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Geometry and topology of the space of Kähler metrics on singular varieties

Eleonora Di Nezza and Vincent Guedj

Abstract

Let Y be a compact Kähler normal space and $\alpha \in H^{1,1}_{BC}(Y)$ a Kähler class. We study metric properties of the space $\mathcal{H}_\alpha$ of Kähler metrics in α using Mabuchi geodesics. We extend several results by Calabi, Chen and Darvas previously established when the underlying space is smooth. As an application we analytically characterize the existence of Kähler-Einstein metrics on $\mathbf{Q}$-Fano varieties, generalizing a result of Tian, and illustrate these concepts in the case of toric varieties.

Introduction

Let Y be a compact Kähler normal space and $\alpha_Y \in H^{1,1}_{BC}(Y)$ a Kähler class, where $H^{1,1}_{BC}(Y)$ denotes the Bott-Chern cohomology space. The space $\mathcal{H}_{\alpha_Y}$ of Kähler metrics ω_Y in α_Y can be seen as an infinite dimensional riemannian manifold whose tangent spaces $T_{\omega_Y}\mathcal{H}_{\alpha_Y}$ can all be identified with $\mathcal{C}^\infty(Y, \mathbf{R})$. When Y is smooth, Mabuchi has introduced in [Mab87] an L^2-metric on $\mathcal{H}_{\alpha_Y}$, by setting

$$\langle f, g\rangle_{\omega_Y} := \int_Y f\, g\, \frac{{\omega_Y}^n}{V_{\alpha_Y}},$$

where $n = \dim_{\mathbf{C}} Y$ and $V_{\alpha_Y} = \int_Y {\omega_Y}^n = \alpha_Y^n$ denotes the volume of α_Y.

Mabuchi studied the corresponding geometry of $\mathcal{H}_{\alpha_Y}$, showing in particular that it can formally be seen as a locally symmetric space of non positive curvature. Semmes [Sem92] re-interpreted the geodesic equation as a complex homogeneous equation, while Donaldson [Don99] strongly motivated the search for smooth geodesics through its connection with the uniqueness of constant scalar curvature Kähler metrics.

In a series of remarkable works [Chen00, CC02, CT08, Chen09, CS12] X.X.Chen and his collaborators have studied the metric and geometric properties of the space $\mathcal{H}_{\alpha_Y}$ when Y is smooth, showing in particular that it is a path metric space (a non trivial assertion in this infinite dimensional setting) of non-positive curvature in the sense of Alexandrov. A key step from [Chen00] has been to produce $\mathcal{C}^{1,\bar{1}}$-geodesics which turn out to minimize the intrinsic distance d. It follows from the work of Lempert-Vivas [LV13], Darvas-Lempert [DL12] and Ross-Witt-Nyström [RWN15] that one can not expect better regularity, but for the toric setting (see Section 6).

The metric study of the space $(\mathcal{H}_{\alpha_Y}, d)$ has been recently pushed further by Darvas in [Dar13, Dar14, Dar15]. He characterizes there the metric completion of $(\mathcal{H}_{\alpha_Y}, d)$ and introduces several Finsler type metrics on $\mathcal{H}_{\alpha_Y}$, which turn out to be quite useful (see [DR15, BBJ15]). For $p \geq 1$,

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we set

$$d_p(\phi_0,\phi_1) := \inf\{\ell_p(\phi) \,|\, \phi \text{ is a path joining } \phi_0 \text{ to } \phi_1\},$$

where

$$\ell_p(\phi) := \int_0^1 |\dot{\phi}_t|_p dt = \int_0^1 \left(\int_Y \left|\dot{\phi}_t\right|^p MA(\phi_t)\right)^{1/p} dt.$$

The goal of this article is to extend these studies to the case when the underlying space has singularities. We fix a base point ω_Y representing α_Y and work with the space of Kähler potentials $\mathcal{H}_{\omega_Y}$. Our first main result extends the main results of [Chen00] and [Dar15, Theorem 1] as follows:

Theorem A.

- $(\mathcal{H}_{\omega_Y}, d_p)$ *is a metric space;*
- $d_p(\phi_0,\phi_1) = \left(\int_Y |\dot{\phi}_0|^p MA(\phi_0)\right)^{1/p} = \left(\int_Y |\dot{\phi}_1|^p MA(\phi_1)\right)^{1/p}$.

Following [Dar14, Dar15] we then study the metric completion of the space $(\mathcal{H}_{\alpha_Y}, d_p)$ and establish the following generalization of [Dar15, Theorem 2]:

Theorem B. *Let Y be a projective normal variety and assume ω_Y is a Hodge form. The metric completion of $(\mathcal{H}_{\omega_Y}, d_p)$ is a geodesic metric space which is bi-Lipschitz equivalent to the finite energy class $(\mathcal{E}^p(Y,\omega_Y), I_p)$.*

Finite energy classes have been introduced in [GZ07] and further studied in [BEGZ10, BBGZ13], we recall their definition in Section 2. The Mabuchi geodesics can be extended to finite energy geodesics which are still metric geodesics. A key technical tool here is Theorem 3.6 which compares d_p and I_p, a natural quantity which defines the "strong topology" on $\mathcal{E}^p(Y,\omega_Y)$

The metric completion of $(\mathcal{H}_{\alpha_Y}, d)$ has been considered by Streets in his study of the Calabi flow [Str16] and also plays an important role in recent works by Berman-Boucksom-Jonsson [BBJ15] and Berman-Darvas-Lu [BDL16]. There is no doubt that the extension to the singular setting will play a leading role in subsequent applications. We illustrate this here by generalizing Tian's analytic criterion [Tian97, PSSW08], using results of [BBEGZ] and an idea of [DR15]:

Theorem C. *Let (Y,D) be a log Fano pair. It admits a unique Kähler-Einstein metric iff there exists $\varepsilon, M > 0$ such that for all $\phi \in \mathcal{H}_{norm}$,*

$$\mathcal{F}(\phi) \leq -\varepsilon d_1(0,\phi) + M.$$

Here $\mathcal{F}$ is a functional whose critical points are Kähler-Einstein potentials (Section 5) and $\mathcal{H}_{norm}$ is the set of normalized potentials. This result has been independently obtained by T.Darvas [Dar16] by a different approach.

Our results should also be useful in analyzing more generally cscK metrics on midly singular varieties (see e.g. the recent construction by Arezzo and Spotti of cscK metrics on crepant resolutions of Calabi-Yau varieties with non-orbifold singularities [AS15]).

A way to establish the above results is to consider a resolution of singularities $\pi : X \to Y$ and to work with the space $\mathcal{H}_\omega$ of potentials associated to the form $\omega = \pi^*\omega_Y$. All the above results actually hold in the more general setting when ω is merely a semi-positive and big form

(i.e. $\int_X \omega^n > 0$). We approximate $\mathcal{H}_\omega$ by spaces of Kähler potentials $\mathcal{H}_{\omega+\varepsilon\omega_X}$ and show that the most important metric properties of $(\mathcal{H}_{\omega+\varepsilon\omega_X}, d_\varepsilon)$ pass to the limit.

The organization of the paper is as follows. *Section 1* starts by a recap on Mabuchi geodesics and metrics. *Theorem A* is proved in *Section 1.2*, where we develop a low-regularity approach for understanding geodesics by approximation. We introduce in *Section 2* classes of finite energy currents and compare their natural topologies with the one induced by the Mabuchi distances in *Section 3*. We study finite energy geodesics in *Section 4* and prove *Theorem B*. We finally prove *Theorem C* in *Section 5* and provide a detailed analysis of the toric setting in *Section 6*.

1. The space of Kähler currents

Let (Y, ω_Y) be a compact Kähler normal space of dimension n. It follows from the definition of $H^{1,1}_{BC}(Y)$ (see for example [BEG, Definition 4.6.2]) that any other Kähler metric on Y in the same Bott-Chern cohomology class of ω_Y can be written as

$$\omega_\phi = \omega_Y + dd^c\phi,$$

where $d = \partial + \overline{\partial}$ and $d^c = \frac{1}{2i\pi}(\partial - \overline{\partial})$. Let $\mathcal{H}_{\omega_Y}$ be the space of *Kähler potentials*

$$\mathcal{H}_{\omega_Y} = \{\phi \in C^\infty(Y, \mathbf{R});\ \omega_\phi = \omega + dd^c\phi > 0\}.$$

This is a convex open subset of the Fréchet vector space $C^\infty(Y) := C^\infty(Y, \mathbf{R})$, thus itself a Fréchet manifold, which is moreover parallelizable :

$$T\mathcal{H}_{\omega_Y} = \mathcal{H}_{\omega_Y} \times C^\infty(Y).$$

For any $\phi \in \mathcal{H}_{\omega_Y}$, each tangent space $T_\phi\mathcal{H}_{\omega_Y}$ is identified with $C^\infty(Y)$.

As two Kähler potentials define the same metric when (and only when) they differ by an additive constant, we set

$$\mathcal{H}_{\alpha_Y} = \mathcal{H}_{\omega_Y}/\mathbf{R}$$

where $\mathbf{R}$ acts on $\mathcal{H}_{\omega_Y}$ by addition. The set $\mathcal{H}_{\alpha_Y}$ is therefore the space of Kähler metrics on Y in the cohomology class $\alpha_Y := \{\omega_Y\} \in H^{1,1}_{BC}(Y)$.

In the whole article we fix $\pi : X \to Y$ a resolution of singularities and set $\omega = \pi^*\omega_Y$, $\alpha = \pi^*\alpha_Y$. Since α is no longer Kähler, we fix ω_X a Kähler form on X and set

$$\omega_\varepsilon := \omega + \varepsilon\omega_X,$$

for $\varepsilon > 0$. We will study the geometry and the topology of the spaces

$$\mathcal{H}_\alpha = \pi^*\mathcal{H}_{\alpha_Y} \quad \text{and} \quad \mathcal{H}_\omega = \pi^*\mathcal{H}_{\omega_Y}$$

by approximating them by the spaces $\mathcal{H}_{\alpha_\varepsilon}, \mathcal{H}_{\omega_\varepsilon}$, where

$$\mathcal{H}_{\omega_\varepsilon} := \{\varphi \in C^\infty(X, \mathbf{R})\,; \omega_\varepsilon + dd^c\varphi > 0\} \quad \text{and} \quad \alpha_\varepsilon := \{\omega_\varepsilon\}.$$

All the properties that we are going to establish actually hold for cohomology classes α that are merely *semi-positive* and *big* (not necessarily the pull-back of a Kähler class under a desingularization).

Our analysis will focus on the ample locus of α:

Definition 1.1 *The ample locus* $\mathrm{Amp}\,(\alpha)$ *of* α *is the Zarisiki open set of those points* $x \in X$ *such that* α *can be represented by a positive closed* $(1,1)$*-current which is a smooth positive form near* x*.*

We then let $\mathcal{H}_\omega$ denote the space of potentials $\varphi \in C^\infty(X, \mathbf{R})$ such that ω_φ is a Kähler form in $\mathrm{Amp}\,(\alpha)$. In our main case of interest, i.e. when $\alpha = \pi^*\alpha_Y$, the ample locus

$$\mathrm{Amp}\,(\alpha) = \pi^{-1}(Y^{reg})$$

is the preimage of the set of regular points of Y.

1.1 The Riemannian structure

1.1.1 *Mabuchi geodesics*

Definition 1.2 *[Mab87] The* Mabuchi metric *is the L^2 Riemannian metric on $\mathcal{H}_\omega$. It is defined by*

$$< \psi_1, \psi_2 >_\varphi = \int_X \psi_1\psi_2 \, \frac{(\omega + dd^c\varphi)^n}{V_\alpha}$$

where $\varphi \in \mathcal{H}_\omega$, $\psi_1, \psi_2 \in C^\infty(X)$ and $(\omega + dd^c\varphi)^n/V_\alpha$ is the volume element, normalized so that it is a probability measure. Here $V_\alpha := \alpha^n = \int_X \omega^n$.

In the sequel we shall also use the notation $\omega_\varphi := \omega + dd^c\varphi$ and

$$MA(\varphi) := V_\alpha^{-1}\omega_\varphi^n.$$

Geodesics between two points φ_0, φ_1 in $\mathcal{H}_\omega$ correspond to the extremals of the Energy functional

$$\varphi \mapsto H(\varphi) = \frac{1}{2}\int_0^1 \int_X (\dot\varphi_t)^2 \, MA(\varphi_t)\, dt.$$

where $\varphi = \varphi_t$ is a smooth path in $\mathcal{H}_\omega$ joining φ_0 and φ_1. The geodesic equation is formally obtained by computing the Euler-Lagrange equation for this Energy functional (with fixed end points). It is given by

$$\ddot\varphi \, MA(\varphi) = \frac{n}{V_\alpha} d\dot\varphi \wedge d^c\dot\varphi \wedge \omega_\varphi^{n-1}. \tag{1}$$

We are interested in the boundary value problem for the geodesic equation: given φ_0, φ_1 two distinct points in $\mathcal{H}_\omega$, can one find a path $(\varphi(t))_{0\le t\le 1}$ in $\mathcal{H}_\omega$ which is a solution of (1) with end points $\varphi(0) = \varphi_0$ and $\varphi(1) = \varphi_1$?

For each path $(\varphi_t)_{t\in[0,1]}$ in $\mathcal{H}_\omega$, we set

$$\varphi\,(x, t + is) = \varphi_t(x), \qquad x \in X, \quad t + is \in S = \{z \in \mathbf{C} \,:\, 0 < \Re(z) < 1\};$$

i.e. we associate to each path (φ_t) a function φ on the complex manifold $M = X \times S$, which only depends on the real part of the stripe coordinate: we consider S as a Riemann surface with boundary and use the complex coordinate $z = t + is$ to parametrize the stripe S. Set $\omega(x, z) := \omega(x)$.

Semmes observed in [Sem92] that the path φ_t is a geodesic in $\mathcal{H}_\omega$ if and only if the associated function φ on $X \times S$ is a ω-psh solution of the homogeneous complex Monge-Ampère equation

$$(\omega + dd^c_{x,z}\varphi)^{n+1} = 0. \tag{2}$$

This motivates the following:

Definition 1.3 *The function*

$$\varphi = \sup\{u\,;\, u \in PSH(M, \omega) \text{ and } u \le \varphi_{0,1} \text{ on } \partial M\}$$

is the Mabuchi geodesic joining φ_0 to φ_1.

Here $PSH(M,\omega)$ denotes the set of ω-psh functions on M: these are functions $u : M \to \mathbf{R} \cap \{-\infty\}$ which are locally the sum of a plurisubharmonic and a smooth function and such that $\omega + dd^c_{x,z} u \geq 0$ in the sense of currents (see section 2.1.1 for more details).

Proposition 1.4 *Let $(\varphi_t)_{0\leq t\leq 1}$ be the Mabuchi geodesic joining φ_0 to φ_1. Then*

(i) $\varphi \in PSH(M,\omega)$ is uniformly bounded on M and continuous on $\mathrm{Amp}\,(\{\omega\}) \times \bar{S}$.

(ii) $|\varphi(x,z) - \varphi(x,z')| \leq A|\Re(z) - \Re(z')|$ with $A = \|\varphi_0 - \varphi_1\|_{L^\infty(X)}$.

(iii) $\varphi_{|\{\Re(z)=0\}} = \varphi_0$, $\varphi_{|\{\Re(z)=1\}} = \varphi_1$ and $(\omega + dd^c_{x,z}\varphi)^{n+1} = 0$.

It is moreover the unique bounded ω-psh solution to this Dirichlet problem.

We thank Hoang Chinh Lu for sharing his ideas on the continuity of φ.

Proof. The proof follows from a classical balayage technique, together with a barrier argument as noted by Berndtsson [Bern15]. Set $A = \|\varphi_1 - \varphi_0\|_{L^\infty(X)}$.

Observe that the function $\varphi_0 - At$, with $t = \Re(z)$, is ω-psh on M and $\varphi_0 - At|_{\partial M} \leq \varphi_{0,1}$. Hence it belongs to the family $\mathcal{F}$ defining the upper envelope φ, so $\varphi_0 - At \leq \varphi_t$.

Similarly $\varphi_0 + At$ is a ω-psh function on M and $\varphi_0 + At|_{\partial M} \geq \varphi_{0,1}$. Since $(\omega + dd^c_{x,z}(\varphi_0 + At))^{n+1} = 0$, it follows from the maximum principle that $u \leq \varphi_0 + At$, for any $u \in \mathcal{F}$ in the family. Therefore

$$\varphi_0 - At \leq \varphi_t \leq \varphi_0 + At.$$

Similar arguments show that

$$\varphi_1 + A(t-1) \leq \varphi_t \leq \varphi_1 - A(t-1).$$

The upper semi-continuous regularization φ^* of φ satisfies the same estimates, showing in particular that $\varphi^*|_{\partial M} = \varphi_{0,1}$. Since φ^* is ω-psh, we infer $\varphi^* \in \mathcal{F}$ hence $\varphi^* = \varphi$. Thus φ is ω-psh and uniformly bounded, proving the first statement in (i). Classical balayage arguments show that $(\omega + dd^c_{x,z}\varphi)^{n+1} = 0$, proving (iii).

We now prove prove (ii). Consider the function

$$\chi_t(x) = \max\{\varphi_0(x) - A\log|z|, \varphi_1(x) + A(\log|z| - 1)\}$$

and note that it belongs to $\mathcal{F}$ and has the right boundary values.

Since $\chi_- = \varphi_0(x) - At \leq \varphi$ with equality at $t = 0$, we infer for all x,

$$-A = \frac{\partial \chi_-}{\partial t}_{|t=0} \leq \dot{\varphi}_0(x).$$

Similarly $\chi_+ = \varphi_1(x) + A(t-1) \leq \varphi$ with equality at $t = 1$ yields for all x, $\dot{\varphi}_1(x) \leq +A = \frac{\partial \chi_+}{\partial t}_{|t=1}$. Since $t \mapsto \varphi_t(x)$ is convex (by subharmonicity in z), we infer that for a.e. t, x, $-A \leq \dot{\varphi}_0(x) \leq \dot{\varphi}_t(x) \leq \dot{\varphi}_1(x) \leq +A$.

It remains to show that φ is continuous on $\mathrm{Amp}\,(\{\omega\}) \times \bar{S}$. We can assume without loss of generality that $\varphi_0 < \varphi_1$. Indeed, given any $\varphi_0, \varphi_1 \in \mathcal{H}_\omega$, there exists $C > 0$ such that $\varphi_0 < \varphi_1 + C$. By Lemma 1.8, the Mabuchi geodesic joining φ_0 and $\varphi_1 + C$ is $\psi_t = \varphi_t + Ct$, $t \in [0,1]$. The continuity of $(x,t) \to \psi_t(x)$ will then imply the continuity of $(x,t) \to \varphi_t(x)$.

We change notations slighlty, replacing the stripe S by the annulus $D := \{z = e^{t+is} \in \mathbf{C} \ : \ 1 \leq |w| \leq e\}$. We are going to express the function φ as a global Θ-psh envelope on the compact manifold $X \times \mathbf{P}^1$, where we view the annulus D as a subset of the Rieman sphere,

$\mathbf{C} \subset \mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$. The form $\Theta(x,z) = \omega(x) + A\omega_{FS}(z)$ is a semi-positive and big form on the compact Kähler manifold $\widetilde{M} := X \times \mathbf{P}^1$, so the viscosity approach of [EGZ16] can be applied showing that the envelope φ is continuous on $\mathrm{Amp}\,(\{\omega\}) \times \bar{S}$. Here ω_{FS} denotes the Fubini-Study metric on $\mathbf{P}^1$ and $A > 0$ is a constant to be chosen below.

Consider $U = \max(U_0, U_1)$, where $U_0(x,z) := \varphi_0(x)$ and

$$U_1(x,z) := \varphi_1(x) + A(\log|z|^2 - \log(|z|^2+1) + \log(e^2+1) - 2).$$

We choose $A > 0$ so large that $U(x,1) \equiv \varphi_0(x)$. Note that $U(x,e) \equiv \varphi_1(x)$ since $\varphi_0 < \varphi_1$. Both U_0 and U_1 are Θ-psh on $\widetilde{M}$, hence so is U.

Fix ρ a local potential of $A\omega_{FS}$ in D such that $\rho|_{\partial D} = 0$ and let F be a continuous S^1-invariant function on $\widetilde{M}$ such that

(a) $F = \varphi_{0,1}$ on $X \times \partial D$,

(b) $F(x,z) \geq U(x,z) \geq \varphi_0(x)$,

(c) $F(x,z) + \rho(z) > \varphi_t(x)$ in $X \times D$, with $t = \log|z|$.

We let the reader check that the function $F = U$ in $\widetilde{M} \setminus X \times D$ and

$$F(x,z) := (1 - \log|z|)\varphi_0(x) + (\log|z|)\varphi_1(x) - \rho(z) + (\log|z|)(1 - \log|z|),$$

for $(x,z) \in X \times D$, does the job.

