3 P_6(z)}{(1 - \tau)^{21} (1 - 4\tau)^{27}}, \tag{7.10}$$

with

$$\begin{aligned}
 P_3(z) = & 28496 \tau^9 - 36888 \tau^8 - 13164 \tau^7 + 61676 \tau^6 - 61872 \tau^5 + 35172 \tau^4 \\
 & - 13168 \tau^3 + 3360 \tau^2 - 552 \tau + 45,
 \end{aligned}$$

$$\begin{aligned}
 P_4(z) = & 32375616 \tau^{14} + 15509760 \tau^{13} - 243313744 \tau^{12} + 442844592 \tau^{11} \\
 & - 389268768 \tau^{10} + 170357328 \tau^9 + 1281984 \tau^8 - 53553072 \tau^7 \\
 & + 39814032 \tau^6 - 17597520 \tau^5 + 5541192 \tau^4 - 1320920 \tau^3 + 239697 \tau^2 \\
 & - 30456 \tau + 2016,
 \end{aligned}$$

$$\begin{aligned}
 P_5(z) = & 61742404608 \tau^{19} + 239043447552 \tau^{18} - 1163002515456 \tau^{17} \\
 & + 1403096348736 \tau^{16} + 338393916800 \tau^{15} - 2962590413376 \tau^{14} \\
 & + 4243997599488 \tau^{13} - 3552865706240 \tau^{12} + 2000782619136 \tau^{11} \\
 & - 761565230016 \tau^{10} + 165542511744 \tau^9 + 7568059872 \tau^8 \\
 & - 23295865824 \tau^7 + 11016156244 \tau^6 - 3336459144 \tau^5 + 761835465 \tau^4 \\
 & - 141393220 \tau^3 + 21738240 \tau^2 - 2490480 \tau + 151200
 \end{aligned}$$

and

$$\begin{aligned}
 P_6(z) = & 178054771302400 \tau^{24} + 1584534210564096 \tau^{23} - 4933663711730688 \tau^{22} \\
 & - 2073822560019456 \tau^{21} + 28025505345377280 \tau^{20} \\
 & - 55010184951564288 \tau^{19} + 54283457920223232 \tau^{18} \\
 & - 22997164994372352 \tau^{17} - 13439214645718272 \tau^{16} \\
 & + 31734000656779264 \tau^{15} - 29719458122609664 \tau^{14} \\
 & + 18704646148809216 \tau^{13} - 8736443315384448 \tau^{12} \\
 & + 3098312828500416 \tau^{11} - 813298324826016 \tau^{10} + 138473163256176 \tau^9 \\
 & - 4043551301232 \tau^8 - 6580517850696 \tau^7 + 2630924485729 \tau^6 \\
 & - 626336383104 \tau^5 + 112079088144 \tau^4 - 17314508592 \tau^3 + 2485496880 \tau^2 \\
 & - 284717376 \tau + 17107200.
 \end{aligned}$$

We have also computed the generating functions for  $7 \leq g \leq 11$ . Their expressions are too large to be included in the present text, but a Maple file is available from the first author on request.

A. Mednykh and R. Nedela used our formulas (7.4) to (7.7) to find explicit formulas for the number of rooted hypermaps for genus  $g = 0, 1, 2$  and  $3$  [19].

### 7.3 Other parameterization

In a private communication to the second author, P. Zograf suggests the parameterization

$$z = \frac{t}{(1 + 2t)^2}. \tag{7.11}$$

After adding the condition that  $t = 0$  when  $z = 0$ , it corresponds to

$$t = \frac{1 - 4z - \sqrt{1 - 8z}}{8z}. \tag{7.12}$$

These two parameterizations are equivalent. The one can be transformed into the other by means of the following substitutions:

$$\tau = \frac{t}{1 + 2t} \tag{7.13}$$

and

$$t = \frac{\tau}{1 - 2\tau}. \tag{7.14}$$

By means of these substitutions, the following parametric expressions in  $t$  can be obtained from the parametric expressions (7.4)–(7.10) for  $H_g(t)$  in  $\tau$ :

$$\begin{aligned} H_0(z) &= t(1-t), \\ H_1(z) &= \frac{t^3}{(1+t)(1-2t)^2}, \\ H_2(z) &= \frac{4t^5(1+2t)(t^4-t^3+6t^2+t+2)}{(1+t)^5(1-2t)^7}, \\ H_3(z) &= 4t^7(1+2t)(1+t)^{-9}(1-2t)^{-12}(80t^9-120t^8+1500t^7+1036t^6 \\ &\quad + 3768t^5+2820t^4+2288t^3+1008t^2+258t+45), \\ H_4(z) &= 4t^9(1+2t)(1+t)^{-13}(1-2t)^{-17}(16768t^{14}-33536t^{13} \\ &\quad + 653776t^{12}+786480t^{11}+4358016t^{10}+6151056t^9+10059552t^8 \\ &\quad + 10217040t^7+8418240t^6+5227024t^5+2365888t^4+800128t^3 \\ &\quad + 181665t^2+25992t+2016), \\ H_5(z) &= 4t^{11}(1+2t)(1+t)^{-17}(1-2t)^{-22}(6732800t^{19}-16832000t^{18} \\ &\quad + 450011520t^{17}+773106240t^{16}+5764983552t^{15}+11910647232t^{14} \\ &\quad + 29130502912t^{13}+46090300928t^{12}+63452543616t^{11} \\ &\quad + 68713116608t^{10}+60654218080t^9+43591208976t^8 \\ &\quad + 25142796864t^7+11637842232t^6+4232899206t^5+1181820745t^4 \\ &\quad + 245635580t^3+35501760t^2+3255120t+151200), \\ H_6(z) &= 4t^{13}(1+2t)(1+t)^{-21}(1-2t)^{-27}(4424052736t^{24}-13272158208t^{23} \\ &\quad + 452750478336t^{22}+1012254206976t^{21}+9488911137792t^{20} \\ &\quad + 25803592571904t^{19}+83891900050944t^{18}+180120643165440t^{17} \\ &\quad + 346626234587904t^{16}+535272874975232t^{15}+701152993531392t^{14} \\ &\quad + 771688966862592t^{13}+716686355273472t^{12}+563018634260736t^{11} \\ &\quad + 372549313187520t^{10}+207088794784752t^9+96021082581732t^8 \\ &\quad + 36765061031004t^7+11475757049569t^6+2863185376896t^5 \\ &\quad + 556090776432t^4+80913152016t^3+8274846384t^2+536428224t \\ &\quad + 17107200). \end{aligned}$$

For  $0 \leq g \leq 3$ , these expressions correspond to  $F_g(t)$  in Zograf’s communication. Moreover, they reveal an extra factorization by  $4(1+2t)$  for  $g \geq 2$ .

### 8 Efficient enumeration of rooted and sensed unrooted hypermaps by number of darts, vertices and hyperedges

We recall that a sensed map or hypermap is an equivalence class of (unrooted) maps or hypermaps under orientation-preserving isomorphism.

Before enumerating sensed hypermaps we first need to enumerate rooted hypermaps. We use an efficient method of counting rooted hypermaps by number of darts, faces, ver-

tices and hyperedges or, equivalently [23], 2-coloured bipartite maps rooted at a white vertex by number of edges, faces, white vertices and black vertices, presented by Kazarian and Zograf [15], and then count sensed 2-coloured bipartite maps and hypermaps with the same parameters using the same method we used [26, 12] to count sensed maps by number of edges, faces and vertices. The recurrence (formula (11) in [15]), with  $f$  changed to  $H$ , is as follows. Define  $H_{g,d}$  to be a homogeneous polynomial in the three variables  $t, u$ , and  $v$ . The coefficient of  $t^f u^b v^w$  in  $H_{g,d}$  is the number of 2-coloured bipartite maps of genus  $g$  with  $d$  edges,  $f$  faces,  $b$  black vertices and  $w$  white vertices rooted at a white vertex or, equivalently, the number of rooted hypermaps of genus  $g$  with  $d$  darts,  $f$  faces,  $b$  hyperedges and  $w$  vertices. Then  $H_{0,1} = tuv$  and

$$\begin{aligned}
 (d + 1)H_{g,d} &= \\
 &(2d - 1)(t + u + v)H_{g,d-1} \\
 &+ (d - 2) (2(tu + tv + uv) - (t^2 + u^2 + v^2)) H_{g,d-2} \tag{8.1} \\
 &+ (d - 1)^2(d - 2)H_{g-1,d-2} + \sum_{i=0}^g \sum_{j=1}^{d-3} (4 + 6j)(d - 2 - j)H_{i,j}H_{g-i,d-2-j}.
 \end{aligned}$$

In [26] we collaborated with Mednykh to enumerate rooted and sensed maps. Mednykh enumerated maps of genus up to 11 by number of edges alone, while we enumerated maps of genus up to 10 by number of edges and vertices. The method we used to enumerate rooted maps is presented in [25]. The method we used to enumerate sensed maps is based on Liskovets’ refinement [17] of the method Mednykh and Nedela used to enumerate sensed map of genus up to 3 by number of edges [18]. Later we used a more efficient method of enumerating rooted maps, presented in [5], to enumerate rooted and sensed maps of genus up to 50 [12].

To describe here the modifications we made to pass from maps to 2-coloured bipartite maps we need to briefly discuss a few of the concepts described in more detail in [26]. All the automorphisms of a map on an orientable surface are periodic. If the period is  $L > 1$ , then the automorphism divides the map into  $L$  isomorphic copies of a smaller map, called the *quotient map*. Most of the *cells* (vertices, edges and faces) are in orbits of length  $L$  under the automorphism; those that aren’t are called *branch points*. For example, if a map is drawn on the surface of a sphere which undergoes a rotation through  $360/L$  degrees, the two cells through which the axis of rotation pass are fixed; so they are each in an orbit of length 1 for any  $L$ . For maps of higher genus, not all the branch points are on orbits of length 1. For example, if a torus is represented as a square with opposite edges identified in pairs, and is rotated by 90 degrees (period 4), then the centre of the square is a branch point of orbit length 1 and so is the point represented by all four corners of the square, but the middle of the sides of the square are two branch points of orbit length 2: the point represented by the middle of both vertical sides of the square is taken by the rotation onto the point represented by the middle of both horizontal sides, and vice versa; so it takes two rotations to take either of these points back onto itself. Also, if the middle of an edge is a branch point, then the quotient map contains half of that edge – a *dangling semi-edge*.

