The equivalence between many-to-one polygraphs and opetopic sets

Cédric Ho Thanh

THE EQUIVALENCE BETWEEN MANY-TO-ONE POLYGRAPHS AND OPETOPIC SETS

CÉDRIC HO THANH

Abstract. From the polynomial approach to the definition of opetopes of Kock et al., we derive a category of opetopes, and show that its set-valued presheaves, or opetopic sets, are equivalent to many-to-one polygraphs. As an immediate corollary, we establish that opetopic sets are equivalent to multitopic sets, introduced and studied by Harnick et al, and we also address an open question of Henry.

Contents

1. Introduction

Opetopes were originally introduced by Baez and Dolan in [BD98] as an algebraic structure to describe compositions and coherence laws in weak higher dimensional categories. They differ from other shapes (such as globular or simplicial) by their (higher) tree structure, giving them the informal designation of "many-to-one". Pasting opetopes give rise to opetopes of higher dimension (it is in fact how they are defined!), and the analogy between opetopes and cells in a free higher category starts to emerge. On the other hand, polygraphs (also called computads) are higher dimensional directed graphs used to generate free higher categories by specifying generators and the way they may be pasted together (by means of source and targets).

In this paper, we relate opetopes and polygraphs in a direct way. Namely, we define a category $\mathbb{O}$ whose objects are opetopes, in such a way that the category of its $\mathcal{S}$et-valued presheaves, or opetopic sets, is equivalent to the category of many-to-one polygraphs. This equivalence was already known from [HMZ02, HMZ08, HMP00], however the proof is very indirect. The recent work of Henry [Hen17] showed the category of many-to-one polygraphs (among many others) to be a presheaf category, but left the equivalence between "opetopic plexes" (serving as shapes for many-to-one polygraphs in his paper) and opetopes open. We establish this in our present work.

The notion of multitope [HMP02, HMZ08] is related to that of opetope, and has been developed based on similar motivations. However the approaches used are different: *ope*topes are based on *ope*rads[1] [Lei04], while *multi*topes are based on (symmetric) *multi*categories. It is known that multitopic sets are equivalent to many-to-one polygraphs [HMZ08, HMZ02], and in particular our present contribution reasserts the equivalence between multitopic and opetopic sets.

2010 *Mathematics Subject Classification.* Primary 18D50; Secondary 18C20.
Key words and phrases. Opetope, Polynomial functor, Polygraph, Computad.

[1]Specifically, T_n-operads, where T_n is a certain sequence of cartesian monads

Plan. We begin by recalling elements of the theory of polynomial functors and polynomial monads section 2. This formalism is at the base of our chosen approach to opetopes, which we present in section 3. In section 4, we review some basic polygraphs theory, and pay special attention to those that are many-to-one. Finally, we state and prove the equivalence between opetopic sets and many-to-one polygraphs in section 5.

Acknowledgments. I would like to thank my PhD advisors, Pierre-Louis Curien and Samuel Mimram, for their kind attention and guidance. This project has received funding from the European Union's Horizon 2020 research and innovation program under the Marie Sklodowska-Curie grant agreement No 665850.

2. Polynomial functors and polynomial monads

We survey elements of the theory of polynomial functors, trees, and monads. For more comprehensive references, see [Koc11, GK13].

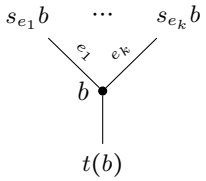
2.1. **Polynomial functors.**

Definition 2.1 (Polynomial functor)**.** A *polynomial (endo)functor P over I*[2] is a diagram in $\mathcal{S}\mathrm{et}$ of the form

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I. \tag{2.2}$$

P is said to be *finitary* if the fibres of $p : E \longrightarrow B$ are finite sets. We will always assume polynomial functors to be finitary.

We use the following terminology for a polynomial functor P as in equation (2.2), which is motivated by the intuition that a polynomial functor encodes a multi-sorted signature of function symbols. The elements of B are called the *nodes* or *operations* of P, and for every node b, the elements of the fibre $E(b) \coloneqq p^{-1}(b)$ are called the *inputs* of b. The elements of I are called the *colours* or *sorts* of P. For every input e of a node b, we denote its colour by $s_e(b) \coloneqq s(e)$.



Definition 2.3 (Morphism of polynomial functor)**.** A morphism from a polynomial functor P over I (as in equation (2.2)) to a polynomial functor P' over I' (on the second row) is a commutative diagram of the form

$$\begin{array}{ccccccc} I & \xleftarrow{s} & E & \xrightarrow{p} & B & \xrightarrow{t} & I \\ {\scriptstyle f_0}\downarrow & & {\scriptstyle f_2}\downarrow & \lrcorner & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_0}\downarrow \\ I' & \xleftarrow{s'} & E' & \xrightarrow{p'} & B' & \xrightarrow{t} & I' \end{array}$$

where the middle square is cartesian (i.e. is a pullback square). If P and P' are both polynomial functors over I, then a morphism from P to P' *over* I is a commutative diagram as above, but where f_0 is required to be the identity. Let $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$ denote the category of polynomial functors and morphisms of polynomial functors, and $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$ the category of polynomial functors over I and morphisms of polynomial functors over I.

2.2. **Trees.**

The combinatorial notion of a tree fits nicely in the framework of polynomial functors. We now state the definition of a polynomial tree, and refer the reader to [Koc11, section 1.0.3] for more details about the intuition behind it.

Definition 2.4 (Polynomial tree)**.** A polynomial functor T given by

$$T_0 \xleftarrow{s} T_2 \xrightarrow{p} T_1 \xrightarrow{t} T_0$$

is a *(polynomial) tree* (or just *tree*) [Koc11, section 1.0.3] if

[2]The denomination "functor" comes from the fact that such a diagram induces a functor $\mathcal{S}\mathrm{et}/I \xrightarrow{s^*} \mathcal{S}\mathrm{et}/E \xrightarrow{\Pi_p} \mathcal{S}\mathrm{et}/B \xrightarrow{\Sigma_t} \mathcal{S}\mathrm{et}/I$ by composition of the pullback along s, dependent product along p, and dependent sum along t, respectively.

(1) the sets T_0, T_1 and T_2 are finite (in particular, each node has finitely many inputs);
(2) the map t is injective;
(3) the map s is injective, and the complement of its image $T_0 - \operatorname{im} s$ has a single element, called the *root*;
(4) let $T_0 = T_2 + \{r\}$, with r the root, and define the *walk-to-root* function σ by $\sigma(r) = r$, and otherwise $\sigma(e) = tp(e)$; then we ask that for all $x \in T_0$, there exists $k \in \mathbb{N}$ such that $\sigma^k(x) = r$.

We call the colours of a tree its *edges* and the inputs of a node the *input edges* of that node.

Let $\mathcal{T}\mathrm{ree}$ be the full subcategory of $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$ whose objects are trees. Note that it is the category of *symmetric* or *non-planar* trees (the automorphism group of a tree is in general non-trivial) and that its morphisms correspond to inclusions of non-planar subtrees. An *elementary tree* is a tree with at most one node. Let $el\mathcal{T}\mathrm{ree}$ be the full subcategory of $\mathcal{T}\mathrm{ree}$ spanned by elementary trees.

Definition 2.5 (P-tree)**.** For $P \in \mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$, the category $\mathrm{tr}\, P$ of *P-trees* is the slice $\mathcal{T}\mathrm{ree}/P$. The fundamental difference between $\mathcal{T}\mathrm{ree}$ and $\mathrm{tr}\, P$ is that the latter is always rigid i.e. it has no non-trivial automorphisms [Koc11, proposition 1.2.3]. In particular, this implies that $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$ does not have a terminal object.

Notation 2.6. Every P-tree $T \in \mathrm{tr}\, P$ corresponds to a morphism from a tree (which we shall denote by $\langle T \rangle$) to P, so that $T : \langle T \rangle \longrightarrow P$. We point out that $\langle T \rangle_1$ is the set of nodes of $\langle T \rangle$, while $T_1 : \langle T \rangle_1 \longrightarrow P_1$ is a decoration of the nodes of $\langle T \rangle$ by nodes of P, and likewise for edges.

Definition 2.7 (Address)**.** Let $T \in \mathcal{T}\mathrm{ree}$ be a polynomial tree and σ be its walk-to-root function (definition 2.4). We define the *address* function & on edges inductively as follows:

(1) if r is the root edge, let $\&r := []$,
(2) if $e \in T_0 - \{r\}$ and if $\&\sigma(e) = [x]$, define $\&e := [xe]$.

The address of a node $b \in T_1$ is defined as $\&b := \&t(b)$. Note that this function is injective since t is. Let $T^\bullet$ denote its image, the set of *node addresses* of T, and let $T^|$ be the set of addresses of leaf edges, i.e. those not in the image of t.

Assume now that $T : \langle T \rangle \longrightarrow P$ is a P-tree. If $b \in \langle T \rangle_1$ has address $\&b = [p]$, write $\mathsf{s}_{[p]}\, T := T_1(b)$. For convenience, we let $T^\bullet := \langle T \rangle^\bullet$, and $T^| := \langle T \rangle^|$.

Remark 2.8. The formalism of addresses is a useful bookkeeping syntax for the operations of grafting and substitution on trees. The syntax of addresses will extend to the category of opetopes and will allow us to give a precise description of the composition of morphisms in the category of opetopes (see definition 3.6) as well as certain constructions on opetopic sets.

Notation 2.9. We denote by $\mathrm{tr}^|\, P$ the set of P-trees with a marked leaf, i.e. endowed with the address of one of its leaves. Similarly, we denote by $\mathrm{tr}^\bullet\, P$ the set of P-trees with a marked node.

Definition 2.10 (Elementary P-trees)**.** Let P be a polynomial endofunctor as in equation equation (2.2). For $i \in I$, define $\mathsf{I}_i \in \mathrm{tr}\, P$ as having underlying tree

$$\{i\} \longleftarrow \varnothing \longrightarrow \varnothing \longrightarrow \{i\}, \tag{2.11}$$

along with the obvious morphism to P, that which maps i to $i \in I$. This corresponds to a tree with no nodes and a unique edge decorated by i. Define $\mathsf{Y}_b \in \mathrm{tr}\, P$, the *corolla* at b, as having underlying tree

$$s(E(b)) + \{*\} \xleftarrow{s} E(b) \longrightarrow \{b\} \longrightarrow s(E(b)) + \{*\}, \tag{2.12}$$

where the right map sends b to $*$, and where the morphism $\mathsf{Y}_b \longrightarrow P$ is the identity on $s(E(b)) \subseteq I$, maps $*$ to $t(b) \in I$, is the identity on $E(b) \subseteq E$, and maps b to $b \in B$. This corresponds to a P-tree with a unique node, decorated by b. Observe that for $T \in \mathrm{tr}\, P$, giving a morphism $\mathsf{I}_i \longrightarrow T$ is equivalent to specifying the address $[p]$ of an edge of T decorated by i. Likewise, morphisms of the form $\mathsf{Y}_b \longrightarrow T$ are in bijection with addresses of nodes of T decorated by b.

Remark 2.13. Let P be a polynomial endofunctor as in equation equation (2.2).

(1) Let $i \in I$ be a color of P. Since I_i does not have any nodes, the set of its node addresses $\mathsf{I}_i^\bullet = \varnothing$. On the other hand, the set of its leaf addresses is $\mathsf{I}_i^| = \{[]\}$, since the unique leaf is the root edge.

(2) Let $b \in B$ be an operation of P. Then $\mathsf{Y}_b^\bullet = \{[]\}$ since the only node is that above the rood edge. For leaves, we have $\mathsf{Y}_b^| = \{[e] \mid e \in E(b)\}$.

Definition 2.14 (Grafting)**.** For $S, T \in \mathrm{tr}\, P$, $[l] \in S^|$ such that the leaf of S at $[l]$ and the root edge of T are decorated by the same $i \in I$, define the *grafting* $S \circ_{[l]} T$ of S and T on $[l]$ by the following pushout (in $\mathrm{tr}\, P$):

$$\begin{array}{ccc} \mathsf{l}_i & \xrightarrow{[]} & T \\ {\scriptstyle [l]}\downarrow & \ulcorner & \downarrow \\ S & \longrightarrow & S \circ_{[l]} T. \end{array} \tag{2.15}$$

Note that if S (resp. T) is a trivial tree, then $S \circ_{[l]} T = T$ (resp. S). We assume, by convention, that the grafting operator $\circ$ associates to the right.