We claim that for all $(x,z) \in X \times D$,

$$P_\Theta(F)(x,z) + \rho(z) = \varphi_{\log|z|}(x)$$

where

$$P_\Theta(F) := \sup\{v \ : \ v \in \mathrm{PSH}(\widetilde{M}, \Theta) \text{ and } v \leq F\}.$$

Indeed $P_\Theta(F) + \rho$ is ω-psh in $X \times D$ and has boundary values $\leq \varphi_{0,1}$. It follows from definition of the geodesic that $P_\Theta(F) + \rho \leq \varphi_t$. On the other hand, $F + \rho \geq U + \rho \in \mathrm{PSH}(X \times D, \omega)$ and $U = \varphi_{0,1}$ on ∂M thus $P_\Theta(F) + \rho = \varphi_{0,1}$ on ∂M. Condition (c) insures that $M = X \times D$ does not meet the contact set $\{P_\Theta(F) = F\}$ since $F + \rho > \varphi_t \geq P_\Theta(F) + \rho$. It thus follows from [BD12] that $(\Theta + dd^c P_\Theta(F))^{n+1} = 0$ in M, and the maximum principle yields

$$P_\Theta(F) + \rho = \varphi_t.$$

The continuity of φ on $\mathrm{Amp}\,(\{\omega\}) \times \bar{S}$ now follows from [EGZ16] together with the following easy observation: the arguments in [EGZ16, Section 2.2] insures that if F is a smooth function on $\widetilde{M}$, then $P_\Theta(F)$ is a Θ-psh function, continuous on $\mathrm{Amp}\,(\{\Theta\})$. The same result holds if F is merely continuous. Indeed, let F_j be a sequence of smooth functions on $\widetilde{M}$ converging uniformly to F. Taking the envelope at both sides of the inequality $F_j \leq F + \|F_j - F\|_{L^\infty(X)}$ we get $P_\Theta(F_j) \leq P_\Theta(F) + \|F_j - F\|_{L^\infty(X)}$. Hence, $\|P_\Theta(F_j) - P_\Theta(F)\|_{L^\infty(X)} \leq \|F_j - F\|_{L^\infty(X)}$. Thus $P_\Theta(F_j)$ converges uniformly to $P_\Theta(F)$, and so $P_\Theta(F)$ is a Θ-psh function that is continuous on $\mathrm{Amp}\,(\{\Theta\}) = \mathrm{Amp}\,(\{\omega\}) \times \bar{S}$. □

Remark 1.5 *If one could choose F smooth in the proof above, it would follow from [BD12] that $\varphi \in \mathcal{C}^{1,\bar{1}}(\mathrm{Amp}\,(\alpha) \times S)$. This would also provide a compact proof of Chen's regularity result.*

We now observe that geodesics in $\mathcal{H}_\omega$ are projection of those in $\mathcal{H}_{\omega_\varepsilon}$:

Proposition 1.6 *Let φ denote the geodesic joining φ_0 to φ_1 in $\mathcal{H}_\omega$ and let φ^ε denote the corresponding geodesic in the space $\mathcal{H}_{\omega_\varepsilon}$. The map $\varepsilon \mapsto \varphi^\varepsilon$ is increasing and φ^ε decreases to φ as ε decreases to zero. Moreover*

$$\varphi = P(\varphi^\varepsilon),$$

where P denotes the projection operator onto the space $PSH(M,\omega)$.

Recall that, for an upper semi-continuous function $u : M \to \mathbf{R}$, its projection $P(u)$ is defined by

$$P(u) := \sup\{v \in PSH(M,\omega)\,;\, v \leq u\}.$$

The function $P(u)$ is either identically $-\infty$ or belongs to $PSH(M,\omega)$. It is the greatest ω-psh function on M that lies below u.

Proof. Set $\psi := P(\varphi^\varepsilon)$. Since $\omega \leq \omega_\varepsilon$, it follows from the envelope point of view that $\varphi \leq \varphi^\varepsilon$. Thus $\varphi = P(\varphi) \leq P(\varphi^\varepsilon) = \psi$ and $\psi \in PSH(M,\omega)$. Now $\psi \leq \varphi$ since $\psi \leq \varphi^\varepsilon = \varphi_0, \varphi_1$ on ∂M and $\psi \in PSH(M,\omega)$. Thus $\psi = P(\varphi^\varepsilon) = \varphi$.

Fix $\varepsilon' \leq \varepsilon$. The inclusion $PSH(M,\omega_{\varepsilon'}) \subset PSH(M,\omega_\varepsilon)$ implies similarly that $\varphi \leq \varphi^{\varepsilon'} \leq \varphi^\varepsilon$. The decreasing limit v of φ^ε, as ε decreases to zero, satsifies both $\varphi \leq v$ and $v \in PSH(M,\omega)$ with boundary values φ_0, φ_1, thus $v = \varphi$. □

It will also be interesting to consider *subgeodesics*:

Definition 1.7 *A subgeodesic is a path (φ_t) of functions in $\mathcal{H}_\omega$ (or in larger classes of ω-psh functions) such that the associated function is a ω-psh function on $X \times S$.*

We shall soon need the following simple observation:

Lemma 1.8 *Fix $c \in \mathbf{R}$, $\varphi, \psi \in \mathcal{H}_\omega$ and let $(\varphi_t)_{0\leq t\leq 1}$ denote the Mabuchi geodesic joining $\varphi = \varphi_0$ to $\varphi_1 = \psi$. Then $\psi_t(x) := \varphi_t(x) - ct, 0 \leq t \leq 1, x \in X$, is the Mabuchi geodesic joining φ to $\psi - c$.*

Proof. The proof follows from Definition 1.3 and the definition of envelopes since $\sup\{v\,;\, v \in PSH(M,\omega)$ and $v \leq \varphi, v \leq \psi - c$ on $\partial M\} = \varphi_t - ct$. □

1.1.2 *Mabuchi and other Finsler distances* When ω is Kähler, the length of a differential path $(\varphi_t)_{t\in[0,1]}$ in $\mathcal{H}_\omega$ is defined in a standard way,

$$\ell(\varphi) := \int_0^1 |\dot\varphi_t| dt = \int_0^1 \sqrt{\int_X \dot\varphi_t^2 MA(\varphi_t)} dt.$$

The distance between two points in $\mathcal{H}_\omega$ is then

$$d(\varphi_0, \varphi_1) := \inf\{\ell(\varphi) \,|\, \varphi \text{ is a smooth path joining } \varphi_0 \text{ to } \varphi_1\}.$$

It is easy to verify that d defines a semi-distance (i.e. non-negative, symmetric and satisfying the triangle inequality). It is however non trivial to check that d is non degenerate (see [MM05] for a striking example).

Observe that d induces a distance on $\mathcal{H}_\alpha$ (that we abusively still denote by d) compatible with the riemannian splitting $\mathcal{H}_\omega = \mathcal{H}_\alpha \times \mathbf{R}$, by setting

$$d(\omega_\varphi, \omega_\psi) := d(\varphi, \psi)$$

whenever the potentials φ, ψ of $\omega_\varphi, \omega_\psi$ are normalized by $E(\varphi) = E(\psi) = 0$ (see section 2.2.1 for the definition of the functional E).

It is rather easy to check that $(\mathcal{H}_\alpha, d)$ is not a complete metric space. We shall describe the metric completion $(\overline{\mathcal{H}}_\alpha, d)$ in section 4. Following Darvas [Dar15] we introduce a family of distances that generalize d:

Definition 1.9 *For $p \geq 1$ and ω Kähler, we set*

$$d_p(\varphi_0, \varphi_1) := \inf\{\ell_p(\varphi) \,|\, \varphi \text{ is a smooth path joining } \varphi_0 \text{ to } \varphi_1\},$$

where $\ell_p(\varphi) := \int_0^1 |\dot{\varphi}_t|_p dt = \int_0^1 \left(\int_X |\dot{\varphi}_t|^p \, MA(\varphi_t)\right)^{1/p} dt$.

Note that $d_2 = d$ is the Mabuchi distance. Mabuchi geodesics have constant speed with respect to all the Finsler structures ℓ_p, as was observed by Berndtsson [Bern09, Lemma 2.1]: for any $\mathcal{C}^1$-function χ,

$$t \mapsto \int_X \chi(\dot{\varphi}_t) MA(\varphi_t)$$

is constant along a geodesic. Indeed

$$\begin{aligned}\frac{d}{dt}\int_X \chi(\dot{\varphi}_t) MA(\varphi_t) &= \int_X \chi'(\dot{\varphi}_t)\ddot{\varphi}_t MA(\varphi_t) + \frac{n}{V_\alpha}\int_X \chi(\dot{\varphi}_t) dd^c\dot{\varphi}_t \wedge \omega_{\varphi_t}^{n-1} \\ &= \int_X \chi'(\dot{\varphi}_t)\left\{\ddot{\varphi}_t MA(\varphi_t) - \frac{n}{V_\alpha} d\dot{\varphi}_t \wedge d^c\dot{\varphi}_t \wedge \omega_{\varphi_t}^{n-1}\right\} = 0\end{aligned}$$

since $\ddot{\varphi}_t MA(\varphi_t) - \frac{n}{V_\alpha} d\dot{\varphi}_t \wedge d^c\dot{\varphi}_t \wedge \omega_{\varphi_t}^{n-1} = 0$. Applying this observation to $\chi(t) = t^p$ shows that Mabuchi geodesics have constant ℓ_p-speed.

When ω is merely semi-positive there are fewer smooth paths within $\mathcal{H}_\omega$. It is natural to consider smooth paths in $\mathcal{H}_{\omega_\varepsilon}$ and pass to the limit in the previous definitions :

Definition 1.10 *Assume ω is semi-positive and big. Let $\varphi_0, \varphi_1 \in \mathcal{H}_\omega$. We define the Mabuchi distance between φ_0 and φ_1 as*

$$d_p(\varphi_0, \varphi_1) := \liminf_{\varepsilon \to 0} d_{p,\varepsilon}(\varphi_0, \varphi_1),$$

where $d_{p,\varepsilon}$ is the distance w.r.t. the Kähler form $\omega_\varepsilon := \omega + \varepsilon\omega_X$.

It is again easy to check that d_p is a semi-distance. We will show in Theorem 1.13 that it is a distance, which moreover does not depend on the way we approximate ω by Kähler classes.

Remark 1.11 *For any smooth path $\psi : [0,1] \to \mathcal{H}_\omega$, we can still define*

$$\ell_p(\psi) := \int_0^1 \left(\frac{1}{V}\int_X |\dot{\psi}_t|^p (\omega + dd^c\psi_t)^n\right)^{1/p} dt$$

when ω is merely semi-positive. Since $PSH(M,\omega) \subset PSH(M,\omega_\varepsilon)$, ψ_t is both in $\mathcal{H}_\omega$ and $\mathcal{H}_{\omega_\varepsilon}$. Observe that

$$\begin{aligned}V_\varepsilon^{-1}\int_X |\dot{\psi}_t|^p (\omega_\varepsilon + dd^c\psi_t)^n &= V_\varepsilon^{-1}\int_X |\dot{\psi}_t|^p (\omega + dd^c\psi_t + \varepsilon\omega_X)^n \\ &\leq V^{-1}\int_X |\dot{\psi}_t|^p (\omega + dd^c\psi_t)^n + A\varepsilon,\end{aligned}$$

hence

$$\ell_{p,\varepsilon}(\psi) \leq \ell_p(\psi) + A'\varepsilon$$

where $\ell_{p,\varepsilon}$ denotes the length in $\mathcal{H}_{\omega_\varepsilon}$. We infer

$$d_p(\varphi_0,\varphi_1) \leq \inf\{\ell_p(\psi) \quad \psi \text{ smooth path joining } \varphi_0 \text{ and } \varphi_1 \text{ in } \mathcal{H}_\omega\}.$$

The converse inequality is however unclear, due to the lack of positivity of ω: it is difficult to smooth out ω-psh functions if ω is not Kähler. This partially explains Definition 1.10.

1.2 Approximation by Kähler classes

Fix $\varphi_0,\varphi_1 \in \mathcal{H}_\omega$. We let $(\varphi_t)_{0\leq t\leq 1}$ denote the Mabuchi geodesic in $\mathcal{H}_\omega$ joining φ_0 to φ_1.

Definition 1.12 *For $t=0,1$ we set*

$$I(t) := \int_X |\dot\varphi_t|^p MA(\varphi_t).$$

Theorem 1.13 *Set $\omega_\varepsilon = \omega + \varepsilon\omega_X$, $\varepsilon>0$. Then $\lim_{\varepsilon\to 0} d_{p,\omega_\varepsilon}(\varphi_0,\varphi_1)$ exists and is independent of ω_X. More precisely,*

$$d^p_{p,\varepsilon}(\varphi_0,\varphi_1) \to I(0) = I(1)$$

for almost every $t\in(0,1)$. In particular $d_p(\varphi_0,\varphi_1) = I(0)^{1/p} = I(1)^{1/p}$ defines a distance on $\mathcal{H}_\omega$.

In the definition of $I(0), I(1)$, the time derivatives $\dot\varphi_0 = \dot\varphi_0^+$, $\dot\varphi_1 = \dot\varphi_1^-$ denote the right and left derivative, respectively.

Proof. Observe that $\varphi_0,\varphi_1 \in \mathcal{H}_{\omega_\varepsilon}$ and let φ_t^ε be the corresponding geodesic. It follows from [Chen00] that

$$d^p_{p,\varepsilon}(\varphi_0,\varphi_1) = V_\varepsilon^{-1}\int_X |\dot\varphi_0^\varepsilon|^p(\omega_\varepsilon + dd^c\varphi_0)^n.$$

Now observe that

$$\dot\varphi_0^+ \leq \dot\varphi_0^\varepsilon \leq \frac{\varphi_t^\varepsilon - \varphi_0}{t} \quad \forall t\in(0,1)$$

where the first inequality follows from the fact that $\varepsilon\to\varphi_t^\varepsilon$ is decreasing (Proposition 1.6), while second uses the convexity of $t\mapsto\varphi_t^\varepsilon$. Thus

$$|\dot\varphi_0^\varepsilon - \dot\varphi_0^+| \leq \left|\frac{\varphi_t^\varepsilon-\varphi_0}{t} - \dot\varphi_0^+\right|.$$

Letting $\varepsilon\searrow 0$ and then $t\to 0$ shows that $|\dot\varphi_0^\varepsilon - \dot\varphi_0^+|$ converges pointwise to zero. Moreover, $(\omega_\varepsilon + dd^c\varphi_0)^n = f_\varepsilon dV$ where dV is the Lebesgue measure and $f_\varepsilon>0$ are smooth densities which converge locally uniformly to $f\geq 0$ with $(\omega+dd^c\varphi_0)^n = f dV$. The dominated convergence theorem thus yields

$$\lim_{\varepsilon\to 0} d^p_{p,\varepsilon}(\varphi_0,\varphi_1) = V^{-1}\int_X |\dot\varphi_0^+|^p(\omega+dd^c\varphi_0)^n = I(0).$$

The argument for $I(1)$ is similar.

This shows in particular that d_p is a distance on $\mathcal{H}_\omega$: if $d_p(\varphi_0,\varphi_1)=0$, then $I(0)=I(1)=0$, hence $\dot\varphi_0(x)=\dot\varphi_1(x)=0$ for a.e. $x\in X$, which implies $\dot\varphi_t(x)=0$ for a.e. $x\in X$, by convexity of $t\mapsto\varphi_t(x)$. Thus, $\varphi_0(x)=\varphi_1(x)$ for a.e. $x\in X$. □

We now extend the definition of the distance d_p for bounded ω-psh potentials.

Definition 1.14 *Let $\varphi_0, \varphi_1 \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$ then*

$$d_p(\varphi_0,\varphi_1) := \liminf_{\varepsilon\to 0} \liminf_{j,k\to+\infty} d_{p,\varepsilon}(\varphi_0^j,\varphi_1^k) = \liminf_{\varepsilon\to 0} d_{p,\varepsilon}(\varphi_0,\varphi_1)$$

where φ_0^j, φ_1^k are smooth sequences of ω_ε-psh functions decreasing to φ_0 and φ_1, respectively.

Observe that $d_{p,\omega_\varepsilon}(\varphi_0,\varphi_1)$ is well defined for potentials in $\mathcal{E}^p(X,\omega_\varepsilon)$ ([Dar15]), and so in particular for bounded ω_ε-psh functions.

Proposition 1.15 *Let $\varphi_0, \varphi_1 \in \mathrm{PSH}(X,\omega)\cap L^\infty(X)$. The limit of $d_{p,\omega_\varepsilon}(\varphi_0,\varphi_1)$ as ε goes to zero exists and it does not depend on the choice of ω_X.*

Proof. Let φ_0^j, φ_1^k be smooth sequences of ω_ε-psh functions decreasing to φ_0 and φ_1, respectively. Fix j,k. By [Dar15, Corollary 4.14] we know that the Pythagore formula holds true, i.e.

$$d_{p,\varepsilon}(\varphi_0^j,\varphi_1^k) = d_{p,\varepsilon}(\varphi_0^j, \varphi_0^j \vee_\varepsilon \varphi_1^k) + d_{p,\varepsilon}(\varphi_0^j \vee_\varepsilon \varphi_1^k, \varphi_1^k),$$

where $\psi := \varphi_0^j \vee_\varepsilon \varphi_1^k$ is the greatest ω_ε-psh function that lies below $\min(\varphi_0^j,\varphi_1^k)$. Fix $\varepsilon \le \varepsilon'$. We claim that

$$d_{p,\varepsilon}(\varphi_0^j,\psi) \le d_{p,\varepsilon'}(\varphi_0^j,\psi) \quad \text{and} \quad d_{p,\varepsilon}(\psi,\varphi_1^k) \le d_{p,\varepsilon'}(\psi,\varphi_1^k).$$

Let ψ_t^ε, $\psi_t^{\varepsilon'}$ denote the ε-geodesic and the ε'-geodesic both joining φ_0^j and ψ. Since $\varepsilon \to \psi_t^\varepsilon$ is increasing (Proposition 1.6) we have that for any $t \in (0,1)$

$$\frac{\psi_t^\varepsilon - \varphi_0^j}{t} \le \frac{\psi_t^{\varepsilon'} - \varphi_0^j}{t}$$

that implies $\dot\psi_0^\varepsilon \le \dot\psi_0^{\varepsilon'}$. Moreover observe that since $\varphi_0^j(x) \le \psi(x)$ for all $x \in X$, Lemma 3.3 yields $\dot\psi_0^\varepsilon(x) \ge 0$ for all $x \in X$. It then follows that

$$d_{p,\varepsilon}^p(\varphi_0^j,\psi) = \int_X |\dot\psi_0^\varepsilon|^p \frac{(\omega_\varepsilon + dd^c\varphi_0^j)^n}{V_\varepsilon} \le \int_X |\dot\psi_0^{\varepsilon'}|^p \frac{(\omega_{\varepsilon'} + dd^c\varphi_0^j)^n}{V_\varepsilon} = d_{p,\varepsilon'}^p(\varphi_0^j,\psi). \tag{3}$$

The same type of arguments give $d_{p,\varepsilon}(\psi,\varphi_1^k) \le d_{p,\varepsilon'}(\psi,\varphi_1^k)$. Hence

$$d_{p,\varepsilon}(\varphi_0^j,\varphi_1^k) \le d_{p,\varepsilon'}(\varphi_0^j, \varphi_0^j \vee_\varepsilon \varphi_1^k) + d_{p,\varepsilon'}(\varphi_0^j \vee_\varepsilon \varphi_1^k, \varphi_1^k).$$

Using again [Dar15, Corollary 4.14] and the triangle inequality we get

$$d_{p,\varepsilon}(\varphi_0^j,\varphi_1^k) \le d_{p,\varepsilon'}(\varphi_0^j,\varphi_1^k) + 2d_{p,\varepsilon'}(\varphi_0^j \vee_\varepsilon \varphi_1^k, \varphi_0^j \vee_{\varepsilon'} \varphi_1^k).$$

Moreover Lemma 3.3 yields $d_{p,\varepsilon'}(\varphi_0^j \vee_\varepsilon \varphi_1^k, \varphi_0^j \vee_{\varepsilon'} \varphi_1^k) \le ||\varphi_0^j \vee_{\varepsilon'} \varphi_1^k - \varphi_0^j \vee_\varepsilon \varphi_1^k||_{L^\infty} \le (\varepsilon' - \varepsilon)$, where the last inequality follows from the fact that $\varphi_0^j \vee_\varepsilon \varphi_1^k, \varphi_0^j \vee_{\varepsilon'} \varphi_1^k$ are continuos functions.