An automorphism of a map  $M$  of genus  $G$  is characterized by the following parameters: the period  $L$ , the genus  $g$  of its quotient map and the number of branch points of each orbit length. If each orbit length is replaced by its *branch index* ( $L$  divided by the orbit length), we obtain what is called an *orbifold signature* in [18]. In [18] a method is presented for determining which orbifold signatures could characterize an automorphism

of a map of genus  $G$  (a  $G$ -admissible orbifold) and how many such automorphisms could be characterized by that orbifold signature; a variant of that method is presented in [17], and this is the one we use except that we deal with orbit lengths instead of branch indices. The method used in [18] to enumerate sensed maps of genus  $G$  with  $E$  edges by number of edges can be roughly described as follows. For each  $G$ -admissible orbifold  $O$ , let  $g$  be the genus of the quotient map,  $L$  be the period and  $q_i$  be the number of branch points with branch index  $i$ . Then the number  $\nu_O(d)$  of rooted maps with  $d$  darts that could serve as a quotient map for an automorphism with that signature once the branch points are pasted onto the map in all possible ways is given by

$$\nu_O(d) = \sum_{s=0}^{q_2} \binom{d}{s} \binom{(d-s)/2+2-2g}{q_2-s, q_3, \dots, q_L} N_g((d-s)/2), \tag{8.2}$$

where  $N_g(n)$  is the number of rooted maps of genus  $g$  with  $n$  edges (0 if  $n$  is not an integer). Here  $s$  is the number of dangling semi-edges in the quotient map  $m$ , all of which must be in orbits of length  $L/2$  so that they represent normal edges in the original map  $M$ . The binomial coefficient is the number of ways of inserting dangling semi-edges into the rooted map multiplied by  $d/(d-s)$  because there are  $d$  ways to root the map once the dangling edges have been inserted and only  $d-s$  ways to root it without the dangling edges. The multinomial coefficient is the number of ways to distribute the branch points with the various branch indices among the non-edges of the quotient map; the number at the top of the multinomial coefficient is the number of non-edges and is given by the Euler-Poincaré formula (1.1). Then the number of sensed maps of genus  $G$  with  $E$  edges is

$$\frac{1}{2E} \sum_{L|E} \sum_O \text{Epi}_0(\pi_1(O), Z_L) \nu_O(2E/L), \tag{8.3}$$

where  $O$  runs over all the  $G$ -admissible orbifolds with period  $L$  and  $\text{Epi}_0(\pi_1(O), Z_L)$  is the number of automorphisms that have the orbifold signature of  $O$ .

In [26] we distributed the branch points that aren't on dangling semi-edges among the vertices and faces separately. The quotient map of a bipartite map can't contain any dangling semi-edges; otherwise the lifted map would have an edge joining two vertices of the same colour. Here we distribute the branch points among the white vertices, black vertices and faces, and, like in [26], we don't use a formula like (8.3); instead we compute the contribution of each orbifold signature to the number of sensed 2-coloured bipartite maps whose number of white vertices, black vertices, faces and edges are allowed to vary within a user-defined upper bound on the number of edges.

Suppose that the quotient map is of genus  $g$  and has  $w$  white vertices,  $b$  black vertices and  $f$  faces. Then the number  $e$  of edges can be calculated from the formula

$$f - e + w + b = 2(1 - g) \tag{8.4}$$

and the number  $d$  of darts is  $2e$ . Suppose also that among the branch points of orbit length  $i$ ,  $w_i$  are on a white vertex,  $b_i$  are on a black vertex and  $f_i$  are in a face. We denote by  $w_L$ ,  $b_L$  and  $f_L$  the number of white vertices, black vertices and faces, respectively, that do not contain a branch point. The original map will have  $W$  white vertices,  $B$  black vertices and  $F$  faces, where

$$W = \sum_{i=1}^L iw_i, \quad B = \sum_{i=1}^L ib_i \quad \text{and} \quad F = \sum_{i=1}^L if_i, \tag{8.5}$$



and the total number  $E$  of edges is equal to  $Le = F + W + B - 2(1 - g)$ .

The binomial coefficient in (8.2) disappears because the quotient map can't contain any dangling semi-edges. The multinomial coefficient must be replaced by the number of ways to distribute the branch points among the white vertices, black vertices and faces. Then (8.2) becomes

$$\nu_O(d, w, b, f) = \binom{w}{w_1, w_2, \dots, w_L} \binom{b}{b_1, b_2, \dots, b_L} \binom{f}{f_1, f_2, \dots, f_L} N_g(d, w, b, f), \quad (8.6)$$

where  $d$  is the number of edges in the quotient maps on both sides of the formula (or the number of darts in the corresponding hypermaps) and  $N_g(d, w, b, f)$  is the number of 2-coloured bipartite maps with  $d$  edges with  $w$  white vertices,  $b$  black vertices and  $f$  faces, rooted at a white vertex. For this number to be positive, the sum of all the  $w_i$  cannot exceed  $w$  with a similar bound on the sum of all the  $b_i$  and the sum of all the  $f_i$ ; so  $w$ ,  $b$  and  $f$  each starts at its respective sum and increases by 1 until the number  $E$  of edges in the original map exceeds a user-defined maximum. With each increase of  $w$ ,  $b$  or  $f$ , one of the multinomial coefficients in (8.6) gets updated using a single multiplication and division. The product of these three multinomial coefficients must be computed for all sets of non-negative integers such that for each  $i$ ,  $w_i + b_i + f_i$  is equal to the total number of branch points of orbit length  $i$ .

Once (8.6) is multiplied by the number of automorphisms with the current orbifold signature, we get the contribution of that signature and the numbers  $w_i$ ,  $b_i$  and  $f_i$  to  $E$  times the number of sensed 2-coloured bipartite maps of genus  $G$  with  $E$  edges,  $F$  faces,  $B$  black vertices and  $W$  white vertices. This contribution is added to the appropriate element of an array, initially 0, and when all the contributions have been tallied, for each  $E$ ,  $F$ ,  $W$  and  $B$  the corresponding array element is divided by  $E$  (not  $2E$  because the root must be incident to a white vertex) to give the number of sensed 2-coloured bipartite maps of genus  $G$  with  $E$  edges,  $F$  faces,  $B$  black vertices and  $W$  white vertices or, equivalently, the number of sensed hypermaps of genus  $G$  with  $E$  darts,  $F$  faces,  $B$  hyperedges and  $W$  vertices.

This enumeration was done with a program written in C++ using CLN to treat big integers. It enumerated rooted and sensed hypermaps of genus up to 24 with up to 50 darts as fast as it could display the numbers on the screen. The numbers coincide with those obtained by generating the hypermaps [24]. The source code is available from the second author on request.

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### A First numbers of rooted hypermaps

The following sections show the numbers  $h$  of rooted hypermaps of genus  $g$  with  $d$  darts,  $v$  vertices,  $e$  edges and  $d - v - e + 2(1 - g)$  faces, for  $g \leq 6$  and  $d \leq 14$ .

#### A.1 Genus 0

d	v	e	f	h	6	1	5	2	15	8	2	5	3	2436
1	1	1	1	1	6	2	4	2	135	8	3	4	3	7500
					6	3	3	2	262	8	4	3	3	7500
			sum	1	6	4	2	2	135	8	5	2	3	2436
					6	5	1	2	15	8	6	1	3	196
2	1	1	2	1	6	1	6	1	1	8	1	7	2	28
2	1	2	1	1	6	2	5	1	15	8	2	6	2	518
2	2	1	1	1	6	3	4	1	50	8	3	5	2	2436
					6	4	3	1	50	8	4	4	2	3985
			sum	3	6	5	2	1	15	8	5	3	2	2436
					6	6	1	1	1	8	6	2	2	518
3	1	1	3	1						8	7	1	2	28
3	1	2	2	3	6			sum	1584	8	1	8	1	1
3	2	1	2	3						8	2	7	1	28
3	1	3	1	1	7	1	1	7	1	8	3	6	1	196
3	2	2	1	3	7	1	2	6	21	8	4	5	1	490
3	3	1	1	1	7	2	1	6	21	8	5	4	1	490
					7	1	3	5	105	8	6	3	1	196
			sum	12	7	2	2	5	280	8	7	2	1	28
					7	3	1	5	105	8	8	1	1	1
4	1	1	4	1	7	1	4	4	175					
4	1	2	3	6	7	2	3	4	889	8			sum	54912
4	2	1	3	6	7	3	2	4	889					
4	1	3	2	6	7	4	1	4	175	9	1	1	9	1
4	2	2	2	17	7	1	5	3	105	9	1	2	8	36
4	3	1	2	6	7	2	4	3	889	9	2	1	8	36
4	1	4	1	1	7	3	3	3	1694	9	1	3	7	336
4	2	3	1	6	7	4	2	3	889	9	2	2	7	882
4	3	2	1	6	7	5	1	3	105	9	3	1	7	336
4	4	1	1	1	7	1	6	2	21	9	1	4	6	1176
					7	2	5	2	280	9	2	3	6	5754
			sum	56	7	3	4	2	889	9	3	2	6	5754
					7	4	3	2	889	9	4	1	6	1176
5	1	1	5	1	7	5	2	2	280	9	1	5	5	1764
5	1	2	4	10	7	6	1	2	21	9	2	4	5	13941
5	2	1	4	10	7	1	7	1	1	9	3	3	5	26004
5	1	3	3	20	7	2	6	1	21	9	4	2	5	13941
5	2	2	3	55	7	3	5	1	105	9	5	1	5	1764
5	3	1	3	20	7	4	4	1	175	9	1	6	4	1176
5	1	4	2	10	7	5	3	1	105	9	2	5	4	13941
5	2	3	2	55	7	6	2	1	21	9	3	4	4	42015
5	3	2	2	55	7	7	1	1	1	9	4	3	4	42015
5	4	1	2	10						9	5	2	4	13941
5	1	5	1	1	7			sum	9152	9	6	1	4	1176
5	2	4	1	10						9	1	7	3	336
5	3	3	1	20	8	1	1	8	1	9	2	6	3	5754
5	4	2	1	10	8	1	2	7	28	9	3	5	3	26004
5	5	1	1	1	8	2	1	7	28	9	4	4	3	42015
					8	1	3	6	196	9	5	3	3	26004
			sum	288	8	2	2	6	518	9	6	2	3	5754
					8	3	1	6	196	9	7	1	3	336
6	1	1	6	1	8	1	4	5	490	9	1	8	2	36
6	1	2	5	15	8	2	3	5	2436	9	2	7	2	882
6	2	1	5	15	8	3	2	5	2436	9	3	6	2	5754
6	1	3	4	50	8	4	1	5	490	9	4	5	2	13941
6	2	2	4	135	8	1	5	4	490	9	5	4	2	13941
6	3	1	4	50	8	2	4	4	3985	9	6	3	2	5754
6	1	4	3	50	8	3	3	4	7500	9	7	2	2	882
6	2	3	3	262	8	4	2	4	3985	9	8	1	2	36
6	3	2	3	262	8	5	1	4	490	9	1	9	1	1
6	4	1	3	50	8	1	6	3	196	9	2	8	1	36

