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# Some quantitative homogenization results in a simple case of interface

Marc Josien\*

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## Abstract

Following a framework initiated by Blanc, Le Bris and Lions, this article aims at obtaining quantitative homogenization results in a simple case of interface between two periodic media. By using Avellaneda and Lin's techniques, we provide pointwise estimates for the gradient of the solution to the multiscale problem and for the associated Green function. Also we generalize the classical two-scale expansion in order to build a pointwise approximation of the gradient of the solution to the multiscale problem (up to the interface), and, adapting Kenig, Lin and Shen's approach, we obtain convergence rates.

**Keywords:** homogenization, interface, two-scale expansion

## 1 Introduction

In this article, we are concerned with the quantitative homogenization of the following elliptic equations in divergence form:

$$-\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\cdot\nabla u^\varepsilon(x)\right)=f(x), \quad (1)$$

in a simple case of interface between two periodic media.

Equation (1) is a prototypical equation for various physical phenomena (like electrostatics or when generalized to systems, elastostatics) set on a material with a microstructure of characteristic scale  $\varepsilon \ll 1$ . Homogenization of (1), which aims at studying the behavior of the solution  $u^\varepsilon$  when  $\varepsilon \rightarrow 0$ , has attracted much attention for half a century. Two particular structures are especially studied: the periodic structure and the stationary ergodic structure (see, *e.g.*, the reference books [1, Chap. 1] for the periodic case, and [13, Chap. 7] for a the stationary stochastic case). Both of these frameworks can be used for actual numerical computations: the homogenization theory is an efficient tool for approximating numerically the solution  $u^\varepsilon$  of (1) and its gradient, for a fixed  $\varepsilon > 0$ .

Recently, Blanc, Le Bris and Lions proposed in [7] two other cases that can be amenable to numerical computations (see [6]). In the first case, the matrix  $A$

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is periodic but perturbed by a defect at the microscopic scale (see also [8] for an extension to the advection-diffusion case, and [5] for quantitative homogenization results). In the second case, which might be a fair model for bicrystals, the matrix  $A$  is obtained by gluing two periodic structures with Hölder continuous coefficients along a planar interface. This particular framework has the specificity that the associated homogenized equation involves a matrix  $A^*$  that is piecewise constant with a discontinuity across the interface (in the generic case). From this perspective, this second case is very different from the aforementioned settings, where the homogenized matrix is constant. The authors of [7] proposed a definition of the correctors and showed that they exist and enjoy some desirable properties of regularity and boundedness. This is a first step in order to obtain quantitative homogenization results. The present article is an attempt to go further, by taking advantage of the literature in periodic homogenization (in particular, the celebrated work of Avellaneda and Lin [2] and the recent article of Kenig, Lin and Shen [15]).

The type of results we show here are familiar to the experts of periodic or stochastic homogenization. But the main idea of this article is the following: in a simple case of bicrystals, the generalized two-scale expansion yields an approximation that possesses the same qualitative and quantitative properties as the two-scale expansion in the periodic setting when considering the gradient of the multiscale solution. From a theoretical point of view, this might be useful for understanding the homogenization of elliptic equations in the case where the homogenized matrix is discontinuous. We also hope this may be of interest for the numerical practitioner.

Our aim is twofold: estimate and approximate the gradient  $\nabla u^\varepsilon$  in  $L^\infty$  norm up to the interface. Obviously, far from the interface, the classical theory of periodic homogenization provides a way to fulfill these goals, first by Avellaneda and Lin's results [2], and then by using the two-scale expansion. Hence, the very difficulty of our study is located close to the interface. This is the reason why we strive for *pointwise* estimates and approximations (for  $u^\varepsilon$  but also on the level of the multiscale Green function).

Our first purpose is to obtain pointwise estimates on the gradient  $\nabla u^\varepsilon$  of the multiscale problem (1). In the periodic setting, such results are provided by Avellaneda and Lin's theory [2]. But, as shown in [11] (see also [4, 5]), the periodicity assumption is not necessary to these local estimates: they can be obtained in various frameworks, as long as the correctors and the potential (defined below by (14) and (22)) associated with the matrix  $A$  are strictly sublinear and as long as the homogenized matrix is constant.

The fact that the homogenized matrix is constant is a useful but mere contingent assumption due to the framework used by the authors (the matrix  $A$  is supposed to be periodic, possibly perturbed by a defect, or stationary ergodic). Actually, the crucial ingredient is that the multiscale problem inherits regularity properties from the homogenized problem, which are very favorable when the homogenized matrix is constant. But the solution of an elliptic equation the coefficient of which is piecewise Hölder continuous with discontinuities only on smooth interfaces also enjoys some regularity properties (see, *e.g.*, [17]), which are sufficient for Avellaneda and Lin's approach. Yet, there is another impediment: in the case of a discontinuous homogenized coefficient  $A^*$ , the  $A^*$ -harmonic functions (*i.e.* satisfying  $-\operatorname{div}(A^* \cdot \nabla u^*) = 0$ ) might have a discon-

tinuous gradient (as a consequence, its second gradient may involve a singular measure supported on the interface). As discussed below, this fact prevents the classical two-scale expansion to work properly. This motivates us to introduce a generalized two-scale expansion. Equipped with this expansion and with the regularity result of [17], we can proceed with Avellaneda and Lin's proof.

Our second purpose is to show to what extent the generalized two-scale expansion yields an accurate pointwise approximation of the gradient  $\nabla u^\varepsilon$ , as does the classical two-scale expansion in the periodic setting, where the convergence rate can be quantified in  $\varepsilon$  (see *e.g.* [15, Lem. 3.5]). We aim at deriving the same type of convergence rate in the case of bicrystals, up to the interface.

Our article is articulated as follows. In Section 2, we describe precisely our mathematical setting. Then, in Section 3, we introduce and motivate the generalized two-scale expansion. It is formulated by appealing to the  $A$ -harmonic functions (which involve the so-called correctors) and to the  $A^*$ -harmonic functions (which are piecewise linear). This expansion is meant to approximate the solution  $u^\varepsilon$  of (2) by means of the solution  $u^*$  of the homogenized problem. As in the classical cases, the residuum solves an elliptic equation with a right-hand side in divergence form. We state our main results in Section 4. They concern first pointwise estimates on  $u^\varepsilon$  and on  $\nabla u^\varepsilon$  and then pointwise approximations of these quantities by the generalized two-scale expansion. These results are also interpreted on the level of the Green functions. We conclude this section by discussing some aspects, limitations and possible extensions of those results. The following sections are devoted to the proofs. More precisely, we collect some elementary results in Section 5 concerning the correctors and the H-convergence of the matrix  $A(\cdot/\varepsilon)$ , and concerning the regularity properties of the solutions of elliptic equations involving discontinuous coefficients. Then, in Section 6, we use Avellaneda and Lin's techniques to prove pointwise estimates on  $u^\varepsilon$  and  $\nabla u^\varepsilon$ . Finally, in Section 7, we follow Kenig, Lin and Shen's approach [15] to estimate the residuum between  $u^\varepsilon$  and the generalized two-scale expansion. There, the Green function plays a central role.

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The main ideas of Sections 6 and 7.1 were found when the author was a PhD student at the École Nationale des Ponts ParisTech.

## 2 Mathematical setting

From now on,  $\mathbb{R}^d$  is endowed with a canonical base  $(e_1, \dots, e_d)$ . Since we want to focus on the interface and avoid the problem of boundaries, we set following the equation on the whole ambient space  $\mathbb{R}^d$ , with  $d \geq 3$ :

$$\begin{cases} -\operatorname{div}(A(x/\varepsilon) \cdot \nabla u^\varepsilon(x)) = f(x) & \text{in } \mathbb{R}^d, \\ \nabla u^\varepsilon \in L^2(\mathbb{R}^d, \mathbb{R}^d), \end{cases} \quad (2)$$

(the more difficult case  $d = 2$  will be mentioned in some results). In the above expression,  $f \in C_c^\infty(\mathbb{R}^d)$  is a smooth function with compact support,  $0 < \varepsilon < 1$ ,

and  $A$  is an elliptic and bounded matrix modeling an interface between two infinite crystals that share a common periodic cell on the interface  $\mathcal{I} := \{0\} \times \mathbb{R}^{d-1}$ . As is also classical in Avellaneda and Lin's theory, we assume that the matrix  $A$  is Hölder continuous on the left and on the right of the interface. These assumptions, formalized below, correspond to the simplest case of interface in [7, Sec. 5]:

*Assumption 1* (Ellipticity and boundedness). There exists a constant  $\mu > 0$  such that, for all  $x, \xi \in \mathbb{R}^d$ , the matrix  $A(x)$  is invertible and

$$\xi \cdot A(x) \cdot \xi \geq \mu |\xi|^2 \quad \text{and} \quad \xi \cdot A^{-1}(x) \cdot \xi \geq \mu |\xi|^2.$$

*Assumption 2* (Periodicity with commensurable periods). The matrix  $A(x)$  satisfies

$$A(x) = \begin{cases} A_+(x) & \text{if } x \cdot e_1 > 0, \\ A_-(x) & \text{if } x \cdot e_1 < 0, \end{cases} \quad (3)$$

where  $A_\pm$  is  $[0, T_1^\pm] \times \cdots \times [0, T_d^\pm]$ -periodic with  $T_i^+/T_i^- \in \mathbb{Q}, \forall i \in \llbracket 2, d \rrbracket$ .

*Assumption 3* (Regularity). For a fixed  $\alpha > 0$ , there holds

$$A_- \in C^{0,\alpha}(\mathbb{R}^d, \mathbb{R}^{d \times d}) \quad \text{and} \quad A_+ \in C^{0,\alpha}(\mathbb{R}^d, \mathbb{R}^{d \times d}).$$

*Remark 1.* The above regularity assumption can be weakened as in [17, Th. 1.9]:  $A_-$  and  $A_+$  can be assumed to be uniformly  $\alpha$ -Hölder continuous everywhere but on the (regular) boundaries of disjoint inclusions.

By using the Lax-Milgram theorem, it can be shown that there exists a solution  $u^\varepsilon \in H_{\text{loc}}^1(\mathbb{R}^d)$  to (3) such that  $\nabla u^\varepsilon \in L^2(\mathbb{R}^d, \mathbb{R}^d)$ . This solution is unique up to the addition of a constant that we set by imposing that the mean of  $u^\varepsilon$  on  $\mathbb{R}^d$  vanishes.

Under Assumptions 1 and 2, the homogenized problem associated with (2) when  $\varepsilon \rightarrow 0$  is the following:

$$\begin{cases} -\operatorname{div}(A^*(x) \cdot \nabla u^*(x)) = f(x) & \text{in } \mathbb{R}^d, \\ \nabla u^* \in L^2(\mathbb{R}^d, \mathbb{R}^d), \end{cases} \quad (4)$$

where the homogenized matrix  $A^*$  is defined by

$$A^*(x) = \begin{cases} A_+^* & \text{if } x \cdot e_1 > 0, \\ A_-^* & \text{if } x \cdot e_1 < 0, \end{cases} \quad (5)$$

and  $A_\pm^*$  are the homogenized matrices associated with the periodic matrices  $A_\pm$ . In general, the matrix  $A^*$  is discontinuous across the interface.

By standard arguments, it can be shown (see Lemma 5.6 below) that the gradient  $\nabla u^\varepsilon$  weakly converges to  $\nabla u^*$  in  $L^2(\mathbb{R}^d, \mathbb{R}^d)$ . In the periodic case (namely if  $A_+ = A_-$ ), obtaining strong convergence is more difficult and requires the so-called two-scale expansion:

$$u^{\varepsilon,1}(x) := u^*(x) + w_i \left( \frac{x}{\varepsilon} \right) \partial_i u^*(x), \quad (6)$$

where here, and in the sequel, the Einstein summation convention is used. The functions  $w_i$  are the so-called correctors, which are the strictly sublinear solutions (unique up to the addition of a constant) to the following equation:

$$-\operatorname{div}(A \cdot (e_i + \nabla w_i)) = 0 \quad \text{in } \mathbb{R}^d. \quad (7)$$

We explain in the next section how to generalize the definition of correctors and the two-scale expansion.

### 3 Definition of the correctors and the two-scale expansion

A fundamental ingredient of Avellaneda and Lin's proof is that the so-called correctors "correct" sublinear  $A^*$ -harmonic functions to  $A$ -harmonic sublinear functions. Hence, the first step is to build the sublinear  $A^*$ -harmonic functions, *i.e.*, the functions  $P_j$  satisfying:

$$-\operatorname{div}(A^*(x) \cdot \nabla P_j(x)) = 0 \quad \text{in } \mathbb{R}^d. \quad (8)$$

They induce a natural definition of correctors, which slightly differs from [7]. Unfortunately, with these correctors, the classical formula (6) for the two-scale expansion is algebraically inadequate. As a consequence, we propose a generalization of this formula which takes into account the fact that the homogenized matrix is not constant and that allows for a divergence-form representation of the residuum  $u^{\varepsilon,1} - u^\varepsilon$ .

#### 3.1 $A^*$ -harmonic functions

When  $A^*$  is constant, the sublinear  $A^*$ -harmonic functions are the affine functions. (We say that a function  $f$  is sublinear if  $\limsup_{|x| \rightarrow +\infty} |x|^{-1}|f(x)| = l < +\infty$  and strictly sublinear if  $l = 0$  in the previous limit.) In our case, the space of sublinear  $A^*$ -harmonic functions is spanned by the constant functions and the following piecewise linear functions:

$$P_j(x) = P(x) \cdot e_j := \begin{cases} x \cdot e_j & \text{if } x \cdot e_1 < 0, \\ x \cdot e_j + \check{a}_j x \cdot e_1 & \text{if } x \cdot e_1 > 0, \end{cases} \quad (9)$$

for  $j \in \llbracket 1, d \rrbracket$ , where  $\check{a}$  is related to the transmission matrix through the interface  $\mathcal{I}$  and reads:

$$\check{a}_j = \frac{(A_-^*)_{1j} - (A_+^*)_{1j}}{(A_+^*)_{11}}. \quad (10)$$

If  $\check{a} = 0$  (which strictly encompasses the case where  $A^*$  is constant), the functions  $P_j$  are linear.

It is straightforward that the functions  $P_j$  are solution to (8). Indeed, by definition, the functions  $P_j$  are continuous and their gradients read

$$\nabla P_j(x) = \begin{cases} e_j & \text{if } x \cdot e_1 < 0, \\ e_j + \check{a}_j e_1 & \text{if } x \cdot e_1 > 0. \end{cases} \quad (11)$$

Hence, the functions  $P_j$  are  $A^*$ -harmonic in  $\mathbb{R}_-^* \times \mathbb{R}^{d-1}$  and in  $\mathbb{R}_+^* \times \mathbb{R}^{d-1}$ , and they satisfy the transmission conditions across the interface:

$$\lim_{h \rightarrow 0^+} [(A^* \cdot \nabla P_j)(x + he_1)] \cdot e_1 = \lim_{h \rightarrow 0^+} [(A^* \cdot \nabla P_j)(x - he_1)] \cdot e_1, \quad (12)$$

$$\lim_{h \rightarrow 0^+} \partial_k P_j(x + he_1) = \lim_{h \rightarrow 0^+} \partial_k P_j(x - he_1), \quad (13)$$

for all  $x \in \mathcal{I}$  and  $k \in \llbracket 2, d \rrbracket$ .