Lemma 2.16. *For $S, T \in \mathrm{tr}\, P$, $[l] \in S^|$ such that the grafting $S \circ_{[l]} T$ is defined, we have*

$$(S \underset{[l]}{\circ} T)^\bullet = S^\bullet + \{[lp] \mid [p] \in T^\bullet\}, \qquad (S \underset{[l]}{\circ} T)^| = S^| - \{[l]\} + \left\{[lp] \mid [p] \in T^|\right\}.$$

Notation 2.17 (Total grafting). Take $T, U_1, \ldots, U_k \in \mathrm{tr}\, P$, where $T^| = \{[l_1], \ldots, [l_k]\}$, and assume the grafting $T \circ_{[l_i]} U_i$ is defined for all i. Then the *total grafting* will be denoted concisely by

$$T \bigcirc_{[l_i]} U_i = (\cdots (T \underset{[l_1]}{\circ} U_1) \underset{[l_2]}{\circ} U_2 \cdots) \underset{[l_k]}{\circ} U_k. \tag{2.18}$$

It is easy to see that the result does not depend on the order in which the graftings are performed.

Proposition 2.19 ([Koc11, proposition 1.1.21])**.** *Every P-tree is either of the form l_i, for some $i \in I$, or obtained by iterated graftings of corollas (i.e. P-trees of the form Y_b for $b \in B$).*

Remark 2.20. As a consequence of [Koc11, proposition 1.1.3], a morphism $T \longrightarrow S$ of P-trees exhibits T as a subtree of S as in

$$S = U \underset{[p]}{\circ} T \bigcirc_{[l]} V_{[l]}$$

where U is spanned by all the edges of S that are either descendant of the root edge of T, or incomparable to it [Koc11, paragraphs 1.0.7 and 1.1.11], and where $[l]$ ranges over $T^|$. Conversely, any such decomposition of S induces a morphism $T \longrightarrow S$.

2.3. **Polynomial monads.**

Definition 2.21 (Polynomial monad)**.** A *polynomial monad over I* is a monoid in $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$. Note that a polynomial monad over I is thus necessarily a cartesian monad on $\mathcal{S}\mathrm{et}/I$.[3] Let $\mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd}(I)$ be the category of monoids in $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$. That is, $\mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd}(I)$ is the category of polynomial monads over I and morphisms of polynomial functors over I that are also monad morphisms.

Definition 2.22 ($(-)^\star$ construction)**.** Given a polynomial endofunctor P as in equation equation (2.2), we define a new polynomial endofunctor $P^\star$ as

$$I \xleftarrow{\ s\ } \mathrm{tr}^|\, P \xrightarrow{\ p\ } \mathrm{tr}\, P \xrightarrow{\ r\ } I \tag{2.23}$$

where s maps a P-tree with a marked leaf to the decoration of that leaf, p forgets the marking, and r maps a tree to the decoration of its root. Remark that for $T \in \mathrm{tr}\, P$ we have $p^{-1}T = T^|$.

Theorem 2.24 ([Koc11, section 1.2.7], [KJBM10, sections 2.7 to 2.9])**.** *The polynomial functor $P^\star$ has a canonical structure of a polynomial monad. Furthermore, the functor $(-)^\star$ is left adjoint to the forgetful functor $\mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd}(I) \longrightarrow \mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$, and the adjunction is monadic.*

[3] We recall that a monad is *cartesian* if its endofunctor preserves pullbacks and its unit and multiplication are cartesian natural transformations.

Definition 2.25 (Readdressing function)**.** We abuse notation slightly by letting $(-)^\star$ denote the associated monad on $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$. Let M be a polynomial monad as on the left below. Buy theorem 2.24, M is a $(-)^\star$-algebra, and we will write its structure map $M^\star \longrightarrow M$ as on the right:

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I, \qquad \begin{array}{ccccccc} I & \longleftarrow & \mathrm{tr}^{|} M & \longrightarrow & \mathrm{tr}\, M & \longrightarrow & I \\ \| & & \downarrow{\scriptstyle \wp} & \lrcorner & \downarrow{\scriptstyle \mathsf{t}} & & \| \\ I & \longleftarrow & E & \longrightarrow & B & \longrightarrow & I. \end{array} \tag{2.26}$$

For $T \in \mathrm{tr}\, M$, we call $\mathsf{t}T \in B$ the *target* of T. We call $\wp_T : T^{|} \xrightarrow{\cong} E(rT)$ the *readdressing* function of T, and $\mathsf{t}T \in B$ is called the *target* of T. If we think of an element $b \in B$ as the corolla Y_b, then the target map t "contracts" a tree to a corolla, and since the middle square is a pullback, the number of leaves is preserved. The map $\wp_T$ establishes a coherent correspondence between the set $T^{|}$ of leaf addresses of a tree T and the elements of $E(\mathsf{t}T)$.

Definition 2.27 (Baez–Dolan $(-)^+$ construction)**.** Let M be a polynomial monad as in equation (2.2), and define its *Baez–Dolan construction* M^+ to be

$$B \xleftarrow{\mathsf{s}} \mathrm{tr}^\bullet M \xrightarrow{p} \mathrm{tr}\, M \xrightarrow{\mathsf{t}} B \tag{2.28}$$

where s maps an M-tree with a marked node to the label of that node, p forgets the marking, and t is the target map of definition 2.25. If $T \in \mathrm{tr}\, M$, remark that $p^{-1}T = T^\bullet$ is the set of node addresses of T. If $[p] \in T^\bullet$, then $\mathsf{s}[p] \coloneqq \mathsf{s}_{[p]}\, T$.

Theorem 2.29 ([KJBM10, section 3.2])**.** *If M a polynomial monad, then M^+ has a canonical structure of a polynomial monad.*

Remark 2.30 (Nested addresses)*.* Let M be a polynomial monad, and $T \in \mathrm{tr}\, M^+$. Then the nodes of T are decorated in M-trees, and its edge by operations of M. Assume that $U \in \mathrm{tr}\, M$ decorates some node of T, say $U = \mathsf{s}_{[p]}\, T$ for some node address $[p] \in T^\bullet$.

(1) The input edges of that node are in bijection with $U^\bullet$. In particular, the address of those input edges are of the form $[p[q]]$, where $[q]$ ranges over $U^\bullet$. This really motivates enclosing addresses in brackets.
(2) On the other hand, the output edge of that node is decorated by $\mathsf{t}U$ (where t is defined in definition 2.25).

Notation 2.31*.* Let M be a polynomial monad, and $T \in \mathrm{tr}\, M^+$. For $[a]$ the address of an edge of T, let $\mathsf{e}_{[a]}\, T$ be the operation of M decorating that edge. Explicitely, if $[a] = []$, then $\mathsf{e}_{[]}\, T \coloneqq \mathsf{t}\, \mathsf{s}_{[]}\, T$. Otherwise, $[a] = [p[q]]$ for some $[p] \in T^\bullet$ (the node below the edge) and $[q] \in (\mathsf{s}_{[p]}\, T)^\bullet$, and let $\mathsf{e}_{[p[q]]}\, T \coloneqq \mathsf{s}_{[q]}\, \mathsf{s}_{[p]}\, T$.

3. Opetopes

3.1. **Definition.**

Definition 3.1 (The $\mathfrak{Z}^n$ monad)**.** Let $\mathfrak{Z}^0$ be the identity polynomial monad on $\mathcal{S}\mathrm{et}$, as depicted on the left below, and let $\mathfrak{Z}^n \coloneqq (\mathfrak{Z}^{n-1})^+$. Write $\mathfrak{Z}^n$ as on right:

$$\{*\} \longleftarrow \{*\} \longrightarrow \{*\} \longrightarrow \{*\}, \qquad \mathbb{O}_n \xleftarrow{\mathsf{s}} E_{n+1} \xrightarrow{p} \mathbb{O}_{n+1} \xrightarrow{\mathsf{t}} \mathbb{O}_n. \tag{3.2}$$

Definition 3.3 (Opetope)**.** An *n-dimensional opetope* (or simply *n-opetope*) ω is by definition an element of $\mathbb{O}_n$, and we write $\dim \omega = n$. If $n \geq 2$, then opetopes are exactly $\mathfrak{Z}^{n-2}$-trees. In this case, an opetope $\omega \in \mathbb{O}_n$ is called *degenerate* if its underlying tree has no nodes (thus consists of a unique edge); it is *non degenerate* otherwise.

Following equation (2.26), for $\omega \in \mathbb{O}_{n+2}$, the structure of polynomial monad $(\mathfrak{Z}^n)^\star \longrightarrow \mathfrak{Z}^n$ gives a bijection $\wp_\omega : \omega^{|} \longrightarrow (\mathsf{t}\omega)^\bullet$ between the leaves of ω and the nodes of $\mathsf{t}\omega$, preserving the decoration by n-opetopes.

Example 3.4.
(1) The unique 0-opetope is denoted $\blacklozenge$ and called the *point*.
(2) The unique 1-opetope is denoted $\blacksquare$ and called the *arrow*.
(3) If $n \geq 2$, then $\omega \in \mathbb{O}_n$ is a $\mathfrak{Z}^{n-2}$-tree, i.e. a tree whose nodes are labeled in $(n-1)$-opetopes, and edges are labeled in $(n-2)$-opetopes. In particular, 2-opetopes are $\mathfrak{Z}^0$-trees, i.e. linear trees, and thus in bijection with $\mathbb{N}$. We will refer to them as *opetopic integers*, and write $\mathbf{n}$ for the 2-opetope having exactly n nodes.

3.2. **The category of opetopes.** Akin to the work of Cheng [Che03], we define a category of opetopes by means of generators and relations. The difference with the aforementioned reference is our use of polynomial opetopes (also equivalent to Leinster's definition [Lei04, KJBM10]), while Cheng uses an approach by multicategorical slicing, yielding "symmetric" opetopes.

Theorem 3.5 (Opetopic identities)**.** *Let $\omega \in \mathbb{O}_n$ with $n \geq 2$.*

(1) (Inner edge) For $[p[q]] \in \omega^\bullet$ (forcing ω to be non degenerate), we have $\mathsf{t}\mathsf{s}_{[p[q]]}\,\omega = \mathsf{s}_{[q]}\,\mathsf{s}_{[p]}\,\omega$.
(2) (Globularity 1) If ω is non degenerate, we have $\mathsf{t}\mathsf{s}_{[]}\,\omega = \mathsf{t}\mathsf{t}\omega$.
(3) (Globularity 2) If ω is non degenerate, and $[p[q]] \in \omega^|$, we have $\mathsf{s}_{[q]}\,\mathsf{s}_{[p]}\,\omega = \mathsf{s}_{\wp\omega[p[q]]}\,\mathsf{t}\omega$.
(4) (Degeneracy) If ω is degenerate, we have $\mathsf{s}_{[]}\,\mathsf{t}\omega = \mathsf{t}\mathsf{t}\omega$.

Proof. (1) (Inner edge) By definition of a $\mathfrak{Z}^{n-2}$-tree.

(2) (Globularity 1 and 2) By theorem 2.24, the monad structure on $\mathfrak{Z}^{n-2}$ amounts to a structure map $(\mathfrak{Z}^{n-2})^\star \longrightarrow \mathfrak{Z}^{n-2}$, which, taking the notations of definition 2.25, is written as

$$\begin{array}{ccccccc}
\mathbb{O}_{n-2} & \xleftarrow{\mathsf{s}} & \mathrm{tr}^| \mathfrak{Z}^{n-2} & \xrightarrow{p} & \mathrm{tr}\,\mathfrak{Z}^{n-2} & \xrightarrow{\mathsf{e}_{[]}} & \mathbb{O}_{n-2} \\
\| & & \wp\downarrow & \lrcorner & \mathsf{t}\downarrow & & \| \\
\mathbb{O}_{n-2} & \xleftarrow{\mathsf{s}} & \mathbb{O}^\bullet_{n-1} & \xrightarrow{p} & \mathbb{O}_{n-1} & \xrightarrow{\mathsf{t}} & \mathbb{O}_{n-2}.
\end{array}$$

The claims follow from the commutativity of the right and left square respectively.