9	3	7	1	336	11	2	1	10	55	12	1	3	10	1210	
9	4	6	1	1176	11	1	3	9	825	12	2	2	10	3135	
9	5	5	1	1764	11	2	2	9	2145	12	3	1	10	1210	
9	6	4	1	1176	11	3	1	9	825	12	1	4	9	9075	
9	7	3	1	336	11	1	4	8	4950	12	2	3	9	43098	
9	8	2	1	36	11	2	3	8	23694	12	3	2	9	43098	
9	9	1	1	1	11	3	2	8	23694	12	4	1	9	9075	
					11	4	1	8	4950	12	1	5	8	32670	
9			sum	339456	11	1	5	7	13860	12	2	4	8	245223	
					11	2	4	7	105435	12	3	3	8	449988	
10	1	1	10	1	11	3	3	7	194304	12	4	2	8	245223	
10	1	2	9	45	11	4	2	7	105435	12	5	1	8	32670	
10	2	1	9	45	11	5	1	7	13860	12	1	6	7	60984	
10	1	3	8	540	11	1	6	6	19404	12	2	5	7	666996	
10	2	2	8	1410	11	2	5	6	216601	12	3	4	7	1936308	
10	3	1	8	540	11	3	4	6	634865	12	4	3	7	1936308	
10	1	4	7	2520	11	4	3	6	634865	12	5	2	7	666996	
10	2	3	7	12180	11	5	2	6	216601	12	6	1	7	60984	
10	3	2	7	12180	11	6	1	6	19404	12	1	7	6	60984	
10	4	1	7	2520	11	1	7	5	13860	12	2	6	6	925190	
10	1	5	6	5292	11	2	6	5	216601	12	3	5	6	3915576	
10	2	4	6	40935	11	3	5	5	931854	12	4	4	6	6195560	
10	3	3	6	75840	11	4	4	5	1482250	12	5	3	6	3915576	
10	4	2	6	40935	11	5	3	5	931854	12	6	2	6	925190	
10	5	1	6	5292	11	6	2	5	216601	12	7	1	6	60984	
10	1	6	5	5292	11	7	1	5	13860	12	1	8	5	32670	
10	2	5	5	60626	11	1	8	4	4950	12	2	7	5	666996	
10	3	4	5	179860	11	2	7	4	105435	12	3	6	5	3915576	
10	4	3	5	179860	11	3	6	4	634865	12	4	5	5	9032898	
10	5	2	5	60626	11	4	5	4	1482250	12	5	4	5	9032898	
10	6	1	5	5292	11	5	4	4	1482250	12	6	3	5	3915576	
10	1	7	4	2520	11	6	3	4	634865	12	7	2	5	666996	
10	2	6	4	40935	11	7	2	4	105435	12	8	1	5	32670	
10	3	5	4	179860	11	8	1	4	4950	12	1	9	4	9075	
10	4	4	4	288025	11	1	9	3	825	12	2	8	4	245223	
10	5	3	4	179860	11	2	8	3	23694	12	3	7	4	1936308	
10	6	2	4	40935	11	3	7	3	194304	12	4	6	4	6195560	
10	7	1	4	2520	11	4	6	3	634865	12	5	5	4	9032898	
10	1	8	3	540	11	5	5	3	931854	12	6	4	4	6195560	
10	2	7	3	12180	11	6	4	3	634865	12	7	3	4	1936308	
10	3	6	3	75840	11	7	3	3	194304	12	8	2	4	245223	
10	4	5	3	179860	11	8	2	3	23694	12	9	1	4	9075	
10	5	4	3	179860	11	9	1	3	825	12	1	10	3	1210	
10	6	3	3	75840	11	1	10	2	55	12	2	9	3	43098	
10	7	2	3	12180	11	2	9	2	2145	12	3	8	3	449988	
10	8	1	3	540	11	3	8	2	23694	12	4	7	3	1936308	
10	1	9	2	45	11	4	7	2	105435	12	5	6	3	3915576	
10	2	8	2	1410	11	5	6	2	216601	12	6	5	3	3915576	
10	3	7	2	12180	11	6	5	2	216601	12	7	4	3	1936308	
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10	6	4	2	40935	11	9	2	2	2145	12	10	1	3	1210	
10	7	3	2	12180	11	10	1	2	55	12	1	11	2	66	
10	8	2	2	1410	11	1	11	1	1	12	2	10	2	3135	
10	9	1	2	45	11	2	10	1	55	12	3	9	2	43098	
10	1	10	1	1	11	3	9	1	825	12	4	8	2	245223	
10	2	9	1	45	11	4	8	1	4950	12	5	7	2	666996	
10	3	8	1	540	11	5	7	1	13860	12	6	6	2	925190	
10	4	7	1	2520	11	6	6	1	19404	12	7	5	2	666996	
10	5	6	1	5292	11	7	5	1	13860	12	8	4	2	245223	
10	6	5	1	5292	11	8	4	1	4950	12	9	3	2	43098	
10	7	4	1	2520	11	9	3	1	825	12	10	2	2	3135	
10	8	3	1	540	11	10	2	1	55	12	11	1	2	66	
10	9	2	1	45	11	11	1	1	1	12	1	12	1	1	
10	10	1	1	1						12	2	11	1	66	
					11				sum	13891584	12	3	10	1	1210
10			sum	2149888							12	4	9	1	9075
					12	1	1	12	1		12	5	8	1	32670
11	1	1	11	1	12	1	2	11	66		12	6	7	1	60984
11	1	2	10	55	12	2	1	11	66		12	7	6	1	60984

12	8	5	1	32670	13	8	4	3	5264545	14	3	7	6	80231508
12	9	4	1	9075	13	9	3	3	960960	14	4	6	6	249321114
12	10	3	1	1210	13	10	2	3	74217	14	5	5	6	360078558
12	11	2	1	66	13	11	1	3	1716	14	6	4	6	249321114
12	12	1	1	1	13	1	12	2	78	14	7	3	6	80231508
					13	2	11	2	4433	14	8	2	6	10701873
12		sum		91287552	13	3	10	2	74217	14	9	1	6	429429
					13	4	9	2	525525	14	1	10	5	143143
13	1	1	13	1	13	5	8	2	1827683	14	2	9	5	4557553
13	1	2	12	78	13	6	7	2	3356522	14	3	8	5	44221632
13	2	1	12	78	13	7	6	2	3356522	14	4	7	5	181925268
13	1	3	11	1716	13	8	5	2	1827683	14	5	6	5	360078558
13	2	2	11	4433	13	9	4	2	525525	14	6	5	5	360078558
13	3	1	11	1716	13	10	3	2	74217	14	7	4	5	181925268
13	1	4	10	15730	13	11	2	2	4433	14	8	3	5	44221632
13	2	3	10	74217	13	12	1	2	78	14	9	2	5	4557553
13	3	2	10	74217	13	1	13	1	1	14	10	1	5	143143
13	4	1	10	15730	13	2	12	1	78	14	1	11	4	26026
13	1	5	9	70785	13	3	11	1	1716	14	2	10	4	1053052
13	2	4	9	525525	13	4	10	1	15730	14	3	9	4	13043030
13	3	3	9	960960	13	5	9	1	70785	14	4	8	4	69432090
13	4	2	9	525525	13	6	8	1	169884	14	5	7	4	181925268
13	5	1	9	70785	13	7	7	1	226512	14	6	6	4	249321114
13	1	6	8	169884	13	8	6	1	169884	14	7	5	4	181925268
13	2	5	8	1827683	13	9	5	1	70785	14	8	4	4	69432090
13	3	4	8	5264545	13	10	4	1	15730	14	9	3	4	13043030
13	4	3	8	5264545	13	11	3	1	1716	14	10	2	4	1053052
13	5	2	8	1827683	13	12	2	1	78	14	11	1	4	26026
13	6	1	8	169884	13	13	1	1	1	14	1	12	3	2366
13	1	7	7	226512						14	2	11	3	122122
13	2	6	7	3356522	13		sum		608583680	14	3	10	3	1919918
13	3	5	7	14019928						14	4	9	3	13043030
13	4	4	7	22089600	14	1	1	14	1	14	5	8	3	44221632
13	5	3	7	14019928	14	1	2	13	91	14	6	7	3	80231508
13	6	2	7	3356522	14	2	1	13	91	14	7	6	3	80231508
13	7	1	7	226512	14	1	3	12	2366	14	8	5	3	44221632
13	1	8	6	169884	14	2	2	12	6097	14	9	4	3	13043030
13	2	7	6	3356522	14	3	1	12	2366	14	10	3	3	1919918
13	3	6	6	19315114	14	1	4	11	26026	14	11	2	3	122122
13	4	5	6	44136820	14	2	3	11	122122	14	12	1	3	2366
13	5	4	6	44136820	14	3	2	11	122122	14	1	13	2	91
13	6	3	6	19315114	14	4	1	11	26026	14	2	12	2	6097
13	7	2	6	3356522	14	1	5	10	143143	14	3	11	2	122122
13	8	1	6	169884	14	2	4	10	1053052	14	4	10	2	1053052
13	1	9	5	70785	14	3	3	10	1919918	14	5	9	2	4557553
13	2	8	5	1827683	14	4	2	10	1053052	14	6	8	2	10701873
13	3	7	5	14019928	14	5	1	10	143143	14	7	7	2	14168988
13	4	6	5	44136820	14	1	6	9	429429	14	8	6	2	10701873
13	5	5	5	64013222	14	2	5	9	4557553	14	9	5	2	4557553
13	6	4	5	44136820	14	3	4	9	13043030	14	10	4	2	1053052
13	7	3	5	14019928	14	4	3	9	13043030	14	11	3	2	122122
13	8	2	5	1827683	14	5	2	9	4557553	14	12	2	2	6097
13	9	1	5	70785	14	6	1	9	429429	14	13	1	2	91
13	1	10	4	15730	14	1	7	8	736164	14	1	14	1	1
13	2	9	4	525525	14	2	6	8	10701873	14	2	13	1	91
13	3	8	4	5264545	14	3	5	8	44221632	14	3	12	1	2366
13	4	7	4	22089600	14	4	4	8	69432090	14	4	11	1	26026
13	5	6	4	44136820	14	5	3	8	44221632	14	5	10	1	143143
13	6	5	4	44136820	14	6	2	8	10701873	14	6	9	1	429429
13	7	4	4	22089600	14	7	1	8	736164	14	7	8	1	736164
13	8	3	4	5264545	14	1	8	7	736164	14	8	7	1	736164
13	9	2	4	525525	14	2	7	7	14168988	14	9	6	1	429429
13	10	1	4	15730	14	3	6	7	80231508	14	10	5	1	143143
13	1	11	3	1716	14	4	5	7	181925268	14	11	4	1	26026
13	2	10	3	74217	14	5	4	7	181925268	14	12	3	1	2366
13	3	9	3	960960	14	6	3	7	80231508	14	13	2	1	91
13	4	8	3	5264545	14	7	2	7	14168988	14	14	1	1	1
13	5	7	3	14019928	14	8	1	7	736164					
13	6	6	3	19315114	14	1	9	6	429429	14		sum		4107939840
13	7	5	3	14019928	14	2	8	6	10701873					