### 3.2 Definition of the correctors

Since the correctors are meant to turn the  $A^*$ -harmonic functions  $P_j$  into  $A$ -harmonic sublinear functions, they should solve the following equation:

$$-\operatorname{div}(A(x) \cdot \nabla(P_j(x) + w_j(x))) = 0 \quad \text{in } \mathbb{R}^d. \quad (14)$$

Using the techniques of [7], we show in Section 5.3 the following proposition:

**Proposition 3.1.** *Suppose that the matrix  $A$  satisfies Assumptions 1, 2 and 3. Then, there exists a solution  $w_j \in H_{\text{loc}}^1(\mathbb{R}^d)$  to (14), which satisfies the following estimates:*

$$\|w_j\|_{L^\infty(\mathbb{R}^d)} < +\infty, \quad (15)$$

$$\text{and } \|\nabla w_j\|_{C_{\text{unif}}^{0,\beta}(\mathbb{R}^d \setminus \mathcal{I})} < +\infty, \quad (16)$$

for any  $0 < \beta < \min(\alpha, 1/4)$ .

If  $\check{a} = 0$ , the definition (14) coincides with the classical one (7) and with [7, (48)], that we recall here:

$$-\operatorname{div}(A(x) \cdot (\nabla w_j(x) + e_j)) = -\operatorname{div}(A^*(x) \cdot e_j). \quad (17)$$

However, in the case where  $\check{a} \neq 0$ , these three definitions lead to different objects. We motivate our choice in the next section.

### 3.3 A possible generalization of the two-scale expansion

Now, we introduce a generalization of the two-scale expansion. From above, it appears clearly that the corrected version of the sublinear  $A^*$ -harmonic functions  $u^*(x) = a_j P_j(x)$  (for  $(a_j) \in \mathbb{R}^d$ ) is the following

$$a_j (P_j(x) + w_j(x)) = u^*(x) + w_j(x) \left( (\nabla P(x))^{-1} \right)_{jk} \partial_k u^*(x),$$

where we use the convention  $(\nabla P)_{ij} := \partial_i P_j$ . This suggests to set, for the solution  $u^*$  to (4), the following generalized two-scale expansion

$$u^{\varepsilon,1}(x) = u^*(x) + \varepsilon w_j \left( \frac{x}{\varepsilon} \right) \left( (\nabla P(x))^{-1} \right)_{jk} \partial_k u^*(x). \quad (18)$$

In (18), the quantity

$$U^*(x) := (\nabla P(x))^{-1} \cdot \nabla u^*(x), \quad (19)$$

is actually a gradient in harmonic coordinates. Indeed, if we set

$$\tilde{u}(z) := u^*(P^{-1}(z)), \quad (20)$$

then, it obviously holds that  $\partial_{z_j} \tilde{u}(z) = U_j^*(P^{-1}(z))$ . Moreover, by the transmission conditions through the interface (see (12) and (13)), the function  $U_j^*$  is continuous across the interface  $\mathcal{I}$  (for  $f$  sufficiently regular).

Notice that we recover the classical two-scale expansion when  $\check{a} = 0$ .

The classical argument for assessing the quality of the two-scale expansion is that it allows for a divergence-form representation of the residuum  $u^{\varepsilon,1} - u^\varepsilon$  (see *e.g.* [13, pp. 26-27]). We justify that this algebraical structure is preserved by the generalized expansion (18), with a right-hand term involving the gradient  $\nabla U^*$ .

In this perspective, it shall be underlined that the formal computation of [13] with the classical two-scale expansion (6) and with the definition (17) of [7] involves the quantity  $\nabla^2 u^*$  (which, in our case, might involve a singular measure supported on the interface  $\mathcal{I}$ ) multiplied by quantities that might be discontinuous across the interface  $\mathcal{I}$ . As a consequence, the mathematical significance of this formal computation is not clear for bicrystals, even when resorting to the theory of distributions.

We now proceed with the computation of  $-\operatorname{div} \left( A \left( \frac{x}{\varepsilon} \right) \cdot \nabla (u^{\varepsilon,1}(x) - u^\varepsilon(x)) \right)$ . For simplicity, we set  $\varepsilon = 1$  and drop the argument  $x$  of the functions below. By (2) and (4), we have

$$-\operatorname{div} \left( A \cdot \nabla (u^{\varepsilon,1} - u^\varepsilon) \right) = -\operatorname{div} \left( A \cdot \nabla u^{\varepsilon,1} \right) + \operatorname{div} \left( A^* \cdot \nabla u^* \right).$$

We now use the definitions (19) and (18) to expand the above right-hand term:

$$\begin{aligned} -\operatorname{div} \left( A \cdot \nabla (u^{\varepsilon,1} - u^\varepsilon) \right) &= -\partial_i \left( A_{ij} (\partial_j u^* + \partial_j w_k U_k^*) - A_{ij}^* \partial_j u^* \right) \\ &\quad - \partial_i \left( A_{ij} w_k \partial_j U_k^* \right). \end{aligned}$$

Next, using once more (19), we obtain:

$$\begin{aligned} &-\partial_i \left( A_{ij} (\partial_j u^* + \partial_j w_k U_k^*) - A_{ij}^* \partial_j u^* \right) \\ &= -\partial_i \left( [A_{ij} (\partial_j P_k + \partial_j w_k) - A_{ij}^* \partial_j P_k] U_k^* \right). \end{aligned}$$

Yet, by definition of  $P_j$  and  $w_j$ , there holds

$$\partial_i \left( A_{ij} (\partial_j P_k + \partial_j w_k) - A_{ij}^* \partial_j P_k \right) = 0. \quad (21)$$

Hence, as will be justified below by Proposition 5.5, there exists a tensor  $B_{ijk}$  that is antisymmetric in its first two indices and that satisfies

$$\partial_i B_{ijk} = A_{jl}^* \partial_l P_k - A_{jl} (\partial_l P_k + \partial_l w_k). \quad (22)$$

Therefore, using the antisymmetry of  $B$ , one can express:

$$-\partial_i \left( [A_{ij} (\partial_j P_k + \partial_j w_k) - A_{ij}^* \partial_j P_k] U_k^* \right) = \partial_l B_{lik} \partial_i U_k^* = \partial_i (B_{ijk} \partial_j U_k^*).$$

As a conclusion, while restoring the scale  $\varepsilon$ , we obtain:

$$-\operatorname{div} \left( A(\cdot/\varepsilon) \cdot \nabla (u^{\varepsilon,1} - u^\varepsilon) \right) = \varepsilon \partial_i \left( (B_{ijk} - A_{ij} w_k) (\cdot/\varepsilon) \partial_j U_k^* \right). \quad (23)$$

In the above expression, it can be seen that every term is well-defined in the weak sense. Moreover, the right-hand term is multiplied by  $\varepsilon$  so that, formally, one can expect that the error  $|\nabla u^{\varepsilon,1} - \nabla u^\varepsilon|$  scales like  $\varepsilon$  in various  $L^p$  norms. This justifies the introduction of the generalized two-scale expansion (18).



## 4 Main results

We are now in a position to state our main results. The first ones concern Lipschitz estimates. They can be used in a second step to quantify the error residuum between the generalized two-scale expansion and the actual solution of the multiscale problem.

### 4.1 Estimation

Our first result is a generalization of the local Lipschitz estimates [2, Lem. 16]:

**Theorem 4.1.** *Suppose that  $d \geq 2$  and that the matrix  $A$  satisfies Assumptions 1, 2 and 3. Let  $\varepsilon > 0$ ,  $x_0 \in \mathbb{R}^d$  and  $R > 0$ . Assume that the function  $u^\varepsilon \in H^1(B(x_0, 2))$  is a solution to*

$$-\operatorname{div}(A(x/\varepsilon) \cdot \nabla u^\varepsilon(x)) = 0 \quad \text{in } B(x_0, 2R). \quad (24)$$

*Then, there exists a constant  $C$  that only depends on  $A$  and  $d$  such that*

$$\sup_{x \in B(x_0, R) \setminus \mathcal{I}} |\nabla u^\varepsilon(x)| \leq \frac{C}{R^{d+1}} \left( \int_{B(x_0, 2R)} |u^\varepsilon|^2 \right)^{1/2}. \quad (25)$$

If the ball  $B(x_0, R)$  does not intersect the interface  $\mathcal{I}$ , the above result concerns nothing but the classical periodic setting. But, in Theorem 4.1 the ball  $B(x_0, R)$  may intersect the interface  $\mathcal{I}$ , where the gradient  $\nabla u^\varepsilon(x)$  might be discontinuous: in this case, a Lipschitz estimate holds up to the interface. On the first hand, this result might seem surprising: one could have expected that the discontinuity of  $A$  through the interface would interact with the oscillations of the small scale so that  $\nabla u^\varepsilon$  would not remain bounded when  $\varepsilon$  goes to 0. But, on the other hand, in the periodic setting, it is known that some Lipschitz estimates can also be obtained up to the boundary of a smooth domain (see e.g. [2, Th. 2]), which, from a geometric point of view, might be seen as a kind of interface. Moreover, the way of building the correctors themselves (see [7, Th. 5.1] and Section 5.3) is reminiscent of boundary layers. However, we have not been able to take this apparent similarity further.

*Remark 2.* Since the function  $u^\varepsilon$  is continuous in  $B(x_0, R)$ , Theorem 4.1 actually induces a local  $L^\infty$  estimate in the following sense:

$$\|\nabla u^\varepsilon\|_{L^\infty(B(x_0, R))} \leq \frac{C}{R^{d+1}} \left( \int_{2B(x_0, R)} |u^\varepsilon|^2 \right)^{1/2}. \quad (26)$$

Similarly, Corollary 4.2 and Theorem 4.5 below can be understood in a local  $L^\infty$  sense.

We prove Theorem 4.1 by using the compactness method of [2]. Two scales should be separated:

- the small scales, where  $R/\varepsilon \ll 1$ , where the Schauder estimates provided by [17] comes into play,
- the large scales, for  $R/\varepsilon \gg 1$ , where we use the compactness method of Avellaneda and Lin.

The large-scale control on  $\nabla u^\varepsilon$  is due to a structural property of the matrix  $A$ , which uniformly H-converges to its associated homogenized matrix  $A^*$  (this statement is made precise in Lemma 5.6). The idea of the proof is to compare  $u^\varepsilon$  to a locally  $A^*$ -harmonic function  $u^*$  (since  $A^*$  is piecewise constant, this function enjoys sufficient regularity properties for our purpose). By the uniform H-convergence,  $u^\varepsilon$  can be made sufficiently close to  $u^*$ , and thus inherit a medium-scale regularity estimate from it. Then, by “linearizing”  $u^\varepsilon$  in the spirit of the two-scale expansion (18) (here we need the correctors  $w_j$  to be strictly sublinear), one can iterate the medium-scale regularity estimate on balls of exponentially increasing radii to obtain a large-scale regularity estimate. There, it is of the uttermost importance to use a  $A(\cdot/\varepsilon)$ -harmonic approximation of  $u^\varepsilon$  in order to iterate the reasoning (this is another motivation for using the correctors defined by (14)). Finally, a blow-up argument turns the large-scale regularity estimate into an estimate on the gradient  $\nabla u^\varepsilon$  by resorting to the Schauder estimates of [17].

As is well-known in the periodic setting (see *e.g.* [15]), pointwise estimates on the Green function can be derived from the Lipschitz estimates. The Green function  $\mathcal{G}(x, y)$  (also called fundamental solution) associated with the operator  $-\operatorname{div}(A \cdot \nabla)$  is a solution of the following equation weak formulation (see [9] for a precise definition):

$$-\operatorname{div}(A(x) \cdot \nabla_x \mathcal{G}(x, y)) = \delta_y(x).$$

If  $d \geq 3$ , since  $A$  is uniformly bounded and coercive, by [9, Th. 1], there exists a Green function which is unique. Moreover, it satisfies the following estimate:

$$|\mathcal{G}(x, y)| \leq C|x - y|^{-d+2}. \quad (27)$$

Remark that the Green function  $\mathcal{G}(x, y)$  is locally  $A$ -harmonic for  $x \neq y$ . Therefore, by applying Theorem 4.1, we deduce the following estimates on the gradient and the mixed gradient of the Green function:

**Corollary 4.2.** *Let  $d \geq 3$ . Suppose that the matrix  $A$  satisfies Assumptions 1, 2, and 3. Let  $\mathcal{G}$  be the Green function of the operator  $-\operatorname{div}(A \cdot \nabla)$  on  $\mathbb{R}^d$ . Then, there exists a constant  $C > 0$  depending only on  $d$  and  $A$  such that, for any  $x \neq y \in \mathbb{R}^d \setminus \mathcal{I}$ , there holds*

$$|\nabla_x \mathcal{G}(x, y)| + |\nabla_y \mathcal{G}(x, y)| \leq C|x - y|^{-d+1}, \quad (28)$$

$$|\nabla_x \nabla_y \mathcal{G}(x, y)| \leq C|x - y|^{-d}. \quad (29)$$

It should be noted that, by a dilatation argument, the Green function  $\mathcal{G}^\varepsilon$  of the operator  $-\operatorname{div}(A(\cdot/\varepsilon) \cdot \nabla)$  can be written as:

$$\mathcal{G}^\varepsilon(x, y) = \varepsilon^{2-d} \mathcal{G}(x/\varepsilon, y/\varepsilon).$$

Whence the Green function  $\mathcal{G}^\varepsilon$  also satisfies (27), (28) and (29), with a constant  $C$  that does not depend on  $\varepsilon$ .

*Remark 3 (Case  $d = 2$ ).* The conclusions of Corollary 4.2 also hold in the case  $d = 2$ . It can be retrieved from the case  $d = 3$  by expressing the 2-dimensional Green function by means of a 3-dimensional Green function with well-chosen coefficients. This is not shown here but can be found in [2, Th. 13] (see also [9, Prop. 5]).

The proofs of the Theorem 4.1 and Corollary 4.2 are respectively postponed until Sections 6.1, and 6.2.

