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Estimation of multivariate asymmetric power GARCH models

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Abstract

It is now widely accepted that volatility models have to incorporate the so-called leverage effect in order to model the dynamics of daily financial returns. We suggest a new class of multivariate power transformed asymmetric models. It includes several functional forms of multivariate GARCH models which are of great interest in financial modeling and time series literature. We provide an explicit necessary and sufficient condition to establish the strict stationarity of the model. We derive the asymptotic properties of the quasi-maximum likelihood estimator of the parameters. These properties are established both when the power of the transformation is known or is unknown. The asymptotic results are illustrated by Monte Carlo experiments. An application to real financial data is also proposed.

Key words: Constant conditional correlation, multivariate asymmetric power GARCH models, quasi-maximum likelihood, threshold models

1. Introduction

The ARCH (AutoRegressive Conditional Heteroscedastic) model has been introduced by [Engle \(1982\)](#) in an univariate context. Since this work a lot of extensions have been proposed. A first one has been suggested four years later, namely the GARCH (Generalised ARCH) model by [Bollerslev \(1986\)](#). This model had for goal to improve modeling by considering the past conditional variance (volatility). Their concept are based on the past conditional heteroscedasticity which depends on the past values of the return. A consequence is the volatility has the same magnitude for a negative or positive return.

Financial series have their own characteristics which are usually difficult to reproduce artificially. An important characteristic is the leverage effect which consider negative returns differently than the positive returns. This is in contradiction with the construction of the GARCH model, because it cannot consider the asymmetry. The TGARCH (Threshold GARCH) model introduced by [Rabemananjara and Zakoïan \(1993\)](#) improve modeling because it considers the asymmetry since the volatility is determined by the past negative observations and the past positive observations with different weights. Various asymmetric GARCH processes are introduced in the econometric literature, for instance the EGARCH (Exponential GARCH) and the log-GARCH models (see [Francq et al. \(2013\)](#) who studied the asymptotic properties of an EGARCH(1, 1) models).

The standard GARCH model is based on the concept that the conditional variance is a linear function of the squared past innovations whereas the TGARCH model is based on the concept of conditional standard deviation (also called volatility). In reality, as mentioned by [Engle \(1982\)](#), other formulations (or several functional forms) of the volatility can be more appropriate. Motivated by Box-Cox power transformations, [Higgins and Bera \(1992\)](#) proposed a general functional form for the ARCH

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model, the so-called PARCH (power-transformed ARCH) models in which the conditional variance is modeled by a power δ (see also [Ding et al. \(1993\)](#)). [Hwang and Kim \(2004\)](#) extend the PARCH models to the class of asymmetric models: the power transformed asymmetric (threshold) ARCH (APARCH for short) model. [Pan et al. \(2008\)](#) generalized the APARCH (APGARCH) model by adding the past realizations of the volatility. [Hamadeh and Zakoian \(2011\)](#) studied the asymptotic properties of the APGARCH models.

In practice, estimate real dataset by the APGARCH model gives important results concerning the power. In fact, as we can see on the [Table 1](#), the power is not necessary equal to 1 or 2 and is different for each series.

Table 1: Estimation of APGARH(1, 1) model for real dataset. We consider the series of the returns of the daily exchange rates of the Dollar (USD), the Yen (JPY), the Pounds (GBP) and the Canadian Dollar (CAD) with respect to the Euro. The observations cover the period from January 4, 1999 to July 13, 2017 which correspond to 4746 observations. The data were obtained from the web site of the National Bank of Belgium.

Exchange rates	$\hat{\omega}$	$\hat{\alpha}^+$	$\hat{\alpha}^-$	$\hat{\beta}$	$\hat{\delta}$
USD	0.00279	0.02618	0.04063	0.96978	1.04728
JPY	0.00740	0.05331	0.08616	0.93580	1.12923
GBP	0.00240	0.06078	0.06337	0.94330	1.41851
CAD	0.00416	0.04054	0.03111	0.96114	1.56085

In the econometric applications, the univariate APGARCH framework is very restrictive. Despite the fact that the volatility of univariate series has been widely studied in the literature, modeling the realizations of several series is of great practical importance. When several series displaying temporal dependencies are available, it is useful to analyze them jointly, by viewing them as the components of a vector-valued (multivariate) process. Contrarily to the class of Vector Auto Regressive Moving Average (VARMA), there is no natural extension of GARCH models for vector series, and many MGARCH (multivariate GARCH) formulations are presented in the literature (see for instance [Nelson \(1991\)](#), [Engle and Kroner \(1995\)](#), [Engle \(2002\)](#) and [McAleer et al. \(2008\)](#)). See also [Bauwens et al. \(2006\)](#), [Silvennoinen and Teräsvirta \(2009\)](#) and [Bauwens et al. \(2012\)](#) for recent surveys on MGARCH processes. These extensions entail numerous specific problems such that identifiability conditions, estimation, . . . Among the numerous specifications of MGARCH models, the most popular seem to be the Constant Conditional Correlations (CCC) model introduced by [Bollerslev \(1990\)](#) and extended by [Jeantheau \(1998\)](#) (denoted CCC-GARCH, in the sequel).

As mentioned before, to model the dynamics of daily financial returns, we need to incorporate the leverage effect to volatility models. Many asymmetric univariate GARCH models have been considered in the literature to capture the leverage effect, extensions to the multivariate setting have not been much developed. To our knowledge, notable exceptions are [McAleer et al. \(2009\)](#) who extend the GJR model (see [Glosten et al. \(1993\)](#)) to the CCC Asymmetric GARCH. Another extension is the Generalized Autoregressive Conditional Correlation (GARCC) model proposed by [McAleer et al. \(2008\)](#). Recently [Francq and Zakoian \(2012\)](#) proposed an asymmetric CCC-GARCH (CCC-AGARCH) model that includes the CCC-GARCH introduced by [Bollerslev \(1990\)](#) and its generalization by [Jeantheau \(1998\)](#). The attractiveness of the CCC-AGARCH models follows from their tractability (see [Francq and Zakoian \(2012\)](#)). [Sucarrat et al. \(2016\)](#) proposed a general framework for the estimation and inference in univariate and multivariate log-GARCH-X models (when covariates or other conditioning variables "X" are added to the volatility equation) with Dynamic Conditional Correlations of unknown form via the VARMA-X representation (Vector Auto Regressive Moving Average).

The main purpose of this paper is to introduce and study a multivariate version of the APGARCH models. In view of the results summarized in Table 1, it appears to be inadequate to consider an unique power for all the m series. Hence we propose the CCC power transformed asymmetric (threshold) GARCH (denoted CCC-APGARCH or $\underline{\delta}_0$ -CCC-GARCH, where $\underline{\delta}_0$ is a m -vector of powers and m is the number of series considered). Our model includes for examples the CCC-AGARCH developed by Francq and Zakoïan (2012) and of course, the most classical MGARCH model, the CCC-GARCH introduced by Bollerslev (1990). An important feature is that the interpretation of the coefficients and the conditional variance is simpler and explicit. We shall give a necessary and sufficient condition for the existence of a strictly stationary solution of the proposed model and we study the problem of estimation of the CCC-APGARCH.

For the estimation of GARCH and MGARCH models, the commonly used estimation method is the quasi-maximum likelihood estimation (QMLE for short). The asymptotic distribution of the Gaussian QMLE is obtained for a wide class of asymmetric GARCH models with exogenous covariates by Francq and Thieu (2018). The asymptotic theory of MGARCH models are well-known in the literature. For instance Jeantreau (1998) gave general conditions for the strong consistency of the QMLE for multivariate GARCH models. Comte and Lieberman (2003) (see also Hafner and Preminger (2009)) have proved the consistency and the asymptotic normality of the QMLE for the BEKK formulation (the acronym comes from synthesized work on multivariate models by Baba, Engle, Kraft and Kroner). Asymptotic results were established by Ling and McAleer (2003) for the CCC formulation of an ARMA-GARCH. See also Bardet and Wintenberger (2009) who studied the asymptotic behavior of the QMLE in a general class of multidimensional causal processes allowing asymmetries. Recently, the quasi-maximum likelihood (QML) results have been established for a MGARCH with stochastic correlations by Francq and Zakoïan (2016) under the assumption that the system is estimable equation-by-equation. See also Darolles et al. (2018), who proved the asymptotic properties of the QML equation-by-equation estimator of Cholesky GARCH models and time-varying conditional betas. Francq and Sucarrat (2017) prove the consistency and asymptotic normality of a least squares equation-by-equation estimator of a multivariate log-GARCH-X model with Dynamic Conditional Correlations by using the VARMA-X representation. Strong consistency and asymptotic normality of CCC-Periodic-GARCH models are established by Bibi (2018). The asymptotic normality of maximum-likelihood estimator of Dynamic Conditional Beta is proved by Engle (2016). In our context, we use the quasi-maximum likelihood estimation. The proofs of our results are quite technical. These are adaptations of the arguments used in Francq and Zakoïan (2012) when the power is known, Hamadeh and Zakoïan (2011) and Pan et al. (2008) when the power is unknown.

This paper is organized as follows. In Section 2 we introduce the CCC-APGARCH model and show that it includes some class of (M)GARCH models. We established the strict stationarity condition and we give an identifiability condition. Section 3 is devoted to the asymptotic properties of the quasi-maximum likelihood estimation when the power $\underline{\delta}_0$ is known. In Section 4, we consider the estimation of $\underline{\delta}_0$. Wald test is developed in Section 5 in order to test the classical MGARCH model against a class of asymmetric MGARCH models. The test can also be used to test the equality between the components of $\underline{\delta}_0$. Simulation studies and an illustrative application on real data are presented in Section 6 and we provide a conclusion in Section 7. The proofs of the main results are collected in the Appendix A.

2. Model and strict stationarity condition

In all this work, we use the following notation $\underline{u}^{\underline{d}} := (u_1^{d_1}, \dots, u_m^{d_m})'$ for $\underline{u}, \underline{d} \in \mathbb{R}^m$.

2.1. Model presentation

The m -dimensional process $\underline{\varepsilon}_t = (\varepsilon_{1,t}, \dots, \varepsilon_{m,t})'$ is called a CCC-APGARCH(p, q) if it verifies

$$\begin{cases} \underline{\varepsilon}_t = H_t^{1/2} \eta_t, \\ H_t = D_t R_0 D_t, \quad D_t = \text{diag}(\sqrt{h_{1,t}}, \dots, \sqrt{h_{m,t}}), \\ \underline{h}_t^{\delta_0/2} = \underline{\omega}_0 + \sum_{i=1}^q \left\{ A_{0i}^+(\underline{\varepsilon}_{t-i}^+)^{\delta_0/2} + A_{0i}^-(\underline{\varepsilon}_{t-i}^-)^{\delta_0/2} \right\} + \sum_{j=1}^p B_{0j} \underline{h}_{t-j}^{\delta_0/2}, \end{cases} \quad (2.1)$$

where $\underline{h}_t = (h_{1,t}, \dots, h_{m,t})'$ and with $x^+ = \max(0, x)$ and $x^- = \min(0, x)$

$$\underline{\varepsilon}_t^+ = \left(\{\varepsilon_{1,t}^+\}^2, \dots, \{\varepsilon_{m,t}^+\}^2 \right)', \quad \underline{\varepsilon}_t^- = \left(\{-\varepsilon_{1,t}^-\}^2, \dots, \{-\varepsilon_{m,t}^-\}^2 \right)',$$

$\underline{\omega}_0$ and $\underline{\delta}_0$ are vectors of size $m \times 1$ with strictly positive coefficients, A_{0i}^+, A_{0i}^- and B_{0j} are matrices of size $m \times m$ with positive coefficients and R_0 is a correlation matrix. The parameters of the model are the coefficients of the vectors $\underline{\omega}_0, \underline{\delta}_0$, the coefficients of the matrices $A_{0i}^+, A_{0i}^-, B_{0j}$ and the coefficients in the lower triangular part excluding the diagonal of the matrix R_0 . The number of unknown parameters is

$$s_0 = 2m + m^2(2q + p) + \frac{m(m-1)}{2}.$$

The innovation process $(\eta_t)_t$ is a vector of size $m \times 1$ and satisfies the assumption:

A0 : (η_t) is an independent and identically distributed (iid for short) sequence of variables on \mathbb{R}^m with identity covariance matrix and $\mathbb{E}[\eta_t] = 0$.

With this assumption, the matrix H_t is interpreted as the conditional variance (volatility) of $\underline{\varepsilon}_t$. The representation (2.1) includes various MGARCH models. For instance, if we assume that $A_{0i}^+ = A_{0i}^-$, the model (2.1) can be viewed as a multivariate extension and generalization of the the so-called PARCH (power-transformed ARCH) models introduced by [Higgins and Bera \(1992\)](#) (denoted CCC-PGARCH, in the sequel). Moreover, if we also fixed the power vector $\underline{\delta}_0 = (2, \dots, 2)'$ we obtain the CCC-GARCH(p, q) model proposed by [Jeantheau \(1998\)](#). Now, if we assume that $A_{0i}^+ \neq A_{0i}^-$, the model CCC-AGARCH(p, q) of [Francq and Zakoïan \(2012\)](#) is retrieved. If we also fixed $m = 1$ (the univariate case, we retrieve the APGARCH introduced by [Pan et al. \(2008\)](#).

As remarked by [Francq and Zakoïan \(2012\)](#), the interest of the APGARCH model (2.1) is to consider the leverage effect observed for most of financial series. It is well known that a negative return has more effect on the volatility as a positive return. In fact, for the same magnitude, the volatility $\underline{h}_t^{\delta_0/2}$ increases more if the return is negative than if it is positive. In general we observe that $A_{0i}^+ < A_{0i}^-$ for some $i > 0$ component by component.

[Bollerslev \(1990\)](#) introduced the CCC-GARCH model with the assumption that the coefficients matrices $A_{0i}^+ = A_{0i}^-$ and B_{0j} are diagonal. This assumption implies that the conditional variance $h_{k,t}$ of the k -th component of $\underline{\varepsilon}_t$ depends only on their own past values and not on the past values of the other components. By contrast, in the model (2.1) (including the CCC-GARCH) the conditional variance $h_{k,t}$ of the k -th component of $\underline{\varepsilon}_t$ depends not only on its past values but also on the past values of the other components. For this reason, Model (2.1) is referred to as the Extended CCC model by [He and Teräsvirta \(2004\)](#).

2.2. Strict stationarity condition

A sufficient condition for strict stationarity of the CCC-AGARCH(p, q) is given by [Francq and Zakoïan \(2012\)](#). For the model (2.1) the strict stationarity condition is established in the same way. For that sake we rewrite the first equation of (2.1) as

$$\underline{\varepsilon}_t = D_t \tilde{\eta}_t, \quad \text{where } \tilde{\eta}_t = (\tilde{\eta}_{1,t}, \dots, \tilde{\eta}_{m,t}) = R_0^{1/2} \eta_t. \quad (2.2)$$

Using the third equation of model (2.1), we may write

$$(\underline{\varepsilon}_t^\pm)^{\delta_0/2} = (\Upsilon_t^{\pm,(\delta_0)}) \underline{h}_t^{\delta_0/2}, \text{ with } \Upsilon_t^{\pm,(\delta_0)} = \text{diag} \left((\pm \tilde{\eta}_{1,t}^\pm)^{\delta_0/2}, \dots, (\pm \tilde{\eta}_{m,t}^\pm)^{\delta_0/2} \right), \quad (2.3)$$

where $\delta_0 = (\delta_{0,1}, \dots, \delta_{0,m})'$. To study the strict stationarity condition, we introduce the matrix expression for the model (2.1)

$$\underline{z}_t = \underline{b}_t + C_t \underline{z}_{t-1},$$

where

$$\underline{z}_t = \left(\left\{ (\underline{\varepsilon}_t^+)^{\delta_0/2} \right\}', \dots, \left\{ (\underline{\varepsilon}_{t-q+1}^+)^{\delta_0/2} \right\}', \left\{ (\underline{\varepsilon}_t^-)^{\delta_0/2} \right\}', \dots, \left\{ (\underline{\varepsilon}_{t-q+1}^-)^{\delta_0/2} \right\}', \left\{ \underline{h}_t^{\delta_0/2} \right\}', \dots, \left\{ \underline{h}_{t-p+1}^{\delta_0/2} \right\}' \right)',$$

$$\underline{b}_t = \left(\left\{ (\Upsilon_t^{+,(\delta_0)}) \underline{\omega}_0 \right\}', 0'_{m(q-1)}, \left\{ (\Upsilon_t^{-,(\delta_0)}) \underline{\omega}_0 \right\}', 0'_{m(q-1)}, \{\underline{\omega}_0\}', 0'_{m(p-1)} \right)'$$

and

$$C_t = \begin{pmatrix} \Upsilon_t^{+,(\delta_0)} A_{01:q}^+ & \Upsilon_t^{+,(\delta_0)} A_{01:q}^- & \Upsilon_t^{+,(\delta_0)} B_{01:p} \\ I_{m(q-1)} & 0_{m(q-1) \times m(p+q+1)} & \\ \Upsilon_t^{-,(\delta_0)} A_{01:q}^+ & \Upsilon_t^{-,(\delta_0)} A_{01:q}^- & \Upsilon_t^{-,(\delta_0)} B_{01:p} \\ 0_{m(q-1) \times mq} & I_{m(q-1)} & 0_{m(q-1) \times m(p-1)} \\ A_{01:q}^+ & A_{01:q}^- & B_{01:p} \\ 0_{m(p+q+1) \times m(q-1)} & & I_{m(p-1)} \end{pmatrix}. \quad (2.4)$$

We have denoted $A_{01:q}^+ = (A_{01}^+ \dots A_{0q}^+)$, $A_{01:q}^- = (A_{01}^- \dots A_{0q}^-)$ and $B_{01:p} = (B_{01} \dots B_{0p})$ (they are $m \times pm$ matrices). The matrix C_t is of size $(p+2q)m \times (p+2q)m$.

Let $\gamma(\mathbf{C}_0)$ the top Lyapunov exponent of the sequence $\mathbf{C}_0 = \{C_t, t \in \mathbb{Z}\}$. It is defined by

$$\gamma(\mathbf{C}_0) := \lim_{t \rightarrow +\infty} \frac{1}{t} \mathbb{E} [\log \|C_t C_{t-1} \dots C_1\|] = \inf_{t \geq 1} \frac{1}{t} \mathbb{E} [\log \|C_t C_{t-1} \dots C_1\|].$$

Now we can state the following results. Their proofs are the same than the one in [Francq and Zakoïan \(2012\)](#) so they are omitted.

Theorem 1. (*Strict stationarity*)

A necessary and sufficient condition for the existence of a strictly stationary and non anticipative solution process to model (2.1) is $\gamma(\mathbf{C}_0) < 0$.

When $\gamma(\mathbf{C}_0) < 0$, the stationary and non anticipative solution is unique and ergodic.

The two following corollaries are consequences of the necessary condition for strict stationarity. Let A a square matrix, $\rho(A)$ is the spectral radius of A (i.e. the greatest modulus of its eigenvalues).

Corollary 1.

Let \mathcal{B}_0 be the matrix polynomial defined by $\mathcal{B}_0(z) = I_m - zB_{01} - \dots - z^p B_{0p}$ for $z \in \mathbb{C}$ and define

$$\mathbf{B}_0 = \begin{pmatrix} & B_{01:p} \\ I_{(p-1)m} & 0_{(p-1)m \times 1} \end{pmatrix}.$$

Then, if $\gamma(\mathbf{C}_0) < 0$ the following equivalent properties hold:

(i) The roots of $\det(\mathcal{B}_0(z))$ are outside the unit disk,

(ii) $\rho(\mathbf{B}_0) < 1$.

Corollary 2.

Suppose $\gamma(\mathbf{C}_0) < 0$. Let $\underline{\varepsilon}_t$ be the strictly stationary and non anticipative solution of model (2.1).

There exists $s > 0$ such that $\mathbb{E} \|\underline{h}_t^{\delta_0/2}\|^s < \infty$ and $\mathbb{E} \|\underline{\varepsilon}_t^{\delta_0/2}\|^s < \infty$.

2.3. Identifiability condition

In this part, we are interested in the identifiability condition to ensure the uniqueness of the parameters in the CCC-APGARCH representation. This is a crucial step before the estimation.

The parameter ν is defined by

$$\nu := (\underline{\omega}', \alpha_1^{+'}, \dots, \alpha_q^{+'}, \alpha_1^{-'}, \dots, \alpha_q^{-'}, \beta_1', \dots, \beta_p', \underline{\tau}', \rho')',$$

where α_i^+ and α_i^- are defined by $\alpha_i^\pm = \text{vec}(A_i^\pm)$ for $i = 1, \dots, q$, $\beta_j = \text{vec}(B_j)$ for $j = 1, \dots, p$, $\underline{\tau}' = (\tau_1, \dots, \tau_m)$ is the vector of powers and $\rho = (\rho_{21}, \dots, \rho_{m1}, \rho_{32}, \dots, \rho_{m2}, \dots, \rho_{mm-1})'$ such that the ρ_{ij} 's are the components of the matrix R . The parameter ν belongs to the parameter space

$$\Delta \subset]0, +\infty[^m \times]0, \infty[^{m^2(2q+p)} \times]0, +\infty[^m \times]-1, 1[^{m(m-1)/2}.$$

The unknown true parameter value is denoted by

$$\nu_0 := (\underline{\omega}'_0, \alpha_{01}^{+'}, \dots, \alpha_{0m}^{+'}, \alpha_{01}^{-'}, \dots, \alpha_{0m}^{-'}, \beta_{01}', \dots, \beta_{0m}', \underline{\delta}'_0, \rho'_0)'$$

We adopt the following notation. For a matrix A (which has to be seen as a parameter of the model), we write A_0 when the coefficients of the matrix are evaluated in the true value $\nu = \nu_0$.

Let $\mathcal{A}^+(L) = \sum_{i=1}^q A_i^+ L^i$, $\mathcal{A}^-(L) = \sum_{i=1}^q A_i^- L^i$ and $\mathcal{B}(L) = I_m - \sum_{j=1}^p B_j L^j$ where L is the backshift operator. By convention $\mathcal{A}^\pm = 0$ if $q = 0$ and $\mathcal{B}(L) = I_m$ if $p = 0$.

If the roots of $\det(\mathcal{B}_0(L)) = 0$ are outside the unit disk, we have from $\mathcal{B}_0(L)h_t^{\delta_0/2} = \underline{\omega}_0 + \mathcal{A}_0^+(L)(\underline{\varepsilon}_t^+)^{\delta_0/2} + \mathcal{A}_0^-(L)(\underline{\varepsilon}_t^-)^{\delta_0/2}$ the compact expression:

$$\underline{h}_t^{\delta_0/2} = \mathcal{B}_0(1)^{-1} \underline{\omega}_0 + \mathcal{B}_0(L)^{-1} \mathcal{A}_0^+(L)(\underline{\varepsilon}_t^+)^{\delta_0/2} + \mathcal{B}_0(L)^{-1} \mathcal{A}_0^-(L)(\underline{\varepsilon}_t^-)^{\delta_0/2}. \quad (2.5)$$

The parameter ν_0 is said to be identifiable if (2.5) does not hold true when ν_0 is replaced by $\nu \neq \nu_0$ belonging to Δ .

The assumption that the polynomials \mathcal{A}_0^+ , \mathcal{A}_0^- and \mathcal{B}_0 have no common roots is not sufficient to consider that there is not another triple $(\mathcal{A}_0^+, \mathcal{A}_0^-, \mathcal{B}_0)$ such that

$$\mathcal{B}^{-1} \mathcal{A}^+ = \mathcal{B}_0^{-1} \mathcal{A}_0^+ \quad \text{and} \quad \mathcal{B}^{-1} \mathcal{A}^- = \mathcal{B}_0^{-1} \mathcal{A}_0^-. \quad (2.6)$$

This condition is equivalent as the existence of an operator $U(B)$ such that

$$\mathcal{A}^+(L) = U(L) \mathcal{A}_0^+(L), \quad \mathcal{A}^-(L) = U(L) \mathcal{A}_0^-(L) \quad \text{and} \quad \mathcal{B}(L) = U(L) \mathcal{B}_0(L).$$

The matrix $U(L)$ is unimodular if $\det(U(L))$ is a constant not equal to zero. If the common factor to both polynomials is unimodular,

$$P(L) = U(L)P_1(L), \quad Q(L) = U(L)Q_1(L) \Rightarrow \det(U(L)) = c,$$

the polynomials $P(L)$ and $Q(L)$ are left-coprimes.

But in the vectorial case, suppose that \mathcal{A}_0^+ , \mathcal{A}_0^- and \mathcal{B}_0 are left-coprimes is not sufficient to consider that (2.6) have no solution for $\nu \neq \nu_0$ (see [Francq and Zakoïan \(2012\)](#)).

To obtain a mild condition, for any column i of the matrix operators \mathcal{A}_0^+ , \mathcal{A}_0^- and \mathcal{B}_0 , we denote by $q_i^+(\nu)$, $q_i^-(\nu)$, and $p_i(\nu)$ their maximal degrees. We suppose that the maximal values of the orders are imposed:

$$\forall \nu \in \Delta, \forall i = 1, \dots, m, \quad q_i^+(\nu) \leq q_i^+, \quad q_i^-(\nu) \leq q_i^-, \quad \text{and} \quad p_i(\nu) \leq p_i \quad (2.7)$$

where $q_i^+ \leq q$, $q_i^- \leq q$ and $p_i \leq p$ are fixed integers.

We denote $a_{q_i^+}^+(i)$ the column vector of the coefficients $L^{q_i^+}$, $a_{q_i^-}^-(i)$ the column vector of the coefficients $L^{q_i^-}$ in the column i of \mathcal{A}_0^+ , respectively \mathcal{A}_0^- and $b_{p_i}(i)$ the column vector of the coefficients L^{p_i} in the column i of \mathcal{B}_0 .

Proposition 1. (*Identifiability condition*)

If the matrix polynomials $\mathcal{A}_0^+(L), \mathcal{A}_0^-(L)$ and $\mathcal{B}_0(L)$ are left-coprime, $\mathcal{A}_0^+(1) + \mathcal{A}_0^-(1) \neq 0$ and if the matrix

$$M(\mathcal{A}_0^+(L), \mathcal{A}_0^-(L), \mathcal{B}_0(L)) = \left[a_{q_1^+}^+(1) \dots a_{q_m^+}^+(m) a_{q_1^-}^-(1) \dots a_{q_m^-}^-(m) b_{p_1}(1) \dots b_{p_m}(m) \right]$$

has full rank m , under the constraints (2.7) with $q_i^+ = q_i^+(\nu_0), q_i^- = q_i^-(\nu_0)$ and $p_i = p_i(\nu_0)$ for any value of i , then

$$\begin{cases} \mathcal{B}(L)^{-1} \mathcal{A}^+(L) = \mathcal{B}_0(L)^{-1} \mathcal{A}_0^+(L) \\ \mathcal{B}(L)^{-1} \mathcal{A}^-(L) = \mathcal{B}_0(L)^{-1} \mathcal{A}_0^-(L) \end{cases} \Rightarrow (\mathcal{A}^+, \mathcal{A}^-, \mathcal{B}) = (\mathcal{A}_0^+, \mathcal{A}_0^-, \mathcal{B}_0).$$

Proof of Proposition 1

The proof of the identifiability condition is identical as Francq and Zakoïan (2012) in the case of the CCC-AGARCH model.

3. Estimation when the power is known

In this section, we assume that $\underline{\delta}_0$ is known. We write $\underline{\delta}_0 = \underline{\delta}$ in order to simplify the writings. For the estimation of GARCH and MGARCH models, the commonly used estimation method is the QMLE, which can also be viewed as a nonlinear least squares estimation (LSE). The QML method is particularly relevant for GARCH models because it provides consistent and asymptotically normal estimators for strictly stationary GARCH processes under mild regularity conditions. For example, no moment assumptions on the observed process are required (see for instance Francq and Zakoïan (2004) or Francq and Zakoïan (2010)).

As remarked in Section 2, some particular cases of Model (2.1) are obtained for $\underline{\delta} = (2, \dots, 2)'$: the CCC-AGARCH introduced by Francq and Zakoïan (2012) and the CCC model introduced by He and Teräsvirta (2004) with $A_{0i}^+ = A_{0i}^-$. This section provides asymptotic results which can, in particular, be applied to those models.