A.2 Genus 1

d	v	e	f	h	8	2	5	1	1470	10	1	8	1	330
3	1	1	1	1	8	3	4	1	4410	10	2	7	1	6930
					8	4	3	1	4410	10	3	6	1	41580
			sum	1	8	5	2	1	1470	10	4	5	1	97020
					8	6	1	1	126	10	5	4	1	97020
4	1	1	2	5						10	6	3	1	41580
4	1	2	1	5	8			sum	131307	10	7	2	1	6930
4	2	1	1	5						10	8	1	1	330
					9	1	1	7	210					
			sum	15	9	1	2	6	3360	10			sum	9713835
					9	2	1	6	3360					
5	1	1	3	15	9	1	3	5	14700	11	1	1	9	495
5	1	2	2	40	9	2	2	5	37035	11	1	2	8	13200
5	2	1	2	40	9	3	1	5	14700	11	2	1	8	13200
5	1	3	1	15	9	1	4	4	23520	11	1	3	7	103950
5	2	2	1	40	9	2	3	4	108285	11	2	2	7	259017
5	3	1	1	15	9	3	2	4	108285	11	3	1	7	103950
			sum	165	9	4	1	4	23520	11	1	4	6	332640
					9	1	5	3	14700	11	2	3	6	1493525
					9	2	4	3	108285	11	3	2	6	1493525
6	1	1	4	35	9	3	3	3	197896	11	4	1	6	332640
6	1	2	3	175	9	4	2	3	108285	11	1	5	5	485100
6	2	1	3	175	9	5	1	3	14700	11	2	4	5	3420835
6	1	3	2	175	9	1	6	2	3360	11	3	3	5	6165478
6	2	2	2	456	9	2	5	2	37035	11	4	2	5	3420835
6	3	1	2	175	9	3	4	2	108285	11	5	1	5	485100
6	1	4	1	35	9	4	3	2	108285	11	1	6	4	332640
6	2	3	1	175	9	5	2	2	37035	11	2	5	4	3420835
6	3	2	1	175	9	6	1	2	3360	11	3	4	4	9684433
6	4	1	1	35	9	1	7	1	210	11	4	3	4	9684433
			sum	1611	9	2	6	1	3360	11	5	2	4	3420835
					9	3	5	1	14700	11	6	1	4	332640
					9	4	4	1	23520	11	1	7	3	103950
7	1	1	5	70	9	5	3	1	14700	11	2	6	3	1493525
7	1	2	4	560	9	6	2	1	3360	11	3	5	3	6165478
7	2	1	4	560	9	7	1	1	210	11	4	4	3	9684433
7	1	3	3	1050						11	5	3	3	6165478
7	2	2	3	2695	9			sum	1138261	11	6	2	3	1493525
7	3	1	3	1050						11	7	1	3	103950
7	1	4	2	560	10	1	1	8	330	11	1	8	2	13200
7	2	3	2	2695	10	1	2	7	6930	11	2	7	2	259017
7	3	2	2	2695	10	2	1	7	6930	11	3	6	2	1493525
7	4	1	2	560	10	1	3	6	41580	11	4	5	2	3420835
7	1	5	1	70	10	2	2	6	104115	11	5	4	2	3420835
7	2	4	1	560	10	3	1	6	41580	11	6	3	2	1493525
7	3	3	1	1050	10	1	4	5	97020	11	7	2	2	259017
7	4	2	1	560	10	2	3	5	440440	11	8	1	2	13200
7	5	1	1	70	10	3	2	5	440440	11	1	9	1	495
			sum	14805	10	4	1	5	97020	11	2	8	1	13200
					10	1	5	4	97020	11	3	7	1	103950
					10	2	4	4	697250	11	4	6	1	332640
8	1	1	6	126	10	3	3	4	1264310	11	5	5	1	485100
8	1	2	5	1470	10	4	2	4	697250	11	6	4	1	332640
8	2	1	5	1470	10	5	1	4	97020	11	7	3	1	103950
8	1	3	4	4410	10	1	6	3	41580	11	8	2	1	13200
8	2	2	4	11199	10	2	5	3	440440	11	9	1	1	495
8	3	1	4	4410	10	3	4	3	1264310					
8	1	4	3	4410	10	4	3	3	1264310	11			sum	81968469
8	2	3	3	20684	10	5	2	3	440440					
8	3	2	3	20684	10	6	1	3	41580	12	1	1	10	715
8	4	1	3	4410	10	1	7	2	6930	12	1	2	9	23595
8	1	5	2	1470	10	2	6	2	104115	12	2	1	9	23595
8	2	4	2	11199	10	3	5	2	440440	12	1	3	8	235950
8	3	3	2	20684	10	4	4	2	697250	12	2	2	8	585585
8	4	2	2	11199	10	5	3	2	440440	12	3	1	8	235950
8	5	1	2	1470	10	6	2	2	104115	12	1	4	7	990990
8	1	6	1	126	10	7	1	2	6930	12	2	3	7	4410120

12	3	2	7	4410120	13	3	4	6	260619268	14	1	6	7	38648610
12	4	1	7	990990	13	4	3	6	260619268	14	2	5	7	375707570
12	1	5	6	1981980	13	5	2	6	93880696	14	3	4	7	1035514340
12	2	4	6	13768300	13	6	1	6	9513504	14	4	3	7	1035514340
12	3	3	6	24695580	13	1	7	5	6936930	14	5	2	7	375707570
12	4	2	6	13768300	13	2	6	5	93880696	14	6	1	7	38648610
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12	1	6	5	1981980	13	4	4	5	582408775	14	2	6	6	512104880
12	2	5	5	19920390	13	5	3	5	374805834	14	3	5	6	2020140430
12	3	4	5	55785870	13	6	2	5	93880696	14	4	4	6	3126887407
12	4	3	5	55785870	13	7	1	5	6936930	14	5	3	6	2020140430
12	5	2	5	19920390	13	1	8	4	2642640	14	6	2	6	512104880
12	6	1	5	1981980	13	2	7	4	47604648	14	7	1	6	38648610
12	1	7	4	990990	13	3	6	4	260619268	14	1	8	5	21471450
12	2	6	4	13768300	13	4	5	4	582408775	14	2	7	5	375707570
12	3	5	4	55785870	13	5	4	4	582408775	14	3	6	5	2020140430
12	4	4	4	87100531	13	6	3	4	260619268	14	4	5	5	4475516612
12	5	3	4	55785870	13	7	2	4	47604648	14	5	4	5	4475516612
12	6	2	4	13768300	13	8	1	4	2642640	14	6	3	5	2020140430
12	7	1	4	990990	13	1	9	3	495495	14	7	2	5	375707570
12	1	8	3	235950	13	2	8	3	11674663	14	8	1	5	21471450
12	2	7	3	4410120	13	3	7	3	85050784	14	1	9	4	6441435
12	3	6	3	24695580	13	4	6	3	260619268	14	2	8	4	145864355
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12	5	4	3	55785870	13	6	4	3	260619268	14	4	6	4	3126887407
12	6	3	3	24695580	13	7	3	3	85050784	14	5	5	4	4475516612
12	7	2	3	4410120	13	8	2	3	11674663	14	6	4	4	3126887407
12	8	1	3	235950	13	9	1	3	495495	14	7	3	4	1035514340
12	1	9	2	235950	13	1	10	2	40040	14	8	2	4	145864355
12	2	8	2	585585	13	2	9	2	1225653	14	9	1	4	6441435
12	3	7	2	4410120	13	3	8	2	11674663	14	1	10	3	975975
12	4	6	2	13768300	13	4	7	2	47604648	14	2	9	3	28283255
12	5	5	2	19920390	13	5	6	2	93880696	14	3	8	3	259750218
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12	8	2	2	585585	13	8	3	2	11674663	14	6	5	3	2020140430
12	9	1	2	235950	13	9	2	2	1225653	14	7	4	3	1035514340
12	1	10	1	715	13	10	1	2	40040	14	8	3	3	259750218
12	2	9	1	235950	13	1	11	1	1001	14	9	2	3	28283255
12	3	8	1	235950	13	2	10	1	40040	14	10	1	3	975975
12	4	7	1	990990	13	3	9	1	495495	14	1	11	2	65065
12	5	6	1	1981980	13	4	8	1	2642640	14	2	10	2	2407405
12	6	5	1	1981980	13	5	7	1	6936930	14	3	9	2	28283255
12	7	4	1	990990	13	6	6	1	9513504	14	4	8	2	145864355
12	8	3	1	235950	13	7	5	1	6936930	14	5	7	2	375707570
12	9	2	1	235950	13	8	4	1	2642640	14	6	6	2	512104880
12	10	1	1	715	13	9	3	1	495495	14	7	5	2	375707570
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					13	11	1	1	1001	14	9	3	2	28283255
12		sum		685888171						14	10	2	2	2407405
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13	1	1	11	1001	13		sum		5702382933	14	1	12	1	1365
13	1	2	10	40040						14	2	11	1	65065
13	2	1	10	40040	14	1	1	12	1365	14	3	10	1	975975
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13	2	2	9	1225653	14	2	1	11	65065	14	5	8	1	21471450
13	3	1	9	495495	14	1	3	10	975975	14	6	7	1	38648610
13	1	4	8	2642640	14	2	2	10	2407405	14	7	6	1	38648610
13	2	3	8	11674663	14	3	1	10	975975	14	8	5	1	21471450
13	3	2	8	11674663	14	1	4	9	6441435	14	9	4	1	6441435
13	4	1	8	2642640	14	2	3	9	28283255	14	10	3	1	975975
13	1	5	7	6936930	14	3	2	9	28283255	14	11	2	1	65065
13	2	4	7	47604648	14	4	1	9	6441435	14	12	1	1	1365
13	3	3	7	85050784	14	1	5	8	21471450			sum		47168678571
13	4	2	7	47604648	14	2	4	8	145864355					
13	5	1	7	6936930	14	3	3	8	259750218					
13	1	6	6	9513504	14	4	2	8	145864355					
13	2	5	6	93880696	14	5	1	8	21471450					