## 4.2 Approximation

We now estimate the residuum  $u^{\varepsilon,1} - u^\varepsilon$  (or equivalently  $u^\varepsilon - u^*$ ) in the  $L^\infty$  norm by combining the algebraical expression (23) and the estimates on the Green function provided by Corollary 4.2:

**Proposition 4.3.** *Let  $d \geq 3$ ,  $x_0 \in \mathbb{R}^d$  and  $\varepsilon > 0$ . Suppose that the matrix  $A$  satisfies Assumptions 1, 2, and 3. Let  $f \in L^p(\mathbb{R}^d)$  with support inside  $B(x_0, 1)$ , for  $p > d$ . Assume that the functions  $u^\varepsilon$  and  $u^*$  are respectively the zero-mean solutions to (2) and (4). Then, there exists a constant  $C$  that only depends on  $A$ ,  $d$  and  $p$  such that*

$$\|u^\varepsilon - u^*\|_{L^\infty(\mathbb{R}^d)} \leq C\varepsilon \|f\|_{L^p(\mathbb{R}^d)}. \quad (30)$$

By a duality argument (see [15, Th. 1.1]), this provides a pointwise error estimate on the level of the Green function:

**Proposition 4.4.** *Let  $d \geq 3$ . Suppose that the matrix  $A$  satisfies Assumptions 1, 2, and 3. Let  $\mathcal{G}$ , respectively  $\mathcal{G}^*$ , be the Green function of the operator  $-\operatorname{div}(A \cdot \nabla)$ , respectively  $-\operatorname{div}(A^* \cdot \nabla)$ , on  $\mathbb{R}^d$ . Then, there exists a constant  $C > 0$  depending only on  $d$  and  $A$  such that, for any  $x \neq y \in \mathbb{R}^d$ , there holds:*

$$|\mathcal{G}(x, y) - \mathcal{G}^*(x, y)| \leq C|x - y|^{-d+1}. \quad (31)$$

For the sake of concise notations, we define the matrices  $W(x)$  and  $W^\dagger(x)$  by

$$W_{ij}(x) := \delta_{ij} + \partial_i w_k(x) (\nabla P(x))_{kj}^{-1}, \quad (32)$$

$$W_{ij}^\dagger(x) := \delta_{ij} + \partial_i w_k^\dagger(x) (\nabla P^\dagger(x))_{kj}^{-1}, \quad (33)$$

where,  $\delta_{ij}$  stands for the Kronecker symbol, and the functions  $P^\dagger$  and  $w^\dagger$  are the analogous of  $P$  and  $w$ , but with respect to the transposed matrix  $A^T$ . Then, the gradient  $\nabla u^{\varepsilon,1}$  can be expressed by means of  $W$  and  $U^*$  respectively defined by (32) and (19) as

$$\nabla u^{\varepsilon,1}(x) = W(x/\varepsilon) \cdot \nabla u^*(x) + \varepsilon w_j(x/\varepsilon) \nabla U_j^*(x).$$

Since the last right-hand term of the above identity scales like  $\varepsilon$ , we expect  $\nabla u^\varepsilon(x)$  to be well approximated by  $W(x/\varepsilon) \cdot \nabla u^*(x)$ .

We justify it first on the level of the Green function, in the same vein as the recent results of [15] (see also [3] in the stationary ergodic case). Indeed, as a consequence of Theorem 4.1 and of Proposition 4.4:

**Theorem 4.5.** *Under the assumptions of Proposition 4.4, there exists a constant  $C > 0$  depending only on  $d$  and  $A$  such that, for all  $x \neq y \in \mathbb{R}^d \setminus \mathcal{I}$ , there holds:*

$$|\nabla_x \mathcal{G}(x, y) - W(x) \cdot \nabla_x \mathcal{G}^*(x, y)| \leq C \frac{\ln(2 + |x - y|)}{|x - y|^d}, \quad (34)$$

$$\left| \nabla_x \nabla_y \mathcal{G}(x, y) - W(x) \cdot \nabla_x \nabla_y \mathcal{G}^*(x, y) \cdot (W^\dagger(y))^T \right| \leq C \frac{\ln(2 + |x - y|)}{|x - y|^{d+1}}. \quad (35)$$

Going backwards to the solutions  $u^*$  and  $u^\varepsilon$ , this implies an  $L^\infty$  estimate on the gradient of the residuum:

**Corollary 4.6.** *Let  $d \geq 3$ ,  $x_0 \in \mathbb{R}^d$  and  $\varepsilon > 0$ . Suppose that the matrix  $A$  satisfies Assumptions 1, 2, and 3. Let  $f \in L^\infty(\mathbb{R}^d)$  with support inside  $B(x_0, 1)$ . Assume that the function  $u^\varepsilon$  is the zero-mean solution to (2) and that  $u^*$  is the zero-mean solution to (4). Then, there exists a constant  $C$  that only depends on  $A$  and  $d$  such that*

$$\|W(\cdot/\varepsilon) \cdot \nabla u^* - \nabla u^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leq C\varepsilon |\ln(2 + \varepsilon^{-1})|^2 \|f\|_{L^\infty(\mathbb{R}^d)}. \quad (36)$$

The proofs of Proposition 4.3 and 4.4, respectively Theorem 4.5 and Corollary 4.6 are postponed until Sections 7.1, respectively 7.2.

### 4.3 Remarks and possible extensions

We conclude this Section by discussing some aspects of this study.

First, we shall underline that the above results concern the problem on  $\mathbb{R}^d$ , so that there is no boundary. In this regard, if we denote the cell  $Q := [-1/2, 1/2] \times [0, T_2] \times \dots \times [0, T_d]$  and set  $\varepsilon := 1/n$  for  $n \in \mathbb{N}$ , then the above results can be generalized to the problem (2) set on  $Q$  with periodic boundary conditions (see [14] for a related work in the case of a periodic coefficient). But it seems more difficult to treat the case where (2) is set on a regular bounded domain  $\Omega$  along with Dirichlet boundary conditions. Indeed, in this case, we need to show boundary estimates, which might not be true in the neighborhood of the intersection point between the boundary  $\partial\Omega$  and the interface  $\mathcal{I}$ . At the moment, it is not clear for the author which results may still hold in this case.

Second, in all the results above, the constant  $C$  of the estimates is said to “depend on  $A$ ”. This rather vague dependence is a consequence of the fact that the compactness method of Avellaneda and Lin relies on a proof by contradiction. However, one can likely be more precise by proceeding with the proof on the class  $E(\mu, \alpha, \tau, (T_i^\pm))$  of matrices  $A \in L^\infty(\mathbb{R}^d, \mathbb{R}^{d \times d})$  satisfying Assumption 1, 2 and 3 with  $\|A_\pm\|_{C^{0,\alpha}(\mathbb{R}^d)} \leq \tau$  (rather than by working on a fixed matrix). Thus, the dependence on  $A$  would be replaced by a dependence on  $(\mu, \alpha, \tau, (T_i^\pm))$ . Such assumptions have been developed in [15] for example.

Once these limitations are left aside, we remark that, as in [11, 5], the main ingredients used here are the long-range behavior of the correctors and the regularity of the homogenized problem. Actually, our proofs only require the fact that  $A$  is uniformly elliptic and bounded and uniformly Hölder continuous up to the interface  $\mathcal{I}$  (Assumptions 1, 3) and that there exist correctors  $w_j$  and a potential  $B$  that are bounded. Therefore, the structural Assumption 2 can certainly be weakened. In particular (see [7, Th. 5.7]), one can reasonably think that assuming that the ratios  $T_i^+/T_i^-$  are not Liouville-Roth numbers would be sufficient to build bounded correctors  $w_j$  and a bounded potential  $B$ .

The regularity of the matrix  $A$  is a key ingredient in the proof of Avellaneda and Lin to show Lipschitz estimates –which encompass the small scales and the large scales. However, as shown in [11], no regularity assumption is necessary to obtain large-scale regularity down to the scale  $\varepsilon$ . Therefore, this assumption could be removed to obtain a weaker version of the above results. In this regard, the approach of [11] could be adapted to obtain regularity estimates (instead of Avellaneda and Lin’s approach). One can optimistically think that this would

pave the way to quantitative homogenization results in the case of “stochastic” bicrystals.

Finally, one could also think of systems of elliptic equations in divergence form, for which Avellaneda and Lin’s approach as well as the regularity results of [17] are adapted. One can extend Theorem 4.1 to the case of systems by a slight adaptation –namely, by showing that the result of  $C^{0,\alpha}$  regularity [2, Th. 1] still holds in our case and then by invoking this regularity estimate instead of the De Giorgi-Nash Moser theorem in the proofs below. Generalizing the other above results would require first to generalize the  $W^{2,p}$  estimates for piecewise constant coefficients in [19, 16] (see Lemma 5.2 below) to the case of systems. To the best of our knowledge, this has not been done yet.

## 5 Preliminary considerations

In this section, we collect some results that will be used throughout this article. First, we introduce a few notations. Then, we state some regularity results concerning elliptic equation with piecewise regular (or constant) coefficients. In particular, we show some estimates on  $U^*$  defined by (19) and we build a procedure for “linearizing” locally  $A^*$ -harmonic functions by appealing to the  $A^*$ -harmonic sublinear functions  $P_j$ . Next, we build the correctors defined by (14) and a solution  $B$  to (22) (that we call the potential) and we show that they enjoy some regularity properties. Finally, we justify that the matrices  $A(\cdot/\varepsilon)$  uniformly H-converge to  $A^*$  when  $\varepsilon \rightarrow 0$ .

### 5.1 Notations

We introduce here some useful notations for building the correctors and the potential. From now on, the matrix  $A$  satisfies Assumptions 1, 2, and 3.

For  $i \in \llbracket 2, d \rrbracket$ , we denote by  $T_i$  the least common multiple of  $T_i^-$  and  $T_i^+$ . We define the domains

$$D := \mathbb{R} \times [0, T_2] \times \cdots \times [0, T_d], \quad \text{and} \quad D_{\pm} := \mathbb{R}_{\pm} \times [0, T_2] \times \cdots \times [0, T_d].$$

We say that  $u$  is  $D$ -periodic if  $u$  is  $T_i$ -periodic in  $x_i$ , for  $i \geq 2$ .

We denote  $w_j^{\pm}$ , respectively  $B^{\pm}$ , the correctors, respectively the potential associated with the periodic matrices  $A^{\pm}$ . By definition,  $B_{ijk}^{\pm}$  is a tensor anti-symmetric in its first two indices that solves

$$-\partial_i B_{ijk}^{\pm} = (A_{\pm})_{jl} (\delta_{lk} + \partial_l w_k^{\pm}) - (A_{\pm}^*)_{jk} \quad \text{in } \mathbb{R}^d.$$

We recall that both the correctors  $w_j^{\pm}$  and the potential  $B^{\pm}$  are  $[0, T_1^{\pm}] \times \cdots \times [0, T_d^{\pm}]$ -periodic and of regularity  $C^{1,\alpha}$ .

Last, if  $\Omega$  is a bounded domain, we define the rescaled integral  $f_{\Omega} u = |\Omega|^{-1} \int u$ , where  $|\Omega|$  is the Lebesgue measure of  $\Omega$ .

### 5.2 Regularity results

We borrow a regularity result from [17] (see also [18]):

**Theorem 5.1** (Local version of Theorem 1.1 of [17]). *Let  $A \in L^\infty(\mathbb{R}^d, \mathbb{R}^{d \times d})$  be a matrix defined by (3), where the matrices  $A_\pm$  satisfy Assumption 3 (but are not necessarily periodic), and that satisfies Assumption 1. Let  $0 < \beta < \min(\alpha, 1/4)$ . Suppose that  $f \in L^\infty(B(x, 2))$ , and that  $g \in C^{0,\beta}(B(x, 2) \setminus \mathcal{I})$ . If  $u$  solves*

$$-\operatorname{div}(A \cdot \nabla u) = f + \operatorname{div}(g) \quad \text{in } B(x, 2),$$

*then there exists a constant  $C$  only depending on  $d, \alpha, \beta, \mu$  and  $\|A\|_{C^{0,\alpha}(B(x,2))}$  such that*

$$\|u\|_{C^{1,\beta}(B(x,2) \setminus \mathcal{I})} \leq C \left( \|u\|_{L^2(B(x,2))} + \|f\|_{L^\infty(B(x,2))} + \|g\|_{C^{0,\beta}(B(x,2) \setminus \mathcal{I})} \right).$$

Then, we provide some  $W^{1,p}$  estimates on the quantity  $U^*$  defined by (19):

**Lemma 5.2.** *Let  $d \geq 3$ ,  $x_0 \in \mathbb{R}^d$ ,  $p \in (d, +\infty)$ , and  $A^*$  be a matrix defined by (5) and satisfying Assumption 1. Suppose that  $f \in L^p(\mathbb{R}^d)$  is supported into  $B(x_0, 1)$ . Let  $u^* \in H_{\text{loc}}^1(\mathbb{R}^d)$  be the zero-mean solution to (4) and define  $U^*$  by (19). Then there exists a constant  $C > 0$  depending only on  $d$  and  $A_\pm^*$  such that*

$$\|U^*\|_{W^{1,p}(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}. \quad (37)$$

Moreover, there holds

$$\|\nabla U^*\|_{L^2(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}. \quad (38)$$

The proof (37) rests on a regularity result [19] on nondivergence elliptic equations with coefficients that are constant on the half-spaces  $\mathbb{R}_- \times \mathbb{R}^{d-1}$  and  $\mathbb{R}_+ \times \mathbb{R}^{d-1}$ . One turns (4) into such an equation by means of the  $A^*$ -harmonic coordinates  $P_j$ . We need to treat separately Estimation (38) since, if  $d = 3$  or  $d = 4$ , it is not guaranteed that  $u^*$  defined above lies in  $L^2(\mathbb{R}^d)$ .

*Proof of Lemma 5.2.* We first show an  $L^p$  estimate on  $u^*$ . By definition, there holds:

$$u^*(x) = \int_{B(x_0, 1)} \mathcal{G}^*(x, y) f(y) dy. \quad (39)$$

Since the Green function  $\mathcal{G}^*$  associated with the operator  $-\operatorname{div}(A^* \cdot \nabla)$  is such that  $|\mathcal{G}^*(x, y)| \leq C|x - y|^{-d+2}$ , and since the function  $f$  is in  $L^q(\mathbb{R}^d)$  for all  $q \in [1, p]$  (by the Hölder inequality, recalling that the support of  $f$  is inside  $B(x_0, 1)$ ), the Young inequality yields

$$\|u^*\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}. \quad (40)$$

Next, we define the function  $\tilde{u}$  by (20). It satisfies the following elliptic equation:

$$-\operatorname{div} \left( |J(z)|^{-1} \tilde{A}(z) \cdot \nabla \tilde{u}(z) \right) = |J(z)|^{-1} f(P^{-1}(z)), \quad (41)$$

where  $\tilde{A}(z)$  is defined by

$$\tilde{A}(z) := (\nabla P(P^{-1}(z)))^T \cdot A(P^{-1}(z)) \cdot \nabla P(P^{-1}(z)),$$

and  $J(z)$  is the Jacobian of  $P$  evaluated on  $P^{-1}(z)$ . By construction,  $\tilde{A}(z)$  is elliptic and constant on the half-spaces  $\mathbb{R}_{\pm}^* \times \mathbb{R}^{d-1}$ , and the product  $|J(z)|^{-1}\tilde{A}(z)$  is divergence-free in  $\mathbb{R}^d$ . Whence, (41) can be rewritten as

$$\tilde{A}_{ij}(z)\partial_{ij}\tilde{u}(z) = f(P^{-1}(z)). \quad (42)$$

As a consequence, we can apply [19, Th.] (see also [16, Lem. 2.4]): there exists a constant  $C$  so that

$$\|\tilde{u}\|_{W^{2,p}(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)} + C \|\tilde{u}\|_{L^p(\mathbb{R}^d)}.$$

Thus, by (40), we deduce

$$\|\tilde{u}\|_{W^{2,p}(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}.$$

A simple change of variable yields the desired estimate (37).