Thus letting j,k go to $+\infty$ we infer that the function $\varepsilon \to d_{p,\omega_\varepsilon}(\varphi_0,\varphi_1) + \varepsilon$ is increasing. Hence the limit exists. Now, let $\omega_X, \widetilde\omega_X$ be two Kähler metrics on X such that

$$\omega_X \le \widetilde\omega_X \le C\omega_X$$

for some $C > 0$. Assume that φ_0, φ_1 are smooth ω_ε-psh functions such that $\varphi_0 \le \varphi_1$. Set $\widetilde\omega_\varepsilon := \omega + \varepsilon\widetilde\omega_X$ and observe that $\omega_\varepsilon \le \widetilde\omega_\varepsilon \le \omega_{\varepsilon'}$ where $\varepsilon' = \varepsilon C$. Let $\varphi_t^\varepsilon, \tilde\varphi_t^\varepsilon$ be the geodesic w.r.t. ω_ε and $\widetilde\omega_\varepsilon$, respectively and observe that $\varphi_t^\varepsilon \le \tilde\varphi_t^\varepsilon \le \varphi_t^{\varepsilon'}$. The same arguments of above give

$$|\dot\varphi_0^\varepsilon|^p \le |\dot{\tilde\varphi}_0^\varepsilon|^p \le |\dot\varphi_0^{\varepsilon'}|^p$$

hence

$$d_{p,\omega_\varepsilon}(\varphi_0,\varphi_1) \le d_{p,\widetilde\omega_\varepsilon}(\varphi_0,\varphi_1) \le d_{p,\omega_{\varepsilon'}}(\varphi_0,\varphi_1).$$

The latter tells us that the limit does not depend on ω_X. The general case, i.e. without the asspumption $\varphi_0 \leq \varphi_1$, can be treated using Pythagore formula as above. $\square$

An adaptation of the classical Perron envelope technique yields the following result due to Berndtsson [Bern15]:

Proposition 1.16 *Assume φ_0, φ_1 are bounded ω-psh functions. Then*

$$\varphi(x,z) := \sup\{u(x,z) \,|\, u \in PSH(X \times S, \omega) \text{ with } \lim_{t\to 0,1} u \leq \varphi_{0,1}\}.$$

is the unique bounded ω-psh function on $X \times S$, which is the solution of the Dirichlet problem $\varphi_{|X\times\partial S} = \varphi_{0,1}$ with

$$(\omega + dd^c_{x,z}\varphi)^{n+1} = 0 \text{ in } X \times S.$$

Moreover $\varphi(x,z) = \varphi(x,t)$ only depends on $\Re(z)$ and $|\dot{\varphi}| \leq \|\varphi_1 - \varphi_0\|_{L^\infty(X)}$.

The proof goes exactly as that of Proposition 1.4. The function φ (or rather the path $\varphi_t \subset PSH(X,\omega) \cap L^\infty(X)$) is called a *bounded geodesic* in [Bern15]. We use the same terminology here, as it turns out that bounded geodesics are geodesics in the metric sense:

Proposition 1.17 *Bounded geodesics are metric geodesics. More precisely, if φ_0, φ_1 are bounded ω-psh functions and $\varphi(x,z) = \varphi_t(x)$ is the bounded geodesic joining φ_0 to φ_1, then for all $t, s \in [0,1]$,*

$$d_p(\varphi_t, \varphi_s) = |t-s|\, d_p(\varphi_0, \varphi_1).$$

Proof. Let $\varphi_0^j, \varphi_1^k \in \mathcal{H}_{\omega_\varepsilon}$ be sequences decreasing respectively to φ_0, φ_1. It follows from the comparison principle and the uniqueness in Proposition 1.16 that $\varphi_{t,j}$ decreases to φ_t as j increases to $+\infty$. From Definition 1.14, Proposition 1.15 and the fact that the identity below holds in the Kähler setting for d_ε we obtain

$$\begin{aligned} d_p(\varphi_t, \varphi_s) &= \liminf_{\varepsilon\to 0} \liminf_{j,k\to+\infty} d_{p,\varepsilon}(\varphi_{t,j}, \varphi_{s,k}) \\ &= |t-s| \liminf_{\varepsilon\to 0} \liminf_{j,k\to+\infty} d_{p,\varepsilon}(\varphi_0^j, \varphi_1^k) = |t-s| d_p(\varphi_0, \varphi_1). \end{aligned}$$

$\square$

Remark 1.18 *One can no longer expect that $d_p(\varphi_0,\varphi_1)^p = \int_X |\dot{\varphi}_t|^p\, MA(\varphi_t)$ for a.e. $t \in [0,1]$ as simple examples show. One can e.g. take $\varphi_0 \equiv 0$ and $\varphi_1 = \max(u, 0)$, where u takes positive values, has isolated singularities and solves $MA(u)$ =Dirac mass at some point: in this case $MA(\varphi_1)$ is concentrated on the contact set $(u = 0)$ while $\dot{\varphi}_1 \equiv 0$ on this set hence $\int_X |\dot{\varphi}_1|^p\, MA(\varphi_1) = 0$. We thank T.Darvas for pointing this to us.*

As the above remark points out we do not have that $d_p^p(\varphi_0,\varphi_1) = I(0) = I(1)$ when φ_0, φ_1 are just bounded ω-psh functions. Nevertheless we can still recover the formula in some special cases.

We start recalling the following:

Theorem 1.19 *Let f be a continuous function such that $dd^c f \leq C\omega_X$ on X, for some $C > 0$. Then $P(f)$ has bounded laplacian on* $\mathrm{Amp}\,(\{\omega\})$ *and*

$$(\omega + dd^c P_\omega(f))^n = \mathbf{1}_{\{P_\omega(f)=f\}}(\omega + dd^c f)^n. \tag{4}$$

The fact that $P(f)$ has locally bounded laplacian in $\mathrm{Amp}(\{\omega\})$ is essentially [Ber, Theorem 1.2]. We do not assume here that f is smooth but one can check that the upper bound on $dd^c f$ is the only estimate needed in order to pursue Berman's approach. One can then argue as in [GZ17, Theorem 9.25] to get identity (4).

Denote

$$\mathcal{H}_{bd} := \{\varphi \in \mathrm{PSH}(X,\omega) \cap L^\infty(X),\ \varphi = P_\omega(f) \text{ for some } f \in C^0(X) \text{ with } dd^c f \leq C\omega_X,\ C > 0\}.$$

Theorem 1.20 *Assume that $\varphi_0, \varphi_1 \in \mathcal{H}_{bd}$. Let φ_t be the Mabuchi geodesic joining φ_0 and φ_1. Then*

$$d_p^p(\varphi_0,\varphi_1) = \int_X |\dot\varphi_0|^p \frac{(\omega + dd^c\varphi_0)^n}{V} = \int_X |\dot\varphi_1|^p \frac{(\omega + dd^c\varphi_1)^n}{V}. \tag{5}$$

Proof. Set $\varphi_{0,\varepsilon} := P_{\omega_\varepsilon}(f_0)$ and $\varphi_{1,\varepsilon} := P_{\omega_\varepsilon}(f_1)$. Clearly $\varphi_{i,\varepsilon}$ decreases pointwise to φ_i, $i = 1, 2$. Combining Chen's formula together with (4) we get

$$V_\varepsilon d_{p,\varepsilon}^p(\varphi_{0,\varepsilon},\varphi_{1,\varepsilon}) = \int_X |\dot\varphi_{0,\varepsilon}|^p (\omega_\varepsilon + dd^c\varphi_{0,\varepsilon})^n = \int_{\{\varphi_{0,\varepsilon}=f_0\}} |\dot\varphi_{0,\varepsilon}|^p (\omega_\varepsilon + dd^c f_0)^n.$$

Denote $U_\varepsilon := \{\varphi_0 < \varphi_{0,\varepsilon}\}$ and note that $\{\varphi_{0,\varepsilon} = f_0\} \subset \{\varphi_0 < \varphi_{0,\varepsilon}\} \cup \{\varphi_0 = f_0\}$. Therefore

$$\left| V_\varepsilon d_{p,\varepsilon}(\varphi_{0,\varepsilon},\varphi_{1,\varepsilon}) - \int_X |\dot\varphi_0|^p \omega_{\varphi_0}^n \right| \leq \left| \int_{\{\varphi_0=f_0\}} |\dot\varphi_{0,\varepsilon}|^p (\omega_\varepsilon + dd^c f_0)^n - \int_{\{\varphi_0=f_0\}} |\dot\varphi_0|^p (\omega + dd^c f_0)^n \right| + C\int_{U_\varepsilon} \omega_X^n$$

where $C > 0$ is such that

$$\int_{U_\varepsilon} |\dot\varphi_{0,\varepsilon}|^p (\omega_\varepsilon + dd^c f_0)^n \leq C \int_{U_\varepsilon} \omega_X^n.$$

The first term can be shown to converge to zero arguing as in Theorem 1.13. The second term goes to zero since $\varphi_{0,\varepsilon}$ converges pointwise to φ_0. Hence the conclusion. □

Observe that if $\varphi_0, \varphi_1 \in \mathcal{H}_\omega$, then $\varphi_0 \vee \varphi_1 \in \mathcal{H}_{bd}$. Indeed since φ_0, φ_1 are smooth, the functions $-\varphi_0, -\varphi_1$ are quasi-plurisubharmonic, i.e. there exists $C > 0$ such that $dd^c(-\varphi_i) \geq -C\omega_X$ for any $i = 1, 2$. Thus $\min(\varphi_0,\varphi_1) = -\max(-\varphi_0,-\varphi_1)$ is such that

$$dd^c \min(\varphi_0,\varphi_1) = -dd^c \max(-\varphi_0,-\varphi_1) \leq C\omega_X.$$

In particular the equality (5) holds for $d_p(\varphi_0, \varphi_0 \vee \varphi_1)$ and $d_p(\varphi_1, \varphi_0 \vee \varphi_1)$.

2. Finite energy classes

We define in this section the set $\mathcal{E}(\alpha)$ (resp. $\mathcal{E}^p(\alpha)$) of positive closed currents $T = \omega + dd^c\varphi$ with full Monge-Ampère mass (resp. finite weighted energy) in α, by defining the corresponding class $\mathcal{E}(X,\omega)$ (resp. $\mathcal{E}^p(X,\omega)$) of finite energy potentials φ.

2.1 The space $\mathcal{E}(\alpha)$

2.1.1 *Bounded quasi-plurisubharmonic functions* Recall that a function is quasi-plurisubharmonic if it is locally given as the sum of a smooth and a psh function. In particular quasi-psh (*qpsh* for short) functions are upper semi-continuous and integrable. They are actually in L^p for

all $p \geq 1$, and the induced topologies are all equivalent. A much stronger integrability property actually holds: Skoda's integrability theorem [Sko72] asserts indeed that $e^{-\varepsilon\varphi} \in L^1(X)$ if $0 < \varepsilon$ is smaller than $2/\nu(\varphi)$, where $\nu(\varphi)$ denotes the maximal logarithmic singularity (Lelong number) of φ on X.

Quasi-plurisubharmonic functions have gradient in L^r for all $r < 2$, but not in L^2 as shown by the local model $\log|z_1|$.

Definition 2.1 *We let $PSH(X,\omega)$ denote the set of all ω-plurisubharmonic functions. These are quasi-psh functions $\varphi : X \to \mathbf{R} \cup \{-\infty\}$ such that*

$$\omega + dd^c\varphi \geq 0$$

in the weak sense of currents.

The set $PSH(X,\omega)$ is a closed subset of $L^1(X)$, for the L^1-topology.

Bedford and Taylor have observed in [BT82] that one can define the complex Monge-Ampère operator

$$MA(\varphi) := {V_\alpha}^{-1}(\omega + dd^c\varphi)^n$$

for all *bounded* ω-psh function: they showed that whenever (φ_j) is a sequence of smooth ω-psh functions locally decreasing to φ, then the smooth probability measures $MA(\varphi_j)$ converge, in the weak sense of Radon measures, towards a unique probability measure that we denote by $MA(\varphi)$.

At the heart of Bedford-Taylor's theory lies the following *maximum principle*: if u, v are bounded ω-plurisubharmonic functions, then

$$(MP) \qquad 1_{\{v<u\}} MA(\max(u,v)) = 1_{\{v<u\}} MA(u).$$

This equality is elementary when u is *continuous*, as the set $\{v < u\}$ is open. When u is merely *bounded*, this set is only open in the plurifine topology. Since Monge-Ampère measures of bounded qpsh functions do not charge pluripolar sets (by the Chern-Levine-Nirenberg inequalities), and since u is nevertheless *quasi-continuous*, this gives a heuristic justification for (MP).

2.1.2 *The class $\mathcal{E}(X,\omega)$* Given $\varphi \in PSH(X,\omega)$, we consider

$$\varphi_j := \max(\varphi, -j) \in PSH(X,\omega) \cap L^\infty(X).$$

It follows from the Bedford-Taylor theory that the $MA(\varphi_j)$'s are well defined probability measures. Since the φ_j's are decreasing, it is natural to expect that these measures converge. The following monotonicity property holds:

Lemma 2.2 *The sequence $\mu_j := \mathbf{1}_{\{\varphi > -j\}} MA(\varphi_j)$ is increasing.*

The proof is an elementary consequence of (MP) (see [GZ07, p.445]).

Remark 2.3 *Note : $t \mapsto \max(\varphi(x), -t)$ is a subgeodesic (Definition 1.7).*

Since the μ_j's all have total mass bounded from above by 1, we consider

$$\mu_\varphi := \lim_{j \to +\infty} \mu_j,$$

which is a positive Borel measure on X, with total mass ≤ 1.

Definition 2.4 *We set*

$$\mathcal{E}(X,\omega) := \{\varphi \in PSH(X,\omega) \mid \mu_\varphi(X) = 1\}.$$

For $\varphi \in \mathcal{E}(X,\omega)$, we set $MA(\varphi) := \mu_\varphi$.

The notation is justified by the following important fact [GZ07]:

Theorem 2.5 *The complex Monge-Ampère operator $\varphi \mapsto MA(\varphi)$ is well defined on the class $\mathcal{E}(X,\omega)$: for every decreasing sequence of bounded ω-psh functions φ_j, the measures $MA(\varphi_j)$ converge towards μ_φ, if $\varphi \in \mathcal{E}(X,\omega)$.*

Every bounded ω-psh function clearly belongs to $\mathcal{E}(X,\omega)$ since in this case $\{\varphi > -j\} = X$ for j large enough, hence $\mu_\varphi \equiv \mu_j = MA(\varphi_j) = MA(\varphi)$. The class $\mathcal{E}(X,\omega)$ also contains many ω-psh functions which are unbounded.

Example 2.6 *If $\varphi \in PSH(X,\omega)$ is normalized so that $\varphi \leq -1$, then $-(-\varphi)^\varepsilon$ belongs to $\mathcal{E}(X,\omega)$ whenever $0 \leq \varepsilon < 1$. The functions which belong to the class $\mathcal{E}(X,\omega)$, although usually unbounded, have relatively mild singularities. In particular they have zero Lelong numbers on* $\mathrm{Amp}\,(\alpha)$.

It is shown in [GZ07] that the maximum principle (MP) continue to hold in the class $\mathcal{E}(X,\omega)$. The latter can be characterized as the largest class for which the complex Monge-Ampère is well defined and the maximum principle holds. We further note that the *domination principle* holds:

Proposition 2.7 *If $\varphi, \psi \in \mathcal{E}(X,\omega)$ are such that*

$$\varphi(x) \leq \psi(x) \text{ for } MA(\psi) - a.e.\ x,$$

then $\varphi(x) \leq \psi(x)$ for all $x \in X$.

It follows from the $\partial\overline{\partial}$-lemma that any positive closed current $T \in \alpha$ can be written $T = \omega + dd^c\varphi$ for some function $\varphi \in PSH(X,\omega)$ which is unique up to an additive constant.

Definition 2.8 *We let $\mathcal{E}(\alpha)$ denote the set of all positive currents in α, $T = \omega + dd^c\varphi$, with $\varphi \in \mathcal{E}(X,\omega)$.*

The definition does not depend on the choice of ω, nor on the choice of φ.

2.2 The class $\mathcal{E}^1(X,\omega)$

2.2.1 *The Aubin-Mabuchi functional* Each tangent space $T_\varphi\mathcal{H}$ admits the following orthogonal decomposition

$$T_\varphi\mathcal{H} = \{\psi \in C^\infty(X);\ \beta_\varphi(\psi) = 0\} \oplus \mathbf{R},$$

where $\beta = MA$ is the 1-form defined on $\mathcal{H}$ by

$$\beta_\varphi(\psi) = \int_X \psi\, MA(\varphi).$$

It is a classical observation due to Mabuchi that the 1-form β is closed. Therefore there exists a unique function E defined on the convex open set $\mathcal{H}$, such that $\beta = dE$ and $E(0) = 0$. It is often called the *Aubin-Mabuchi functional* and can be expressed (after integration along affine paths) by

$$E(\varphi) = \frac{1}{(n+1)V_\alpha} \sum_{j=0}^{n} \int_X \varphi\,(\omega + dd^c\varphi)^j \wedge \omega^{n-j}.$$

Lemma 2.9 *The Aubin-Mabuchi functional E is concave along euclidean segments, increasing, and satisfies the cocycle condition*

$$E(\varphi) - E(\psi) = \frac{1}{(n+1)V_\alpha} \sum_{j=0}^{n} \int_X (\varphi - \psi)\, (\omega + dd^c\varphi)^j \wedge (\omega + dd^c\psi)^{n-j}$$

It is affine along geodesics and convex along subgeodesics in $\mathcal{H}$.

Proof. These properties are well-known when ω is a Kähler form.

The monotonicity property follows from the definition since the first derivative of E is $dE = \beta = MA \geq 0$, a probability measure: if φ_t is an arbitrary path, then

$$\frac{d}{dt} E(\varphi_t) = \int_X \dot{\varphi}_t MA(\varphi_t).$$

It follows from Stokes theorem that

$$\begin{aligned} \frac{d^2}{dt^2} E(\varphi_t) &= \int_X \ddot{\varphi}_t\, MA(\varphi_t) + \frac{n}{V_\alpha} \int_X \dot{\varphi}_t\, dd^c\dot{\varphi}_t \wedge \omega_\varphi^{n-1} \\ &= \int_X \left\{ \ddot{\varphi}_t\, MA(\varphi_t) - \frac{n}{V_\alpha} d\dot{\varphi}_t \wedge d^c\dot{\varphi}_t \wedge \omega_{\varphi_t}^{n-1} \right\}. \end{aligned}$$

Thus E is concave along euclidean segments ($\ddot{\varphi}_t = 0$), affine along Mabuchi geodesics, and convex along Mabuchi subgeodesics. The cocycle condition follows by differentiating $E(t\varphi + (1-t)\psi)$.

These computations are mereley heuristic as $t \to \varphi_t(x)$ is poorly regular when φ_t is a geodesic or a subgeodesic. We can however approximate ω by $\omega_\varepsilon = \omega + \varepsilon\omega_X$, consider (φ_t^ε) the corresponding geodesic and

$$E_{\omega_\varepsilon}(\varphi_t^\varepsilon) = \frac{1}{(n+1)V_\varepsilon} \sum_{j=0}^{n} \int_X \varphi_t^\varepsilon (\omega_\varepsilon + dd^c\varphi_t^\varepsilon) \wedge \omega_\varepsilon^{n-j}.$$

It follows from Proposition 1.6 that $\varepsilon \mapsto \varphi_t^\varepsilon$ decreases to φ_t, hence $t \mapsto E(\varphi_t)$ is affine, being the limit of the affine maps $t \mapsto E_{\omega_\varepsilon}(\varphi_t^\varepsilon)$.

For subgeodesics we approximate again ω by ω_ε and we proceed as in the Kähler case. □

Observe that $E(\varphi + t) = E(\varphi) + t$. Given $\varphi \in \mathcal{H}$ there exists a unique $c \in \mathbf{R}$ such that $E(\varphi + c) = 0$. The restriction of the Mabuchi metric to the fiber $E^{-1}(0)$ induces a Riemannian structure on the quotient space $\mathcal{H}_\alpha = \mathcal{H}/\mathbf{R}$ and allows to decompose $\mathcal{H} = \mathcal{H}_\alpha \times \mathbf{R}$ as a product of Riemannian manifolds.

Definition 2.10 *For $\varphi \in PSH(X, \omega)$, we set*

$$E(\varphi) := \inf\{E(\psi)\,;\, \varphi \leq \psi \text{ and } \psi \in PSH(X,\omega) \cap L^\infty(X)\} \in [-\infty, +\infty[$$

and $\mathcal{E}^1(X, \omega) := \{\varphi \in PSH(X, \omega)\,;\, E(\varphi) > -\infty\}$.

