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# Students' use of movement in the exploration of dynamic functions

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*The concept of function has a central role both at school and in everyday situations. A dynamic algebra and geometry software program allows students to experience the dependence relation and to explore functions as covariation. In this paper we propose a description of different dragging modalities and the analysis of a protocol in which two students work together on a problem that involves coordinating two covarying quantities. The analysis has been carried out through this classification of dragging modalities, that can be efficiently used to observe, describe and analyze students' processes involved in the exploration and solution of dynamic activities.*

*Keywords: Function, variation, dynamic algebra and geometry software, dragging.*

## Introduction and conceptual framework

The concept of function is very important both in secondary school and university mathematics but it also has a central role in everyday situations. For a long time, this notion has been at the core of several studies in mathematics education, and a rich literature has revealed students' difficulties in understanding the concept in all its aspects (Vinner & Dreyfus, 1989; Tall, 1991; Dubinsky & Harel, 1992; Carlson & Oehrtman, 2005). Difficulties in interpreting the dependence relation as a dynamic relation between covarying quantities are widely reported (Goldenberg et al., 1992; Carlson et al., 2002). Falcade et al. (2007) suggest that the use of a dynamic algebra/geometry software, such as GeoGebra, allows students to experience functions as covariation, that is a crucial aspect of the idea of function (Confrey & Smith, 1995; Tall, 1996). According to these assumptions we are interested in studying students' cognitive processes involved in approaching functions represented in a specific dynamic environment.

Our study is an exploratory study aimed at analyzing students' use of movement in the exploration processes of the dynamic functions. We have adopted the idea of analyzing the movement because several studies have revealed that it can support a cognitive analysis of students' reasoning processes.

In order to analyze students' appropriation of movement Arzarello et al. (2002) identified different types of dragging which students use investigating a geometric problem, according to their different purposes. Antonini & Martignone (2009) proposed a similar classification in the case of physical artifacts. They introduced a classification of students' utilization schemes of pantographs, that are particular mathematical machines designed for geometrical transformations. Although the differences due to the different nature of the instruments these two studies concern, there are certain similarities. Especially the common purpose is to identify students' utilization schemes in order to analyze the cognitive processes involved in the investigation of geometric problems.

In this paper we shall present the first steps of our research: a classification of dragging modalities and the analysis of a protocol, that has been carried out through this classification. Some of our descriptions echo the classifications presented in the above-cited studies, but they have been completely transformed in order to suit the use of particular function representations in a specific dynamic environment. This is the original contribution of this paper. Indeed, while in literature we

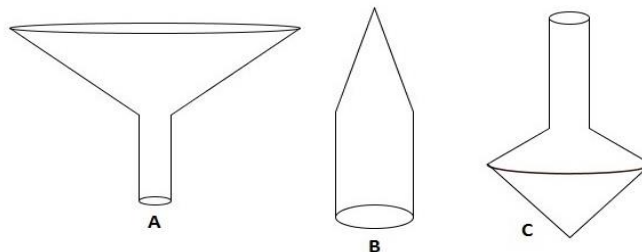
can also find other studies on the use of dragging (Baccaglioni & Mariotti, 2010; Robutti, 2013) they concern in particular the dynamic geometry.

## Contextualization of the study

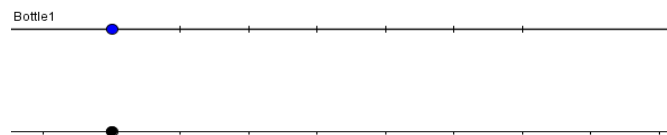
We analysed a sequence of classroom activities, 14 hours in total, implemented in an Italian high school for Math and Science, in which students explore the functional relationship in dynamic interactive files (in GeoGebra). The subjects of this investigation are 16 years old students who never met the concept of function before. The activities were led by the teacher and they have been video-recorded by three cameras present in the classroom simultaneously. The analysis is mainly based on the transcripts of the activities and it was led by paying a special attention to the use of dragging, the language employed and the gestures.

