# Students' reasoning on linear transformations in a DGS: A semiotic perspective

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*The aim of this paper is to analyze students' reasoning on linear transformations while using a Dynamic Geometry System* (*DGS*) *from a semiotic mediation perspective. Considering design heuristics of Realistic Mathematics Education and the semiotic potential of certain tools and functions of DGS, I have developed a hypothetical learning trajectory and have designed a task for inventing fundamental properties of linear transformations. The task was field-tested in a case study with pair of undergraduate linear algebra students. An analysis of the task-based interviews, with a semiotic mediation lens, shows that the students managed to* (*re–*)*invent the fundamental properties of linear transformations.*

*Keywords: Semiotic mediation, linear algebra, DGS.*

## Introduction

One major issue in the teaching and learning of linear algebra is providing students with ready–made mathematics using different representations and different contexts (Dorier, 1998) without considering the students' intellectual needs for learning (Harel, 1998). An example might be to introduce the notion of linear transformations with two fundamental properties as in numerous textbooks, where such introduction to the topic could trigger epistemological issues for students' conceptualization of non–linear transformations (Dreyfus, Hillel, & Sierpinska, 1998). In this paper, I acknowledge a contrary introduction to the topic and consider a research question: Is it possible for students to (re–)invent fundamental properties of linear transformations? To answer this question, I consider a dynamic geometry system (DGS), which invite students into a progressive process of epistemic exploring, conjecturing and generalizing (Leung, Baccaglini-Frank, & Mariotti, 2013). Consequently, I focus on specific tools and functions of GeoGebra, such as dragging and grid functions, ApplyMatrix command and slider tool of the DGS as a tool of semiotic mediation for students' reinvention of proposed mathematics.

## Theoretical Perspectives

In this work, I consider two theoretical insights: (i) Realistic Mathematics Education and (ii) Theory of Semiotic Mediation, for designing instructional activity and analyzing the teaching–learning process.

### Realistic Mathematics Education (RME)

RME is a domain–specific instructional theory developed by Dutch researchers (Van den Heuvel-Panhuizen & Drijvers, 2014). The word *realistic*, here, does not directly refer to real–world task situations, but to paradigmatic situations that invite the development of meaningful mathematics. The problem situations do not necessarily come from real life directly, they can be related to an imaginary world or to real mathematics that students experience as meaningful: task situations have to be *experientially real* (Gravemeijer, 1999) to students. In parallel to such views, RME offers three interacting design heuristics for curriculum developers and educational designers

(Freudenthal, 1983; Gravemeijer, 1999; Van den Heuvel-Panhuizen & Drijvers, 2014): guided reinvention, didactic phenomenology, and emergent modelling. Guided reinvention means providing students with an environment for their exploration, elaboration and inventing of mathematics. Didactic phenomenology refers to finding certain experientially real phenomena, which might form an environment where students create mathematics. The objective of emergent modelling is to enable students to shift from informal task situations to formal mathematics through support, enabling them to create their own informal mathematics.

### Theory of Semiotic Mediation (TSM)

TSM was presented by Bartolini Bussi and Mariotti (2008) with the following main idea: to construct mathematical meanings, the teacher intentionally uses *artefacts* as a tool of semiotic mediation, which are used in carefully–designed tasks. The aim of the TSM is to transform students' personal meanings to mathematical meanings. The teacher exploits the *semiotic potential* of the artefact, in which he or she uses an epistemological and didactical analysis to picture out possible learning steps from personal meanings to shared conventional mathematical meanings. Here, taking into account the didactic goals, the teacher considers what students know, what their experience with the artefact is and how they will accomplish the task by using the artefact. As a next step, the teacher designs a *didactic cycle* for classroom interventions.

Students' interaction with the artefact produces a complex semiotic process. Artefact signs (aS) appear when students who use the artefact relate in some way to the activity; specifically, to the use of artefact. Mathematical signs (mS) appear, when the students make a definition, conjecture, generalization or proof corresponding to didactic goals. Pivot signs (pS) have an interpretative link between personal meanings and mathematical signs and can appear in the accomplishment of the task. In the application of the didactic cycles, the teacher's role is orchestrating students' learning.