(3) (Degeneracy) Let $\omega = \mathsf{I}_\phi$, for $\phi \in \mathbb{O}_{n-2}$. Then $\mathsf{t}\omega = \mathsf{Y}_\phi$, and $\mathsf{s}_{[]}\,\mathsf{Y}_\phi = \mathsf{s}_{[]}\,\mu\mathsf{Y}_\phi = \mathsf{t}\,\mathsf{Y}_\phi$ where μ is the monad law of $\mathfrak{Z}^{n-2}$. □

Definition 3.6 (The category $\mathbb{O}$ of opetopes)**.** With the identities of theorem 3.5, we define the category $\mathbb{O}$ of opetopes by generators and relations as follows.

(1) (Objects) We set $\mathrm{ob}\,\mathbb{O} = \sum_{n\in\mathbb{N}} \mathbb{O}_n$.
(2) (Generating morphisms) Let $\omega \in \mathbb{O}_n$ with $n \geq 1$. We introduce a generator $\mathsf{t} : \mathsf{t}\omega \longrightarrow \omega$, called the *target embedding*. If $[p] \in \omega^\bullet$, then we introduce a generator $\mathsf{s}_{[p]} : \mathsf{s}_{[p]}\,\omega \longrightarrow \omega$, called a *source embedding*. A *face embedding* is either a source or the target embedding.
(3) (Relations) We impose 4 relations described by the following commutative squares, that are well defined thanks to theorem 3.5. Let $\omega \in \mathbb{O}_n$ with $n \geq 2$

(a) **(Inner)** for $[p[q]] \in \omega^\bullet$ (forcing ω to be non degenerate), the following square must commute:

$$\begin{array}{ccc}
\mathsf{s}_{[q]}\,\mathsf{s}_{[p]}\,\omega & \xrightarrow{\mathsf{s}_{[q]}} & \mathsf{s}_{[p]}\,\omega \\
{\scriptstyle\mathsf{t}}\downarrow & & \downarrow{\scriptstyle\mathsf{s}_{[p]}} \\
\mathsf{s}_{[p[q]]}\,\omega & \xrightarrow{\mathsf{s}_{[p[q]]}} & \omega
\end{array}$$

(b) **(Glob1)** if ω is non degenerate, the following square must commute:

$$\begin{array}{ccc}
\mathsf{t}\mathsf{t}\omega & \xrightarrow{\mathsf{t}} & \mathsf{t}\omega \\
{\scriptstyle\mathsf{t}}\downarrow & & \downarrow{\scriptstyle\mathsf{t}} \\
\mathsf{s}_{[]}\,\omega & \xrightarrow{\mathsf{s}_{[]}} & \omega.
\end{array}$$

(c) **(Glob2)** if ω is non degenerate, and for $[p[q]] \in \omega^|$, the following square must commute:

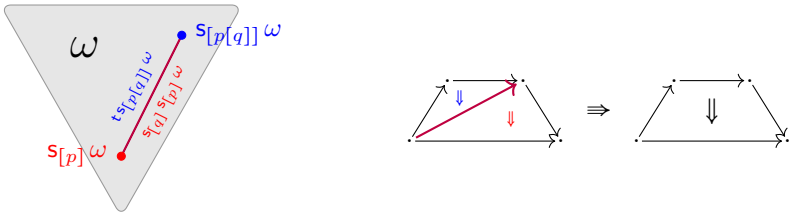
$$\begin{array}{ccc}
\mathsf{s}_{\wp\omega[p[q]]}\,\mathsf{t}\omega & \xrightarrow{\mathsf{s}_{\wp\omega[p[q]]}} & \mathsf{t}\omega \\
{\scriptstyle\mathsf{s}_{[q]}}\downarrow & & \downarrow{\scriptstyle\mathsf{t}} \\
\mathsf{s}_{[p]}\,\omega & \xrightarrow{\mathsf{s}_{[p]}} & \omega.
\end{array}$$

(d) **(Degen)** if ω is degenerate, the following square must commute:

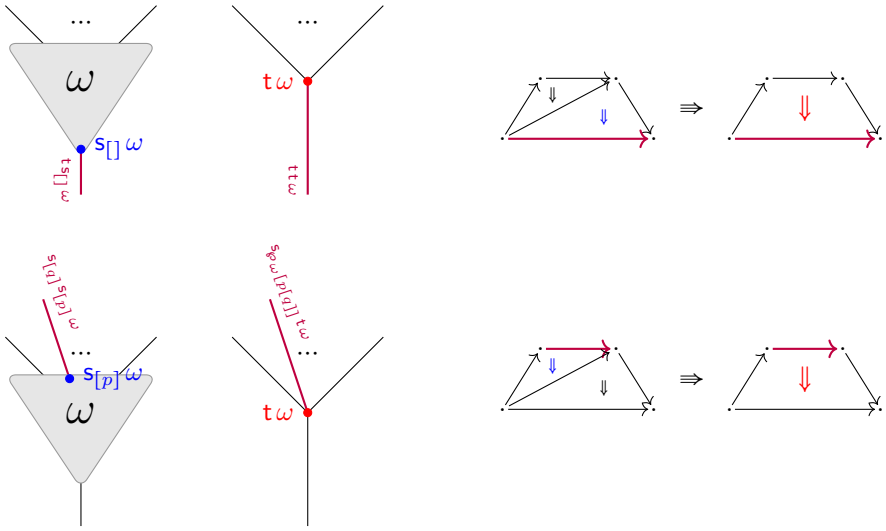
$$\begin{array}{ccc} \mathsf{t}\,\mathsf{t}\,\omega & \xrightarrow{\mathsf{t}} & \mathsf{t}\,\omega \\ {\scriptstyle \mathsf{s}_{[]}}\downarrow & & \downarrow{\scriptstyle \mathsf{t}} \\ \mathsf{t}\,\omega & \xrightarrow{\mathsf{t}} & \omega. \end{array}$$

Remark 3.7. Let us explain this definition a little more. Opetopes are trees whose nodes (and edges) are decorated by opetopes. The decoration is now interpreted as a geometrical feature, namely as an embedding of a lower dimensional opetope. Further, the target of an opetope, while not an intrinsic data, is also represented as an embedding. The relations can be understood as follows.

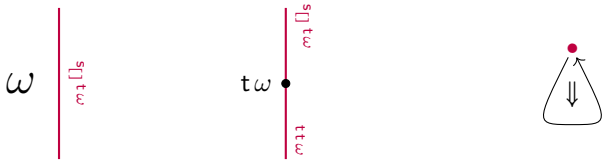
(1) **(Inner)** The inner edge at $[p[q]] \in \omega^\bullet$ is decorated by the target of the decoration of the node "above" it (here $\mathsf{s}_{[p[q]]}\,\omega$), and in the $[q]$-source of the node "below" it (here $\mathsf{s}_{[p]}\,\omega$). By construction, those two decorations match, and this relation makes the two corresponding embeddings $\mathsf{s}_{[q]}\,\mathsf{s}_{[p]}\,\omega \longrightarrow \omega$ match as well. On the left is an informal diagram about ω as a tree (reversed gray triangle), and on the right is an example of pasting diagram represented by an opetope, with the relevant features of the **(Inner)** relation colored or thickened.



(2) **(Glob1-2)** If we consider the underlying tree of ω (which really is ω itself) as its "geometrical source", and the corolla $\mathsf{Y}_{\mathsf{t}\,\omega}$ as its "geometrical target", then they should be parallel. The relation **(Glob1)** expresses this idea by "gluing" the root edges of ω and $\mathsf{Y}_{\mathsf{t}\,\omega}$ together, while **(Glob2)** glues the leaves according to $\wp_\omega$.



(3) **(Degen)** If ω is a degenerate opetope, depicted as on the right, then its target should be a "loop", i.e. its only source and target should be glued together.



Notation 3.8. For $n \in \mathbb{N}$, let $\mathbb{O}_{\leq n}$ be the full subcategory of $\mathbb{O}$ spanned by opetopes of dimension at most n. We define $\mathbb{O}_{<n}$, $\mathbb{O}_{\geq n}$, and $\mathbb{O}_{>n}$ similarly. We also consider the set $\mathbb{O}_n$ as a discrete category.

4. Polygraphs

4.1. **Reminders.** We review some elements of the theory of polygraphs. For a more complete introduction, we refer to [M03] or [HMZ08].

Definition 4.1 (Polygraph)**.** A *polygraph* (also called a *computad*) P consists of a small ω-category P^* and sets $P_n \subseteq P_n^*$ for all $n \in \mathbb{N}$, such that P_0 is the set of objects of P^*, and such that the underlying $(n+1)$-category $P^*|_{n+1}$ is freely generated by P_{n+1} over its underlying n-category $P^*|_n$, for all $n \geq 1$. Write P_n^* the set of n-cells of P, and

$$\mathsf{s}, \mathsf{t} : P_{n+1}^* \longrightarrow P_n^*$$

the *source* and *target* maps, respectively. A polygraph P is an *n-polygraph* if $P_k = \varnothing$ whenever $k > n$.

Definition 4.2 (Morphism of polygraph)**.** A *morphism of polygraphs* is an ω-functor mapping generators to generators. Let $\mathcal{P}\mathrm{ol}$ be the category of polygraphs and morphisms between them.

Definition 4.3 (Parallel cells)**.** For $n \geq 1$, two n-cells $x, y \in P_n^*$ are *parallel*, denoted by $x \parallel y$, if $\mathsf{s}x = \mathsf{s}y$ and $\mathsf{t}x = \mathsf{t}y$. By convention, 0-cells are pairwise parallel.

Definition 4.4 (Counting function)**.** Let P be an arbitrary n-polygraph, and take $k \leq n$. Define a k-category $\mathbb{N}^{(k)}$ by

$$0 \xleftarrow{\mathsf{s},\mathsf{t}} \cdots \xleftarrow{\mathsf{s},\mathsf{t}} 0 \xleftarrow{\mathsf{s},\mathsf{t}} \mathbb{N}^{(k)},$$

where all compositions correspond to the addition of integers. For $x \in P_k$, define a *counting function* $\#_x : P_k \longrightarrow \mathbb{N}^{(k)}$ that maps x to 1, and all other generators to 0. This extends to a k-functor $P^*|_k \longrightarrow \mathbb{N}^{(k)}$, and let $\#_x : P_k^* \longrightarrow \mathbb{N}^{(k)}$ be its k-th component. Similarly, let $\# : P_k \longrightarrow \mathbb{N}^{(k)}$ be the map sending all generators to 1, and extend it as $\# : P_k^* \longrightarrow \mathbb{N}^{(k)}$.

Definition 4.5 (Context [GM09, definition 2.1.1])**.** Let $P \in \mathcal{P}\mathrm{ol}$ and $n \in \mathbb{N}$. Consider a polygraph Q obtained by adding a new n-generator ? to P (with arbitrary source and target). An n-context of P is an n-cell $C \in Q_n^*$ such that $\#_? C = 1$. One may think of C as a cell of P with a "hole", and we sometime write $C = C[?]$.

If $x \in P_n^*$ is parallel to ? in Q, let $C[x] := pC$, where $p : Q \longrightarrow P$ is the ω-functor[4] preserving all the generators of P, but mapping ? to x.

Definition 4.6 (Category of contexts [GM09, definition 2.1.2])**.** The category of *n-contexts* $\mathcal{C}\mathrm{tx}_n Q$ of a n-polygraph Q has objects the n-cells of Q, and a morphism $C : x \longrightarrow y$ is an *n-context* $C = C[?]$ such that $C[x] = y$.

Definition 4.7 (Irreducible context)**.** Let $x \in P_n^*$. An *irreducible context over* x is a context $C : a \longrightarrow x$ where $a \in P_n$ (as opposed to just P_n^*).

Remark 4.8. If $C : a \longrightarrow x$ is an irreducible context, then for every decomposition $C = D[E]$, either D or E is the trivial context ?. This motivates the terminology of definition 4.7.

Definition 4.9 (Partial composition [HMZ08, definition 3.8])**.** Let P be a polygraph, $x, y \in P_n^*$ be n-cells, and $C : \mathsf{t}y \longrightarrow \mathsf{s}x$ be a context. The *partial composition* (called *placed composition* in [HMZ08, definition 3.8]) $x \circ_D y$ is defined as follows:

$$x \underset{C}{\circ} y := x \underset{n-1}{\circ} C[y],$$

where $\circ_{n-1}$ is the $(n-1)$-composition of P. Let $\mathcal{P}\mathrm{ol}_{\mathrm{mto}}$ be the full subcategory of $\mathcal{P}\mathrm{ol}$ spanned by many-to-one polygraphs. $C[y]$ is the C-whisker of y.

