



HAL
open science

Interactive Virtual Math: A tool to support self-construction graphs by dynamic events

Sonia Palha, Stephan Koopman

► **To cite this version:**

Sonia Palha, Stephan Koopman. Interactive Virtual Math: A tool to support self-construction graphs by dynamic events. CERME 10, Feb 2017, Dublin, Ireland. hal-01946333

HAL Id: hal-01946333

<https://hal.science/hal-01946333>

Submitted on 5 Dec 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Interactive Virtual Math: A tool to support self-construction graphs by dynamic events

Sonia Palha¹ and Stephan Koopman¹

¹University of Applied Sciences of Amsterdam, The Netherlands; s.abrantes.garcez.palha@hva.nl; s.a.h.koopman@hva.nl

An essential condition to use mathematics to solve problems is the ability to recognize, imagine and represent relations between quantities. In particular, covariational reasoning has been shown to be very challenging for students at all levels. The aim of the project Interactive Virtual Math (IVM) is to develop a visualization tool that supports students' learning of covariation graphs. In this paper we present the initial development of the tool and we discuss its main features based on the results of one preliminary study and one exploratory study. The results suggest that the tool has potential to help students to engage in covariational reasoning by affording construction and explanation of different representations and comparison, relation and generalization of these ones. The results also point to the importance of developing tools that elicit and build upon students' self-productions.

Keywords: Visualization, virtual reality, interactive tool, mathematical modeling, reasoning.

Introduction

Students' difficulties with constructing graphs that model dynamic events are well documented in literature (e.g. Thompson, 2011; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Carlson, Oehrtman, & Engelke, 2010). When modeling a dynamic situation into a graph (e.g. the speed varying with time or the height of water in a bottle varying with volume), it has to be conceptualized as a covariation relation, that is a relationship between two variables that vary simultaneously (Thompson, 2011; Carlson et al., 2002). However, students have a tendency to view functions in terms of symbolic manipulations and procedures rather than as relationships of dependency between two variables. These students might encounter difficulties imagining how the output values of a function are changing while imagining changes in function input values. And therefore they might fail in successfully construct a graph of a function modeling a dynamic situation.

Research has revealed that traditional approaches have not been successful in overcoming the above described difficulties. Technological tools can however afford alternative approaches to the subject. Also, most of the research that provides insight in students difficulties with understanding graphical situations is done in clinical environments. We need to develop a better understanding of students learning in classroom settings.

In our research we developed a tool that intends to provide an alternative way to approach the learning of graphs by dynamic events and an opportunity for examining its learning in the classroom. The tool Interactive Virtual Math (IVM), which can be found at <https://virtualmath.hva.nl>, is designed to support 14-17 years old students at secondary school to understand the graphical representation of relations between variables in dynamic situations. IVM supports this process by addressing the visualization of these relationships. The aims of this paper are to introduce a prototype of the tool, its

main features and design and, to discuss its added value for students' learning based on the results of one preliminary study and one exploratory study.

Theoretical framework

Covariational reasoning

An example of a mathematical task that requires understanding of covariational reasoning is Task A from Figure 1. The task is about a dynamic situation involving the height of water in a bowl and the volume and, it was taken from Carlson et al. (2010), who used it to diagnose students' understanding of graphs of this type of events.

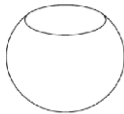
Figure 1: tasks used in preliminary study

Task A

Imagine this bowl is steadily being filled with water.

Sketch a graph of the water height in the bowl as a function of the amount of water in the bowl.

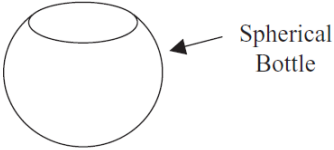
Explain the thinking you used to construct your graph.

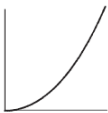



Task B


Assume that water is poured into a spherical bowl at a constant rate.


- Which of the following graphs best represents the height of water in the bowl as a function of the amount of water in the bowl?
- Explain the thinking you used to make your choice.




a) 

b) 

c) 

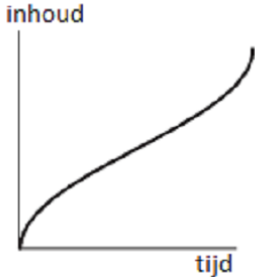
d) 

e) 

Task C

Assume that water is poured into a bowl at a constant rate. The graph in the figure represents the height of water in the bowl as a function of the amount of water in the bowl. Describe the filling in of the bowl in words,

- Explain the thinking you used to make the description.
- Draw a possible bowl



To solve task A, students will need to consider how the dependent variable (height) changes while imagining changes in the independent variable (volume). The coordination of such changes requires the ability to represent and interpret relevant features in the shape of the graph (Carlson et al., 2010).