Remark 2.11 *The functional E can be used to characterize the class $\mathcal{E}(X, \omega)$. For $\varphi \in PSH(X, \omega)$, we set $\varphi_t = \max(\varphi, -t)$. Observe that $t \mapsto E(\varphi_t)$ is convex since $t \mapsto \varphi_t$ is a subgeodesic ray and $E(\varphi_t) = O(t)$. Moreover $E(\varphi_t) = O(1)$ if and only if $\varphi \in \mathcal{E}^1(X, \omega)$. Following Darvas [Dar13] we now claim that $\varphi \in \mathcal{E}(X, \omega) \iff E(\varphi_t) = o(t)$.*

We provide an alternative proof of independent interest. Observe that

$$\int_X \varphi\, \omega_\varphi^{j+1} \wedge \omega^{n-j-1} = \int_X \varphi\, \omega_\varphi^j \wedge \omega^{n-j} + \int_X \varphi\, dd^c\varphi \wedge \omega_\varphi^j \wedge \omega^{n-j-1}$$
$$\leq \int_X \varphi\, \omega_\varphi^j \wedge \omega^{n-j},$$

since $\int_X \varphi\, dd^c\varphi \wedge \omega_\varphi^j \wedge \omega^{n-j-1} = -\int_X d\varphi \wedge d^c\varphi \wedge \omega_\varphi^j \wedge \omega^{n-j-1} \leq 0$. *For* $\varphi \leq 0$, *we infer* $\int_X \varphi\, MA(\varphi) \leq E(\varphi) \leq (n+1)^{-1} \int_X \varphi\, MA(\varphi)$ *so our claim is equivalent to showing that* $\varphi \in \mathcal{E}(X,\omega) \Longleftrightarrow t^{-1} \int_X \varphi_t MA(\varphi_t) \to 0$. *Observe now that*

$$t^{-1} \int_X \varphi_t MA(\varphi_t) = -MA(\varphi_t)(\varphi \leq -t) + t^{-1} \int_{(\varphi > -t)} \varphi\, d\mu_\varphi.$$

Since $\mu_\varphi(\varphi = -\infty) = 0$, *there exists* $\chi : \mathbf{R} \to \mathbf{R}$, *a convex increasing function such that* $\chi(-\infty) = -\infty$ *and* $\chi \circ \varphi \in L^1(\mu_\varphi)$. *Therefore* $t^{-1} \int_{\{\varphi > -t\}} \varphi\, d\mu_\varphi = O(\chi(-t)^{-1}) \to 0$, *hence*

$$\int_X \varphi_t MA(\varphi_t) = o(t) \Longleftrightarrow MA(\varphi_t)(\varphi \leq -t) = o(1) \Longleftrightarrow \varphi \in \mathcal{E}(X,\omega).$$

2.2.2 *Strong topology on* $\mathcal{E}^1(\alpha)$ Set

$$I(\varphi,\psi) = \int_X (\varphi - \psi)\left(MA(\psi) - MA(\varphi)\right).$$

It has been shown in [BBEGZ] that I defines a complete metrizable uniform structure on $\mathcal{E}^1(\alpha)$. More precisely we identify $\mathcal{E}^1(\alpha)$ with the set

$$\mathcal{E}^1_{norm}(X,\omega) = \{\varphi \in \mathcal{E}^1(X,\omega) \mid \sup_X \varphi = 0\}$$

of normalized potentials. Then

- I is symmetric and positive on $\mathcal{E}^1_{norm}(X,\omega)^2 \setminus \{\text{diagonal}\}$;
- I satisfies a quasi-triangle inequality [BBEGZ, Theorem 1.8];
- I induces a uniform structure which is metrizable [Bourbaki];
- the metric space $(\mathcal{E}^1(\alpha), d_I)$ is complete [BBEGZ, Proposition 2.4], where d_I denotes one of the distances induced by the uniform structure I.

Definition 2.12 *The strong topology on* $\mathcal{E}^1(\alpha)$ *is the metrizable topology defined by* I.

The corresponding notion of convergence is the *convergence in energy* previously introduced in [BBGZ13] (see [BBEGZ, Proposition 2.3]). It is the coarsest refinement of the weak topology such that E becomes continuous. In particular if $T_j \longrightarrow T$ in $(\mathcal{E}^1(\alpha), d_I)$, then

$$T_j \longrightarrow T \text{ weakly and } T_j^n \longrightarrow T^n$$

in the weak sense of Radon measures, while the Monge-Ampère operator is usually discontinuous for the weak topology of currents.

Example 2.13 *When* $\dim_{\mathbf{C}} X = n = 1$, $\mathcal{E}^1(X,\omega) = PSH(X,\omega) \cap W^{1,2}(X)$ *is the set of* ω-*subharmonic functions with square integrable gradient. The strong topology on* $\mathcal{E}^1(\alpha)$ *is the one induced by the Sobolev norm.*

2.2.3 *Yet another distance* To fit in with the notations of the next section, we introduce yet another notion of convergence in $\mathcal{E}^1(X,\omega)$. We set

$$I_1(\varphi,\psi) := \int_X |\varphi-\psi| \left[\frac{MA(\varphi)+MA(\psi)}{2}\right]$$

This symmetric quantity is non-negative. It follows from the Proposition 2.7 that it only vanishes on the diagonal of $\mathcal{E}^1(X,\omega)^2$, while Theorem 3.6 will insure that it satisfies a quasi-triangle inequality. For $C>0$, we set

$$\mathcal{E}^1_C(X,\omega) := \{\varphi \in \mathcal{E}^1(X,\omega)\,;\, E(\varphi) \geq -C \text{ and } \varphi \leq C\}.$$

It follows from Hartogs lemma, the upper-semi continuity and the concavity of E along euclidean segments (Lemma 2.9) that this set is a compact and convex subset of $PSH(X,\omega)$, when endowed with the L^1-topology (see [BBGZ13, Lemma 2.6]).

Proposition 2.14 *For all $\varphi,\psi \in \mathcal{E}^1(X,\omega)$, $I(\varphi,\psi) \leq 2I_1(\varphi,\psi)$. Conversely for each $C>0$, there exists $A>0$ such that for all $\varphi,\psi \in \mathcal{E}^1_C(X,\omega)$*

$$I_1(\varphi,\psi) \leq \int_X [2\max(\varphi,\psi)-(\varphi+\psi)]\, MA(0) + A\, I(\varphi,\psi)^{1/2^n}.$$

In particular the distances induced by I, I_1 on $\mathcal{E}^1_{norm}(X,\omega)$ are equivalent.

Observe that I_1 induces a distance on $\mathcal{E}^1(X,\omega)$, but I is merely defined on $\mathcal{E}^1_{norm}(X,\omega)$, as $I(\varphi+c,\psi+c') = I(\varphi,\psi)$, for any $c,c' \in \mathbf{R}$.

Proof. The first inequality is obvious, as

$$I(\varphi,\psi) = \int_X (\varphi-\psi)\,(MA(\psi)-MA(\varphi)) \leq \int_X |\varphi-\psi|\,(MA(\psi)+MA(\varphi)).$$

It follows from Proposition 2.19 below that

$$I_1(\varphi,\psi) = I_1(\varphi,\max(\varphi,\psi)) + I_1(\max(\varphi,\psi),\psi),$$

hence it suffices to establish the second inequality when $\varphi \leq \psi$. In this case

$$I_1(\varphi,\psi) \leq \int_X (\psi-\varphi) MA(\varphi),$$

by Lemma 2.18, while Cauchy-Schwarz inequality yields

$$\begin{aligned}
&\int_X (\psi-\varphi)MA(\varphi)\\
&= \int_X (\psi-\varphi)MA(0) + \int_X d(\varphi-\psi)\wedge d^c\varphi \wedge S_\varphi\\
&\leq \int_X (\psi-\varphi)MA(0) + I(\varphi,0)^{1/2}\left(\int_X d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge S_\varphi\right)^{1/2},
\end{aligned}$$

where we have set $S_\varphi := \sum_{j=0}^{n-1} \omega_\varphi^j \wedge \omega^{n-1-j}$. Observing that $S_\varphi \leq 2^{n-1}\omega^{n-1}_{\varphi/2}$, we can invoke [BBEGZ, Lemma 1.9] to obtain

$$\int_X d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge S_\varphi \leq c_n I(\varphi,\psi)^{1/2^{n-1}}\left\{I\left(\varphi,\frac{\varphi}{2}\right)^{1-1/2^{n-1}} + I\left(\psi,\frac{\varphi}{2}\right)^{1-1/2^{n-1}}\right\}.$$

Now $I(\varphi,\varphi/2)\leq a_n I(\varphi,0)\leq C'$ and [BBEGZ, Theorem 1.3] yields

$$I(\psi,\varphi/2)\leq b_n\left\{I(\psi,0)+I(\varphi/2,0)\right\}\leq b_n'\left\{I(\psi,0)+I(\varphi,0)\right\}\leq C''.$$

The conclusion follows. □

2.3 The complete metric spaces $\mathcal{E}^p(\alpha)$

Fix $p\geq 1$. Following [GZ07, BEGZ10] we consider the following finite energy classes:

Definition 2.15 *We set*

$$\mathcal{E}^p(X,\omega):=\left\{\varphi\in\mathcal{E}(X,\omega)\,/\,|\varphi|^p\in L^1(MA(\varphi))\right\}$$

and let $\mathcal{E}^p(\alpha)=\{T=\omega+dd^c\varphi\,|\,\varphi\in\mathcal{E}^p(X,\omega)\}$ *denote the corresponding sets of finite energy currents.*

We introduce a strong topology on the class $\mathcal{E}^p(\alpha)$, $p\geq 1$, by setting

$$I_p(\varphi,\psi):=\left(\int_X|\varphi-\psi|^p\left[\frac{MA(\varphi)+MA(\psi)}{2}\right]\right)^{1/p}$$

This quantity is well-defined by [GZ07, Proposition 3.6]. It is obviously non-negative and symmetric. It follows from the domination principle (Proposition 2.7) that

$$I_p(\varphi,\psi)=0\Longrightarrow\varphi=\psi.$$

Definition 2.16 *The strong topology on* $\mathcal{E}^p(\alpha)$ *is the one induced by* I_p.

By [BEGZ10, Theorem 2.17], a decreasing sequence converges strongly. We also have good convergence properties if we approximate by slightly larger finite energy classes $\mathcal{E}^p(X,\omega_\varepsilon)$:

Proposition 2.17 *Fix* $\omega_\varepsilon=\omega+\varepsilon\omega_X$, $\varepsilon>0$. *If* $\varphi,\psi\in\mathcal{E}^p(X,\omega)\cap L^\infty(X)$, *then* $\varphi,\psi\in\mathcal{E}^p(X,\omega_\varepsilon)\cap L^\infty(X)$ *and* $I_{p,\omega_\varepsilon}(\varphi,\psi)\to I_{p,\omega}(\varphi,\psi)$ *as* $\varepsilon\to 0$.

Moreover, if $\varphi,\psi\in\mathcal{E}^p(X,\omega)$ *and* φ_j,ψ_j *are sequences of smooth* ω_{ε_j}*-psh functions decreasing to* φ,ψ *with* $\varepsilon_j\to 0$, *then*

$$I_{p,\omega_{\varepsilon_j}}(\varphi_j,\psi_j)\to I_{p,\omega}(\varphi,\psi)$$

as j *goes to* $+\infty$.

Proof. The first assertion follows from the fact that $(\omega_\varepsilon+dd^c\varphi)^n$ and $(\omega_\varepsilon+dd^c\psi)^n$ converges weakly to $(\omega+dd^c\varphi)^n$ and $(\omega+dd^c\psi)^n$ as $\varepsilon\to 0$, respectively. For the second statement, we observe that by symmetry it suffices to prove that

$$\int_X|\varphi_j-\psi_j|^p(\omega_{\varepsilon_j}+dd^c\varphi_j)^n\to\int_X|\varphi-\psi|^p(\omega+dd^c\varphi)^n,\quad\text{as } j\to+\infty.$$

Given a bounded function f on X, we set

$$|f|_p:=\left(\int_X|f|^p(\omega_{\varepsilon_j}+dd^c\varphi_j)^n\right)^{1/p}.$$

The triangle inequality yields

$$|\varphi_j-\psi_j|_p\leq|\varphi-\psi|_p+|(\varphi_j-\varphi)|+|(\psi-\psi_j)|_p$$

and similarly

$$|\varphi_j-\psi_j|_p\geq|\varphi-\psi|_p-|(\varphi_j-\varphi)|-|(\psi-\psi_j)|_p.$$

Since $\varphi-\psi$ is a quasi-continuous function on X, it follows from the continuity of the Monge-Ampère operator along decreasing sequence [GZ07, Theorem 1.9] and [Kol05, Corollary 1.14] that

$$|\varphi-\psi|_p^p=\int_X|\varphi-\psi|^p(\omega_{\varepsilon_j}+dd^c\varphi_j)^n\to\int_X|\varphi-\psi|^p(\omega+dd^c\varphi)^n$$

as $j\to+\infty$. Moreover, we claim that the terms $|(\varphi_j-\varphi)|_p$ and $|(\psi-\psi_j)|_p$ goes to 0 as $j\to+\infty$. Lemma 2.18 together with the fact that $\omega_{\varepsilon_j}\leq\omega+\omega_X$ yields

$$\int_X(\varphi_j-\varphi)^p(\omega_{\varepsilon_j}+dd^c\varphi_j)^n\leq\int_X(\varphi_j-\varphi)^p(\omega+\omega_X+dd^c\varphi)^n.$$

Since φ_j is decreasing to φ, it then follows from the dominated convergence theorem that $|(\varphi_j-\varphi)|_p^p\to 0$ as $j\to+\infty$. Fix $j_0<j$. Then

$$\int_X(\psi_j-\psi)^p(\omega_{\varepsilon_j}+dd^c\varphi_j)^n\leq\int_X(\psi_{j_0}-\psi)^p(\omega+\omega_X+dd^c\varphi_j)^n.$$

It follows again from the continuity of the Monge-Ampère operator along decreasing sequence, [Kol05, Corollary 1.14] and the dominated convergence theorem that letting $j\to+\infty$ and then $j_0\to+\infty$ we get

$$\int_X(\psi_{j_0}-\psi)^p(\omega+\omega_X+dd^c\varphi_j)^n\to 0.$$

Thus $|(\psi_j-\psi)|_p^p\to 0$ as $j\to+\infty$. Hence the conclusion. □

It follows from Hölder inequality that the strong topology on $\mathcal{E}^p(\alpha)$ is stronger than the one on $\mathcal{E}^1(\alpha)$: if a sequence $(\varphi_j)\in\mathcal{E}^p(X,\omega)$ is a Cauchy sequence for I_p, then it is a Cauchy sequence in $(\mathcal{E}^1(X,\omega),d_I)$ since

$$0\leq I(\varphi,\psi)=\int_X(\varphi-\psi)\left[MA(\psi)-MA(\varphi)\right]\leq 2^{1/p}I_p(\varphi,\psi).$$

Since $(\mathcal{E}^1(X,\omega),d_I)$ is complete, there is $\varphi\in\mathcal{E}^1(X,\omega)$ s.t. $d_I(\varphi_j,\varphi)\to 0$. Now $I_p(\varphi_j,0)$ is bounded and $MA(\varphi_j)$ converges to $MA(\varphi)$ (by [BBGZ13, Proposition 5.6]). Thus $\varphi\in\mathcal{E}^p(X,\omega)$ by Fatou's and Hartogs' lemma.

One would now like to prove that $I_p(\varphi_j,\varphi)\to 0$ and conclude that the space $(\mathcal{E}^p(X,\omega),I_p)$ is complete, arguing as in [BBEGZ, Proposition 2.4]. There is an abuse of terminology here as we haven't checked that I_p induces a uniform structure. This follows from Theorem 3.6 which shows in particular that I_p satisfies a quasi-triangle inequality (like I does, see [BBEGZ, Theorem 1.8]). We refer the reader to Theorem 4.2 for a neat treatment.

Lemma 2.18 *Let φ,ψ be bounded ω-psh functions and S be a positive closed current of bidimension $(1,1)$ on X. If $\varphi\leq\psi$, then*

$$\int_X(\psi-\varphi)^p\omega_\psi\wedge S\leq\int_X(\psi-\varphi)^p\omega_\varphi\wedge S.$$

In particular $V_\alpha^{-1}\int_X(\psi-\varphi)^p\omega_\psi^j\wedge\omega_\varphi^{n-j}\leq\int_X(\psi-\varphi)^pMA(\varphi)$.

Proof. By Stokes' theorem,

$$\int_X(\psi-\varphi)^p\omega_\varphi\wedge S-\int_X(\psi-\varphi)^p\omega_\psi\wedge S=p\int_X(\psi-\varphi)^{p-1}d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge S$$

is non-negative if $(\psi-\varphi)\geq 0$.

The second assertion follows by applying the first one inductively. □

We now establish a few useful properties of I_p that will notably allow to compare I_p to d_p in the next section.

Proposition 2.19 *For $\varphi, \psi \in \mathcal{E}^p(X, \omega)$,*

$$I_p(\varphi, \psi)^p = I_p(\varphi, \max(\varphi, \psi))^p + I_p(\max(\varphi, \psi), \psi)^p.$$

Proof. Recall that the maximum principle insures that

$$\mathbf{1}_{\{\varphi<\psi\}} MA(\max(\varphi, \psi)) = \mathbf{1}_{\{\varphi<\psi\}} MA(\psi),$$

while $(\varphi - \max(\varphi, \psi))^p = 0$ on $(\varphi \geq \psi)$, thus

$$2I_p(\varphi, \max(\varphi, \psi))^p = \int_{\{\varphi<\psi\}} |\varphi - \psi|^p \left[MA(\varphi) + MA(\psi)\right].$$

Similarly $2I_p(\psi, \max(\varphi, \psi))^p = \int_{\{\varphi>\psi\}} |\varphi - \psi|^p \left[MA(\varphi) + MA(\psi)\right]$ and the result follows since

$$I_p(\varphi, \psi)^p = \frac{1}{2} \int_{\{\varphi\neq\psi\}} |\varphi - \psi|^p \left[MA(\varphi) + MA(\psi)\right].$$

□

Corollary 2.20 *For all $\varphi, \psi \in \mathcal{E}^p(X, \omega)$,*

$$I_p\left(\frac{\varphi + \psi}{2}, \psi\right) \leq I_p(\varphi, \psi).$$

Proof. By approximating φ, ψ from above by a decreasing sequences, it suffices to treat the case when $\varphi, \psi \in \mathcal{H}_\omega$. Changing ω in ω_ψ, we can further assume that $\psi = 0$. It follows from Proposition 2.19 that

$$I_p\left(0, \varphi/2\right)^p = I_p(0, \max(0, \varphi/2))^p + I_p(\max(0, \varphi/2), \varphi/2)^p.$$

It follows from Lemma 2.18 that

$$\begin{aligned} I_p(0, \max(0, \varphi/2))^p &\leq \int_X \max(0, \varphi/2)^p MA(0) \\ &= 2^{-p} \int_X \max(0, \varphi)^p MA(0) \leq I_p(0, \max(0, \varphi))^p. \end{aligned}$$

We claim that for all $0 \leq j \leq n$,

$$\int_X (\max(0, \varphi) - \varphi)^p \omega_\varphi^j \wedge \omega^{n-j} \leq \int_X (\max(0, \varphi) - \varphi)^p \omega_\varphi^n.$$

Assuming this for the moment, it follows again from Lemma 2.18 that

$$\begin{aligned} I_p(\max(0, \varphi/2), \varphi/2)^p &\leq \int_X (\max(0, \varphi/2) - \varphi/2)^p MA(\varphi/2) \\ &= \frac{1}{2^{n+p} V_\alpha} \sum_{j=0}^n C_n^j \int_X (\max(0, \varphi) - \varphi)^p \omega_\varphi^j \wedge \omega^{n-j} \\ &\leq \frac{1}{2} \int_X (\max(0, \varphi) - \varphi)^p MA(\varphi) \leq I_p(\varphi, \max(0, \varphi))^p. \end{aligned}$$

We infer

$$I_p\left(0, \varphi/2\right)^p \leq I_p(0, \max(0, \varphi))^p + I_p(\max(0, \varphi), \varphi)^p = I_p(0, \varphi)^p,$$

by using Proposition 2.19 again.

It remains to justify our claim. Set $S = \omega^{j-1} \wedge \omega_\varphi^{n-j}$. It suffices, by induction, to establish the following inequality:

$$\begin{aligned}
&\int_X (\max(0,\varphi) - \varphi)^p \omega \wedge S \\
&= \int_X (\max(0,\varphi) - \varphi)^p \omega_\varphi \wedge S - \int_X (\max(0,\varphi) - \varphi)^p dd^c\varphi \wedge S \\
&\leq \int_X (\max(0,\varphi) - \varphi)^p \omega_\varphi \wedge S.
\end{aligned}$$

This follows by observing that

$$\begin{aligned}
-\int_X (\max(0,\varphi) - \varphi)^p dd^c\varphi \wedge S &= p \int_X (\max(0,\varphi) - \varphi)^{p-1} d(\max(0,\varphi) - \varphi) \wedge d^c\varphi \wedge S \\
&= -p \int_{\{\varphi<0\}} (-\varphi)^{p-1} d\varphi \wedge d^c\varphi \wedge S \leq 0.
\end{aligned}$$

□

3. Comparing distances

In this section we show that I_p is equivalent to d_p (Theorem 3.6). For notational convenience we let $\mathcal{H}$ denote the set $\mathcal{H}_{bd}$ defined in Section 1.2.