Lemma 4.10 ([HMZ08, definition 3.8])**.** *With x, y, and C as in definition 4.9, we have $\mathsf{s}(x \circ_C y) = C[\mathsf{s}y]$ and $\mathsf{t}(x \circ_C y) = \mathsf{t}x$.*

[4]Note that unless x is a generator, p is not a morphism of polygraphs in the sense of definition 4.2.

Notation 4.11 (Total composition). Assume now that P is many-to-one, and take $z \in P_n^{\mathrm{mto}}$. For $C_i : a_i \longrightarrow \mathtt{s}\, z$, $1 \leq i \leq k$, ranging over the irreducible contexts over $\mathtt{s}\, z$, and $w_1, \dots, w_k \in P_n^{\mathrm{mto}}$ cells such that $\mathtt{t}\, w_i = a_i$ (so that the partial composition $z \circ_{C_i} w_i$ is well-defined), define the *total composition*[5]

$$z \bigcirc_{C_i} w_i = (\cdots (z \underset{C_1}{\circ} w_1) \underset{C_2}{\circ} w_2 \cdots) \underset{C_k}{\circ} w_k.$$

By [HMZ08, theorem 3.9], the result does not depend in the order in which the partial compositions are computed.

4.2. **Many-to-one polygraphs.**

Definition 4.12 (Many-to-one polygraph)**.** Let $P \in \mathcal{P}\mathrm{ol}$ be a polygraph. For $n \geq 1$, an n-cell $x \in P_n^*$ is said *many-to-one* of $\mathtt{t}\, x \in P_{n-1}$ (instead of P_{n-1}^*), and we write P_n^{mto} the set of many-to-one n-cells of P. By convention, all 0-cells are many-to-one. In turn, the polygraph P is *many-to-one* if all its generators are, or equivalently if the target of a generator is also a generator.

Let $\mathcal{P}\mathrm{ol}_{\mathrm{mto}}$ be the full subcategory of $\mathcal{P}\mathrm{ol}$ spanned by many-to-one polygraphs. Let $\mathcal{C}\mathrm{tx}_n^{\mathrm{mto}} P$ be the full subcategory of $\mathcal{C}\mathrm{tx}_n P$ generated by many-to-one n-cells. Necessarily, the morphisms of $\mathcal{C}\mathrm{tx}_n^{\mathrm{mto}} P$ are many-to-one contexts.

Theorem 4.13 (Structure theorem for many-to-one polygraphs)**.** *(1) The category $\mathcal{P}\mathrm{ol}_{\mathrm{mto}}$ is a presheaf category.*

(2) In particular, $\mathcal{P}\mathrm{ol}_{\mathrm{mto}}$ cocomplete, and for $F : \mathcal{J} \longrightarrow \mathcal{P}\mathrm{ol}_{\mathrm{mto}}$ a small diagram, and $n \in \mathbb{N}$, $(\mathrm{colim}_{i \in \mathcal{J}} Fi)_n \cong \mathrm{colim}_{i \in \mathcal{J}} (Fi)_n$.

(3) If $P \in \mathcal{P}\mathrm{ol}_{\mathrm{mto}}$ and $x \in P_n^*$, then x is obtained as a unique *composition of n-generators. Further, if $\# x = k$ (definition 4.4), then k generators are needed.*

Proof. (1) This is [Hen17, corollary 2.4.6].

(2) By [Hen17, proposition 2.2.3], there is a set X_n of *representable* many-to-one polygraphs such that for all $P \in \mathcal{P}\mathrm{ol}_{\mathrm{mto}}$,

$$P_n \cong \sum_{x \in X_n} \mathcal{P}\mathrm{ol}_{\mathrm{mto}}(x, P).$$

The result follows since if x is representable, $\mathcal{P}\mathrm{ol}_{\mathrm{mto}}(x, -)$ preserves colimits.

(3) The first claim is [Hen17, corollary 2.3.11]. The second is clear, as the counting function $\# : P^*|_n \longrightarrow \mathbb{N}^{(n)}$ of definition 4.4 maps n-generators to n-generators. □

The following result comes as a polygraphic analogue to proposition 2.19.

Corollary 4.14 (Induction principle for many-to-one polygraphs)**.** *Let $P \in \mathcal{P}\mathrm{ol}_{\mathrm{mto}}$. Then a many-to-one n-cell of P is of either of the following forms:*

(1) id_a for $a \in P_{n-1}$,

(2) $x \in P_n$,

(3) $y \circ_C x$, for a unique y a many-to-one n-cell, a unique irreducible context C over $\mathtt{s}\, y$, and a unique n-generator $x \in P_n$.

Proof. Follows directly from theorem 4.13. □

Remark 4.15. Let $P \in \mathcal{P}\mathrm{ol}_{\mathrm{mto}}$ and $C : x \longrightarrow y$ be an n-context of P between many-to-one cells. On particular, C is a many-to-one cell in the extended category Q of definition 4.5, so in virtue of corollary 4.14, it uniquely decomposes as

$$C = z \underset{D}{\circ} ? \bigcirc_{E} t_E$$

where E ranges over all irreducible contexts over $\mathtt{s}?$. Further, since $\#_? C = 1$, z and the t_Es are cells of P. Since $C[x] = y$, we have

$$y = z \underset{D}{\circ} x \bigcirc_{E} t_E.$$

Conversely, any decomposition of y of the form above induces a context $C : x \longrightarrow y$.

[5] In fact, the formula has a slight abuse of notation: for instance, C_2 is not a context $\mathtt{t}\, w_2 \longrightarrow \mathtt{s}(z \circ_{C_1} w_1)$, wich technically makes the gafting $(z \circ_{C_1} w_1) \circ_{C_2} w_2$ ill-defined. One needs to consider the context C_2' that selects the adequate generator of $\mathtt{s}\, z \circ_{C_1} w_1 = C_1[\mathtt{s}\, w_1]$ instead. This is a minor technicality that tends to clutter notations significantly, so we omit it.

4.3. **Composition trees.** Let $P \in \mathcal{P}\mathrm{ol}_{\mathrm{mto}}$ and $x \in P_n^{\mathrm{mto}}$. Then x is a composition of n-generators of P, which are many-to-one, so intuitively, x is a "tree of n-generators". In this section, we make this idea formal. We first define a polynomial functor $\nabla_n P$ whose operations are the n-generators of P (definition 4.16), and then construct the *composition tree* $\mathrm{ct}\, x \in \mathrm{tr}\, \nabla_n P$ of x (definition 4.20). In proposition 4.21, we show that there is in fact a bijective correspondence between the many-to-one n-cells of P, and the trees $\nabla_n P$.

Definition 4.16 (The ∇ construction)**.** For $P \in \mathcal{P}\mathrm{ol}_{\mathrm{mto}}$ and $n \geq 1$, let $\nabla_n P$ be the following polynomial endofunctor:

$$P_{n-1} \xleftarrow{\mathsf{s}} P_n^\bullet \xrightarrow{p} P_n \xrightarrow{\mathsf{t}} P_{n-1}, \quad ,$$

where for $x \in P_n$, the fiber $P_n^\bullet(x)$ is the set of irreducible contexts over $\mathsf{s}\, x$, and for $C : a \longrightarrow \mathsf{s}\, x$ in $P_n^\bullet(x)$, $\mathsf{s}\, C \coloneqq a$, $pC \coloneqq x$, and t is the target map of P. By theorem 4.13, $\# P_n^\bullet(x) = \# x$, and in particular, $P_n^\bullet(x)$ is finite.

Lemma 4.17. *Let $P, Q \in \mathcal{P}\mathrm{ol}_{\mathrm{mto}}$, and $f : P \longrightarrow Q$. For $x \in P_n^{\mathrm{mto}}$, write $E(x)$ for the set of irreducible contexts over x, and likewise for many-to-one cells of Q. Then f_n induced a bijection $E(x) \longrightarrow E(f_n x)$.*

Proof. We proceed by induction on x (corollary 4.14).

(1) If x is an identity (resp. a generator), then so is $f_n x$, thus $E(x)$ and $E(f_n x)$ are both empty (resp. singletons). Trivially, $f_n : E(x) \longrightarrow E(f_n x)$ is a bijection.
(2) Assume that x decomposes as $x = y \circ z$ with $\# y \geq 1$ and $z \in P_n$. Then an irreducible context over x is either $y \circ ?$ or of the form $C[?] \circ z$ for $C \in E(y)$, and $E(x) \cong 1 + E(y)$. Likewise, $E(f_n x) \cong 1 + E(f_n y)$, and it is straightforward to check that $f_n : E(x) \longrightarrow E(f_n x)$ is indeed a bijection. □

Proposition 4.18. *Let $P, Q \in \mathcal{P}\mathrm{ol}_{\mathrm{mto}}$. A morphism $f : P \longrightarrow Q$ induces a morphism of polynomial functors $\nabla_n f : \nabla_n P \longrightarrow \nabla_n Q$ for all $n \geq 1$, such that $(\nabla_n f)_1 = f_n : P_n \longrightarrow Q_n$. Thus, for each $n \geq 1$, we have a functor $\nabla_n : \mathcal{P}\mathrm{ol}_{\mathrm{mto}} \longrightarrow \mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$.*

Proof. Consider

$$\begin{array}{ccccccc} P_{n-1} & \xleftarrow{\mathsf{s}} & P_n^\bullet & \xrightarrow{p} & P_n & \xrightarrow{\mathsf{t}} & P_{n-1} \\ \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_n^\bullet} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\ Q_{n-1} & \xleftarrow{\mathsf{s}} & Q_n^\bullet & \xrightarrow{p} & Q_n & \xrightarrow{\mathsf{t}} & Q_{n-1} \end{array}$$

where $f_n^\bullet$ maps a context $C : a \longrightarrow \mathsf{s}\, x$ to $f_{n-1} C : f_{n-1} a \longrightarrow f_{n-1}\, \mathsf{s}\, x$. Clearly, all squares commute, and by lemma 4.17, the middle one is cartesian. □

Definition 4.19 (Composition)**.** We define a functor $(-)^\circ : \mathrm{tr}\, \nabla_n P \longrightarrow \mathcal{C}\mathrm{tx}_n^{\mathrm{mto}} P$. On objects, we define T° by induction on $T \in \mathrm{tr}\, \nabla_n P$ (proposition 2.19). At the same time, we establish a bijection between $T^|$ and the irreducible contexts over $\mathsf{s} T^\circ$.

(1) If $a \in P_{n-1}$, then $(\mathsf{l}_a)^\circ \coloneqq \mathrm{id}_a$. Note that the only irreducible context over $\mathsf{s}\, \mathrm{id}_a$ is $? : a \longrightarrow a$, abd let $C_{[]} \coloneqq ?$.
(2) If $x \in P_n$, then $(\mathsf{Y}_x)^\circ \coloneqq x$. Note that by definition of $\nabla_n P$ (definition 4.16) we have $\mathsf{Y}_x^| = \{[D] \mid D \text{ irred. ctx. over } \mathsf{s}\, x\}$ (see remark 2.13), and let $C_{[D]} \coloneqq D$.
(3) Consider a tree of the form $S = T \circ_{[l]} \mathsf{Y}_x$, with $T \in \mathrm{tr}\, \nabla_n P$ having at least one node, $[l] \in T^|$ and $x \in P_n$. By indiction, the leaf $[l]$ corresponds to an irreducible context $C_{[l]} : a \longrightarrow \mathsf{s} T^\circ$, and moreover, $a = \mathsf{t}\, x$. Let $S^\circ \coloneqq T^\circ \circ_{C_{[l]}} x$. Let $[l'] \in S^|$. If $[l']$ is of the form $[lD]$, for some $[D] \in \mathsf{Y}_x^|$, let $C_{[l']} \coloneqq C_{[l]}[D]$. Otherwise, $[l']$ is a leaf of T, and so $C_{[l']}$ is already defined.

This defines $(-)^\circ : \mathrm{tr}\, \nabla_n P \longrightarrow \mathcal{C}\mathrm{tx}_n^{\mathrm{mto}} P$ on objects. Let $f : T \longrightarrow S$ be a morphism in $\mathrm{tr}\, \nabla_n P$. It corresponds to a decomposition of S as on the left (remark 2.20)

$$S = U \underset{[p]}{\circ} T \bigcirc_{[l] \in T^|} V_{[l]}, \qquad f^\circ \coloneqq U^\circ \underset{C_{[p]}}{\circ} ? \bigcirc_{C_{[l]} \in T^|} V_{[l]}^\circ,$$

and let f° be the context $T^\circ \longrightarrow S^\circ$ on the right.