We now show (38). Since  $f$  is compactly supported in  $B(x_0, 1)$ , then  $\tilde{u}$  is  $\tilde{A}$ -harmonic on  $\mathbb{R}^d \setminus B(z_0, \rho)$ , where  $\rho := \|(\nabla P)^{-1}\|_{L^\infty(\mathbb{R}^d)}$  and  $z_0 := P^{-1}(x_0)$ . Therefore, for  $z_1 \in \mathbb{R}^d$  such that  $|z_0 - z_1| > 2\rho$ , one can apply [17, Prop. 1.7] so that, on  $B(z_1, |z_0 - z_1|/2)$  so that

$$\|\nabla^2 \tilde{u}\|_{L^\infty(B(z_1, |z_0 - z_1|/4))} \leq C |z_0 - z_1|^{-2} \left( \int_{B(z_1, |z_0 - z_1|/2)} |\tilde{u}|^2 \right)^{1/2}. \quad (43)$$

Now, recalling that  $u^*$  satisfies (40), then, by using (27) and the Cauchy-Schwartz inequality, we obtain that, if  $|x - x_0| > 2$ , there holds

$$|u^*(x)| \leq C |x - x_0|^{-d+2} \|f\|_{L^1(B(x_0, 1))}.$$

Transposing it on the level of  $\tilde{u}$  yields that, for any  $z \in B(z_1, |z_0 - z_1|/2)$ , we have

$$|\tilde{u}(z)| \leq C |z - z_0|^{-d+2}.$$

Therefore, we deduce from (43) that

$$\|\nabla^2 \tilde{u}\|_{L^\infty(B(z_1, |z_0 - z_1|/4))} \leq C |z_0 - z_1|^{-d}. \quad (44)$$

As a consequence, since we already know that  $\nabla^2 \tilde{u} \in L^p(\mathbb{R}^d)$  for  $p > 2$ , we finally obtain that  $\nabla^2 \tilde{u} \in L^2(\mathbb{R}^d)$ . This proves (38).  $\square$

We now explain how locally  $A^*$ -harmonic functions can be “linearized” by using the sublinear  $A^*$ -harmonic functions  $P_j$ .

**Lemma 5.3.** *Let  $A^*$  be a matrix defined by (5) and satisfying Assumption 1. Let  $x_0 \in \mathbb{R}^d$ , and assume that the function  $u^* \in H^1(B(x_0, 1))$  satisfies*

$$-\operatorname{div}(A^*(x) \cdot \nabla u^*(x)) = 0 \quad (45)$$

*in  $B(x_0, 1)$ . Then, there exists a constant  $C$  depending only on  $d$  and  $\mu$  such that, for all  $\theta \in (0, 1/2)$ , there holds*

$$\begin{aligned} & \sup_{x \in B(x_0, \theta)} \left| u^*(x) - u^*(x_0) - (P(x) - P(x_0)) \cdot \int_{B(x_0, \theta)} (\nabla P)^{-1} \cdot \nabla u^* \right| \\ & \leq C \theta^2 \left( \int_{B(x_0, 1)} |u^*|^2 \right)^{1/2}. \end{aligned} \quad (46)$$

We underline that the above formula (46) gives a first-order approximation of  $u^*$  that is also  $A^*$ -harmonic. In this regard, it is a generalization of [2, (3.5)]. This estimates will play a central role in the proof of Theorem 4.1 by encapsulating some regularity properties of the homogeneous problem (4).

The (simple) proof below interprets the  $A^*$ -harmonic functions  $P_j$  as new coordinates, in which (46) appears as a first-order Taylor expansion.

*Proof of Lemma 5.3.* The key ingredient of the proof is that the function  $\tilde{u}$  defined by (20) satisfies

$$\|\nabla^2 \tilde{u}\|_{L^\infty(P^{-1}(B(x_0, 1/2)))} \leq C \|u^*\|_{L^2(B(x_0, 1))}. \quad (47)$$

Indeed, by the same argument as for establishing (41) above, we obtain that  $\tilde{u}$  satisfies

$$-\operatorname{div} \left( |J(z)|^{-1} \tilde{A}(z) \cdot \nabla \tilde{u}(z) \right) = 0 \quad \text{in } P^{-1}(B(x_0, 1)), \quad (48)$$

with  $J$  and  $\tilde{A}$  defined as in the proof of Lemma 5.2. Since the matrix  $|J|^{-1}A$  is piecewise constant, as a consequence of [17, Prop. 1.7], there holds

$$\sup_{z \in P^{-1}(B(x_0, 1/2) \setminus \mathcal{I})} |\nabla^2 \tilde{u}(z)| \leq C \|\tilde{u}\|_{L^2(P^{-1}(B(x_0, 1)))}. \quad (49)$$

Moreover, since the matrix  $|J|^{-1}\tilde{A}$  is divergence-free, the gradient  $\nabla \tilde{u}$  is continuous across the interface (inside  $B(x_0, 1/2)$ ). Hence, (49) can be improved as (47).

Therefore, a first-order Taylor expansion on  $\tilde{u}$  yields

$$\begin{aligned} & \left| \tilde{u}(P(x)) - \tilde{u}(P(x_0)) - (P(x) - P(x_0)) \cdot \int_{B(x_0, \theta)} \nabla \tilde{u}(P(z)) dz \right| \\ & \leq C\theta^2 \|\nabla^2 \tilde{u}\|_{L^\infty(P^{-1}(B(x_0, 1/2)))} \\ & \leq C\theta^2 \|u^*\|_{L^2(B(x_0, 1))}. \end{aligned}$$

Finally, since  $\nabla \tilde{u}(P(x)) = (\nabla P(x))^{-1} \cdot \nabla u^*(x)$ , we obtain (46).  $\square$

### 5.3 Correctors and potential

Proposition 3.1 is shown by appealing to Theorem 5.1 and to the following result, which is inspired by [7, Th. 5.1]:

**Proposition 5.4** (Adaptation of Th. 5.1 of [7]). *Suppose that the matrix  $A$  satisfies Assumptions 1, 2, and 3. Then:*

(i) *There exists a solution  $w_j$  to Equation (14). This solution satisfies*

$$\begin{cases} \nabla(w_j - w_j^-) \in L^2(D_-), \\ \nabla(w_j - w_j^+ - \check{a}_j \nabla w_1^+) \in L^2(D_+), \\ w_j \text{ is } D\text{-periodic.} \end{cases} \quad (50)$$



The function  $w_j$  satisfying both (14) and (50) is unique up to the addition of a constant.

(ii) There exist constants  $C > 0$  and  $\kappa > 0$  such that

$$|\nabla w_j(x) - \nabla w_j^-(x)| \leq C \exp(-\kappa|x \cdot e_1|) \quad \text{if } x \cdot e_1 < -1, \quad (51)$$

$$|\nabla w_j(x) - \nabla w_j^+(x) - \check{a}_j \nabla w_1^+(x)| \leq C \exp(-\kappa|x \cdot e_1|) \quad \text{if } x \cdot e_1 > 1. \quad (52)$$

We now build a potential  $B$ :

**Proposition 5.5.** *Suppose that the matrix  $A$  satisfies Assumptions 1, 2, and 3. Then, there exists a D-periodic potential  $B \in L^\infty(\mathbb{R}^d, \mathbb{R}^{d^3})$  associated with  $A$ . Namely,  $B_{ijk}$  is antisymmetric in its first two indices and satisfies (22). Moreover, it lies in  $C_{\text{unif}}^{0,\beta}(\mathbb{R}^d, \mathbb{R}^{d^3})$  for any  $\beta \in (0, 1)$ .*

Since the proofs of Propositions 5.4 and 5.4 closely follows the proof of [7, Th. 5.1], we postpone them until Appendix A.

## 5.4 Uniform H-convergence

Equipped with the correctors, we are in a position to state a first qualitative homogenization result:

**Lemma 5.6.** *Suppose that the matrix  $A$  satisfies Assumptions 1, 2, and 3. Let sequences  $x_n \in \mathbb{R}^d$  and  $\varepsilon_n \in \mathbb{R}_+^*$  satisfy  $x_n \cdot e_1 \rightarrow l \in \mathbb{R}$  and  $\varepsilon_n \rightarrow 0$ . Then, the sequence  $A_n := A((\cdot - x_n)/\varepsilon_n)$  H-converges to  $A^*(\cdot - le_1)$  on every regular bounded domain of  $\mathbb{R}^d$ .*

The proof is classical and relies on the div-curl lemma [20, Lem. 1.1 p. 4]. Therefore, we only emphasize on its main ingredient: the matrix  $A$  admits correctors  $w_j$  such that

$$\nabla w_j \in L_{\text{unif}}^2(\mathbb{R}^d, \mathbb{R}^d), \quad (53)$$

and that satisfy the following weak convergences in  $L^2(\Omega, \mathbb{R}^d)$ :

$$\nabla w_j((\cdot - x_n)/\varepsilon_n) \rightharpoonup 0, \quad (54)$$

$$(A \cdot (\nabla P_j + \nabla w_j))((\cdot - x_n)/\varepsilon_n) - (A^* \cdot \nabla P_j)(\cdot - x_n) \rightharpoonup 0, \quad (55)$$

for any bounded domain  $\Omega$ , for any  $j \in \llbracket 1, d \rrbracket$  and for all sequences  $x_n \in \mathbb{R}^d$  and  $\varepsilon_n \rightarrow 0$ . The above facts (53), (54) and (55) are consequences of Proposition 5.4, using the properties of the periodic correctors  $w_j^\pm$ .

## 6 Estimation

This section is devoted to proving the Lipschitz estimates of Theorem 4.1, from which we derived the estimates on the multiscale Green function of Corollary 4.2.

## 6.1 Lipschitz estimates

Our proof of Lipschitz estimates closely follows the proof of Avellaneda and Lin [2]. It is based on the method of compactness and it is done in three steps:

1. the initialization step (see Section 6.1.1), in which we take advantage of the uniform H-convergence (Lemma 5.6) of the multiscale problem to the homogeneous problem (4). Thus, the multiscale solution  $u^\varepsilon$  inherits the medium-scale regularity property of the solution  $u^*$  of (4) encapsulated in (46). This property is reinterpreted in terms of a “linearization” of  $u^\varepsilon$  by  $A(\cdot/\varepsilon)$ -harmonic functions (here, it is crucial that the correctors  $w_j$  are strictly sublinear).
2. the iteration step (see Section 6.1.2), in which the previous estimates are iterated to obtain Lipschitz regularity of  $u^\varepsilon$  down to scale  $\varepsilon$  (this is also called “excess decay” in [11, Lem. 2]). In this step, it is crucial to resort to an  $A(\cdot/\varepsilon)$ -harmonic approximation of  $u^\varepsilon$  (otherwise, we could not iterate).
3. a blow-up step (see Section 6.1.3), in which we use the regularity result Theorem 5.1 to obtain Lipschitz regularity on scales smaller than  $\varepsilon$ .

### 6.1.1 Initialization: “linearization” of locally $A(\cdot/\varepsilon)$ -harmonic functions

For the sake of conciseness, we define the  $A$ -harmonic coordinates  $\chi$  by

$$\chi_j(x) = P_j(x) + w_j(x).$$

We prove first that the multiscale problem inherits regularity from the homogenized problem:

**Lemma 6.1** (See Lemma 14 in [2]). *Suppose that the matrix  $A$  satisfies Assumptions 1, 2, and 3. Let  $\gamma \in (0, 1)$  and  $x_0 \in \mathbb{R}^d$ . Then, there exists  $\theta \in (0, 1/4)$ , which only depends on  $A_\pm^*$  and  $\gamma$ , and  $\varepsilon_0$ , which only depends on  $A$ ,  $d$ ,  $\gamma$  and  $\theta$ , such that, if  $u^\varepsilon \in H^1(B(x_0, 1))$  satisfies*

$$-\operatorname{div}(A(x/\varepsilon) \cdot \nabla u^\varepsilon(x)) = 0, \tag{56}$$

in  $B(x_0, 1)$  for  $\varepsilon \leq \varepsilon_0$ , then

$$\begin{aligned} & \sup_{x \in B(x_0, \theta)} \left| u^\varepsilon(x) - u^\varepsilon(x_0) - \varepsilon \left( \chi\left(\frac{x}{\varepsilon}\right) - \chi\left(\frac{x_0}{\varepsilon}\right) \right) \cdot \int_{B(x_0, \theta)} (\nabla P)^{-1} \cdot \nabla u^\varepsilon \right| \\ & \leq \theta^{1+\gamma} \left( \int_{B(x_0, 1)} |u^\varepsilon|^2 \right)^{1/2}. \end{aligned} \tag{57}$$

*Proof of Lemma 6.1.* By Theorem 3.1, the correctors  $w_j$  are bounded. Moreover, by the Cauchy-Schwartz inequality and the Cacciopoli estimate, there holds

$$\left| \int_{B(x_0, \theta)} (\nabla P)^{-1} \cdot \nabla u^\varepsilon \right| \leq C \left( \int_{B(x_0, \theta)} |\nabla u^\varepsilon|^2 \right)^{1/2} \leq C \left( \int_{B(x_0, 1)} |u^\varepsilon|^2 \right)^{1/2}.$$

Therefore, proving (57) amounts to establishing a similar estimate, in which  $\chi_j$  is replaced by  $P_j$  (up to taking a smaller  $\varepsilon_0$ ).

By Lemma 5.3, we set  $\theta \in (0, 1/4)$  sufficiently small so that, for any  $x_\infty \in \mathbb{R}^d$ , if  $u^*$  satisfies (45) in  $B(x_\infty, 1/2)$ , then

$$\begin{aligned} & \sup_{x \in B(x_\infty, \theta)} \left| u^*(x) - u^*(x_\infty) - (P(x) - P(x_\infty)) \cdot \int_{B(x_\infty, \theta)} (\nabla P)^{-1} \nabla u^* \right| \\ & \leq \frac{\theta^{1+\gamma}}{3 \cdot 2^d} \left( \int_{B(x_\infty, 1/2)} |u^*|^2 \right)^{1/2}. \end{aligned} \quad (58)$$

Now, by absurd, we assume that there exist  $\varepsilon_n \rightarrow 0$ ,  $x_n \in \mathbb{R}^d$  and  $u^{\varepsilon_n}$  satisfying (56) in  $B(x_n, 1)$  and such that, for any  $n \in \mathbb{N}$ ,

$$\begin{aligned} & \sup_{x \in B(x_n, \theta)} \left| u^{\varepsilon_n}(x) - u^{\varepsilon_n}(x_n) - (P(x) - P(x_n)) \cdot \int_{B(x_n, \theta)} (\nabla P)^{-1} \cdot \nabla u^{\varepsilon_n} \right| \\ & \geq \frac{\theta^{1+\gamma}}{2} \left( \int_{B(x_n, 1)} |u^{\varepsilon_n}|^2 \right)^{1/2}. \end{aligned} \quad (59)$$

(We recall that  $\varepsilon P(\frac{x}{\varepsilon}) = P(x)$  for all  $x \in \mathbb{R}^d$  and  $\varepsilon > 0$ .) We renormalize  $u^{\varepsilon_n}$  by

$$\left( \int_{B(x_n, 1)} |u^{\varepsilon_n}(x)|^2 dx \right)^{1/2} = 1. \quad (60)$$

Up to a subsequence, there holds  $x_n \cdot e_1 \rightarrow l \in \overline{\mathbb{R}}$ . Since the cases  $l = \pm\infty$  are the classical periodic cases, we assume that  $l \in \mathbb{R}$ . We denote  $x_\infty := le_1$ .

The sequence  $u^{\varepsilon_n}(\cdot + x_n)$  is bounded in the space  $L^2(B(0, 1))$  and, by the Cacciopoli estimate, in the space  $H^1(B(0, 1/2))$ . Therefore, up to a subsequence (that we do not relabel), it weakly converges to  $u^*(\cdot + x_\infty) \in H^1(B(0, 1/2))$  and in  $L^2(B(0, 1))$ .