A.3 Genus 2

d v e f h	10 2 5 1	167013	12 1 8 1	88803
5 1 1 1 8	10 3 4 1	471240	12 2 7 1	1585584
	10 4 3 1	471240	12 3 6 1	8654646
5	10 5 2 1	167013	12 4 5 1	19324305
sum 8	10 6 1 1	16401	12 5 4 1	19324305
			12 6 3 1	8654646
6 1 1 2 84			12 7 2 1	1585584
6 1 2 1 84	10	sum 13545216	12 8 1 1	88803
6 2 1 1 84				
	11 1 1 7	39963		
6	11 1 2 6	550011	12	sum 1805010948
sum 252	11 2 1 6	550011		
	11 1 3 5	2221065	13 1 1 9	183183
7 1 1 3 469	11 2 2 5	5409019	13 1 2 8	4114110
7 1 2 2 1183	11 3 1 5	2221065	13 2 1 8	4114110
7 2 1 2 1183	11 1 4 4	3465000	13 1 3 7	29135106
7 1 3 1 469	11 2 3 4	15014846	13 2 2 7	70367479
7 2 2 1 1183	11 3 2 4	15014846	13 3 1 7	29135106
7 3 1 1 469	11 4 1 4	3465000	13 1 4 6	87933846
	11 1 5 3	2221065	13 2 3 6	374127663
7	11 2 4 3	15014846	13 3 2 6	374127663
sum 4956	11 3 3 3	26717482	13 4 1 6	87933846
	11 4 2 3	15014846	13 1 5 5	125855730
8 1 1 4 1869	11 5 1 3	2221065	13 2 4 5	824962502
8 1 2 3 8526	11 1 6 2	550011	13 3 3 5	1453414846
8 2 1 3 8526	11 2 5 2	5409019	13 4 2 5	824962502
8 1 3 2 8526	11 3 4 2	15014846	13 5 1 5	125855730
8 2 2 2 21229	11 4 3 2	15014846	13 1 6 4	87933846
8 3 1 2 8526	11 5 2 2	5409019	13 2 5 4	824962502
8 1 4 1 1869	11 6 1 2	550011	13 3 4 4	2239280420
8 2 3 1 8526	11 1 7 1	39963	13 4 3 4	2239280420
8 3 2 1 8526	11 2 6 1	550011	13 5 2 4	824962502
8 4 1 1 1869	11 3 5 1	2221065	13 6 1 4	87933846
	11 4 4 1	3465000	13 1 7 3	29135106
8	11 5 3 1	2221065	13 2 6 3	374127663
sum 77992	11 6 2 1	550011	13 3 5 3	1453414846
	11 7 1 1	39963	13 4 4 3	2239280420
9 1 1 5 5985			13 5 3 3	1453414846
9 1 2 4 42588	11	sum 160174960	13 6 2 3	374127663
9 2 1 4 42588			13 7 1 3	29135106
9 1 3 3 77028	12 1 1 8	88803	13 1 8 2	4114110
9 2 2 3 189999	12 1 2 7	1585584	13 2 7 2	70367479
9 3 1 3 77028	12 2 1 7	1585584	13 3 6 2	374127663
9 1 4 2 42588	12 1 3 6	8654646	13 4 5 2	824962502
9 2 3 2 189999	12 2 2 6	20981337	13 5 4 2	824962502
9 3 2 2 189999	12 3 1 6	8654646	13 6 3 2	374127663
9 4 1 2 42588	12 1 4 5	19324305	13 7 2 2	70367479
9 1 5 1 5985	12 2 3 5	82897296	13 8 1 2	4114110
9 2 4 1 42588	12 3 2 5	82897296	13 1 9 1	183183
9 3 3 1 77028	12 4 1 5	19324305	13 2 8 1	4114110
9 4 2 1 42588	12 1 5 4	19324305	13 3 7 1	29135106
9 5 1 1 5985	12 2 4 4	128420004	13 4 6 1	87933846
	12 3 3 4	227256510	13 5 5 1	125855730
9	12 4 2 4	128420004	13 6 4 1	87933846
sum 1074564	12 5 1 4	19324305	13 7 3 1	29135106
	12 1 6 3	8654646	13 8 2 1	4114110
10 1 1 6 16401	12 2 5 3	82897296	13 9 1 1	183183
10 1 2 5 167013	12 3 4 3	227256510		
10 2 1 5 167013	12 4 3 3	227256510	13	sum 19588944336
10 1 3 4 471240	12 5 2 3	82897296		
10 2 2 4 1154095	12 6 1 3	8654646	14 1 1 10	355355
10 3 1 4 471240	12 1 7 2	1585584	14 1 2 9	9798789
10 1 4 3 471240	12 2 6 2	20981337	14 2 1 9	9798789
10 2 3 3 2068070	12 3 5 2	82897296	14 1 3 8	87291204
10 3 2 3 2068070	12 4 4 2	128420004	14 2 2 8	210164227
10 4 1 3 471240	12 5 3 2	82897296	14 3 1 8	87291204
10 1 5 2 167013	12 6 2 2	20981337	14 1 4 7	341825484
10 2 4 2 1154095	12 7 1 2	1585584	14 2 3 7	1444432612
10 3 3 2 2068070				
10 4 2 2 1154095				
10 5 1 2 167013				
10 1 6 1 16401				

14	3	2	7	1444432612	14	5	3	4	164274711172	14	7	3	2	1444432612
14	4	1	7	341825484	14	6	2	4	4286172247	14	8	2	2	210164227
14	1	5	6	661320660	14	7	1	4	341825484	14	9	1	2	9798789
14	2	4	6	4286172247	14	1	8	3	87291204	14	1	10	1	355355
14	3	3	6	7523770016	14	2	7	3	1444432612	14	2	9	1	9798789
14	4	2	6	4286172247	14	3	6	3	7523770016	14	3	8	1	87291204
14	5	1	6	661320660	14	4	5	3	164274711172	14	4	7	1	341825484
14	1	6	5	661320660	14	5	4	3	164274711172	14	5	6	1	661320660
14	2	5	5	6100939726	14	6	3	3	7523770016	14	6	5	1	661320660
14	3	4	5	16427471172	14	7	2	3	1444432612	14	7	4	1	341825484
14	4	3	5	16427471172	14	8	1	3	87291204	14	8	3	1	87291204
14	5	2	5	6100939726	14	1	9	2	9798789	14	9	2	1	9798789
14	6	1	5	661320660	14	2	8	2	210164227	14	10	1	1	355355
14	1	7	4	341825484	14	3	7	2	1444432612	14				
14	2	6	4	4286172247	14	4	6	2	4286172247	14				
14	3	5	4	16427471172	14	5	5	2	6100939726	14				
14	4	4	4	25199010256	14	6	4	2	4286172247	14				
												sum		206254571236

### A.4 Genus 3

d	v	e	f	h						13	1	7	1	8691683
7	1	1	1	180	11	sum	112868844			13	2	6	1	108452916
										13	3	5	1	414918075
7			sum	180	12	1	1	6	2641925	13	4	4	1	636184120
					12	1	2	5	24656775	13	5	3	1	414918075
8	1	1	2	3044	12	2	1	5	24656775	13	6	2	1	108452916
8	1	2	1	3044	12	1	3	4	66805310	13	7	1	1	8691683
8	2	1	1	3044	12	2	2	4	159762815					
					12	3	1	4	66805310	13			sum	28540603884
8			sum	9132	12	1	4	3	66805310					
					12	2	3	3	280514670	14	1	1	8	25537655
9	1	1	3	26060	12	3	2	3	280514670	14	1	2	7	409732895
9	1	2	2	63600	12	4	1	3	66805310	14	2	1	7	409732895
9	2	1	2	63600	12	1	5	2	24656775	14	1	3	6	2096068975
9	1	3	1	26060	12	2	4	2	159762815	14	2	2	6	4973691275
9	2	2	1	63600	12	3	3	2	280514670	14	3	1	6	2096068975
9	3	1	1	26060	12	4	2	2	159762815	14	1	4	5	4538348815
					12	5	1	2	24656775	14	2	3	5	18733893115
9			sum	268980	12	1	6	1	2641925	14	3	2	5	18733893115
					12	2	5	1	24656775	14	4	1	5	4538348815
10	1	1	4	152900	12	3	4	1	66805310	14	1	5	4	4538348815
10	1	2	3	659340	12	4	3	1	66805310	14	2	4	4	28579309570
10	2	1	3	659340	12	5	2	1	24656775	14	3	3	4	49719495672
10	1	3	2	659340	12	6	1	1	2641925	14	4	2	4	28579309570
10	2	2	2	1595480						14	5	1	4	4538348815
10	3	1	2	659340	12		sum	1877530740		14	1	6	3	2096068975
10	1	4	1	152900	13	1	1	7	8691683	14	2	5	3	18733893115
10	2	3	1	659340	13	1	2	6	108452916	14	3	4	3	49719495672
10	3	2	1	659340	13	2	1	6	108452916	14	4	3	3	49719495672
10	4	1	1	152900	13	2	1	6	108452916	14	5	2	3	18733893115
					13	1	3	5	414918075	14	6	1	3	2096068975
10			sum	6010220	13	2	2	5	988043771	14	1	7	2	409732895
					13	3	1	5	414918075	14	2	6	2	4973691275
11	1	1	5	696905	13	1	4	4	636184120	14	3	5	2	18733893115
11	1	2	4	4606910	13	2	3	4	2646424729	14	4	4	2	28579309570
11	2	1	4	4606910	13	3	2	4	2646424729	14	5	3	2	18733893115
11	1	3	3	8141100	13	4	1	4	636184120	14	6	2	2	4973691275
11	2	2	3	19571123	13	1	5	3	414918075	14	7	1	2	409732895
11	3	1	3	8141100	13	2	4	3	2646424729	14	1	8	1	25537655
11	1	4	2	4606910	13	3	3	3	4623070842	14	2	7	1	409732895
11	2	3	2	19571123	13	4	2	3	2646424729	14	3	6	1	2096068975
11	3	2	2	19571123	13	5	1	3	414918075	14	4	5	1	4538348815
11	4	1	2	4606910	13	1	6	2	108452916	14	5	4	1	4538348815
11	1	5	1	696905	13	2	5	2	988043771	14	6	3	1	2096068975
11	2	4	1	4606910	13	3	4	2	2646424729	14	7	2	1	409732895
11	3	3	1	8141100	13	4	3	2	2646424729	14	8	1	1	25537655
11	4	2	1	4606910	13	5	2	2	988043771					
11	5	1	1	696905	13	6	1	2	108452916	14			sum	404562365316



## B First numbers of unrooted hypermaps

The following sections show the numbers  $H$  of unrooted hypermaps of genus  $g$  with  $d$  darts,  $v$  vertices,  $e$  edges and  $d - v - e + 2(1 - g)$  faces, for  $g \leq 6$  and  $d \leq 14$ .

### B.1 Genus 0

d	v	e	f	H	6	1	5	2	3	8	2	5	3	309
1	1	1	1	1	6	2	4	2	24	8	3	4	3	946
			sum	1	6	3	3	2	46	8	4	3	3	946
					6	4	2	2	24	8	5	2	3	309
2	1	1	2	1	6	5	1	2	3	8	6	1	3	26
2	1	2	1	1	6	1	6	1	1	8	1	7	2	4
2	2	1	1	1	6	2	5	1	3	8	2	6	2	67
			sum	3	6	3	4	1	10	8	3	5	2	309
					6	4	3	1	10	8	4	4	2	505
3	1	1	3	1	6	5	2	1	3	8	5	3	2	309
3	1	2	2	1	6	6	1	1	1	8	6	2	2	67
3	2	1	2	1				sum	291	8	7	1	2	4
3	1	3	1	1	7	1	1	7	1	8	1	8	1	1
3	2	2	1	1	7	1	2	6	3	8	2	7	1	4
3	3	1	1	1	7	2	1	6	3	8	3	6	1	26
			sum	6	7	1	3	5	15	8	4	5	1	64
					7	2	2	5	40	8	5	4	1	64
4	1	1	4	1	7	1	3	5	15	8	6	3	1	26
4	1	2	3	2	7	2	2	5	40	8	7	2	1	4
4	2	1	3	2	7	3	1	5	15	8	8	1	1	1
4	1	3	2	2	7	1	4	4	25				sum	6975
4	2	2	2	5	7	2	3	4	127	8				
4	3	1	2	2	7	3	2	4	127					
4	1	4	1	1	7	4	1	4	25	9	1	1	9	1
4	2	3	1	2	7	1	5	3	15	9	1	2	8	4
4	1	4	1	1	7	2	4	3	127	9	2	1	8	4
4	2	3	1	2	7	3	3	3	242	9	1	3	7	38
4	3	2	1	2	7	4	2	3	127	9	2	2	7	98
4	4	1	1	1	7	5	1	3	15	9	3	1	7	38
			sum	20	7	1	6	2	3	9	1	4	6	132
					7	2	5	2	40	9	2	3	6	640
5	1	1	5	1	7	3	4	2	127	9	3	2	6	640
5	1	2	4	2	7	4	3	2	127	9	4	1	6	132
5	2	1	4	2	7	5	2	2	40	9	1	5	5	196
5	1	3	3	4	7	6	1	2	3	9	2	4	5	1549
5	2	2	3	11	7	1	7	1	1	9	3	3	5	2890
5	3	1	3	4	7	2	6	1	3	9	4	2	5	1549
5	1	4	2	2	7	3	5	1	15	9	5	1	5	196
5	2	3	2	11	7	4	4	1	25	9	1	6	4	132
5	3	2	2	11	7	5	3	1	15	9	2	5	4	1549
5	4	1	2	2	7	6	2	1	3	9	3	4	4	4671
5	1	5	1	1	7	7	1	1	1	9	4	3	4	4671
5	2	4	1	2				sum	1310	9	5	2	4	1549
5	3	3	1	4	8	1	1	8	1	9	6	1	4	132
5	4	2	1	2	8	1	2	7	4	9	1	7	3	38
5	5	1	1	1	8	2	1	7	4	9	2	6	3	640
			sum	60	8	1	3	6	26	9	3	5	3	2890
					8	2	2	6	67	9	4	4	3	4671
6	1	1	6	1	8	3	1	6	26	9	5	3	3	2890
6	1	2	5	3	8	2	3	5	309	9	6	2	3	640
6	2	1	5	3	8	3	2	5	309	9	7	1	3	38
6	1	3	4	10	8	4	1	5	64	9	2	8	2	4
6	2	2	4	24	8	1	5	4	64	9	3	6	2	98
6	3	1	4	10	8	2	4	4	505	9	4	5	2	1549
6	1	4	3	10	8	3	3	4	946	9	5	4	2	1549
6	2	3	3	46	8	4	2	4	505	9	6	3	2	640
6	3	2	3	46	8	5	1	4	64	9	7	2	2	98
6	4	1	3	10	8	1	6	3	26	9	8	1	2	4
					8	5	1	4	64	9	1	9	1	1
					8	1	6	3	26	9	2	8	1	4

