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# Using dynamic worksheets to support functional thinking in lower secondary school

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*This paper reports on a current case study about the use of dynamic worksheets in a middle school in Austria. These worksheets were designed based on typical problems and misconceptions outlined in the literature concerning functional thinking, and they focus on the representational transfer between situational model and graphical representation. Grade 7 students were video recorded while working on these worksheets, pre- and post diagnostic tests and diagnostic interviews were conducted to examine their conceptions in relation to functions. This case study particularly pays attention to the intuitive conceptions of students, the influence of the dynamic worksheets on these conceptions, and whether or not dynamic worksheets are able to support students in developing appropriate mathematical conceptions. In this paper, some preliminary results are to be discussed.*

*Keywords: Functional thinking, technology, representational transfer, lower secondary school.*

## Introduction

Functional thinking is an important concept in mathematics education. For students, a variety of problems arise while working with functions and thus functional thinking has been widely investigated by numerous researchers. Considering the development of dynamic mathematics software, additional aspects of functional thinking appear. It needs to be examined whether or not dynamic mathematics tasks are able to support students in an early stage of learning functions in developing appropriate conceptions. Based on these issues, we developed dynamic worksheets visualizing the transfer between situational and graphical representations and integrated them into a qualitative case study to examine their influence on students' conceptions.

## Theoretical background

Working with functions is a usual activity in mathematics lessons in school. Vollrath (1989) describes functional thinking as a typical way of thinking when dealing with function and he mentions different aspects of functional thinking. Malle (2000) refers to it and specifies the following aspects in a slightly altered version, which is better suited than Vollrath's (1989) description for the purposes of this research project: Relational aspect (each argument  $x$  is associated with exactly one function value  $f(x)$ ) and co-variational aspect (if the argument  $x$  is changed, the function value  $f(x)$  will change in a specific way and vice versa). The relational aspect represents a static perspective of functional thinking whereas the co-variational aspect describes dynamic processes; particularly in this project, functional thinking comprises of both aspects.

In the context of functional thinking, various difficulties have been found and examined in the research literature. The graph-as-picture error occurs in various forms and means that students see function graphs as photographic images of a real situation (Clement, 1989; Schlöglhofer, 2000). Illusion of linearity means the preferable use of linear or direct proportional models for the description of relations, even if they are not appropriate (De Bock, Van Dooren, Janssens, & Verschaffel, 2002).

Difficulties arise also in the interpretation of slope and growth, for example, if the point of maximum growth is confused with the largest function value. This slope-height-confusion leads also to difficulties in the interpretation of path-time graphs (Clement, 1989). These problems can cause students' misinterpretations of functions and especially of graphs of functions. Vosniadou and Vamvakoussi (2006) suggest – to avoid that intuitive conceptions develop to misconceptions – considering the introduction of mathematical concepts at an earlier stage in mathematics education.

Vogel (2007) stresses that multiple representations of functions, such as graphs, situational representations, terms, and tables are able to represent aspects of functional thinking (relational as well as co-variational aspect) externally, and they have the potential to support students' ability to interpret functions. According to Duval (2006), only the flexible alternation between different representations allows a differentiated approach to mathematical content and forms the basis for sustainable acquisition of skills. But representations have to be considered critically as they influence ways of thinking, they may constrain students' thinking about the concepts involved and are interpreted by students according to their prior knowledge (Vosniadou & Vamvakoussi, 2006).

Dynamic mathematics software such as GeoGebra may support students' development of functional thinking, because it is suitable to emphasize different functional aspects through interactive representations (Barzel & Greefrath, 2015). Research findings in relation to the use of technology in teaching often only show small positive effects on students' learning achievements (Drijvers et al., 2016). Results concerning dynamic representations are more encouraging, because these representations can help students in understanding mathematical concepts (Hoyles, Noss, Vahey, & Roschelle, 2013). Thus, we need to examine in more detail the influence of technology on students' individual conceptions.

### **Conceptions in a dynamic mathematics environment**

Based on problems and examples concerning misconceptions mentioned in literature several dynamic GeoGebra worksheets were designed reflecting multimedia design criteria (Clark & Mayer, 2011). We chose GeoGebra for this study, because it is the most widely employed mathematics software in Austrian schools. Due to the prior knowledge of selected students (experiences mainly with path-time diagrams, direct and inverse proportionality including their graphical representations, but none with the explicit function concept), these worksheets primarily address the representational transfer between situational model and graphical representation. Figure 1 displays a typical worksheet based on a task of Schlöglhofer (2000) addressing a graph-as-picture error.