On the one hand, by the De Giorgi-Nash Moser theorem [10, Th. 8.24 p. 202], there exists  $\beta \in (0, 1)$  such that the sequence  $u^{\varepsilon_n}(\cdot + x_n)$  is bounded in  $C^{0, \beta}(B(0, 1/4))$ . By weak convergence, we also have

$$\left( \int_{B(x_n, 1)} |u^{\varepsilon_n}|^2 \right)^{1/2} \geq \left( \int_{B(x_\infty, 1)} |u^*|^2 \right)^{1/2}.$$

Moreover, the quantity  $P(x_n + z) - P(x_n)$  only depends on  $z$  and  $x_n \cdot e_1$  and  $\nabla P(z)$  only depends on  $\text{sign}(z \cdot e_1)$ . As a consequence, one can take the limit  $n \rightarrow +\infty$  in (59). This yields

$$\begin{aligned} & \sup_{x \in B(x_\infty, \theta)} \left| u^*(x) - u^*(x_\infty) - (P(x) - P(x_\infty)) \cdot \int_{B(x_\infty, \theta)} (\nabla P)^{-1} \nabla u^* \right| \\ & \geq \frac{\theta^{1+\gamma}}{2} \left( \int_{B(x_\infty, 1)} |u^*|^2 \right)^{1/2}. \end{aligned} \quad (61)$$

On the other hand, by Lemma 5.6,  $u^*$  satisfies (45) in  $B(x_\infty, 1/2)$ . Therefore, it also satisfies (58). This is in contradiction with (61) (since  $u^*$  cannot be uniformly equal to 0 on  $B(x_\infty, 1/2)$  by (59) and (60)). As a consequence, our supposition (59) was absurd. This establishes the existence of  $\varepsilon_0$  such that (46) is valid for any  $\varepsilon < \varepsilon_0$  and  $x_0 \in \mathbb{R}^d$ .  $\square$

### 6.1.2 Iteration

We iterate Lemma 6.1 to obtain the following:

**Lemma 6.2** (See Lemma 15 in [2]). *Suppose that the matrix  $A$  satisfies Assumptions 1, 2, and 3. Let  $\gamma \in (0, 1)$ . Let  $\theta$  and  $\varepsilon_0$  as in Lemma 6.1. Assume that  $u^\varepsilon$  satisfies (56) in  $B(x_0, 1)$ , for  $x_0 \in \mathbb{R}^d$ , and  $\varepsilon \leq \theta^n \varepsilon_0$ . Then, there exist a constant  $C$  that only depends on  $d$ ,  $\theta$  and  $\mu$ , and a sequence  $\kappa(n) \in \mathbb{R}^d$  such that*

$$\begin{aligned} & \sup_{x \in B(x_0, \theta^{n+1})} \left| u^\varepsilon(x) - u^\varepsilon(x_0) - \varepsilon \left( \chi\left(\frac{x}{\varepsilon}\right) - \chi\left(\frac{x_0}{\varepsilon}\right) \right) \cdot \kappa(n) \right| \\ & \leq \theta^{(1+n)(1+\gamma)} \|u^\varepsilon\|_{L^\infty(B(x_0, 1))}, \end{aligned} \quad (62)$$

$$|\kappa(n)| \leq C \left( \sum_{j=0}^n \theta^{j\gamma} \right) \|u^\varepsilon\|_{L^\infty(B(x_0, 1))}. \quad (63)$$

A central argument of the proof is that the functions  $\chi_j$  are  $A$ -harmonic, so that Lemma 6.1 can be iterated.

*Proof.* We proceed by induction.

If  $n = 0$ , we set

$$\kappa(0) = \int_{B(x_0, \theta)} (\nabla P)^{-1} \cdot \nabla u^\varepsilon.$$

By Lemma 6.1, (62) is satisfied. Moreover, since  $\nabla P$  only takes two values, we have:

$$\begin{aligned} \int_{B(x_0, \theta)} (\nabla P)^{-1} \nabla u^\varepsilon &= \frac{1}{|B(x_0, \theta)|} \left[ \nabla P(-e_1) \cdot \int_{B(x_0, \theta) \cap (\mathbb{R}_- \times \mathbb{R}^{d-1})} \nabla u^\varepsilon \right. \\ & \quad \left. + \nabla P(e_1) \cdot \int_{B(x_0, \theta) \cap (\mathbb{R}_+ \times \mathbb{R}^{d-1})} \nabla u^\varepsilon \right]. \end{aligned}$$

and, by the Stokes' theorem

$$\int_{B(x_0, \theta) \cap (\mathbb{R}_- \times \mathbb{R}^{d-1})} \nabla u^\varepsilon = \int_{\partial(B(x_0, \theta) \cap \mathbb{R}_- \times \mathbb{R}^{d-1})} u^\varepsilon(x) d\vec{S}(x).$$

A similar formula is obtained for the other part of the ball  $B(x_0, \theta) \cap (\mathbb{R}_+ \times \mathbb{R}^{d-1})$ . As a consequence, (63) is satisfied for  $n = 0$ .

We assume now that Lemma 6.2 is true for  $n \geq 0$ . Let  $0 < \varepsilon \leq \theta^{n+1} \varepsilon_0$  and  $u^\varepsilon \in H_{\text{loc}}^1(B(x_0, 1))$  satisfying (56) in  $B(x_0, 1)$ . Applying Lemma 6.2, there

exists  $\kappa_j(n)$  associated to  $u^\varepsilon$  such that (62) and (63) are satisfied. We set  $\tilde{\varepsilon} := \varepsilon\theta^{-n-1} \leq \varepsilon_0$ ,  $\tilde{x}_0 := \theta^{-n-1}x_0$  and

$$v(z) := u^\varepsilon(\theta^{n+1}z) - u^\varepsilon(x_0) - \theta^{n+1}\tilde{\varepsilon} \left( \chi\left(\frac{z}{\tilde{\varepsilon}}\right) - \chi\left(\frac{\tilde{x}_0}{\tilde{\varepsilon}}\right) \right) \cdot \kappa(n). \quad (64)$$

Since the functions  $\chi_j$  are  $A$ -harmonic and by (56), we deduce that the function  $v$  is  $A(\cdot/\tilde{\varepsilon})$ -harmonic in  $B(\tilde{x}_0, 1)$ . Hence, thanks to Lemma 6.1,

$$\begin{aligned} & \sup_{z \in B(\tilde{x}_0, \theta)} \left| v(z) - v(\tilde{x}_0) - \tilde{\varepsilon} \left( \chi\left(\frac{z}{\tilde{\varepsilon}}\right) - \chi\left(\frac{\tilde{x}_0}{\tilde{\varepsilon}}\right) \right) \cdot \int_{B(\tilde{x}_0, \theta)} (\nabla P)^{-1} \cdot \nabla v \right| \\ & \leq \theta^{1+\gamma} \|v\|_{L^\infty(B(\tilde{x}_0, 1))}. \end{aligned} \quad (65)$$

Yet, by the induction hypothesis (62) and by definition (64),

$$\|v\|_{L^\infty(B(\tilde{x}_0, 1))} \leq \theta^{(1+n)(1+\gamma)} \|u^\varepsilon\|_{L^\infty(B(x_0, 1))}. \quad (66)$$

We set

$$\kappa(n+1) := \kappa(n) + \theta^{-n-1} \int_{B(\tilde{x}_0, \theta)} (\nabla P)^{-1} \cdot \nabla v, \quad (67)$$

so that inserting (64) and (67) in (65) and using (66) yields (62) for the  $n+1$ -th step. Moreover, thanks to Stokes' theorem (see above) and to (66),

$$\begin{aligned} |\kappa_j(n+1)| & \leq |\kappa_j(n)| + C\theta^{-n-2} \|v\|_{L^\infty(B(\tilde{x}_0, 1))} \\ & \leq |\kappa_j(n)| + C\theta^{(1+n)\gamma} \|u^\varepsilon\|_{L^\infty(B(x_0, 1))}, \end{aligned}$$

where the constant  $C$  only depends on  $d$  and  $\theta$  (but not on  $n$ ). This proves (63) for the  $n+1$ -th step and concludes the proof of Lemma 6.2.  $\square$

### 6.1.3 Blow-up

We proceed with the last part of the proof of Theorem 4.1.

*Proof of Theorem 4.1.* The proof is done by a blow-up argument, in two steps: the first aims at controlling the oscillation of  $u^\varepsilon$  down to the scale  $\varepsilon$ . It relies on Lemma 6.2 and on the fact that the correctors are strictly sublinear; the second step uses the first step along with the regularity of the operator  $-\operatorname{div}(A(\cdot/\varepsilon) \cdot \nabla)$  at a scale finer than  $\varepsilon$  -the latter being provided by Theorem 5.1.

Without loss of generality, we assume that  $R = 4$  and that  $\varepsilon < \varepsilon_0$ .

**Step 1:** We set  $\gamma = 1/2$ , and obtain  $\varepsilon_0$  and  $\theta$  from Lemma 6.1. Let  $x_1 \in B(x_0, 2) \setminus \mathcal{I}$ . We first show that, if  $1 \geq r \geq \varepsilon/\varepsilon_0$ , there holds

$$\sup_{x \in B(x_1, r)} |u^\varepsilon(x) - u^\varepsilon(x_1)| \leq Cr \|u^\varepsilon\|_{L^\infty(B(x_1, 1))}. \quad (68)$$

We set  $n \in \mathbb{N}$  such that  $\theta^{n+1} \leq r \leq \theta^n$ , and  $x \in B(x_1, r)$ . Thanks to Lemma 6.2, we obtain

$$\begin{aligned} |u^\varepsilon(x) - u^\varepsilon(x_1)| & \leq C \left| \varepsilon \left( \chi\left(\frac{x}{\varepsilon}\right) - \chi\left(\frac{x_1}{\varepsilon}\right) \right) \right| \|u^\varepsilon\|_{L^\infty(B(x_1, 1))} \\ & \quad + \theta^{(1+n)(1+\gamma)} \|u^\varepsilon\|_{L^\infty(B(x_1, 1))}. \end{aligned} \quad (69)$$

By Proposition 3.1, the correctors  $w_j$  are bounded. Therefore, we deduce from the above estimate (69) that

$$|u^\varepsilon(x) - u^\varepsilon(x_1)| \leq C(|x - x_1| + \varepsilon + r^{1+\gamma}) \|u^\varepsilon\|_{L^\infty(B(x_1,1))},$$

which yields (68).

**Step 2:** Let  $v(z) = u^\varepsilon(\varepsilon z/\varepsilon_0) - u^\varepsilon(x_1)$ . By definition, the function  $v$  satisfies

$$-\operatorname{div}(A(z/\varepsilon_0) \cdot \nabla v(z)) = 0 \quad \text{in } B(\varepsilon_0 x_1/\varepsilon, 1).$$

By Theorem 5.1, there exists a constant  $C > 0$  independent of  $\varepsilon$  such that

$$|\nabla v(\varepsilon_0 x_1/\varepsilon)| \leq C \|v\|_{L^\infty(B(\varepsilon_0 x_1/\varepsilon, 1))}.$$

Rescaling the above estimates yields

$$|\nabla u^\varepsilon(x_1)| \leq C \varepsilon^{-1} \varepsilon_0 \|u^\varepsilon\|_{L^\infty(B(x_1, \varepsilon/\varepsilon_0))}.$$

Appealing to (68) applied with  $r := \varepsilon/\varepsilon_0$  and to the De Giorgi-Nash Moser theorem [10, Th. 8.24 p. 202], we deduce that

$$|\nabla u^\varepsilon(x_1)| \leq C \|u^\varepsilon\|_{L^\infty(B(x_1,1))} \leq C \|u^\varepsilon\|_{L^2(B(x_1,2))}.$$

By a covering argument, this implies

$$\sup_{x \in B(x_0, 2) \setminus \mathcal{I}} |\nabla u^\varepsilon(x)| \leq C \|u^\varepsilon\|_{L^2(B(x_0, 4))}$$

and establishes Theorem 4.1.  $\square$

## 6.2 Estimates on the Green function

We prove Corollary 4.2 by appealing to the Lipschitz estimate of Theorem 4.1.

*Proof of Corollary 4.2.* Let  $x \neq y \in \mathbb{R}^d \setminus \mathcal{I}$ . By [12, Th. 1.3] we have  $\mathcal{G}(y, x) = \mathcal{G}^\dagger(y, x)$ , where  $\mathcal{G}^\dagger$  is the Green function associated with the transposed operator  $-\operatorname{div}(A^T \cdot \nabla)$ . Therefore, without loss of generality, it is sufficient to estimate  $\nabla_x \mathcal{G}(x, y)$  in order to establish (28). By definition,  $\mathcal{G}(\cdot, y)$  is  $A$ -harmonic in  $B(x, |x - y|/2)$ :

$$-\operatorname{div}(A \cdot \nabla_x \mathcal{G}(\cdot, y)) = 0 \quad \text{in } B(x, |x - y|/2). \quad (70)$$

Hence, applying Theorem 4.1 and using (27) yields (28) as follows:

$$|\nabla_x \mathcal{G}(x, y)| \leq C|x - y|^{-1} \left( \int_{B(x, |x - y|/2)} |\mathcal{G}(x', y)|^2 dx' \right)^{1/2} \leq C|x - y|^{-d+1}.$$

Finally, differentiating (70) with respect to  $y$  implies that  $\nabla_y \mathcal{G}(\cdot, y)$  is also  $A$ -harmonic in  $B(x, |x - y|/2)$ . Therefore, as a consequence of Theorem 4.1, we obtain

$$|\nabla_x \nabla_y \mathcal{G}(x, y)| \leq C|x - y|^{-1} \left( \int_{B(x, |x - y|/2)} |\nabla_y \mathcal{G}(x', y)|^2 dx' \right)^{1/2},$$

which implies (29), by resorting to (28).  $\square$

## 7 Approximation

In this section, we prove Proposition 4.3, Proposition 4.4, Theorem 4.5 and Corollary 4.6. The proofs of this section follow the strategy of [15]. For simplicity, we denote henceforth the residuum:

$$R^\varepsilon(x) := u^{\varepsilon,1}(x) - u^\varepsilon(x) = u^*(x) + w_i(x/\varepsilon) \partial_i u^*(x) - u^\varepsilon(x). \quad (71)$$

### 7.1 Pointwise approximation

This section is concerned with the proof of the pointwise approximation of the function  $u^\varepsilon$  and of the Green function  $\mathcal{G}$  (that is, Propositions 4.3 and 4.4). The first step is to show a global pointwise estimate on  $|u^\varepsilon(x) - u^*(x)|$ , namely (30). It relies on the identity (23) combined with the estimates on the multiscale Green function and its derivatives provided by Corollary 4.2. Then, by a duality argument (and by rescaling), the first step yields an estimate on  $\|\mathcal{G}(x, \cdot) - \mathcal{G}^*(x, \cdot)\|_{L^{p'}}$  for  $p' < d/(d-1)$ . By establishing a local counterpart of Proposition 4.3, one finally obtains a pointwise estimate on  $|\mathcal{G}(x, y) - \mathcal{G}^*(x, y)|$ .