3.1. QML estimation

The procedure of estimation and the asymptotic properties are similar to those of the model CCC-AGARCH introduced by Francq and Zakoïan (2012).

The parameters are the coefficients of the vector $\underline{\omega}_0$, the matrices A_{0i}^+, A_{0i}^- and B_{0j} , and the coefficients of the lower triangular part without the diagonal of the correlation matrix R_0 . The number of parameters is

$$s_0 = m + m^2(p + 2q) + \frac{m(m-1)}{2}.$$

The goal is to estimate the s_0 coefficients of the model (2.1). In this section, we note the parameter

$$\theta := (\underline{\omega}', \alpha_1^{+'}, \dots, \alpha_m^{+'}, \alpha_1^{-'}, \dots, \alpha_m^{-'}, \beta_1', \dots, \beta_m', \rho')',$$

where α_i^+ and α_i^- are define by $\alpha_i^\pm = \text{vec}(A_i^\pm)$ for $i = 1, \dots, q, \beta_j = \text{vec}(B_j)$ for $j = 1, \dots, p$ and $\rho = (\rho_{21}, \dots, \rho_{m1}, \rho_{32}, \dots, \rho_{m2}, \dots, \rho_{mm-1})'$. The parameter θ belongs to the parameter space

$$\Theta \subset]0, +\infty[^m \times [0, \infty[^{m^2(2q+p)} \times]-1, 1[^{m(m-1)/2}.$$

The unknown true value of the parameter is denoted by

$$\theta_0 := (\underline{\omega}'_0, \alpha_{01}^{+'}, \dots, \alpha_{0m}^{+'}, \alpha_{01}^{-'}, \dots, \alpha_{0m}^{-'}, \beta_{01}', \dots, \beta_{0m}', \rho'_0)'$$

The determinant of a square matrix A is denoted by $\det(A)$ or $|A|$.

Let $(\varepsilon_1, \dots, \varepsilon_n)$ be a realization of length n of the unique non-anticipative strictly stationary solution (ε_t) of Model (2.1). Conditionally to nonnegative initial values $\varepsilon_0, \dots, \varepsilon_{1-q}, \tilde{h}_0, \dots, \tilde{h}_{1-p}$, the Gaussian quasi-likelihood writes

$$L_n(\theta) = L_n(\theta; \varepsilon_1, \dots, \varepsilon_n) = \prod_{t=1}^n \frac{1}{(2\pi)^{m/2} |\tilde{H}_t|^{1/2}} \exp\left(-\frac{1}{2} \varepsilon_t' \tilde{H}_t^{-1} \varepsilon_t\right),$$

where the \tilde{H}_t are recursively defined, for $t \geq 1$, by

$$\begin{cases} \tilde{H}_t = \tilde{D}_t R \tilde{D}_t, & \tilde{D}_t = \text{diag}\left(\sqrt{\tilde{h}_{1,t}}, \dots, \sqrt{\tilde{h}_{m,t}}\right) \\ \tilde{h}_t^{\delta/2} := \tilde{h}_t^{\delta/2}(\theta) = \underline{\omega} + \sum_{i=1}^q A_i^+(\varepsilon_{t-i}^+)^{\delta/2} + A_i^-(\varepsilon_{t-i}^-)^{\delta/2} + \sum_{j=1}^p B_j \tilde{h}_{t-j}^{\delta/2}. \end{cases}$$

A quasi-likelihood estimator of θ is defined as any measurable solution $\hat{\theta}_n$ of

$$\hat{\theta}_n = \arg \max_{\theta \in \Theta} L_n(\theta) = \arg \min_{\theta \in \Theta} \tilde{\mathcal{L}}_n(\theta), \quad (3.1)$$

where

$$\tilde{\mathcal{L}}_n(\theta) = \frac{1}{n} \sum_{t=1}^n \tilde{l}_t, \quad \tilde{l}_t = \tilde{l}_t(\theta) = \varepsilon_t' \tilde{H}_t^{-1} \varepsilon_t + \log |\tilde{H}_t|.$$

3.2. Asymptotic properties

To establish the strong consistency, we need the following assumptions:

A1 : $\theta_0 \in \Theta$ and Θ is compact,

A2 : $\gamma(\mathbf{C}_0) < 0$ and $\forall \theta \in \Theta, \det(\mathcal{B}(z)) = 0 \Rightarrow |z| > 1$,

A3 : For $i = 1, \dots, m$ the distribution of $\tilde{\eta}_{it}$ is not concentrated on 2 points and $\mathbb{P}(\tilde{\eta}_{it} > 0) \in (0, 1)$.

A4 : if $p > 0, \mathcal{A}_0^+(1) + \mathcal{A}_0^-(1) \neq 0, \mathcal{A}_0^+(z), \mathcal{A}_0^-(z)$ and $\mathcal{B}_0(z)$ are left-coprime and the matrix $M(\mathcal{A}_0^+, \mathcal{A}_0^-, \mathcal{B}_0)$ has full rank m .

If the space Θ is constrained by (2.7), Assumption **A4** can be replaced by the more general condition

A4' : **A4** with $M(\mathcal{A}_0^+, \mathcal{A}_0^-, \mathcal{B}_0)$ replaced by $[A_{0q}^+ A_{0q}^- B_{0p}]$.

A5 : R is a positive-definite correlation matrix for all $\theta \in \Theta$.

To ensure the strong consistency of the QMLE, a compactness assumption is required (i.e **A1**). The assumption **A2** makes reference to the condition of strict stationarity for the model (2.1). This assumption implies that for the true parameter θ_0 , Model (2.1) admits a strictly stationary solution but is less restrictive concerning the other values $\theta \in \Theta$. The second part of Assumption **A2** implies that the roots of $\det(\mathcal{B}(z))$ are outside the unit disk. Assumptions **A3** and **A4** or **A4'** are made for identifiability reasons. In particular $\mathbb{P}(\tilde{\eta}_{it} > 0) \in (0, 1)$ ensures that the process (ε_{it}) (for $i = 1, \dots, m$) takes positive and negative values with a positive probability (if, for instance, the (ε_{it}) were a.s. positive, the parameters α_{0j}^- for $j = 1, \dots, q$ could not be identified).

We are now able to state the following strong consistency theorem.

Theorem 2. *Let $(\hat{\theta}_n)$ be a sequence of QMLE satisfying (3.1). Then, under **A0**, \dots , **A5** or **A0**, \dots , **A4'** and **A5**, we have $\hat{\theta}_n \rightarrow \theta_0$, almost surely as $n \rightarrow \infty$.*

The proof of this result is postponed to Subsection A.2 in the Appendix A.

It will be useful to approximate the sequence (\tilde{l}_t) by an ergodic and stationary sequence. Under Assumption **A2** there exists a strictly stationary, non anticipative and ergodic solution $(h_t)_t = (h_t(\theta))_t$ of

$$\tilde{h}_t^{\delta/2} := \tilde{h}_t^{\delta/2}(\theta) = \underline{\omega} + \sum_{i=1}^q A_i^+(\varepsilon_{t-i}^+)^{\delta/2} + A_i^-(\varepsilon_{t-i}^-)^{\delta/2} + \sum_{j=1}^p B_j \tilde{h}_{t-j}^{\delta/2}. \quad (3.2)$$

We denote $D_t = D_t(\theta) = \text{diag}(\sqrt{h_{1,t}}, \dots, \sqrt{h_{m,t}})$ and $H_t = H_t(\theta) = D_t(\theta)RD_t(\theta)$ and we define

$$\mathcal{L}_n(\theta) = \frac{1}{n} \sum_{t=1}^n l_t, \quad l_t = l_t(\theta) = \underline{\varepsilon}'_t H_t^{-1} \underline{\varepsilon}_t + \log |H_t|.$$

To establish the asymptotic normality, the following additional assumptions are required:

A6 : $\theta_0 \in \overset{\circ}{\Theta}$, where $\overset{\circ}{\Theta}$ is the interior of Θ .

A7 : $\mathbb{E}\|\eta_t \eta'_t\|^2 < \infty$.

Assumption **A6** prevents the situation where certain components of θ_0 are equal to zero (more precisely the coefficients of the matrices in our model). The second main result of this section is the following asymptotic normality theorem.

Theorem 3. *Under the assumptions of Theorem 2 and **A6–A7**, when $n \rightarrow \infty$, we have*

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{\mathcal{L}} \mathcal{N}(0, J^{-1}IJ^{-1}),$$

where J is a positive-definite matrix and I is a positive semi-definite matrix, defined by

$$I := I(\theta_0) = \mathbb{E} \left[\frac{\partial l_t(\theta_0)}{\partial \theta} \frac{\partial l_t(\theta_0)}{\partial \theta'} \right], \quad J := J(\theta_0) = \mathbb{E} \left[\frac{\partial^2 l_t(\theta_0)}{\partial \theta \partial \theta'} \right].$$

The proof of this result is postponed to Subsection [A.3](#) in the Appendix [A](#).

Remark 1. *In the one dimensional case (when $m = 1$), we have $I = (\mathbb{E}\eta_t^4 - 1)J$ with*

$$J = \frac{4}{\underline{\delta}_0^2} \mathbb{E} \left(\frac{1}{\underline{h}_t^{\underline{\delta}_0}(\theta_0)} \frac{\partial \underline{h}_t^{\underline{\delta}_0/2}(\theta_0)}{\partial \theta} \frac{\partial \underline{h}_t^{\underline{\delta}_0/2}(\theta_0)}{\partial \theta'} \right).$$

This expression is in accordance with the one of Theorem 2.2 in [Hamadeh and Zakoïan \(2011\)](#).

4. Estimation when the power is unknown

In this section, it is assumed that $\underline{\delta}_0$ is unknown.

4.1. QML estimation

Now we consider the case when the power is unknown. Thus parameter is jointly estimated with the others parameters. In practice the coefficients $\underline{\delta}_0$ is difficult to identified, as it was remarked in [Hamadeh and Zakoïan \(2011\)](#) for the APGARCh model. For that sake, we add the following assumption:

A8 : η_t has a positive density on some neighbourhood of zero.

To define the QML estimator of ν , we replace H_t by \mathcal{H}_t in the expression of the criterion defined in [\(3.1\)](#) and we obtain recursively $\tilde{\mathcal{H}}_t$, for $t \geq 1$,

$$\left\{ \begin{array}{l} \tilde{\mathcal{H}}_t = \tilde{D}_t R \tilde{D}_t, \quad \tilde{D}_t = \text{diag} \left(\sqrt{\tilde{h}_{1,t}}, \dots, \sqrt{\tilde{h}_{m,t}} \right) \\ \tilde{\underline{h}}_t := \tilde{\underline{h}}_t(\nu) = \left(\underline{\omega} + \sum_{i=1}^q A_i^+ (\underline{\varepsilon}_{t-i}^+)^{\tau/2} + A_i^- (\underline{\varepsilon}_{t-i}^-)^{\tau/2} + \sum_{j=1}^p B_j \tilde{\underline{h}}_{t-j}^{\tau/2} \right)^{2/\tau}. \end{array} \right.$$

A quasi-maximum likelihood estimator of ν is defined as any measurable solution $\hat{\nu}_n$ of

$$\hat{\nu}_n = \arg \min_{\nu \in \Delta} \tilde{\mathcal{L}}_n(\nu), \tag{4.1}$$

where

$$\tilde{\mathcal{L}}_n(\nu) = \frac{1}{n} \sum_{t=1}^n \tilde{l}_t, \quad \tilde{l}_t = \tilde{l}_t(\nu) = \underline{\varepsilon}'_t \tilde{\mathcal{H}}_t^{-1} \underline{\varepsilon}_t + \log |\tilde{\mathcal{H}}_t|.$$

4.2. Asymptotic properties

To establish the consistency and the asymptotic normality, we need some assumptions similar to those we assumed when the power is known. We will assume **A1**, ..., **A6** with the parameter θ which is replaced by ν and the space parameter Θ is replaced by Δ .

Theorem 4. (Strong consistency)

Let $(\hat{\nu}_n)$ be a sequence of QMLE satisfying (3.1). Then, under **A0**, ..., **A5** or **A0**, ..., **A4'** and **A5**, we have $\hat{\nu}_n \rightarrow \nu_0$, almost surely when $n \rightarrow +\infty$.

The proof of this result is postponed to Subsection A.4 in the Appendix A.

Theorem 5. (Asymptotic normality)

Under the assumptions of Theorem 4, **A6**, **A7** and **A8**, when $n \rightarrow \infty$, we have

$$\sqrt{n}(\hat{\nu}_n - \nu_0) \xrightarrow{\mathcal{L}} \mathcal{N}(0, J^{-1}IJ^{-1}),$$

where J is positive-definite matrix and I is a positive semi-definite matrix, defined by

$$I := I(\nu_0) = \mathbb{E} \left[\frac{\partial l_t(\nu_0)}{\partial \nu} \frac{\partial l_t(\nu_0)}{\partial \nu'} \right], \quad J := J(\nu_0) = \mathbb{E} \left[\frac{\partial^2 l_t(\nu_0)}{\partial \nu \partial \nu'} \right].$$

The proof of this result is postponed to Subsection A.5 in the Appendix A.

Remark 2. As in Remark 1, in the one dimensional case, we have $I = (\mathbb{E}\eta_t^4 - 1)J$ with

$$J = \mathbb{E} \left(\frac{\partial \log h_t^2(\nu_0)}{\partial \nu} \frac{\partial \log h_t^2(\nu_0)}{\partial \nu'} \right).$$

This expression is again in accordance with the one of Theorem 3.1 in Hamadeh and Zakoïan (2011).

5. Linear tests

The asymptotic normality results from Theorem 3 and Theorem 5 are used to test linear constraints on the parameter. We thus consider a null hypothesis of the form

$$H_0 : C\nu_0 = c, \tag{5.1}$$

where C is a known $s \times s_0$ matrix of rank s and c is a known vector of size $s \times 1$. The Wald test is a standard parametric test for testing H_0 and it is particularly appropriate in the heavy-tailed case (which is the context of financial series). Let \hat{I} and \hat{J} be weakly consistent estimators of I and J involved in the asymptotic normality of the QMLE. For instance, I and J can be estimated by their empirical or observable counterparts given by

$$\hat{I} = \frac{1}{n} \sum_{t=1}^n \frac{\partial \tilde{l}_t(\hat{\nu}_n)}{\partial \nu} \frac{\partial \tilde{l}_t(\hat{\nu}_n)}{\partial \nu'} \quad \text{and} \quad \hat{J} = \frac{1}{n} \sum_{t=1}^n \frac{\partial^2 \tilde{l}_t(\hat{\nu}_n)}{\partial \nu \partial \nu'}.$$

Under the assumptions of Theorem 5, and the assumption that the matrix I is invertible, the Wald test statistic is defined as follows

$$W_n(s) = (C\hat{\nu}_n - c)'(C\hat{J}^{-1}\hat{I}\hat{J}^{-1}C')^{-1}(C\hat{\nu}_n - c).$$

We reject the null hypothesis for large values of $W_n(s)$. If $W_n(s) > \chi_s^2(\alpha)$ (here $\chi_s^2(\alpha)$ correspond to the quantile of the Khi^2 distribution of asymptotic level α), we could conclude that the powers of the

model (2.1) are equals. Thus we can consider a model with equal powers and so we are able to reduce the dimension of the parameter space.

We can also test the equality between the components of the power $\underline{\nu}$. For instance, in the bivariate case with the orders fixed at $p = q = 1$, we take $C = (0_{14}, 1, 0, 0)$ and $c = \tau_2$ in (5.1) (remind that $\nu_0 = (\underline{\omega}', \alpha^{+'}, \alpha^{-'}, \beta', \tau_1, \tau_2, \rho)'$). Via this kind of test, we can also test the asymmetric property. If we reject the asymmetric assumption, we can consider the standard CCC-PGARCH model and reduce the dimension of the parameter space.

We strength the fact that our normality results can not be used in order to determine the orders (p, q) of the model (2.1). Indeed, this can be usually done when one tests the nullity of the coefficients of the matrix B_p . If they are all equal to zero, we can consider a smaller order p (and similarly with the matrices A_q^+ and A_q^- with the order q). Unfortunately, the assumption **A6** excludes the case of vanishing coefficients so we can not apply our results in this situation. One refers to Section 8.2 of Francq and Zakoïan (2010) for a discussion on this topic.

6. Numerical illustrations

In this section, we make some simulations with $\eta_t \sim \mathcal{N}(0, I_2)$ and we compute the QML estimator to estimate the coefficients of the model (2.1). The simulations are made with the open source statistical software R (see R Development Core Team, 2017) or (see <http://cran.r-project.org/>). We use the function `nlminb()` to minimize the log-quasi-likelihood.

6.1. Estimation when the power is known

We fixed the orders of the model (2.1) at $q = 1$ and $p = 0$. We computed the estimator on $N = 100$ independent simulated trajectories to assess the performance of the QMLE in finite sample. The trajectories are made in dimension 2 with a length $n = 500$ and $n = 5,000$.

The parameter used to simulate the trajectories are given in first row of the Table 2 and the space Θ associated is chosen to satisfied the assumptions of Theorem 3. As expected, Table 2 shows that the bias and the RMSE decrease when the size of the sample increases. Figures 1 and 2 summarize via box-plot, the distribution of the QMLE for these simulations. Of course the precision around the estimated coefficients is better when the size of the sample increases (see Figures 1 and 2).

Table 2: Sampling distribution of the QMLE of θ_0 for the CCC-APGARCH(0,1) model (2.1) for $\underline{\delta} = (2, 2)'$.

Length	True val.	ω_0	A_0^+		A_0^-		ρ_0
		1 1	0.25 0.05	0.05 0.25	0.5 0.5	0.5 0.5	0.5
$n = 500$	Bias	-0.00498 -0.00090	0.00180 0.00683	0.00789 -0.02083	-0.00525 -0.01505	0.00485 0.00122	-0.00471
	RMSE	0.10714 0.12073	0.08290 0.04308	0.04520 0.07730	0.12666 0.12738	0.12379 0.15095	0.03774
$n = 5,000$	Bias	0.00105 0.00044	-0.00063 -0.00175	-0.00024 0.00188	0.00346 0.00175	-0.00324 0.00121	0.00010
	RMSE	0.03526 0.03745	0.02129 0.01437	0.01464 0.02220	0.04219 0.03883	0.03950 0.03890	0.01143

δ_0 is known

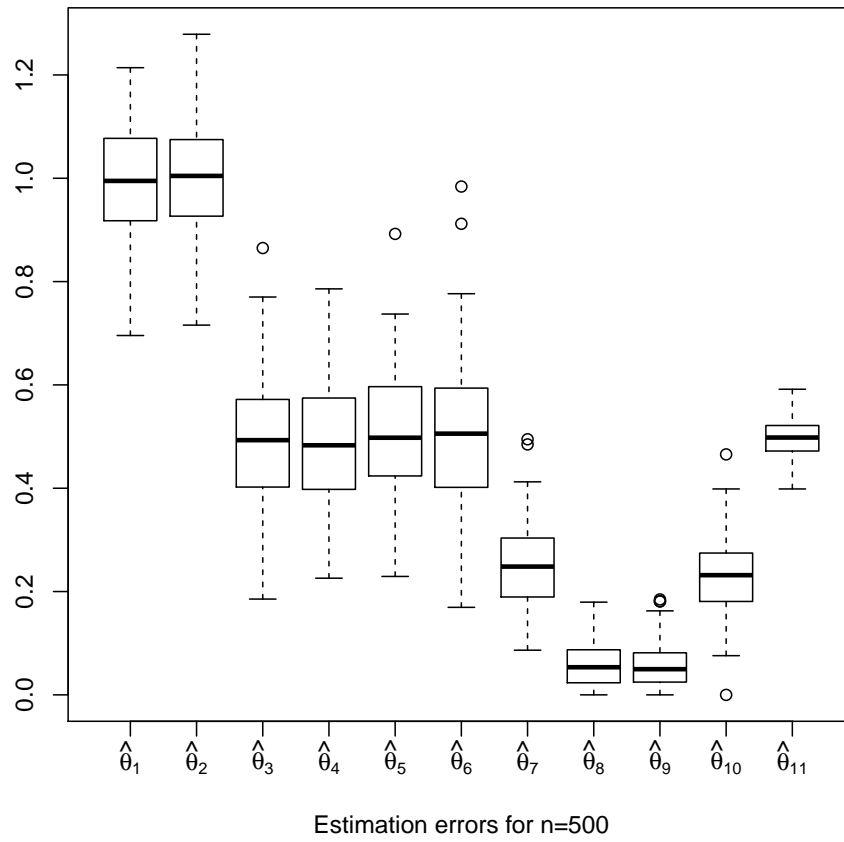


Figure 1: Box-plot of the QMLE distribution for $n = 500$

δ_0 is known

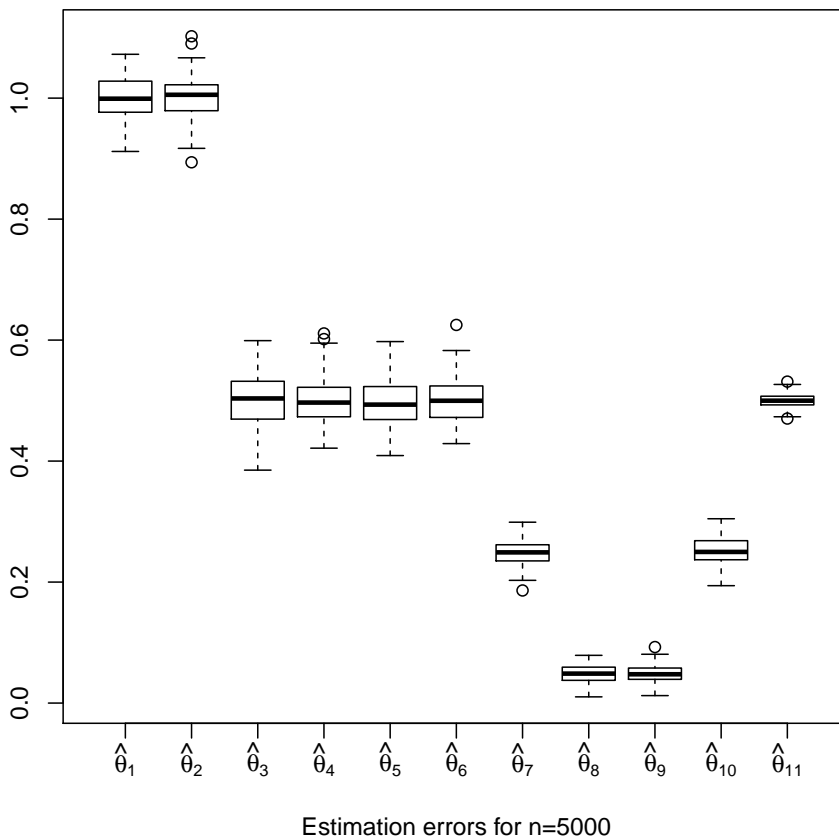


Figure 2: Box-plot of the QMLE distribution for $n = 5,000$

6.2. Estimation when the power is unknown

In this section, we also fixed the orders $q = 1$ and $p = 0$ of Model (2.1) and we estimate the parameter ν_0 . We compute, as the previous section, the estimator on $N = 100$ independent simulated trajectories in dimension 2 with sizes $n = 500$ and $n = 5,000$. Table 3 presents the results of QMLE of the parameters including the powers $\underline{\delta} = (\delta_1, \delta_2)'$. The results are satisfactory even for the parameter $\underline{\delta}$ which is difficult to identify in practice (see Section 4.1 of Hamadeh and Zakoian (2011)) and which is more difficult to estimate than the others parameters. The conclusions are similar as in the case where $\underline{\delta}$ is known. The dispersion around the parameter $\underline{\delta}$ is much more precise when the sample size is $n = 5,000$. The bias and the RMSE decrease when the size of sample increases. Figure 3 shows the distribution of the parameter $\hat{\nu}_n$ for the size $n = 500$. We remark that the estimation of the parameter $\underline{\delta}$ has an important dispersion as regard to the other parameters. When the size of the sample is $n = 5,000$ (see Figure 4), the estimation of the power is more accurate.

Table 4 displays the relative percentages of rejection of the Wald test proposed in Section 5 for testing the null hypothesis $H_0 : C\nu_0 = (1, 1)'$ in the case of bivariate CCC-APGARCH(0, 1), where C is a 2×13 matrix with 1 in positions (1, 11) and (2, 12) and 0 elsewhere and $\nu_0 = (0.2, 0.3, 0.25, 0.05, 0.05, 0.25, 0.5, 0.5, 0.5, 0.5, 1, 1, 0.5)'$. We simulated $N = 100$ independent trajectories of size $n = 1,000$ and $n = 10,000$ of Model (2.1) with $m = 2$, $q = 1$ and $p = 0$. The nominal asymptotic level of the tests is $\alpha = 5\%$. The values in bold correspond to the null hypothesis H_0 . We remark that the relative rejection frequencies of the Wald test are close to the nominal 5% level under the null, and are close to 100% under the alternative. We draw the conclusion that the proposed test

well controls the error of first kind.

Table 3: Sampling distribution of the QMLE of ν_0 for the CCC-APGARCH(0, 1) model (2.1) with $\underline{\delta}$ unknown.

Length	True val.	$\underline{\omega}_0$	A_0^+		A_0^-		$\underline{\delta}_0$	ρ_0
		1	0.25	0.05	0.5	0.5	2	0.5
		1	0.05	0.25	0.5	0.5	2	
$n = 500$	Bias	0.12600	-0.03211	0.00370	-0.00853	-0.00155	0.27015	-0.00112
		0.11629	0.00937	-0.02054	0.04720	0.00177	0.33354	
$n = 500$	RMSE	0.45385	0.08449	0.06808	0.14385	0.27142	1.13662	0.03772
		0.48839	0.07917	0.09275	0.26265	0.16417	1.14144	
$n = 5,000$	Bias	0.00921	-0.00490	0.00082	-0.00526	-0.00574	0.00000	-0.00013
		0.00072	-0.00052	0.00183	0.01204	0.00196	0.00000	
$n = 5,000$	RMSE	0.05023	0.02434	0.01783	0.04278	0.07725	0.16761	0.01067
		0.04825	0.01935	0.02661	0.08433	0.04402	0.16004	

Table 4: Empirical size (in %) of $W_n(s)$ for testing the null hypothesis $H_0 : C\nu_0 = (1, 1)'$ in the case of a CCC-APGARCH(0, 1) model (2.1), where C is a 2×13 matrix with 1 in positions (1, 11) and (2, 12) and 0 elsewhere. The number of replications is $N = 100$ and the nominal asymptotic level is $\alpha = 5\%$.

(τ_1, τ_2)	(0.5, 0.5)		(0.5, 1)		(1, 1)	
Length n	1,000	10,000	1,000	10,000	1,000	10,000
$W_n(2)$	100.0	100.0	100.0	100.0	6.0	5.0
(τ_1, τ_2)	(0.5, 1.5)		(1, 1.5)		(2, 2)	
Length n	1,000	10,000	1,000	10,000	1,000	10,000
$W_n(2)$	100.0	100.0	83.0	100.0	80.0	100.0

δ_0 is unknown

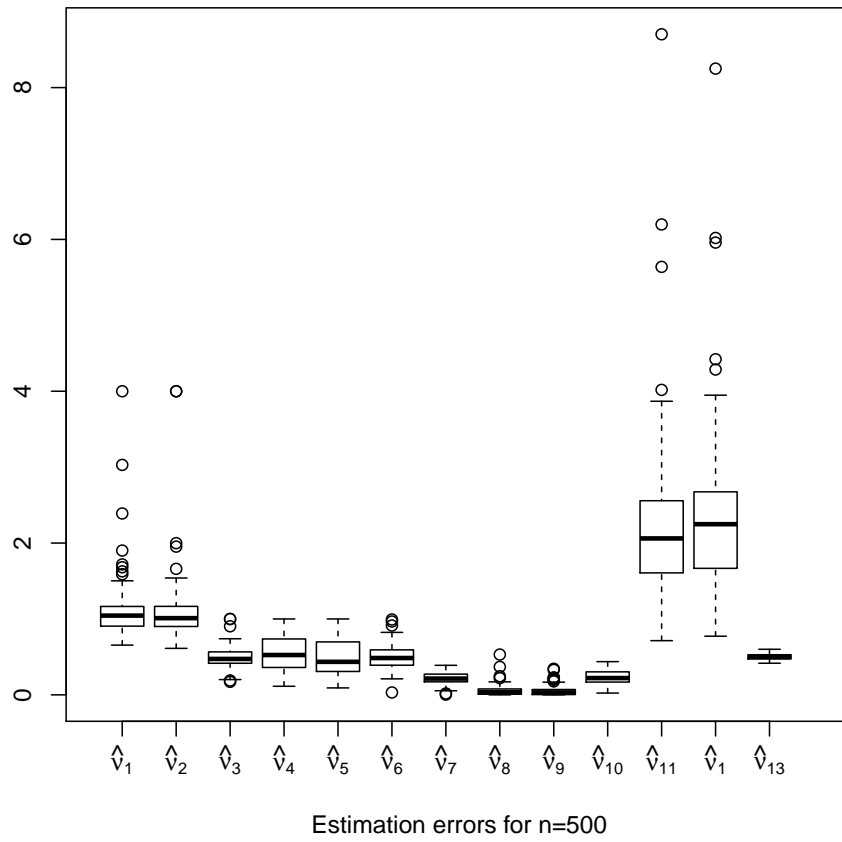


Figure 3: Box-plot of the QMLE distribution for these simulations experiments, with $n = 500$.