This GeoGebra worksheet consists of a situational model, in particular an iconic representation of the situation, as well as a Cartesian coordinate system displaying the corresponding graph. In the situational representation on the left side a triangle is displayed. The shaded area left of the dotted line inside the triangle is treated as a function of  $x$ , which is the horizontal distance between the vertex A and the dotted line. Students can move the line and change the size of the coloured area. Afterwards they should formulate a hypothesis about the shape of the graph. In the diagram the coloured area is a function of the distance  $x$ . After clicking the checkbox, the size of the area is displayed. At the end, students should display the graph in order to examine their assumptions about the shape of the graph. In this research project, accompanying tasks assist the students in working with the representational transfer, which is considered particularly difficult conceptually.

## Triangle

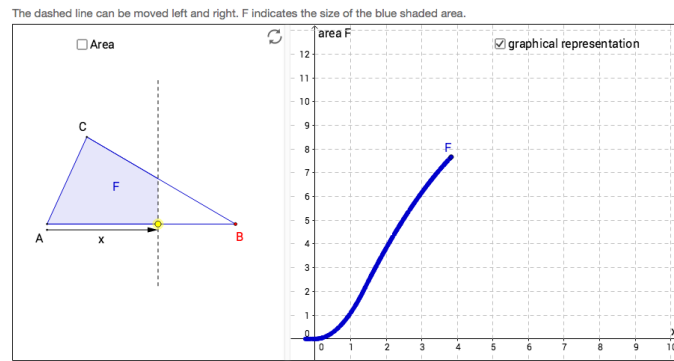


Figure 1: “Triangle”, <http://ggbm.at/GYeY4ayO>

As the situational model shows an iconic representation, the corresponding task is likely to trigger a graph-as-picture error (Schlöghofer, 2000). It is especially interesting if the dynamic worksheet has the potential to support students’ ability to comprehend the graph.

Such problems concerning functional thinking and theoretical considerations have led to the research questions below. The first two offer a basis for research question three, as we believe this third question contributes the most to the field of inquiry about technology use related to misconceptions. Due to space restriction this paper focuses only on the first and third research question. Future papers will offer further details on research question two as well as more in-depth analyses.

- 1: What conceptions, with particular attention to pre- or intuitive conceptions, emerge concerning functional thinking of students in an early phase of learning functions (grade 7/8)?
- 2: How should dynamic materials addressing to this topic be designed to support students in developing appropriate mathematical conceptions?
- 3: What kinds of influence of these dynamic materials exist on conceptions and internal representations of students of lower secondary school concerning functional thinking?

## Research design

To offer answers to the research questions, we selected a qualitative research approach. The overarching methodology for this research project is an exploratory and collective case study research, but integrating elements of Grounded Theory (Eisenhardt, 1989). The study was conducted in a 7<sup>th</sup> grade classroom of a rural middle school<sup>1</sup> in Austria with 28 students aged 12 to 13, who had some experience in working with graphs (mainly distance-time-graphs) but none with the function concept itself.

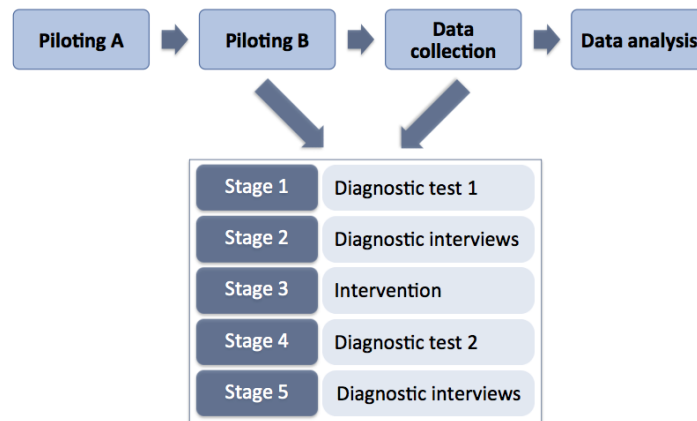
Figure 2 shows an overview of the research design. Piloting A was the first phase of the study aimed to evaluate the technical details of the recording procedure, to choose the tasks for the diagnostic tests and the worksheets for the intervention. The second phase – piloting B – consisted of one complete

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<sup>1</sup> Rural middle schools in Austria usually have the most diverse student population concerning achievement levels. This is especially true for the selected school in this study, thus this choice offers us the possibility to examine as many different students and their conceptions as possible.

data collection process. After the data collection, the data was transcribed and analysed with qualitative methods described later in detail.