9	3	7	1	38	11	2	1	10	5	12	1	3	10	104
9	4	6	1	132	11	1	3	9	75	12	2	2	10	265
9	5	5	1	196	11	2	2	9	195	12	3	1	10	104
9	6	4	1	132	11	3	1	9	75	12	1	4	9	765
9	7	3	1	38	11	1	4	8	450	12	2	3	9	3605
9	8	2	1	4	11	2	3	8	2154	12	3	2	9	3605
9	9	1	1	1	11	3	2	8	2154	12	4	1	9	765
9					11	4	1	8	450	12	1	5	8	2736
9		sum		37746	11	1	5	7	1260	12	2	4	8	20472
10	1	1	10	1	11	2	4	7	9585	12	3	3	8	37545
10	1	2	9	5	11	3	3	7	17664	12	4	2	8	20472
10	2	1	9	5	11	4	2	7	9585	12	5	1	8	2736
10	1	3	8	56	11	5	1	7	1260	12	1	6	7	5102
10	2	2	8	144	11	1	6	6	1764	12	2	5	7	55633
10	3	1	8	56	11	2	5	6	19691	12	3	4	7	161455
10	1	4	7	256	11	3	4	6	57715	12	4	3	7	161455
10	2	3	7	1226	11	4	3	6	57715	12	5	2	7	55633
10	3	2	7	1226	11	5	2	6	19691	12	6	1	7	5102
10	4	1	7	256	11	6	1	6	1764	12	1	7	6	5102
10	1	5	6	536	11	1	7	5	1260	12	2	6	6	77174
10	2	4	6	4111	11	2	6	5	19691	12	3	5	6	326432
10	3	3	6	7606	11	3	5	5	84714	12	4	4	6	516507
10	4	2	6	4111	11	4	4	5	134750	12	5	3	6	326432
10	5	1	6	536	11	5	3	5	84714	12	6	2	6	77174
10	1	6	5	536	11	6	2	5	19691	12	7	1	6	5102
10	2	5	5	6081	11	7	1	5	1260	12	1	8	5	2736
10	3	4	5	18019	11	1	8	4	450	12	2	7	5	55633
10	4	3	5	18019	11	2	7	4	9585	12	3	6	5	326432
10	5	2	5	6081	11	3	6	4	57715	12	4	5	5	752940
10	6	1	5	536	11	4	5	4	134750	12	5	4	5	752940
10	1	7	4	256	11	5	4	4	134750	12	6	3	5	326432
10	2	6	4	4111	11	6	3	4	57715	12	7	2	5	55633
10	3	5	4	18019	11	7	2	4	9585	12	8	1	5	2736
10	4	4	4	28852	11	8	1	4	450	12	1	9	4	765
10	5	3	4	18019	11	1	9	3	75	12	2	8	4	20472
10	6	2	4	4111	11	2	8	3	2154	12	3	7	4	161455
10	7	1	4	256	11	3	7	3	17664	12	4	6	4	516507
10	1	8	3	56	11	4	6	3	57715	12	5	5	4	752940
10	2	7	3	1226	11	5	5	3	84714	12	6	4	4	516507
10	3	6	3	7606	11	6	4	3	57715	12	7	3	4	161455
10	4	5	3	18019	11	7	3	3	17664	12	8	2	4	20472
10	5	4	3	18019	11	8	2	3	2154	12	9	1	4	765
10	6	3	3	7606	11	9	1	3	75	12	1	10	3	104
10	7	2	3	1226	11	1	10	2	5	12	2	9	3	3605
10	8	1	3	56	11	2	9	2	195	12	3	8	3	37545
10	1	9	2	5	11	3	8	2	2154	12	4	7	3	161455
10	2	8	2	144	11	4	7	2	9585	12	5	6	3	326432
10	3	7	2	1226	11	5	6	2	19691	12	6	5	3	326432
10	4	6	2	4111	11	6	5	2	19691	12	7	4	3	161455
10	5	5	2	6081	11	7	4	2	9585	12	8	3	3	37545
10	6	4	2	4111	11	8	3	2	2154	12	9	2	3	3605
10	7	3	2	1226	11	9	2	2	195	12	10	1	3	104
10	8	2	2	144	11	10	1	2	5	12	1	11	2	6
10	9	1	2	5	11	1	11	1	1	12	2	10	2	265
10	1	10	1	1	11	2	10	1	5	12	3	9	2	3605
10	2	9	1	5	11	3	9	1	75	12	4	8	2	20472
10	3	8	1	56	11	4	8	1	450	12	5	7	2	55633
10	4	7	1	256	11	5	7	1	1260	12	6	6	2	77174
10	5	6	1	536	11	6	6	1	1764	12	7	5	2	55633
10	6	5	1	536	11	7	5	1	1260	12	8	4	2	20472
10	7	4	1	256	11	8	4	1	450	12	9	3	2	3605
10	8	3	1	56	11	9	3	1	75	12	10	2	2	265
10	9	2	1	5	11	10	2	1	5	12	11	1	2	6
10	10	1	1	1	11	11	1	1	1	12	1	12	1	1
10					11					12	2	11	1	6
10		sum		215602	11		sum		1262874	12	3	10	1	104
11	1	1	11	1	12	1	1	12	1	12	4	9	1	765
11	1	2	10	5	12	1	2	11	6	12	5	8	1	2736
					12	2	1	11	6	12	6	7	1	5102
										12	7	6	1	5102

12	8	5	1	2736	13	8	4	3	404965	14	3	7	6	5731330
12	9	4	1	765	13	9	3	3	73920	14	4	6	6	17809776
12	10	3	1	104	13	10	2	3	5709	14	5	5	6	25720986
12	11	2	1	6	13	11	1	3	132	14	6	4	6	17809776
12	12	1	1	1	13	1	12	2	6	14	7	3	6	5731330
					13	2	11	2	341	14	8	2	6	764633
12		sum		7611156	13	3	10	2	5709	14	9	1	6	30711
					13	4	9	2	40425	14	1	10	5	10247
13	1	1	13	1	13	5	8	2	140591	14	2	9	5	325652
13	1	2	12	6	13	6	7	2	258194	14	3	8	5	3159069
13	2	1	12	6	13	7	6	2	258194	14	4	7	5	12995424
13	1	3	11	132	13	8	5	2	140591	14	5	6	5	25720986
13	2	2	11	341	13	9	4	2	40425	14	6	5	5	25720986
13	3	1	11	132	13	10	3	2	5709	14	7	4	5	12995424
13	1	4	10	1210	13	11	2	2	341	14	8	3	5	3159069
13	2	3	10	5709	13	12	1	2	6	14	9	2	5	325652
13	3	2	10	5709	13	1	13	1	1	14	10	1	5	10247
13	4	1	10	1210	13	2	12	1	6	14	1	11	4	1868
13	1	5	9	5445	13	3	11	1	132	14	2	10	4	75283
13	2	4	9	40425	13	4	10	1	1210	14	3	9	4	931845
13	3	3	9	73920	13	5	9	1	5445	14	4	8	4	4960016
13	4	2	9	40425	13	6	8	1	13068	14	5	7	4	12995424
13	5	1	9	5445	13	7	7	1	17424	14	6	6	4	17809776
13	1	6	8	13068	13	8	6	1	13068	14	7	5	4	12995424
13	2	5	8	140591	13	9	5	1	5445	14	8	4	4	4960016
13	3	4	8	404965	13	10	4	1	1210	14	9	3	4	931845
13	4	3	8	404965	13	11	3	1	132	14	10	2	4	75283
13	5	2	8	140591	13	12	2	1	6	14	11	1	4	1868
13	6	1	8	13068	13	13	1	1	1	14	1	12	3	172
13	1	7	7	17424						14	2	11	3	8741
13	2	6	7	258194	13		sum		46814132	14	3	10	3	137217
13	3	5	7	1078456						14	4	9	3	931845
13	4	4	7	1699200	14	1	1	14	1	14	5	8	3	3159069
13	5	3	7	1078456	14	1	2	13	7	14	6	7	3	5731330
13	6	2	7	258194	14	2	1	13	7	14	7	6	3	5731330
13	7	1	7	17424	14	1	3	12	172	14	8	5	3	3159069
13	1	8	6	13068	14	2	2	12	440	14	9	4	3	931845
13	2	7	6	258194	14	3	1	12	172	14	10	3	3	137217
13	3	6	6	1485778	14	1	4	11	1868	14	11	2	3	8741
13	4	5	6	3395140	14	2	3	11	8741	14	12	1	3	172
13	5	4	6	3395140	14	3	2	11	8741	14	1	13	2	7
13	6	3	6	1485778	14	4	1	11	1868	14	2	12	2	440
13	7	2	6	258194	14	1	5	10	10247	14	3	11	2	8741
13	8	1	6	13068	14	2	4	10	75283	14	4	10	2	75283
13	1	9	5	5445	14	3	3	10	137217	14	5	9	2	325652
13	2	8	5	140591	14	4	2	10	75283	14	6	8	2	764633
13	3	7	5	1078456	14	5	1	10	10247	14	7	7	2	1012271
13	4	6	5	3395140	14	1	6	9	30711	14	8	6	2	764633
13	5	5	5	4924094	14	2	5	9	325652	14	9	5	2	325652
13	6	4	5	3395140	14	3	4	9	931845	14	10	4	2	75283
13	7	3	5	1078456	14	4	3	9	931845	14	11	3	2	8741
13	8	2	5	140591	14	5	2	9	325652	14	12	2	2	440
13	9	1	5	5445	14	6	1	9	30711	14	13	1	2	7
13	1	10	4	1210	14	1	7	8	52634	14	1	14	1	1
13	2	9	4	40425	14	2	6	8	764633	14	2	13	1	7
13	3	8	4	404965	14	3	5	8	3159069	14	3	12	1	172
13	4	7	4	1699200	14	4	4	8	4960016	14	4	11	1	1868
13	5	6	4	3395140	14	5	3	8	3159069	14	5	10	1	10247
13	6	5	4	3395140	14	6	2	8	764633	14	6	9	1	30711
13	7	4	4	1699200	14	7	1	8	52634	14	7	8	1	52634
13	8	3	4	404965	14	1	8	7	52634	14	8	7	1	52634
13	9	2	4	40425	14	2	7	7	1012271	14	9	6	1	30711
13	10	1	4	1210	14	3	6	7	5731330	14	10	5	1	10247
13	1	11	3	132	14	4	5	7	12995424	14	11	4	1	1868
13	2	10	3	5709	14	5	4	7	12995424	14	12	3	1	172
13	3	9	3	73920	14	6	3	7	5731330	14	13	2	1	7
13	4	8	3	404965	14	7	2	7	1012271	14	14	1	1	1
13	5	7	3	1078456	14	8	1	7	52634					
13	6	6	3	1485778	14	1	9	6	30711	14		sum		293447817
13	7	5	3	1078456	14	2	8	6	764633					