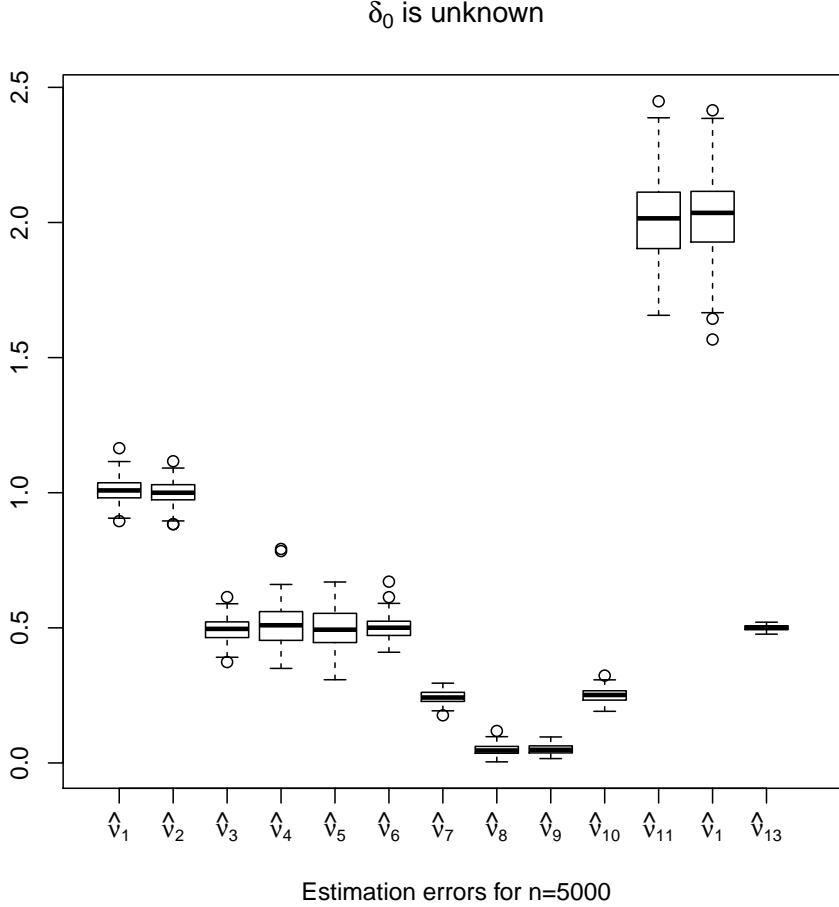


Figure 4: Box-plot of the QMLE distribution for these simulations experiments, with $n = 5,000$.

6.3. Estimation on a real dataset

In this section, we propose to estimate the bivariate series returns of the daily exchange rates between the Dollar (USD) and the Yen (JPY) against the Euro (EUR). The observations cover the period from January 4, 1999 to July 13, 2017 which correspond to 4746 observations. The data were obtained from the website of the National Bank of Belgium (<https://www.nbb.be>). We divide the full period in three subperiods with equal length ($n = 1581$): the first period runs from 1999-01-04 to 2005-03-02, the second from 2005-03-03 to 2011-05-05 and the last one from 2011-05-06 to 2017-07-13.

First, we consider the univariate case and an APGARCh(1, 1) model is estimated by QML on the full period and the three subperiods. Tables 5 and 6 show the univariate analysis of the two series (when $\underline{\delta}_0$ is known and estimated, respectively) and give an idea of the behaviour of the modeling. We observe:

1. a strong leverage effect (resp. no leverage effect) of the JPY with respect to its own past return for the full, first and second periods (resp. for the third periods) confirmed by the Wald test proposed in Section 5.
2. almost no leverage effect is also confirmed by Wald test in the volatility of the USD for some periods
3. a strong persistence of past volatility for the USD and JPY
4. different values of the power $\hat{\tau}$ for the USD in the three subperiods.

To give an idea of the reliability of the parameter estimates (in the bivariate case) obtained from the full period (4746 observations), the period has been divided into 3 subperiods as in the univariate analysis (see Tables 5 and 6). Tables 7 and 8 present the estimation of the model (2.1) for the bivariate series $(USD_t, JPY_t)'$ when $\underline{\delta}_0$ is known and estimated, respectively. The conclusion is the same as in Tables 5 and 6. But we also remark a strong correlation between the two exchange rates. The power $\underline{\delta}_0$ is equivalent to the univariate case and differs for each subperiod. We also note that none is equal to the others. The persistence matrix B_{01} has a diagonal form and the persistence of past volatility is strong as in the univariate case. The coefficients corresponding to the negative returns are generally bigger than those for the positive returns and some of them are equal to zero. Since we obtain null estimated coefficients, we think that the assumption **A6** is not satisfied and thus our asymptotic normality theorems do not apply. In future works we intent to study how the identification (orders p and q selection) procedure should be adapted in the CCC-APGARCH(p, q) framework considered in the present paper by extending Section 8.2 of [Francq and Zakoian \(2010\)](#).

Table 5: Estimation of daily exchange rates of the Dollar and Yen against the Euro with the APGARCH(1,1) model in dimension 1.

$\underline{\delta}_0$	Parameters	Full Period		First Period		Second Period		Third Period	
		USD	JPY	USD	JPY	USD	JPY	USD	JPY
0.5	$\hat{\omega}$	0.00492	0.01107	0.01098	0.01245	0.00245	0.01192	0.00363	0.01115
	$\hat{\alpha}_1^+$	0.02573	0.05211	0.02132	0.05832	0.03822	0.03087	0.00984	0.05656
	$\hat{\alpha}_1^-$	0.03512	0.06995	0.02919	0.07707	0.04447	0.06687	0.02780	0.06050
	$\hat{\beta}_1$	0.96960	0.93866	0.96626	0.93106	0.96405	0.94787	0.98016	0.94095
1	$\hat{\omega}$	0.00293	0.00803	0.00803	0.00993	0.00184	0.00826	0.00176	0.00826
	$\hat{\alpha}_1^+$	0.02628	0.05377	0.02205	0.06009	0.04306	0.02422	0.00127	0.06211
	$\hat{\alpha}_1^-$	0.04059	0.08504	0.03498	0.09537	0.04933	0.08989	0.03315	0.06808
	$\hat{\beta}_1$	0.96972	0.93620	0.96588	0.92635	0.96180	0.94543	0.98318	0.93977
1.5	$\hat{\omega}$	0.00177	0.00597	0.00600	0.00815	0.00149	0.00564	0.00103	0.00637
	$\hat{\alpha}_1^+$	0.02447	0.05075	0.02048	0.05620	0.04050	0.01979	0.00205	0.05886
	$\hat{\alpha}_1^-$	0.03803	0.08468	0.03274	0.09880	0.04617	0.09510	0.03102	0.06150
	$\hat{\beta}_1$	0.97041	0.93463	0.96675	0.92254	0.96131	0.94293	0.98273	0.94044
2	$\hat{\omega}$	0.00113	0.00466	0.00456	0.00693	0.00120	0.00396	0.00064	0.00509
	$\hat{\alpha}_1^+$	0.02100	0.04517	0.01672	0.04936	0.03408	0.01669	0.00406	0.05209
	$\hat{\alpha}_1^-$	0.03156	0.07586	0.02646	0.08730	0.03970	0.08730	0.02402	0.04845
	$\hat{\beta}_1$	0.97110	0.93242	0.96856	0.94046	0.96131	0.94046	0.98233	0.94082
2.5	$\hat{\omega}$	0.00078	0.00378	0.00347	0.00597	0.00096	0.00285	0.00044	0.00425
	$\hat{\alpha}_1^+$	0.01652	0.03838	0.01164	0.04137	0.02680	0.01340	0.00386	0.04405
	$\hat{\alpha}_1^-$	0.02438	0.06435	0.01945	0.08541	0.03246	0.07338	0.01624	0.03406
	$\hat{\beta}_1$	0.97161	0.92895	0.97126	0.91149	0.96114	0.93840	0.98284	0.94060

Table 6: Estimation of daily exchange rates of the Dollar and Yen against the Euro with the APGARCH(1,1) model in dimension 1.

Parameters	Full Period		First Period		Second Period		Third Period	
	USD	JPY	USD	JPY	USD	JPY	USD	JPY
$\hat{\omega}$	0.00279	0.00740	0.01070	0.00896	0.00127	0.00810	0.00185	0.00777
$\hat{\alpha}_1^+$	0.02618	0.05331	0.02150	0.05864	0.03599	0.02394	0.00161	0.06189
$\hat{\alpha}_1^-$	0.04063	0.08616	0.02992	0.09854	0.04158	0.09056	0.03302	0.06754
$\hat{\beta}_1$	0.96978	0.93580	0.96620	0.92450	0.96130	0.94531	0.98309	0.93990
$\hat{\tau}$	1.04728	1.12923	0.53666	1.24883	1.86581	1.02499	0.96029	1.10951

Table 7: Estimation of daily exchange rates of the (Dollar,Yen) against the Euro with the CCC-APGARCH(1,1) model in dimension 2 when $\hat{\delta}_0$ is fixed.

$\hat{\delta}_0$	Parameters	Full Period	First Period	Second Period	Third Period
(2, 1)	$\hat{\omega}$	0.00535	0.33950	0.01813	0.06369
		0.14680	0.33068	0.08347	0.09406
	$\hat{\alpha}_1^+$	0.04265	0.01283	0.10252	0.14253
		0.00000	0.00000	0.04520	0.00361
		0.00484	0.11529	0.06218	0.02385
		0.04232	0.03573	0.00000	0.07928
	$\hat{\alpha}_1^-$	0.02825	0.00000	0.05721	0.02225
		0.00000	0.00000	0.00000	0.00000
		0.02203	0.03164	0.00000	0.06317
		0.11239	0.17432	0.08536	0.04928
	$\hat{\beta}_1$	0.94718	0.37445	0.86761	0.73823
		0.04727	0.00000	0.02670	0.00000
0.00000		0.00000	0.00000	0.00000	
0.79038		0.63165	0.86991	0.87176	
$\hat{\rho}_{21}$	0.68316	0.74252	0.66079	0.69230	
(2, 2)	$\hat{\omega}$	0.00600	0.06562	0.01928	0.04892
		0.04538	0.12019	0.01354	0.02421
	$\hat{\alpha}_1^+$	0.03022	0.02592	0.07476	0.09216
		0.00526	0.00000	0.06385	0.00262
		0.00000	0.00000	0.02459	0.00734
		0.06315	0.04730	0.00443	0.07986
	$\hat{\alpha}_1^-$	0.02421	0.02062	0.03516	0.03387
		0.00000	0.0000	0.00000	0.00000
		0.00808	0.00986	0.00000	0.01241
		0.14923	0.25040	0.13401	0.05892
	$\hat{\beta}_1$	0.95080	0.82370	0.87099	0.77251
		0.00000	0.00000	0.00000	0.00000
0.00000		0.00000	0.00528	0.00000	
0.81186		0.65577	0.88146	0.88318	
$\hat{\rho}_{21}$	0.55335	0.61313	0.52596	0.56145	

Table 8: Estimation of daily exchange rates of the (Dollar,Yen) against the Euro with the CCC-APGARCH(1,1) model in dimension 2.

Parameters	Full Period	First Period	Second Period	Third Period
$\hat{\omega}$	0.00136	0.00117	0.02407	0.11245
	0.06124	0.15275	0.01453	0.07930
$\hat{\alpha}_1^+$	0.03050	0.00719	0.06434	0.04457
	0.00000	0.00000	0.06129	0.01818
	0.00000	0.05530	0.01975	0.00000
	0.05368	0.04459	0.00292	0.05135
$\hat{\alpha}_1^-$	0.02351	0.00000	0.03112	0.02393
	0.00000	0.00613	0.00000	0.00000
	0.01072	0.00994	0.00000	0.00000
	0.12207	0.15270	0.12752	0.01499
$\hat{\beta}_1$	0.95326	0.41630	0.89072	0.90158
	0.03182	0.00000	0.00000	0.00000
	0.00000	0.00000	0.00382	0.00410
	0.80512	0.70082	0.88657	0.87580
$\hat{\rho}_{21}$	0.55106	0.61076	0.52380	0.57556
$\hat{\tau}_1$	2.01916	3.93035	1.95333	1.53699
$\hat{\tau}_2$	1.88965	1.72984	1.98917	1.77340

7. Conclusion

In this paper we propose a class of multivariate asymmetric GARCH models which includes numerous functional forms of MGARCH. We provide an explicit necessary and sufficient condition to the strict stationary of the proposed model. In addition the asymptotic properties of the QMLE are investigated in the two cases of $\underline{\delta}$ known and unknown) We remark that moment conditions are not needed. A Wald test is proposed to test s linear constraints on the parameter. In Monte Carlo experiments we demonstrated that the QMLE of the CCC-APGARCH models provide some satisfactory results, at least for the models considered in our study.

A. Appendix : Proofs of the main results

A.1. Preliminaries

In the following technical proofs we will use the following notations. We will use the multiplicative norm defined as:

$$\|A\| := \sup_{\|x\| \leq 1} \|Ax\| = \rho^{1/2}(A'A),$$

where A is a matrix of size $d_1 \times d_2$ and $\|x\|$ is the Euclidian norm of vector $x \in \mathbb{R}^{d_2}$ and $\rho(\cdot)$ is the spectral radius. We recall that this norm satisfies

$$\|A\|^2 \leq \sum_{i,j} a_{i,j}^2 = Tr(A'A) \leq d_2 \|A\|^2, \quad |A'A| \leq \|A\|^{2d_2}.$$

Moreover we have the following relation

$$|Tr(AB)| \leq (d_1 d_2)^{1/2} \|A\| \|B\|$$

as long as the matrix product is well defined (actually B is a $d_2 \times d_1$ matrix).

We recall some useful derivation rules for matrix valued functions. If we consider $f(A)$ a scalar function of a matrix A , where all the a_{ij} are considered as a function of an one real variable x , we have

$$\frac{\partial f(A)}{\partial x} = \sum_{i,j} \frac{\partial f(A)}{\partial a_{ij}} \frac{\partial a_{ij}}{\partial x} = \text{Tr} \left(\frac{\partial f(A)}{\partial A'} \frac{\partial A}{\partial x} \right). \quad (\text{A.1})$$

For a non singular matrix A , we have the following relations:

$$\frac{\partial c' A c}{\partial A'} = c c' \quad (\text{A.2})$$

$$\frac{\partial \text{Tr}(C A' B A')}{\partial A'} = C' A B' + B' A C' \quad (\text{A.3})$$

$$\frac{\partial \log |\det(A)|}{\partial A'} = A^{-1} \quad (\text{A.4})$$

$$\frac{\partial A^{-1}}{\partial x} = -A^{-1} \frac{\partial A}{\partial x} A^{-1} \quad (\text{A.5})$$

$$\frac{\partial \text{Tr}(C A^{-1} B)}{\partial A'} = -A^{-1} B C A^{-1} \quad (\text{A.6})$$

$$\frac{\partial \text{Tr}(C A B)}{\partial A'} = B C \quad (\text{A.7})$$

A.2. Proof of Theorem 2

We prove the consistency of the QMLE result when δ_0 known following the same lines as in [Francq and Zakoïan \(2012\)](#).

We first rewrite Equation (3.2) in the matrix form as follows:

$$\mathbb{H}_t(\theta) = \underline{c}_t(\theta) + \mathbb{B}(\theta) \cdot \mathbb{H}_{t-1}(\theta), \quad (\text{A.8})$$

with

$$\mathbb{H}_t(\theta) = \begin{pmatrix} \underline{h}_t^{\delta/2}(\theta) \\ \underline{h}_{t-1}^{\delta/2}(\theta) \\ \vdots \\ \underline{h}_{t-p+1}^{\delta/2}(\theta) \end{pmatrix}, \quad \underline{c}_t(\theta) = \begin{pmatrix} \underline{\omega} + \sum_{i=1}^q A_i^+ (\underline{\varepsilon}_{t-i}^+)^{\delta/2} + A_i^- (\underline{\varepsilon}_{t-1}^-)^{\delta/2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$\mathbb{B}(\theta) = \begin{pmatrix} B_1 & B_2 & \dots & B_p \\ I_m & & & 0 \\ 0 & I_m & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & I_m & 0 \end{pmatrix}.$$

For simplicity, one writes \mathbb{H}_t instead of $\mathbb{H}_t(\theta)$ when there is no possible confusion (and analogously one writes \underline{c}_t and \mathbb{B}). We iterate the expression (A.8) and we obtain

$$\mathbb{H}_t = \underline{c}_t + \mathbb{B} \underline{c}_{t-1} + \mathbb{B}^2 \underline{c}_{t-2} + \dots + \mathbb{B}^{t-1} \underline{c}_1 + \mathbb{B}^t \mathbb{H}_0 = \sum_{k=0}^{\infty} \mathbb{B}^k \underline{c}_{t-k}. \quad (\text{A.9})$$

The proof is decomposed in the four following points which will be treated in separate subsections.

- (i) Initial values do not influence quasi-likelihood: $\lim_{n \rightarrow \infty} \sup_{\theta \in \Theta} |\mathcal{L}_n(\theta) - \tilde{\mathcal{L}}_n(\theta)| = 0$, almost surely.
- (ii) Identifiability: if there exists $t \in \mathbb{Z}$ such that $\underline{h}_t^{\delta_0/2}(\theta) = \underline{h}_t^{\delta_0/2}(\theta_0)$ almost surely and $R = R_0$, then $\theta = \theta_0$
- (iii) Minimization of the quasi-likelihood on the true value: $\mathbb{E}_{\theta_0} |l_t(\theta_0)| < \infty$, and if $\theta \neq \theta_0$, $\mathbb{E}_{\theta_0} [l_t(\theta)] > \mathbb{E}_{\theta_0} [l_t(\theta_0)]$.
- (iv) For any $\theta \neq \theta_0$ there exists a neighborhood $V(\theta)$ such that

$$\liminf_{n \rightarrow \infty} \inf_{\theta^* \in V(\theta)} \tilde{\mathcal{L}}_n(\theta^*) > \mathbb{E}_{\theta_0} l_1(\theta_0), \quad a.s. \quad (\text{A.10})$$

A.2.1. Initial values do not influence quasi-likelihood

We define the vectors $\tilde{\mathbb{H}}_t$ by replacing the variables $\underline{h}_{t-i}^{\delta/2}$, $i = 0, \dots, p-1$, in \mathbb{H}_t by $\tilde{\underline{h}}_{t-i}^{\delta/2}$ and we have

$$\tilde{\mathbb{H}}_t = \underline{c}_t + \mathbb{B}\underline{c}_{t-1} + \dots + \mathbb{B}^{t-q-1}\underline{c}_{q+1} + \mathbb{B}^{t-q}\tilde{\underline{c}}_q + \dots + \mathbb{B}^{t-1}\tilde{\underline{c}}_1 + \mathbb{B}^t\mathbb{H}_0, \quad (\text{A.11})$$

where the vector $\tilde{\underline{c}}_t$ is obtained by replacing $\underline{\varepsilon}_0, \dots, \underline{\varepsilon}_{1-q}$ in \underline{c}_t by the initial values.

From the Assumption **A2** and Corollary 1, we have $\rho(\mathbb{B}) < 1$ and we deduce from the compactness of Θ that we have $\sup_{\theta \in \Theta} \rho(\mathbb{B}) < 1$. Using the two iterative equations (A.8) and (A.11), we obtain almost surely that for any t :

$$\sup_{\theta \in \Theta} \left\| \mathbb{H}_t - \tilde{\mathbb{H}}_t \right\| = \sup_{\theta \in \Theta} \left\| \sum_{k=1}^q \mathbb{B}^{t-k} (\underline{c}_k - \tilde{\underline{c}}_k) + \mathbb{B}^t (\mathbb{H}_0 - \tilde{\mathbb{H}}_0) \right\| \leq K \rho^t, \quad (\text{A.12})$$

where K is a random constant that depends on the past values of $\{\varepsilon_t, t \leq 0\}$. We may write (A.12) as

$$\sup_{\theta \in \Theta} \left\| \underline{h}_t^{\delta/2}(\theta) - \tilde{\underline{h}}_t^{\delta/2}(\theta) \right\| \leq K \rho^t. \quad (\text{A.13})$$

Thus, for $i_1 = 1, \dots, m$, since $\min \left(h_{i_1,t}^{\delta_{i_1}/2}(\theta), \tilde{h}_{i_1,t}^{\delta_{i_1}/2}(\theta) \right) \geq \omega = \inf_{1 \leq i \leq m} \omega(i)$, the mean-value theorem implies that

$$\begin{aligned} \sup_{\theta \in \Theta} \left| h_{i_1,t}(\theta) - \tilde{h}_{i_1,t}(\theta) \right| &\leq \frac{2}{\delta_{i_1}} \sup_{\theta \in \Theta} \max \left(h_{i_1,t}^{1-\delta_{i_1}/2}(\theta), \tilde{h}_{i_1,t}^{1-\delta_{i_1}/2}(\theta) \right) \sup_{\theta \in \Theta} \left| h_{i_1,t}^{\delta_{i_1}/2}(\theta) - \tilde{h}_{i_1,t}^{\delta_{i_1}/2}(\theta) \right| \\ &\leq \frac{2K}{\delta_{i_1}} \left(\sup_{\theta \in \Theta} \frac{1}{\omega} \right) \sup_{\theta \in \Theta} \max \left(h_{i_1,t}(\theta), \tilde{h}_{i_1,t}(\theta) \right) \rho^t \leq K \rho^t, \end{aligned} \quad (\text{A.14})$$

and similarly

$$\begin{aligned} \sup_{\theta \in \Theta} \left| h_{i_1,t}^{1/2}(\theta) - \tilde{h}_{i_1,t}^{1/2}(\theta) \right| &\leq \frac{1}{\delta_{i_1}} \sup_{\theta \in \Theta} \max \left(h_{i_1,t}^{(1-\delta_{i_1})/2}(\theta), \tilde{h}_{i_1,t}^{(1-\delta_{i_1})/2}(\theta) \right) \sup_{\theta \in \Theta} \left| h_{i_1,t}^{\delta_{i_1}/2}(\theta) - \tilde{h}_{i_1,t}^{\delta_{i_1}/2}(\theta) \right| \\ &\leq \frac{K}{\delta_{i_1}} \left(\sup_{\theta \in \Theta} \frac{1}{\omega} \right) \sup_{\theta \in \Theta} \max \left(h_{i_1,t}^{1/2}(\theta), \tilde{h}_{i_1,t}^{1/2}(\theta) \right) \rho^t \leq K \rho^t. \end{aligned} \quad (\text{A.15})$$

From (A.14) we can deduce that, almost surely, we have

$$\sup_{\theta \in \Theta} \left\| H_t - \tilde{H}_t \right\| \leq K \rho^t, \quad \forall t. \quad (\text{A.16})$$

Since $\|R^{-1}\|$ is the inverse of the eigenvalue of smaller module of R and $\|\tilde{D}_t^{-1}\| = [\min_i(h_{ii,t}^{1/\delta_i})]^{-1}$, we have

$$\sup_{\theta \in \Theta} \|\tilde{H}_t^{-1}\| \leq \sup_{\theta \in \Theta} \|\tilde{D}^{-1}\|^2 \|R^{-1}\| \leq \sup_{\theta \in \Theta} \left[\min_i(\underline{\omega}(i)^{1/\delta_i}) \right]^{-1} \|R^{-1}\| \leq K, \quad (\text{A.17})$$

by using the fact that R is a positive-definite matrix (see assumption **A5**), the compactness of Θ and the strict positivity of the components of $\underline{\omega}$. Similarly, we have

$$\sup_{\theta \in \Theta} \|H_t^{-1}\| \leq K. \quad (\text{A.18})$$

One may writes

$$\begin{aligned} \sup_{\theta \in \Theta} |\mathcal{L}_t(\theta) - \tilde{\mathcal{L}}_t(\theta)| &\leq \frac{1}{n} \sum_{t=1}^n \sup_{\theta \in \Theta} \left| \underline{\varepsilon}'_t (H_t^{-1} - \tilde{H}_t^{-1}) \underline{\varepsilon}_t + \log |H_t| - \log |\tilde{H}_t| \right| \\ &\leq \frac{1}{n} \sum_{t=1}^n \sup_{\theta \in \Theta} \left| \underline{\varepsilon}'_t (H_t^{-1} - \tilde{H}_t^{-1}) \underline{\varepsilon}_t \right| + \frac{1}{n} \sum_{t=1}^n \sup_{\theta \in \Theta} \left| \log |H_t| - \log |\tilde{H}_t| \right| \\ &\leq S_1 + S_2. \end{aligned}$$

We can rewrite the first term S_1 in the right hand side of the above inequality as

$$\begin{aligned} S_1 &= \frac{1}{n} \sum_{t=1}^n \sup_{\theta \in \Theta} \left| \underline{\varepsilon}'_t H_t^{-1} (H_t - \tilde{H}_t) \tilde{H}_t^{-1} \underline{\varepsilon}_t \right| \\ &= \frac{1}{n} \sum_{t=1}^n \sup_{\theta \in \Theta} \left| \text{Tr}(H_t^{-1} (H_t - \tilde{H}_t) \tilde{H}_t^{-1} \underline{\varepsilon}_t \underline{\varepsilon}'_t) \right| \\ &\leq K \frac{1}{n} \sum_{t=1}^n \sup_{\theta \in \Theta} \|H_t^{-1}\| \|H_t - \tilde{H}_t\| \|\tilde{H}_t^{-1}\| \|\underline{\varepsilon}_t \underline{\varepsilon}'_t\| \\ &\leq K \frac{1}{n} \sum_{t=1}^n \rho^t \|\underline{\varepsilon}_t \underline{\varepsilon}'_t\| \end{aligned}$$

where we have used (A.16), (A.17) and (A.18). Using the Borel-Cantelli lemma and Corollary 2, we deduce that $\rho^t \|\underline{\varepsilon}_t \underline{\varepsilon}'_t\| = \rho^t \underline{\varepsilon}'_t \underline{\varepsilon}_t$ goes to zero almost surely. Consequently, the Cesàro lemma implies that $n^{-1} \sum_{t=1}^n \rho^t \|\underline{\varepsilon}_t \underline{\varepsilon}'_t\| \rightarrow 0$ when n goes to infinity.