The data collection included five stages. First, all students participated in a diagnostic test based on ten different tasks from literature concerning conceptions (Schlöglhofer, 2000; De Bock et al., 2002) as well as a test instrument called CODI (Nitsch, 2015). Afterwards, eight students were chosen for diagnostic interviews (Hunting, 1997) depending on their test responses so that their – incorrect – results represent a wide range of different conceptions related to the various test tasks to obtain an in-depth view of their individual conceptions.



**Figure 2: Research design overview**

During the three-lesson-intervention, students worked in pairs with GeoGebra worksheets addressing different topics guided by accompanying tasks. While working, ten students were audio- and videotaped and the screens of their laptops were recorded. Also, students' paper worksheets were collected. After completing the intervention, another diagnostic test with slightly altered tasks was conducted. Based on the observational data and an analysis of the test results, eight students were selected for diagnostic interviews to investigate the influence of the worksheets on the students' conceptions.

### **Data analysis and preliminary results**

The collected data is divided by the data source to address different research questions (the first test results and the corresponding interview data to approach the first research question, the recordings from the intervention and students' paper worksheets, the second test results and the corresponding interview data to focus on the second and the third research question).

Based on the research methodology, we conducted, for each student or pair of students, a within-case analysis using initial (or open) coding, then compared cases and searched for cross-case patterns using focused coding (Eisenhardt, 1989; Saldaña, 2013).

Further qualitative analysis of the observational data and the interview recordings will give an insight into the conceptions of the students concerning functional thinking as well as the influence of dynamic worksheets on these conceptions. In this section, preliminary results concerning the task "Area" from both diagnostic tests are to be presented, because both task and results exemplify the process of the research.

### Diagnostic test 1

Figure 3 displays a task from the diagnostic test 1 that is similar to the GeoGebra worksheet in Figure 1 and was based on a standard test example concerning the graph-as-picture error from Nitsch (2015). The picture shows a trapezoid, and in the exercise students had to choose one diagram out of four that showed the grey marked area left of the dotted line as a function of the distance  $x$ , and to explain their decisions.

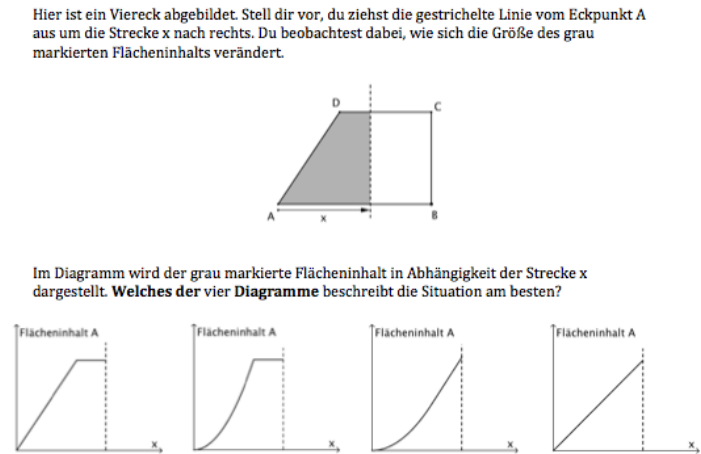


Figure 3: Screenshot task “Area” from diagnostic test 1

Students’ explanations reveal different levels of conceptual understanding. A categorization of students’ solutions and argumentations is visualized in Figure 4. The arrows represent the direction of the representational transfer from the situational model to the function graph, and the categories are arranged according to the correctness and elaborateness of students’ understanding.

The first two categories represent the choice of the first graph addressing the graph-as-picture error. Either the students marked the similarity between both representations, or they already recognized an increase of area but ‘remained’ at the shape of a trapezoid. These answers reveal reasoning from a situational perspective of students, who did not manage to transfer the situational model into a function graph.