We proceed with the:

*Proof of Proposition 4.3.* By (23), there holds

$$R^\varepsilon(x) = -\varepsilon \int_{\mathbb{R}^d} \partial_{y_i} \mathcal{G}^\varepsilon(x, y) ((B_{ijk} - A_{ij} w_k)(y/\varepsilon) \partial_j U_k^*(y)) dy, \quad (72)$$

where  $U^*$  is defined by (45). By Propositions 3.1 and 5.5, the quantity  $B_{ijk} - A_{ij} w_k$  is uniformly bounded on  $\mathbb{R}^d$ . Hence, applying the Hölder inequality on (72) for a suitable decomposition of  $\mathbb{R}^d$  and invoking (28) yields

$$\begin{aligned} |R^\varepsilon(x)| &\leq C\varepsilon \left( \int_{|y-x|<2} |\nabla \mathcal{G}^\varepsilon(x, y)|^{\frac{p}{p-1}} dy \right)^{\frac{p-1}{p}} \|\nabla U^*\|_{L^p(\mathbb{R}^d)} \\ &\quad + C\varepsilon \left( \int_{|y-x|>2} |\nabla \mathcal{G}^\varepsilon(x, y)|^2 dy \right)^{\frac{1}{2}} \|\nabla U^*\|_{L^2(\mathbb{R}^d)} \\ &\leq C\varepsilon \left( \int_{|z|<2} |z|^{-\frac{(d-1)p}{p-1}} dz \right)^{\frac{p-1}{p}} \|\nabla U^*\|_{L^p(\mathbb{R}^d)} \\ &\quad + C\varepsilon \left( \int_{|z|>2} |z|^{-2(d-1)} dz \right)^{\frac{1}{2}} \|\nabla U^*\|_{L^2(\mathbb{R}^d)}. \end{aligned} \quad (73)$$

Since  $(d-1)p/(p-1) < d$  and  $2(d-1) > d$ , then the above integrals converge. Moreover, by Lemma 5.2, and since  $f$  is supported in  $B(x_0, 1)$  there holds

$$\|U^*\|_{W^{1,p}(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)} \quad \text{and} \quad \|\nabla U^*\|_{L^2(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}.$$

Therefore, (73) yields

$$\|u^{\varepsilon,1} - u^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leq C\varepsilon \|f\|_{L^p(\mathbb{R}^d)}. \quad (74)$$

Furthermore, by a Sobolev injection (recall that  $p > d$ ), we estimate

$$\|\nabla u^*\|_{L^\infty(\mathbb{R}^d)} \leq C \|U^*\|_{L^\infty(\mathbb{R}^d)} \leq C \|U^*\|_{W^{1,p}(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}. \quad (75)$$

As a consequence of (74) and (75), and since the correctors  $w_j$  are bounded, the definition (18) of  $u^{\varepsilon,1}$  implies that

$$\|u^\varepsilon - u^*\|_{L^\infty(\mathbb{R}^d)} \leq C\varepsilon \|\nabla u^*\|_{L^\infty(\mathbb{R}^d)} + \|u^{\varepsilon,1} - u^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leq C\varepsilon \|f\|_{L^p(\mathbb{R}^d)}.$$

□

We now show a localized version of (30), which is a key step to prove point-wise error estimate on the Green function (27):

**Lemma 7.1** (Adaptation of Lemma 4.2 of [15]). *Assume that  $A$  satisfies Assumptions 1, 2, and 3. Let  $\varepsilon > 0$ ,  $x_0 \in \mathbb{R}^d$ ,  $q \in (1, \infty)$ . Suppose that  $u^\varepsilon, u^* \in H^1(B(x_0, 1))$  satisfies*

$$-\operatorname{div}(A^*(x) \cdot \nabla u^*(x)) = -\operatorname{div}(A(x/\varepsilon) \cdot \nabla u^\varepsilon(x)) \quad (76)$$

in  $B(x_0, 1)$ . Then, there exists a constant  $C$  independent of  $\varepsilon$  so that

$$\|R^\varepsilon\|_{L^\infty(B(x_0, 1/2))} \leq C \|R^\varepsilon\|_{L^q(B(x_0, 1))} + C\varepsilon \|\nabla U^*\|_{L^\infty(B(x_0, 1))}, \quad (77)$$

for  $R^\varepsilon$  and  $U^*$  respectively defined by (71) and by (19).

*Proof.* We decompose  $R^\varepsilon := R_1^\varepsilon + R_2^\varepsilon$  where  $R_1^\varepsilon$  is the zero-mean solution on  $\mathbb{R}^d$  to the following equation:

$$-\operatorname{div}(A(x/\varepsilon) \cdot \nabla R_1^\varepsilon(x)) = \varepsilon \operatorname{div}(H^\varepsilon(x)) \quad \text{and} \quad \nabla R^\varepsilon \in L^2(\mathbb{R}^d, \mathbb{R}^d), \quad (78)$$

and where the vector-valued function  $H^\varepsilon$  is defined by

$$H_i^\varepsilon(x) := \varepsilon \chi_{B(x_0, 1)}(x) (B_{ijk} - A_{ij} w_k)(x/\varepsilon) \partial_j U_k^*(x). \quad (79)$$

By definition

$$R_1^\varepsilon(x) = - \int_{B(x_0, 1)} \nabla_y \mathcal{G}^\varepsilon(x, y) \cdot H^\varepsilon(y) dy + \int_{B(x_0, 1)} \mathcal{G}^\varepsilon(x, y) H^\varepsilon(y) \cdot d\vec{S}(y).$$

As a consequence of (27) and (28), and since the quantity  $B_{ijk} - A_{ij} w_k$  is bounded, there holds

$$\|R_1^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leq C \|H^\varepsilon\|_{L^\infty(B(x_0, 1))} \leq C\varepsilon \|\nabla U^*\|_{L^\infty(B(x_0, 1))}. \quad (80)$$

We now estimate the function  $R_2^\varepsilon$ . By (23) and (78), it satisfies

$$-\operatorname{div}(A(x/\varepsilon) \cdot \nabla R_2^\varepsilon(x)) = 0 \quad \text{in} \quad B(x_0, 1). \quad (81)$$

Hence, by [10, Th. 8.25 p. 202],  $R_2^\varepsilon$  can be estimated as follows

$$\|R_2^\varepsilon\|_{L^\infty(B(x_0, 1/2))} \leq C \|R_2^\varepsilon\|_{L^q(B(x_0, 1))}. \quad (82)$$



Therefore, by applying the triangular inequality and then (80) and (82), we get

$$\begin{aligned} \|R^\varepsilon\|_{L^\infty(B(x_0, 1/2))} &\leq \|R_1^\varepsilon\|_{L^\infty(B(x_0, 1/2))} + \|R_2^\varepsilon\|_{L^\infty(B(x_0, 1/2))} \\ &\leq C\varepsilon \|\nabla U^*\|_{L^\infty(B(x_0, 1))} + C \|R_2^\varepsilon\|_{L^q(B(x_0, 1))}. \end{aligned} \quad (83)$$

The triangular inequality and then (80) yield

$$\begin{aligned} \|R_2^\varepsilon\|_{L^q(B(x_0, 1))} &\leq \|R_1^\varepsilon\|_{L^q(B(x_0, 1))} + \|R^\varepsilon\|_{L^q(B(x_0, 1))} \\ &\leq C\varepsilon \|\nabla U^*\|_{L^\infty(B(x_0, 1))} + \|R^\varepsilon\|_{L^q(B(x_0, 1))}. \end{aligned} \quad (84)$$

As a consequence, we obtain (77) by combining (83) and (84).  $\square$

Proposition 4.4 is then obtained by a duality argument involving Proposition 4.3 coupled with the local  $L^\infty$  estimate of Lemma 7.1:

*Proof of Proposition 4.4.* If  $|x-y| < 1$ , then the result is deduced by a triangular inequality and by (27). Hence, we restrict to the case  $|x-y| > 1$ .

On the one hand, by Proposition 4.3 (used with a scaling argument), for all  $f \in L^p(\mathbb{R}^d)$ , for  $p > d$ , with support inside  $B(y, |x-y|/2)$ , there holds

$$\begin{aligned} |u(x) - u^*(x)| &= \left| \int_{B(y, \frac{|x-y|}{2})} (\mathcal{G}(x, z) - \mathcal{G}^*(x, z)) f(z) dz \right| \\ &\leq C|x-y|^{1-\frac{d}{p}} \|f\|_{L^p(\mathbb{R}^d)}, \end{aligned}$$

where  $u$  and  $u^*$  are respectively the zero-mean solutions to (2) (with  $\varepsilon = 1$ ) and (4). Hence, by duality,

$$\left( \int_{B(y, \frac{|x-y|}{2})} |\mathcal{G}(x, z) - \mathcal{G}^*(x, z)|^{\frac{p}{p-1}} dz \right)^{\frac{p-1}{p}} \leq C|x-y|^{-d+1}, \quad (85)$$

which scales like (31), but involves a weaker norm.

On the other hand, by [12, Th. 1.3] the functions  $\mathcal{G}(x, \cdot)$  and  $\mathcal{G}^*(x, \cdot)$  are respectively  $A^T$ -harmonic and  $(A^*)^T$ -harmonic. Therefore, by Lemma 7.1 and by (85), there holds

$$|\mathcal{G}(x, y) - \mathcal{G}^*(x, y)| \leq C|x-y|^{-d+1} + C|x-y|^2 \|\nabla U^*\|_{L^\infty(B(y, \frac{|x-y|}{2}))}, \quad (86)$$

for  $U^*$  defined by

$$U^*(y) := (\nabla P^\dagger(y))^{-1} \cdot \nabla_y \mathcal{G}^*(x, y).$$

By applying [17, Prop. 1.7] in a ball  $B(y, |x-y|/2)$ , in which  $\mathcal{G}^*(x, \cdot)$  is  $(A^*)^T$ -harmonic, there holds

$$\|\nabla U^*\|_{L^\infty(B(y, \frac{|x-y|}{2}))} \leq C|x-y|^{-2} \|\mathcal{G}^*\|_{L^\infty(B(y, \frac{3|x-y|}{4}))} \leq C|x-y|^{-d}.$$

Injecting the above inequality in (86) yields (31).  $\square$

## 7.2 Pointwise approximation of the gradient

In this Section, we approximate the gradients  $\nabla_x \mathcal{G}$  and  $\nabla_x \nabla_y \mathcal{G}$  of the multi-scale Green function by means of the two-scale expansion applied on  $\mathcal{G}^*$  (*i.e.* Theorem 4.5). It relies on Lemma 7.2 below, which estimates the gradient of the residuum associated with locally  $A(\cdot/\varepsilon)$ -harmonic functions. Applying it on the Green function and invoking Proposition 4.4 yields (34). Iterating once more the same reasoning, we obtain (35). Finally, Corollary 4.6 is a consequence of (34) and of the Hölder inequality (with a small technical argument required by the non-integrability in  $x = y$  of the right-hand side of (34)).

Theorem 4.5 relies on the following:

**Lemma 7.2.** *Let  $d \geq 3$ ,  $x_0 \in \mathbb{R}^d \setminus \mathcal{I}$  and  $\varepsilon > 0$ . Suppose that the matrix  $A$  satisfies Assumptions 1, 2, and 3. Suppose that  $u^\varepsilon$  and  $u^*$  are respectively  $A(\cdot/\varepsilon)$ -harmonic and  $A^*$ -harmonic in  $B(x_0, 2)$ . Then, there exists a constant  $C > 0$  depending only on  $A$  and  $d$  such that*

$$\begin{aligned} \|\nabla u^\varepsilon - W(\cdot/\varepsilon) \cdot \nabla u^*\|_{L^\infty(B(x_0, 1/2))} &\leq C \|u^\varepsilon - u^*\|_{L^\infty(B(x_0, 2))} \\ &\quad + C\varepsilon \ln(2 + \varepsilon^{-1}) \|u^*\|_{L^\infty(B(x_0, 2))}, \end{aligned} \quad (87)$$

where  $W$  is defined by (32).

The proof is divided in four steps.

The first step concerns the case where  $x \in B(x_0, 1/2)$  is far from the interface: we suppose  $\text{dist}(x, \mathcal{I}) \geq \delta$  (where  $\delta \in (0, \varepsilon/2)$  will be fixed at the end of the proof). We define  $R^\varepsilon$  by (71). In this case, thanks to the estimates on the Green function provided by Corollary 4.2 combined with the identity (23), we show that

$$\begin{aligned} |\nabla R^\varepsilon(x)| &\leq C \|R^\varepsilon\|_{L^\infty(B(x_0, 1))} + C\varepsilon |\ln(\delta)| \|\nabla U^*\|_{L^\infty(B(x_0, 1))} \\ &\quad + C\varepsilon \delta \|\nabla^2 U^*\|_{L^\infty(B(x, \delta))}. \end{aligned} \quad (88)$$

This step closely follows the proof of [15, Lem. 3.5]. However, two points should be underlined: First, the function  $\nabla^2 U^*$  might involve a singular measure supported on  $\mathcal{I}$ , so that it is necessary to assume that  $\text{dist}(x, \mathcal{I}) \geq \delta$ . Second, we shall play with the extra parameter  $\delta$  (not present in [15, Lem. 3.5]) to get sufficiently close to the interface  $\mathcal{I}$  (the salient point is that right-hand side of (88) blows up very slowly when  $\delta \rightarrow 0$ ). The second step is concerned with  $x \in B(x_0, 1/2)$  close to the interface (*i.e.* at a distance smaller than  $\delta$ ). Then we use a regularity result at the scale  $\varepsilon$  (namely Theorem 5.1) to compare  $\nabla R^\varepsilon(x)$  with  $\nabla R^\varepsilon(x')$ , for  $x'$  farther from the interface. Appealing to the previous step for  $x'$  and using a triangular inequality provides the desired bound. In the third step, we estimate the derivatives of  $U^*$  in (88) by invoking the regularity results of [17]. Finally, in the fourth step, we choose an optimal parameter  $\delta$  and establish (87) by means of the two previous steps.

*Proof.* Without loss of generality, we assume that  $\varepsilon < 1/8$ . Let  $x \in B(x_0, 1/2) \setminus \mathcal{I}$ . The parameter  $\delta \in (0, \varepsilon/2)$  will be set in Step 4.

**Step 1: Estimates far from the interface** In this step, we assume that the distance  $\text{dist}(x, \mathcal{I})$  between  $x$  and the interface  $\mathcal{I}$ , is larger than  $\delta$  and we show (88). As in the proof of Lemma 7.1, we decompose  $R^\varepsilon := R_1^\varepsilon + R_2^\varepsilon$  where  $R_1^\varepsilon$  is the solution on  $\mathbb{R}^d$  to (78) and  $R_2^\varepsilon$  solves (81).