Definition 4.20 (Composition tree)**.** We define a functor $\mathrm{ct} : \mathcal{C}\mathrm{tx}_n^{\mathrm{mto}} P \longrightarrow \mathrm{tr}\, \nabla_n P$. On object, we define $\mathrm{ct}\, x$ by induction on $x \in P_n^*$ (corollary 4.14). At the same time, we establish a bijection between the irreducible contexts over $\mathsf{s}\, x$ and the leaves of $\mathrm{ct}\, x$.

(1) If $a \in P_{n-1}$, let $\operatorname{ct} \mathrm{id}_a \coloneqq \mathsf{I}_a$. The only context of the form above is $? : a \longrightarrow a$, which corresponds to the leaf $[l_?] \coloneqq []$ of I_a.
(2) If $x \in P_n$, let $\operatorname{ct} x \coloneqq \mathsf{Y}_x$. By definition of $\nabla_n P$ (definition 4.16) we have $\mathsf{Y}_x^{\mid} = \{[D] \mid D \text{ irred. ctx. over } \mathsf{s}x\}$, and let $[l_D] \coloneqq [D]$.
(3) Consider an n-cell $z = y \circ_C x$, with $x \in P_n$. Since P is many-to-one, $\mathsf{t}x$ is an $(n-1)$-generator and the context $C : \mathsf{t}x \longrightarrow \mathsf{s}y$ is irreducible. Thus, it corresponds to a leaf $[l_C] \in \operatorname{ct} y^{\mid}$, and furthermore, $\mathsf{e}_{[l_C]} \operatorname{ct} y = \mathsf{t}x$ (notation 2.31). Let $\operatorname{ct} z \coloneqq \operatorname{ct} y \circ_{[l_C]} \mathsf{Y}_x$. Let $D : a \longrightarrow \mathsf{s}z = C[\mathsf{s}x]$ be an $(n-1)$-context, where $a \in P_{n-1}$. If D decomposes as $D = C[E]$, for some context $E : a \longrightarrow \mathsf{s}x$, let $[l_D] \coloneqq [l_C E]$. Otherwise, D exhibits a as a cell in the source of y, and $[l_D]$ is already defined.

This defines $\operatorname{ct} : \mathcal{C}\mathrm{tx}_n^{\mathrm{mto}} P \longrightarrow \operatorname{tr} \nabla_n P$ on objects. Let $C : x \longrightarrow y$ be an n-context. It corresponds to a decomposition of y as on the left (remark 4.15)

$$y = z \underset{D}{\circ} x \bigcirc_{E} t_E, \qquad \operatorname{ct} y = \operatorname{ct} z \underset{[l_D]}{\circ} \operatorname{ct} x \bigcirc_{[l_E]} \operatorname{ct} t_E,$$

where E ranges over all irreducible contexts over $\mathsf{s}x$, and let $\operatorname{ct} C : \operatorname{ct} x \longrightarrow \operatorname{ct} y$ be given by the decomposition of $\operatorname{ct} y$ on the right.

One readily checks the following:

Proposition 4.21 (Composition tree duality)**.** *The functors* $(-)^\circ$ *and* ct *are mutually inverse isomorphisms of categories.*

Corollary 4.22. *For* $n \geq 2$ *and* $x \in P_n$*, the functor* ct *induces a natural bijection over* P_{n-1}

$$P_n^\bullet(x) \cong \sum_{a \in P_{n-1}} (\operatorname{tr} \nabla_{n-1} P)(\mathsf{Y}_a, \operatorname{ct} \mathsf{s}x).$$

Proof. Direct consequence of proposition 4.21. □

Notation 4.23. Let $x \in P_n$, and $[p] \in (\operatorname{ct} \mathsf{s}x)^\bullet$. Then we write $\mathsf{s}_{[p]} x \coloneqq \mathsf{s}_{[p]} \operatorname{ct} \mathsf{s}x \in P_{n-1}$.

5. The equivalence

We now aim to prove that the category of opetopic sets, i.e. $\mathcal{S}\mathrm{et}$-presheaves over the category $\mathbb{O}$ defined previously, is equivalent to the category of many-to-one polygraphs $\mathcal{P}\mathrm{ol}_{\mathrm{mto}}$. We achieve this by first constructing the *polygraphic realization* functor $|-| : \mathbb{O} \longrightarrow \mathcal{P}\mathrm{ol}_{\mathrm{mto}}$ in section 5.1. This functor "realizes" an opetope as a polygraph in that it freely implements all its tree structure by means of adequately chosen generators in each dimension. Secondly, writing $\mathcal{P}\mathrm{sh}(\mathbb{O}) = \mathcal{S}\mathrm{et}^{\mathbb{O}^{\mathrm{op}}}$ we consider the left Kan extension $|-| : \mathcal{P}\mathrm{sh}(\mathbb{O}) \longrightarrow \mathcal{P}\mathrm{ol}_{\mathrm{mto}}$ along the Yoneda embedding. This functor has a right adjoint, the "opetopic nerve" $N : \mathcal{P}\mathrm{ol}_{\mathrm{mto}} \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O})$, and we prove this adjunction to be an adjoint equivalence. This is done using the shape function, defined in section 5.2, which to any generator x of a many-to-one polygraph P associates an opetope $x^\natural$ along with a canonical morphism $\tilde{x} : |x^\natural| \longrightarrow P$.

5.1. **Polygraphic realization.** An opetope $\omega \in \mathbb{O}_n$, with $n \geq 1$, has one target $\mathsf{t}\omega$, and sources $\mathsf{s}_{[p]}\omega$ laid out in a tree. If the sources $\mathsf{s}_{[p]}\omega$ happened to be generators in some polygraph, then that tree would describe a way to compose them. With this in mind, we define a many-to-one polygraph $|\omega|$, whose generators are essentially iterated faces (i.e. sources or targets) of ω (hypothesis **(PR1)** below). Moreover, $|\omega|$ will be "maximally unfolded" (or "free"), in that two (iterated) faces that are the same opetope, but located at different addresses, will correspond to distinct generators.

The rest of this section is devoted to inductively define the realization functor $|-|$ together with its *boundary* $\partial|-|$. We bootstrap the process with definition 5.1, state our induction hypothesis in axiom 5.2.

Definition 5.1 (Low dimensional cases)**.** For $\blacklozenge$ the unique 0-opetope, let $\partial|\blacklozenge|$ be the empty polygraph, and $|\blacklozenge|$ be the polygraph with a unique generator in dimension 0, which we denote by $\blacklozenge$. For $\blacksquare$ the unique 1-opetope, let $\partial|\blacksquare| \coloneqq |\blacklozenge| + |\blacklozenge|$, and let $|\blacksquare|$ be the cellular extension

$$\partial|\blacksquare| \xleftarrow{\mathsf{s},\mathsf{t}} \{\blacksquare\},$$

where s and t map $\blacksquare$ to distinct 0-generators. There are obvious functors $|\mathsf{s}_{[]}|, |\mathsf{t}| : |\blacklozenge| \longrightarrow |\blacksquare|$, mapping $\blacklozenge$ to $\mathsf{s}\blacksquare$ and $\mathsf{t}\blacksquare$, respectively.

At this stage, we defined a functor $|-| : \mathbb{O}_{\leq 1} \longrightarrow \mathcal{P}\mathrm{ol}_{\mathrm{mto}}$ (notation 3.8). Let $N \geq 2$ and assume by induction that $\partial|-|$ and $|-|$ are defined on $\mathbb{O}_{<n}$. Assume further that the following induction hypothesis hold (they are easily verified for $N = 2$).

Axiom 5.2. For all $\psi \in \mathbb{O}_k$ with $k < N$, the following hold

(1) **(PR0)** the polygraphs $\partial|\psi|$ and $|\psi|$ are many-to-one, and $\partial|\psi|$ is the sub-polygraph of $|\psi|$ spanned by its k-generators;
(2) **(PR1)** for all $j \in \mathbb{N}$, the set $|\psi|_j$ of j-generators of $|\psi|$ is the slice $\mathbb{O}_j/\psi$, i.e. of the form $\left(\phi \xrightarrow{\mathsf{a}} \psi\right)$ for $\phi \in \mathbb{O}_j$ and $f : \phi \longrightarrow \psi$ a morphism in $\mathbb{O}$;
(3) **(PR2)** for $\left(\phi \xrightarrow{\mathsf{a}} \psi\right)$ a generator of $|\psi|$, its target is $\mathsf{t}\left(\phi \xrightarrow{\mathsf{a}} \psi\right) = \left(\mathsf{t}\,\phi \xrightarrow{\mathsf{t}} \phi \xrightarrow{\mathsf{a}} \psi\right)$;
(4) **(PR3)** for $l \leq k$, and for $\left(\phi \xrightarrow{\mathsf{a}} \psi\right)$ a l-generator of $|\psi|$, the composition tree of its source $\mathrm{ct}\,\mathsf{s}\left(\phi \xrightarrow{\mathsf{a}} \psi\right) \in \mathrm{tr}\,\nabla_{l-1}|\psi|$ is

$$\begin{aligned} \langle \phi \rangle &\longrightarrow \nabla_{l-1}|\psi| \\ [p] &\longmapsto \left(\mathsf{s}_{[p]}\,\phi \xrightarrow{\mathsf{s}_{[p]}} \phi \xrightarrow{\mathsf{a}} \psi\right). \end{aligned}$$

Recall that by proposition 4.21, this completely determines $\mathsf{s}\left(\phi \xrightarrow{\mathsf{a}} \psi\right) \in |\psi|^*_{l-1}$.

We now define $\partial|\omega|$ and $|\omega|$ when $\omega \in \mathbb{O}_N$. Defining the former is easy, and done in definition 5.3. The latter is defined in definition 5.9 as a cellular extension

$$\partial|\omega| \xleftarrow{\mathsf{s},\mathsf{t}} \{\omega\}$$

of $\partial|\omega|$, where the target and source of the new generator are given by **(PR2)** and **(PR3)**. Lastly, we check the inductive hypothesis in proposition 5.10.

Definition 5.3 (Inductive step for $\partial|-|$)**.** For $\omega \in \mathbb{O}_N$, let

$$\partial|\omega| := \operatorname*{colim}_{\psi \in \mathbb{O}_{<N}/\omega} |\psi|.$$

For $\mathsf{a} : \psi \longrightarrow \omega$ in $\mathbb{O}_{<N}/\omega$, this colimit comes with a corresponding coprojection $|\mathsf{a}| : |\psi| \longhookrightarrow \partial|\omega|$.

Lemma 5.4. *For $\omega \in \mathbb{O}_N$, and $j < N$, the set $\partial|\omega|_j$ of j-generators of $\partial|\omega|$ is the slice $\mathbb{O}_j/\omega$.*

Proof. Follows from the induction hypothesis **(PR1)** and theorem 4.13. □

Corollary 5.5. *For $\omega \in \mathbb{O}_N$ and $1 \leq k < N$, the polynomial functor $\nabla_k \partial|\omega|$ is described as follows:*

$$\mathbb{O}_{k-1}/\omega \xleftarrow{\mathsf{s}} E \xrightarrow{p} \mathbb{O}_k/\omega \xrightarrow{\mathsf{t}} \mathbb{O}_{k-1}/\omega$$

where for $\left(\psi \xrightarrow{\mathsf{a}} \omega\right) \in \mathbb{O}_k\omega$,

(1) the fiber $E\left(\psi \xrightarrow{\mathsf{a}} \omega\right)$ is simply $\psi^\bullet$;
(2) for $[p] \in E\left(\psi \xrightarrow{\mathsf{a}} \omega\right) \cong \psi^\bullet$, we have $\mathsf{s}[p] = \left(\mathsf{s}_{[p]}\,\psi \xrightarrow{\mathsf{s}_{[p]}} \psi \xrightarrow{\mathsf{a}} \omega\right)$;
(3) $\mathsf{t}\left(\psi \xrightarrow{\mathsf{a}} \omega\right) = \left(\mathsf{t}\,\psi \xrightarrow{\mathsf{t}} \psi \xrightarrow{\mathsf{a}} \omega\right)$.