In this paper we present one of the activities carried out by two students. The activity selected concerns The Bottle Problem: an open problem about bottles filled with water (the task is reported in Figure 1), which involves coordinating the variations of two quantities. Students are asked to work in pairs so that they can form conjectures and explain their reasoning to each other. They have no time limits and are video-recorded through a camera behind them pointing at the computer screen. They are given the following task with an interactive dynamic file (in GeoGebra) for the explorations and some sheets of paper for the answer.

In the dynamic file in GeoGebra you can drag the black point. Starting from zero (left side), the position of the black point represents the volume of water filled in 5 bottles (the same volume is filled simultaneously in each bottle to the top). The blue points represent the height of the water in each of the 5 bottles. Link the lines with blue points to the relative bottle in the figure below and if there is no suitable figure, draw the bottle with the right shape.



**Figure 1: The task of the Bottle Problem**



**Figure 2: The dynamic file**

Figure 2 shows a part of the GeoGebra file in which are presented the graphs of five functions representing the height with respect to the volume of water. They are not the “usual” graphs in the Cartesian plane: there is an unnamed horizontal line with a black point attached to it that represents the x-axis and five other horizontal lines, parallel to it and labelled “Bottle1, Bottle2, ...” with blue points moving on them. The motion of the blue points, bounded at the lines, is an indirect motion

because these points can not be dragged directly: they represent the dependent variables so their movement is determined by the dragging of the black point, that represents the independent variable. The height of each bottle is fixed equal to six, for this reason the blue points move in the interval  $[0, 6]$  and there are six notches on the lines which they move on. The black point can be dragged everywhere along the line without the magnetism, that is a property that GeoGebra allows to give to a point and makes it move on the real axis as if it has a magnet that attaches it to the whole numbers; and disabling this tool the dragging of the point is more uniform.

### Dragging modalities

In this section we introduce a classification of dragging observed during students' exploration of dynamic interactive files. It can be efficiently used to observe, describe and analyze students' cognitive processes, involved in the exploration and solution of problems about functions represented in a specific dynamic environment.

The identified dragging modalities are divided into two families: the first one describes the quality of the movement, this type of dragging could be also recognized by a computer that captures how the mouse moves on the screen (Table 1) and the second one describes the use of dragging with regard to an aim, that is associated through the study of the language employed, the sight and the gestures (Table 2).

One of the potentialities of this classification is the fact that the two families of dragging modalities can be combined and, for example, keeping an element from the first one and an element from the second one allows a complete description of a students' process in solving problems.

First of all we observe that in our cases it is always a *bound dragging*, that according to Arzarello et al. (2002) consists of moving a semi-dragable point (a point which it is already linked to an object). Because the only point that students can move is bound to the x-axis, all the other points move depending on it.

	Description
Continuous dragging	Continuous movement
Discrete dragging	Movement with jumps, associated with counting
Impossible dragging	Trying to move a dependent <sup>1</sup> point that can not be dragged

**Table 1: Types of dragging**

	Description
Wandering dragging	Random movement, exploring the construction
Dragging test	Movement aimed at testing a possibly implicit conjecture

<sup>1</sup> We use this term to identify the point but we do not know if the students are aware of this dependence relation.

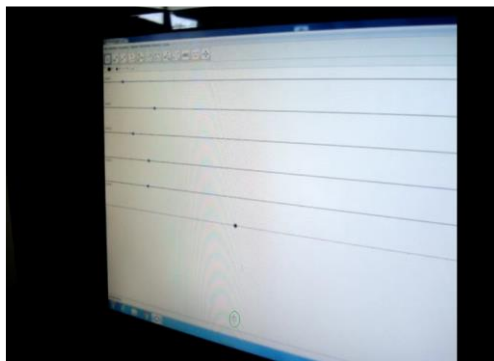
Handle dragging	Movement of the object as if it was a handle, in order to observe other objects' movements
Guided dragging	Movement aimed at reaching a particular configuration

**Table 2: Dragging with an aim**

### A protocol

In this section we present an activity in which two students, Luca and Mara, work together at the bottle problem and we can identify some of the dragging modalities described before.