For the second term S_2 we use $\log(1+x) \leq x$ for $x > -1$ and the inequality $|\det(A)| \leq \rho(A)^m \leq \|A\|^m$ and we obtain

$$\begin{aligned} \log |H_t| - \log |\tilde{H}_t| &= \log |H_t \tilde{H}_t^{-1}| \\ &= \log |I_m + (H_t - \tilde{H}_t) \tilde{H}_t^{-1}| \\ &\leq m \log \|I_m + (H_t - \tilde{H}_t) \tilde{H}_t^{-1}\| \\ &\leq m \|H_t - \tilde{H}_t\| \|\tilde{H}_t^{-1}\|, \end{aligned}$$

and, by symmetry, we have

$$\log |\tilde{H}_t| - \log |H_t| \leq m \|H_t - \tilde{H}_t\| \|H_t^{-1}\|.$$

Using (A.16), (A.17) and (A.18), we have

$$\left| \log |H_t| - \log |\tilde{H}_t| \right| \leq m \|H_t - \tilde{H}_t\| \left(\|\tilde{H}_t^{-1}\| + \|H_t^{-1}\| \right) \leq K \rho^t.$$

Using the same arguments as for S_1 , we conclude that S_2 goes to 0. We have shown that $\sup_{\theta \in \Theta} |\mathcal{L}_t(\theta) - \tilde{\mathcal{L}}_t(\theta)| \xrightarrow[n \rightarrow \infty]{} 0$ almost surely and thus (i) is proved.

A.2.2. Identifiability

We suppose that for some $\theta \neq \theta_0$ we have

$$\underline{h}_t(\theta) = \underline{h}_t(\theta_0), \quad \mathbb{P}_{\theta_0} - p.s., \quad \text{and} \quad R(\theta) = R(\theta_0).$$

So we have $\rho = \rho_0$. Remind that by (2.5) we have

$$\underline{h}_t^{\delta/2}(\theta_0) = \mathcal{B}_0^{-1}(1)\underline{\omega} + \mathcal{B}_0^{-1}(L)\mathcal{A}_0^+(L)(\underline{\varepsilon}_t^+)^{\delta/2} + \mathcal{B}_0^{-1}(L)\mathcal{A}_0^-(L)(\underline{\varepsilon}_t^-)^{\delta/2}.$$

We have a similar expression with the parameter θ :

$$\underline{h}_t^{\delta/2}(\theta) = \mathcal{B}^{-1}(1)\underline{\omega} + \mathcal{B}^{-1}(L)\mathcal{A}^+(L)(\underline{\varepsilon}_t^+)^{\delta/2} + \mathcal{B}^{-1}(L)\mathcal{A}^-(L)(\underline{\varepsilon}_t^-)^{\delta/2}$$

Consequently we have

$$\begin{aligned} 0 &= \mathcal{B}^{-1}(1)\underline{\omega} - \mathcal{B}_0^{-1}(1)\underline{\omega}_0 + \mathcal{B}^{-1}(L)\mathcal{A}^+(L)(\underline{\varepsilon}_t^+)^{\delta/2} - \mathcal{B}_0^{-1}(L)\mathcal{A}_0^+(L)(\underline{\varepsilon}_t^+)^{\delta/2} \\ &\quad + \mathcal{B}^{-1}(L)\mathcal{A}^-(L)(\underline{\varepsilon}_t^-)^{\delta/2} - \mathcal{B}_0^{-1}(L)\mathcal{A}_0^-(L)(\underline{\varepsilon}_t^-)^{\delta/2} \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}^{-1}(1)\underline{\omega} - \mathcal{B}_0^{-1}(1)\underline{\omega}_0 &= [\mathcal{B}^{-1}(L)\mathcal{A}^+(L) - \mathcal{B}_0^{-1}(L)\mathcal{A}_0^+(L)] (\underline{\varepsilon}_t^+)^{\delta/2} + [\mathcal{B}^{-1}(L)\mathcal{A}^-(L) - \mathcal{B}_0^{-1}(L)\mathcal{A}_0^-(L)] (\underline{\varepsilon}_t^-)^{\delta/2} \\ &= \mathcal{P}^+(L)(\underline{\varepsilon}_t^+)^{\delta/2} + \mathcal{P}^-(L)(\underline{\varepsilon}_t^-)^{\delta/2}, \end{aligned}$$

where

$$\mathcal{P}^{\pm}(L) = \mathcal{B}_0^{-1}(L)\mathcal{A}_0^{\pm}(L) - \mathcal{B}^{-1}(L)\mathcal{A}^{\pm}(L) = \sum_{i=0}^{\infty} P_i^{\pm} L^i. \quad (\text{A.19})$$

We remark that $P_0^{\pm} = \mathcal{P}^{\pm}(0) = 0$ by the identifiability conditions. Using (2.2) and (2.3) we can write

$$\begin{aligned} \mathcal{P}^+(L)(\underline{\varepsilon}_t^+)^{\delta/2} + \mathcal{P}^-(L)(\underline{\varepsilon}_t^-)^{\delta/2} &= P_1^+(\underline{\varepsilon}_{t-1}^+)^{\delta/2} + P_1^-(\underline{\varepsilon}_{t-1}^-)^{\delta/2} + Z_{t-2} \\ &= P_1^+ \Upsilon_{t-1}^{+,(\delta)} \underline{h}_{t-1}^{\delta/2} + P_1^- \Upsilon_{t-1}^{-,(\delta)} \underline{h}_{t-1}^{\delta/2} + Z_{t-2} \quad a.s., \end{aligned}$$

where Z_{t-2} is measurable with respect to the σ -field \mathcal{F}_{t-2} generated by $\{\eta_{t-2}, \eta_{t-3}, \dots\}$. Hence we have

$$P_1^+ \Upsilon_{t-1}^{+,(\delta)} \underline{h}_{t-1}^{\delta/2} + P_1^- \Upsilon_{t-1}^{-,(\delta)} \underline{h}_{t-1}^{\delta/2} = \mathcal{B}^{-1}(1)\underline{\omega} - \mathcal{B}_0^{-1}(1)\underline{\omega}_0 - Z_{t-2} = \tilde{Z}_{t-2}$$

where \tilde{Z}_{t-2} is another \mathcal{F}_{t-2} -measurable random matrix. Since $P_1^+ \Upsilon_{t-1}^{+,(\delta)} + P_1^- \Upsilon_{t-1}^{-,(\delta)}$ is independent from $(Z_{t-2}, \underline{h}_{t-1})$ and since $\underline{h}_{t-1} > 0$, $P_1^+ \Upsilon_{t-1}^{+,(\delta)} + P_1^- \Upsilon_{t-1}^{-,(\delta)} = C$ for some constant matrix C . Since the matrices $\Upsilon_{t-1}^{\pm,(\delta)}$ are diagonal (see (2.3)), the element (i, j) of the matrix C satisfies

$$C(i, j) = P_1^+(i, j)(\tilde{\eta}_{j,t-1}^+)^{\delta_j} + P_1^-(i, j)(-\tilde{\eta}_{j,t-1}^-)^{\delta_j}.$$

If $P_1^+(i, j)P_1^-(i, j) \neq 0$, then $\tilde{\eta}_{j,t-1}$ takes at most two different values, which is in contradiction with **A3**. If $P_1^+(i, j) \neq 0$ and $P_1^-(i, j) = 0$, then $P_1^+(i, j)(\tilde{\eta}_{j,t-1}^+)^{\delta_j} = C(i, j)$ which entails $C(i, j) = 0$, since $\mathbb{P}(\tilde{\eta}_{j,t-1}^+ > 0) \neq 0$, and then $\tilde{\eta}_{j,t-1} = 0$, a.s., which is also in contradiction with **A3**. We thus have $P_1^+ = P_1^- = 0$. We argue similarly for P_i^{\pm} for $i \geq 2$ and by (A.19) we obtain that $\mathcal{P}^+(L) = \mathcal{P}^-(L) = 0$. Therefore, in view of (A.19), we may apply Proposition 1 because we assumed **A4** (or **A4'**). Thus we have $\theta = \theta_0$. We have thus established (ii).

The proof of (iii) and (iv) is strictly identical to the one given in Francq and Zakoian (2012). Therefore it is omitted and the proof of Theorem 2 is complete. \square

A.3. Proof of Theorem 3

Here, we prove the asymptotic normality result when $\underline{\delta}$ is known. The prove is based on the standard Taylor expansion. We have

$$\sqrt{n}(\hat{\theta}_n - \theta_0) = - \left[\frac{1}{n} \sum_{t=1}^n \frac{\partial^2 \tilde{l}_t(\theta_{ij}^*)}{\partial \theta_i \partial \theta_j} \right]^{-1} \left(\frac{1}{\sqrt{n}} \sum_{t=1}^n \frac{\partial \tilde{l}_t(\theta_0)}{\partial \theta} \right),$$

where the parameter θ_{ij}^* is between $\hat{\theta}_n$ and θ_0 . To establish the asymptotic normality result when the power is known, we will decomposed the proof in six intermediate points as in [Francq and Zakoïan \(2012\)](#).

- (a) First derivative of the quasi log-likelihood.
- (b) Existence of moments at any order of the score.
- (c) Asymptotic normality of the score vector:

$$\frac{1}{\sqrt{n}} \sum_{t=1}^n \frac{\partial l_t(\theta_0)}{\partial \theta} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, I). \quad (\text{A.20})$$

- (d) Convergence to J :

$$\frac{1}{n} \sum_{t=1}^n \frac{\partial^2 l_t(\theta_{ij}^*)}{\partial \theta_i \partial \theta_j} \xrightarrow[n \rightarrow \infty]{} J(i, j) \text{ in probability.} \quad (\text{A.21})$$

- (e) Invertibility of the matrix J .
- (f) Forgetting of the initial values.

We shall need the following notations.

- * $s_0 = m + (p + 2q)m^2 + m(m - 1)/2$,
- * $s_1 = m + (p + 2q)m^2$,
- * $s_2 = m + 2qm^2$,
- * $s_3 = m + qm^2$.

These notations will help us to point out the different entries of our parameter θ .

A.3.1. First derivative of log-likelihood

The aim of this subsection is to establish the expressions of the first order derivatives of the quasi log-likelihood. We shall use the following notations: $D_{0t} = D_t(\theta_0)$, $R_0 = R(\theta_0)$,

$$D_{0t}^{(i)} = \frac{\partial D_t}{\partial \theta_i}(\theta_0), \quad R_0^{(i)} = \frac{\partial R}{\partial \theta_i}(\theta_0),$$

$$D_{0t}^{(i,j)} = \frac{\partial^2 D_t}{\partial \theta_i \partial \theta_j}(\theta_0), \quad R_0^{(i,j)} = \frac{\partial^2 R}{\partial \theta_i \partial \theta_j}(\theta_0),$$

and $\underline{\varepsilon}_t = D_{0t} \tilde{\eta}_t$, where $\tilde{\eta}_t(\theta) = R^{1/2} \eta_t(\theta)$ with $\tilde{\eta}_t = \tilde{\eta}_t(\theta_0) = R_0^{1/2} \eta_t$.

We recall the expression:

$$\begin{aligned}
l_t(\theta) &= \underline{\varepsilon}'_t H_t^{-1} \underline{\varepsilon}_t + \log(\det(H_t)) \\
&= \underline{\varepsilon}'_t (D_t R D_t)^{-1} \underline{\varepsilon}_t + \log(\det(D_t R D_t)) \\
&= \underline{\varepsilon}'_t D_t^{-1} R^{-1} D_t^{-1} \underline{\varepsilon}_t + \log(\det(D_t) \det(R) \det(D_t)) \\
&= \underline{\varepsilon}'_t D_t^{-1} R^{-1} D_t^{-1} \underline{\varepsilon}_t + 2 \log(\det(D_t)) + \log(\det(R)).
\end{aligned}$$

- We differentiate with respect to θ_i for $i = 1, \dots, s_1$ (that is with respect to $(\underline{\omega}', \alpha_1^{+'}, \dots, \alpha_q^{+'}, \alpha_1^{-'}, \dots, \alpha_q^{-'}, \beta'_1, \dots, \beta'_p)'$). We have

$$\frac{\partial l_t(\theta)}{\partial \theta_i} = -Tr \left((\underline{\varepsilon}_t \underline{\varepsilon}'_t D_t^{-1} R^{-1} + R^{-1} D_t^{-1} \underline{\varepsilon}_t \underline{\varepsilon}'_t) D_t^{-1} \frac{\partial D_t}{\partial \theta_i} D_t^{-1} \right) + 2Tr \left(D_t^{-1} \frac{\partial D_t}{\partial \theta_i} \right) \quad (\text{A.22})$$

$$\frac{\partial l_t(\theta_0)}{\partial \theta_i} = Tr \left[(I_m - R_0^{-1} \tilde{\eta}_t \tilde{\eta}'_t) D_{0t}^{(i)} D_{0t}^{-1} + (I_m - \tilde{\eta}_t \tilde{\eta}'_t R_0^{-1}) D_{0t}^{-1} D_{0t}^{(i)} \right]. \quad (\text{A.23})$$

- We differentiate with respect to θ_i for $i = s_1 + 1, \dots, s_0$ (that is with respect to ρ'). We have

$$\frac{\partial l_t(\theta)}{\partial \theta_i} = -Tr \left(R^{-1} D_t^{-1} \underline{\varepsilon}_t \underline{\varepsilon}'_t D_t^{-1} R^{-1} \frac{\partial R}{\partial \theta_i} \right) + Tr \left(R^{-1} \frac{\partial R}{\partial \theta_i} \right) \quad (\text{A.24})$$

$$\frac{\partial l_t(\theta_0)}{\partial \theta_i} = Tr \left[(I_m - R^{-1} \tilde{\eta}_t \tilde{\eta}'_t) R^{-1} R^{(i)} \right]. \quad (\text{A.25})$$

A.3.2. Existence of moments of any order of the score

- (i) For $i = 1, \dots, s_1$, in view of (A.23) we have

$$\begin{aligned}
\left| \frac{\partial l_t(\theta_0)}{\partial \theta_i} \right| &= \left| Tr \left[(I_m - R_0^{-1} \tilde{\eta}_t \tilde{\eta}'_t) D_{0t}^{(i)} D_{0t}^{-1} + (I_m - \tilde{\eta}_t \tilde{\eta}'_t R_0^{-1}) D_{0t}^{-1} D_{0t}^{(i)} \right] \right| \\
&\leq 2m \underbrace{\| (I_m - R_0^{-1} \tilde{\eta}_t \tilde{\eta}'_t) \|}_{\text{constant}} \| D_{0t}^{(i)} D_{0t}^{-1} \| \leq K \| D_{0t}^{(i)} D_{0t}^{-1} \|.
\end{aligned}$$

- (ii) For $i = s_1 + 1, \dots, s_0$, in view of (A.25) we have

$$\begin{aligned}
\left| \frac{\partial l_t(\theta_0)}{\partial \theta_i} \right| &= \left| Tr \left[(I_m - R_0^{-1} \tilde{\eta}_t \tilde{\eta}'_t) R_0^{-1} R_0^{(i)} \right] \right| \\
&\leq m \underbrace{\| (I_m - R_0^{-1} \tilde{\eta}_t \tilde{\eta}'_t) \|}_{\text{constant}} \underbrace{\| R_0^{-1} R_0^{(i)} \|}_{\text{constant}} \leq K.
\end{aligned}$$

- (iii) For $i, j = 1, \dots, s_1$, in view of (A.23) we have

$$\begin{aligned}
\mathbb{E} \left| \frac{\partial l_t(\theta_0)}{\partial \theta_i} \frac{\partial l_t(\theta_0)}{\partial \theta_j} \right| &= \mathbb{E} \left| Tr \left[(I_m - R_0^{-1} \tilde{\eta}_t \tilde{\eta}'_t) D_{0t}^{(i)} D_{0t}^{-1} + (I_m - \tilde{\eta}_t \tilde{\eta}'_t R_0^{-1}) D_{0t}^{-1} D_{0t}^{(i)} \right] \right. \\
&\quad \left. \times Tr \left[(I_m - R_0^{-1} \tilde{\eta}_t \tilde{\eta}'_t) D_{0t}^{(j)} D_{0t}^{-1} + (I_m - \tilde{\eta}_t \tilde{\eta}'_t R_0^{-1}) D_{0t}^{-1} D_{0t}^{(j)} \right] \right| \\
&\leq \mathbb{E} \left[K_1 \| D_{0t}^{(i)} D_{0t}^{-1} \| \times K_2 \| D_{0t}^{(j)} D_{0t}^{-1} \| \right] \\
&\leq K \left(\mathbb{E} \| D_{0t}^{(i)} D_{0t}^{-1} \|^2 \mathbb{E} \| D_{0t}^{(j)} D_{0t}^{-1} \|^2 \right)^{1/2}.
\end{aligned}$$

by using the Cauchy-Schwarz inequality.

(iv) In view of (A.23) and (A.25), we also have for $i = 1, \dots, s_1$ and $j = s_1 + 1, \dots, s_0$,

$$\begin{aligned} \mathbb{E} \left| \frac{\partial l_t(\theta_0)}{\partial \theta_i} \frac{\partial l_t(\theta_0)}{\partial \theta_j} \right| &= \mathbb{E} \left| \text{Tr} \left[(I_m - R_0^{-1} \tilde{\eta}_t \tilde{\eta}_t') D_{0t}^{(i)} D_{0t}^{-1} + (I_m - \tilde{\eta}_t \tilde{\eta}_t' R_0^{-1}) D_{0t}^{-1} D_{0t}^{(i)} \right] \right. \\ &\quad \left. \times \text{Tr} \left[(I_m - R_0^{-1} \tilde{\eta}_t \tilde{\eta}_t') R_0^{-1} R_0^{(j)} \right] \right| \\ &\leq \mathbb{E} \left[K_1 \left\| D_{0t}^{(i)} D_{0t}^{-1} \right\| \times K_2 \right] \leq K \mathbb{E} \left\| D_{0t}^{(i)} D_{0t}^{-1} \right\|. \end{aligned}$$

(v) For $i, j = s_1 + 1, \dots, s_0$, in view of (A.25) we have,

$$\begin{aligned} \mathbb{E} \left| \frac{\partial l_t(\theta_0)}{\partial \theta_i} \frac{\partial l_t(\theta_0)}{\partial \theta_j} \right| &= \mathbb{E} \left| \text{Tr} \left[(I_m - R_0^{-1} \tilde{\eta}_t \tilde{\eta}_t') R_0^{-1} R_0^{(i)} \right] \right. \\ &\quad \left. \times \text{Tr} \left[(I_m - R_0^{-1} \tilde{\eta}_t \tilde{\eta}_t') R_0^{-1} R_0^{(j)} \right] \right| \leq \mathbb{E}[K_1 \times K_2] \leq K. \end{aligned}$$

To have the finiteness of the moments of the first derivative of the log-likelihood, it remains to treat the cases (i), (iii) and (iv) above. Thus, we have to control the term $\|D_{0t}^{(i)} D_{0t}^{-1}\|$. Since

$$D_{0t} = \text{Diag}(h_t^{1/2}(\theta_0)) = \text{Diag} \left((h_t^{\delta/2}(\theta_0))^{1/\delta} \right),$$

we can work component by component. We have for $i_1 = 1, \dots, m$ and $i = 1, \dots, s_1$

$$\frac{\partial D_{0t}(i_1, i_1)}{\partial \theta_i} = \frac{\partial \left(h_{i_1, t}^{\delta_{i_1}/2} \right)^{1/\delta_{i_1}}}{\partial \theta_i} = \frac{1}{\delta_{i_1}} h_{i_1, t}^{1/2} \times \frac{1}{h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial h_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_i}(\theta_0). \quad (\text{A.26})$$

Control the term $\|D_{0t}^{(i)} D_{0t}^{-1}\|$ is equivalent to control $1/h_{i_1, t}^{\delta_{i_1}/2} \partial h_{i_1, t}^{\delta_{i_1}/2} / \partial \theta_i$ in θ_0 . So it is sufficient to prove that for any $r_0 \geq 1$

$$\mathbb{E} \left| \frac{1}{h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial h_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_i}(\theta_0) \right|^{r_0} < \infty. \quad (\text{A.27})$$

For this purpose, we shall use the matrix expression (A.9). Three kinds of computations (listed (a), (b) and (c) below) are necessary according to the parameter with respect to which we differentiate.

(a) We first differentiate with respect to $\underline{\omega}$ and we obtain

$$\frac{\partial \mathbb{H}_t}{\partial \theta_i} = \sum_{k=0}^{\infty} \mathbb{B}^k \frac{\partial \underline{c}_{t-k}}{\partial \theta_i}, \text{ for } i = 1, \dots, m,$$

and since $\partial \underline{c}_{t-k}(j_1) / \partial \theta_i = (0, \dots, 1, \dots, 0)'$ (the vector composed with 0 and 1 at the j_1 -th position for $j_1 = 1, \dots, m$), we have

$$\theta_i \frac{\partial \mathbb{H}_t(i_1)}{\partial \theta_i} = \sum_{k=0}^{\infty} \sum_{j_1=1}^m \mathbb{B}^k(i_1, j_1) \theta_i \frac{\partial \underline{c}_{t-k}(j_1)}{\partial \theta_i} \leq \sum_{k=0}^{\infty} \sum_{j_1=1}^m \mathbb{B}^k(i_1, j_1) \underline{c}_{t-k}(j_1) = \mathbb{H}_t(i_1)$$

where $\mathbb{H}_t(i_1)$ is the i_1 -component of \mathbb{H}_t . So we have

$$\frac{1}{\mathbb{H}_t(i_1)} \frac{\partial \mathbb{H}_t(i_1)}{\partial \theta_i} \leq \frac{1}{\theta_i}. \quad (\text{A.28})$$

(b) We differentiate with respect to $(\alpha_1^{+'}, \dots, \alpha_q^{+'}, \alpha_1^{-'}, \dots, \alpha_q^{-'})'$. We have

$$\frac{\partial \mathbb{H}_t}{\partial \theta_i} = \sum_{k=0}^{\infty} \mathbb{B}^k \frac{\partial \underline{c}_{t-k}}{\partial \theta_i} \text{ for } i = m+1, \dots, s_2,$$

with

$$\frac{\partial \underline{c}_{t-k}(j_1)}{\partial \theta_i} = \sum_{l=1}^q \sum_{j_2=1}^m \frac{\partial A_l^{\pm}(j_1, j_2)}{\partial \theta_i} (\pm \varepsilon_{j_2, t-l}^{\pm})^{\delta_{j_2}}, \text{ for } j_1 = 1, \dots, m$$

where $\partial A_l^{\pm}(j_1, j_2)/\partial \theta_i$ is a null matrix or a matrix whose entries are all zero except the one (equal to 1) which is located at the same place of θ_i . Thus

$$\theta_i \frac{\partial \mathbb{H}_t(i_1)}{\partial \theta_i} = \sum_{k=0}^{\infty} \sum_{j_1=1}^m \mathbb{B}^k(i_1, j_1) \theta_i \frac{\partial \underline{c}_{t-k}(j_1)}{\partial \theta_i} \leq \sum_{k=0}^{\infty} \sum_{j_1=1}^m \mathbb{B}^k(i_1, j_1) \underline{c}_{t-k}(j_1) = \mathbb{H}_t(i_1),$$

and we have

$$\frac{1}{\mathbb{H}_t(i_1)} \frac{\partial \mathbb{H}_t(i_1)}{\partial \theta_i} \leq \frac{1}{\theta_i}. \quad (\text{A.29})$$

(c) We now differentiate with respect to $(\beta'_1, \dots, \beta'_p)'$ and we have

$$\frac{\partial \mathbb{H}_t}{\partial \theta_i} = \sum_{k=1}^{\infty} \left\{ \sum_{j=1}^k \mathbb{B}^{j-1} \frac{\partial \mathbb{B}}{\partial \theta_i} \mathbb{B}^{k-j} \right\} \underline{c}_{t-k} \text{ for } i = s_2 + 1, \dots, s_1. \quad (\text{A.30})$$

The matrix $\partial \mathbb{B}/\partial \theta_i$ is a matrix whose entries are all 0, apart from a 1 located at the same place as θ_i in \mathbb{B} . Thus $\theta_i \partial \mathbb{B}/\partial \theta_i \leq \mathbb{B}$ and using (A.30), for all $i = s_2 + 1, \dots, s_1$ we obtain

$$\theta_i \frac{\partial \mathbb{H}_t}{\partial \theta_i} \leq \sum_{k=1}^{\infty} \left\{ \sum_{j=1}^k \mathbb{B}^{j-1} \mathbb{B} \mathbb{B}^{k-j} \right\} \underline{c}_{t-k} = \sum_{k=1}^{\infty} k \mathbb{B}^k \underline{c}_{t-k}.$$

Reasoning component by component, we have

$$\theta_i \frac{\mathbb{H}_t(i_1)}{\partial \theta_i} \leq \sum_{k=1}^{\infty} k \sum_{j_1=1}^m \mathbb{B}^k(i_1, j_1) \underline{c}_{t-k}(j_1). \quad (\text{A.31})$$

We use $\mathbb{H}_t(i_1) \geq \omega + \sum_{j_1=1}^m \mathbb{B}^k(i_1, j_1) \underline{c}_{t-k}(j_1)$, $\forall k$, with $\omega = \inf_{1 \leq i \leq m} \omega(i)$ and the relation $x/(1+x) \leq x^s$ which is valid for all $x \geq 0$ and $s \in [0, 1]$. In view of (A.31), we obtain

$$\begin{aligned} \frac{\theta_i}{\mathbb{H}_t(i_1)} \frac{\partial \mathbb{H}_t(i_1)}{\partial \theta_i} &\leq \sum_{j_1=1}^m \sum_{k=1}^{\infty} \frac{k \mathbb{B}^k(i_1, j_1) \underline{c}_{t-k}(j_1)}{\omega + \mathbb{B}^k(i_1, j_1) \underline{c}_{t-k}(j_1)} \\ &\leq \sum_{j_1=1}^m \sum_{k=1}^{\infty} k \left(\frac{\mathbb{B}^k(i_1, j_1) \underline{c}_{t-k}(j_1)}{\omega} \right)^{s/r_0} \\ &\leq K \sum_{j_1=1}^m \sum_{k=1}^{\infty} k \rho^{sk/r_0} \underline{c}_{t-k}^{s/r_0}(j_1), \end{aligned}$$

with the constant ρ which belongs to the interval $[0, 1)$. Since $\underline{c}_{t-k}(\theta_0)$ has moments of any order (see Corollary 2) we have proved that

$$\mathbb{E} \left| \frac{\theta_i}{\mathbb{H}_t(i_1)} \frac{\partial \mathbb{H}_t(i_1)}{\partial \theta_i} \right|^{r_0} < \infty. \quad (\text{A.32})$$

Since (A.28) (as well as (A.29) and (A.32)) is an equivalent writing of (A.27) (as well as (A.29) and (A.32)), we deduce that (A.27) is true. By continuity of the functions that are involved on our estimations, the above inequalities are uniform on a neighborhood $V(\theta_0)$ of $\theta_0 \in \overset{\circ}{\Theta}$: for all $i_1 = 1, \dots, m$ and all $i = 1, \dots, s_1$ we have

$$\mathbb{E} \sup_{\theta \in V(\theta_0)} \left| \frac{1}{h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial h_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_i}(\theta) \right|^{r_0} < \infty. \quad (\text{A.33})$$

A.3.3. Asymptotic normality of the score vector.

The proof is the same than the one in [Francq and Zakoïan \(2012\)](#). It follows the following arguments:

- the process $\{\partial l_t(\theta_0)/\partial \theta\}_t$ is stationary,
- $\partial l_t(\theta_0)/\partial \theta$ is measurable with respect to the σ -field generated by $\{\eta_u, u < t\}$,
- $\mathbb{E} \left[\frac{\partial l_t(\theta_0)}{\partial \theta} | \mathcal{F}_{t-1} \right] = 0$ thus we have a martingale-difference sequence,
- Subsection A.3.2 implies that the matrix $I := \mathbb{E} \left[\frac{\partial l_t(\theta_0)}{\partial \theta} \frac{\partial l_t(\theta_0)}{\partial \theta'} \right]$ is well defined.

Using the central limit theorem from [Billingsley \(1995\)](#) we obtain (A.20).