## Methodology

This paper, in which I focus and present the results of a single task, is part of an extensive Design–Based Research (project) (Bakker & van Eerde, 2015). Due to page limitation, I will present a case limited to a pair of students (A male, B female), who were sophomore level undergraduate linear algebra students, aged twenty. The students had experience solving linear equations, matrix algebra, (geometric) vector spaces and subspaces, and they had learned that every linear transformation can be represented through matrices. They also had experience in the use of GeoGebra's main functions, specifically forming a slider and a matrix, and applying the ApplyMatrix construction tool from previous task sequences, where they constructed meaning of a transformation and linear transformation. However, the students did not know the fundamental properties of linear transformations. Task–based interviews were video–recorded and lasted around half an hour. The data was analyzed through a semiotic lens using categories of signs (Bartolini Bussi & Mariotti, 2008): aS, mS and pS.

### Mathematical context, semiotic potential of DGS and task design

A linear transformation is a specific transformation between $V$ and $W$ can be represented as $T:V\to W$ for vector spaces $V$ and $W$, where $T$ satisfies: (i) $T(\mathbf{u}+\mathbf{v})=T(\mathbf{u})+T(\mathbf{v})$ for all vectors $\mathbf{u},\mathbf{v}\in V$, and (ii) $T(k\mathbf{u})=kT(\mathbf{u})$ for all $\mathbf{u}\in V$ and all scalars $k\in\mathbf{R}$ (Lay, 2006). Here, I

took $V = W = \mathbf{R}^2$ because of DGS availability (for example as in GeoGebra) and considered the semiotic potential of the following tools and functions of DGS for students' (re–)invention of the fundamental properties above: (i) the *dragging* function allows the user to manipulate figures and explore independency–dependency of drawings and constructed objects, (ii) the *grid* function activates specific lines for integer values on the *x* and *y* axes and this function enables the user to observe variations of the coordinates of the objects in different windows, (iii) the *slider* tool offers a means to define a parameter and this may evoke meaning for dynamic (co)variation (Turgut & Drijvers, 2016), (iv) the *ApplyMatrix* tool works through an input line that enables the user to apply certain matrix transformations to geometric figures. I postulate that students' dragging sliders connected to a matrix and applying matrix transformations to arbitrary vectors could provide an understanding for a meaning: matrix (and therefore linear) transformations preserve vector addition and scalar multiplication.

The synergy between the definitions of guided reinvention and didactic phenomenology heuristics and the notion of semiotic potential in TSM implies the construction of a possible learning route, in other words, a Hypothetical Learning Trajectory (HLT) (Simon, 1995) which has to be elaborated on by the designer before the experiment by following four points (Bakker & van Eerde, 2015): (i) learning goals, (ii) students' pre-knowledge, (iii) assumptions for students' learning, and (iv) the teacher's role (also in our case, the role of artefacts). Therefore, in Table 1, I express (i), (ii) and (iii) points of a HLT for invention of fundamental properties of linear transformations in a DGS.

| Associated Concepts | Expected Steps in the DGS | Exemplary Task | Epistemic Artefacts in DGS | Expected Mathematical Meanings |
|---|---|---|---|---|
| –Geometric vectors<br>–Addition of vectors<br>–Multiplication with scalars<br>–Matrix transformations<br>–Fundamental properties of linear transformations | –Exploring the effects of sliders on (arbitrary) linear transformations of arbitrary vectors<br>–Comparing the initial and final versions of vectors while moving sliders or dragging the objects | –Form sliders<br>–Construct $2 \times 2$ matrix<br>–Form arbitrary vectors<br>–Use Apply Matrix command<br>–Move the sliders and drag the objects | –Dragging<br>–Grid function<br>–Apply Matrix construction tool<br>–Slider tool | –Comprehending that the situation is independent from matrix entries or vectors<br>–Formulating the first rule situation, $T(\mathbf{u}+\mathbf{v})=T(\mathbf{u})+T(\mathbf{v})$<br>–Formulating the second $T(k\mathbf{u})=kT(\mathbf{u})$<br>–Proving such results in terms of matrix representations |