Carlson et al., (2002, 2010), developed a framework that allows to investigate students' covariational reasoning abilities when responding to dynamic function tasks. The framework describes covariational reasoning as entailing five mental actions, which are successively more complex: (M1) coordinating the value of one quantity with changes in the other; (M2) coordinating the direction of the change; (M3) coordinating the amount of change of one quantity while imagining successive changes in the other quantity; (M4) coordinating the average rate of change of the function with uniform increments of change in the input variable; (M5) coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function. We used this framework to evaluate the quality of students' graphs and explanations in our study.

Guiding principles and main features of the tool

There are many technological tools available for learning graphs from dynamic events, but very few request students' own productions. They are often simulation-tools, which involve whole figures or part of figures that have to be moved, changed or dragged. When students are asked to construct a graph with these kind of tools, construction actually means using representations that are already given or can be synthesized by putting parts together. In this case there is not a true visualization of students' concept image (Vinner, 1983), since part of the representation is already given. A distinguishing feature of the IVM is that it builds solely on students' graphical productions.

The tool Interactive Virtual Math allows students to draw, analyze and compare graphs for themselves and improve the graphs if they conclude this improvement is needed. At CERME 10 we presented a second prototype version of the tool in which the students work on an assignment involving a single graphic situation: the dynamic event described in task A (Figure 1). In later versions we expect it to be possible to use more contexts and varied assignments so that all students can practice at their own level. In Table 1 we present a short description of the main features of the tool: *Self-construction*, *Contrast*, *Help 1 and Help 2*, *Reward* and *flow*. These features are based on general learning principles that include building on students' previous knowledge, interaction and feedback. We expect that the use of the tool will challenge students to create their own graphs and explanations, to make assumptions, conjectures and to reflect upon these (feature *Flow*).

The tool was also built according to topic specific learning principles. Thompson (2011) states that it is critical for students to first engage in mental activity to visualize a situation and construct relevant quantitative relationships prior to determining formulas or graphs. Therefore, the graphs in the tool must be drawn by the student themselves and the tool elicit students to imagine relationships from scratch, without presenting any (partial) graphical representation that has not been drawn by the student themselves (feature *Self-construction*).

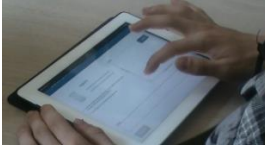


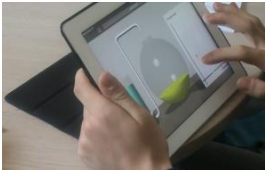
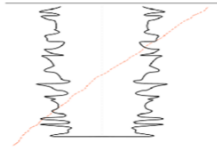
A second guiding idea behind the tool-design is the focus on visualizing quantities. Results from Ellis (2007) indicate that instruction encouraging a focus on quantities can support generalizations about relationships, connections between situations, and dynamic phenomena. To help students to focus on the relation between the height of the water and the volume we provide two kinds of help with the tool: the features *Help* and *Help 2*. In *Help 1* the student visualizes the increasing height of the water in the bowl and he can start and stop the water falling in bowl. In *Help 2* students must assume the height of the water in the bowl and represent it in the graph with dots. We expect that the students,

while guessing where to put the dot for the height, will notice that the difference in height between consecutive dots (values of the height) decreases in certain situations and increases in others.

Another guiding principle was to provide constructive feedback to the students' final graph and to give them a way to evaluate their production. The students get to see, after submitting their graph, the corresponding bowl-figure to the graph they draw (feature *Reward*).

Finally, the tool also includes the use of Virtual Reality (VR), which is still limited to Help 1. Here the use of VR (sound, movement, interaction) is expected to improve the experience of the graphic situation.

Table 1: main features of Interactive Virtual Math

Feature	Description
	<p><i>Self-construction</i></p> <p>The student is given two assignments. The first assignment is task A from Fig.1 and the second assignment is a variation of the same task with a cylinder instead of a bowl. In both assignments they are requested to draw a graph that describes the relationship between two variables in the corresponding dynamic situation. The student constructs the graph with a finger, a digital pen or a mouse.</p>
	<p><i>Contrast</i></p> <p>The student compares her/his own graph and explanation of the two situations, referred to as <i>a</i> and <i>b</i>. The student can then submit the graphs or improve them.</p>
	<p><i>Help 1</i></p> <p>The student visualizes the increasing height of the water in the bowl. He listens to the water he moves the platform with the ball and he can start and stop the water falling. Using a mobile device and a cardboard, Help 1 can be experienced as Virtual Reality</p>
	<p><i>Help 2</i></p> <p>The student connects the graphical representation to the context representation. A Cartesian coordinate system in the plane and the bowl appear next to each other. The student must construct a dot graph that represents the height of the water in the Cartesian graph. He does this by dragging and dropping dots into the graph.</p>
	<p><i>Reward</i></p> <p>The student gets the corresponding form of the bowl.</p>

Methodology

Preliminary study

Previous to the development of the first version of the IVM tool, we conducted a preliminary study to explore students' knowledge, skills and difficulties with constructing covariation graphs. The study (February-March 2016) involved $N=98$ students from 4 classes age 15-17 years old and we used three versions of the same task with different questioning (Figure 1). The students in each of the four classes were divided into three groups and each group was presented with one of the three versions.