3.1 Kiselman transform and geodesics

Let $(\varphi_t)_{0\leq t\leq 1}$ be the Mabuchi geodesic. For all $x \in X$, $t \in [0,1] \mapsto \varphi_t(x) \in \mathbf{R}$ is convex. It is natural to consider its Legendre transform $u_s(x) : s \mapsto \sup_{t\in[0,1]}\{st - \varphi_t(x)\}$. This function is convex in s, but the dependence in x is $-\omega$-psh, so we rather consider $-u_s$. We finally change s in $-s$ to obtain a more elegant formula,

$$\psi_s(x) := \inf_{0\leq t\leq 1}\{st + \varphi_t(x)\}.$$

Proposition 3.1 *The functions $x \mapsto \psi_s(x)$ are ω-plurisubharmonic. In particular $x \mapsto \psi_0(x) = \inf_{0\leq t\leq 1} \varphi_t(x)$ is ω-psh.*

This is the minimum principle of Kiselman [Kis78]. For $\varphi_0, \varphi_1 \in \mathcal{H}$ we let $\varphi_0 \vee \varphi_1$ denote the greatest ω-psh function that lies below φ_0 and φ_1. In the notations of Berman-Demailly [BD12]

$$\varphi_0 \vee \varphi_1 = P(\min(\varphi_0, \varphi_1)),$$

while $\varphi_0 \vee \varphi_1$ is denoted by $P(\varphi_0, \varphi_1)$ in [Dar14].

An important consequence of Kiselman minimum principle [Kis78] is the following observation due to Darvas and Rubinstein [DR14]:

Proposition 3.2 *The function $\varphi_0 \vee \varphi_1$ is a bounded ω-psh which has locally bounded Laplacian on the ample locus of $\alpha = \{\omega\}$ and its Monge-Ampère measure $MA(\varphi_0 \vee \varphi_1)$ is supported on the coincidence set*

$$\{x \in X \mid \varphi_0 \vee \varphi_1(x) = \min(\varphi_0, \varphi_1)(x)\}.$$

Moreover $MA(\varphi_0 \vee \varphi_1) = \mathbf{1}_{\{\varphi_0\vee\varphi_1=\varphi_0\}} MA(\varphi_0) + \mathbf{1}_{\{\varphi_0\vee\varphi_1=\varphi_1<\varphi_0\}} MA(\varphi_1)$.

Let (φ_t) be the Mabuchi geodesic joining φ_0 and φ_1. Then for all $x \in X$,

$$\varphi_0 \vee \varphi_1(x) = \inf_{t\in[0,1]} \varphi_t(x).$$

Proof. It follows from a classical balayage procedure that goes back to Bedford and Taylor [BT82] that $MA(\varphi_0 \vee \varphi_1)$ is supported on the coincidence set $\{x \in X \,|\, \varphi_0 \vee \varphi_1(x) = \min(\varphi_0, \varphi_1)(x)\}$ This holds true more generally for the Monge-Ampère measure of any envelope, namely

$$\mathbf{1}_{\{P(h)<h\}} MA(P(h)) \equiv 0,$$

where h is a bounded lower semcontinuous function.

We have observed in Proposition 3.1 that $x \mapsto \inf_{t\in[0,1]} \varphi_t(x)$ is a ω-psh function. Since it lies both below φ_0 and φ_1, we infer

$$\inf_{t\in[0,1]} \varphi_t \leq \varphi_0 \vee \varphi_1.$$

Conversely $(t,x) \mapsto \varphi_0 \vee \varphi_1(x)$ is a subgeodesic (independent of t), hence for all t, x, $\varphi_0 \vee \varphi_1(x) \leq \varphi_t(x)$. Thus $\psi := \varphi_0 \vee \varphi_1 = \inf_{t\in[0,1]} \varphi_t$, hence ψ is bounded thanks to Proposition 1.4.

By Proposition 3.1, ψ is ω-psh, hence $A\omega_X$-psh for some Kähler form ω_X and $A > 0$. Thus $\sup_X \Delta_{\omega_X} \psi \geq -C$ for some $C > 0$.

It follows from the work of Berman and Demailly [BD12] that for any compact subset $K \subset \mathrm{Amp}\,(\alpha)$, there exists $C_K > 0$ such that for all $t \in [0,1]$,

$$\sup_K \Delta_{\omega_X} \varphi_t < C_K n.$$

Thus $(-\varphi_t)$ is a family of $C_K\omega_X$-psh functions in a neighborhood of K, which are uniformly bounded from above. Thus

$$-\psi = \sup_{0\leq t\leq 1} (-\varphi_t) = -\inf_{0\leq t\leq 1} \varphi_t$$

is $C_K\omega_X$-psh near K, in particular $\Delta_{\omega_X}\psi < C_K n$. This means that ψ has locally bounded laplacian on $\mathrm{Amp}\,(\alpha)$.

It follows then from classical arguments that the measure $MA(\varphi_0 \vee \varphi_1)$ is absolutely continuous with respect to Lebesgue measure. Since $\varphi_0 \vee \varphi_1, \varphi_0$ (resp. $\varphi_0 \vee \varphi_1, \varphi_1$) have locally bounded Laplacian in $\mathrm{Amp}\,(\alpha)$, it follows from [GT83, Lemma 7.7] that their second partial derivatives agree on $\{\varphi_0 \vee \varphi_1 = \varphi_0\}$ (resp. on $\{\varphi_0 \vee \varphi_1 = \varphi_1\}$), hence

$$MA(\varphi_0 \vee \varphi_1) = \mathbf{1}_{\{\varphi_0\vee\varphi_1=\varphi_0\}} MA(\varphi_0) + \mathbf{1}_{\{\varphi_0\vee\varphi_1=\varphi_1<\varphi_0\}} MA(\varphi_1).$$

We have used here the fact that none of the measures $MA(\varphi_0 \vee \varphi_1), MA(\varphi_0)$, $MA(\varphi_1)$ charges the pluripolar set $X \setminus \mathrm{Amp}\,(\alpha)$. □

A basic observation that we shall use on several occasions is the following:

Lemma 3.3 *Assume $\varphi_0, \varphi_1 \in \mathcal{H}$ and let $(\varphi_t)_{0\leq t\leq 1}$ be the Mabuchi geodesic joining φ_0 to φ_1. Then:*

$$d_p(\varphi_0, \varphi_1) \leq ||\varphi_1 - \varphi_0||_{L^\infty(X)}.$$

Moreover,

(i) If $\varphi_0(x) \leq \varphi_1(x)$ for some $x \in X$, then $\dot{\varphi}_1(x) \geq 0$.

(ii) If $\varphi_0(x) \leq \varphi_1(x)$ for all $x \in X$ then $\dot{\varphi}_t(x) \geq 0$ for all $x \in X$ and a.e $t \in [0,1]$.

By symmetry, if $\varphi_1(x) \leq \varphi_0(x)$, it follows that $\dot{\varphi}_0(x) \leq 0$. Moreover, if $\varphi_1(x) \leq \varphi_0(x)$ for all $x \in X$ then $\dot{\varphi}_t(x) \leq 0$ for a.e. x, t. Here and in the sequel $\dot{\varphi}_0$, $\dot{\varphi}_1$ denote the right and left derivative, respectively while we recall that $\dot{\varphi}_t(x)$ is well defined for a.e (x,t).

Proof. From Theorem 1.13 we know that $d_p^p(\varphi_0,\varphi_1)=\int_X|\dot\varphi_0|^pMA(\varphi_0)$. Moreover, Proposition 1.4 insures that $|\dot\varphi_0|\leq||\varphi_1-\varphi_0||_{L^\infty(X)}$. Hence, the first statement.

Assume $\dot\varphi_1(x)<0$. Since $t\mapsto\varphi_t(x)$ is convex we infer $\dot\varphi_t(x)\leq\dot\varphi_1(x)<0$. Thus $t\mapsto\varphi_t(x)$ is decreasing, hence $\varphi_1(x)<\varphi_0(x)$, a contradiction. This proves (i).

Assume now that $\varphi_0(x)\leq\varphi_1(x)$ for all $x\in X$. Then

$$\varphi_0\leq\varphi_t\leq\varphi_1.$$

The first of the inequalities above follows from the fact that by Proposition 1.4

$$\varphi=\sup\{u\quad u\in PSH(M,\omega)\ :\ u\leq\varphi_{0,1}\text{ on }M\}$$

with $\varphi(x,t+is)=\varphi_t(x)$ and that $\varphi_0(x,t+is)=\varphi_0(x)$ is a subsolution (i.e. a candidate in the envelope). The other inequality follows from the fact that $\varphi_1(x,t+is)=\varphi_1(x)$ is a supersolution of (2) since $(\omega+dd^c_{x,z}\varphi_1)^{n+1}=0$ and $\varphi_1\geq\varphi_{0,1}$. The same argument shows that $\varphi_0\leq\varphi_s\leq\varphi_t$ for all $0<s<t$ and $x\in X$, hence $\dot\varphi_t(x)\geq0$ for all $x\in X$ and a.e $t\in[0,1]$ since the derivative in time of φ_t is well defined for a.e. t. □

We now establish a very useful relation established by Darvas [Dar14, Proposition 8.1] when ω is Kähler (see also [Dar15, Corollary 4.14]).

Proposition 3.4 *Assume $\varphi_0,\varphi_1\in\mathcal{H}$. Then for all $p\geq1$,*

$$d_p^p(\varphi_0,\varphi_1)=d_p^p(\varphi_0,\varphi_0\vee\varphi_1)+d_p^p(\varphi_0\vee\varphi_1,\varphi_1).$$

Proof. We proceed by approximation, so as to reduce to the Kähler case. The identity is known to hold for $d_{p,\varepsilon}$ and $\varphi_0\vee_\varepsilon\varphi_1$, where $d_{p,\varepsilon}$ denotes the distance associated to the Kähler form $\omega_\varepsilon=\omega+\varepsilon\omega_X$ and $\varphi_0\vee_\varepsilon\varphi_1$ is the greatest ω_ε-psh function that lies below $\min(\varphi_0,\varphi_1)$.

Using Theorem 1.13 and the triangle inequality, the proof boils down to check that $d_{p,\varepsilon}(\varphi_0\vee\varphi_1,\varphi_0\vee_\varepsilon\varphi_1)\to0$ as $\varepsilon\to0$. The same arguments used in the proof of Proposition 1.15 yield

$$d_{p,\varepsilon}(\varphi_0\vee\varphi_1,\varphi_0\vee_\varepsilon\varphi_1)\leq d_{p,\varepsilon'}(\varphi_0\vee\varphi_1,\varphi_0\vee_\varepsilon\varphi_1),\quad\varepsilon<\varepsilon'.$$

We claim that $d_{p,\varepsilon'}(\varphi_0\vee\varphi_1,\varphi_0\vee_\varepsilon\varphi_1)$ goes to zero as ε goes to zero since $\varphi_0\vee_\varepsilon\varphi_1$ decreases to $\varphi_0\vee\varphi_1$ as $\varepsilon\to0$. Indeed, observe that $\varphi_0\vee\varphi_1,\varphi_0\vee_\varepsilon\varphi_1\in\mathcal{E}^p(X,\omega'_\varepsilon)\cap L^\infty(X)$ and by Proposition 3.8 we know that

$$d_{p,\varepsilon'}(\varphi_0\vee\varphi_1,\varphi_0\vee_\varepsilon\varphi_1)\leq2I_{p,\varepsilon'}(\varphi_0\vee\varphi_1,\varphi_0\vee_\varepsilon\varphi_1).$$

The same arguments in the proof of Proposition 2.17 then show that $I_{p,\varepsilon'}(\varphi_0\vee\varphi_1,\varphi_0\vee_\varepsilon\varphi_1)\to0$ as ε goes to zero. The conclusion then follows from (3). □

We note for later use the following consequence:

Corollary 3.5 *If $\varphi_0,\varphi_1\in\mathcal{H}$ then*

$$d_p(\varphi_0,\varphi_0\vee\varphi_1)\leq d_p(\varphi_0,\varphi_1).$$

3.2 Comparing d_p and I_p

The goal of this section is to establish that d_p and I_p are equivalent, extending [Dar15, Theorem 5.5]:

Theorem 3.6 *For all $\varphi_0,\varphi_1\in\mathcal{H}$,*

$$2^{-1}d_p(\varphi_0,\varphi_1)\leq I_p(\varphi_0,\varphi_1)\leq2^{4+(2n-1)/p}d_p(\varphi_0,\varphi_1).$$

It follows from Definition 1.10 and Proposition 2.17 that

$$d_p(\varphi_0,\varphi_1)=\lim_{\varepsilon\to 0} d_{p,\varepsilon}(\varphi_0,\varphi_1) \text{ and } I_p(\varphi_0,\varphi_1)=\lim_{\varepsilon\to 0} I_{p,\varepsilon}(\varphi_0,\varphi_1),$$

so it suffices to establish these inequalities when ω is a Kähler form.

We nevertheless give a direct proof, valid when ω is merely semi-positive, with several intermediate results of independent interest. Several of these results have been obtained by Darvas in [Dar13, Dar14, Dar15] when ω is Kähler.

Lemma 3.7 *Assume $\varphi_0,\varphi_1\in\mathcal{H}$ satisfy $\varphi_0\le\varphi_1$. Then*

1) $d_p\left(\varphi_1,\frac{\varphi_0+\varphi_1}{2}\right)\le d_p(\varphi_0,\varphi_1)$;

2) $d_p(\varphi_0,\varphi_1)\le 2^{1+n/p}d_p(\varphi_0/2,\varphi_1/2)$;

3) if $\varphi_1=0$ then $d_p(\varphi_0,0)\ge 2d_p(\varphi_0/2,0)$;

4) If $\psi\in\mathcal{H}$ is such that $\varphi_0\le\psi\le\varphi_1$, then

$$\max\{d_p(\varphi_0,\psi);d_p(\psi,\varphi_1)\}\le d_p(\varphi_0,\varphi_1).$$

Proof. Let φ_t (resp. ψ_t) denote the Mabuchi geodesic joining φ_0 (resp. $(\varphi_0+\varphi_1)/2$) to φ_1. Since $\varphi_0\le\varphi_1$, it follows from Lemma 3.3.ii that $t\mapsto\varphi_t$, $t\mapsto\psi_t$ are increasing and $\varphi_t\le\psi_t$ hence

$$\frac{\varphi_t-\varphi_1}{t-1}\ge\frac{\psi_t-\psi_1}{t-1}$$

since $\varphi_1=\psi_1$. Therefore $\dot\varphi_1\ge\dot\psi_1\ge 0$ and we infer

$$\int_X|\dot\psi_1|^pMA(\psi_1)=d_p\left(\varphi_1,\frac{\varphi_0+\varphi_1}{2}\right)^p\le d_p(\varphi_0,\varphi_1)^p=\int_X|\dot\varphi_1|^pMA(\varphi_1).$$

This proves 1).

Let now (φ_t) (resp. (ψ_t)) denote the geodesic joining φ_0 to φ_1 (resp. $\varphi_0/2$ to $\varphi_1/2$). Observe that $t\mapsto\varphi_t,\psi_t$ are increasing hence $\dot\varphi_0\ge 0$. The family $(\varphi_t/2)$ is a subgeodesic joining $\varphi_0/2$ to $\varphi_1/2$, hence $\varphi_t/2\le\psi_t$ and

$$0\le\frac{\dot\varphi_0}{2}\le\dot\psi_0\Longrightarrow|\dot\varphi_0|^p\le 2^p|\dot\psi_0|^p.$$

Moreover $MA(\varphi_0)\le 2^nMA(\varphi_0/2)$, so we infer

$$d_p(\varphi_0,\varphi_1)^p=\int_X|\dot\varphi_0|^p\,MA(\varphi_0)\le 2^{n+p}d_p(\varphi_0/2,\varphi_1/2)^p,$$

which proves 2). A similar argument shows that

$$0\le\dot\psi_1\le\frac{\dot\varphi_1}{2}\Longrightarrow\left|\dot\psi_1\right|^p\le 2^{-p}|\dot\varphi_1|^p.$$

Now $MA(\varphi_1/2)=MA(\varphi_1)=MA(0)$ when $\varphi_1=0$, hence

$$d_p(\varphi_0,0)^p=\int_X|\dot\varphi_1|^p\,MA(0)\ge 2^pd_p(\varphi_0/2,0)^p,$$

which yields 3).

It remains to prove 4). Let $(\varphi_t)_{0\le t\le 1}$ (resp. $(\psi_t)_{0\le t\le 1}$) be the geodesic joining φ_0 to φ_1 (resp. φ_0 to ψ). Observe that $\varphi_0=\psi_0$ and $\psi_t\le\varphi_t$, hence $\dot\psi_0\le\dot\varphi_0$. Moreover $0\le\dot\psi_0$ since $t\mapsto\psi_t(x)$

is increasing. We infer

$$d_p(\varphi_0,\psi)^p=\int_X|\dot\psi_0|^pMA(\varphi_0)\le\int_X|\dot\varphi_0|^pMA(\varphi_0)=d_p(\varphi_0,\varphi_1)^p.$$

The other inequality is proved similarly. □

Proposition 3.8 *For all $\varphi_0,\varphi_1\in\mathcal{H}$,*

$$0\le d_p(\varphi_0,\varphi_1)\le 2I_p(\varphi_0,\varphi_1).$$

Moreover if $\varphi_0\le\varphi_1$ then $I_p(\varphi_0,\varphi_1)\le\big(\int_X(\varphi_1-\varphi_0)^pMA(\varphi_0)\big)^{1/p}$ and

$$d_p(\varphi_0,\varphi_1)\le\left(\int_X(\varphi_1-\varphi_0)^pMA(\varphi_0)\right)^{1/p}\le 2^{1+n/p}d_p(\varphi_0,\varphi_1).$$

Proof. We first assume that $\varphi_0\le\varphi_1$. The inequality

$$I_p(\varphi_0,\varphi_1)\le\left(\int_X(\varphi_1-\varphi_0)^pMA(\varphi_0)\right)^{1/p}$$

follows from Lemma 2.18. Let (φ_t) be the geodesic joining φ_0 to φ_1. It follows from Lemma 3.3 that $0\le\dot\varphi_0\le\varphi_1-\varphi_0\le\dot\varphi_1$ hence

$$\int_X(\varphi_1-\varphi_0)^pMA(\varphi_1)\le\int_X(\dot\varphi_1)^pMA(\varphi_1)=d_p(\varphi_0,\varphi_1)^p \tag{6}$$

and similarly $d_p(\varphi_0,\varphi_1)^p\le\int_X(\varphi_1-\varphi_0)^pMA(\varphi_0)$.

We give an alternative proof of this upper bound which could be of interest in more singular contexts. We can join φ_0 to φ_1 by a straight line $\varphi_t=t\varphi_1+(1-t)\varphi_0$. This is a smooth path both in $\mathcal{H}_\omega$ and $\mathcal{H}_{\omega_\varepsilon}$, hence its length dominates the distance d_p (see Remark 1.11). Hölder inequality yields

$$\begin{aligned}d_p(\varphi_0,\varphi_1)\le\ell_p(\varphi)&=\int_0^1\left(\int_X(\varphi_1-\varphi_0)^pMA(\varphi_t)\right)^{1/p}dt\\&\le\left(\int_0^1\int_X(\varphi_1-\varphi_0)^pMA(\varphi_t)dt\right)^{1/p}.\end{aligned}$$

Now $MA(\varphi_t)=V_\alpha^{-1}\sum_{j=0}^n\begin{pmatrix}n\\j\end{pmatrix}t^j(1-t)^{n-j}\omega_{\varphi_1}^j\wedge\omega_{\varphi_0}^{n-j}$ and for $0\le j\le n$, $\int_0^1t^j(1-t)^{n-j}dt=(n+1)^{-1}\begin{pmatrix}n\\j\end{pmatrix}^{-1}$, hence

$$\frac{1}{(n+1)V_\alpha}\sum_{j=0}^n\int_X(\varphi_1-\varphi_0)^p\omega_{\varphi_1}^j\wedge\omega_{\varphi_0}^{n-j}\le\int_X(\varphi_1-\varphi_0)^pMA(\varphi_0),$$

as follows from Lemma 2.18, yielding

$$d_p(\varphi_0,\varphi_1)\le\left(\int_X(\varphi_1-\varphi_0)^pMA(\varphi_0)\right)^{1/p}.$$

We now show that $\int_X(\varphi_1-\varphi_0)^pMA(\varphi_0)\le 2^{n+p}d(\varphi_0,\varphi_1)^p$. Observe that $\frac{\varphi_0+\varphi_1}{2}\in\mathcal{H}$ with

$MA(\varphi_0) \leq 2^n\, MA\left(\frac{\varphi_0+\varphi_1}{2}\right)$ hence

$$\begin{aligned}\int_X (\varphi_1-\varphi_0)^p MA(\varphi_0) &= 2^p \int_X \left(\frac{\varphi_0+\varphi_1}{2}-\varphi_0\right)^p MA(\varphi_0)\\ &\leq 2^{n+p}\int_X \left(\frac{\varphi_0+\varphi_1}{2}-\varphi_0\right)^p MA\left(\frac{\varphi_0+\varphi_1}{2}\right)\\ &\leq 2^{n+p} d_p\left(\varphi_0, \frac{\varphi_0+\varphi_1}{2}\right)^p,\end{aligned}$$

as follows from the first step of the proof since $\varphi_0 \leq \varphi_1$. Lemma 3.7.4 yields

$$d_p\left(\varphi_0, \frac{\varphi_0+\varphi_1}{2}\right) \leq d_p(\varphi_0,\varphi_1)$$

hence $\int_X (\varphi_1-\varphi_0)^p MA(\varphi_0) \leq 2^{n+p} d_p(\varphi_0,\varphi_1)^p$.