**Table 1: HLT for the inventing of fundamental rules for linear transformations**

As aforementioned before, students worked on GeoGebra interface in the previous didactic cycles, which were about transformation of geometric vectors, figures, and constructing meaning for linear transformation. Consequently, the tools and functions of GeoGebra and proposed concepts were experientially real for them. Following Table 1 and considering guided reinvention heuristic, the task was formulated as follows (a possible interface for the task steps is presented in Figure 1), and also for students' making their own models (cf. emergent modeling).

*Step 1*: Open GeoGebra and activate grid function. Next, form two sliders *a* and *b* and, using *a* and *b*, form an arbitrary $2 \times 2$ matrix. *Step 2*: Form two arbitrary vectors $\mathbf{u}, \mathbf{v}$ and construct $\mathbf{u} + \mathbf{v}$

through an Input line. *Step 3*: Apply matrix transformation to $\mathbf{u}, \mathbf{v}$ and $\mathbf{u}+\mathbf{v}$. Name these vectors, respectively: $\mathbf{u}'$, $\mathbf{v}'$ and $\mathbf{w}$ respectively, and then calculate $\mathbf{u}'+\mathbf{v}'$. Move the sliders and drag $\mathbf{u}$ and $\mathbf{v}$ in itself. Discuss with your pair and explain your observations. *Step 4*: Form a new slider $k$. Now, obtain matrix transformation of $k\mathbf{u}$ and also compute $k\mathbf{u}'$. Drag the vector $\mathbf{u}$ and explain your observations, and make conjectures. What happens when you move the sliders?

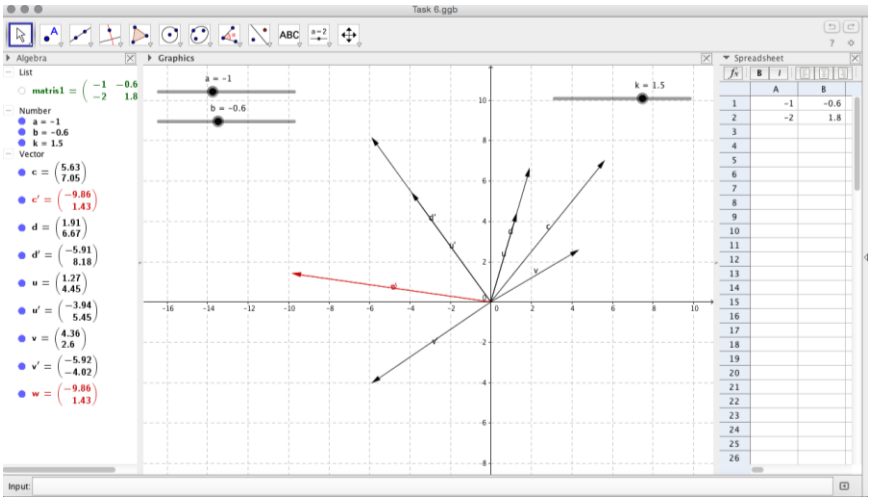



**Figure 1: An expected DGS interface for the task**

Teacher's (possible) underpinning questions in the interview are: *What is the role of sliders here*? *What is the role of the matrix*? *What are the relationships between initial vectors and transformations*? *How do you prove this?* [In case they make a generalization with matrix notation]. Within this task, I hypothesized that students would observe that the transformation of $\mathbf{u}+\mathbf{v}$, denoted by $T(\mathbf{u}+\mathbf{v})$, always overlaps on the $T(\mathbf{u})+T(\mathbf{v})$ vector and similarly, $T(k\mathbf{u})$ also always overlaps on $kT(\mathbf{u})$ vector, where situations were independent from the choice of matrix and/or choice of vector. This could be made possible through the semiotic potential of the aforementioned functions and tools of DGS and teacher's (T) guidance role for reinvention of the mathematics.