Analyses of students' written answers showed that the majority of the students (64%) failed to successfully solve task A (see also Table 2). Nineteen of them presented an increasing but incorrect graph, suggesting that they understand that the water increases or that the height increases with the amount of water but they don't have a consistent concept image of this process. Most of these students (13 out 19) produced one straight line (9 students) or a combination of two/three straight lines (4 students). These findings point that the majority of students that solved the self-construction tasks (tasks A and C) could not construct for themselves an acceptable representation. These results motivated the importance of engaging students in self-construction assignments and the development of the IVM-tool.

Table 2: results of preliminary study

	Task A (self-construction graph)	Task B (multiple choice)	Task C (self-construction bowl)
Acceptable	12 (36%)	25 (66%)	3 (11%)
Incorrect	19 (58%)	11 (29%)	22 (79%)
No answer	2 (6%)	2 (5%)	1 (4%)

Exploratory study about the first version of the tool

The first version of the tool was developed in February –April 2016 by a team composed by one researcher-math educator (first author), a high school teacher (second author) and ICT -designers. We decided to use task A (Fig.1) that we considered suitable to explore students' understanding of covariation and within a broad age group. To explore its learning potential and usability we investigated through a small qualitative study the learning of four students age 14-15 years old (two boys and two girls) with different school performance for mathematics. Kevin¹ has high grades for mathematics, Lisa and Anton have average grades and Wilma has low grades. We observed and interviewed the students while working with the tool. The aims of the exploratory study were: (i) to understand how the students construct a graphical representation with IVM; (ii) to identify features of the tool that support or constrain students' successful construction; (iii) to get a better understanding about how the guiding principles work and can be used to develop later versions of the tool. The collected data consisted of video records and students' written work and it was collected at two different moments in April 2016. In both situations the students were asked to go first through the whole application on their own. Lisa was the first student to be interviewed; she used the application on a computer. The other three students Kevin, Wilma and Anton were interviewed together at their

¹ The real names of the students were modified

school. Wilma and Anton use a tablet and Kevin a mobile device. The data was first organized chronologically with relation to each student's attempt to construct the graph and use of the tool. Secondly, a global description of how each student attempted to construct and transform the graph was made and how they used the main features of the tool. We used the covariational framework (Carlson et al., 2002) to get insight in students' covariational reasoning abilities. A summary of the results are presented in Table 3. These results and the data were shared and discussed with the ICT-team and used to evaluate the tool and to make decisions for the development of a next version.

Results and discussion

As we can see in Table 3, all four students improved their graphs on basis of the tool. Kevin produced in the first trial an incorrect graph with three straight lines and he improved it in second trial after comparing the form of the bowl he got in the *Reward* with the bowl in the bowl-assignment. Wilma produced in the bowl-assignment, in the first trial two incorrect graphs: a straight line and afterwards a raising curve. She 'improved' the graph after seeing the cylinder- assignment (*Contrast*). Through consulting *Help 1* and *Help 2* she constructed in a second trial a final acceptable graph. Anton produced in the bowl-assignment several incorrect graphs. His final graph in the first trial is a curve raising slowly. He consulted *Help 1* several times and, based on that, he produced a graph with three straight lines and adapted the length of the line segments. Anton's improvement did not lead to a final acceptable solution and the student remained in doubt whether the pieces of the graph should be curved or not.

Table 3: students' use of the features of the tool during the exploratory study

Features	Kevin	Wilma	Anton	Lisa
Construction (round bowl)	Acceptable final graph after two trials	Acceptable final graph after two trials	Incorrect final graph after two trials	Acceptable final graph after two trials
Construction (cylinder bowl)	All students have produced an acceptable graph at first trial (straight line)			
Contrast	First, all students draw a straight line at assignment one but improve their drawing after constructing the graph of assignment two.			
Help 1: Bowl is being filled up	Doesn't consult help 1 in first trial	changes a straight line into a rising curve	changes the middle line of the graph,	Consults but doesn't improve the graph
Help 2: relation figure - graph	Doesn't consult Help 2 in first trial	changes a rising curve in an acceptable curve	Consults Help 2	Does not understand how it works
Reward	Improves straight line to a curve.	Not observed	Not observed	Does not understand the reward
Flow	Constructs graphs without consulting Help 1 and 2.	Consults Help 1 and Help 2	Consults Help 1 and Help 2 several times	Consults Help 1 and Help 2
VR (Help 1 with cardboard)	Not used	Not used	Not used	rich experience