A.3.4. Convergence to J

\rightsquigarrow *Expression of the second order derivatives of the log-likelihood*

We start from the expression (A.22) and (A.24) in order to compute the second order derivatives of the log-likelihood. According to the index of θ_i , we have three cases:

- (i) For all $i, j = 1, \dots, s_1$ we deduce from (A.22) that

$$\frac{\partial^2 l_t(\theta_0)}{\partial \theta_i \partial \theta_j} = \text{Tr}(c_1 + c_2 + c_3) \quad (\text{A.34})$$

with

$$\begin{aligned} c_1 &= D_{0t}^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' D_{0t}^{-1} D_{0t}^{(i)} D_{0t}^{-1} R^{-1} D_{0t}^{-1} D_{0t}^{(j)} + D_{0t}^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' D_{0t}^{-1} R^{-1} D_{0t}^{-1} D_{0t}^{(i)} D_{0t}^{-1} D_{0t}^{(j)} \\ &\quad - D_{0t}^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' D_{0t}^{-1} R^{-1} D_{0t}^{-1} D_{0t}^{(i, j)} + D_t^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' D_{0t}^{-1} R^{-1} D_{0t}^{-1} D_{0t}^{(j)} D_{0t}^{-1} D_{0t}^{(i)}, \\ c_2 &= D_{0t}^{-1} R^{-1} D_{0t}^{-1} D_{0t}^{(i)} D_{0t}^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' D_{0t}^{-1} D_{0t}^{(j)} + D_{0t}^{-1} R^{-1} D_{0t}^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' D_{0t}^{-1} D_{0t}^{(i)} D_{0t}^{-1} D_{0t}^{(j)} \\ &\quad - D_{0t}^{-1} R^{-1} D_{0t}^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' D_{0t}^{-1} D_{0t}^{(i, j)} + D_t^{-1} R^{-1} D_{0t}^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' D_{0t}^{-1} D_{0t}^{(j)} D_{0t}^{-1} D_{0t}^{(i)}, \\ c_3 &= -2D_{0t}^{-1} D_{0t}^{(i)} D_{0t}^{-1} D_{0t}^{(j)} + 2D_{0t}^{-1} D_{0t}^{(i, j)}. \end{aligned}$$

- (ii) For all $i, = 1, \dots, s_1$ and $j = s_1 + 1, \dots, s_0$ we have

$$\frac{\partial^2 l_t(\theta_0)}{\partial \theta_i \partial \theta_j} = \text{Tr}(c_4 + c_5) \quad (\text{A.35})$$

with

$$\begin{aligned} c_4 &= R^{-1} D_{0t}^{-1} D_{0t}^{(i)} D_{0t}^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' D_{0t}^{-1} R^{-1} R^{(j)} \text{ and} \\ c_5 &= R^{-1} D_{0t}^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' D_{0t}^{-1} D_{0t}^{(i)} D_{0t}^{-1} R^{-1} R^{(j)}. \end{aligned}$$

This comes from the following computations. Starting from (A.22), we use (A.1) and (A.6) and we obtain

$$\begin{aligned} -\frac{\partial}{\partial\theta_j} \text{Tr} \left(\underline{\varepsilon}_t \underline{\varepsilon}'_t D_{0t}^{-1} R^{-1} D_{0t}^{-1} D_{0t}^{(i)} D_{0t}^{-1} \right) &= \text{Tr} \left(R^{-1} D_{0t}^{-1} D_{0t}^{(i)} D_{0t}^{-1} \underline{\varepsilon}_t \underline{\varepsilon}'_t D_{0t}^{-1} R^{-1} R^{(j)} \right) \\ -\frac{\partial}{\partial\theta_j} \text{Tr} \left(R^{-1} D_{0t}^{-1} \underline{\varepsilon}_t \underline{\varepsilon}'_t D_{0t}^{-1} D_{0t}^{(i)} D_{0t}^{-1} \right) &= \text{Tr} \left(R^{-1} D_{0t}^{-1} \underline{\varepsilon}_t \underline{\varepsilon}'_t D_{0t}^{-1} D_{0t}^{(i)} D_{0t}^{-1} R^{-1} R^{(j)} \right) \text{ and} \\ 2 \frac{\partial}{\partial\theta_j} \text{Tr} \left(D_{0t}^{-1} \frac{\partial D_{0t}}{\partial\theta_i} \right) &= 0. \end{aligned}$$

(iii) For all $i, j = s_1 + 1, \dots, s_0$ we have from (A.24)

$$\frac{\partial^2 l_t(\theta_0)}{\partial\theta_i \partial\theta_j} = \text{Tr}(c_6) \quad (\text{A.36})$$

with

$$\begin{aligned} c_6 &= R^{-1} R^{(i)} R^{-1} D_{0t}^{-1} \underline{\varepsilon}_t \underline{\varepsilon}'_t D_{0t}^{-1} R^{-1} R^{(j)} + R^{-1} D_{0t}^{-1} \underline{\varepsilon}_t \underline{\varepsilon}'_t D_{0t}^{-1} R^{-1} R^{(i)} R^{-1} R^{(j)} \\ &\quad - R^{-1} D_{0t}^{-1} \underline{\varepsilon}_t \underline{\varepsilon}'_t D_{0t}^{-1} R^{-1} R^{(i,j)} - R^{-1} R^{(i)} R^{-1} R^{(j)} + R^{-1} R^{(i,j)}. \end{aligned}$$

Remark that, with our parametrization, $R^{(i,j)} = \partial^2 R / \partial\theta_i \partial\theta_j = 0$.

Thanks to the three above cases, we remark that to control the second order derivatives, it is sufficient to control the new term $\|D_{0t}^{-1} D_{0t}^{(i,j)}\|$. Indeed, all the other terms in c_1, \dots, c_6 can be controlled thanks to the results from Subsection A.3.3.

\rightsquigarrow *Existence of the moments of the second order derivatives of the log-likelihood*

We take derivatives in the expression (A.26) and we obtain

$$\frac{\partial^2 D_t(i_1, i_1)}{\partial\theta_i \partial\theta_j} = \frac{1}{\delta_{i_1}} h_{i_1, t}^{1/2}(\theta) \left[\frac{1}{h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial h_{i_1, t}^{\delta_{i_1}/2}}{\partial\theta_j} \frac{1}{h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial h_{i_1, t}^{\delta_{i_1}/2}}{\partial\theta_i} \left(\frac{1}{\delta_{i_1}} - 1 \right) + \frac{1}{h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial^2 h_{i_1, t}^{\delta_{i_1}/2}}{\partial\theta_i \partial\theta_j} \right] (\theta).$$

It only remains to control the last term in the right hand side of the above identity. We will prove that

$$\mathbb{E} \left| \frac{1}{h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial^2 h_{i_1, t}^{\delta_{i_1}/2}}{\partial\theta_i \partial\theta_j} (\theta_0) \right|^{r_0} < \infty. \quad (\text{A.37})$$

By (A.8), we have for $i_1 = 1, \dots, m$:

$$\mathbb{H}_t(i_1) = \sum_{k=0}^{\infty} \sum_{j_1=1}^m \mathbb{B}^k(i_1, j_1) \underline{c}_{t-k}(j_1)$$

and we remark that

$$\begin{aligned} 0 &= \frac{\partial^2 \mathbb{H}_t(i_1)}{\partial\omega_i^2} = \frac{\partial^2 \mathbb{H}_t(i_1)}{\partial\omega_i \partial\omega_j} = \frac{\partial^2 \mathbb{H}_t(i_1)}{\partial\omega_i \partial\alpha_j^\pm} \\ 0 &= \frac{\partial^2 \mathbb{H}_t(i_1)}{\partial\alpha_i^\pm \partial\alpha_j^\pm} \text{ and} \\ 0 &= \frac{\partial^2 \mathbb{H}_t(i_1)}{\partial(\beta_i)^2}. \end{aligned}$$

We also have

$$\frac{\partial^2 \mathbb{H}_t(i_1)}{\partial \omega_i \partial \beta_j} = \sum_{k=1}^{\infty} \sum_{j_1=1}^m \left\{ \sum_{h=1}^k \mathbb{B}^{h-1}(i_1, j_1) \frac{\partial \mathbb{B}(i_1, j_1)}{\partial \beta_j} \mathbb{B}^{k-h}(i_1, j_1) \right\} \frac{\partial \mathfrak{c}_{t-k}(j_1)}{\partial \omega_i} \quad (\text{A.38})$$

$$\frac{\partial^2 \mathbb{H}_t(i_1)}{\partial \alpha_i^{\pm} \partial \beta_j} = \sum_{l=1}^q \sum_{k=1}^{\infty} \sum_{j_1=1}^m \sum_{j_2=1}^m \left\{ \sum_{h=1}^k \mathbb{B}^{h-1}(i_1, j_1) \frac{\partial \mathbb{B}(i_1, j_1)}{\partial \beta_j} \mathbb{B}^{k-h}(i_1, j_1) \right\} \frac{\partial A_l^{\pm}(j_1, j_2)}{\partial \alpha_i^{\pm}} (\pm \varepsilon_{j_2, t-l}^{\pm})^{\delta_{j_2}} \quad (\text{A.39})$$

and

$$\begin{aligned} \frac{\partial^2 \mathbb{H}_t}{\partial \beta_i \partial \beta_j} &= \sum_{k=2}^{\infty} \left[\sum_{l=2}^k \left\{ \left(\sum_{r=1}^{l-1} \mathbb{B}^{r-1} \frac{\partial \mathbb{B}}{\partial \beta_j} \mathbb{B}^{l-1-r} \right) \frac{\partial \mathbb{B}}{\partial \beta_i} \mathbb{B}^{k-l} \right\} \right. \\ &\quad \left. + \sum_{l=1}^{k-1} \left\{ \mathbb{B}^{l-1} \frac{\partial \mathbb{B}}{\partial \beta_i} \left(\sum_{r=1}^{k-l} \mathbb{B}^{r-1} \frac{\partial \mathbb{B}}{\partial \beta_j} \mathbb{B}^{k-l-r} \right) \right\} \right] \mathfrak{c}_{t-k}. \end{aligned} \quad (\text{A.40})$$

We recall that the matrix $\partial \mathbb{B} / \partial \beta_j$ is a matrix whose entries are all 0, apart from a 1 located at the same place as β_j in \mathbb{B} .

Now we treat separately the three different expressions of the derivatives.

(a) For $i = 1, \dots, m$ we deduce from (A.38) that

$$\begin{aligned} \omega_i \beta_j \frac{\partial^2 \mathbb{H}_t(i_1)}{\partial \omega_i \partial \beta_j} &= \sum_{k=1}^{\infty} \sum_{j_1=1}^m \left\{ \sum_{h=1}^k \mathbb{B}^{j_1-1}(i_1, j_1) \beta_j \frac{\partial \mathbb{B}(i_1, j_1)}{\partial \beta_j} \mathbb{B}^{k-h}(i_1, j_1) \right\} \omega_i \frac{\partial \mathfrak{c}_{t-k}(j_1)}{\partial \omega_i} \\ &\leq \sum_{k=1}^{\infty} \sum_{j_1=1}^m k \mathbb{B}^k(i_1, j_1) \mathfrak{c}_{t-k}(j_1). \end{aligned}$$

Using the same arguments as for (A.32), we obtain that

$$\mathbb{E} \left| \frac{\omega_i \beta_j}{\mathbb{H}_t(i_1)} \frac{\partial^2 \mathbb{H}_t(i_1)}{\partial \omega_i \partial \beta_j} \right|^{r_0} < \infty. \quad (\text{A.41})$$

(b) From (A.39) we have

$$\begin{aligned} \alpha_i^{\pm} \beta_j \frac{\partial^2 \mathbb{H}_t(i_1)}{\partial \alpha_i^{\pm} \partial \beta_j} &= \sum_{l=1}^q \sum_{k=1}^{\infty} \sum_{j_1=1}^m \sum_{j_2=1}^m \left\{ \sum_{h=1}^k \mathbb{B}^{h-1}(i_1, j_1) \beta_j \frac{\partial \mathbb{B}(i_1, j_1)}{\partial \beta_j} \mathbb{B}^{k-h}(i_1, j_1) \right\} \alpha_i^{\pm} \frac{\partial A_l^{\pm}(j_1, j_2)}{\partial \alpha_i^{\pm}} (\pm \varepsilon_{j_2, t-l}^{\pm})^{\delta_{j_2}} \\ &\leq \sum_{l=1}^q \sum_{k=1}^{\infty} \sum_{j_1=1}^m \sum_{j_2=1}^m k \mathbb{B}^k(i_1, j_1) A_l^{\pm}(j_1, j_2) (\pm \varepsilon_{j_2, t-l}^{\pm})^{\delta_{j_2}} \\ &\leq \sum_{k=1}^{\infty} \sum_{j_1=1}^m k \mathbb{B}^k(i_1, j_1) \mathfrak{c}_{t-k}(j_1) \end{aligned}$$

and we may proceed as in the proof of (A.32) and we obtain that

$$\mathbb{E} \left| \frac{\alpha_i^{\pm} \beta_j}{\mathbb{H}_t(i_1)} \frac{\partial^2 \mathbb{H}_t(i_1)}{\partial \alpha_i^{\pm} \partial \beta_j} \right|^{r_0} < \infty. \quad (\text{A.42})$$

(c) Using (A.40) we write

$$\begin{aligned}
\beta_i \beta_j \frac{\partial^2 \mathbb{H}_t(i_1)}{\partial \beta_i \partial \beta_j} &= \sum_{j_1=1}^m \sum_{k=2}^{\infty} \left[\sum_{l=2}^k \left\{ \left(\sum_{r=1}^{l-1} \mathbb{B}^{r-1}(i_1, j_1) \beta_j \frac{\partial \mathbb{B}(i_1, j_1)}{\partial \beta_j} \mathbb{B}^{l-1-r}(i_1, j_1) \right) \beta_i \frac{\partial \mathbb{B}(i_1, j_1)}{\partial \beta_i} \mathbb{B}^{k-l}(i_1, j_1) \right\} \right. \\
&\quad \left. + \sum_{l=1}^{k-1} \left\{ \mathbb{B}^{l-1}(i_1, j_1) \beta_i \frac{\partial \mathbb{B}(i_1, j_1)}{\partial \beta_i} \left(\sum_{r=1}^{k-l} \mathbb{B}^{r-1}(i_1, j_1) \beta_j \frac{\partial \mathbb{B}(i_1, j_1)}{\partial \beta_j} \mathbb{B}^{k-l-r}(i_1, j_1) \right) \right\} \right] \underline{c}_{t-k}(i_1) \\
&\leq \sum_{j_1=1}^m \sum_{k=2}^{\infty} \left[\sum_{l=2}^k (l-1) \mathbb{B}^k(i_1, j_1) + \sum_{l=1}^{k-1} (k-l) \mathbb{B}^k(i_1, j_1) \right] \underline{c}_{t-k}(j_1) \\
&\leq \sum_{j_1=1}^m \sum_{k=2}^{\infty} k(k-1) \mathbb{B}^k(i_1, j_1) \underline{c}_{t-k}(j_1).
\end{aligned}$$

Arguing as before we deduce that

$$\mathbb{E} \left| \frac{\beta_i \beta_j}{\mathbb{H}_t(i_1)} \frac{\partial^2 \mathbb{H}_t(i_1)}{\partial \beta_i \partial \beta_j} \right|^{r_0} < \infty. \quad (\text{A.43})$$

We deduce from (A.41), (A.42) and (A.43) that (A.37) is true.

Once again, by continuity of the involved functions, the above inequalities are uniform on a neighborhood $V(\theta_0)$ of $\theta_0 \in \overset{\circ}{\Theta}$: for all $i_1 = 1, \dots, m$ and all $i, j, k = 1, \dots, s_1$ we have

$$\mathbb{E} \sup_{\theta \in V(\theta_0)} \left| \frac{1}{\underline{h}_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial^2 \underline{h}_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_i \partial \theta_j}(\theta) \right|^{r_0} < \infty. \quad (\text{A.44})$$

\rightsquigarrow *Existence of the moments of the third order derivatives of the log-likelihood*

First we write the quite heavy expressions of the third order derivatives derivatives with respect to the different parameters.

- (i) For $i, j, k = 1, \dots, s_1$, the derivatives with respect to θ_i, θ_j and θ_k will correspond to the derivatives with respect to the parameters $\underline{\omega}'$, $\text{vec}(A_t^{\pm})'$ et $\text{vec}(B_t)'$. We obtain from (A.34) that

$$\frac{\partial^3 l_t(\theta)}{\partial \theta_i \partial \theta_j \partial \theta_k} = \text{Tr} \left((\sum_{p=1}^4 c_{1p} + c_{2p}) + c_{31} + c_{32} \right) \quad (\text{A.45})$$

with

$$\begin{aligned}
c_{61} &= -R^{-1}R^{(i)}R^{-1}R^{(j)}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}R^{-1}R^{(k)} + R^{-1}R^{(i,j)}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}R^{-1}R^{(k)} \\
&\quad - R^{-1}R^{(j)}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}R^{-1}R^{(i)}R^{-1}R^{(k)} + R^{-1}R^{(j)}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}R^{-1}R^{(i,k)}, \\
c_{62} &= -R^{-1}R^{(i)}R^{-1}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}R^{-1}R^{(j)}R^{-1}R^{(k)} - R^{-1}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}R^{-1}R^{(i)}R^{-1}R^{(j)}R^{-1}R^{(k)} \\
&\quad + R^{-1}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}R^{-1}R^{(i,j)}R^{-1}R^{(k)} - R^{-1}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}R^{-1}R^{(j)}R^{-1}R^{(i)}R^{-1}R^{(k)} \\
&\quad + R^{-1}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}R^{-1}R^{(j)}R^{-1}R^{(i,k)}, \\
c_{63} &= R^{-1}R^{(i)}R^{-1}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tR^{-1}R^{(j,k)} + R^{-1}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tR^{-1}R^{(i)}R^{-1}R^{(j,k)} - R^{-1}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tR^{-1}R^{(i,j,k)}, \\
c_{64} &= R^{-1}R^{(i)}R^{-1}R^{(j)}R^{-1}R^{(k)} - R^{-1}R^{(i,j)}R^{-1}R^{(k)} + R^{-1}R^{(j)}R^{-1}R^{(i)}R^{-1}R^{(k)} - R^{-1}R^{(j)}R^{-1}R^{(i,k)}, \\
c_{65} &= -R^{-1}R^{(i)}R^{-1}R^{(j,k)} + R^{-1}R^{(i,j,k)}.
\end{aligned}$$

- (iii) For $i, j = 1, \dots, s_1$ (differentiation with respect to $\underline{\omega}'$, $\text{vec}(A_l^\pm)'$ and $\text{vec}(B_l)'$) and for $k = s_1 + 1, \dots, s_0$ (differentiation with respect to ρ), we obtain from (A.35)

$$\frac{\partial^3 l_t(\theta)}{\partial \theta_i \partial \theta_j \partial \theta_k} = \text{Tr}(c_{41} + c_{51}) \quad (\text{A.47})$$

where

$$\begin{aligned}
c_{41} &= -R^{-1}D_t^{-1}D_t^{(i)}D_t^{-1}D_t^{(j)}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}R^{-1}R^{(k)} + R^{-1}D_t^{-1}D_t^{(i,j)}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}R^{-1}R^{(k)} \\
&\quad - R^{-1}D_t^{-1}D_t^{(j)}D_t^{-1}D_t^{(i)}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}R^{-1}R^{(k)} - R^{-1}D_t^{-1}D_t^{(j)}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}D_t^{(i)}D_t^{-1}R^{-1}R^{(k)}, \\
c_{51} &= -R^{-1}D_t^{-1}D_t^{(i)}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}D_t^{(j)}D_t^{-1}R^{-1}R^{(k)} - R^{-1}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}D_t^{(i)}D_t^{-1}D_t^{(j)}D_t^{-1}R^{-1}R^{(k)} \\
&\quad + R^{-1}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}D_t^{(i,j)}D_t^{-1}R^{-1}R^{(k)} - R^{-1}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}D_t^{(j)}D_t^{-1}D_t^{(i)}D_t^{-1}R^{-1}R^{(k)}.
\end{aligned}$$

- (iv) In the same way, from (A.36) we obtain for $i = 1, \dots, s_1$ (derivatives with respect to $\underline{\omega}'$, $\text{vec}(A_l^\pm)'$ and $\text{vec}(B_l)'$) and for $j, k = s_1 + 1, \dots, s_0$ (derivatives with respect to ρ):

$$\frac{\partial^3 l_t(\theta)}{\partial \theta_i \partial \theta_j \partial \theta_k} = \text{Tr}(c'_{61} + c'_{62} + c'_{63}) \quad (\text{A.48})$$

with

$$\begin{aligned}
c'_{61} &= -R^{-1}R^{(j)}D_t^{-1}D_t^{(i)}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}R^{-1}R^{(k)} - R^{-1}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}D_t^{(i)}D_t^{-1}R^{-1}R^{(j)}R^{-1}R^{(k)}, \\
c'_{62} &= -R^{-1}D_t^{-1}D_t^{(i)}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}R^{-1}R^{(j)}R^{-1}R^{(k)} - R^{-1}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}D_t^{(i)}D_t^{-1}R^{-1}R^{(j)}R^{-1}R^{(k)}, \\
c'_{63} &= R^{-1}D_t^{-1}D_t^{(i)}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}R^{-1}R^{(i,j)} + R^{-1}D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}D_t^{(i)}D_t^{-1}R^{-1}R^{(i,j)}.
\end{aligned}$$

We remark that the last two terms of c_6 in (A.36) are not composed with the matrix D_t and thus their derivatives vanishes.

In order to estimate the moments of order r_0 of the third order derivatives, we need to study the term $D_t^{-1}D_{0t}$ which appears, for example, in the third term of c_{11} namely $c_{11}^{(3)} = D_t^{-1}\underline{\varepsilon}_t\underline{\varepsilon}'_tD_t^{-1}D_t^{(i,j)}D_t^{-1}R^{-1}D_t^{-1}D_t^{(k)}$. Indeed, using the fact that $\underline{\varepsilon}_t = D_{0t}\tilde{\eta}_t$, we may write

$$\begin{aligned}
\mathbb{E} \left[\sup_{\theta \in V(\theta_0)} \left| \text{Tr} \left(c_{11}^{(3)} \right) \right| \right] &\leq K \mathbb{E} \|\tilde{\eta}_t \tilde{\eta}'_t\| \mathbb{E} \left[\sup_{\theta \in V(\theta_0)} \|D_t^{-1}D_{0t}\|^2 \|R^{-1}\| \|D_t^{-1}D_t^{(i,j)}\| \|D_t^{-1}D_t^{(k)}\| \right] \\
&\leq K \mathbb{E} \left[\sup_{\theta \in V(\theta_0)} \|D_t^{-1}D_{0t}\|^2 \|D_t^{-1}D_t^{(i,j)}\| \|D_t^{-1}D_t^{(k)}\| \right]. \quad (\text{A.49})
\end{aligned}$$

Consequently we have to prove that for any $r_0 \geq 1$,

$$\mathbb{E} \left[\sup_{\theta \in V(\theta_0)} \|D_t^{-1} D_{0t}\|^{r_0} \right] < \infty. \quad (\text{A.50})$$

Letting $\mathbb{B}_0 = \mathbb{B}(\theta_0)$, by (A.9) the i_1^{th} component of $h_t^{\delta/2}$ equals

$$h_{i_1,t}^{\delta_{i_1}/2}(\theta_0) = c_0 + \sum_{k=0}^{\infty} \sum_{j_1=1}^m \sum_{j_2=1}^m \sum_{i=1}^q \mathbb{B}_0^k(i_1, j_1) \left\{ A_{0i}^+(j_1, j_2) (\varepsilon_{j_2,t-k-i}^+)^{\delta_{j_2}} + A_{0i}^-(j_1, j_2) (-\varepsilon_{j_2,t-k-i}^-)^{\delta_{j_2}} \right\}$$

where c_0 is a strictly positive constant. For a sufficiently small neighborhood $V(\theta_0)$ of θ_0 , we have

$$\sup_{\theta \in V(\theta_0)} \frac{A_{0i}^+(j_1, j_2)}{A_i^+(j_1, j_2)} < K, \quad \sup_{\theta \in V(\theta_0)} \frac{A_{0i}^-(j_1, j_2)}{A_i^-(j_1, j_2)} < K, \quad \sup_{\theta \in V(\theta_0)} \frac{\mathbb{B}_0^k(i_1, j_1)}{\mathbb{B}^k(i_1, j_1)} \leq (1 + \xi)^k$$

for any $i_1, j_1, j_2 \in \{1, \dots, m\}$ and for all integer k and all $\xi > 0$. Moreover, the coefficients $A_{0i}^+(j_1, j_2)$ et $A_{0i}^-(j_1, j_2)$ are bounded below by a constant $c > 0$ uniformly in θ on $V(\theta_0)$. We thus obtain that

$$\begin{aligned} \sup_{\theta \in V(\theta_0)} \frac{h_{i_1,t}^{\delta_{i_1}/2}(\theta_0)}{h_{i_1,t}^{\delta_{i_1}/2}(\theta)} &\leq K + K \sum_{k=0}^{\infty} \sum_{j_1=1}^m \sum_{j_2=1}^m \sum_{i=1}^q \left\{ \frac{(1 + \xi)^k \mathbb{B}^k(i_1, j_1) (\varepsilon_{j_2,t-k-i}^+)^{\delta_{j_2}}}{\omega + c \mathbb{B}^k(i_1, j_1) (\varepsilon_{j_2,t-k-i}^+)^{\delta_{j_2}}} \right. \\ &\quad \left. + \frac{(1 + \xi)^k \mathbb{B}^k(i_1, j_1) (-\varepsilon_{j_2,t-k-i}^-)^{\delta_{j_2}}}{\omega + c \mathbb{B}^k(i_1, j_1) (-\varepsilon_{j_2,t-k-i}^-)^{\delta_{j_2}}} \right\} \\ &\leq K + K \sum_{j_2=1}^m \sum_{i=1}^q \sum_{k=0}^{\infty} (1 + \xi)^k \rho^{ks} |\varepsilon_{j_2,t-k-i}|^{\delta_{j_2}s}, \end{aligned}$$

for some $\rho \in [0, 1[$, all $\xi > 0$ and all $s \in [0, 1]$. We then deduce that

$$\sup_{\theta \in V(\theta_0)} \frac{h_{i_1,t}^{1/2}(\theta_0)}{h_{i_1,t}^{1/2}(\theta)} = \sup_{\theta \in V(\theta_0)} \left(\frac{h_{i_1,t}^{\delta_{i_1}/2}(\theta_0)}{h_{i_1,t}^{\delta_{i_1}/2}(\theta)} \right)^{1/\delta_{i_1}} \leq \left(K + K \sum_{j_2=1}^m \sum_{i=1}^q \sum_{k=0}^{\infty} (1 + \xi)^k \rho^{ks} |\varepsilon_{j_2,t-k-i}|^{\delta_{j_2}s} \right)^{1/\delta_{i_1}}.$$

We distinguish two cases. If $\delta_{i_1} > 1$, the concavity of the function $x \mapsto x^{\delta_{i_1}}$ on $[0, \infty[$ implies, by the Jensen inequality, that

$$\mathbb{E} \left[\sup_{\theta \in V(\theta_0)} \frac{h_{i_1,t}^{1/2}(\theta_0)}{h_{i_1,t}^{1/2}(\theta)} \right] \leq \left(K + K \sum_{j_2=1}^m \sum_{i=1}^q \sum_{k=0}^{\infty} (1 + \xi)^k \rho^{ks} \mathbb{E} |\varepsilon_{j_2,t-k-i}|^{\delta_{j_2}s} \right)^{1/\delta_{i_1}} < \infty, \quad (\text{A.51})$$

by using Corollary 2. Now, when $\delta_{i_1} = 1$, Corollary 2 entails that

$$\mathbb{E} \left[\sup_{\theta \in V(\theta_0)} \frac{h_{i_1,t}^{1/2}(\theta_0)}{h_{i_1,t}^{1/2}(\theta)} \right] \leq K + K \sum_{j_2=1}^m \sum_{i=1}^q \sum_{k=0}^{\infty} (1 + \xi)^k \rho^{ks} \mathbb{E} |\varepsilon_{j_2,t-k-i}|^{\delta_{j_2}s} < \infty. \quad (\text{A.52})$$

In view of (A.51) and (A.52), for all $r_0 \geq 1$, we deduce that

$$\mathbb{E} \left[\sup_{\theta \in V(\theta_0)} \left| \frac{h_{i_1,t}^{1/2}(\theta_0)}{h_{i_1,t}^{1/2}(\theta)} \right|^{r_0} \right] < \infty \quad (\text{A.53})$$

and (A.50) is true.