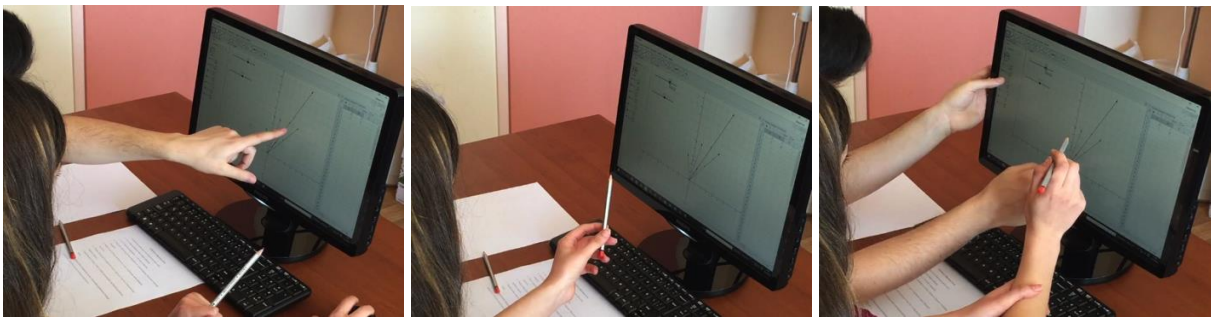
## Analysis: Emergence of signs

Students followed the line of the task. First they constructed two sliders, *a* and *b*. Next, using such values in the spreadsheet window of Figure 1, they defined a $2\times 2$ matrix as $\mathbf{A}=\begin{pmatrix} a & b \\ b & a \end{pmatrix}$. Through the Input line, they formed two vectors $\mathbf{u}=(1,2)$ and $\mathbf{v}=(-1,3)$. They first obtained the sum of the vectors and thereafter applied matrix transformation by the ApplyMatrix command. The software assigned $\mathbf{u}'$ for $T(\mathbf{u})$, similarly, $\mathbf{v}'$ for $T(\mathbf{v})$, and $\mathbf{u}+\mathbf{v}=\mathbf{w}$ and $\mathbf{u}'+\mathbf{v}'=\mathbf{d}$. For a while, the students discussed the steps of the task to determine which matrix application is the first, the second or the third, which seemed rather confusing for them. After they had completed the three steps, while dragging the sliders, they were surprised because a number of vectors and some transformation vectors overlapped. At this moment a few aS appeared (see 18–20):

18 A: … [*pointing on the grid* (see Figure 2a)] look, how this happened, these are overlapping…

19 B: No. I think, it is because of matrix, look, [*dragging sliders and pointing matrix entries with pencil* (see Figure 2b)] it is changing.

20 A: Let's analyze them, which is which and why overlaps… [*They are trying to separate the vectors* (see Figure 2c) *and taking notes*]



(a) (b) (c)

**Figure 2: a, b, c Emergence of aS during analyzing the overlapping situation**

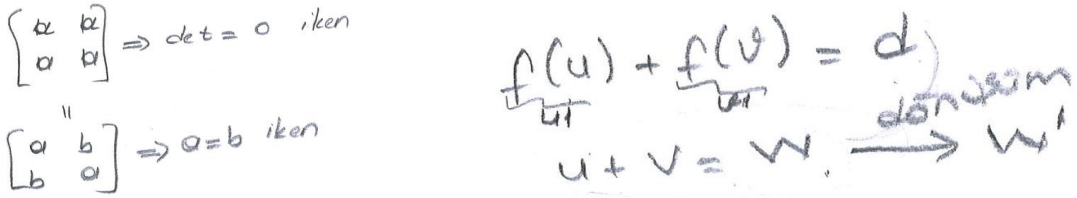
Next, the teacher intervened to make the students focus on the transformations of the vectors, because they had spent a lot of time dragging sliders, changing matrix entries (i.e., trying a unit or zero matrix and so on) to figure out why some overlapped (see Figure 3a). Then students re–checked the steps and wrote up the findings in their own way. Some pS appeared here, reflecting the students' new meanings through the semiotic potential of the artefact (37–38), and also appeared on the students' productions (Figure 3b).