Their first approach to the problem involves dragging the black point, representing the volume of water filling the bottle, with a continuous movement (*continuous dragging*) and without apparently paying attention to the dragging of such point: it is used as a sort of handle that allows them to see the movement of the blue point, representing the height of the water in the bottle (*handle dragging*). Indeed, as we can see in Figure 3, during the dragging the arrow representing the mouse does not overlap the black point in every moment, suggesting a weak haptic control because the students' attention seems not on the dragged point.



**Figure 3: An example of handle dragging**

The students do not express, through their words and gestures, awareness that as one variable changes, the other variable changes; they seem more concentrated on the differences between the movements of the blue points than on the relation that links the movement of a blue point to the movement of the black point. For example, they look for which one is the fastest in order to associate it to the tightest bottle, because the speed of blue points represents the speed at which the height increases if the water is poured in at a constant volume per time, and the tighter the bottle is, the faster the height increases; in the same way the slowest blue point will be associated to the widest bottle.

For example, the following dialogue takes place while students explore the file, dragging the black point very slowly and trying to keep a constant speed (*continuous dragging*):

- Luca: The bottle three is the steepest in the lower part.
- Mara: The bottle one goes very slow, also the bottle two.
- Luca: Also the fifth, the bottle two is the slowest respect all the others.
- Mara: No the five, the five does not move!
- Luca: Yes and then it is steeper at the end.

What we can infer from this excerpt is that the students' attention is on the blue points and the independent black point is used as a handle (*handle dragging*), they compare the speed of these points, observing for example that the second is the slowest, or probably the fifth. Luca, in the last sentence, says "is steeper" instead of "goes faster" and this suggests that he mixes up the trend of the height of water in the bottle with the shape of the bottle.

Their initial approach changes: when they have to decide which one of the blue points represents the bottle B, shown in Figure 1 (that in the lower part has a cylindrical shape). They search for a point that has a constant speed and, in doing this, they compare the movements of the black and the blue points. So, first of all, they look at the picture of the bottle on the sheet of paper and imagine how the height of the water in the bottle should evolve, then they drag the point representing the volume of water in order to see whether there is a point, representing the height, with the needed properties (*guided dragging*).

In particular, they count how many notches of volume are necessary to let the blue point reach the first notch of height and then to let it reach the second and finally the third and finally they compare these numbers: if they are equal to each other they conclude that the bottle has a cylindrical shape. It is an example in which the two quantities that are varying are coordinated in order to establish the average speed of the blue point. This seems an attempt to make a continuous situation discrete and it is also suggested from their use of dragging: they drag the black point with jumps, while counting the notches (*discrete dragging*).

The following excerpt shows this combination of *discrete* and *guided dragging*. Luca summarizes their idea about how the blue point representing the bottle B should behave and searches for it:

Luca:            So we have to find a point that is constant till the third notch and then it goes faster. I would see the bottle one, look: first, second, third more or less goes in the same way.

He drags the black point counting 1,2,3 and stops, 1,2,3 and stops, finally 1,2,3 and stops and during this process the mouse makes some jumps (*discrete dragging*).

Luca:            We could say that it is constant till the third notch and then...

He drags the black point again, this time with a continuous movement and an almost constant speed (*continuous dragging*).

Luca:            Then it goes faster!

The last part of the analysis reports students' explorations and conjectures when they have to draw the bottle looking at the movements of the points: the black point seems no longer to be only a handle for them. Indeed, as the next excerpts show, the students relate the changing values of height and volume in order to find whether the speed of the blue point is constant; their question is: how many notches of volume are necessary to have one notch of height? They fix the amount of change of the height (uniform increments) and find out the relative rate of change of the volume. In doing this, they consider the average rate of change locally, for a specific interval of the domain of the function.