We also need to control the term $\|D_t^{-1}D_t^{(i,j,k)}\|$. There are many other terms that involve derivatives of order one and two that we may control thanks our previous estimations. Moreover $R^{(i,j,k)} = 0$ because $R^{(i,j)} = 0$.

The third order derivative of the matrix D_t with respect to the parameters θ_i , θ_j and θ_k has the following expression for $i, j, k = 1, \dots, s_1$

$$\begin{aligned} \frac{\partial^3 D_t(i_1, i_1)}{\partial \theta_i \partial \theta_j \partial \theta_k} &= \frac{1}{\delta_{i_1}^2} h_{i_1, t}^{1/2} \frac{1}{h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial h_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_k} \left[\frac{1}{h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial h_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_j} \frac{1}{h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial h_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_i} \left(\frac{1}{\delta_{i_1}} - 1 \right) + \frac{1}{h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial^2 h_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_i \partial \theta_j} \right] \\ &+ \frac{1}{\delta_{i_1}} h_{i_1, t}^{1/2} \left[-\frac{1}{h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial h_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_k} \frac{1}{h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial h_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_j} \frac{1}{h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial h_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_i} \left(\frac{1}{\delta_{i_1}} - 1 \right) \right. \\ &+ \frac{1}{h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial^2 h_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_j \partial \theta_k} \frac{1}{h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial h_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_i} \left(\frac{1}{\delta_{i_1}} - 1 \right) + \frac{1}{h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial h_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_j} \frac{1}{h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial^2 h_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_i \partial \theta_k} \left(\frac{1}{\delta_{i_1}} - 1 \right) \\ &\left. - \frac{1}{h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial h_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_k} \frac{1}{h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial^2 h_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_i \partial \theta_j} + \frac{1}{h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial^3 h_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_i \partial \theta_j \partial \theta_k} \right]. \end{aligned}$$

The terms in which the first and second order derivatives of $\frac{h_t^{\delta_{i_1}/2}}{h_t}$ are involved are already controlled thanks to (A.33) and (A.44). Thus it remains to prove that

$$\mathbb{E} \left| \frac{1}{h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial^3 h_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_i \partial \theta_j \partial \theta_k}(\theta) \right|^{r_0} < \infty. \quad (\text{A.54})$$

Starting from (A.9)

$$\mathbb{H}_t(i_1) = \sum_{k=0}^{\infty} \sum_{j_1=1}^m \mathbb{B}^k(i_1, j_1) \underline{c}_{t-k}(j_1),$$

one may express the derivatives with respect to the different parameters. We only have to treat the derivatives

$$\frac{\partial^3 \mathbb{H}_t(i_1)}{\partial \theta_i \partial \beta_j \partial \beta_k}$$

when $\theta_i \neq \beta_j$ and $\theta_i \neq \beta_k$ because the other derivatives vanish.

There are three cases.

- (i) For $i = 1, \dots, m$ (this means that we differentiate with respect to the parameter ω) and for fixed j and k , it holds

$$\begin{aligned} \frac{\partial^3 \mathbb{H}_t}{\partial \omega_i \partial \beta_j \partial \beta_k} &= \sum_{k'=2}^{\infty} \left[\sum_{l=2}^{k'} \left\{ \left(\sum_{r=1}^{l-1} \mathbb{B}^{r-1} \frac{\partial \mathbb{B}}{\partial \beta_k} \mathbb{B}^{l-1-r} \right) \frac{\partial \mathbb{B}}{\partial \beta_j} \mathbb{B}^{k'-l} \right\} \right. \\ &\left. + \sum_{l=1}^{k'-1} \left\{ \mathbb{B}^{l-1} \frac{\partial \mathbb{B}}{\partial \beta_j} \left(\sum_{r=1}^{k-l} \mathbb{B}^{r-1} \frac{\partial \mathbb{B}}{\partial \beta_k} \mathbb{B}^{k'-l-r} \right) \right\} \right] \frac{\partial \underline{c}_{t-k'}}{\partial \omega_i}. \quad (\text{A.55}) \end{aligned}$$

Arguing as we did for the second order derivatives, we obtain

$$\begin{aligned}
\omega_i \beta_j \beta_k \frac{\partial^3 \mathbb{H}_t(i_1)}{\partial \omega_i \partial \beta_j \partial \beta_k} &= \sum_{j_1=1}^m \sum_{k'=2}^{\infty} \left[\sum_{l=2}^{k'} \left\{ \left(\sum_{r=1}^{l-1} \mathbb{B}^{r-1}(i_1, j_1) \beta_k \frac{\partial \mathbb{B}(i_1, j_1)}{\partial \beta_k} \mathbb{B}^{l-1-r}(i_1, j_1) \right) \beta_j \frac{\partial \mathbb{B}(i_1, j_1)}{\partial \beta_j} \mathbb{B}^{k-l}(i_1, j_1) \right\} \right. \\
&\quad \left. + \sum_{l=1}^{k'-1} \left\{ \mathbb{B}^{l-1}(i_1, j_1) \beta_j \frac{\partial \mathbb{B}(i_1, j_1)}{\partial \beta_j} \left(\sum_{r=1}^{k'-l} \mathbb{B}^{r-1}(i_1, j_1) \beta_k \frac{\partial \mathbb{B}(i_1, j_1)}{\partial \beta_k} \mathbb{B}^{k'-l-r}(i_1, j_1) \right) \right\} \right] \omega_i \frac{\partial \mathcal{C}_{t-k'}(j_1)}{\partial \omega_i}. \\
&\leq \sum_{j_1=1}^m \sum_{k'=2}^{\infty} \left[\sum_{l=2}^{k'} (l-1) \mathbb{B}^{k'}(i_1, j_1) + \sum_{l=1}^{k'-1} (k'-l) \mathbb{B}^{k'}(i_1, j_1) \right] \mathcal{C}_{t-k'}(j_1) \\
&\leq \sum_{j_1=1}^m \sum_{k'=2}^{\infty} k'(k'-1) \mathbb{B}^{k'}(i_1, j_1) \mathcal{C}_{t-k'}(j_1).
\end{aligned}$$

Consequently, it holds

$$\mathbb{E} \left| \frac{\omega_i \beta_j \beta_k}{\mathbb{H}_t(i_1)} \frac{\partial^3 \mathbb{H}_t(i_1)}{\partial \omega_i \partial \beta_j \partial \beta_k} \right|^{r_0} < \infty. \quad (\text{A.56})$$

- (ii) For $i = m+1, \dots, s_2$ (corresponding to differentiation with respect to $\text{vec}(A_t^\pm)$) and for fixed j and k , we have

$$\begin{aligned}
\frac{\partial^3 \mathbb{H}_t(i_1)}{\partial \alpha_i^\pm \partial \beta_j \partial \beta_k} &= \sum_{l'=1}^q \sum_{k'=2}^{\infty} \sum_{j_1=1}^m \sum_{j_2=1}^m \left[\sum_{l=2}^{k'} \left\{ \left(\sum_{r=1}^{l-1} \mathbb{B}^{r-1}(i_1, j_1) \frac{\partial \mathbb{B}(i_1, j_1)}{\partial \beta_k} \mathbb{B}^{l-1-r}(i_1, j_1) \right) \frac{\partial \mathbb{B}(i_1, j_1)}{\partial \beta_j} \mathbb{B}^{k-l}(i_1, j_1) \right\} \right. \\
&\quad \left. + \sum_{l=1}^{k-1} \left\{ \mathbb{B}^{l-1}(i_1, j_1) \frac{\partial \mathbb{B}(i_1, j_1)}{\partial \beta_j} \left(\sum_{r=1}^{k'-l} \mathbb{B}^{r-1}(i_1, j_1) \frac{\partial \mathbb{B}(i_1, j_1)}{\partial \beta_k} \mathbb{B}^{k'-l-r}(i_1, j_1) \right) \right\} \right] \frac{\partial A_i^\pm(j_1, j_2)}{\partial \alpha_i^\pm} (\pm \varepsilon_{j_2, t-l'}^{\pm})^{\delta_{j_2}}, \\
&\quad (\text{A.57})
\end{aligned}$$

and we write that

$$\begin{aligned}
\alpha_i^\pm \beta_j \beta_k \frac{\partial^3 \mathbb{H}_t(i_1)}{\partial \alpha_i^\pm \partial \beta_j \partial \beta_k} &= \sum_{l'=1}^q \sum_{k'=2}^{\infty} \sum_{j_1=1}^m \sum_{j_2=1}^m \left[\sum_{l=2}^{k'} \left\{ \left(\sum_{r=1}^{l-1} \mathbb{B}^{r-1}(i_1, j_1) \beta_k \frac{\partial \mathbb{B}(i_1, j_1)}{\partial \beta_k} \mathbb{B}^{l-1-r}(i_1, j_1) \right) \beta_j \frac{\partial \mathbb{B}(i_1, j_1)}{\partial \beta_j} \mathbb{B}^{k-l}(i_1, j_1) \right\} \right. \\
&\quad \left. + \sum_{l=1}^{k-1} \left\{ \mathbb{B}^{l-1}(i_1, j_1) \beta_j \frac{\partial \mathbb{B}(i_1, j_1)}{\partial \beta_j} \left(\sum_{r=1}^{k'-l} \mathbb{B}^{r-1}(i_1, j_1) \beta_k \frac{\partial \mathbb{B}(i_1, j_1)}{\partial \beta_k} \mathbb{B}^{k'-l-r}(i_1, j_1) \right) \right\} \right] \alpha_i^\pm \frac{\partial A_i^\pm(j_1, j_2)}{\partial \alpha_i^\pm} (\pm \varepsilon_{j_2, t-l'}^{\pm})^{\delta_{j_2}} \\
&\leq \sum_{l'=1}^q \sum_{k'=2}^{\infty} \sum_{j_1=1}^m \sum_{j_2=1}^m \left[\sum_{l=2}^{k'} (l-1) \mathbb{B}^{k'}(i_1, j_1) + \sum_{l=1}^{k'-1} (k'-l) \mathbb{B}^{k'}(i_1, j_1) \right] A_i^\pm(j_1, j_2) (\pm \varepsilon_{j_2, t-l'}^{\pm})^{\delta_{j_2}} \\
&\leq \sum_{j_1=1}^m \sum_{k'=2}^{\infty} k'(k'-1) \mathbb{B}^{k'}(i_1, j_1) \mathcal{C}_{t-k'}(j_1).
\end{aligned}$$

Hence we deduce that

$$\mathbb{E} \left| \frac{\alpha_i^\pm \beta_j \beta_k}{\mathbb{H}_t(i_1)} \frac{\partial^3 \mathbb{H}_t(i_1)}{\partial \alpha_i^\pm \partial \beta_j \partial \beta_k} \right|^{r_0} < \infty. \quad (\text{A.58})$$

- (iii) For $i = s_2+1, \dots, s_1$ (that corresponds to the parameters $\text{vec}(B_\ell)$) and for fixed j and k such

that $\beta_i \neq \beta_j \neq \beta_k$, we have

$$\begin{aligned}
\frac{\partial^3 \mathbb{H}_t(i_1)}{\partial \beta_i \partial \beta_j \partial \beta_k} &= \sum_{j_1=1}^m \sum_{k'=3}^{\infty} \left[\sum_{l=3}^{k'} \left[\sum_{r=2}^{l-1} \left\{ \left(\sum_{a=1}^{r-1} \mathbb{B}_{i_1, j_1}^{a-1} \mathbb{B}_{i_1, j_1}^{(k)} \mathbb{B}_{i_1, j_1}^{r-1-a} \right) \mathbb{B}_{i_1, j_1}^{(j)} \mathbb{B}_{i_1, j_1}^{l-1-r} \right\} \mathbb{B}_{i_1, j_1}^{(i)} \mathbb{B}_{i_1, j_1}^{k'-l} \right. \right. \\
&\quad \left. \left. + \sum_{r=1}^{l-2} \left\{ \mathbb{B}_{i_1, j_1}^{r-1} \mathbb{B}_{i_1, j_1}^{(j)} \left(\sum_{a=1}^{l-1-r} \mathbb{B}_{i_1, j_1}^{a-1} \mathbb{B}_{i_1, j_1}^{(k)} \mathbb{B}_{i_1, j_1}^{l-1-r-a} \right) \right\} \mathbb{B}_{i_1, j_1}^{(i)} \mathbb{B}_{i_1, j_1}^{k'-l} \right] \right. \\
&\quad \left. + \sum_{l=2}^{k'-1} \left\{ \left(\sum_{r=1}^{l-1} \mathbb{B}_{i_1, j_1}^{r-1} \mathbb{B}_{i_1, j_1}^{(j)} \mathbb{B}_{i_1, j_1}^{l-1-r} \right) \mathbb{B}_{i_1, j_1}^{(i)} \left(\sum_{a=1}^{k'-l} \mathbb{B}_{i_1, j_1}^{a-1} \mathbb{B}_{i_1, j_1}^{(k)} \mathbb{B}_{i_1, j_1}^{k'-l-a} \right) \right\} \right. \\
&\quad \left. + \sum_{l=3}^{k'-1} \left[\sum_{r=2}^l \left\{ \left(\sum_{a=1}^{r-1} \mathbb{B}_{i_1, j_1}^{a-1} \mathbb{B}_{i_1, j_1}^{(k)} \mathbb{B}_{i_1, j_1}^{r-1-a} \right) \mathbb{B}_{i_1, j_1}^{(i)} \left(\sum_{m=1}^{k'-r} \mathbb{B}_{i_1, j_1}^{m-1} \mathbb{B}_{i_1, j_1}^{(j)} \mathbb{B}_{i_1, j_1}^{k'-r-m} \right) \right\} \right] \right. \\
&\quad \left. + \sum_{l=1}^{k'-2} \left[\mathbb{B}_{i_1, j_1}^{l-1} \mathbb{B}_{i_1, j_1}^{(i)} \sum_{r=2}^{k'-l} \left\{ \left(\sum_{a=1}^{r-1} \mathbb{B}_{i_1, j_1}^{a-1} \mathbb{B}_{i_1, j_1}^{(k)} \mathbb{B}_{i_1, j_1}^{r-1-a} \right) \mathbb{B}_{i_1, j_1}^{(j)} \mathbb{B}_{i_1, j_1}^{k'-l-r} \right\} \right. \right. \\
&\quad \left. \left. + \mathbb{B}_{i_1, j_1}^{l-1} \mathbb{B}_{i_1, j_1}^{(i)} \sum_{r=1}^{k'-l-1} \left\{ \mathbb{B}_{i_1, j_1}^{r-1} \mathbb{B}_{i_1, j_1}^{(j)} \left(\sum_{a=1}^{k'-l-r} \mathbb{B}_{i_1, j_1}^{a-1} \mathbb{B}_{i_1, j_1}^{(k)} \mathbb{B}_{i_1, j_1}^{k'-l-r-a} \right) \right\} \right] \right] \mathfrak{C}_{t-k}(j_1),
\end{aligned} \tag{A.59}$$

where $\mathbb{B}_{i_1, j_1} = \mathbb{B}(i_1, j_1)$ denotes the (i_1, j_1) -th component of the matrix \mathbb{B} . By multiplying by β_i , β_j and β_k we obtain

$$\begin{aligned}
\beta_i \beta_j \beta_k \frac{\partial^3 \mathbb{H}_t(i_1)}{\partial \beta_i \partial \beta_j \partial \beta_k} &= \sum_{j_1=1}^m \sum_{k'=3}^{\infty} \left[\sum_{l=3}^{k'} \left[\sum_{r=2}^{l-1} \left\{ \left(\sum_{a=1}^{r-1} \mathbb{B}_{i_1, j_1}^{a-1} \beta_k \mathbb{B}_{i_1, j_1}^{(k)} \mathbb{B}_{i_1, j_1}^{r-1-a} \right) \beta_j \mathbb{B}_{i_1, j_1}^{(j)} \mathbb{B}_{i_1, j_1}^{l-1-r} \right\} \beta_i \mathbb{B}_{i_1, j_1}^{(i)} \mathbb{B}_{i_1, j_1}^{k'-l} \right. \right. \\
&\quad \left. \left. + \sum_{r=1}^{l-2} \left\{ \mathbb{B}_{i_1, j_1}^{r-1} \beta_j \mathbb{B}_{i_1, j_1}^{(j)} \left(\sum_{a=1}^{l-1-r} \mathbb{B}_{i_1, j_1}^{a-1} \beta_k \mathbb{B}_{i_1, j_1}^{(k)} \mathbb{B}_{i_1, j_1}^{l-1-r-a} \right) \right\} \beta_i \mathbb{B}_{i_1, j_1}^{(i)} \mathbb{B}_{i_1, j_1}^{k'-l} \right] \right. \\
&\quad \left. + \sum_{l=2}^{k'-1} \left\{ \left(\sum_{r=1}^{l-1} \mathbb{B}_{i_1, j_1}^{r-1} \beta_j \mathbb{B}_{i_1, j_1}^{(j)} \mathbb{B}_{i_1, j_1}^{l-1-r} \right) \beta_i \mathbb{B}_{i_1, j_1}^{(i)} \left(\sum_{a=1}^{k'-l} \mathbb{B}_{i_1, j_1}^{a-1} \beta_k \mathbb{B}_{i_1, j_1}^{(k)} \mathbb{B}_{i_1, j_1}^{k'-l-a} \right) \right\} \right. \\
&\quad \left. + \sum_{l=3}^{k'-1} \left[\sum_{r=2}^l \left\{ \left(\sum_{a=1}^{r-1} \mathbb{B}_{i_1, j_1}^{a-1} \beta_k \mathbb{B}_{i_1, j_1}^{(k)} \mathbb{B}_{i_1, j_1}^{r-1-a} \right) \beta_i \mathbb{B}_{i_1, j_1}^{(i)} \left(\sum_{m=1}^{k'-r} \mathbb{B}_{i_1, j_1}^{m-1} \beta_j \mathbb{B}_{i_1, j_1}^{(j)} \mathbb{B}_{i_1, j_1}^{k'-r-m} \right) \right\} \right] \right. \\
&\quad \left. + \sum_{l=1}^{k'-2} \left[\mathbb{B}_{i_1, j_1}^{l-1} \beta_i \mathbb{B}_{i_1, j_1}^{(i)} \sum_{r=2}^{k'-l} \left\{ \left(\sum_{a=1}^{r-1} \mathbb{B}_{i_1, j_1}^{a-1} \beta_k \mathbb{B}_{i_1, j_1}^{(k)} \mathbb{B}_{i_1, j_1}^{r-1-a} \right) \beta_j \mathbb{B}_{i_1, j_1}^{(j)} \mathbb{B}_{i_1, j_1}^{k'-l-r} \right\} \right. \right. \\
&\quad \left. \left. + \mathbb{B}_{i_1, j_1}^{l-1} \beta_i \mathbb{B}_{i_1, j_1}^{(i)} \sum_{r=1}^{k'-l-1} \left\{ \mathbb{B}_{i_1, j_1}^{r-1} \beta_j \mathbb{B}_{i_1, j_1}^{(j)} \left(\sum_{a=1}^{k'-l-r} \mathbb{B}_{i_1, j_1}^{a-1} \beta_k \mathbb{B}_{i_1, j_1}^{(k)} \mathbb{B}_{i_1, j_1}^{k'-l-r-a} \right) \right\} \right] \right] \mathfrak{C}_{t-k}(j_1) \\
&\leq \sum_{j_1=1}^m \sum_{k'=3}^{\infty} \left[\sum_{l=3}^{k'} (l-2) \mathbb{B}_{i_1, j_1}^{k'} + \sum_{l=2}^{k'-1} (l-1)(k'-l) \mathbb{B}_{i_1, j_1}^{k'} + \sum_{l=1}^{k'-2} (k'-l-1) \mathbb{B}_{i_1, j_1}^{k'} \right] \mathfrak{C}_{t-k}(j_1) \\
&\leq \sum_{j_1=1}^m \sum_{k'=3}^{\infty} k'(k'-1)(k'-2) \mathbb{B}_{i_1, j_1}^{k'} \mathfrak{C}_{t-k}(j_1).
\end{aligned}$$

So we deduce that

$$\mathbb{E} \left| \frac{\beta_i \beta_j \beta_k}{\mathbb{H}_t(i_1)} \frac{\partial^3 \mathbb{H}_t(i_1)}{\partial \beta_i \partial \beta_j \partial \beta_k} \right|^{r_0} < \infty. \tag{A.60}$$

The three above cases prove the existence of the moments of the third order derivatives. As before, our estimations are in fact uniform and we may write that on a neighborhood $V(\theta_0)$ of $\theta_0 \in \overset{\circ}{\Theta}$, for all $i_1 = 1, \dots, m$ and $i, j, k = 1, \dots, s_1$, we have

$$\mathbb{E} \sup_{\theta \in V(\theta_0)} \left| \frac{1}{h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial^3 h_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_i \partial \theta_j \partial \theta_k}(\theta) \right|^{r_0} < \infty. \quad (\text{A.61})$$

These estimations imply that

$$\mathbb{E} \left[\sup_{\theta \in V(\theta_0)} \left| \frac{\partial^3 l_t(\theta)}{\partial \theta_i \partial \theta_j \partial \theta_k} \right| \right] < \infty. \quad (\text{A.62})$$

By a Taylor expansion around θ_0 , for all i and j it holds that

$$\frac{1}{n} \sum_{t=1}^n \frac{\partial^2 l_t}{\partial \theta_i \partial \theta_j}(\theta_{ij}^*) = \frac{1}{n} \sum_{t=1}^n \frac{\partial^2 l_t}{\partial \theta_i \partial \theta_j}(\theta_0) + \frac{1}{n} \sum_{t=1}^n \frac{\partial}{\partial \theta'} \left\{ \frac{\partial^2 l_t}{\partial \theta_i \partial \theta_j}(\tilde{\theta}_{ij}) \right\} (\theta_{ij}^* - \theta_0), \quad (\text{A.63})$$

where $\tilde{\theta}_{ij}$ lies between θ_{ij}^* et θ_0 . Using the almost sure convergence of $\tilde{\theta}_{ij}$ to θ_0 , the ergodic theorem and (A.62), we imply that almost-surely

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \left\| \frac{1}{n} \sum_{t=1}^n \frac{\partial}{\partial \theta'} \left\{ \frac{\partial^2 l_t}{\partial \theta_i \partial \theta_j}(\tilde{\theta}_{ij}) \right\} \right\| &\leq \limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{t=1}^n \sup_{\theta \in V(\theta_0)} \left\| \frac{\partial}{\partial \theta'} \left\{ \frac{\partial^2 l_t}{\partial \theta_i \partial \theta_j}(\theta_{ij}) \right\} \right\| \\ &= \mathbb{E}_{\theta_0} \sup_{\theta \in V(\theta_0)} \left\| \frac{\partial}{\partial \theta'} \left\{ \frac{\partial^2 l_t}{\partial \theta_i \partial \theta_j}(\theta_{ij}) \right\} \right\| < \infty. \end{aligned}$$

Since $\|\theta_{ij}^* - \theta_0\| \xrightarrow[n \rightarrow \infty]{} 0$, the second term in the right hand side of (A.63) converges to 0 almost-surely. By the ergodic theorem and using the same arguments than in the proof of theorem 2.2 in [Francq and Zakoian \(2004\)](#) it follows that

$$\frac{1}{n} \sum_{t=1}^n \frac{\partial^2 l_t}{\partial \theta_i \partial \theta_j}(\theta_{ij}^*) \longrightarrow J(i, j) \text{ in probability.} \quad (\text{A.64})$$

A.3.5. Invertibility of the matrix J

To prove the invertibility of the matrix J we calculate the derivatives of the criterion $\{\partial l_t(\theta)\}\{\partial \theta_i\}$ and $\{\partial^2 l_t(\theta)\}\{\partial \theta_i \partial \theta_j\}$ as functions of H_t . We start from

$$l_t(\theta) = \underline{\varepsilon}_t' H_t^{-1} \underline{\varepsilon}_t + \log(\det(H_t))$$

and we have the first derivative

$$\frac{\partial l_t(\theta)}{\partial \theta_i} = \text{Tr} \left[(H_t^{-1} - H_t^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' H_t^{-1}) \frac{\partial H_t}{\partial \theta_i} \right],$$

and the second derivative

$$\begin{aligned} \frac{\partial^2 l_t(\theta)}{\partial \theta_i \partial \theta_j} &= \text{Tr} \left[-H_t^{-1} \frac{\partial H_t}{\partial \theta_j} H_t^{-1} \frac{\partial H_t}{\partial \theta_i} + H_t^{-1} \frac{\partial H_t}{\partial \theta_i \partial \theta_j} + H_t^{-1} \frac{\partial H_t}{\partial \theta_j} H_t^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' H_t^{-1} \frac{\partial H_t}{\partial \theta_i} \right. \\ &\quad \left. + H_t^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' H_t^{-1} \frac{\partial H_t}{\partial \theta_j} H_t^{-1} \frac{\partial H_t}{\partial \theta_i} - H_t^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' H_t^{-1} \frac{\partial^2 H_t}{\partial \theta_i \partial \theta_j} \right]. \end{aligned}$$

Since

$$J = \mathbb{E} \left[\frac{\partial^2 l_t(\theta_0)}{\partial \theta_i \partial \theta_j} \right] = \mathbb{E} \left(\mathbb{E} \left[\frac{\partial^2 l_t(\theta_0)}{\partial \theta_i \partial \theta_j} \middle| \mathcal{F}_{t-1} \right] \right),$$

we compute the conditional expectation as follows (with the convention that $H_{0t} = H_t(\theta_0)$):

$$\begin{aligned}
\mathbb{E} \left[\frac{\partial^2 l_t}{\partial \theta_i \partial \theta_j}(\theta_0) \middle| \mathcal{F}_{t-1} \right] &= \mathbb{E} \left[-Tr \left(H_{0t}^{-1} \frac{\partial H_{0t}}{\partial \theta_j} H_{0t}^{-1} \frac{\partial H_{0t}}{\partial \theta_i} \right) - Tr \left(H_{0t}^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' H_{0t}^{-1} \frac{\partial^2 H_{0t}}{\partial \theta_i \partial \theta_j} \right) \right. \\
&\quad + Tr \left(H_{0t}^{-1} \frac{\partial H_{0t}}{\partial \theta_j} H_{0t}^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' H_{0t} \frac{\partial H_{0t}}{\partial \theta_i} \right) + Tr \left(H_{0t}^{-1} \frac{\partial^2 H_{0t}}{\partial \theta_i \partial \theta_j} \right) \\
&\quad \left. + Tr \left(H_{0t}^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' H_{0t}^{-1} \frac{\partial H_{0t}}{\partial \theta_i} H_{0t} \frac{\partial H_{0t}}{\partial \theta_j} \right) \middle| \mathcal{F}_{t-1} \right] \\
&= Tr \left(H_{0t}^{-1} \frac{\partial^2 H_{0t}}{\partial \theta_i \partial \theta_j} \right) - Tr \left(H_{0t}^{-1} \frac{\partial H_{0t}}{\partial \theta_j} H_{0t}^{-1} \frac{\partial H_{0t}}{\partial \theta_i} \right) \\
&\quad + Tr \left(H_{0t}^{-1} \frac{\partial H_{0t}}{\partial \theta_j} H_{0t}^{-1} \frac{\partial H_{0t}}{\partial \theta_i} \right) \mathbb{E} [\eta_t \eta_t' | \mathcal{F}_{t-1}] \\
&\quad + Tr \left(H_{0t}^{-1} \frac{\partial H_{0t}}{\partial \theta_i} H_{0t}^{-1} \frac{\partial H_{0t}}{\partial \theta_j} \right) \mathbb{E} [\eta_t \eta_t' | \mathcal{F}_{t-1}] \\
&\quad - Tr \left(H_{0t}^{-1} \frac{\partial^2 H_{0t}}{\partial \theta_i \partial \theta_j} \right) \mathbb{E} [\eta_t \eta_t' | \mathcal{F}_{t-1}] \\
&= Tr \left(H_{0t}^{-1} \frac{\partial H_{0t}}{\partial \theta_j} H_{0t}^{-1} \frac{\partial H_{0t}}{\partial \theta_i} \right)
\end{aligned}$$

By the relation $Tr(A'B) = (\text{vec}A)' \text{vec}B$ we have

$$Tr \left(H_{0t} H_{0t}^{(i)} H_{0t} H_{0t}^{(j)} \right) = \mathbf{h}_i' \mathbf{h}_j,$$

where $\mathbf{h}_i = \text{vec}(H_{0t}^{-1/2} H_{0t}^{(i)} H_{0t}^{-1/2})$ and $\mathbf{h}_j = \text{vec}(H_{0t}^{-1/2} H_{0t}^{(j)} H_{0t}^{-1/2})$. In view of $\text{vec}(ABC) = (C' \otimes A) \text{vec}B$ we have $\mathbf{h}_i = ((H_{0t}^{-1/2})' \otimes H_{0t}^{-1/2}) \mathbf{d}_i$ with $\mathbf{d}_i = \text{vec}(H_{0t}^{(i)})$.