26 T: … what about the transformations of vectors? What did you observe?

…

37 B: … I think we will find a relationship between these [*pointing on the notes* (see Figure 3b)]. Here, we have the sum vector's transformation and sum of each vector's transformation.

38 A: However, this could be dependent on the choice of matrix? What will happen for the matrices where their determinants are zero? …




(a) (b)

**Figure 3: a, b Students' productions as pS**

Interestingly, once more, they focused on the entries of the matrix, because in the previous step they had employed a unit or a zero matrix, and they began to check other possibilities for the cause of the overlapping situation. Consequently B figured 'they always overlap'. Here, aS 'overlaps' in the previous analyses, and can now be considered a pS (see 63, 86), because it is mediating the transformation of personal meanings to mathematical meanings.

63 B: It is clear that they always overlap … Why is this happening?

64 A: Exactly… but why?

…

86 B: $\mathbf{d}$ and $\mathbf{w}'$ always overlap and they are the same. I could not analyze the others.

87 A: … because of matrix transformation, I think.

As a next step, aS and pS interlaced with the students' personal meanings associated with matrix transformations. They re–analyzed their findings, and finally, mS corresponding to students' reinvention of the fundamental properties appeared (93-97).

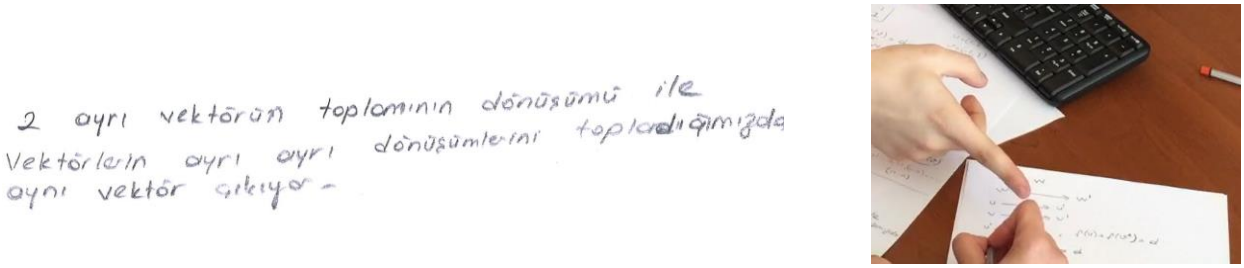
93 B: Just a second. What was the meaning of $\mathbf{w}'$? It was a transformation of $\mathbf{u}+\mathbf{v}$? … [*moving sliders and thinking*]…

94 A: We also applied matrix transformation to $\mathbf{u}+\mathbf{v}$?

95 B: Because, they are overlapping, this means, we have obtained the same vector. … Does transformation of the sum vector [*meaning* $\mathbf{u}+\mathbf{v}$] equals the sum of the separate transformations?

96 A: Absolutely, right. …

97 B: … [*First, she is writing her conclusion, but not mathematically* (see Figure 4a), *then she is trying to write it mathematically with her partner's help* (see Figure 4b)] …




(a) (b)

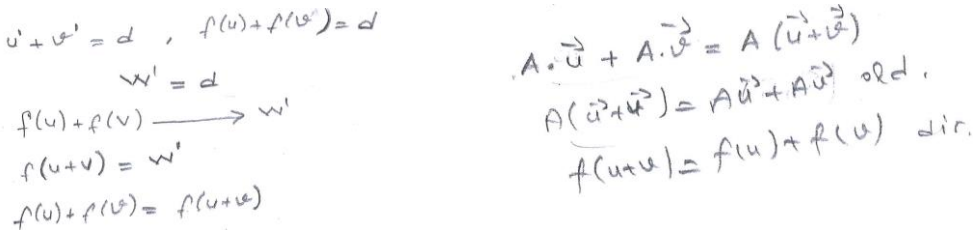
**Figure 4: a, b Students' conclusions for the first fundamental property as mS**