Students with explanations of the next three categories achieved transfer to a graphical representation by recognizing an increasing function value, and these explanations were essentially correct. The third category of students, who selected the linear function, did not recognize the irregular change of the function value. Students who chose a correct graph form the last two categories. They either reasoned their choice with an increasing area or – the most elaborated explanation – with an irregular growth of area.

In the next section, we present three student answers to the corresponding task from the second diagnostic test. Students were chosen from the first category (Graph as Picture, Similarity) to

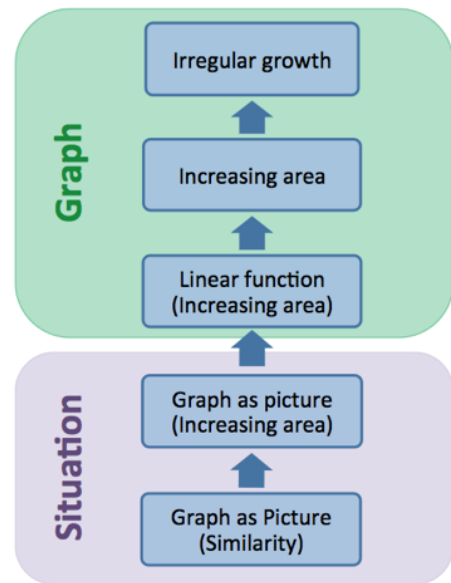
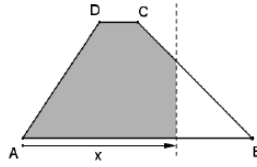


Figure 4: Categorized solutions task “Area”

demonstrate the range of possible developments of their conceptions. The answers represent different extent of influence, also based on the achievement level of these students.

### Diagnostic test 2

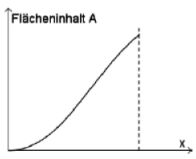
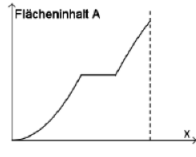

In the second diagnostic test after the intervention, the corresponding task was slightly altered – a trapezoid of another shape is displayed (see Figure 5).



**Figure 5: Trapezoid of task “Area” of diagnostic test 2**

Corresponding to the four function graphs in diagnostic test 1, there are four possible choices for the solution. These function graphs represent the graph-as-picture error, the correct solution, a combination of graph-as-picture and correct graph, and a linear function.

Table 1 presents three students’ solutions and explanations from the diagnostic test 1 category “Graph as Picture (Similarity)” exemplifying a possible diverse influence of the applets. These students were chosen because all three students changed their answers after working with the dynamic worksheets, and due to their different achievement levels they gave a wide range of changes in their results.

Achievement level	Solution	Explanation (translated from German)
High (student 1)		“In the beginning it [the area] increases strongly, then a bit more slowly, ...”
Average (student 2)		“The area is always increasing, except in ... the middle of the trapezoid, it [the area] remains the same.”
Low (student 3)		“You have to consider the x-axis, and because the x-axis is straight, the last ... ought to be correct.”

**Table 1: Students’ answers concerning task “Area” of diagnostic test 2**

Student 1 (high achiever) chose the correct solution, and the explanation reveals a correct understanding of irregular changes of the function value. Also student 2 (average achiever) described the change of area correctly, but he decided for the graph representing a combination of correct graph and graph-as-picture error.

Unlike in the diagnostic test, the GeoGebra worksheet displayed only a triangle and not a trapezoid. Student 2 managed to translate corresponding parts of the situation correctly to the graphical

representation, but was not able to transfer his knowledge to the part of the situational model where the dotted line is moved over the ‘horizontal line’ of the trapezoid.

The explanation of student 3 (low achiever) reveals a lack of understanding concerning the representational transfer and the meaning of Cartesian coordinates. It refers to the students’ look for visual similarities, a solution strategy sometimes used by students during the intervention when confronted with a problem. Further results about students’ test answers, their discussion during the work with the dynamic worksheet (Figure 1) as well as a detailed description of this worksheet and included instructions will be presented in upcoming research papers.

## Discussion

For each task in the diagnostic test, several intuitive and incorrect conceptions appeared. For example, different levels of students’ conceptual understanding have emerged during analysis of the diagnostic test answers. These levels are representing the translation process from situational to graphical representation. Concerning the graph-as-picture error, explanations also made visible that standardized multiple-choice test items were not always able to detect a corresponding incorrect conception. Other results reveal different tendencies of students to use relational or co-variational aspects of functions for explanations. Also the influence of everyday experience is visible in the data, and the influence of formal and informal language (e.g., when students interpret ‘highest speed’ as ‘leading’ or ‘winning’ in distance-time-diagrams) is especially interesting.