We define the $m^2 \times s_0$ matrices $\mathbf{h} = (\mathbf{h}_1 | \dots | \mathbf{h}_{s_0})$ and $\mathbf{d} = (\mathbf{d}_1 | \dots | \mathbf{d}_{s_0})$, we have $\mathbf{h} = \mathbf{H} \mathbf{d}$ with $\mathbf{H} = (H_{0t}^{-1/2})' \otimes H_{0t}^{-1/2}$.

Reasoning by contradiction, we suppose that $J = \mathbb{E} [\mathbf{h}' \mathbf{h}]$ is singular. There exists a non-zero vector $\mathbf{c} \in \mathbb{R}^{s_0}$, such that $\mathbf{c}' J \mathbf{c} = \mathbb{E} [\mathbf{c}' \mathbf{h}' \mathbf{h} \mathbf{c}] = 0$. Since $\mathbf{c}' \mathbf{h}' \mathbf{h} \mathbf{c} \geq 0$ almost surely, it means that $\mathbf{c}' \mathbf{h}' \mathbf{h} \mathbf{c} = \mathbf{c}' \mathbf{d}' \mathbf{H}^2 \mathbf{d} \mathbf{c} = 0$, almost surely.

The matrix H_{0t} is definite-positive, then $H_{0t}^{-1/2}$ is too. This entails that $(H_{0t}^{-1/2})' \otimes H_{0t}^{-1/2}$ is definite positive. This implies that \mathbf{H}^2 is a definite-positive matrix with probability 1, and consequently $\mathbf{d} \mathbf{c} = 0_{m^2}$ with probability 1.

We write \mathbf{c} as $\mathbf{c} = (\mathbf{c}'_1, \mathbf{c}'_2)'$ where $\mathbf{c}_1 \in \mathbb{R}^{s_1}$ and $\mathbf{c}_2 \in \mathbb{R}^{s_4}$ where $s_4 = s_0 - s_1 = m(m-1)/2$ (which is the dimension of the parameters ρ). The rows $1, m+1, \dots, m^2$ of the following equations

$$\begin{aligned}
\mathbf{d} \mathbf{c} &= \sum_{i=1}^{s_0} c_i \frac{\partial}{\partial \theta_i} \text{vec}(H_{0t}) = \sum_{i=1}^{s_0} c_i \frac{\partial}{\partial \theta_i} \text{vec}(D_{0t} R_0 D_{0t}) = \sum_{i=1}^{s_0} c_i \frac{\partial}{\partial \theta_i} [(D_{0t} \otimes D_{0t}) \text{vec}(R_0)] \\
&= \sum_{i=1}^{s_1} c_i \frac{\partial}{\partial \theta_i} [(D_{0t} \otimes D_{0t})] \text{vec}(R_0) + \sum_{i=s_1+1}^{s_0} c_i (D_{0t} \otimes D_{0t}) \frac{\partial}{\partial \theta_i} [\text{vec}(R_0)] = 0_{m^2} \tag{A.65}
\end{aligned}$$

yield

$$\sum_{i=1}^{s_1} c_i \frac{\partial}{\partial \theta_i} h_t^{\delta/2}(\theta_0) = 0_m, \quad a.s. \tag{A.66}$$

by using (A.26). Differentiating the equation (2.1), we obtain that

$$\begin{aligned}
\sum_{k=1}^{s_1} c_k \frac{\partial}{\partial \theta_k} \underline{h}_t^{\delta/2}(\theta_0) &= \sum_{k=1}^{s_1} c_k \frac{\partial \underline{\omega}_0}{\partial \theta_k} + \sum_{i=1}^q \sum_{k=1}^{s_1} c_k \frac{\partial A_{0i}^+}{\partial \theta_k} (\underline{\varepsilon}_{t-i}^+)^{\delta/2} + \sum_{i=1}^q \sum_{k=1}^{s_1} c_k \frac{\partial A_{0i}^-}{\partial \theta_k} (\underline{\varepsilon}_{t-i}^-)^{\delta/2} \\
&\quad + \sum_{j=1}^p \sum_{k=1}^{s_1} c_k \frac{\partial B_{0j}}{\partial \theta_k} \underline{h}_{t-j}^{\delta/2}(\theta_0) + \sum_{j=1}^p \sum_{k=1}^{s_1} c_k B_{0j} \frac{\partial \underline{h}_{t-j}^{\delta/2}}{\partial \theta_k}(\theta_0) \\
&= \underline{\omega}_0^* + \sum_{i=1}^q A_{0i}^{+,*} (\underline{\varepsilon}_{t-i}^+)^{\delta/2} + \sum_{i=1}^q A_{0i}^{-,*} (\underline{\varepsilon}_{t-i}^-)^{\delta/2} + \sum_{j=1}^p B_{0j}^* \underline{h}_{t-j}^{\delta/2} \\
&\quad + \sum_{j=1}^p B_{0j} \sum_{k=1}^{s_1} c_k \frac{\partial \underline{h}_{t-j}^{\delta/2}}{\partial \theta_k}(\theta_0)
\end{aligned}$$

where

$$\underline{\omega}_0^* = \sum_{k=1}^{s_1} c_k \frac{\partial \underline{\omega}_0}{\partial \theta_k}, \quad A_{0i}^{+,*} = \sum_{k=1}^{s_1} c_k \frac{\partial A_{0i}^+}{\partial \theta_k}, \quad A_{0i}^{-,*} = \sum_{k=1}^{s_1} c_k \frac{\partial A_{0i}^-}{\partial \theta_k}, \quad B_{0j}^* = \sum_{k=1}^{s_1} c_k \frac{\partial B_{0j}}{\partial \theta_k}.$$

Because Equation (A.66) is satisfied for all t , we have

$$\underline{\omega}_0^* + \sum_{i=1}^q A_{0i}^{+,*} (\underline{\varepsilon}_{t-i}^+)^{\delta/2} + \sum_{i=1}^q A_{0i}^{-,*} (\underline{\varepsilon}_{t-i}^-)^{\delta/2} + \sum_{j=1}^p B_{0j}^* \underline{h}_{t-j}^{\delta/2} = 0.$$

It follows that

$$\begin{aligned}
\underline{h}_t^{\delta/2}(\theta_0) &= (\underline{\omega}_0 - \underline{\omega}_0^*) + \sum_{i=1}^q (A_{0i}^+ - A_{0i}^{+,*}) (\underline{\varepsilon}_{t-i}^+)^{\delta/2} + \sum_{i=1}^q (A_{0i}^- - A_{0i}^{-,*}) (\underline{\varepsilon}_{t-i}^-)^{\delta/2} \\
&\quad + \sum_{j=1}^p (B_{0j} - B_{0j}^*) \underline{h}_{t-j}^{\delta/2}(\theta_0).
\end{aligned}$$

Finally, we introduce the vector θ_1 for which the first s_1 components are

$$\text{vec}(\underline{\omega}_0 - \underline{\omega}_0^* | A_{01}^+ - A_{01}^{+,*} | \dots | A_{01}^- - A_{01}^{-,*} | \dots | B_{01} - B_{01}^* | \dots).$$

One may obtain $\underline{h}_t^{\delta/2}(\theta_0) = \underline{h}_t^{\delta/2}(\theta_1)$ by choosing \mathbf{c}_1 small enough in such a way that $\theta_1 \in \Theta$. If $\mathbf{c}_1 \neq 0$ then $\theta_1 \neq \theta_0$. This is in contradiction with the identifiability assumption and thus $\mathbf{c}_1 = 0$. Consequently, Equation (A.65) becomes

$$(D_{0t} \otimes D_{0t}) \sum_{k=s_1+1}^{s_0} c_k \frac{\partial}{\partial \theta_k} \text{vec} R_0 = 0_{m^2}, \quad a.s.$$

and then

$$\sum_{i=s_1+1}^{s_0} c_i \frac{\partial}{\partial \theta_i} \text{vec} R_0 = 0_{m^2}.$$

Since the vectors $\partial \text{vec} R_0 / \partial \theta_k$, $k = s_1 + 1, \dots, s_0$ are linearly independent, the vector $\mathbf{c}_2 = (c_{s_1+1}, \dots, c_{s_0})'$ is null and thus $\mathbf{c} = 0$. This is in contradiction with $\mathbf{c}' \mathbf{h}' \mathbf{h} \mathbf{c} = \mathbf{c}' \mathbf{d}' \mathbf{H}^2 \mathbf{d} \mathbf{c} = 0$ almost-surely. Therefore the assumption that J is not singular is absurd.

A.3.6. Forgetting of the initial values

To conclude the proof, we have to deduce (A.20) and (A.21) from (A.20) and (A.21). For this, we must show that the initial values have asymptotically no effect on the derivatives of the quasi likelihood. More precisely we may prove that

$$\left\| \frac{1}{\sqrt{n}} \sum_{t=1}^n \left[\frac{\partial l_t(\theta_0)}{\partial \theta} - \frac{\partial \tilde{l}_t(\theta_0)}{\partial \theta'} \right] \right\| = o_{\mathbb{P}}(1) \quad (\text{A.67})$$

and

$$\frac{1}{n} \sum_{t=1}^n \sup_{\theta \in V(\theta_0)} \left\| \frac{\partial^2 l_t(\theta)}{\partial \theta \partial \theta'} - \frac{\partial^2 \tilde{l}_t(\theta)}{\partial \theta \partial \theta'} \right\| = o_{\mathbb{P}}(1), \quad (\text{A.68})$$

for some neighbourhood $V(\theta_0)$.

The arguments are the same than in [Francq and Zakoïan \(2012\)](#).

By (A.22), it holds

$$\frac{\partial l_t(\theta)}{\partial \theta_i} - \frac{\partial \tilde{l}_t(\theta)}{\partial \theta_i} = \text{Tr}(d_1 + d_2 + d_3),$$

with

$$\begin{aligned} d_1 &= -D_t^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' D_t^{-1} D_t (D_t^{-1} - \tilde{D}_t^{-1}) R^{-1} D_t^{-1} D_t^{(i)} \\ d_2 &= -D_t^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' \tilde{D}_t^{-1} R^{-1} D_t^{-1} (D_t^{(i)} - \tilde{D}_t^{(i)}) \\ d_3 &= -\left(D_t^{-1} - \tilde{D}_t^{-1} \right) \underline{\varepsilon}_t \underline{\varepsilon}_t' \tilde{D}_t^{-1} R^{-1} \tilde{D}_t^{-1} \tilde{D}_t^{(i)} - D_t^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' \tilde{D}_t^{-1} R^{-1} \left(D_t^{-1} - \tilde{D}_t^{-1} \right) \tilde{D}_t^{(i)} \\ &\quad - D_t^{-1} R^{-1} D_t^{-1} D_t (D_t^{-1} - \tilde{D}_t^{-1}) \underline{\varepsilon}_t \underline{\varepsilon}_t' D_t^{-1} D_t^{(i)} - D_t^{-1} R^{-1} \tilde{D}_t^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' D_t^{-1} \left(D_t^{(i)} - \tilde{D}_t^{(i)} \right) \\ &\quad - \left(D_t^{-1} - \tilde{D}_t^{-1} \right) R^{-1} \tilde{D}_t^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' \tilde{D}_t^{-1} \tilde{D}_t^{(i)} - D_t^{-1} R^{-1} \tilde{D}_t^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' \left(D_t^{-1} - \tilde{D}_t^{-1} \right) \tilde{D}_t^{(i)} \\ &\quad + 2 \left[D_t^{-1} \left(D_t^{(i)} - \tilde{D}_t^{(i)} \right) + \left(D_t^{-1} - \tilde{D}_t^{-1} \right) \tilde{D}_t^{(i)} \right]. \end{aligned}$$

The term d_3 is a sum of term which can be handled as d_1 and d_2 . Thus we need to prove that $\sup_{\theta \in V(\theta_0)} \|D_t^{-1} \underline{\varepsilon}_t\| < \infty$, $\sup_{\theta \in V(\theta_0)} \|D_t(D_t^{-1} - \tilde{D}_t^{-1})\| < \infty$, $\sup_{\theta \in V(\theta_0)} \|D_t^{-1} D_t^{(i)}\| < \infty$ and $\sup_{\theta \in V(\theta_0)} \|D_t^{-1} (D_t^{(i)} - \tilde{D}_t^{(i)})\| < \infty$. From (A.15), (A.16), (A.17) and (A.18), we deduce that for any t

$$\begin{aligned} \sup_{\theta \in \Theta} \|D_t - \tilde{D}_t\| &\leq K \rho^t, \quad \sup_{\theta \in \Theta} \|D_t^{-1}\| \leq K, \quad \sup_{\theta \in \Theta} \|\tilde{D}_t^{-1}\| \leq K, \\ \sup_{\theta \in \Theta} \left| \frac{\tilde{h}_{i_1, t}^{1/2}(\theta)}{h_{i_1, t}^{1/2}(\theta)} \right| &\leq 1 + K \rho^t \quad \text{and} \quad \sup_{\theta \in \Theta} \left| \frac{h_{i_1, t}^{1/2}(\theta)}{\tilde{h}_{i_1, t}^{1/2}(\theta)} \right| \leq 1 + K \rho^t, \quad \text{for } i_1 = 1, \dots, m. \end{aligned} \quad (\text{A.69})$$

We remark that $D_t(D_t^{-1} - \tilde{D}_t^{-1}) = (\tilde{D}_t - D_t) \tilde{D}_t^{-1}$. Thus the above estimations yield

$$\sup_{\theta \in \Theta} \|D_t(D_t^{-1} - \tilde{D}_t^{-1})\| = \sup_{\theta \in \Theta} \|(\tilde{D}_t - D_t) \tilde{D}_t^{-1}\| \leq K \rho^t. \quad (\text{A.70})$$

By the matrix expressions (A.9) and (A.11), we have

$$\mathbb{H}_t = \sum_{k=0}^{t-r-1} \mathbb{B}^k \underline{c}_{t-k} + \mathbb{B}^{t-r} \mathbb{H}_r, \quad \tilde{\mathbb{H}}_t = \sum_{k=0}^{t-r-1} \mathbb{B}^k \tilde{c}_{t-k} + \mathbb{B}^{t-r} \tilde{\mathbb{H}}_r,$$

where $r = \max\{p, q\}$. Since, $\underline{c}_t = \tilde{c}_t$ for all $t > r$, we have

$$\mathbb{H}_t - \tilde{\mathbb{H}}_t = \mathbb{B}^{t-r} (\mathbb{H}_r - \tilde{\mathbb{H}}_r),$$

and

$$\frac{\partial}{\partial \theta_i} (\mathbb{H}_t - \tilde{\mathbb{H}}_t) = \mathbb{B}^{t-r} \frac{\partial}{\partial \theta_i} (\mathbb{H}_r - \tilde{\mathbb{H}}_r) + \sum_{j=1}^{t-r} \mathbb{B}^{j-1} \mathbb{B}^{(i)} \mathbb{B}^{t-r-j} (\mathbb{H}_r - \tilde{\mathbb{H}}_r).$$

Since $\sup_{\theta \in \Theta} \rho(\mathbb{B}) < 1$ and (A.12) or (A.13), we obtain

$$\sup_{\theta \in \Theta} \left\| \frac{\partial}{\partial \theta_i} (\mathbb{H}_t - \tilde{\mathbb{H}}_t) \right\| \leq K \rho^t$$

or equivalently

$$\sup_{\theta \in \Theta} \left\| \frac{\partial}{\partial \theta_i} \left(\underline{h}_t^{\delta/2}(\theta) - \tilde{\underline{h}}_t^{\delta/2}(\theta) \right) \right\| \leq K \rho^t. \quad (\text{A.71})$$

By (A.26) we have the expression of the derivative of D_t (and analogously for the derivative of \tilde{D}_t). Thus we may write, for $i_1 = 1, \dots, m$ that

$$\begin{aligned} D_t^{-1} \frac{\partial}{\partial \theta_i} (D_t - \tilde{D}_t)(i_1, i_1) &= D_t^{-1} (h_{i_1, t}^{1/2} - \tilde{h}_{i_1, t}^{1/2}) \frac{1}{\delta_{i_1} h_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial h_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_i}(\theta) \\ &\quad + D_t^{-1} \left(h_{i_1, t}^{\delta_{i_1}/2} - \tilde{h}_{i_1, t}^{\delta_{i_1}/2} \right) \frac{\tilde{h}_{i_1, t}^{1/2}}{\delta_{i_1} h_{i_1, t}^{\delta_{i_1}/2} \tilde{h}_{i_1, t}^{\delta_{i_1}/2}} \frac{\partial h_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_i}(\theta) \\ &\quad + D_t^{-1} \frac{\tilde{h}_{i_1, t}^{1/2}}{\delta_{i_1} \tilde{h}_{i_1, t}^{\delta_{i_1}/2}} \left(\frac{\partial h_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_i}(\theta) - \frac{\partial \tilde{h}_{i_1, t}^{\delta_{i_1}/2}}{\partial \theta_i}(\theta) \right). \end{aligned}$$

Using, for $i_1 = 1, \dots, m$, (A.13), (A.15), (A.69), (A.70), (A.71) and in view of (A.33), we obtain

$$\sup_{\theta \in \Theta} \left\| D_t^{-1} \frac{\partial}{\partial \theta_i} (D_t - \tilde{D}_t)(i_1, i_1) \right\| \leq K \rho^t u_t, \quad (\text{A.72})$$

where u_t is a squared integrable variable. Using (2.2), from (A.53) and (A.69), we deduce

$$\sup_{\theta \in V(\theta_0)} \|D_t^{-1} \underline{\varepsilon}_t\| = \sup_{\theta \in V(\theta_0)} \|D_t^{-1} D_{0t} \tilde{\eta}_t\| \leq v_t \|\tilde{\eta}_t\|, \quad (\text{A.73})$$

$$\sup_{\theta \in V(\theta_0)} \|\tilde{D}_t^{-1} \underline{\varepsilon}_t\| = \sup_{\theta \in V(\theta_0)} \|\tilde{D}_t^{-1} D_t\| \|D_t^{-1} \underline{\varepsilon}_t\| \leq (1 + K \rho^t) v_t \|\tilde{\eta}_t\|, \quad (\text{A.74})$$

where the random variable v_t admits a fourth-order moment. Now, using (A.69)–(A.74) and the Cauchy-Schwarz inequality, we obtain

$$\sup_{\theta \in V(\theta_0)} \left| \frac{\partial l_t(\theta)}{\partial \theta_i} - \frac{\partial \tilde{l}_t(\theta)}{\partial \theta_i} \right| \leq K \rho^t w_t,$$

where w_t is an integrable variable. From the Markov inequality, we have

$$\frac{1}{\sqrt{n}} \sum_{t=1}^n \rho^t w_t = o_{\mathbb{P}}(1),$$

which implies (A.67). By exactly the same arguments, we obtain

$$\sup_{\theta \in V(\theta_0)} \left| \frac{\partial^2 l_t(\theta)}{\partial \theta_i \partial \theta_j} - \frac{\partial^2 \tilde{l}_t(\theta)}{\partial \theta_i \partial \theta_j} \right| \leq K \rho^t w_t^*,$$

where w_t^* is an integrable random variable. Using the Borel-Cantelli lemma and the Markov inequality, we deduce that $\rho^t w_t^*$ goes to zero almost surely. Consequently, the Cesàro lemma implies that $n^{-1} \sum_{t=1}^n \rho^t w_t^* \rightarrow 0$ when n goes to infinity, which entails (A.68).

The proof of Theorem 3 is completed. \square

A.4. Proof of Theorem 4

In the sequel, we will use the version of the matrix representation (A.8) when the parameter δ is unknown. We write

$$\mathbb{H}_t^{(\tau)} = \underline{c}_t^{(\tau)} + \mathbb{B} \cdot \mathbb{H}_{t-1}^{(\tau)}, \quad (\text{A.75})$$

with

$$\mathbb{H}_t^{(\tau)} = \begin{pmatrix} h_t^{\tau/2} \\ h_{t-1}^{\tau/2} \\ \vdots \\ h_{t-p+1}^{\tau/2} \end{pmatrix}, \quad \underline{c}_t^{(\tau)} = \begin{pmatrix} \omega + \sum_{i=1}^q A_i^+ (\varepsilon_{t-i}^+)^{\tau/2} + A_i^- (\varepsilon_{t-1}^-)^{\tau/2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$\mathbb{B} = \begin{pmatrix} B_1 & B_2 & \dots & B_p \\ I_m & & & 0 \\ 0 & I_m & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & I_m & 0 \end{pmatrix},$$

and we can iterate the expression and we have

$$\mathbb{H}_t^{(\tau)} = \underline{c}_t^{(\tau)} + \mathbb{B} \underline{c}_{t-1}^{(\tau)} + \mathbb{B}^2 \underline{c}_{t-2}^{(\tau)} + \dots + \mathbb{B}^{t-1} \underline{c}_1^{(\tau)} + \mathbb{B}^t \mathbb{H}_0^{(\tau)} = \sum_{k=0}^{\infty} \mathbb{B}^k \underline{c}_{t-k}^{(\tau)}. \quad (\text{A.76})$$

We prove our consistency statement when $\underline{\delta}_0$ is unknown. As in the case where $\underline{\delta}_0$ was known (see Section A.2), the proof is decomposed in the four following points which will be treated in separate subsections.

A.4.1. Initial values do not influence quasi-likelihood: $\lim_{n \rightarrow \infty} \sup_{\nu \in \Delta} |\mathcal{L}_n(\nu) - \tilde{\mathcal{L}}_n(\nu)| = 0$ a.s.

A.4.2. Identifiability: If there exists $\in \mathbb{Z}$ such that $h_t(\nu) = h_t(\nu_0)$ almost surely and $R = R_0$, then $\nu = \nu_0$.

A.4.3. Minimisation of the quasi log-likelihood on the true value: $\mathbb{E}_{\nu_0} |l_t(\nu_0)| < \infty$, and if $\nu \neq \nu_0$, $\mathbb{E}_{\nu_0} [l_t(\nu)] > \mathbb{E}_{\nu_0} [l_t(\nu_0)]$

A.4.4. For any $\nu \neq \nu_0$ there exists a neighborhood $V(\nu)$ such that

$$\liminf_{n \rightarrow \infty} \inf_{\nu^* \in V(\nu)} \tilde{\mathcal{L}}_n(\nu^*) > \mathbb{E}_{\nu_0} l_1(\nu_0), \quad a.s. \quad (\text{A.77})$$

There are many similarities with the proof of Theorem 2. We only indicates where the fact that the power is estimated has an importance is our reasoning.

A.4.1. Initial values do not influence quasi-likelihood

The proof is the same than the one done in Subsection A.2.1 when the power is assumed to be known.

A.4.2. Identifiability

As regard to the proof of identifiability from Subsection A.2.2, it only remains to prove that if for $i_1 = 1, \dots, m$, $h_{i_1,t}(\nu)/h_{i_1,t}(\nu_0) = 1$, a.s, then $\tau = \delta_0$. Let δ_{0,i_1} (resp. τ_{i_1}) the i_1^{th} element of δ_0 (resp. of τ). We denote $Q^\pm(L) = \mathcal{B}(L)^{-1}\mathcal{A}^\pm(L) = \sum_{i \geq 1} Q_i^\pm L^i$ and $Q_0^\pm(L) = \mathcal{B}_0(L)^{-1}\mathcal{A}_0^\pm(L) = \sum_{i \geq 1} Q_{0i}^\pm L^i$. Under Assumption A4, by Proposition 1, for any $i_2 = 1, \dots, m$, one may find $i_0 \geq 1$ and $i_1 \in \{1, \dots, m\}$ such that $Q_{0i_0}^+ + Q_{0i_0}^- \neq 0$. Since the coefficients of the matrix are positive, we denote by (i_1, i_2) the position of a non zero element, for $i_2 = 1, \dots, m$. By (2.3) and (2.5) we have

$$\begin{aligned} h_{i_1,t}^{\delta_{0,i_1}/2}(\nu_0) &= \sum_{j_1=1}^m \mathcal{B}_0(1)^{-1}(i_1, j_1)\omega_0(j_1) + \sum_{j_1=1}^m \sum_{l=1}^{\infty} Q_{0l}^+(i_1, j_1)(\varepsilon_{j_1,t-l}^+)^{\delta_{0,j_1}} + \sum_{j_1=1}^m \sum_{l=1}^{\infty} Q_{0l}^-(i_1, j_1)(-\varepsilon_{j_1,t-l}^-)^{\delta_{0,j_1}} \\ &= C_{i_1,t-i_0-1}(\nu_0) + \sum_{j_1=1}^m Q_{0i_0}^+(i_1, j_1)(\varepsilon_{j_1,t-i_0}^+)^{\delta_{0,j_1}} + \sum_{j_1=1}^m Q_{0i_0}^-(i_1, j_1)(-\varepsilon_{j_1,t-i_0}^-)^{\delta_{0,j_1}} \end{aligned}$$

where the quantities indexed by $t - i_0 - 1$ are \mathcal{F}_{t-i_0-1} -measurable. In the same way we have

$$\begin{aligned} h_{i_1,t}^{\tau_{i_1}/2}(\nu) &= \sum_{j_1=1}^m \mathcal{B}(1)^{-1}(i_1, j_1)\omega(j_1) + \sum_{j_1=1}^m \sum_{l=1}^{\infty} Q_l^+(i_1, j_1)(\varepsilon_{j_1,t-l}^+)^{\tau_{j_1}} + \sum_{j_1=1}^m \sum_{l=1}^{\infty} Q_l^-(i_1, j_1)(-\varepsilon_{j_1,t-l}^-)^{\tau_{j_1}} \\ &= C_{i_1,t-i_0-1}(\nu) + \sum_{j_1=1}^m Q_{i_0}^+(i_1, j_1)(\varepsilon_{j_1,t-i_0}^+)^{\tau_{j_1}} + \sum_{j_1=1}^m Q_{i_0}^-(i_1, j_1)(-\varepsilon_{j_1,t-i_0}^-)^{\tau_{j_1}}. \end{aligned}$$

Since $h_{i_1,t}(\nu)/h_{i_1,t}(\nu_0) = 1$, a.s, we have

$$\frac{\left\{ C_{i_1,t-i_0-1}(\nu) + \sum_{j_1=1}^m Q_{i_0}^+(i_1, j_1)(\varepsilon_{j_1,t-i_0}^+)^{\tau_{j_1}} + \sum_{j_1=1}^m Q_{i_0}^-(i_1, j_1)(-\varepsilon_{j_1,t-i_0}^-)^{\tau_{j_1}} \right\}^{\delta_{0,i_1}/\tau_{i_1}}}{C_{i_1,t-i_0-1}(\nu_0) + \sum_{j_1=1}^m Q_{0i_0}^+(i_1, j_1)(\varepsilon_{j_1,t-i_0}^+)^{\delta_{0,j_1}} + \sum_{j_1=1}^m Q_{0i_0}^-(i_1, j_1)(-\varepsilon_{j_1,t-i_0}^-)^{\delta_{0,j_1}}} = 1 \text{ a.s.} \quad (\text{A.78})$$

We denote $r_{j_1} = \tau_{j_1}/\delta_{0,j_1}$ and we introduce the function

$$V(x_1, \dots, x_m) = \frac{\left\{ C_{i_1,t-i_0-1}(\nu) + \sum_{j_1=1}^m Q_{i_0}^+(i_1, j_1)(x_{j_1}^+)^{r_{j_1}\delta_{0,j_1}} + \sum_{j_1=1}^m Q_{i_0}^-(i_1, j_1)(-x_{j_1}^-)^{r_{j_1}\delta_{0,j_1}} \right\}^{1/r_{i_1}}}{C_{i_1,t-i_0-1}(\nu_0) + \sum_{j_1=1}^m Q_{0i_0}^+(i_1, j_1)(x_{j_1}^+)^{\delta_{0,j_1}} + \sum_{j_1=1}^m Q_{0i_0}^-(i_1, j_1)(-x_{j_1}^-)^{\delta_{0,j_1}}}.$$

By (A.78), $V(\eta_{t-1}) = 1$ almost-surely hence V is almost surely constant on some neighborhood of zero (see Assumption A8). Hence for any $y = x_{j_1}^{\delta_{0,j_1}} \in [a, b] \subset [0, +\infty[$:

$$V(0, \dots, 0, y, 0, \dots, 0) = \frac{\left\{ C_{i_1,t-i_0-1}(\nu) + Q_{i_0}^+(i_1, i_2)(y)^{r_{i_2}} \right\}^{1/r_{i_1}}}{C_{i_1,t-i_0-1}(\nu_0) + Q_{0i_0}^+(i_1, i_2)(y)} = 1$$

almost-surely. Since the coefficients $C_{i_1,t-i_0-1}(\nu)$, $Q_{i_0}^+(i_1, i_2)$, $C_{i_1,t-i_0-1}(\nu_0)$ and $Q_{0i_0}^+(i_1, i_2)$ are positive, we deduce that $r_{i_2} = r_{i_1} := r$ after differentiate twice the above equation. Starting now from

$$V(0, \dots, 0, y, 0, \dots, 0) = \frac{\left\{ C_{i_1,t-i_0-1}(\nu) + Q_{i_0}^+(i_1, i_2)(y)^r \right\}^{1/r}}{C_{i_1,t-i_0-1}(\nu_0) + Q_{0i_0}^+(i_1, i_2)(y)} = 1,$$

we can deduce by differentiating twice again, as in Hamadeh and Zakoian (2011), that $r = r_{i_1} = 1$. Hence we have $\tau_{i_2} = \delta_{0,i_2}$. This is done for any $i_2 = 1, \dots, m$ so the result is proved.