Based on the analyses of students reasoning while constructing and explaining their graphs, we identified a number of aspects through which students could be brought to a better understanding of

graphical situations, while working with the tool. One aspect is students engagement in covariational reasoning and their progression through the mental actions (Carlson et al., 2002). For instance, Wilma identifies and represents the two quantities changing together (M1). She draws initially a straight line which suggests that she attends only to the direction in which the height changed while imagining increases in the amount of water (M2). *After consulting help 1* she changes her straight line into a rising curve and then into a curve-down followed by a curve- up graph and she is able to explain how changes in the amount of water were related to changes in the height of the water at various locations in the bottle (M3).

Another aspect is students' involvement in actions that underpin mathematical reasoning such as the construction and explanation of different representations and, comparing, relating and generalizing these ones. Examples that we observed include students *comparing* their own graph and bowl filling up with water, which was the case of Wilma when she used Help 1 or Anton switching from Help 1 to his own graph several times; students *evaluating the relation* between the reward and initial graph. Visualizing the bowl of the reward made Kevin to think about the relation between the form of the bowl and the form of the graph. He used the reward to improve the smoothness of the graph curve; students *contrast* the relation between graphical situations of assignment one and two. For instance, Anton switches between one and two and adapt the graph one after seeing assignment two.

As Table 3 shows, different students used different features to improve their graph, which suggests that tool with possibilities to choose to view additional help or not and to be able to switch between the graphical situations, allows for diversity. Furthermore, all students had difficulty with constructing a graph, even with the tool support. This result suggests that self-construction tasks are needed to reveal these difficulties, which can remain unnoticed when using simulation-tools or tools in which the representations are already given.

A final aspect concerns the usability of the tool. Students valued the opportunity of choice and the interactivity of Help 2 (one can drag and decide where to put the point). And, one student (Lisa) who view Help 1 in VR with the cardboard valued this experience as a more enriching one.

There are also some critical issues with regard to the methodology of the study and the tool design. The small amount of students involved in the use of the IVM tool allowed for a fairly detailed study of their interaction with the tool. But, we should carefully interpret our findings since they regard only 4 students. We need to experiment more with the tool in classrooms, in combination with other tasks and forms of interaction and teacher support to better understand its potential and to what extend these findings can be generalized. With regard to the tool design, a number of aspects should be improved in follow up versions. One challenge concerns the self construction- and reward-features. It is left to the tool to decide what is an acceptable representation and how accurate it can be. We programmed the tool to accept any sketch of concave up followed by a concave down graphs starting at the origin. And, for the graph to be considered accurate, the line must be smoothly drawn. Sometimes the tool rejects answers that are accepted by the researchers and teachers. Another concern is the amount of variables involved in the assignments (height, accumulated volume, time, volume per unit of time, shape of the bottle). It is reasonable that the students should focus on one or two variables but not so many that are changing simultaneously. At the CERME conference we also received useful suggestions to improve the tool. For instance the time-counter in Help 1 can be replaced by a volume-counter and, students could fill the bowl by adding themselves cups of water.

This could help students to focus on the relation between height and volume rather than height and time. Another suggestion was allowing students to change the shape of the bottle as this might afford students' awareness of the phenomenon.

Concluding, this paper reports on the experiences of students learning graphical representations by dynamic events with the aid of a new learning technology (IVM); a topic which many students struggle to understand. We have learned that the prototype-tool has potential to engage students in covariational reasoning and we identified a number of aspects that could bring the students, while working with the tool, to a better understanding of graphical situations. Namely, the tool affords construction and explanation of different representations and, comparison, relation and generalization of these ones. The results also point to the importance of elicit and build upon students self-productions.

References

- Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study: A Framework and a Study. *Journal for Research in Mathematics Education*, 33(5), 352-378.
- Carlson, M., Oehrtman, M., & Engelke, N. (2010). The precalculus concept assessment: a tool for assessing students' reasoning abilities and understandings. *Cognition and Instruction*, 28(2), 113-145.
- Ellis (2007). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction*, 25 (4), 439–478
- Thompson, P. W. (2011). *Quantitative reasoning and mathematical modeling*. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education*. WISDOMe Monographs (Vol. 1, pp. 33-57). Laramie, WY: University of Wyoming"
- Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology*, 14(3), 293-305.