The dynamic worksheets have different purposes, for example, visualization, experimentation, and testing hypotheses. The visualization function is supporting students in translating the text into a situational model or into a correct identification of the interesting variable (e.g., the meaning of ‘area’ in the corresponding task in Figure 3). Also, the worksheets have an adaptational influence on students’ conception (e.g., from linear to non-linear increase of function value). In other words they did not alter the conceptions but changed them partly to the direction of a correct conception.

Preliminary results seem to reveal that the extent of influence of these worksheets on students’ conceptions depends on the intuitive conceptions of students and their achievement level. The interpretation as well as the perception of the GeoGebra worksheets is based on the prior knowledge of the students. The observational data repeatedly demonstrated that students tried to connect new content to their experience and knowledge. Considering that students worked without teacher instructions, for high achieving students the dynamic worksheets seem to be more appropriate, whereas lower achieving students would probably profit of teachers’ assistance to reflect their perceptions and interpretations or to draw the attention to the important features of the worksheets.

## References

- Barzel, B., & Greefrath, G. (2015). Digitale Werkzeuge sinnvoll integrieren [Meaningful integration of digital tools]. In W. Blum, S. Vogel, C. Drücke-Noe, & A. Roppelt (Eds.), *Bildungsstandards aktuell: Mathematik in der Sekundarstufe II* (pp. 145–157). Braunschweig: Westermann.
- Clark, R., & Mayer, R. (2011). *E-learning and the science of instruction*. San Francisco: John Wiley & Sons.
- Clement, J. (1989). The concept of variation and misconceptions in Cartesian graphing. *Focus on Learning Problems in Mathematics*, 11(1–2), 77–87.



- De Bock, D., Van Dooren, W., Janssens, D., & Verschaffel, L. (2002). Improper use of linear reasoning: an in-depth study of the nature and the irresistibility of secondary school students' errors. *Educational Studies in Mathematics*, *50*(3), 311–334.
- Drijvers, P., Ball, L., Barzel, B., Heid, M. K., Cao, Y., & Maschietto, M. (2016). *Uses of technology in lower secondary mathematics education*. Cham: Springer International Publishing
- Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, *61*(1), 103–131.
- Eisenhardt, K. M. (1989). Building theories from case study research. *Academy of Management Review*, *14*(4), 532–550.
- Hoyles, C., Noss, R., Vahey, P., & Roschelle, J. (2013). Cornerstone Mathematics: Designing digital technology for teacher adaptation and scaling. *ZDM – International Journal on Mathematics Education*, *45*, 1057–1070.
- Hunting, R. P. (1997). Clinical interview methods in mathematics education research and practice. *Journal of Mathematical Behavior*, *16*(2), 145–165.
- Malle, G. (2000). Zwei Aspekte von Funktionen: Zuordnung und Kovariation [Two aspects of functions: Relation and co-variation]. *Mathematik lehren*, *103*, 8–11.
- Nitsch, R. (2015). *Diagnose von Lernschwierigkeiten im Bereich funktionaler Zusammenhänge* [Diagnosis of learning difficulties in the context of functions]. Springer Fachmedien, Wiesbaden.
- Saldaña, J. (2013). *The coding manual for qualitative researchers* (2<sup>nd</sup> ed.). London: Sage.
- Schlöglhofer, F. (2000). Vom Foto-Graph zum Funktions-Graph [From photo-graph to function graph]. *Mathematik lehren*, *103*, 16–17.
- Vogel, M. (2007). Multimediale Unterstützung zum Lesen von Funktionsgraphen [Multimedia support of reading graphs]. *Mathematica Didactica*, *30*(7), 3–28.
- Vollrath, H.-J. (1989). Funktionales Denken [Functional thinking]. *Journal für Mathematikdidaktik*, *10*(1), 3–37.
- Vosniadou, S., & Vamvakoussi, X. (2006). Examining mathematics learning from a conceptual change point of view. In L. Verschaffel, F. Dochy, M. Boekaerts, & S. Vosniadou (Eds.), *Instructional Psychology: Past, Present, and Future Trends* (pp. 55–70). Amsterdam: Elsevier.