It is not so clear how they conclude that "it is constant till the first notch" and this could be considered as an advanced statement because it requires an awareness that the instantaneous rate of change results

from smaller and smaller refinements of the average rate of change. From what they say it seems that at the beginning they observe a constant speed from the zero to the second notch:

Luca: Slowly at the beginning, it is wide, then it seems a constant velocity, then it is tighter and then wider again: this is a clepsydra. But a clepsydra that in the upper part is wider than in the tighter part. Wait, go back for a moment (she goes back with the black point onto the zero again: *dragging test*). How many notches of volume do you have to do, to have one notch of height?

Mara drags the black point very slowly and they count how many notches it crosses till the blue point reaches the first notch.

Luca: Five and a half, say five. Are these (notches) five again to reach the second notch?

The black point is dragged slowly again and they count how many notches it crosses till the blue point reaches the second notch. They count five notches, more or less. Therefore they conclude:

Luca: Yes, at the beginning it has a constant velocity.

Then by a similar process (*discrete dragging*) they observe that from the first to the second notch the average speed of the blue point is greater than from the zero to the first notch; so they decide that the bottle has to shrink at the first notch of height and before that point it has a cylindrical shape.

Luca: Now, count how many notches of volume: one, two, three, four, five let's round off (for a moment he stops dragging the black point, the blue point is on the first notch). Then from the first notch: one, two, three, four so it is tighter (for a moment he stops dragging the black point, the blue point is on the second notch), I mean the lower part is bigger than...

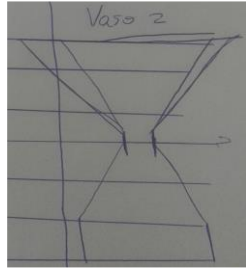
Finally, they check what they found out and they start drawing the bottle on the sheet of paper:

Luca: Therefore, constant till the first notch, then it is tighter and the third notch is the point in which it is the tightest. So: the first notch constant like this (he draws a vertical segment) then it starts to be tighter (he draws an oblique segment) up here. This is the tightest point (indicating the third notch on the sheet of paper).

Mara drags the black point (*continuous dragging*) without apparently paying attention to its movement, indeed the arrow representing the mouse is far from the point (*handle dragging*). She probably wants to find out where the blue point moves faster, because she puts the point on the second notch and explores a neighborhood of the third notch, that is the point suggested by Luca (*dragging test*).

Mara: It is in this passage that it is steeper (she stops dragging and indicates with the arrow of the mouse an interval between the third and the fourth notch).

Therefore they agree that the bottle has a choke point at half height. They conclude that in the upper part the bottle widens and it is wider than in the lower part because the height increases ever slower.



**Figure 4: Luca's drawing of the bottle**

## Discussion

The studies on the interaction between a subject and a software have to take into account a variety of aspects because several components are involved. In this paper we have presented a study to better understand the explorations of functional dependence in a dynamic algebra and geometry environment: in particular, we have identified different dragging modalities and we have shown an analysis carried out through this classification. The analysis highlights how the proposed description of dragging modalities allows an insight into students' problem solving processes.

We noticed that the *handle dragging* is often recognizable through the observation of the mouse's position: if the attention is placed on an object that is not draggable, it is possible that the arrow representing the mouse does not overlap the dragged point in every moment, suggesting a weak haptic control of the solver. But this is not a generalization, because there could be some cases of *handle dragging*, recognizable for example from student's words, in which the student seems to reveal a good haptic control. We observe that there are two types of *continuous dragging*, in some cases it reveals a movement of the object trying to keep a constant velocity, in other cases the object is dragged with a continuous movement, without jumps, but with a variable velocity, for example a point that is dragged back and forth on a line. In the selected protocol there are no examples of *impossible dragging*, probably because the task says explicitly that the only draggable point is the black one; but we identified various examples of this type of dragging in other activities that we analyzed.

One of the potentialities of this classification is that, in order to better describe students' problem solving processes, it is possible to combine two dragging modalities, one indicating the quality of the movement and the other associated with an aim. It could be interesting to develop this study to investigate how a description of students' use of movement in a dynamic algebra and geometry environment is intertwined with the processes involved in conceptualization of functions, that could give an insight into covariation in the concept of function.

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