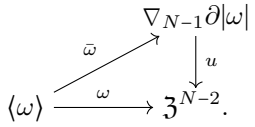
Proof. Direct consequence of lemma 5.4 and **(PR1)**, **(PR2)**, and **(PR3)**. □

Definition 5.6. For $\omega \in \mathbb{O}_N$ and $1 \leq k < N$, we have a morphism $u : \nabla_k \partial|\omega| \longrightarrow \mathfrak{Z}^{k-1}$

$$\begin{array}{ccccccc} \mathbb{O}_{k-1}/\omega & \xleftarrow{\mathsf{s}} & E & \xrightarrow{p} & \mathbb{O}_k/\omega & \xrightarrow{\mathsf{t}} & \mathbb{O}_{k-1}/\omega \\ \downarrow{\scriptstyle u_0} & & \downarrow{\scriptstyle u_2} & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_0} \\ \mathbb{O}_{k-1} & \xleftarrow{\mathsf{s}} & \mathbb{O}_k^\bullet & \xrightarrow{p} & \mathbb{O}_k & \xrightarrow{\mathsf{t}} & \mathbb{O}_{k-1} \end{array}$$

induced by the forgetful maps $\mathbb{O}_{k-1}/\omega \longrightarrow \mathbb{O}_{k-1}$ and $\mathbb{O}_k/\omega \longrightarrow \mathbb{O}_k$.

Lemma 5.7. *Let $\omega \in \mathbb{O}_N = \mathrm{tr}\,\mathfrak{Z}^{N-2}$. The map $\omega : \langle\omega\rangle \longrightarrow \mathfrak{Z}^{N-2}$ factors through $u : \nabla_{N-1}\partial|\omega| \longrightarrow \mathfrak{Z}^{N-2}$ (definition 5.6):*



Proof. Let $\bar{\omega}$ map $[p] \in \omega^\bullet$ to $\left(\mathsf{s}_{[p]}\,\omega \xrightarrow{\mathsf{s}_{[p]}} \omega\right) \in \mathbb{O}_{N-1}/\omega$, and map an edge $[l]$ to $\left(\mathsf{e}_{[l]}\,\omega \xrightarrow{\mathsf{e}_{[l]}} \omega\right) \in \mathbb{O}_{N-2}/\omega$ (notation 2.31). □

Proposition 5.8. *On the one hand, consider the tree $\nabla_{N-1}\partial|\omega|$-tree $\bar{\omega}$ of lemma 5.7, and on the other hand, recall that there is a $(N-1)$-generator $\left(\mathsf{t}\omega \xrightarrow{\mathsf{t}} \omega\right)$ of $\partial|\omega|$ corresponding to the target embedding of ω. Then, in $\partial|\omega|$, the composite $\bar{\omega}^\circ$ (definition 4.19) and the generator $\left(\mathsf{t}\omega \xrightarrow{\mathsf{t}} \omega\right)$ are parallel.*

Proof. If ω is degenerate, say $\omega = \mathsf{I}_\phi$ for some $\phi \in \mathbb{O}_{N-2}$, then $\bar{\omega}^\circ = \mathrm{id}_{\left(\phi \xrightarrow{\mathsf{tt}} \omega\right)}$, while $\left(\mathsf{t}\omega \xrightarrow{\mathsf{t}} \omega\right) = \left(\mathsf{Y}_\phi \xrightarrow{\mathsf{t}} \omega\right)$. By **(Degen)**, those two cells are parallel.

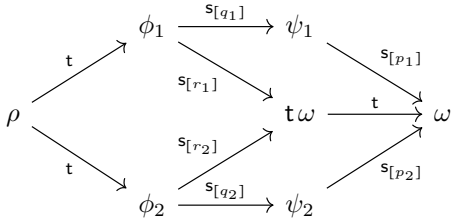
For the rest of the proof, we assume that ω is not degenerate. First, we have

$$\mathsf{t}\bar{\omega}^\circ = \mathsf{t}\,\mathsf{s}_{[]}\,\bar{\omega} = \mathsf{t}\left(\mathsf{t}\mathsf{s}_{[]}\,\omega \xrightarrow{\mathsf{t}\mathsf{s}_{[]}} \omega\right) = \left(\mathsf{t}\mathsf{t}\,\omega \xrightarrow{\mathsf{t}\mathsf{t}} \omega\right) = \mathsf{t}\left(\mathsf{t}\omega \xrightarrow{\mathsf{t}} \omega\right).$$

Then, in order to show that $\mathsf{s}\bar{\omega}^\circ = \mathsf{s}(\mathsf{t}\omega \xrightarrow{\mathsf{t}} \omega)$, we show that the $(N-2)$-generators occurring both sides are the same, and that the way to compose them is unique.

(1) Generators in $\mathsf{s}\bar{\omega}^\circ$ are of the form $\left(\phi \xrightarrow{[q]} \psi \xrightarrow{[p]} \omega\right)$, for $[p[q]] \in \omega^|$. By **(Glob2)**, those are equal to $\left(\phi \xrightarrow{\wp_\omega[p[q]]} \mathsf{t}\omega \xrightarrow{\mathsf{t}} \omega\right)$, which are exactly the generators in $\mathsf{s}\left(\mathsf{t}\omega \xrightarrow{\mathsf{t}} \omega\right)$.

(2) To show that there is a unique way to compose all the $(N-2)$-generators of the form $(\phi \xrightarrow{\mathsf{s}_{[q]}} \psi \xrightarrow{\mathsf{s}_{[p]}} \omega)$, where $[p[q]]$ ranges over $\omega^|$, it is enough to show that no two have the same target. Assume $(\phi_i \xrightarrow{\mathsf{s}_{[q_i]}} \psi_i \xrightarrow{\mathsf{s}_{[p_i]}} \omega)$, with $i = 1, 2$, are $(N-2)$-generators occuring in $\mathsf{s}\bar{\omega}^\circ$ with the same target. Consider the following diagram:



where $[r_i] := \wp_\omega[p_i[q_i]] \in \mathsf{t}\omega^\bullet$. The outer hexagon commutes by assumption, the two squares on the right are instances of **(Glob2)**, and the left square commutes as $\mathsf{t} : \mathsf{t}\omega \longrightarrow \omega$ is a mono, since ω is non degenerate. By inspection of the opetopic identities, the only way for the left square to commute is the trivial way, i.e. $[r_1] = [r_2]$. Since $\wp_\omega$ is a bijection, we have $[p_1[q_1]] = [p_2[q_2]]$, thus $[p_1] = [p_2]$ and $[q_1] = [q_2]$. □

Definition 5.9 (Inductive step for $|-|$)**.** For $\omega \in \mathbb{O}_N$, let $|\omega|$ be the cellular extension

$$\partial|\omega| \xleftarrow{\mathsf{s},\mathsf{t}} \{\omega\},$$

where t maps ω to the $(N-1)$-generator $\left(\mathsf{t}\omega \xrightarrow{\mathsf{t}} \omega\right)$, and where the composition tree of $\mathsf{s}\omega$ is $\bar{\omega}$ (lemma 5.7). For consistency, we also write $\left(\omega \xrightarrow{\mathrm{id}} \omega\right)$ for the unique N-generator of $|\omega|$. This is well defined by proposition 5.8, and gives a functor $|-| : \mathbb{O}_{\leq N} \longrightarrow \mathcal{P}\mathrm{ol}$.

Proposition 5.10. *For $\omega \in \mathbb{O}_N$, the polygraphs $\partial|\omega|$ (definition 5.3) and $|\omega|$ (definition 5.9) satisfy the inductive hypothesis of axiom 5.2.*

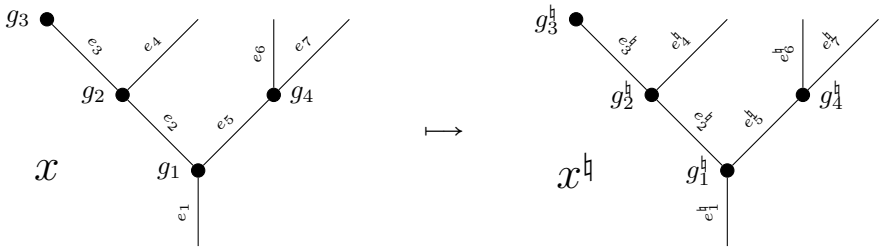
Proof. (1) **(PR0)** By definition.

(2) **(PR1)** For $j < N$, by lemma 5.4, we already have $|\omega|_j = \partial|\omega|_j = \mathbb{O}_j/\omega$. In dimension N, the only element of $\mathbb{O}_N/\omega$ is $\mathrm{id} : \omega \longrightarrow \omega$, which corresponds to the unique N-generator of $|\omega|$. If $j > N$, then both $\mathbb{O}_j/\omega$ and $|\omega|_j$ are empty.

(3) **(PR2)** and **(PR3)**. By definition, those hypothesis hold for the unique N-generator $\left(\omega \xrightarrow{\mathrm{id}} \omega\right)$ of $|\omega|$. By induction, they also hold on the other generators. □

To conclude, we defined a functor $|-| : \mathbb{O} \longrightarrow \mathcal{P}\mathrm{ol}_{\mathrm{mto}}$ which satisfies axiom 5.2 for all $N \in \mathbb{N}$.

5.2. **The shape function.** This subsection is devoted to define the *shape function* $(-)^\natural$. We first sketch the idea. Take $P \in \mathcal{P}\mathrm{ol}_{\mathrm{mto}}$ and define $(-)^\natural : P_n \longrightarrow \mathbb{O}_n$ by induction. The cases $n = 0, 1$ are trivial, since there is a unique 0-opetope and a unique 1-opetope. Assume $n \geq 2$, and take $x \in P_n$. Then the composition tree of $\mathtt{s}x$ is a coherent tree whose nodes are $(n-1)$-generators, and edges are $(n-2)$-generators. Replacing those $(n-1)$ and $(n-2)$-generators by their respective shape, we obtain a coherent tree whose nodes are $(n-1)$-opetopes, and edges are $(n-2)$-opetopes, in other words, we obtain an n-opetope, which we shall denote by $x^\natural$.



The fact that $x^\natural$ corresponds to the intuitive notion of "shape" of x is justified by theorem 5.16. The rest of this subsection makes this sketch formal.

We first define a many-to-one polygraph $\mathtt{1}$, that will turn out in proposition 5.13 to be terminal in $\mathcal{P}\mathrm{ol}_{\mathrm{mto}}$. We then proceed to define the shape function for $\mathtt{1}$, before stating the general case.

Definition 5.11 (The $\mathtt{1}$ polygraph)**.** We define a polygraph $\mathtt{1} \in \mathcal{P}\mathrm{ol}_{\mathrm{mto}}$ as follows: we first set $\mathtt{1}_0 := \{\blacklozenge\}$, and inductively, $\mathtt{1}_{n+1} := \left\{(u, v) \in \mathtt{1}_n^{\mathrm{mto}} \times \mathtt{1}_n \mid u \parallel v\right\}$ (definitions 4.3 and 4.12), with $\mathtt{s}(u, v) := u$ and $\mathtt{t}(u, v) := v$.

Lemma 5.12. *If $x, y \in \mathtt{1}_n$ are two parallel generators, then they are equal.*

Proof. We have $x = (\mathtt{s}x, \mathtt{t}x) = (\mathtt{s}y, \mathtt{t}y) = y$. □

Proposition 5.13. *The polygraph $\mathtt{1}$ is terminal in $\mathcal{P}\mathrm{ol}_{\mathrm{mto}}$.*

Proof. For $P \in \mathcal{P}\mathrm{ol}_{\mathrm{mto}}$, we show that there exists a unique $! : P \longrightarrow \mathtt{1}$.

(1) (Existence) For $x \in P_0$ let $!_0 x := \blacklozenge$, and for $x \in P_n$ with $n \geq 1$, let $!_n x = (!_{n-1}\mathtt{s}x, !_{n-1}\mathtt{t}x)$. The source and target compatibility is trivial.