**B.2 Genus 1**

d	v	e	f	H	8	2	5	1	187	10	1	8	1	34
3	1	1	1	1	8	3	4	1	557	10	2	7	1	698
					8	4	3	1	557	10	3	6	1	4172
			sum	1	8	5	2	1	187	10	4	5	1	9724
					8	6	1	1	17	10	5	4	1	9724
4	1	1	2	2						10	6	3	1	4172
4	1	2	1	2	8			sum	16533	10	7	2	1	698
4	2	1	1	2						10	8	1	1	34
			sum	6	9	1	1	7	24					
					9	1	2	6	374	10		sum		972441
					9	2	1	6	374					
5	1	1	3	3	9	1	3	5	1634	11	1	1	9	45
5	1	2	2	8	9	2	2	5	4115	11	1	2	8	1200
5	2	1	2	8	9	3	1	5	1634	11	2	1	8	1200
5	1	3	1	3	9	1	4	4	2616	11	1	3	7	9450
5	2	2	1	8	9	2	3	4	12033	11	2	2	7	23547
5	3	1	1	3	9	3	2	4	12033	11	3	1	7	9450
			sum	33	9	4	1	4	2616	11	1	4	6	30240
					9	1	5	3	1634	11	2	3	6	135775
					9	2	4	3	12033	11	3	2	6	135775
6	1	1	4	7	9	3	3	3	21990	11	4	1	6	30240
6	1	2	3	31	9	4	2	3	12033	11	1	5	5	44100
6	2	1	3	31	9	5	1	3	1634	11	2	4	5	310985
6	1	3	2	31	9	1	6	2	374	11	3	3	5	560498
6	2	2	2	78	9	2	5	2	4115	11	4	2	5	310985
6	3	1	2	31	9	3	4	2	12033	11	5	1	5	44100
6	1	4	1	7	9	4	3	2	12033	11	1	6	4	30240
6	2	3	1	31	9	5	2	2	4115	11	2	5	4	310985
6	3	2	1	31	9	6	1	2	374	11	3	4	4	880403
6	4	1	1	7	9	1	7	1	24	11	4	3	4	880403
			sum	285	9	2	6	1	374	11	5	2	4	310985
					9	3	5	1	1634	11	6	1	4	30240
					9	4	4	1	2616	11	1	7	3	9450
7	1	1	5	10	9	5	3	1	1634	11	2	6	3	135775
7	1	2	4	80	9	6	2	1	374	11	3	5	3	560498
7	2	1	4	80	9	7	1	1	24	11	4	4	3	880403
7	1	3	3	150						11	5	3	3	560498
7	2	2	3	385	9			sum	126501	11	6	2	3	135775
7	3	1	3	150						11	7	1	3	9450
7	1	4	2	80	10	1	1	8	34	11	1	8	2	1200
7	2	3	2	385	10	1	2	7	698	11	2	7	2	23547
7	3	2	2	385	10	2	1	7	698	11	3	6	2	135775
7	4	1	2	80	10	1	3	6	4172	11	4	5	2	310985
7	1	5	1	10	10	2	2	6	10434	11	5	4	2	310985
7	2	4	1	80	10	3	1	6	4172	11	6	3	2	135775
7	3	3	1	150	10	1	4	5	9724	11	7	2	2	23547
7	4	2	1	80	10	2	3	5	44091	11	8	1	2	1200
7	5	1	1	10	10	3	2	5	44091	11	1	9	1	45
			sum	2115	10	4	1	5	9724	11	2	8	1	1200
					10	1	5	4	9724	11	3	7	1	9450
					10	2	4	4	69790	11	4	6	1	30240
8	1	1	6	17	10	3	3	4	126519	11	5	5	1	44100
8	1	2	5	187	10	4	2	4	69790	11	6	4	1	30240
8	2	1	5	187	10	5	1	4	9724	11	7	3	1	9450
8	1	3	4	557	10	1	6	3	4172	11	8	2	1	1200
8	2	2	4	1409	10	2	5	3	44091	11	9	1	1	45
8	3	1	4	557	10	3	4	3	126519					
8	1	4	3	557	10	4	3	3	126519	11		sum		7451679
8	2	3	3	2597	10	5	2	3	44091					
8	3	2	3	2597	10	6	1	3	4172	12	1	1	10	62
8	4	1	3	557	10	1	7	2	698	12	1	2	9	1976
8	1	5	2	187	10	2	6	2	10434	12	2	1	9	1976
8	2	4	2	1409	10	3	5	2	44091	12	1	3	8	19694
8	3	3	2	2597	10	4	4	2	69790	12	2	2	8	48846
8	4	2	2	1409	10	5	3	2	44091	12	3	1	8	19694
8	5	1	2	187	10	6	2	2	10434	12	1	4	7	82652
8	1	6	1	17	10	7	1	2	698	12	2	3	7	367645

12	3	2	7	367645	13	3	4	6	20047636	14	1	6	7	2760990
12	4	1	7	82652	13	4	3	6	20047636	14	2	5	7	26837442
12	1	5	6	165262	13	5	2	6	7221592	14	3	4	7	73967488
12	2	4	6	1147628	13	6	1	6	731808	14	4	3	7	73967488
12	3	3	6	2058329	13	1	7	5	533610	14	5	2	7	26837442
12	4	2	6	1147628	13	2	6	5	7221592	14	6	1	7	2760990
12	5	1	6	165262	13	3	5	5	28831218	14	1	7	6	2760990
12	1	6	5	165262	13	4	4	5	44800675	14	2	6	6	36580432
12	2	5	5	1660331	13	5	3	5	28831218	14	3	5	6	144298902
12	3	4	5	4649379	13	6	2	5	7221592	14	4	4	6	223353280
12	4	3	5	4649379	13	7	1	5	533610	14	5	3	6	144298902
12	5	2	5	1660331	13	1	8	4	203280	14	6	2	6	36580432
12	6	1	5	165262	13	2	7	4	3661896	14	7	1	6	2760990
12	1	7	4	82652	13	3	6	4	20047636	14	1	8	5	1533950
12	2	6	4	1147628	13	4	5	4	44800675	14	2	7	5	26837442
12	3	5	4	4649379	13	5	4	4	44800675	14	3	6	5	144298902
12	4	4	4	7259140	13	6	3	4	20047636	14	4	5	5	319684549
12	5	3	4	4649379	13	7	2	4	3661896	14	5	4	5	319684549
12	6	2	4	1147628	13	8	1	4	203280	14	6	3	5	144298902
12	7	1	4	82652	13	1	9	3	38115	14	7	2	5	26837442
12	1	8	3	19694	13	2	8	3	898051	14	8	1	5	1533950
12	2	7	3	367645	13	3	7	3	6542368	14	1	9	4	460245
12	3	6	3	2058329	13	4	6	3	20047636	14	2	8	4	10419653
12	4	5	3	4649379	13	5	5	3	28831218	14	3	7	4	73967488
12	5	4	3	4649379	13	6	4	3	20047636	14	4	6	4	223353280
12	6	3	3	2058329	13	7	3	3	6542368	14	5	5	4	319684549
12	7	2	3	367645	13	8	2	3	898051	14	6	4	4	223353280
12	8	1	3	19694	13	9	1	3	38115	14	7	3	4	73967488
12	1	9	2	1976	13	1	10	2	3080	14	8	2	4	10419653
12	2	8	2	48846	13	2	9	2	94281	14	9	1	4	460245
12	3	7	2	367645	13	3	8	2	898051	14	1	10	3	69765
12	4	6	2	1147628	13	4	7	2	3661896	14	2	9	3	2020530
12	5	5	2	1660331	13	5	6	2	7221592	14	3	8	3	18554641
12	6	4	2	1147628	13	6	5	2	7221592	14	4	7	3	73967488
12	7	3	2	367645	13	7	4	2	3661896	14	5	6	3	144298902
12	8	2	2	48846	13	8	3	2	898051	14	6	5	3	144298902
12	9	1	2	1976	13	9	2	2	94281	14	7	4	3	73967488
12	1	10	1	62	13	10	1	2	3080	14	8	3	3	18554641
12	2	9	1	1976	13	1	11	1	77	14	9	2	3	2020530
12	3	8	1	19694	13	2	10	1	3080	14	10	1	3	69765
12	4	7	1	82652	13	3	9	1	38115	14	1	11	2	4659
12	5	6	1	165262	13	4	8	1	203280	14	2	10	2	172040
12	6	5	1	165262	13	5	7	1	533610	14	3	9	2	2020530
12	7	4	1	82652	13	6	6	1	731808	14	4	8	2	10419653
12	8	3	1	19694	13	7	5	1	533610	14	5	7	2	26837442
12	9	2	1	1976	13	8	4	1	203280	14	6	6	2	36580432
12	10	1	1	62	13	9	3	1	38115	14	7	5	2	26837442
					13	10	2	1	3080	14	8	4	2	10419653
					13	11	1	1	77	14	9	3	2	2020530
										14	10	2	2	172040
										14	11	1	2	4659
										14	1	12	1	99
										14	2	11	1	4659
										14	3	10	1	69765
										14	4	9	1	460245
										14	5	8	1	1533950
										14	6	7	1	2760990
										14	7	6	1	2760990
										14	8	5	1	1533950
										14	9	4	1	460245
										14	10	3	1	69765
										14	11	2	1	4659
										14	12	1	1	99
										14				
											sum			3369276867
13	1	1	11	77	13				sum	438644841				
13	1	2	10	3080										
13	2	1	10	3080	14	1	1	12	99					
13	1	3	9	38115	14	1	2	11	4659					
13	2	2	9	94281	14	2	1	11	4659					
13	3	1	9	38115	14	1	3	10	69765					
13	1	4	8	203280	14	2	2	10	172040					
13	2	3	8	898051	14	3	1	10	69765					
13	3	2	8	898051	14	1	4	9	460245					
13	4	1	8	203280	14	2	3	9	2020530					
13	1	5	7	533610	14	3	2	9	2020530					
13	2	4	7	3661896	14	4	1	9	460245					
13	3	3	7	6542368	14	1	5	8	1533950					
13	4	2	7	3661896	14	2	4	8	10419653					
13	5	1	7	533610	14	3	3	8	18554641					
13	1	6	6	731808	14	4	2	8	10419653					
13	2	5	6	7221592	14	5	1	8	1533950					