We finally treat the first upper bound of the Proposition which does not require φ_0 to lie below φ_1. It follows from the triangle inequality that

$$\begin{aligned}&d_p(\varphi_0,\varphi_1) \leq d_p(\varphi_0, \max(\varphi_0,\varphi_1)) + d_p(\max(\varphi_0,\varphi_1),\varphi_1)\\ &\leq \left(\int_{\{\varphi_0<\varphi_1\}} (\varphi_1-\varphi_0)^p MA(\varphi_0)\right)^{1/p} + \left(\int_{\{\varphi_0>\varphi_1\}} (\varphi_0-\varphi_1)^p MA(\varphi_1)\right)^{1/p}\\ &\leq 2^{1-1/p}\left(\int_X |\varphi_1-\varphi_0|^p\left[MA(\varphi_0)+MA(\varphi_1)\right]\right)^{1/p}\\ &= 2\left(\int_X |\varphi_1-\varphi_0|^p \frac{[MA(\varphi_0)+MA(\varphi_1)]}{2}\right)^{1/p}\end{aligned}$$

by using the elementary inequality $a^{1/p}+b^{1/p} \leq 2^{1-1/p}(a+b)^{1/p}$. □

Remark 3.9 *Working with $\psi = t\varphi_0 + (1-t)\varphi_1$, $0<t<1$, instead of $\frac{\varphi_0+\varphi_1}{2}$, one can improve the above inequality and obtain*

$$\left(\int_X (\varphi_1-\varphi_0)^p MA(\varphi_0)\right)^{1/p} \leq \frac{(n+p)^{1+n/p}}{p\, n^{n/p}} d_p(\varphi_0,\varphi_1).$$

We now extend Lemma 3.7.1, following [Dar15, Lemma 5.3]:

Lemma 3.10 *For all $\varphi_0,\varphi_1 \in \mathcal{H}$,*

$$d_p\left(\varphi_0, \frac{\varphi_0+\varphi_1}{2}\right) \leq 2^{2+n/p} d_p(\varphi_0,\varphi_1).$$

Proof. When $\varphi_0 \leq \varphi_1$, this follows from Lemma 3.7.1. Replacing ω by $\omega + dd^c\varphi_0$, we can assume without loss of generality that $\varphi_0 = 0$. The triangle inequality yields

$$d_p\left(0, \frac{\varphi_1}{2}\right) \leq d_p\left(0, 0\vee\frac{\varphi_1}{2}\right) + d_p\left(0\vee\frac{\varphi_1}{2}, \frac{\varphi_1}{2}\right).$$

Observe that $0\vee\varphi_1 \leq 0\vee\frac{\varphi_1}{2} \leq \min(0,\frac{\varphi_1}{2})$. It follows therefore from Lemma 3.7.4 that

$$d_p\left(0, 0\vee\frac{\varphi_1}{2}\right) + d_p\left(0\vee\frac{\varphi_1}{2}, \frac{\varphi_1}{2}\right) \leq d_p\left(0, 0\vee\varphi_1\right) + d_p\left(0\vee\varphi_1, \frac{\varphi_1}{2}\right).$$

Since $0 \vee \varphi_1 \leq 0$ and $0 \vee \varphi_1 \leq \frac{\varphi_1}{2}$, we can invoke Proposition 3.8 to obtain

$$\begin{aligned}
&d_p\left(0, 0 \vee \varphi_1\right) + d_p\left(0 \vee \varphi_1, \frac{\varphi_1}{2}\right) \\
&\leq \left(\int_X |0 \vee \varphi_1|^p MA(0 \vee \varphi_1)\right)^{1/p} + \left(\int_X |0 \vee \varphi_1 - \frac{\varphi_1}{2}|^p MA(0 \vee \varphi_1)\right)^{1/p} \\
&\leq 2^{1-1/p}\left(\int_X \left[|0 \vee \varphi_1|^p + |0 \vee \varphi_1 - \frac{\varphi_1}{2}|^p\right] MA(0 \vee \varphi_1)\right)^{1/p}.
\end{aligned}$$

Recall now that the measure $MA(0 \vee \varphi_1)$ is supported on the contact set $S := \{x \in X \; ; \; 0 \vee \varphi_1(x) = \min(0, \varphi_1)(x)\}$. On this set we have

$$|0 \vee \varphi_1|^p + |0 \vee \varphi_1 - \frac{\varphi_1}{2}|^p \leq 2|\varphi_1|^p = 2\left[|0 \vee \varphi_1|^p + |0 \vee \varphi_1 - \varphi_1|^p\right],$$

while Proposition 3.8 yields

$$\begin{aligned}
&\int_X \left[|0 \vee \varphi_1|^p + |0 \vee \varphi_1 - \varphi_1|^p\right] MA(0 \vee \varphi_1) \\
&\leq 2^{p+n}\left[d_p(0, 0 \vee \varphi_1)^p + d_p(0 \vee \varphi_1, \varphi_1)^p\right] = 2^{p+n} d_p(0, \varphi_1)^p,
\end{aligned}$$

where the last equality follows from Proposition 3.4. Altogether this yields $d_p\left(0, \frac{\varphi_1}{2}\right) \leq 2^{2+n/p} d_p(0, \varphi_1)$, as claimed. □

We are now ready to prove Theorem 3.6.

Proof. We have already observed that $d_p(\varphi_0, \varphi_1) \leq 2I_p(\varphi_0, \varphi_1)$ in Proposition 3.8, so we focus on the reverse control. Lemma 3.10 and Proposition 3.4 yield

$$\begin{aligned}
2^{2p+n} d_p^p(\varphi_0, \varphi_1) &\geq d_p^p\left(\varphi_0, \frac{\varphi_0 + \varphi_1}{2}\right) \\
&= d_p^p\left(\varphi_0, \varphi_0 \vee \frac{\varphi_0 + \varphi_1}{2}\right) + d_p^p\left(\frac{\varphi_0 + \varphi_1}{2}, \varphi_0 \vee \frac{\varphi_0 + \varphi_1}{2}\right)
\end{aligned}$$

It follows from (6) together with the fact that $2^n \mathrm{MA}\left(\frac{\varphi_0 + \varphi_1}{2}\right) \geq \mathrm{MA}(\varphi_0)$ that

$$d_p^p\left(\varphi_0, \varphi_0 \vee \frac{\varphi_0 + \varphi_1}{2}\right) \geq \int_X \left(\varphi_0 - \frac{\varphi_0 + \varphi_1}{2} \vee \varphi_0\right)^p \mathrm{MA}(\varphi_0)$$

and

$$d_p^p\left(\frac{\varphi_0 + \varphi_1}{2}, \varphi_0 \vee \frac{\varphi_0 + \varphi_1}{2}\right) \geq 2^{-n} \int_X \left(\frac{\varphi_0 + \varphi_1}{2} - \varphi_0 \vee \frac{\varphi_0 + \varphi_1}{2}\right)^p \mathrm{MA}(\varphi_0).$$

Hence

$$\begin{aligned}
d_p^p(\varphi_0, \varphi_1) &\geq 2^{-2(p+n)} \int_X \left[\left(\varphi_0 - \frac{\varphi_0 + \varphi_1}{2} \vee \varphi_0\right)^p + \left(\frac{\varphi_0 + \varphi_1}{2} - \frac{\varphi_0 + \varphi_1}{2} \vee \varphi_0\right)^p\right] \mathrm{MA}(\varphi_0) \\
&\geq 2^{1-3p-2n} \int_X \left|\varphi_0 - \frac{\varphi_0 + \varphi_1}{2}\right|^p \mathrm{MA}(\varphi_0) \\
&= 2^{1-4p-2n} \int_X |\varphi_0 - \varphi_1|^p \mathrm{MA}(\varphi_0)
\end{aligned}$$

where in the last inequality we used the fact that $|a - b|^p \leq 2^{p-1}(a^p + b^p)$, for any $a, b \in \mathbf{R}^+$.

Reversing the role of φ_0 adn φ_1 we get

$$d_p^p(\varphi_0,\varphi_1) \geq 2^{1-4p-2n}\int_X |\varphi_1-\varphi_0|^p \mathrm{MA}(\varphi_1)$$

from which it follows $d_p^p(\varphi_0,\varphi_1) \geq 2^{1-4p-2n} I_p^p(\varphi_0,\varphi_1)$. □

3.3 Controlling the sup

It follows from previous results that the supremum of a bounded potential with locally bounded laplacian in $\mathrm{Amp}\,(\alpha)$ is controlled by the distance to the base point:

Lemma 3.11 *There exists $C>0$ such that for all $\varphi \in \mathcal{H}$,*

$$-2^{4+2n} d_1(0,\varphi) \leq \sup_X \varphi \leq 2^{4+2n}(n+1)d_1(0,\varphi)+C$$

Proof. If $\sup_X \varphi \leq 0$, then $\sup_X \varphi \leq 0 \leq (n+1)d_1(0,\varphi)+C$, while

$$-d_1(0,\varphi) = E(\varphi) \leq \sup_X \varphi,$$

as follows from Proposition 3.12. We therefore assume in the sequel that $\sup_X \varphi \geq 0$. If $\varphi \geq 0$, then Proposition 3.12 yields

$$\frac{1}{n+1}\int_X \varphi MA(0) \leq E(\varphi) = d_1(0,\varphi).$$

It is a classical consequence of the ω-plurisubharmonicity [GZ05, Proposition 2.7] that there exists $C>0$ such that such that for all $\varphi \in PSH(X,\omega)$,

$$\sup_X \varphi \leq \int_X \varphi\, MA(0) + C.$$

Thus $\sup_X \varphi \leq (n+1)d_1(0,\varphi)+C$.

When $\sup_X \varphi \geq 0$ but φ takes both positive and negative values, we set $\psi = \max(0,\varphi)$ and observe that $\sup_X \psi = \sup_X \varphi$. Using Propositions 2.19, 3.8 and Theorem 3.6 we obtain

$$d_1(0,\max(0,\varphi)) \leq 2I_1(0,\max(0,\varphi)) \leq 2I_1(0,\varphi) \leq 2^{5-(2n-1)/p} d_1(0,\varphi).$$

The conclusion follows therefore from the previous case. □

Proposition 3.12 *Assume $\varphi,\psi \in \mathcal{H}$. Then*

$$d_1(\varphi,\psi) = E(\varphi)+E(\psi)-2E(\varphi\vee\psi).$$

Proof. Assume first that $\varphi \leq \psi$ and let $(\varphi_t)_{0\leq t\leq 1}$ denote the geodesic joining φ to ψ. Then $\dot\varphi_t(x) \geq 0$ for all t,x, hence

$$d_1(\varphi,\psi) = \int_0^1\int_X \dot\varphi_t MA(\varphi_t)\,dt = \int_0^1 \frac{d}{dt}E(\varphi_t)dt = E(\psi)-E(\varphi).$$

To treat the general case we use Proposition 3.4, which yields

$$d_1(\varphi,\psi) = d_1(\varphi,\varphi\vee\psi)+d_1(\varphi\vee\psi,\psi) = E(\varphi)-E(\varphi\vee\psi)+E(\psi)-E(\varphi\vee\psi),$$

as claimed. □

4. The complete geodesic space $(\mathcal{E}^p(X,\omega), d_p)$

4.1 Metric completion

For $\varphi, \psi \in \mathcal{E}^p(X,\omega)$ we let φ_j, ψ_k denote sequences of elements in $\mathcal{H}_{bd}$ decreasing to φ, ψ respectively, and set

$$D_p(\varphi,\psi) := \liminf_{j,k\to+\infty} d_p(\varphi_j,\psi_k).$$

We list in the proposition below various properties of this extension.

Proposition 4.1

i) D_p is a distance on $\mathcal{E}^p(X,\omega)$ which coincides with d_p on $\mathcal{H}_{bd}$;

ii) the definition of D_p is independent of the choice of the approximants;

iii) D_p is continuous along decreasing sequences in $\mathcal{E}^p(X,\omega)$.

Moreover all previous inequalities comparing d_p and I_p on $\mathcal{H}_{bd}$ extend to inequalities between D_p and I_p on $\mathcal{E}^p(X,\omega)$.

In the sequel we will therefore denote D_p by d_p.

Proof. It is a tedious exercise to verify that D_p defines a "semi-distance", i.e. satisfies all properties of a distance but for the separation property. It follows from the definition of D_p and Proposition 2.17 that Theorem 3.6 extends in a natural way to potentials in $\mathcal{E}^p(X,\omega)$. If $D_p(\varphi,\psi)=0$, it follows therefore that $I_p(\varphi,\psi)=0$ hence $\varphi=\psi$ by the domination principle.

One can check that D_p coincides with d_p on $\mathcal{H}_{bd}$ as follows: using ii) one can use the constant sequences $\varphi_j \equiv \varphi$ and $\psi_k \equiv \psi$ to obtain this equality.

We now prove ii). Let φ_j, u_j (resp. ψ_k, v_k) denote two sequences of elements of $\mathcal{H}_{bd}$ decreasing to φ (resp. ψ). We can assume without loss of generality that these sequences are intertwining, i.e. for all $j,k \in \mathbf{N}$, there exists $\ell, q \in \mathbf{N}$ such that $\varphi_j \le u_\ell$ and $\psi_k \le v_q$, with similar reverse inequalities. It follows from Proposition 3.8 and the triangle inequality that

$$\begin{aligned}|d_p(\varphi_j,\psi_k) - d_p(u_\ell,v_q)| &\le d_p(\varphi_j,u_\ell) + d_p(\psi_k,v_q)\\ &\le 2I_p(\varphi_j,u_\ell) + 2I_p(\psi_k,v_q).\end{aligned}$$

Now, again by Proposition 3.8 we get

$$I_p(\varphi_j,u_\ell)^p \le \int_X (u_\ell-\varphi_j)^p MA(\varphi_j) \le (p+1)^n \int_X (u_\ell-\varphi)^p MA(\varphi)$$

where the last inequality follows from [GZ07, Lemma 3.5]. The monotone convergence theorem therefore yields $I_p(\varphi_j,u_\ell)+I_p(\psi_k,v_q)\to 0$ as $\ell,q\to+\infty$, proving ii).

One shows iii) with similar arguments. The extension of the inequalities comparing d_p and I_p follows from [BEGZ10, Theorem 2.17]. □

Proposition 4.2 *The metric spaces $(\mathcal{E}^p_{norm}(X,\omega), d_p)$ and $(\mathcal{E}^p(X,\omega), d_p)$ are complete. The Mabuchi topology d_p dominates the topology induced by I: if a sequence converges for d_p, then it converges in energy.*

Proof. Let $(\varphi_j) \in \mathcal{E}^p_{norm}(X,\omega)^{\mathbf{N}}$ be a Cauchy sequence for d_p. Since $\sup_X \varphi_j$ is bounded, the sequence is relatively compact for the (weak) L^1-topology. Let ψ be a cluster point for the L^1-topology. We claim that $\psi \in \mathcal{E}^p_{norm}(X,\omega)$,

$$d_p(\varphi_j,\psi)\to 0 \text{ and } I(\psi,\varphi_j)\to 0.$$

Extracting and relabelling, we can assume that

$$\varphi_j \xrightarrow{L^1} \psi \quad \text{and} \quad d_p(\varphi_j, \varphi_{j+1}) \leq 2^{-j}.$$

Set $\varphi_{-1} \equiv 0$ and for $k \geq j$, $\psi_{j,k} := \varphi_j \vee \varphi_{j+1} \vee \cdots \vee \varphi_k$. Observe that

$$\begin{aligned} d_p(0, \psi_{j,k}) &\leq \sum_{\ell=-1}^{j-1} d_p(\varphi_\ell, \varphi_{\ell+1}) + d_p(\varphi_j, \psi_{j,k}) \\ &\leq \sum_{\ell=-1}^{j} d_p(\varphi_\ell, \varphi_{\ell+1}) + d_p(\varphi_{j+1}, \psi_{j+1,k}) \leq 4, \end{aligned}$$

as

$$d_p(\varphi_j, \psi_{j,k}) = d_p(\varphi_j, \varphi_j \vee \psi_{j+1,k}) \leq d_p(\varphi_j, \psi_{j+1,k}) \leq 2^{-j} + d_p(\varphi_{j+1}, \psi_{j+1,k}).$$

It follows from Proposition 3.8 and Theorem 3.6 that $I_p(0, \psi_{j,k})$ is uniformly bounded hence $\psi_j := \lim_{k\to+\infty} \psi_{j,k} \in \mathcal{E}^p(X, \omega)$. Now ψ_j increases a.e. towards ψ, hence $\psi \in \mathcal{E}^p_{norm}(X, \omega)$ and [BEGZ10, Theorem 2.17] yields

$$I(\psi, \psi_j) + I_p(\psi_j, \psi) \longrightarrow 0.$$

It follows therefore from Proposition 3.8 that $d_p(\psi, \psi_j) \to 0$ and

$$d_p(\psi, \varphi_j) \leq d_p(\psi, \psi_j) + d_p(\psi_j, \varphi_j) \leq d_p(\psi, \psi_j) + 2^{1-j} \to 0.$$

Recalling that $\psi_j \leq \varphi_j$, it follows from the quasi-triangle inequality, Proposition 2.14 and Theorem 3.6 that

$$I(\psi, \varphi_j) \leq c_n \left\{ I(\psi, \psi_j) + I(\psi_j, \varphi_j) \right\} \leq c_{n,p} \left\{ I(\psi, \psi_j) + d_p(\psi_j, \varphi_j) \right\} \to 0.$$

It remains to treat the case of a Cauchy sequence $(\varphi_j) \in \mathcal{E}^p(X, \omega)^{\mathbf{N}}$. The only extra information we need to add is that $(\sup_X \varphi_j)_j$ is a bounded sequence of real numbers. This follows from Lemma 3.11, the fact that $d_p(0, \varphi_j) \leq 4$ and Hölder inequality, which guarantees that d_p dominates d_1. □

Recall that the *precompletion* of a metric space (X, d) is the set of all Cauchy sequences C_X of X, together with the semi-distance

$$\delta(\{x_j\}, \{y_j\}) = \lim_{j\to+\infty} d(x_j, y_j).$$

The metric *completion* $(\overline{X}, d)$ of (X, d) is the quotient space $C_X/\sim$, where

$$\{x_j\} \sim \{y_j\} \Longleftrightarrow \delta(\{x_j\}, \{y_j\}) = 0,$$

equipped with the induced distance that we still denote by d.

Recall that a *path metric space* is a metric space for which the distance between any two points coincides with the infimum of the lengths of rectifiable curves joining the two points. By construction the space $(\mathcal{H}, d)$ is a path metric space. For such metric spaces, an alternative description of the metric completion can be obtained as follows: consider C'_X the set of all rectifiable curves $\gamma : (0, 1] \to X$ equipped with the semi-distance

$$\delta(\gamma, \tilde{\gamma}) := \lim_{t\to 0} d(\gamma(t), \tilde{\gamma}(t)).$$

The metric completion $(\overline{X}, d)$ is then the quotient space $C'_X/\sim$ which identifies zero-distance curves $\gamma, \tilde{\gamma}$.

Both constructions yield a rather abstract view on the metric completion. We are now taking advantage of the fact that $\mathcal{H}_{bd}$ lives inside the complete metric space $(\mathcal{E}^p(\alpha), d_p)$ to conclude that:

Theorem 4.3 *The metric completion of $(\mathcal{H}_{bd}, d_p)$ is isometric to $(\mathcal{E}^p(X,\omega), d_p)$.*

Thanks to Theorem 3.6, an equivalent formulation of the above statement is that the metric completion of $(\mathcal{H}_{bd}, d_p)$ is bi-Lipschitz equivalent to $(\mathcal{E}^p(X,\omega), I_p)$.