On the one hand, by Theorem 4.1, there holds

$$\|\nabla R_2^\varepsilon\|_{L^\infty(B(x_0, 1/4))} \leq C \|R_2^\varepsilon\|_{L^2(B(x_0, 1/2))}.$$

Whence, by triangular inequality, and by appealing to (80),

$$\begin{aligned} \|\nabla R_2^\varepsilon\|_{L^\infty(B(x_0, 1/4))} &\leq C \|R^\varepsilon\|_{L^\infty(B(x_0, 1/2))} + C \|R_1^\varepsilon\|_{L^\infty(B(x_0, 1/2))} \\ &\leq C \|R^\varepsilon\|_{L^\infty(B(x_0, 1))} + C\varepsilon \|\nabla U^*\|_{L^\infty(B(x_0, 1))}. \end{aligned} \quad (89)$$

On the other hand, by (78), there holds

$$\begin{aligned} \nabla R^{\varepsilon, 1}(x) &= \int_{\partial(B(x_0, 1))} \nabla_x \mathcal{G}^\varepsilon(x, y) (H^\varepsilon(y) - H^\varepsilon(x)) \cdot d\vec{S}(y) \\ &\quad - \int_{B(x_0, 1)} \nabla_x \nabla_y \mathcal{G}^\varepsilon(x, y) \cdot (H^\varepsilon(y) - H^\varepsilon(x)) dy, \end{aligned} \quad (90)$$

where the vector-valued function  $H^\varepsilon$  is defined by (79). The first integral of (90) is easily bounded thanks to (28):

$$\left| \int_{\partial(B(x_0, 1))} \nabla_x \mathcal{G}^\varepsilon(x, y) (H^\varepsilon(y) - H^\varepsilon(x)) \cdot d\vec{S}(y) \right| \leq C \|H^\varepsilon\|_{L^\infty(B(x_0, 1))}.$$

By resorting to (29), we estimate the second integral in (90).

$$\begin{aligned} &\left| \int_{B(x_0, 1)} \nabla_x \nabla_y \mathcal{G}^\varepsilon(x, y) \cdot (H^\varepsilon(y) - H^\varepsilon(x)) dy \right| \\ &\leq \left| \int_{B(x_0, 1)} |x - y|^{-d} \cdot (H^\varepsilon(y) - H^\varepsilon(x)) dy \right|. \end{aligned}$$

We cut the ball  $B(x_0, 1) = B(x, \delta) \cup (B(x_0, 1) \setminus B(x, \delta))$ . On the small ball, we use the Hölder regularity of  $H^\varepsilon$ , and on the remaining part, we use the  $L^\infty$  norm of  $H^\varepsilon$ :

$$\begin{aligned} &\left| \int_{B(x_0, 1)} \nabla_x \nabla_y \mathcal{G}^\varepsilon(x, y) \cdot (H^\varepsilon(y) - H^\varepsilon(x)) dy \right| \\ &\leq C \left| \int_{B(x, \delta)} |x - y|^\alpha |x - y|^{-d} dy \right| \sup_{y \in B(x, \delta)} \frac{|H^\varepsilon(y) - H^\varepsilon(x)|}{|y - x|^\alpha} \\ &\quad + C \left| \int_{B(x_0, 1) \setminus B(x, \delta)} |y - x|^{-d} dy \right| \|H^\varepsilon\|_{L^\infty(B(x_0, 1))} \\ &\leq C \delta^\alpha \sup_{y \in B(x, \delta)} \frac{|H^\varepsilon(y) - H^\varepsilon(x)|}{|y - x|^\alpha} + C |\ln(\delta)| \|H^\varepsilon\|_{L^\infty(B(x_0, 1))}. \end{aligned}$$

Now, by Propositions 3.1 and 5.5, there holds

$$\|H^\varepsilon\|_{L^\infty(B(x_0, 1))} \leq C\varepsilon \|\nabla U^*\|_{L^\infty(B(x_0, 1))} \quad (91)$$

and (recall that  $\delta < \varepsilon$ ):

$$\begin{aligned}
& \delta^\alpha \sup_{y \in \mathbb{B}(x, \delta)} \frac{|H^\varepsilon(y) - H^\varepsilon(x)|}{|y - x|^\alpha} \\
& \leq C\varepsilon^{1-\alpha} \delta^\alpha \|\nabla U^*\|_{L^\infty(\mathbb{B}(x_0, 1))} + C\varepsilon \delta^\alpha \sup_{y \in \mathbb{B}(x, \delta)} \frac{|\nabla U^*(y) - \nabla U^*(x)|}{|y - x|^\alpha} \\
& \leq C\varepsilon \|\nabla U^*\|_{L^\infty(\mathbb{B}(x_0, 1))} + C\varepsilon \delta \|\nabla^2 U^*\|_{L^\infty(\mathbb{B}(x, \delta))}. \tag{92}
\end{aligned}$$

As a consequence,

$$|\nabla R^{\varepsilon, 1}(x)| \leq C\varepsilon |\ln(\delta)| \|\nabla U^*\|_{L^\infty(\mathbb{B}(x_0, 1))} + C\varepsilon \delta \|\nabla^2 U^*\|_{L^\infty(\mathbb{B}(x, \delta))}, \tag{93}$$

and, by a triangular inequality involving (89) and (93), we show (88).

**Step 2: Estimates close to the interface** Assume that  $\text{dist}(x, \mathcal{I}) \leq \delta$ . We set  $\beta < \min(\alpha, 1/4)/2$ . Without loss of generality, we assume that  $x \cdot e_1 < 0$  and denote by  $\pi_r(x)$  the orthogonal projection of  $x$  on  $-re_1 + \mathcal{I}$ .

By a rescaling argument, one can apply Theorem 5.1 for  $R^\varepsilon$  on  $\mathbb{B}(\pi_\varepsilon(x), 2\varepsilon)$ . Thus, there exists a constant  $C$  independent of  $\delta$  such that for all  $y \neq z \in \mathbb{B}(\pi_\varepsilon(x), 2\varepsilon) \setminus \mathcal{I}$  such that  $y \cdot e_1$  and  $z \cdot e_1$  have the same sign:

$$\begin{aligned}
\frac{|\nabla R^\varepsilon(x) - \nabla R^\varepsilon(\pi_\delta(x))|}{|x - \pi_\delta(x)|^\beta} & \leq C\varepsilon^{-1-\beta} \|R^\varepsilon\|_{L^\infty(\mathbb{B}(\pi_\varepsilon(x), 2\varepsilon))} + C \frac{|H^\varepsilon(z) - H^\varepsilon(y)|}{|z - y|^\beta} \\
& \quad + C\varepsilon^{-\beta} \|H^\varepsilon\|_{L^\infty(\mathbb{B}(\pi_\varepsilon(x), 2\varepsilon))}.
\end{aligned}$$

By a reasoning similar to the one producing (92) (with  $\delta := \varepsilon$ ), we deduce that

$$\begin{aligned}
& \|H^\varepsilon\|_{L^\infty(\mathbb{B}(\pi_\varepsilon(x), 2\varepsilon))} + \varepsilon^\beta \frac{|H^\varepsilon(z) - H^\varepsilon(y)|}{|z - y|^\beta} \\
& \leq C\varepsilon \|\nabla U^*\|_{L^\infty(\mathbb{B}(\pi_\varepsilon(x), 2\varepsilon))} + C\varepsilon^2 \sup_{y \in \mathbb{B}(\pi_\varepsilon(x), 2\varepsilon) \setminus \mathcal{I}} |\nabla^2 U^*(y)|,
\end{aligned}$$

(where we underline that  $y \cdot e_1$  and  $z \cdot e_1$  have the same sign). Therefore,

$$\begin{aligned}
|\nabla R^\varepsilon(x) - \nabla R^\varepsilon(\pi_\delta(x))| & \leq C\delta^\beta \varepsilon^{-1-\beta} \|R^\varepsilon\|_{L^\infty(\mathbb{B}(x_0, 1))} \\
& \quad + C\delta^\beta \varepsilon^{1-\beta} \|\nabla U^*\|_{L^\infty(\mathbb{B}(\pi_\varepsilon(x), 2\varepsilon))} \\
& \quad + C\delta^\beta \varepsilon^{2-\beta} \sup_{y \in \mathbb{B}(\pi_\varepsilon(x), 2\varepsilon) \setminus \mathcal{I}} |\nabla^2 U^*(y)|. \tag{94}
\end{aligned}$$

Hence, invoking (88) for  $\pi_\delta(x)$ , by a triangular inequality, we get

$$\begin{aligned}
|\nabla R^\varepsilon(x)| & \leq C(1 + \delta^\beta \varepsilon^{-1-\beta}) \|R^\varepsilon\|_{L^\infty(\mathbb{B}(x_0, 1))} \\
& \quad + C\varepsilon(\delta^\beta \varepsilon^{-\beta} + |\ln(\delta)|) \|\nabla U^*\|_{L^\infty(\mathbb{B}(x_0, 1))} \\
& \quad + C\varepsilon^2(1 + \delta^\beta \varepsilon^{-\beta}) \sup_{y \in \mathbb{B}(x_0, 1) \setminus \mathcal{I}} |\nabla^2 U^*(y)|. \tag{95}
\end{aligned}$$

**Step 3: Estimates on  $U^*$**  We collect two useful estimates concerning  $\nabla U^*$  and  $\nabla^2 U^*$ . By [17, Prop. 1.7], we have

$$\sup_{x \in B(x_0, 1) \setminus \mathcal{I}} [|\nabla^2 u^*(x)| + |\nabla^3 u^*(x)|] \leq C \|u^*\|_{L^\infty(B(x_0, 2))}. \quad (96)$$

By definition (19),  $U^*$  is continuous through the interface  $\mathcal{I}$  and there holds

$$\|\nabla U^*\|_{L^\infty(B(x_0, 1))} \leq C \|u^*\|_{L^\infty(B(x_0, 2))} \quad (97)$$

$$\sup_{x \in B(x_0, 1) \setminus \mathcal{I}} \|\nabla^2 U^*\| \leq C \|u^*\|_{L^\infty(B(x_0, 2))}. \quad (98)$$

**Step 4: Conclusion** From Steps 1 and 2, we know that Estimate (95) is satisfied for any  $x \in B(x_0, 1/2) \setminus \mathcal{I}$ . Invoking (97) and (98), it implies that, for any  $x \in B(x_0, 1/2) \setminus \mathcal{I}$ , there holds

$$\begin{aligned} |\nabla R^\varepsilon(x)| &\leq C(1 + \delta^\beta \varepsilon^{-1-\beta}) \|R^\varepsilon\|_{L^\infty(B(x_0, 1))} \\ &\quad + C\varepsilon(\delta^\beta \varepsilon^{-\beta} + |\ln(\delta)|) \|u^*\|_{L^\infty(B(x_0, 2))}. \end{aligned} \quad (99)$$

Recall that  $\delta \in (0, \varepsilon)$  is still a free parameter. Now, we set  $\delta := \varepsilon^{1/\beta+1}$ . Therefore, (99) yields

$$|\nabla R^\varepsilon(x)| \leq C \|R^\varepsilon\|_{L^\infty(B(x_0, 1))} + C\varepsilon |\ln(\varepsilon)| \|u^*\|_{L^\infty(B(x_0, 2))}. \quad (100)$$

By Proposition 3.1, and then by (97),

$$\|R^\varepsilon - (u^* - u^\varepsilon)\|_{L^\infty(B(x_0, 1))} \leq \varepsilon \|\nabla U^*\|_{L^\infty(B(x_0, 1))} \leq \varepsilon \|u^*\|_{L^\infty(B(x_0, 2))}, \quad (101)$$

so that  $R^\varepsilon$  can be replaced by  $u^\varepsilon - u^*$  in the right-hand side of Estimate (100). Since

$$\nabla R^\varepsilon = [W(\cdot/\varepsilon) \cdot \nabla u^* - \nabla u^\varepsilon] + \varepsilon w_j(\cdot/\varepsilon) \partial_j \nabla U^*,$$

by Proposition 3.1 and by (98), the quantity  $\nabla R^\varepsilon$  in (100) can be replaced by  $W(\cdot/\varepsilon) \cdot \nabla u^* - \nabla u^\varepsilon$  so that we get (87).  $\square$

We are now in a position to proceed with the:

*Proof of Theorem 4.5.* Let  $x \neq y \in \mathbb{R}^d \setminus \mathcal{I}$ . Recall that  $x' \mapsto \mathcal{G}(x', y)$  and  $x' \mapsto \mathcal{G}^*(x', y)$  are respectively  $A$ -harmonic and  $A^*$ -harmonic on  $B(x, |x-y|/2)$ . As a consequence of Lemma 7.2, (87) properly rescaled yields

$$\begin{aligned} &\|\nabla_x \mathcal{G}(\cdot, y) - W \cdot \nabla_x \mathcal{G}^*(\cdot, y)\|_{L^\infty(B(x, |x-y|/4))} \\ &\leq C|x-y|^{-1} \|\mathcal{G}(\cdot, y) - \mathcal{G}^*(\cdot, y)\|_{L^\infty(B(x, |x-y|/2))} \\ &\quad + C|x-y|^{-2} \ln(2 + |x-y|) \|\mathcal{G}^*(\cdot, y)\|_{L^\infty(B(x, |x-y|/2))}. \end{aligned}$$

Since  $\mathcal{G}^*(x, y) \leq C|x-y|^{-d+2}$ , we obtain (34) by invoking (31).

The function  $y' \mapsto \nabla_x \mathcal{G}(x, y')$  (and similarly  $y' \mapsto W(x) \cdot \nabla_x \mathcal{G}^*(x, y')$ ) is  $A^T$ -harmonic (respectively  $(A^*)^T$ -harmonic on  $B(y, |x-y|/2)$ ). Hence, as a consequence of Lemma 7.2, (87) properly rescaled yields

$$\begin{aligned} &\left| \nabla_x \nabla_y \mathcal{G}(x, y) - W(x) \cdot \nabla_x \nabla_y \mathcal{G}^*(x, y) \cdot (W^\dagger)^T(y) \right| \\ &\leq C|x-y|^{-1} \|\nabla_x \mathcal{G}(x, \cdot) - W(x) \cdot \nabla_x \mathcal{G}^*(x, \cdot)\|_{L^\infty(B(x, |x-y|/2))} \\ &\quad + C|x-y|^{-2} \ln(2 + |x-y|) \|W(x) \cdot \nabla_x \mathcal{G}^*(x, \cdot)\|_{L^\infty(B(x, |x-y|/2))}. \end{aligned}$$

By appealing to (34) and then by using a Lipschitz estimate on  $\mathcal{G}^*(x, \cdot)$ , we finally obtain (35).  $\square$

Corollary 4.6 is a consequence of Theorem 4.5 and of the Hölder inequality.