Next, while trying to express the situation mathematically, which was under the teacher's orchestration, the students reinvented the first fundamental property (see Figure 5a). However, the teacher was orienting students to prove their result considering their pre–knowledge on representing linear transformations with matrices. Student B immediately related the situation with her pre-knowledge and proved her conclusion (119 and Figure 5b).

114 T: Ok right. Please remember the matrix representations of linear transformations. Considering this, how do you prove your result?

…

119 B: … [*She is writing matrix representations* (see Figure 5b), *then explains*], yes … I now realize why this is happening. We can show every linear transformation with a matrix and matrix algebra has distribution property. Then I can do like this [*writing expressions* in Figure 5b]…



(a) (b)

**Figure 5: a, b Emergence of mS in relation to the task's goal**

As a final step, the teacher asked the students to consider Step 4. As expected they placed a slider for *k*, and applied a matrix transformation to $k\mathbf{u}$ and also computed vector $k\mathbf{u}'$. As soon as one student saw that the transformation of $k\mathbf{u}$ and $k\mathbf{u}'$ overlapped and were exactly the same, by the help of the first property she invented, B expressed her views. The final mS emerged in the discourse (129).

129 B: … Oh yes, I think this is obvious; this is also a result of property of matrix algebra. For a matrix, *k* can be multiplied with each entry of a matrix or it can be expressed a factor [writing $k\mathbf{u}' = f(k\mathbf{u})$]. Therefore, their [*meaning* $k\mathbf{u}$ and $k\mathbf{u}'$] transformations are the same.

## Conclusions

In this paper, I consider the research question, 'Is it possible for students to (re–)invent fundamental properties of linear transformations?' Students work on the task formed through the design heuristics of RME and the semiotic potential of some tools and functions of GeoGebra provide an affirmative answer, but with some doubts and limitations. For instance, the students spent much time determining vectors when they overlapped. This issue to be considered is the students' frequent analysis of matrix entries, where they think that an overlapping situation depended on this. I think that such frequent analysis of matrix entries stems from previous experience, where the students were continually trying to find matrices of linear transformations. Interestingly, in the students' analyses for characterizing the matrix, different semiotic resources beyond aS, pS and mS appeared; for instance, gestures and mimics attached to students' analyses process. A multimodal perspective (Arzarello, 2006) could provide a detailed view for our case. However, in the present case within a TSM perspective, I observe a semiotic chain (Bartolini Bussi & Mariotti, 2008), which shows the connection between semiotic resources of students' learning, for inventing fundamental properties of linear transformations as follows (Figure 6).

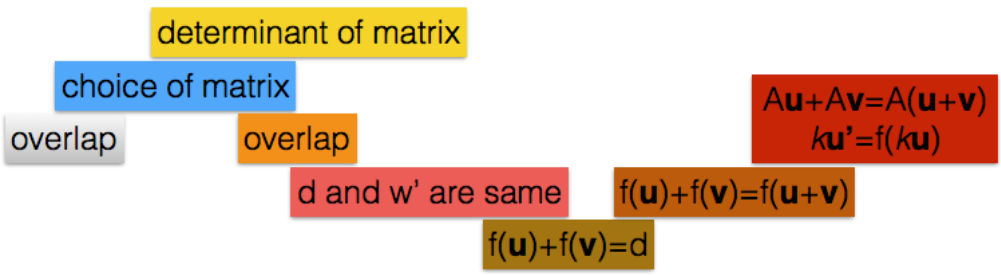



**Figure 6: A semiotic chain for inventing fundamental properties**

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