Proof. We work at the level of normalized potentials,

$$\mathcal{E}^p_0(X,\omega) = \{\varphi \in \mathcal{E}^p(X,\omega) \,|\, E(\varphi) = 0\}$$

and $\mathcal{H}_0 := \{\varphi \in \mathcal{H}_{bd} \,|\, \omega + dd^c\varphi \geq 0 \text{ and } E(\varphi) = 0\}$.

Since $(\mathcal{E}^p_0(X,\omega), d_p)$ is a complete metric space that contains $\mathcal{H}_0$, it suffices to show that the latter is dense in $\mathcal{E}^p_0(X,\omega)$. Fix $\varphi \in \mathcal{E}^p_0(X,\omega)$ and let $(\varphi_j) \in \mathcal{H}_0^{\mathbf{N}}$ be a sequence quasi-decreasing to φ : the normalization condition $E(\varphi_j) = 0$ prevents from getting a truly decreasing sequence, however $\varphi_j + \varepsilon_j$ is decreasing where ε_j is a sequence of real numbers decreasing to zero. It follows from Proposition 3.8 that

$$d_p(\varphi_{j+\ell} + \varepsilon_{j+l}, \varphi_j + \varepsilon_j)^p \leq \int_X (\varphi_j - \varphi_{j+\ell})^p MA(\varphi_{j+\ell}) + \varepsilon_j.$$

Now [GZ07, Lemma 3.5] shows that the latter is bounded from above by

$$(p+1)^n \int_X (\varphi_j - \varphi)^p MA(\varphi) + \varepsilon_j$$

which converges to zero as $j \to +\infty$, as follows from the monotone convergence theorem. Therefore (φ_j) is a Cauchy sequence in $(\mathcal{H}_0, d_p)$ which converges to φ since

$$0 \leq d_p(\varphi, \varphi_j + \varepsilon_j) \leq \liminf_{\ell \to +\infty} d_p(\varphi_{j+\ell}, \varphi_j) \leq 2(1+p)^{n/p} I_p(\varphi_j, \varphi) + \varepsilon_j^{1/p} \to 0$$

by Proposition 3.8 and [BEGZ10, Theorem 2.17].

We note the following alternative approach of independent interest. One first shows that $\mathcal{H}_0$ is dense in the set of all bounded ω-psh functions. Given $\varphi \in \mathcal{E}^p_0(X,\omega)$ one then considers its "canonical approximants"

$$\varphi_j = \max(\varphi, -j) + \varepsilon_j \in PSH_0(X,\omega) \cap L^\infty(X)$$

which decrease towards $\varphi \in \mathcal{E}^p_0(X,\omega)$. It follows from Proposition 3.8 that

$$\begin{aligned}
d_p(\varphi_{j+\ell}, \varphi_j)^p &\leq o(1) + \int_X (\varphi_j - \varphi_{j+\ell})^p MA(\varphi_{j+\ell}) \\
&= o(1) + \int_{(\varphi \leq -j-\ell)} \ell^p MA(\varphi_{j+\ell}) + \int_{(-j-\ell < \varphi < -j)} (\varphi_j - \varphi_{j+\ell})^p MA(\varphi) \\
&= o(1) + \int_{(\varphi \leq -j-\ell)} \ell^p MA(\varphi) + \int_{(-j-\ell < \varphi < -j)} (\varphi_j - \varphi_{j+\ell})^p MA(\varphi) \\
&\leq o(1) + \int_{(\varphi < -j)} \varphi^p MA(\varphi),
\end{aligned}$$

where we have used the maximum principle together with the fact that

$$\int_{(\varphi \leq -k)} MA(\varphi_k) = \int_X MA(\varphi_k) - \int_{(\varphi > -k)} MA(\varphi_k) = \int_{(\varphi \leq -k)} MA(\varphi),$$

since $\varphi \in \mathcal{E}(X,\omega)$, as follows again from the maximum principle. We infer that (φ_j) is a Cauchy sequence which converges to φ. □

We are now in position to prove Theorem B of the introduction:

Corollary 4.4 *Assume $\omega = \pi^\star \omega_Y$, where ω_Y is a Hodge form. Then the metric completion $(\overline{\mathcal{H}}_\alpha, d_p)$ is isometric to $(\mathcal{E}^p(\alpha), d_p)$. Similarly the metric completion $(\overline{\mathcal{H}}_\omega, d_p)$ is isometric to $(\mathcal{E}^p(X,\omega), d_p)$.*

Proof. Thanks to [CGZ, Corollary C] we can insure that the space $\mathcal{H}$ is dense in $\mathcal{H}_{bd}$. The result then follows from Theorem 4.3. □

4.2 Weak geodesics

4.2.1 *Finite energy geodesics* We now define finite energy geodesics joining two finite energy endpoints $\varphi_0, \varphi_1 \in \mathcal{E}^1(X,\omega)$. Fix $j \in \mathbf{N}$ and consider φ_0^j, φ_1^j bounded ω-psh functions decreasing to φ_0, φ_1. We let $\varphi_{t,j}$ denote the bounded geodesic joining φ_0^j to φ_1^j. It follows from the maximum principle that $j \mapsto \varphi_{t,j}$ is non-increasing. We can thus set

$$\varphi_t := \lim_{j\to+\infty} \varphi_{t,j}.$$

Definition 4.5 *The map $(t,x) \mapsto \varphi_t(x)$ is the (finite energy) Mabuchi geodesic joining φ_0 to φ_1.*

The φ_t's form indeed a family of finite energy functions : since $t \mapsto E(\varphi_{t,j})$ is affine (Lemma 2.9), we infer for all $j \in \mathbf{N}$,

$$(1-t)E(\varphi_0) + tE(\varphi_1) \leq (1-t)E(\varphi_0^{(j)}) + tE(\varphi_1^{(j)}) = E(\varphi_{t,j}),$$

hence $\varphi_t \in \mathcal{E}^1(X,\omega)$ with $(1-t)E(\varphi_0) + tE(\varphi_1) = E(\varphi_t)$.

It follows from the maximum principle that φ_t is independent of the choice of the approximants φ_0^j, φ_1^j: if we set $\varphi(x,z) := \varphi_t(x)$, $z = t + is$, then φ is a maximal ω-psh function in $X \times S$, as a decreasing limit of maximal ω-psh functions. It is thus the unique maximal ω-psh function in $X \times S$ with boundary values φ_0, φ_1.

When φ_0, φ_1 belong to $\mathcal{E}^p(X,\omega)$, these weak geodesics are again *metric geodesics* in the complete metric space $(\mathcal{E}^p(X,\omega), d_p)$:

Proposition 4.6 *Given $\varphi_0, \varphi_1 \in \mathcal{E}^p(X,\omega)$, the Mabuchi geodesic φ joining φ_0 to φ_1 lies in $\mathcal{E}^p(X,\omega)$ and satisfies, for all $t, s \in [0,1]$,*

$$d_p(\varphi_t, \varphi_s) = |t-s|\, d_p(\varphi_0, \varphi_1).$$

Thus $(\mathcal{E}^p(X,\omega), d_p)$ is a geodesic space.

Proof. We can assume without loss of generality that $\varphi_0, \varphi_1 \leq 0$. Fix $j \in \mathbf{N}$ and consider φ_0^j, φ_1^j bounded ω-psh functions decreasing to φ_0, φ_1. We let $\varphi_{t,j}$ denote the bounded geodesic joining φ_0^j to φ_1^j, which decreases towards φ_t as j increases to $+\infty$. Observe that

$$\varphi_0 \vee \varphi_1 \leq \varphi_0^j \vee \varphi_1^j \leq \varphi_{t,j}.$$

It follows therefore from [GZ07, Lemma 3.5] and Lemma 4.7 that

$$\int_X (-\varphi_{t,j})^p MA(\varphi_{t,j}) \leq (p+1)^n \int_X (-\varphi_0 \vee \varphi_1)^p MA(\varphi_0 \vee \varphi_1) < +\infty$$

hence the monotone convergence theorem yields $\int_X (-\varphi_t)^p MA(\varphi_t) < +\infty$, for all t, i.e. $\varphi_t \in \mathcal{E}^p(X,\omega)$.

The remaining assertion is proved as in the case of bounded geodesics (Proposition 1.17). □

Lemma 4.7 *Assume $0 \geq \varphi_0, \varphi_1 \in \mathcal{E}^p(X,\omega)$. Then $\varphi_0 \vee \varphi_1 \in \mathcal{E}^p(X,\omega)$ and*

$$\int_X (-\varphi_0 \vee \varphi_1)^p MA(\varphi_0 \vee \varphi_1) \leq \int_X (-\varphi_0)^p MA(\varphi_0) + \int_X (-\varphi_1)^p MA(\varphi_1).$$

Proof. It suffices to establish the claimed inequality when $\varphi_0, \varphi_1 \in \mathcal{H}$ and then proceed by approximation. It follows from Proposition 3.2 that

$$MA(\varphi_0 \vee \varphi_1) \leq \mathbf{1}_{\{\varphi_0 \vee \varphi_1 = \varphi_0\}} MA(\varphi_0) + \mathbf{1}_{\{\varphi_0 \vee \varphi_1 = \varphi_1\}} MA(\varphi_1).$$

The inequality follows since $\varphi_0, \varphi_1 \leq 0$. □

4.2.2 *(Non) uniqueness of geodesics* Fix $\varphi_0, \varphi_1 \in \mathcal{E}^1(X,\omega)$. If the sets $(\varphi_0 < \varphi_1)$ and $(\varphi_0 > \varphi_1)$ are both non empty, the function $\varphi_0 \vee \varphi_1$ differs from φ_0 and φ_1 and it follows from Proposition 3.4 that

$$d_1(\varphi_0, \varphi_1) = d_1(\varphi_0, \varphi_0 \vee \varphi_1) + d_1(\varphi_0 \vee \varphi_1, \varphi_1),$$

thus the concatenation of the geodesic joining φ_0 to $\varphi_0 \vee \varphi_1$ and that joining $\varphi_0 \vee \varphi_1$ to φ_1 gives another minimizing path joining φ_0 to φ_1.

When $\varphi_0 \leq \varphi_1$, this argument does not work anymore, but there are nevertheless very many minimizing paths, as shown by the following result:

Lemma 4.8 *Assume $\varphi_0, \varphi_1 \in \mathcal{H}$ are such that $\varphi_0 \leq \varphi_1$. Let $(\psi_t)_{0 \leq t \leq 1}$ be a path joining φ_0 to φ_1. Then*

$$\ell_1(\psi) = d_1(\varphi_0, \varphi_1) \Longleftrightarrow \dot{\psi}_t(x) \geq 0, \text{ for a.e. } t, x.$$

In particular $t \mapsto t\varphi_1(x) + (1-t)\varphi_0$ is a minimizing path for d_1 which is not a Mabuchi geodesic, unless $\varphi_1 - \varphi_0$ is constant.

Proof. Observe that

$$\begin{aligned} \ell_1(\psi) &= \int_0^1 \int_X \left|\dot{\psi}_t(x)\right| MA(\psi_t)\, dt \geq \left| \int_0^1 \int_X \dot{\psi}_t(x) MA(\psi_t)\, dt \right| \\ &= \left| \int_0^1 \frac{d}{dt} E(\psi_t)\, dt \right| = |E(\varphi_1) - E(\varphi_0)| = d_1(\varphi_0, \varphi_1) \end{aligned}$$

where the last identity follows from Proposition 3.12. There is equality iff $|\dot{\psi}_t(x)| = \dot{\psi}_t(x) \geq 0$ for a.e. (t,x) (the sign has to be positive because $\psi_0 = \varphi_0 \leq \varphi_1 = \psi_1$).

In particular $t \mapsto \psi_t = t\varphi_1(x) + (1-t)\varphi_0$ has this property, since $\dot{\psi}_t = \varphi_1 - \varphi_0 \geq 0$. We recall that, since ψ_t is a smooth path, the geodesic equation can be written as

$$\ddot{\psi}_t \mathrm{MA}(\psi_t) = \frac{n}{V} d\dot{\psi}_t \wedge d^c \dot{\psi}_t \wedge \omega_{\psi_t}^{n-1}$$

(see Section 1.1.1). Now $\ddot{\psi}_t = 0$ hence $t \mapsto \psi_t$ is not a Mabuchi geodesic, unless $d(\varphi_1 - \varphi_0) \wedge d^c(\varphi_1 - \varphi_0) \wedge \omega_{\psi_t}^{n-1} = 0$ for all t, i.e. $\varphi_1 - \varphi_0$ is contant. □

On the other hand it follows from the work of Calabi-Chen [CC02] that minimizing geodesics are unique in $\mathcal{E}^2(X,\omega)$:

Theorem 4.9 *Assume $\omega = \pi^\star \omega_Y$, where ω_Y is a Hodge form. Then the space $(\mathcal{E}^2(X,\omega), d_2)$ is a CAT(0) space.*

Complete CAT(0) spaces are also called Hadamard spaces. Recall that a CAT(0) space is a geodesic space which has non positive curvature in the sense of Alexandrov. Hadamard spaces enjoy many interesting properties (uniqueness of geodesics, contractibility, convexity properties,...see [BH99]).

Proof. By Corollary 4.4 we know that $(\mathcal{E}^2(X,\omega), d_2)$ is the completion of $(\mathcal{H}_\omega, d_2)$. Note that $(\overline{\mathcal{H}_\omega}, d_2)$ is a complete path metric space, being the completion of the path metric space $(\mathcal{H}_\omega, d_2)$. The Hopf-Rinow-Cohn-Vossen theorem (see [BH99, Proposition I.3.7]) insures that a complete *locally compact* path metric space is automatically a geodesic space. Here $(\overline{\mathcal{H}}_\omega, d_2)$ is not locally compact (it is merely locally weakly compact), but we have a natural candidate for the minimizing geodesics.

[BH99, Exercise 1.9.1.c (p. 193)] insures that $(\overline{\mathcal{H}}_\omega, d_2)$ is a CAT(0) space if and only if the CN inequality of Bruhat-Tits [BT72] holds, i.e. $\forall P, Q, R \in \overline{\mathcal{H}}_\omega$ and for any $M \in \overline{\mathcal{H}}_\omega$ such that $d_2(Q,M) = d_2(R,M) = d_2(Q,R)/2$ one has

$$d_2(M,P)^2 \leq \frac{1}{2} d_2(P,Q)^2 + \frac{1}{2} d_2(P,R)^2 - \frac{1}{4} d_2(Q,R)^2. \tag{7}$$

Calabi and Chen proved in [CC02, Theorem 1.1] that $(\mathcal{H}_\omega, d_2)$ satisfies the CN inequality (7) in the case when the reference form ω is Kähler. The result extends to our present setting by approximation (Theorem 1.13).

Moreover, the CN inequality extends to $\mathcal{E}^2(X,\omega)$ by density. It follows therefore from [BH99, Corollary II.3.11] that $(\overline{\mathcal{H}}_\omega, d_2)$ is a CAT(0) space. □

5. Singular Kähler-Einstein metrics of positive curvature

The existence of singular Kähler-Einstein metrics of non-positive curvature has been established in [EGZ09], generalizing the fundamental work of Aubin [Aub78] and Yau [Yau78]. They always exist, provided the underlying variety has mild singularities and the first Chern class is non-positive.

Singular Kähler-Einstein metrics of positive curvature are more difficult to construct. It is already so in the smooth case [CDS15]. Their first properties have been obtained in [BBGZ13, BBEGZ]. In Section 5.3, pushing further these works, we provide a necessary and sufficient analytic condition for their existence, generalizing a result of Tian [Tian97] and Phong-Song-Sturm-Weinkove [PSSW08].

5.1 Log terminal singularities

A *pair* (Y, D) is the data of a connected normal compact complex variety Y and an effective $\mathbf{Q}$-divisor D such that $K_Y + D$ is $\mathbf{Q}$-Cartier. We write

$$Y_0 := Y_{\text{reg}} \setminus \text{Supp} D.$$

Given a log resolution $\pi : X \to Y$ of (Y, D) (which may be chosen to be an isomorphism over Y_0), there exists a unique $\mathbf{Q}$-divisor $\sum_i a_i E_i$ whose push-forward to Y is $-D$ and such that

$$K_X = \pi^*(K_Y + D) + \sum_i a_i E_i.$$

Definition 5.1 *The pair* (Y, D) *is* klt *if* $a_j > -1$ *for all* j.

The same condition will then hold for all log resolutions of Y. When $D = 0$, one says that Y is *log terminal* when the pair $(Y, 0)$ is klt. We have the following analytic interpretation. Fix $r \in \mathbf{N}^*$ such that $r(K_Y + D)$ is Cartier. If σ is a nowhere vanishing section of the corresponding line bundle over a small open set U of Y then

$$\left(i^{rn^2}\sigma \wedge \bar{\sigma}\right)^{1/r} \tag{8}$$

defines a smooth, positive volume form on $U_0 := U \cap Y_0$. If f_j is a local equation of E_j around a point of $\pi^{-1}(U)$, then

$$\pi^* \left(i^{rn^2}\sigma \wedge \bar{\sigma}\right)^{1/r} = \prod_i |f_i|^{2a_i} dV$$

locally on $\pi^{-1}(U)$ for some local volume form dV. Since $\sum_i E_i$ has normal crossings, this shows that (Y, D) is klt iff each volume form of the form (8) has locally finite mass near singular points of Y.

The previous construction globalizes as follows:

Definition 5.2 *Let* (Y, D) *be a pair and let* ϕ *be a smooth Hermitian metric on the* $\mathbf{Q}$*-line bundle* $-(K_Y + D)$*. The corresponding* adapted measure mes_ϕ *on* Y_{reg} *is locally defined by choosing a nowhere zero section* σ *of* $r(K_Y + D)$ *over a small open set* U *and setting*

$$\mathrm{mes}_\phi := \left(i^{rn^2}\sigma \wedge \overline{\sigma}\right)^{1/r} / |\sigma|^{2/r}_{r\phi}.$$

The point is that the measure mes_ϕ does not depend on the choice of σ, hence is globally defined. The above discussion shows that

$$(Y, D) \text{ is klt } \iff \mathrm{mes}_\phi \text{ has finite total mass on } Y,$$

in which case we view it as a Radon measure on the whole of Y.

5.2 Kähler-Einstein metrics on log Fano pairs

Definition 5.3 *A* log Fano pair *is a klt pair* (Y, D) *such that* Y *is projective and* $-(K_Y + D)$ *is ample.*

Let (Y, D) be a log Fano pair. Fix a reference smooth strictly psh metric ϕ_0 on $-(K_Y + D)$, with curvature ω_0 and adapted measure $\mu_0 = \mathrm{mes}_{\phi_0}$. We normalize ϕ_0 so that μ_0 is a probability measure. The volume of (Y, D) is

$$V := c_1(Y, D)^n = \int_X \omega_0^n.$$

Definition 5.4 *A* Kähler-Einstein metric T *for the log Fano pair* (Y, D) *is a finite energy current* $T \in c_1(Y, D)$ *such that* $T^n = V \cdot \mu_T$.

We now list some important properties of these objects established in [BBGZ13, Bern15, BBEGZ]:

– A Kähler-Einstein metric ω is automatically smooth on Y_0, with continuous potentials on Y, and it satisfies

$$\mathrm{Ric}(\omega_{KE}) = \omega_{KE} + [D] \text{ on } Y_{\mathrm{reg}}.$$

- The definition of a log Fano pair requires the singularities to be klt. This condition is in fact necessary to obtain K-E metrics on Y_{reg}.
- The Kähler-Einstein equation reads $(\omega_0 + dd^c\phi)^n = e^{-\phi+c}\mu_0$ for some constant $c \in \mathbf{R}$. If we choose a log resolution, the equation becomes $(\omega + dd^c\varphi)^n = e^{-\varphi+c}\widetilde{\mu}_0$, where $\omega = \pi^*\omega_0$ is semipositive and big and $\widetilde{\mu}_0 = \prod_i |f_i|^{2a_i} dV$.
- The potential φ belongs to $\mathcal{H}$ and maximizes the functional
$$\mathcal{F}(\varphi) := E(\varphi) + \log\left[\int_{\tilde{X}} e^{-\varphi} d\widetilde{\mu}_0\right].$$
Conversely any maximizer of $\mathcal{F}$ is a Kähler-Einstein metric.
- Two Kähler-Einstein metrics are connected by the flow of a holomorphic vector field that leaves D invariant.
- If the functional $\mathcal{F}$ is *proper* (i.e. if $E(\varphi_j) \to -\infty \Rightarrow \mathcal{F}(\varphi_j) \to -\infty$), then there exists a unique Kähler-Einstein metric.