*Proof of Corollary 4.6.* By definition, and since  $f$  is supported inside  $B(x_0, 1)$ , there holds:

$$\begin{aligned} & W(x/\varepsilon) \cdot \nabla u^*(x) - \nabla u^\varepsilon(x) \\ &= \int_{B(x_0, 1)} (W(x/\varepsilon) \cdot \nabla_x \mathcal{G}^*(x, y) - \nabla_x \mathcal{G}^\varepsilon(x, y)) f(y) dy. \end{aligned} \quad (102)$$

We separate  $B(x_0, 1) = B(x, \varepsilon) \cup (B(x_0, 1) \setminus B(x, \varepsilon))$ . On  $B(x_0, 1) \setminus B(x, \varepsilon)$ , the integrand of (102) is estimated thanks to (34) (rescaled by  $\varepsilon$ ). On  $B(x, \varepsilon)$ , the integrand of (102) is dealt with by appealing to (28) and is counterpart for the homogeneous problem. Thus,

$$\begin{aligned} & \left| \int_{B(x_0, 1)} (W(x/\varepsilon) \cdot \nabla_x \mathcal{G}^*(x, y) - \nabla_x \mathcal{G}^\varepsilon(x, y)) f(y) dy \right| \\ & \leq C \left[ \int_{B(x_0, 1) \setminus B(x, \varepsilon)} \frac{\varepsilon \ln(2 + \varepsilon^{-1}|x - y|)}{|x - y|^d} dy + \int_{B(x, \varepsilon)} |y - x|^{-d+1} dy \right] \|f\|_{L^\infty(\mathbb{R}^d)} \\ & \leq \varepsilon |\ln(2 + \varepsilon^{-1})|^2 \|f\|_{L^\infty(\mathbb{R}^d)}. \end{aligned}$$

This establishes (36).  $\square$

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## A Proof of Proposition 5.4

For the sake of clarity, we prove successively and separately points (i), and (ii) of Proposition 5.4. The proof closely follows the proof of [7, Th. 5.1]:

*Proof of Proposition 5.4(i).* The proof consists in three steps. First, we build a function  $v$  that reflects the difference  $w_j(x) - w_j^-(x)$  for  $x \cdot e_1 < 0$  and  $w_j(x) - w_j^+(x) - \check{a}_j w_1^+(x)$  for  $x \cdot e_1 > 0$  by means of a suitable cut-off function. This function  $v$  satisfies an elliptic equation, from which we deduce that  $w_j$  exists and is unique.

**Step 1:** We set a smooth cut-off function  $\phi_+(x)$  only depending on  $x \cdot e_1$  that vanishes on  $\mathbb{R}_- \times \mathbb{R}^{d-1}$  and that is equal to 1 on  $[1, +\infty) \times \mathbb{R}^{d-1}$ , and we define  $\phi_-(x) = \phi_+(-x)$ . Next, we define

$$v(x) = w_j(x) - \phi_+(x) (w_j^+(x) + \check{a}_j w_1^+(x)) - \phi_-(x) w_j^-(x). \quad (103)$$

Therefore, by (14),

$$\begin{aligned} -\operatorname{div}(A \cdot \nabla v) &= \operatorname{div}(A \cdot \nabla P_j) + \operatorname{div}(A \cdot \nabla \{\phi_+(w_j^+ + \check{a}_j w_1^+) + \phi_-(w_j^-)\}) \\ &= \operatorname{div}(f) + \operatorname{div}(g), \end{aligned} \quad (104)$$

where, (by adding  $A_-^* \cdot e_j$ ),

$$\begin{aligned} f &:= (1 - \phi_+ - \phi_-) (A \cdot \nabla P_j - A_-^* \cdot e_j) \\ &\quad + A \cdot \nabla \phi_+ (w_j^+ + \check{a}_j w_1^+) + A \cdot \nabla \phi_- w_j^-, \end{aligned}$$

and, using (11),

$$\begin{aligned} g &:= \phi_+ (A \cdot (e_j + \nabla w_j^+ + \check{a}_j (e_1 + \nabla w_1^+)) - A_-^* \cdot e_j) \\ &\quad + \phi_- (A \cdot (e_j + \nabla w_j^-) - A_-^* \cdot e_j). \end{aligned}$$

**Step 2:** Now, we treat separately  $f$  and  $g$  in (104) to show that  $v$  exists.

Since  $A, A^*, \nabla \phi_\pm$  and  $w^\pm$  are uniformly bounded there holds:

$$\|f\|_{L^\infty(\mathbb{R}^d)} \leq C.$$

Moreover, the support of the D-periodic function  $f$  is inside  $[-1, 1] \times \mathbb{R}^{d-1}$ , whence  $f \in L^2(D, \mathbb{R}^d)$ . Therefore, by the Lax-Milgram theorem, there exists a D-periodic function  $v_1$  such that

$$-\operatorname{div}(A \cdot \nabla v_1) = \operatorname{div}(f) \quad \text{in } \mathbb{R}^d \quad \text{and} \quad \nabla v_1 \in L^2(D, \mathbb{R}^d). \quad (105)$$

We should now build a D-periodic function  $v_2$  satisfying

$$-\operatorname{div}(A \cdot \nabla v_2) = \operatorname{div}(g) \quad \text{in } \mathbb{R}^d \quad \text{and} \quad \nabla v_2 \in L^2(D, \mathbb{R}^d). \quad (106)$$

The function  $g$  satisfies the following properties:

- (i)  $g$  is D-periodic,
- (ii) if  $|x \cdot e_1| > 1$ , then  $\operatorname{div}(g) = 0$ ,
- (iii)  $g$  is  $T_1^+$ -periodic in  $x_1$ , if  $x \cdot e_1 > 1$ , and  $T_1^-$ -periodic in  $x_1$ , if  $x \cdot e_1 < -1$
- (iv) there holds

$$e_1 \cdot \int_{x+[0, T_1^+] \times [0, T_2] \times \dots \times [0, T_d]} g(y) dy = 0 \quad \text{if } x \cdot e_1 > 1, \quad (107)$$

$$e_1 \cdot \int_{x-[0, T_1^-] \times [0, T_2] \times \dots \times [0, T_d]} g(y) dy = 0 \quad \text{if } x \cdot e_1 < -1. \quad (108)$$



The last point (iv) is a consequence of the formula satisfied by the homogenized matrix. We prove below the case  $x \cdot e_1 > 1$ :

$$\begin{aligned} & e_1 \cdot \int_{x+[0, T_1^+] \times [0, T_2] \times \cdots \times [0, T_d]} g(y) dy \\ &= \left( \prod_{j=1}^d T_j \right) \left[ (A_+^*)_{1j} + \check{a}_j (A_+^*)_{11} - (A_-^*)_{1j} \right] = 0 \end{aligned}$$

by definition (10) of  $\check{a}_j$ .

We remark that the proof of [7, Lem. 5.2] only requires Assumptions (i), (ii), (iii), and (iv) above, although its original statement requires that the following equality holds instead of (107):

$$\int_{x+[0, T_1^+] \times [0, T_2] \times \cdots \times [0, T_d]} g(y) dy = 0. \quad (109)$$

Therefore, by a straightforward generalization of [7, Lem. 5.2], there holds

$$e_1 \cdot \int_{\{x \cdot e_1\} \times [0, T_2] \times \cdots \times [0, T_d]} g(y) dy = 0 \quad \text{if } x \cdot e_1 > 1.$$

Hence, the result [7, Lem. 5.3] can be adapted to the less stringent Assumptions (i), (ii), (iii) and (iv) above. As a consequence, there exists a  $D$ -periodic function  $\tilde{g}$  such that

$$-\Delta \tilde{g} = \text{div}(g) \quad \text{and} \quad \nabla \tilde{g} \in L^2(D, \mathbb{R}^d).$$

This turns (106) into

$$-\text{div}(A \cdot \nabla v_2) = -\text{div}(\nabla \tilde{g}).$$

Therefore, by the Lax-Milgram theorem, there exists a  $D$ -periodic solution  $v_2$  to (106).

As a consequence  $v := v_1 + v_2$  is a solution to (14) and satisfies (50).

**Step 3:** Proving uniqueness amounts to showing that if the function  $v$  is  $D$ -periodic and satisfies

$$-\text{div}(A \cdot \nabla v) = 0 \quad \text{in } \mathbb{R}^d \quad \text{and} \quad \nabla v \in L^2(D, \mathbb{R}^d),$$

then  $v$  is a constant function. This fact is a straightforward corollary of the proof of uniqueness in [7, Th. 5.1] (which is similar to the proof of Proposition 5.4 (ii) below).  $\square$

The proof of Proposition 5.4(ii) is a simple adaptation of [7, Th. 5.1].

*Proof of Proposition 5.4 (ii).* We only prove (52), since the proof also applies for (51).

Let  $v$  be defined by (103). Whence,

$$-\text{div}(A \cdot \nabla v) = 0 \quad \text{if } |x \cdot e_1| > 0. \quad (110)$$

Therefore, testing Equation (110) against  $v$  yields, for  $1 < R < R'$ ,

$$\begin{aligned}
& \int_{[R,R'] \times [0,T_2] \times \cdots \times [0,T_d]} A(x) \cdot \nabla v(x) \cdot \nabla v(x) dx \\
&= \int_{\{R'\} \times [0,T_2] \times \cdots \times [0,T_d]} v(x) A(x) \cdot \nabla v(x) \cdot e_1 dx \\
&\quad - \int_{\{R\} \times [0,T_2] \times \cdots \times [0,T_d]} v(x) A(x) \cdot \nabla v(x) \cdot e_1 dx. \tag{111}
\end{aligned}$$

Remark that, by the divergence theorem, the quantity

$$\int_{\{x_1\} \times [0,T_2] \times \cdots \times [0,T_d]} e_1 \cdot A(x) \cdot \nabla v(x) dx$$

does not depend upon  $x_1$ . Therefore, it shall vanish, since  $\nabla v \in L^2(\mathbb{D}, \mathbb{R}^d)$ . Hence, we deduce from (111) that, for any constants  $C_1, C_2 \in \mathbb{R}$ , there holds

$$\begin{aligned}
& \int_{[R,R'] \times [0,T_2] \times \cdots \times [0,T_d]} A(x) \cdot \nabla v(x) \cdot \nabla v(x) dx \\
&= \int_{\{R'\} \times [0,T_2] \times \cdots \times [0,T_d]} (v(x) - C_1) A(x) \cdot \nabla v(x) \cdot e_1 dx \\
&\quad - \int_{\{R\} \times [0,T_2] \times \cdots \times [0,T_d]} (v(x) - C_2) A(x) \cdot \nabla v(x) \cdot e_1 dx.
\end{aligned}$$

By the Cauchy-Schwartz and the Poincaré inequalities, and using ellipticity and boundedness of  $A$ , we obtain

$$\begin{aligned}
\int_{[R,R'] \times [0,T_2] \times \cdots \times [0,T_d]} |\nabla v(x)|^2 dx &\leq C \int_{\{R'\} \times [0,T_2] \times \cdots \times [0,T_d]} |\nabla v(x)|^2 dx \\
&\quad + C \int_{\{R\} \times [0,T_2] \times \cdots \times [0,T_d]} |\nabla v(x)|^2 dx.
\end{aligned}$$

Now, since  $\nabla v \in L^2(\mathbb{D}, \mathbb{R}^d)$ , letting  $R' \rightarrow +\infty$  in the above expression yields

$$\int_{[R,+\infty[ \times [0,T_2] \times \cdots \times [0,T_d]} |\nabla v(x)|^2 dx \leq C \int_{\{R\} \times [0,T_2] \times \cdots \times [0,T_d]} |\nabla v(x)|^2 dx.$$

By the Grönwall Lemma, this implies that there exists constants  $C, \kappa > 0$  such that

$$\int_{[R,+\infty[ \times [0,T_2] \times \cdots \times [0,T_d]} |\nabla v(x)|^2 dx \leq C \exp(-\kappa R).$$

Then, by Schauder regularity [10, Cor. 8.36 p. 212], we finally obtain (52).  $\square$

We finally proceed with the

*Proof of Proposition 5.5.* Recall that, in the periodic case, there exist  $[0, T_1^\pm] \times [0, T_2] \times \cdots \times [0, T_d]$ -periodic potentials  $B_\pm$  associated with  $A_\pm$ , that is, satisfying. They read

$$(B_\pm)_{ijk} = \partial_i (N_\pm)_{jk} - \partial_j (N_\pm)_{ik},$$

where  $N_{\pm}$  are  $[0, T_1^{\pm}] \times [0, T_2] \times \cdots \times [0, T_d]$ -periodic solutions to

$$\Delta(N_{\pm})_{ik} = (A_{\pm}^*)_{ik} - (A_{\pm})_{il} (\delta_{lk} + \partial_l w_k^{\pm}) \quad \text{in } \mathbb{R}^d. \quad (112)$$

Note that, by Schauder regularity, the functions  $N_{\pm}$  belong to  $C_{\text{unif}}^{2,\alpha}(\mathbb{R}^d, \mathbb{R}^{d \times d})$ .

Similarly, if we build a D-periodic function  $N$  satisfying

$$\Delta N_{ik} = A_{il}^* \partial_l P_k - A_{il} (\partial_l P_k + \partial_l w_k), \quad (113)$$

and set  $B_k^{ij} = \partial_i N_k^j - \partial_j N_k^i$ , then  $B$  satisfies (22) (recall that the right-hand side of (113) is divergence-free, in the sense of (21)). Building such a function  $N$  is the goal of what follows.

We proceed in the same manner as in the proof of Proposition 5.4 by using techniques of [7]. We decompose

$$N = \phi_+ N_+ \cdot \nabla P + \phi_- N_- \cdot \nabla P + \tilde{N}. \quad (114)$$

Recall that  $\nabla P$  is piecewise constant and possibly discontinuous only across the interface, where  $\phi_{\pm}$  vanishes. Hence, by definition,

$$\begin{aligned} \Delta \tilde{N} &= \Delta N - \phi_+ \Delta N_+ \cdot \nabla P - \phi_- \Delta N_- \cdot \nabla P \\ &\quad - 2(\nabla \phi_+ \cdot \nabla N_+ \cdot \nabla P + \nabla \phi_- \cdot \nabla N_- \cdot \nabla P) \\ &\quad - \Delta \phi_+ N_+ \cdot \nabla P - \Delta \phi_- N_- \cdot \nabla P. \end{aligned} \quad (115)$$

Using (112) yields

$$\begin{aligned} &\Delta N_{ij} - \phi_+ \Delta((N_+)_{ik}) \partial_k P_j - \phi_- \Delta((N_-)_{ik}) \partial_k P_j \\ &= (1 - \phi_+ - \phi_-) (A_{ik}^* \partial_k P_j - A_{ik} (\partial_k P_j + \partial_k w_j)) \\ &\quad + \phi_+ A_{ik} (\partial_k w_l^+ \partial_l P_j - \partial_k w_j) + \phi_- A_{ik} (\partial_k w_l^- \partial_l P_j - \partial_k w_j). \end{aligned}$$

As a consequence, the right-hand term of (115) is D-periodic and bounded. Moreover, it is in  $L^1(\mathbb{D})$ . Indeed, the functions  $(1 - \phi_+ - \phi_-)$  and  $\nabla \phi_{\pm}$  are supported in  $[-1, 1] \times \mathbb{R}^{d-1}$  and, by Proposition 5.4, the quantities

$$(\partial_k w_l^{\pm} \partial_l P_j - \partial_k w_j)$$

decrease exponentially when  $x \cdot e_1 \rightarrow \pm\infty$  (we recall the formula (11) for the gradient  $\nabla P$ ). Hence the right-hand side of (115) is bounded in all  $L^p$  for  $p \in [1, +\infty]$ . Therefore, by the Lax-Milgram theorem, there exists a D-periodic solution  $\tilde{N}$  to (115) so that  $\nabla \tilde{N} \in L^2(\mathbb{D}, \mathbb{R}^{d \times d \times d})$ . Moreover, by elliptic regularity (see [10, Th. 8.32]), for any  $\beta \in (0, 1)$ , there holds  $\nabla \tilde{N} \in C_{\text{unif}}^{0,\beta}(\mathbb{R}^d, \mathbb{R}^{d^3})$ .

As a conclusion, we have built a D-periodic potential  $B$  that is  $\beta$ -Hölder continuous, for any  $\beta \in (0, 1)$ .  $\square$