(2) (Uniqueness) Let $f : P \longrightarrow \mathtt{1}$ be a morphism different from ! defined above. Necessarily $f_0 = !_0$ as $\mathtt{1}_0$ is a singleton. Let $x \in P_n$ be such that $f_n x \neq !_n x$, with n minimal. Then $n \geq 1$, and we have $!_n x = (!_{n-1}\mathtt{s}x, !_{n-1}\mathtt{t}x) = (f_{n-1}\mathtt{s}x, f_{n-1}\mathtt{t}x) \parallel f_n x$. By lemma 5.12, $!_n x = f_n x$, a contradiction. □

Proposition 5.14. *For $x \in \mathtt{1}_n$ there exists a unique $x^\natural \in \mathbb{O}_n$ such that the terminal morphism $!^{x^\natural} : |x^\natural| \longrightarrow \mathtt{1}$ maps $x^\natural$ (the unique n-generator of $|x^\natural|$) to x.*

Proof. (1) (Uniqueness) Assume that there exists two distinct opetopes $\phi, \phi' \in \mathbb{O}_k$ such that $!_k^\phi(\phi) = !_k^{\phi'}(\phi')$, with k minimal for this property. Then necessarily, $k \geq 2$. On the one hand, we have $\langle\phi\rangle = \langle \mathrm{ct}\,\mathtt{s}\,\phi\rangle =$

$\langle \mathrm{ct}\, \mathsf{s}\, \phi' \rangle = \langle \phi' \rangle$. On the other hand, for $[p] \in \phi^\bullet = (\phi')^\bullet$, we have

$$\begin{aligned}
!^{\mathsf{s}_{[p]}\,\phi}_{k-1}\, \mathsf{s}_{[p]}\, \phi &= !^{\phi}_{k-1}\, \mathsf{s}_{[p]}\, \phi && \text{since } !^{\mathsf{s}_{[p]}\,\phi} \text{ is also } |\mathsf{s}_{[p]}\, \phi| \hookrightarrow |\phi| \to \mathtt{1} \\
&= \mathsf{s}_{[p]} !^{\phi}_{k} \phi && \text{since } !^{\phi} \text{ is a mor. of polygraphs} \\
&= \mathsf{s}_{[p]} !^{\phi'}_{k} \phi' && \text{by assumption} \\
&= !^{\phi'}_{k-1}\, \mathsf{s}_{[p]}\, \phi' && \text{since } !^{\phi'} \text{ is a mor. of polygraphs} \\
&= !^{\mathsf{s}_{[p]}\,\phi'}_{k-1}\, \mathsf{s}_{[p]}\, \phi' && \text{since } !^{\mathsf{s}_{[p]}\,\phi'} \text{ is also } |\mathsf{s}_{[p]}\, \phi'| \hookrightarrow |\phi'| \to \mathtt{1},
\end{aligned}$$

and by minimality of k, we have $\mathsf{s}_{[p]}\, \phi = \mathsf{s}_{[p]}\, \phi'$, for all address $[p]$. Consequently, $\phi = \phi'$, a contradiction.

(2) (Existence) The cases $n = 0, 1$ are trivial, so assume $n \geq 2$, and that by induction, the result holds for all $k < n$, i.e. that for $g \in P_k$, there is a unique opetope $g^\natural \in \mathbb{O}_k$ such that $!^{g^\natural}_k g^\natural = g$. In particular the following two triangles commute:

$$\begin{array}{ccc}
|\mathsf{s}_{[p]}\, g^\natural| & \xrightarrow{|\mathsf{s}_{[p]}|} & |g^\natural| \\
& {\scriptstyle !^{\mathsf{s}_{[p]}\, g^\natural}} \searrow & \downarrow {\scriptstyle !^{g^\natural}} \\
& & P,
\end{array}
\qquad\qquad
\begin{array}{ccc}
|\mathsf{t}\, g^\natural| & \xrightarrow{|\mathsf{t}|} & |g^\natural| \\
& {\scriptstyle !^{\mathsf{t}\, g^\natural}} \searrow & \downarrow {\scriptstyle !^{g^\natural}} \\
& & P,
\end{array}$$

where $[p] \in (g^\natural)^\bullet$. Consequently, $(\mathsf{s}_{[p]}\, g)^\natural = \mathsf{s}_{[p]}(g^\natural)$ and $(\mathsf{t}\, g)^\natural = \mathsf{t}(g^\natural)$, and the following displays an isomorphism $\nabla_{n-1}\mathtt{1} \longrightarrow \mathfrak{Z}^{n-2}$:

$$\begin{array}{ccccccc}
\mathtt{1}_{n-2} & \xleftarrow{\mathsf{s}} & \mathtt{1}^\bullet_{n-1} & \xrightarrow{p} & \mathtt{1}_{n-1} & \xrightarrow{\mathsf{t}} & \mathtt{1}_{n-2} \\
{\scriptstyle (-)^\natural}\downarrow & & \downarrow & & {\scriptstyle (-)^\natural}\downarrow & & {\scriptstyle (-)^\natural}\downarrow \\
\mathbb{O}_{n-2} & \xleftarrow{\mathsf{s}} & \mathbb{O}^\bullet_{n-1} & \xrightarrow{p} & \mathbb{O}_{n-1} & \xrightarrow{\mathsf{t}} & \mathbb{O}_{n-2}.
\end{array}$$

Hence, the composite $\langle \mathrm{ct}\, \mathsf{s}\, x \rangle \xrightarrow{\mathrm{ct}\, \mathsf{s}\, x} \nabla_{n-1}\mathtt{1} \xrightarrow{(-)^\natural} \mathfrak{Z}^{n-2}$ defines an n-opetope $x^\natural$ with $\langle x^\natural \rangle = \langle \mathrm{ct}\, \mathsf{s}\, x \rangle$.

We claim that $!^{x^\natural}_n x^\natural = x$. We first show that $!^{x^\natural}_{n-1}\, \mathsf{s}\, x^\natural = \mathsf{s}\, x$. We have

$$\begin{aligned}
\langle \mathrm{ct}\, \mathsf{s}\, x \rangle &= \langle x^\natural \rangle && \text{by definition} \\
&= \langle \mathrm{ct}\, \mathsf{s}\, x^\natural \rangle && \text{by } \textbf{(PR3)} \\
&= \langle \mathrm{ct} !^{x^\natural}_{n-1}\, \mathsf{s}\, x^\natural \rangle && \text{since } !^{x^\natural} \text{ is a mor. of polygraphs.}
\end{aligned}$$

Then, for any address $[p]$ in $\langle \mathrm{ct}\, \mathsf{s}\, x \rangle$, we have

$$\begin{aligned}
\mathsf{s}_{[p]}\, x &= !^{x^\natural}_{n-1} (\mathsf{s}_{[p]}\, x)^\natural && \text{by induction} \\
&= !^{x^\natural}_{n-1}\, \mathsf{s}_{[p]}\, x^\natural && \text{by definition of } x^\natural \\
&= \mathsf{s}_{[p]} !^{x^\natural}_{n} x^\natural && \text{since } !^{x^\natural} \text{ is a mor. of polygraphs,}
\end{aligned}$$

and therefore, by proposition 4.21, $\mathsf{s}\, x = \mathsf{s}(!^{x^\natural}_n x^\natural)$. Next,

$$\begin{aligned}
\mathsf{t}(!^{x^\natural}_n x^\natural) &= !^{x^\natural}_{n-1}\, \mathsf{t}\, x^\natural && \text{by induction} \\
&\parallel !^{x^\natural}_{n-1}\, \mathsf{s}\, x^\natural && \text{since } \mathsf{s}\, x^\natural \parallel \mathsf{t}\, x^\natural \\
&= \mathsf{s}\, x && \text{showed above} \\
&\parallel \mathsf{t}\, x,
\end{aligned}$$

and therefore, $\mathsf{t} !^{x^\natural}_n x^\natural = \mathsf{t}\, x$. Finally, $!^{x^\natural}_n x^\natural \parallel x$, and by lemma 5.12, $!^{x^\natural}_n x^\natural = x$. □

Notation 5.15. In the light of proposition 5.14, we identify $\mathtt{1}_n$ with $\mathbb{O}_n$. This identification is compatible with faces, i.e. $\mathsf{s}_{[p]}$ and t. Then, $!^\omega : |\omega| \longrightarrow \mathtt{1}$ maps a generator $(\phi \to \omega)$ to ϕ.

Theorem 5.16. *For $P \in \mathcal{P}\mathrm{ol}_{\mathrm{mto}}$ and $x \in P_n$, there exists a unique pair*[6]

$$\left(x^\natural, |x^\natural| \xrightarrow{\tilde{x}} P\right) \in |-|/P$$

such that $\tilde{x}_n(x^\natural) = x$. Moreover, $x^\natural = !_n^P x$, where $!^P$ is the terminal map $P \longrightarrow \mathtt{1}$. In particular, the map

$$\widetilde{(-)} : P_n \longrightarrow \sum_{\omega \in \mathbb{O}_n} \mathcal{P}\mathrm{ol}_{\mathrm{mto}}(|\omega|, P)$$

is an isomorphism.

Proof. (1) (Uniqueness) Assume $|\omega| \xrightarrow{f} P \xleftarrow{f'} |\omega'|$ are different morphisms such that $f_n(\omega) = x = f'_n(\omega')$. Then $!^\omega_n(\omega) = !^P f_n(\omega) = !^P f'_n(\omega') = !^{\omega'}(\omega')$, and by proposition 5.14, $\omega = \omega'$. Let $\left(\phi \xrightarrow{\mathsf{a}} \omega\right) \in |\omega|_k$ be such that $f_k\left(\phi \xrightarrow{\mathsf{a}} \omega\right) \neq f'_k\left(\phi \xrightarrow{\mathsf{a}} \omega\right)$, with k minimal for this property. Then $k < n$ (since by assumption $f_n(\omega) = x = f'_n(\omega')$), and a factorizes as $\left(\phi \xrightarrow{\mathsf{j}} \psi \xrightarrow{\mathsf{b}} \omega\right)$, where j is a face embedding, i.e. either $\mathtt{t}$ or $\mathtt{s}_{[p]}$ for some $[p] \in \omega^\bullet$. Then by assumption,

$$\begin{aligned} f_k\left(\phi \xrightarrow{\mathsf{a}} \omega\right) &= \mathsf{j}\left(f_{k+1}\left(\psi \xrightarrow{\mathsf{b}} \omega\right)\right) \\ &= \mathsf{j}\left(f'_{k+1}\left(\psi \xrightarrow{\mathsf{b}} \omega\right)\right) && \text{by minimality of } k \\ &= f'_k\left(\phi \xrightarrow{\mathsf{a}} \omega\right), \end{aligned}$$

a contradiction.

(2) (Existence) The cases $n = 0, 1$ are trivial, so assume $n \geq 2$, and that by induction, the result holds for all $k < n$. Let $x^\natural = !^P_n x \in \mathbb{O}_n$. We wish to construct a morphism $O[x^\natural] \xrightarrow{\tilde{x}} P$ having x in its image. For $\left(\psi \xrightarrow{\mathsf{j}} x^\natural\right)$ a face of $x^\natural$ (i.e. $\mathtt{t}$ or $\mathtt{s}_{[p]}$ for some $[p] \in (x^\natural)^\bullet$), we have $(\mathsf{j}\, x)^\natural = \psi$, so that by induction, there exists a morphism $|\psi| \xrightarrow{\widetilde{\mathsf{j}x}} P$ having $\mathsf{j}\, x$ in its image, providing a commutative square

$$\begin{array}{ccc} |\psi| & \xrightarrow{\widetilde{\mathsf{j}x}} & P \\ {\scriptstyle |\mathsf{j}|}\downarrow & & \downarrow{\scriptstyle !^P} \\ |x^\natural| & \xrightarrow{!^{x^\natural}} & \mathtt{1}. \end{array}$$

To alleviate upcoming notations, write $\bar{\mathsf{j}} := \widetilde{\mathsf{j}x} : |\psi| \longrightarrow P$. Let $\left(\phi \xrightarrow{\mathsf{a}} x^\natural\right) \in \mathbb{O}_{<n}/x^\natural$. If a is a face embedding, define $\bar{\mathsf{a}}$ as before. If not, then it factors through a face embedding as $\mathsf{a} = \left(\phi \xrightarrow{\mathsf{j}} \psi \xrightarrow{\mathsf{b}} \omega\right)$, and let $\bar{\mathsf{a}} := \bar{\mathsf{b}} \cdot |\mathsf{j}|$. Then the left square commutes, and passing to the colimit over $\mathbb{O}_{<n}/x^\natural$, we obtain the right square:

$$\begin{array}{ccc} |\phi| & \xrightarrow{\bar{\mathsf{a}}} & P \\ {\scriptstyle |a|}\downarrow & & \downarrow{\scriptstyle !^P} \\ |x^\natural| & \xrightarrow{!^{x^\natural}} & \mathtt{1}, \end{array} \qquad\qquad \begin{array}{ccc} \partial|x^\natural| & \xrightarrow{f} & P \\ \downarrow & & \downarrow{\scriptstyle !^P} \\ |x^\natural| & \xrightarrow{!^{x^\natural}} & \mathtt{1}. \end{array}$$

We want a diagonal filler of the right square. Since $|x^\natural|$ is a one-generator cellular extension of $\partial|x^\natural|$ (definition 5.9), it is enough to check that $f_{n-1}\, \mathtt{s}\, x^\natural = \mathtt{s}\, x$, and $f_{n-1}\, \mathtt{t}\, x^\natural = \mathtt{t}\, x$. The latter is clear, as f extends $\bar{\mathtt{t}} : |\mathtt{t}\, x^\natural| \longrightarrow P$, and $f_{n-1}\, \mathtt{t}\, x^\natural = \bar{\mathtt{t}}_{n-1}\, \mathtt{t}\, x^\natural = \mathtt{t}\, x$ by definition. We now proceed to prove the former. First, $\left\langle \mathrm{ct}\, \mathtt{s}\, x^\natural \right\rangle = \langle \mathrm{ct}\, \mathtt{s}\, x \rangle$ since both are mapped to the same element of $\mathtt{1}_n$. Then, for $[p]$ a node address of $\mathrm{ct}\, \mathtt{s}\, x^\natural$, we have $f_{n-1}\, \mathtt{s}_{[p]}\, x^\natural = \overline{\mathtt{s}_{[p]}}_{n-1}\, \mathtt{s}_{[p]}\, x^\natural = \mathtt{s}_{[p]}\, x$. Hence $f_{n-1}\, \mathtt{s}\, x^\natural = \mathtt{s}\, x$. □

[6] In [Hen17, proposition 2.2.3], $x^\natural$ is written $[x]$.