Here $[D]$ is the integration current on $D|_{Y_{\mathrm{reg}}}$. Writing $\mathrm{Ric}(\omega_{KE})$ on Y_{reg} implicitely means that the positive measure $\omega_{KE}^n|_{Y_{\mathrm{reg}}}$ corresponds to a singular metric on $-K_{Y_{\mathrm{reg}}}$, whose curvature is then $\mathrm{Ric}(\omega_{KE})$ by definition.

5.3 The analytic criterion

Following and idea of Darvas-Rubinstein [DR15], we now extend [Tian97, Theorem 1.6] and [PSSW08] by proving the following:

Theorem 5.5 *Let (Y, D) be a log Fano pair. It admits a unique Kähler-Einstein metric iff there exists $\varepsilon, M > 0$ such that for all $\varphi \in \mathcal{H}_{norm}$,*
$$\mathcal{F}(\varphi) \leq -\varepsilon d_1(0, \varphi) + M.$$

This is Theorem D of the introduction.

Proof. We are going to use Theorem B. Note that $\omega_Y \in c_1(-K_X - D)$ is a Hodge form. One implication is due to [BBEGZ, Theorems 4.8 and 5.4]: if
$$\mathcal{F}(\varphi) \leq -\varepsilon d_1(0, \varphi) + M,$$
then $\mathcal{F}$ is proper, hence there exists a unique Kähler-Einstein metric.

So we assume now that there exists ω a unique Kähler-Einstein metric, which we take as our base point of $\mathcal{H}$. It is the unique maximizer of $\mathcal{F}$ on $\mathcal{E}^1(X, \omega)$,
$$\mathcal{F}(0) = \sup_{\varphi \in \mathcal{E}^1(X,\omega)} \mathcal{F}(\varphi),$$
as follows from [BBGZ13, Theorem 6.6], [BBEGZ, Theorems 4.8 and 5.3].

Note that $\mathcal{F}$ is invariant by translations, so we actually consider the restriction of $\mathcal{F}$ on $\mathcal{E}^1_{norm}(X, \omega) = \{\varphi \in \mathcal{E}^1(X, \omega),\ \sup_X \varphi = 0\}$. Assume for contradiction that there is no $\varepsilon > 0$ such that $\mathcal{F}(\varphi) \leq -\varepsilon d_1(0, \varphi) + M$ for all $\varphi \in \mathcal{H}_{norm}$, where we set $M := \mathcal{F}(0) + 1$. Then we can find a sequence $(\varphi_j) \in \mathcal{H}^{\mathbf{N}}$ such that $\sup_X \varphi_j = 0$ and
$$\mathcal{F}(\varphi_j) > -\frac{d_1(0, \varphi_j)}{j+1} + \mathcal{F}(0) + 1.$$

If $E(\varphi_j)$ does not blow up to $-\infty$, we reach a contradiction: up to extracting and relabelling, we can assume that $E(\varphi_j)$ is bounded and φ_j converges to some $\psi \in \mathcal{E}^1(X,\omega)$. Since $\mathcal{F}$ is upper semi-continuous, we infer $\mathcal{F}(\psi) \geq \mathcal{F}(0)+1$, a contradiction.

So we assume now that $E(\varphi_j) \to -\infty$. It follows from Lemma 3.12 that $d_j := d_1(0,\varphi_j) = -E(\varphi_j) \to +\infty$. We let $(\varphi_{t,j})_{0\leq t\leq d_j}$ denote the Mabuchi geodesic with unit speed joining 0 to φ_j and set $\psi_j := \varphi_{1,j}$. Note that the arguments in Lemma 3.3 show that $t \mapsto \varphi_{t,j}$ is decreasing, hence $\varphi_j \leq \psi_j \leq 0$. In particular $\sup_X \psi_j = 0$, while by definition $d_1(0,\psi_j) = 1 = -E(\psi_j)$.

It follows now from Berndtsson's convexity result [Bern15, Section 6.2] and its generalization to the singular context [BBEGZ, Theorem 11.1] that the map $t \mapsto \mathcal{F}(\varphi_{t,j})$ is concave. We infer

$$0 \geq \mathcal{F}(\varphi_{1,j}) - \mathcal{F}(\varphi_{0,j}) \geq \frac{\mathcal{F}(\varphi_{d_j,j}) - \mathcal{F}(\varphi_{0,j})}{d_j} > -\frac{1}{j+1} + \frac{1}{d_j},$$

thus $\mathcal{F}(\psi_j) \to \mathcal{F}(0)$. This shows that (ψ_j) is a maximizing sequence for $\mathcal{F}$ which therefore strongly converges to 0, by [BBEGZ, Theorem 5.3.3]. This yields a contradiction since $d_1(0,\psi_j) = 1$. $\square$

6. The toric case

Recall that a compact Kähler *toric* manifold (X,ω,T) is an equivariant compactification of the torus $T = (\mathbf{C}^*)^n$ equipped with a $(S^1)^n$-invariant Kähler metric ω which can be written

$$\omega = dd^c\psi \text{ in } (\mathbf{C}^*)^n,$$

with ψ $(S^1)^n$-invariant hence $\psi(z) = F \circ L(z)$ where

$$L : z \in (\mathbf{C}^*)^n \mapsto (\log|z_1|, \cdots, \log|z_n|) \in \mathbf{R}^n$$

and $F : \mathbf{R}^n \to \mathbf{R}$ is strictly convex.

The celebrated Atiyah-Guillemin-Sternberg theorem asserts that the moment map $\nabla F : \mathbf{R}^n \to \mathbf{R}^n$ sends $\mathbf{R}^n$ to a bounded convex polytope

$$P = \{\ell_i(s) \geq 0,\ 1 \leq i \leq d\} \subset \mathbf{R}^n$$

where $d \geq n+1$ is the number of $(n-1)$-dimensional faces of P,

$$\ell_i(s) = \langle s, u_i\rangle - \lambda_i,$$

with $\lambda_i \in \mathbf{R}$ and u_i is a primitive element of $\mathbf{Z}^n$, normal to the ith $(n-1)$-dimensional face of P.

Delzant observed in [Del88] that in this case P is "Delzant", i.e. there are exactly n faces of dimension $(n-1)$ meeting at each vertex, and the corresponding u_j's form a $\mathbf{Z}$-basis of $\mathbf{Z}^n$. He conversely showed that there is exactly one (up to symplectomorphism) compact toric Kähler manifold $(X_P, \{\omega_P\}, T)$ associated to a Delzant polytope $P \subset \mathbf{R}^n$. Here $\{\omega_P\}$ denotes the cohomology class of the T-invariant Kähler form ω_P. Let

$$G(s) := \sup_{x\in\mathbf{R}^n} \{\langle x, s\rangle - F(x)\}$$

denote the Legendre transform of F. Observe that $G = +\infty$ in $\mathbf{R}^n \setminus P$ and for $s \in P = \nabla F(\mathbf{R}^n)$,

$$G(s) = \langle x, s\rangle - F(x) \text{ with } \nabla F(x) = s \Leftrightarrow \nabla G(s) = x.$$

Guillemin observed in [Gui94] that a "natural" representative of the cohomology class $\{\omega_P\}$ is

given by

$$G_{ref}(s) = \frac{1}{2}\left\{\sum_{i=1}^{d} \ell_i(s)\log \ell_i(s) + \ell_\infty(s)\log \ell_\infty(s)\right\}$$

where $\ell_\infty(s) = \sum_{i=1}^{d}\langle s, u_i\rangle$. We refer the reader to [CDG03] for a neat proof of this beautiful formula of Guillemin.

Example 6.1 *When $X = \mathbf{CP}^n$ and ω is the Fubini-Study Kähler form, then $F_{ref}(x) = \frac{1}{2}\log\left[1 + \sum_{i=1}^{n} e^{2x_i}\right]$, $P = \nabla F_{ref}(\mathbf{R}^n)$ is the simplex*

$$P = \left\{s_i \geq 0,\ 1 \leq i \leq n \text{ and } \sum_{i=1}^{n} s_i \leq 1\right\},$$

thus $d = n+1$, $\ell_i(s) = s_i$, $\lambda_i = 0$, $u_i = e_i$ for $1 \leq i \leq n$, $\ell_{n+1}(s) = 1 - \sum_{i=1}^{n} s_i$, $\lambda_{n+1} = -1$, $e_{n+1} = -\sum_{j=1}^{n} e_j$ and $\ell_\infty \equiv 0$ so that

$$G_{ref}(s) = \frac{1}{2}\left\{\sum_{i=1}^{n} s_i \log s_i + \left(1 - \sum_{j=1}^{n} s_j\right)\log\left(1 - \sum_{j=1}^{n} s_j\right)\right\}.$$

6.1 Toric geodesics

Let (X, ω, T) be a compact toric manifold. Here (and through all the section) ω is a genuine Kähler form. In the sequel we let $PSH_{tor}(X,\omega), \mathcal{E}_{tor}(X,\omega), \mathcal{H}_{tor}$ denote the $(S^1)^n$-invariant versions of the classes of ω-psh functions we have considered so far.

If $\varphi_0, \varphi_1 \in \mathcal{H}_{tor}$ are both $(S^1)^n$-invariant, it follows from the uniqueness that the geodesic $(\varphi_t)_{0\leq t\leq 1}$ consists of $(S^1)^n$-invariant functions. Let F_t denote the corresponding potentials in $\mathbf{R}^n$ so that

$$F_t \circ L = F_{ref} \circ L + \varphi_t \text{ in } (\mathbf{C}^*)^n.$$

Proposition 6.2 *[Guan99] The map $(x,t) \mapsto \varphi_t(x)$ is smooth and corresponds to the Legendre transform of an affine path on P. In other words the Legendre transform G_t of F_t is affine in t.*

We include the proof for the reader's convenience.

Proof. Recall that

$$G_t(s) = \sup_{x\in\mathbf{R}^n}\{\langle x, s\rangle - F_t(x)\} = \langle x_t, s\rangle - F_t(x_t),$$

where $x_t = x_t(s)$ is such that $\nabla F_t(x_t) = s$. Taking derivatives of this identity with respect to t yields

$$\left[\frac{\partial^2 F_t}{\partial x_i \partial x_j}\right]\cdot[\dot{x}_t] = -\nabla \dot{F}_t$$

hence $\dot{G}_t(s) = -\dot{F}_t(x_t)$ and

$$\begin{aligned}\ddot{G}_t(s) = -\ddot{F}_t(x) - \langle \dot{x}_t, \nabla \dot{F}_t\rangle &= -\ddot{F}_t(x) + \left\langle \left[\frac{\partial^2 F_t}{\partial x_i \partial x_j}\right]^{-1}\cdot \nabla \dot{F}_t, \nabla \dot{F}_t\right\rangle \\ &= -\ddot{F}_t(x) + \left|\nabla \dot{F}_t\right|^2_{\omega_t}.\end{aligned}$$

Therefore (φ_t) is a geodesic if and only if $\ddot{G}_t \equiv 0$. □

In a similar vein we obtain an explicit formula for the Mabuchi distance between φ_0 and φ_1:

Proposition 6.3 *For all $q \geq 1$ and $\varphi_0, \varphi_1 \in \mathcal{H}_{tor}$,*

$$d_q(\varphi_0, \varphi_1) = (\frac{\pi}{2})^{n/q} ||G_1 - G_0||_{L^q(P)} = (\frac{\pi}{2})^{n/q} \left(\int_P |G_1 - G_0|^q(s) ds \right)^{1/q}.$$

Proof. Recall that

$$d_q(\varphi_0, \varphi_1) = \left(\int_X |\dot{\varphi}_0|^q MA(\varphi_0) \right)^{1/q}.$$

Now $F_t \circ L = F_{ref} \circ L + \varphi_t$ has Legendre transform $G_t = tG_1 + (1-t)G_0$. Thus $\dot{\varphi}_t = \dot{F}_t \circ L$ with $G_t(s) = \langle x_t, s \rangle - F_t(x_t)$ with $s = \nabla F_t(x_t)$ hence $\dot{G}_t(s) = -\dot{F}_t(x)$ and we infer

$$d_q(\varphi_0, \varphi_1)^q = \int_{(\mathbf{C}^*)^n} |\dot{F}_0 \circ L|^q MA(F_0 \circ L).$$

Observe that

$$\frac{\partial^2 (F_0 \circ L)}{\partial z_i \partial \overline{z_j}} = \frac{1}{4} \frac{1}{\overline{z_i} z_j} \cdot \frac{\partial^2 F_0}{\partial x_i \partial x_j} \circ L \ \text{ in } \ (\mathbf{C}^*)^n$$

hence

$$\det \left(\frac{\partial^2 (F_0 \circ L)}{\partial z_i \partial \overline{z_j}} \right) = \left(\frac{1}{4} \right)^n \frac{1}{\Pi_j |z_j|^2} \cdot MA_{\mathbf{R}}(F_0) \circ L,$$

where $MA_{\mathbf{R}}$ denotes the real Monge-Ampère measure (in the sense of Alexandrov, see [Gut01]) of the convex function F_0. Thus

$$\int_{(\mathbf{C}^*)^n} |\dot{F}_0 \circ L|^q MA(F_0 \circ L) = \left(\frac{\pi}{2} \right)^n \int_{\mathbf{R}^n} |\dot{F}_0|^q MA_{\mathbf{R}}(F_0).$$

Now $\dot{F}_0 = -\dot{G}_0 \circ \nabla F_0$ and $MA_{\mathbf{R}}(F_0) = (\nabla F_0)^* ds$ therefore

$$\int_{\mathbf{R}^n} |\dot{F}_0|^q MA_{\mathbf{R}}(F_0) = \int_P |\dot{G}_0|^q(s) ds = \int_P |G_1 - G_0|^q(s) ds.$$

□

Example 6.4 *Assume $X = \mathbf{CP}^1$ is the Riemann sphere and ω is the Fubini-Study Kähler form. Let φ_0 be the toric function associated to the convex potential*

$$F_0(x) = \max(x, 0) \text{ so that } G_0(s) \equiv 0 \text{ on the simplex } P = [0, 1].$$

Observe that $\omega_0 = dd^c F_0 \circ L$ is the (normalized) Lebesgue measure on the unit circle $S^1 \subset \mathbf{C}^* \subset \mathbf{CP}^1$. We consider $\varphi_1 = \varphi_j$ a sequence of toric potentials defined by the convex functions

$$F_j(x) = (1 - \varepsilon_j) F_0(x) + \varepsilon_j \max(x, -C_j),$$

where ε_j decreases to 0, while C_j increases to $+\infty$. A straightforward computation yields $G_j(s) = \max(C_j[\varepsilon_j - s], 0)$. Therefore

$$d_q(\varphi_j, \varphi_0) = \frac{C_j \varepsilon_j^{1+1/q}}{(q+1)^{1/q}} \left(\frac{\pi}{2} \right)^{n/q}$$

We thus obtain in this case, as $j \to +\infty$,

- *$\varphi_j \longrightarrow \varphi_0$ in L^1 iff $\varepsilon_j \to 0$;*
- *$\varphi_j \longrightarrow \varphi_0$ in L^∞ iff $\varepsilon_j C_j \to 0$;*

– $\varphi_j \longrightarrow \varphi_0$ *in* $(\mathcal{E}^q(X,\omega), d_q)$ *iff* $\varepsilon_j^{1+1/q} C_j \to 0$*;*

The convergence in $(\mathcal{E}^1(X,\omega), d_1)$ *is here the convergence in the Sobolev norm* $W^{1,2}$*. For* $\varepsilon_j = 1/j$*,* $C_j = j^{3/2}$ *we therefore obtain an example of a sequence which converges in the Sobolev sense but not in the Mabuchi metric* d_2*. Observe that this example also shows that the convergence in the Mabuchi sense is weaker than the uniform convergence.*

6.2 Toric singularities

Let $\varphi \in \mathcal{H}_\omega$ be a toric potential. We are going to read off the singular behavior of φ from the integrability properties of the Legendre transform of its associated convex potential.

We let F_φ and G_φ denote the corresponding convex function and its Legendre transform. The function φ is bounded if and only if so is $F_\varphi - F_{ref}$ on $\mathbf{R}^n$, since $F_\varphi \circ L = F_{ref} \circ L + \varphi$, if and only if so is G_φ on P, as G_{ref} (Guillemin's potential) is continuous on P. The same conclusion holds if we take as a reference potential the support function F_P of P, defined by

$$F_P(x) := \sup_{s \in P} \langle s, x \rangle.$$

It is the Legendre transform of the function G_P which is identically 0 on P and $+\infty$ in $\mathbf{R}^n \setminus P$. We can similarly understand finite energy classes:

Proposition 6.5

$$\varphi \in PSH_{tor}(X,\omega) \cap L^\infty(X) \Longleftrightarrow G_\varphi \in L^\infty(P).$$
$$\varphi \in \mathcal{E}^q_{tor}(X,\omega) \Longleftrightarrow G_\varphi \in L^q(P).$$

We refer the reader to [BerBer13, Proposition 2.9] for an elegant proof of this result when $q = 1$.

Proof. We first show that $\varphi \in \mathcal{E}^q_{tor}(X,\omega) \Longrightarrow G_\varphi \in L^q(P)$. Approximating φ from above by a decreasing sequence of smooth strictly ω-psh toric functions, this boils down to show a uniform a priori bound

$$||G_\varphi||_{L^q(P)} \leq C \left(\int_X |\varphi_P - \varphi|^q MA(\varphi) \right)^{1/q}.$$

for some uniform constant $C > 0$. We can assume without loss of generality that $F_\varphi \leq F_P$ (since φ is upper semi-continuous hence bounded from above on X which is compact). Recall that $\varphi = (F_\varphi - F_{ref}) \circ L$ in $(\mathbf{C}^*)^n$, where F_{ref} denotes a reference potential associated to ω. Changing variables and using the Legendre transform yields

$$\begin{aligned}\int_{(\mathbf{C}^*)^n} |\varphi - \varphi_P|^q MA(\varphi) &= \left(\frac{\pi}{2}\right)^n \int_{\mathbf{R}^n} |F_\varphi - F_P|^q MA_{\mathbf{R}}(F_\varphi) \\ &= \left(\frac{\pi}{2}\right)^n \int_P |F_\varphi \circ \nabla G_\varphi(s) - F_P \circ \nabla G_\varphi(s)|^q ds,\end{aligned}$$

where $F_\varphi(x) = \langle x, s \rangle - G_\varphi(s)$, with $\nabla G_\varphi(s) = x$. Therefore

$$F_\varphi(\nabla G_\varphi(s)) = \langle \nabla G_\varphi(s), s \rangle - G_\varphi(s)$$

and

$$\begin{aligned}F_P(\nabla G_\varphi(s)) - F(\nabla G_\varphi(s)) &= G_\varphi(s) - \{\langle \nabla G_\varphi(s), s \rangle - F_P \circ \nabla G_\varphi(s)\} \\ &\geq G_\varphi(s) - G_P(s) = G_\varphi(s) \geq 0,\end{aligned}$$

since $G_P(s) = \sup_{x\in\mathbf{R}^n}\{\langle x,s\rangle - F_P(x)\} = 0$ for $s \in P$. We infer

$$||G_\varphi||^q_{L^q(P)} \leq \int_P |F_P(\nabla G_\varphi(s)) - F_\varphi(\nabla G_\varphi(s))|^q\, ds$$
$$\leq \left(\frac{2}{\pi}\right)^n \int_X |\varphi_P - \varphi|^q MA(\varphi).$$

We now take care of the converse implication. Assume $\varphi \in PSH_{tor}(X,\omega)$ is such that $\varphi \leq 0$ and $G_\varphi \in L^q(P)$. It follows then from Proposition 3.8 and Proposition 6.3 that

$$\int_X (-\varphi)^q MA(\varphi) \leq 2^{q+n} d_q(0,\varphi)^q = 2^{q+n} C(n)||G_\varphi - G_0||^q_{L^q(P)} < +\infty,$$

hence $\varphi \in \mathcal{E}^q_{tor}(X,\omega)$, as claimed. □

It also follows from the previous arguments that:

Theorem 6.6 *The metric completion of $(\mathcal{H}_{tor}, d_q)$ is $(\mathcal{E}^q_{tor}, d_p)$.*

Remark 6.7 *We let the reader check that the Legendre transform $G_{\varphi\vee\psi}$ of the minimum of two convex functions is*

$$G_{\varphi\vee\psi} = \max(G_\varphi, G_\psi).$$

The orthogonality relation from Proposition 3.4 thus translates here

$$d_p(\varphi, \varphi\vee\psi)^p = \int_P (G_{\varphi\vee\psi} - G_\varphi)^p = \int_{\{G_\varphi < G_\psi\}} (G_\psi - G_\varphi)^p\,,$$

while $d_p(\psi, \varphi\vee\psi)^p = \int_{\{G_\varphi > G_\psi\}} (G_\varphi - G_\psi)^p$ so that

$$d_p(\varphi, \varphi\vee\psi)^p + d_p(\varphi\vee\psi, \psi)^p = d_p(\varphi,\psi)^p.$$

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