**B.3 Genus 2**

d	v	e	f	H	10	2	5	1	16725	12	1	8	1	7417
5	1	1	1	4	10	3	4	1	47164	12	2	7	1	132202
					10	4	3	1	47164	12	3	6	1	721382
5			sum	4	10	5	2	1	16725	12	4	5	1	1610617
					10	6	1	1	1649	12	5	4	1	1610617
6	1	1	2	16						12	6	3	1	721382
6	1	2	1	16	10			sum	1355400	12	7	2	1	132202
6	2	1	1	16						12	8	1	1	7417
					11	1	1	7	3633					
6			sum	48	11	1	2	6	50001	12			sum	150429819
					11	2	1	6	50001					
7	1	1	3	67	11	1	3	5	201915	13	1	1	9	14091
7	1	2	2	169	11	2	2	5	491729	13	1	2	8	316470
7	2	1	2	169	11	3	1	5	201915	13	2	1	8	316470
7	1	3	1	67	11	1	4	4	315000	13	1	3	7	2241162
7	2	2	1	169	11	2	3	4	1364986	13	2	2	7	5412883
7	3	1	1	67	11	3	2	4	1364986	13	3	1	7	2241162
					11	4	1	4	315000	13	1	4	6	6764142
7			sum	708	11	1	5	3	201915	13	2	3	6	28779051
					11	2	4	3	1364986	13	3	2	6	28779051
8	1	1	4	237	11	3	3	3	2428862	13	4	1	6	6764142
8	1	2	3	1072	11	4	2	3	1364986	13	1	5	5	9681210
8	2	1	3	1072	11	5	1	3	201915	13	2	4	5	63458654
8	1	3	2	1072	11	1	6	2	50001	13	3	3	5	111801142
8	2	2	2	2664	11	2	5	2	491729	13	4	2	5	63458654
8	3	1	2	1072	11	3	4	2	1364986	13	5	1	5	9681210
8	1	4	1	237	11	4	3	2	1364986	13	1	6	4	6764142
8	2	3	1	1072	11	5	2	2	491729	13	2	5	4	63458654
8	3	2	1	1072	11	6	1	2	50001	13	3	4	4	172252340
8	4	1	1	237	11	1	7	1	3633	13	4	3	4	172252340
					11	2	6	1	50001	13	5	2	4	63458654
8			sum	9807	11	3	5	1	201915	13	6	1	4	6764142
					11	4	4	1	315000	13	1	7	3	2241162
9	1	1	5	667	11	5	3	1	201915	13	2	6	3	28779051
9	1	2	4	4736	11	6	2	1	50001	13	3	5	3	111801142
9	2	1	4	4736	11	7	1	1	3633	13	4	4	3	172252340
9	1	3	3	8560						13	5	3	3	111801142
9	2	2	3	21113	11			sum	14561360	13	6	2	3	28779051
9	3	1	3	8560						13	7	1	3	2241162
9	1	4	2	4736	12	1	1	8	7417	13	1	8	2	316470
9	2	3	2	21113	12	1	2	7	132202	13	2	7	2	5412883
9	3	2	2	21113	12	2	1	7	132202	13	3	6	2	28779051
9	4	1	2	4736	12	1	3	6	721382	13	4	5	2	63458654
9	1	5	1	667	12	2	2	6	1748723	13	5	4	2	63458654
9	2	4	1	4736	12	3	1	6	721382	13	6	3	2	28779051
9	3	3	1	8560	12	1	4	5	1610617	13	7	2	2	5412883
9	4	2	1	4736	12	2	3	5	6908644	13	8	1	2	316470
9	5	1	1	667	12	3	2	5	6908644	13	1	9	1	14091
					12	4	1	5	1610617	13	2	8	1	316470
9			sum	119436	12	1	5	4	1610617	13	3	7	1	2241162
					12	2	4	4	10702449	13	4	6	1	6764142
10	1	1	6	1649	12	3	3	4	18938994	13	5	5	1	9681210
10	1	2	5	16725	12	4	2	4	10702449	13	6	4	1	6764142
10	2	1	5	16725	12	5	1	4	1610617	13	7	3	1	2241162
10	1	3	4	47164	12	1	6	3	721382	13	8	2	1	316470
10	2	2	4	115478	12	2	5	3	6908644	13	9	1	1	14091
10	3	1	4	47164	12	3	4	3	18938994					
10	1	4	3	47164	12	4	3	3	18938994	13			sum	1506841872
10	2	3	3	206895	12	5	2	3	6908644					
10	3	2	3	206895	12	6	1	3	721382	14	1	1	10	25405
10	4	1	3	47164	12	1	7	2	132202	14	1	2	9	700045
10	1	5	2	16725	12	2	6	2	1748723	14	2	1	9	700045
10	2	4	2	115478	12	3	5	2	6908644	14	1	3	8	6235526
10	3	3	2	206895	12	4	4	2	10702449	14	2	2	8	15012496
10	4	2	2	115478	12	5	3	2	6908644	14	3	1	8	6235526
10	5	1	2	16725	12	6	2	2	1748723	14	1	4	7	24417030
10	1	6	1	1649	12	7	1	2	132202	14	2	3	7	103175785

14	3	2	7	103175785	14	5	3	4	1173398706	14	7	3	2	103175785
14	4	1	7	24417030	14	6	2	4	306159286	14	8	2	2	15012496
14	1	5	6	47238510	14	7	1	4	24417030	14	9	1	2	700045
14	2	4	6	306159286	14	1	8	3	6235526	14	1	10	1	25405
14	3	3	6	537417269	14	2	7	3	103175785	14	2	9	1	700045
14	4	2	6	306159286	14	3	6	3	537417269	14	3	8	1	6235526
14	5	1	6	47238510	14	4	5	3	1173398706	14	4	7	1	24417030
14	1	6	5	47238510	14	5	4	3	1173398706	14	5	6	1	47238510
14	2	5	5	435785878	14	6	3	3	537417269	14	6	5	1	47238510
14	3	4	5	1173398706	14	7	2	3	103175785	14	7	4	1	24417030
14	4	3	5	1173398706	14	8	1	3	6235526	14	8	3	1	6235526
14	5	2	5	435785878	14	1	9	2	700045	14	9	2	1	700045
14	6	1	5	47238510	14	2	8	2	15012496	14	10	1	1	25405
14	1	7	4	24417030	14	3	7	2	103175785					
14	2	6	4	306159286	14	4	6	2	306159286	14			sum	14732613116
14	3	5	4	1173398706	14	5	5	2	435785878					
14	4	4	4	1799940644	14	6	4	2	306159286					

### B.4 Genus 3

d	v	e	f	H						13	1	7	1	668591
7	1	1	1	30	11	sum	10260804			13	2	6	1	8342532
			sum	30						13	3	5	1	31916775
7					12	1	1	6	220244	13	4	4	1	48937240
					12	1	2	5	2054974	13	5	3	1	31916775
8	1	1	2	385	12	2	1	5	2054974	13	6	2	1	8342532
8	1	2	1	385	12	1	3	4	5567550	13	7	1	1	668591
8	2	1	1	385	12	2	2	4	13314231					
			sum	1155	12	3	1	4	5567550	13			sum	2195431068
8					12	1	4	3	5567550					
					12	2	3	3	23377106	14	1	1	8	1824323
9	1	1	3	2900	12	3	2	3	23377106	14	1	2	7	29267487
9	1	2	2	7070	12	4	1	3	5567550	14	2	1	7	29267487
9	2	1	2	7070	12	1	5	2	2054974	14	1	3	6	149721473
9	1	3	1	2900	12	2	4	2	13314231	14	2	2	6	355267058
9	2	2	1	7070	12	3	3	2	23377106	14	3	1	6	149721473
9	3	1	1	2900	12	4	2	2	13314231	14	1	4	5	324171185
			sum	29910	12	5	1	2	2054974	14	2	3	5	1338142324
9					12	1	6	1	220244	14	3	2	5	1338142324
					12	2	5	1	2054974	14	4	1	5	324171185
10	1	1	4	15308	12	3	4	1	5567550	14	1	5	4	324171185
10	1	2	3	65972	12	4	3	1	5567550	14	2	4	4	2041388556
10	2	1	3	65972	12	5	2	1	2054974	14	3	3	4	3551405485
10	1	3	2	65972	12	6	1	1	220244	14	4	2	4	2041388556
10	2	2	2	159608						14	5	1	4	324171185
10	3	1	2	65972	12					14	1	6	3	149721473
10	1	4	1	15308	12		sum	156469887		14	2	5	3	1338142324
10	2	3	1	65972	13	1	1	7	668591	14	3	4	3	3551405485
10	3	2	1	65972	13	1	2	6	8342532	14	4	3	3	3551405485
10	4	1	1	15308	13	2	1	6	8342532	14	5	2	3	1338142324
			sum	601364	13	1	3	5	31916775	14	6	1	3	149721473
10					13	2	2	5	76003367	14	1	7	2	29267487
					13	3	1	5	31916775	14	2	6	2	355267058
11	1	1	5	63355	13	1	4	4	48937240	14	3	5	2	1338142324
11	1	2	4	418810	13	2	3	4	203571133	14	4	4	2	2041388556
11	2	1	4	418810	13	3	2	4	203571133	14	5	3	2	1338142324
11	1	3	3	740100	13	4	1	4	48937240	14	6	2	2	355267058
11	2	2	3	1779193	13	1	5	3	31916775	14	7	1	2	29267487
11	3	1	3	740100	13	2	4	3	203571133	14	1	8	1	1824323
11	1	4	2	418810	13	3	3	3	355620834	14	2	7	1	29267487
11	2	3	2	1779193	13	4	2	3	203571133	14	3	6	1	149721473
11	3	2	2	1779193	13	5	1	3	31916775	14	4	5	1	324171185
11	4	1	2	418810	13	1	6	2	8342532	14	5	4	1	324171185
11	1	5	1	63355	13	2	5	2	76003367	14	6	3	1	149721473
11	2	4	1	418810	13	3	4	2	203571133	14	7	2	1	29267487
11	3	3	1	740100	13	4	3	2	203571133	14	8	1	1	1824323
11	4	2	1	418810	13	5	2	2	76003367					
11	5	1	1	63355	13	6	1	2	8342532	14			sum	28897471080