5.3. **The adjoint equivalence.**

Definition 5.17 (Polygraphic realization-nerve adjunction)**.** The polygraphic realization functor $|-| : \mathbb{O} \longrightarrow \mathcal{P}\mathrm{ol}_{\mathrm{mto}}$ of section 5.1 extends to a left adjoint

$$|-| : \mathcal{P}\mathrm{sh}(\mathbb{O}) \rightleftarrows \mathcal{P}\mathrm{ol}_{\mathrm{mto}} : N,$$

by left Kan extension of along the Yoneda embedding $\mathsf{y} : \mathbb{O} \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O})$. Explicitly, the polygraphic realization of an opetopic set $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})$ can be computed with the coend on the left, while the and the *polygraphc nerve* NP of a polygraph $P \in \mathcal{P}\mathrm{ol}_{\mathrm{mto}}$ is given on the right:

$$|X| = \int^{\omega \in \mathbb{O}} X_\omega \times |\omega|, \qquad\qquad NP = \mathcal{P}\mathrm{ol}_{\mathrm{mto}}(|-|, P) : \mathbb{O}^{\mathrm{op}} \longrightarrow \mathcal{S}\mathrm{et}.$$

We note $\eta : \mathrm{id}_{\mathcal{P}\mathrm{sh}(\mathbb{O})} \longrightarrow N|-|$ the unit, $\varepsilon : |N| \longrightarrow \mathrm{id}_{\mathcal{P}\mathrm{ol}_{\mathrm{mto}}}$ the counit, and $\Phi : \mathcal{P}\mathrm{sh}(\mathbb{O})(-, N) \longrightarrow \mathcal{P}\mathrm{ol}_{\mathrm{mto}}(|-|, -)$ the natural hom-set isomorphism.

Notation 5.18. Unfolding the coend formula of definition 5.17, for $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})$, its realization can be written as the quotient

$$|X| = \frac{\sum_{\omega \in \mathbb{O}} X_\omega \times |\omega|}{\binom{f^* x}{\phi \xrightarrow{h} \psi} \sim \binom{x}{\phi \xrightarrow{h} \psi \xrightarrow{f} \omega} \text{ with } x \in X_\omega, h : \phi \to \psi, f : \psi \to \omega}.$$

For $\binom{x}{\psi} \in \sum_{\omega \in \mathbb{O}} X_\omega \times |\omega|$, let $\left[\begin{smallmatrix} x \\ \psi \end{smallmatrix}\right]$ be its equivalence class in $|X|$. Note that all classes have a representative of the form $\left[\begin{smallmatrix} y \\ \mathrm{id}_\phi \end{smallmatrix}\right]$, for some $\phi \in \mathbb{O}$ and $y \in X_\phi$.

Proposition 5.19. *Take $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})$, $P \in \mathcal{P}\mathrm{ol}_{\mathrm{mto}}$, and $f : X \longrightarrow NP$. The unit η at X, the transpose Φf of f, and the counit ε at P are respectively given by:*

$$\begin{array}{lll} \eta : X_\omega \longrightarrow N|X|_\omega & \Phi f : |X|_\omega \longrightarrow P_\omega & \varepsilon : |NP|_\omega \longrightarrow P_\omega \\ x \longmapsto \widetilde{\left[\begin{smallmatrix} x \\ \mathrm{id}_\omega \end{smallmatrix}\right]}, & \left[\begin{smallmatrix} x \\ \mathrm{id}_\omega \end{smallmatrix}\right] \longmapsto f(x)(\omega), & \left[\begin{smallmatrix} \tilde{x} \\ \mathrm{id}_\omega \end{smallmatrix}\right] \longmapsto x. \end{array}$$

Proof. (1) (Unit and transpose) We have to check that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\eta} & N|X| \\ {\scriptstyle f}\downarrow & \swarrow_{N\Phi f} & \\ NP, & & \end{array}$$

and that f is unique for that property. For $x \in X_\omega$ we have

$$(N\Phi f)(\eta(x)) = (N\Phi f)\left(\widetilde{\left[\begin{smallmatrix} x \\ \mathrm{id}_\omega \end{smallmatrix}\right]}\right) = (\Phi f)\widetilde{\left[\begin{smallmatrix} x \\ \mathrm{id}_\omega \end{smallmatrix}\right]},$$

which maps ω to $f(x)(\omega)$. Since a map $|\omega| \longrightarrow P$ is uniquely determined by the image of ω, we have $(N\Phi f)\eta = f$. Let $g : X \longrightarrow NP$ be another morphism such that $(N\Phi g) \cdot \eta = f$. Then for $x \in X_\omega$ we have

$$f(x)(\omega) = \left((N\Phi g) \cdot \eta\right)(x)(\omega) = (N\Phi g)\widetilde{\left[\begin{smallmatrix} x \\ \mathrm{id}_\omega \end{smallmatrix}\right]}(\omega) = g(x)(\omega),$$

whence $f = g$.

(2) (Counit) The counit is given by $\varepsilon = \Phi(\mathrm{id}_{NP})$, so that

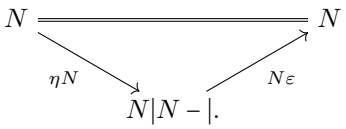
$$\varepsilon\left[\begin{smallmatrix} \tilde{x} \\ \mathrm{id}_\omega \end{smallmatrix}\right] = (\Phi\,\mathrm{id}_{NP})\left[\begin{smallmatrix} \tilde{x} \\ \mathrm{id}_\omega \end{smallmatrix}\right] = \tilde{x}(\omega) = x.$$

□

Theorem 5.20. *The unit and counit are natural isomorphisms. Consequently, the polygrahic realization-nerve adjunction of definition 5.17 is an adjoint equivalence between $\mathcal{P}\mathrm{sh}(\mathbb{O})$ and $\mathcal{P}\mathrm{ol}_{\mathrm{mto}}$.*

Proof. (1) (Unit) Remark that for $x, y \in X_\omega$, if $\left[\begin{smallmatrix} x \\ \mathrm{id}_\omega \end{smallmatrix}\right] = \left[\begin{smallmatrix} y \\ \mathrm{id}_\omega \end{smallmatrix}\right]$, then $x = y$, which shows that η is injective. Take $f \in N|X|_\omega$. Then $f(\omega)$ is of the form $\left[\begin{smallmatrix} x \\ \mathrm{id}_\omega \end{smallmatrix}\right]$, hence $f = \eta(x)$, and η is surjective.

(2) (Counit) The following triangle identity shows that $N\varepsilon$ is a natural isomorphism:



It is easy to check that the following square commutes, and since $\widetilde{(-)}$ is a bijection by theorem 5.16, ε is a natural isomorphism:

$$\begin{array}{ccc} |NP| & \xrightarrow{\varepsilon} & P \\ {\scriptstyle\widetilde{(-)}}\downarrow & & \downarrow{\scriptstyle\widetilde{(-)}} \\ N|NP| & \xrightarrow{N\varepsilon} & NP. \end{array}$$

□

Many-to-one polygraphs have been the subject of other work [HMZ02, HMZ08], and proved to be equivalent to the notion of *multitopic sets*. This, together with our present contribution, proves the following:

Corollary 5.21. *The category* $\mathcal{P}\mathrm{sh}(\mathbb{O})$ *of opetopic sets is equivalent to the category of multitopic sets.*

In [Hen17, corollary 2.4.6 and paragraph 2.4.7], Henry shows that $\mathcal{P}\mathrm{ol}_{\mathrm{mto}}$ is a presheaf category over some category $\mathbb{O}$plex of *opetopic plexes*, and asks wether they are the same as opetope. We now answer positively to this question.

Corollary 5.22. *The category* $\mathbb{O}$plex *of opetopic plexes is equivalent to* $\mathbb{O}$*.*

Proof. Opetopic plexes are proved to be generators of the terminal many-to-one polygraph $\mathtt{1}$ in [Hen17, Proposition 2.2.3], and so together with proposition 5.14, we have that opetopic plexes are exactly opetopes. On the other hand, morphisms of opetopic plexes are by definition morphisms of polygraphs between the representables they induce, which by the Yoneda lemma are exactly morphisms of opetopes. □

References

[BD98] John C. Baez and James Dolan. Higher-dimensional algebra. III. n-categories and the algebra of opetopes. *Advances in Mathematics*, 135(2):145–206, 1998.

[Che03] Eugenia Cheng. The category of opetopes and the category of opetopic sets. *Theory and Applications of Categories*, 11:No. 16, 353–374, 2003.

[GK13] Nicola Gambino and Joachim Kock. Polynomial functors and polynomial monads. *Mathematical Proceedings of the Cambridge Philosophical Society*, 154(1):153–192, 2013.

[GM09] Yves Guiraud and Philippe Malbos. Higher-dimensional categories with finite derivation type. *Theory and Applications of Categories*, 22:No. 18, 420–478, 2009.

[Hen17] Simon Henry. Non-unital polygraphs form a presheaf categories. *arXiv e-prints*, November 2017. `arXiv:1711.00744 [math.CT]`.

[HMP00] Claudio Hermida, Michael Makkai, and John Power. On weak higher dimensional categories. I. 1. *Journal of Pure and Applied Algebra*, 154(1-3):221–246, 2000. Category theory and its applications (Montreal, QC, 1997).

[HMP02] Claudio Hermida, Michael Makkai, and John Power. On weak higher-dimensional categories. I. 3. *Journal of Pure and Applied Algebra*, 166(1-2):83–104, 2002.

[HMZ02] Victor Harnik, Michael Makkai, and Marek Zawadowski. Multitopic sets are the same as many-to-one computads. Available at `http://www.math.mcgill.ca/makkai/m_1_comp/Manytoonecomputads.pdf`, 2002.

[HMZ08] Victor Harnik, Michael Makkai, and Marek Zawadowski. Computads and multitopic sets. *arXiv e-prints*, 2008. `arXiv:0811.3215 [math.CT]`.

[KJBM10] Joachim Kock, André Joyal, Michael Batanin, and Jean-François Mascari. Polynomial functors and opetopes. *Advances in Mathematics*, 224(6):2690–2737, 2010.

[Koc11] Joachim Kock. Polynomial functors and trees. *International Mathematics Research Notices*, 2011(3):609–673, January 2011.

[Lei04] Tom Leinster. *Higher Operads, Higher Categories*. Cambridge University Press, 2004.

[M03] François Métayer. Resolutions by polygraphs. *Theory and Applications of Categories*, 11:No. 7, 148–184, 2003.

Research Institute for Foundations of Computer Science (IRIF), Paris University, Paris, France