A.4.3. Minimisation of the likelihood on the true value

Replacing θ_0 by ν_0 , the proof is the same than the one when the power is assumed to be known.

A.4.4. *Proof of (A.77)*

Once again, the proof is the same than the one when the power is assumed to be known.

A.4.5. *Conclusion*

The proof Theorem 4 follows the argument from Theorem 2.

A.5. *Proof of Theorem 5*

Now we deal with the asymptotic normality result when \underline{d}_0 is unknown. We follow the arguments and the different steps that we used in the proof of Theorem 3 in Section A.3. To establish the asymptotic normality result when the power is known, the proof is again decomposed in six intermediates points.

A.5.1. First derivative of the quasi log-likelihood

A.5.2. Existence of moments at any order of the score

A.5.3. Asymptotic normality of the score vector:

$$\frac{1}{\sqrt{n}} \sum_{t=1}^n \frac{\partial l_t(\theta_0)}{\partial \theta} \xrightarrow{\mathcal{L}} \mathcal{N}(0, I). \quad (\text{A.79})$$

A.5.4. Convergence to J :

$$\frac{1}{n} \sum_{t=1}^n \frac{\partial^2 l_t(\theta_{ij}^*)}{\partial \theta_i \partial \theta_j} \longrightarrow J(i, j) \text{ in probability,} \quad (\text{A.80})$$

A.5.5. Invertibility of the matrix J

A.5.6. Forgetting of the initial values

We introduce the following notations:

- * $s_0 = 2m + (p + 2q)m^2 + m(m - 1)/2$,
- * $s_1 = 2m + (p + 2q)m^2$,
- * $s_2 = m + (p + 2q)m^2$,
- * $s_3 = m + 2qm^2$,
- * $s_4 = m + qm^2$.

A.5.1. *First derivative of the quasi log-likelihood*

The aim of this subsection is to establish the expressions of the first order derivatives of the quasi log-likelihood. We may argue as in subsection A.3.1.

We denote $D_{0t} = D_t(\nu_0)$, $R_0 = R(\nu_0)$,

$$D_{0t}^{(i)} = \frac{\partial D_t}{\partial \nu_i}(\nu_0), \quad R_0^{(i)} = \frac{\partial R}{\partial \nu_i}(\nu_0),$$

$$D_{0t}^{(i,j)} = \frac{\partial^2 D_t}{\partial \nu_i \partial \nu_j}(\nu_0), \quad R_0^{(i,j)} = \frac{\partial^2 R}{\partial \nu_i \partial \nu_j}(\nu_0),$$

and $\underline{\varepsilon}_t = D_{0t} \tilde{\eta}_t$, where $\tilde{\eta}_t(\nu) = R^{1/2} \eta_t(\nu)$ with $\tilde{\eta}_t = \tilde{\eta}_t(\nu_0) = R_0^{1/2} \eta_t$.

When we differentiate with respect to ν_i for $i = 1, \dots, s_1$ (that is with respect to $(\underline{\omega}', \alpha_1^{+'}, \dots, \alpha_q^{+'}, \alpha_1^{-'}, \dots, \alpha_q^{-'}, \beta_1', \dots, \beta_p', \underline{\tau}')'$) we obtain:

$$\frac{\partial l_t(\nu)}{\partial \nu_i} = -Tr \left((\underline{\varepsilon}_t \underline{\varepsilon}_t' D_t^{-1} R^{-1} + R^{-1} D_t^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t') D_t^{-1} \frac{\partial D_t}{\partial \nu_i} D_t^{-1} \right) + 2Tr \left(D_t^{-1} \frac{\partial D_t}{\partial \nu_i} \right) \quad (\text{A.81})$$

$$\frac{\partial l_t(\nu_0)}{\partial \nu_i} = Tr \left[(I_m - R_0^{-1} \tilde{\eta}_t \tilde{\eta}_t') D_{0t}^{(i)} D_{0t}^{-1} + (I_m - \tilde{\eta}_t \tilde{\eta}_t' R_0^{-1}) D_{0t}^{-1} D_{0t}^{(i)} \right]. \quad (\text{A.82})$$

We differentiate with respect to ν_i for $i = s_1 + 1, \dots, s_0$ (that is with respect to ρ'). We have

$$\frac{\partial l_t(\nu)}{\partial \nu_i} = -Tr \left(R^{-1} D_t^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' D_t^{-1} R^{-1} \frac{\partial R}{\partial \nu_i} \right) + Tr \left(R^{-1} \frac{\partial R}{\partial \nu_i} \right) \quad (\text{A.83})$$

$$\frac{\partial l_t(\nu_0)}{\partial \nu_i} = Tr \left[(I_m - R_0^{-1} \tilde{\eta}_t \tilde{\eta}_t') R_0^{-1} R_0^{(i)} \right]. \quad (\text{A.84})$$

A.5.2. Existence of moments at any order for the quasi log-likelihood

Arguing as in the beginning of Subsection A.3.2 we have:

(i) for $i = 1, \dots, s_1$

$$\left| \frac{\partial l_t(\nu_0)}{\partial \nu_i} \right| \leq K \left\| D_{0t}^{(i)} D_{0t}^{-1} \right\|,$$

(ii) for $i = s_1 + 1, \dots, s_0$

$$\left| \frac{\partial l_t(\nu_0)}{\partial \nu_i} \right| \leq K,$$

(iii) for $i, j = 1, \dots, s_1$

$$\mathbb{E} \left| \frac{\partial l_t(\nu_0)}{\partial \nu_i} \frac{\partial l_t(\nu_0)}{\partial \nu_j} \right| \leq K \left(\mathbb{E} \| D_{0t}^{(i)} D_{0t}^{-1} \|^2 \mathbb{E} \| D_{0t}^{(j)} D_{0t}^{-1} \|^2 \right)^{1/2},$$

(iv) for $i = 1, \dots, s_1$ and $j = s_1 + 1, \dots, s_0$

$$\mathbb{E} \left| \frac{\partial l_t(\nu_0)}{\partial \nu_i} \frac{\partial l_t(\nu_0)}{\partial \nu_j} \right| \leq K \mathbb{E} \left\| D_{0t}^{(i)} D_{0t}^{-1} \right\|,$$

(v) and finally for $i, j = s_1 + 1, \dots, s_0$, we have

$$\mathbb{E} \left| \frac{\partial l_t(\nu_0)}{\partial \nu_i} \frac{\partial l_t(\nu_0)}{\partial \nu_j} \right| \leq K.$$

To have the finiteness of the moments of the first derivative of the quasi log-likelihood, it remains to treat the cases (i), (iii) and (iv) above. Thus, we have to control the term $\| D_{0t}^{(i)} D_{0t}^{-1} \|$. Since

$$D_{0t} = \text{Diag}(\underline{h}_t^{1/2}(\nu_0)) = \text{Diag}(\underline{h}_t^{\underline{\tau}/2}(\nu_0))^{1/\underline{\tau}},$$

we can work component wise.

All the computations that we have done in Subsection A.3.2 are valid. This means that we have the same estimations on the derivatives as long as we differentiate with respect to ν_i for $i \in \{1, \dots, s_2\}$ (that is when we do not differentiate with respect to $\underline{\tau}_j$ for $j = 1, \dots, m$). Indeed, for $i_1 = 1, \dots, m$ and $i = 1, \dots, s_2$, we have

$$\frac{\partial D_{0t}(i_1, i_1)}{\partial \nu_i} = \frac{1}{\tau_{i_1}} h_{i_1, t}^{1/2} \times \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial h_{i_1, t}^{\tau_{i_1}/2}}{\partial \nu_i}(\nu_0) \quad (\text{A.85})$$

and the reasonings are unchanged.

So we can focus ourselves on the derivatives with respect to $\underline{\tau}$:

$$\frac{\partial D_t(i_1, i_1)}{\partial \tau_j} = h_{i_1, t}^{1/2} \left[-\delta_{j, i_1} \frac{1}{\tau_{i_1}^2} \log \left(h_{i_1, t}^{\tau_{i_1}/2} \right) + \frac{1}{\tau_{i_1}} \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial h_{i_1, t}^{\tau_{i_1}/2}}{\partial \tau_j} (\nu_0) \right], \quad (\text{A.86})$$

where δ_{j, i_1} denotes the Kronecker symbol. Using the matrix expression (A.76), we calculate the derivatives $\partial \mathbb{H}_t^{(\tau)} / \partial \nu_i$ for $i = s_2 + 1, \dots, s_2 + m$ (with $s_1 = s_2 + m$) and for $i_1 = 1, \dots, m$:

$$\frac{\partial \mathbb{H}_t^{(\tau)}(i_1)}{\partial \nu_i} = \sum_{k=0}^{\infty} \sum_{j_1=1}^m \mathbb{B}^k(i_1, j_1) \frac{\partial \underline{c}_{t-k}^{(\tau)}(j_1)}{\partial \nu_i},$$

with

$$\frac{\partial \underline{c}_{t-k}^{(\tau)}(j_1)}{\partial \nu_i} = \sum_{l=1}^q \sum_{j_2=1}^m \left\{ A_l^+(j_1, j_2) \frac{\partial (\varepsilon_{j_2, t-l}^+)^{\tau_{j_2}}}{\partial \nu_i} + A_l^-(j_1, j_2) \frac{\partial (-\varepsilon_{j_2, t-l}^-)^{\tau_{j_2}}}{\partial \nu_i} \right\}.$$

Differentiating with respect to ν_i corresponds to a differentiation with respect to τ_{i_0} for an index $i_0 \in \{1, \dots, m\}$. In the following computations, it is easy to work with an arbitrary order of derivation. So we write, for an order of derivation $\kappa \in \{1, 2, 3\}$, that

$$\frac{\partial^\kappa (\pm \varepsilon_{j_2, t-l}^\pm)^{\tau_{j_2}}}{\partial \nu_i^\kappa} = \frac{\partial^\kappa (\pm \varepsilon_{j_2, t-l}^\pm)^{\tau_{j_2}}}{\partial \tau_{i_0}^\kappa} = \begin{cases} 0 & \text{if } j_2 \neq i_0 \\ \log^\kappa (\pm \varepsilon_{i_0, t-l}^\pm) (\pm \varepsilon_{i_0, t-l}^\pm)^{\tau_{i_0}} & \text{if } j_2 = i_0, \end{cases}$$

and we have

$$\frac{\partial^\kappa \underline{c}_{t-\kappa}^{(\tau)}(j_1)}{\partial \tau_{i_0}^\kappa} = \sum_{l=1}^q A_l^+(j_1, i_0) \log^\kappa (\varepsilon_{i_0, t-l}^+) (\varepsilon_{i_0, t-l}^+)^{\tau_{i_0}} + A_l^-(j_1, i_0) \log^\kappa (-\varepsilon_{i_0, t-l}^-) (-\varepsilon_{i_0, t-l}^-)^{\tau_{i_0}}.$$

By convention, we consider $\log(x^+) = 0$ when x is negative and $\log(x^-) = 0$ when x is positive.

$$\begin{aligned} \left| \frac{1}{\underline{c}_t^{(\tau)}(j_1)} \frac{\partial^\kappa \underline{c}_t^{(\tau)}(j_1)}{\partial \tau_{i_0}^\kappa} \right| &\leq \frac{\sum_{l=1}^q A_l^+(j_1, i_0) |\log(\varepsilon_{i_0, t-l}^+)|^\kappa (\varepsilon_{i_0, t-l}^+)^{\tau_{i_0}} + A_l^-(j_1, i_0) |\log(-\varepsilon_{i_0, t-l}^-)|^\kappa (-\varepsilon_{i_0, t-l}^-)^{\tau_{i_0}}}{\underline{\omega}(j_1) + \sum_{l=1}^q \sum_{j_2=1}^m \left\{ A_l^+(j_1, j_2) (\varepsilon_{j_2, t-l}^+)^{\tau_{j_2}} + A_l^-(j_1, j_2) (-\varepsilon_{j_2, t-l}^-)^{\tau_{j_2}} \right\}} \\ &\leq \frac{\sum_{l=1}^q A_l^+(j_1, i_0) |\log(\varepsilon_{i_0, t-l}^+)|^\kappa (\varepsilon_{i_0, t-l}^+)^{\tau_{i_0}} + A_l^-(j_1, i_0) |\log(-\varepsilon_{i_0, t-l}^-)|^\kappa (-\varepsilon_{i_0, t-l}^-)^{\tau_{i_0}}}{\underline{\omega}(j_1) + \sum_{l=1}^q \left\{ A_l^+(j_1, i_0) (\varepsilon_{i_0, t-l}^+)^{\tau_{i_0}} + A_l^-(j_1, i_0) (-\varepsilon_{i_0, t-l}^-)^{\tau_{i_0}} \right\}} \\ &\leq \sum_{l=1}^q \left| \log(\varepsilon_{i_0, t-l}^+) \right|^\kappa + \left| \log(-\varepsilon_{i_0, t-l}^-) \right|^\kappa. \end{aligned}$$

Using the inequality

$$\mathbb{H}_t^{(\tau)}(i_1) \geq \omega + \sum_{k'=1}^{\infty} \sum_{j_1=1}^m \mathbb{B}^{k'}(i_1, j_1) \underline{c}_{t-k'}^{(\tau)}(j_1) \geq \omega + \mathbb{B}^k(i_1, j_1) \underline{c}_{t-k}^{(\tau)}(j_1)$$

valid for any $k \geq 0$ where $\omega = \inf_{1 \leq i \leq m} \underline{\omega}(i)$, and the fact that $x/(1+x) \leq x^s$ for all $x \geq 0$, we obtain

$$\begin{aligned}
\sup_{\nu \in \Delta} \left| \frac{1}{\mathbb{H}_t^{(\tau)}(i_1)} \frac{\partial^\kappa \mathbb{H}_t^{(\tau)}(i_1)}{\partial \nu_i^\kappa} \right| &\leq \sup_{\nu \in \Delta} \sum_{k=0}^{\infty} \sum_{j_1=1}^m \frac{\mathbb{B}^k(i_1, j_1)}{\omega + \mathbb{B}^k(i_1, j_1) \underline{c}_{t-k}^{(\tau)}(j_1)} \left| \frac{\partial^\kappa \underline{c}_{t-k}^{(\tau)}(j_1)}{\partial \nu_i^\kappa} \right| \\
&\leq \sup_{\nu \in \Delta} \sum_{k=0}^{\infty} \sum_{j_1=1}^m \frac{\mathbb{B}^k(i_1, j_1) \underline{c}_{t-k}^{(\tau)}(j_1)}{\omega + \mathbb{B}^k(i_1, j_1) \underline{c}_{t-k}^{(\tau)}(j_1)} \left| \frac{1}{\underline{c}_{t-k}^{(\tau)}(j_1)} \frac{\partial^\kappa \underline{c}_{t-k}^{(\tau)}(j_1)}{\partial \nu_i^\kappa} \right| \\
&\leq \sup_{\nu \in \Delta} \sum_{k=0}^{\infty} \sum_{j_1=1}^m \frac{\mathbb{B}^k(i_1, j_1) \underline{c}_{t-k}^{(\tau)}(j_1)}{\omega + \mathbb{B}^k(i_1, j_1) \underline{c}_{t-k}^{(\tau)}(j_1)} \left(\sum_{j=1}^q \left\{ \left| \log(\varepsilon_{i,t-k-j}^+) \right|^\kappa + \left| \log(-\varepsilon_{i,t-k-j}^-) \right|^\kappa \right\} \right) \\
&\leq \sum_{k=0}^{\infty} \sum_{j_1=1}^m \sup_{\nu \in \Delta} \left(\frac{\mathbb{B}^k(i_1, j_1) \underline{c}_{t-k}^{(\tau)}(j_1)}{\omega} \right)^s \left(\sum_{j=1}^q \left\{ \left| \log(\varepsilon_{i,t-k-j}^+) \right|^\kappa + \left| \log(-\varepsilon_{i,t-k-j}^-) \right|^\kappa \right\} \right) \\
&\leq K \sum_{k=0}^{\infty} \sum_{j_1=1}^m \rho^{sk} \sup_{\nu \in \Delta} \left(\underline{c}_{t-k}^{(\tau)}(j_1) \right)^s \left(\sum_{j=1}^q \left\{ \left| \log(\varepsilon_{i,t-k-j}^+) \right|^\kappa + \left| \log(-\varepsilon_{i,t-k-j}^-) \right|^\kappa \right\} \right).
\end{aligned}$$

We have $\mathbb{E}_{\nu_0} \left(\sup_{\nu \in \Delta} (\underline{c}_{t-k}^{(\tau)}(j_1))^{2s} \right) < \infty$, for all $s > 0$ (see Corollary 2). So we obtain

$$\mathbb{E}_{\nu_0} \left[\sup_{\nu \in \Delta} \left| \frac{1}{\mathbb{H}_t^{(\tau)}(i_1)} \frac{\partial^\kappa \mathbb{H}_t^{(\tau)}(i_1)}{\partial \nu_i^\kappa} \right| \right] \leq K \sum_{j=1}^q (S_{i,j}^+ + S_{i,j}^-)$$

with for $j \geq 1$:

$$S_{i,j}^\pm = \sum_{k=0}^{\infty} \rho^{sk} \left(\mathbb{E}_{\nu_0} \left| \log(\pm \varepsilon_{i,t-k-j}^\pm) \right|^{2\kappa} \right)^{1/2}.$$

By stationarity, we treat only the terms $S_{i,1}^\pm$ and the computations are identical when one replaces $\underline{\varepsilon}^+$ by $\underline{\varepsilon}^-$ so we will only to treat $S_{i,1}^+$. We have for any $A > 0$

$$\mathbb{E}_{\nu_0} \left| \log(\varepsilon_{i,t-k-1}^+) \right|^{2\kappa} \leq A + \mathbb{E}_{\nu_0} \left| \log(\varepsilon_{i,t-k-1}^+) \mathbb{1}_{\left| \log(\varepsilon_{i,t-k-1}^+) \right|^{2\kappa} \geq A} \right|^{2\kappa}.$$

It follows that

$$\begin{aligned}
S_{i,1}^+ &\leq A + \sum_{k=0}^{\infty} \rho^{sk} \left(\int_A^\infty \mathbb{P} \left(\left| \log(\varepsilon_{i,t-k-1}^+) \right|^{2\kappa} > x \right) dx \right)^{1/2} \\
&\leq A + \sum_{k=0}^{\infty} \rho^{sk} \left(\int_A^\infty \mathbb{P} \left(\left| \log(\varepsilon_{i,t-k-1}^+) \right| > x^{1/2\kappa} \right) dx \right)^{1/2} \\
&\leq A + \sum_{k=0}^{\infty} \rho^{sk} \left(\int_A^\infty 2\kappa x^{2\kappa-1} \mathbb{P} \left(\left| \log(\varepsilon_{i,t-k-1}^+(j_1)) \right| > x \right) dx \right)^{1/2} \\
&\leq A + \sum_{k=0}^{\infty} \rho^{sk} \left(\int_A^\infty 2\kappa x^{2\kappa-1} \left[\mathbb{P} \left(\log(\varepsilon_{i,t-k-1}^+) > x \right) + \mathbb{P} \left(\log(\varepsilon_{i,t-k-1}^+) < -x \right) \right] dx \right)^{1/2} \\
&\leq A + \sum_{k=0}^{\infty} \rho^{sk} \left(\int_A^\infty 2\kappa x^{2\kappa-1} \left[\mathbb{P} \left(\varepsilon_{i,t-k-1}^+ \geq \exp\{x\} \right) + \mathbb{P} \left(\varepsilon_{i,t-k-1}^+ < \exp\{-x\} \right) \right] dx \right)^{1/2} \\
&\leq A + \sum_{k=0}^{\infty} \rho^{sk} (T_1^+ + T_2^+)^{1/2}, \tag{A.87}
\end{aligned}$$

with obvious notations. Using the Markov inequality one has

$$T_1^+ = \int_A^\infty 2\kappa x^{2\kappa-1} \mathbb{P}\left(\varepsilon_{i,t-k-1}^+ \geq \exp\{x\}\right) dx \leq C. \quad (\text{A.88})$$

The second term T_2^+ is more difficult. One has to use the property

$$\lim_{y \rightarrow 0^+} \frac{1}{y} \mathbb{P}(\varepsilon_{i,t-k-1}^+ \leq y) = 0. \quad (\text{A.89})$$

This is the Assumption A1 in [Pan et al. \(2008\)](#) and in our case, it is a consequence of the fact that η_t has a positive density on some neighborhood of zero (see Assumption **A8**). We apply this property to $\mathbb{P}(\varepsilon_{i,t-k-1}^+ < \exp\{-x\})$. Indeed one may find some $A > 0$ such that $\exp\{-x\}$ is small. Thus for any $x \geq A$:

$$\mathbb{P}\left(\varepsilon_{i,t-k-1}^+ < \exp\{-x\}\right) \leq c \exp\{-x\}. \quad (\text{A.90})$$

Hence, the second term T_2^+ satisfies

$$T_2^+ = \int_A^\infty 2\kappa x^{2\kappa-1} \mathbb{P}\left(\varepsilon_{j_1,t-k}^+ < \exp\{-x\}\right) dx \leq c \int_A^\infty 2\kappa x^{2\kappa-1} \exp\{-x\} dx \leq C < \infty. \quad (\text{A.91})$$

We use [\(A.88\)](#) and [\(A.91\)](#) in [\(A.87\)](#) and we obtain that

$$S_{i,1}^+ \leq A + C \sum_{k=0}^{\infty} \rho^{sk} < \infty.$$

This yields

$$\mathbb{E}_{\nu_0} \left[\sup_{\nu \in \Delta} \left| \frac{1}{\mathbb{H}_t^{(\tau)}(i_1)} \frac{\partial^\kappa \mathbb{H}_t^{(\tau)}(i_1)}{\partial \nu_i^\kappa} \right| \right] < \infty$$

and it can be similarly be shown that, for any $r \geq 1$:

$$\mathbb{E}_{\nu_0} \left[\sup_{\nu \in \Delta} \left| \frac{1}{\mathbb{H}_t^{(\tau)}(i_1)} \frac{\partial^\kappa \mathbb{H}_t^{(\tau)}(i_1)}{\partial \nu_i^\kappa} \right|^r \right] < \infty \quad (\text{A.92})$$

or equivalently that for any $i_1 = 1, \dots, m$

$$\mathbb{E} \sup_{\nu \in V(\nu_0)} \left| \frac{1}{h_{i_1,t}^{\tau_{i_1}/2}} \frac{\partial^\kappa h_{i_1,t}^{\tau_{i_1}/2}}{\partial \nu_i^\kappa}(\nu) \right|^r < \infty. \quad (\text{A.93})$$

A.5.3. Asymptotic normality of the score vector

The arguments are the same than in the case of known power (see subsection [A.3.3](#)).

A.5.4. Convergence to J

\rightsquigarrow *Expression of the second order derivatives of the log-likelihood*

First one may remark that the algebraic expressions of the second order derivatives [\(A.34\)](#), [\(A.35\)](#) and [\(A.36\)](#) are unchanged (even if the values of s_0 and s_1 take into account the parameter $\underline{\tau}$). To control the derivates at the second order it is sufficient to control the term $\|D^{-1}D_{0t}^{(i,j)}\|$. For this, it necessary to calculate the second order derivatives of D_{0t} .

↪ *Existence of the moments of the second order derivatives of the log-likelihood*

We give now the expressions of the second order derivatives of D_{0t} .

(i) For $i, j = 1, \dots, s_2$, we have

$$\frac{\partial^2 D_{0t}(i_1, j_1)}{\partial \nu_i \partial \nu_j} = \frac{1}{\tau_{i_1}} h_{i_1, t}^{1/2} \left[\frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial h_{i_1, t}^{\tau_{i_1}/2}}{\partial \nu_j} \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial h_{i_1, t}^{\tau_{i_1}/2}}{\partial \nu_i} \left(\frac{1}{\tau_{i_1}} - 1 \right) + \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial^2 h_{i_1, t}^{\tau_{i_1}/2}}{\partial \nu_i \partial \nu_j} \right]$$

(ii) for $i_1 = 1, \dots, m$ and $j = 1, \dots, s_2$, we obtain

$$\begin{aligned} \frac{\partial^2 D_{0t}(i_1, j_1)}{\partial \tau_{i_1} \partial \nu_j} &= \frac{1}{\tau_{i_1}} h_{i_1, t}^{1/2} \left[-\frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial h_{i_1, t}^{\tau_{i_1}/2}}{\partial \nu_j} + \frac{1}{\tau_{i_1}} \left(\frac{1}{\tau_{i_1}} \log(h_{i_1, t}^{\tau_{i_1}/2}) + \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial h_{i_1, t}^{\tau_{i_1}/2}}{\partial \tau_{i_1}} \right) \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial h_{i_1, t}^{\tau_{i_1}/2}}{\partial \nu_j} \right. \\ &\quad \left. - \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial h_{i_1, t}^{\tau_{i_1}/2}}{\partial \tau_{i_1}} \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial h_{i_1, t}^{\tau_{i_1}/2}}{\partial \nu_i} + \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial^2 h_{i_1, t}^{\tau_{i_1}/2}}{\partial \tau_{i_1} \partial \nu_j} \right]. \end{aligned}$$

(iii) for $i_1 = 1, \dots, m, j = 1, \dots, s_2$ and $i_0 = 1, \dots, m$ such that $i_0 \neq i_1$, we have

$$\begin{aligned} \frac{\partial^2 D_{0t}(i_1, j_1)}{\partial \tau_{i_0} \partial \nu_j} &= \frac{1}{\tau_{i_1}} h_{i_1, t}^{1/2} \left[\frac{1}{\tau_{i_0}} \left(-\frac{1}{\tau_{i_0}} \log(h_{i_1, t}^{\tau_{i_1}/2}) + \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial h_{i_1, t}^{\tau_{i_1}/2}}{\partial \tau_{i_0}} \right) \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial h_{i_1, t}^{\tau_{i_1}/2}}{\partial \nu_j} \right. \\ &\quad \left. - \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial h_{i_1, t}^{\tau_{i_1}/2}}{\partial \tau_{i_0}} \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial h_{i_1, t}^{\tau_{i_1}/2}}{\partial \nu_j} + \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial^2 h_{i_1, t}^{\tau_{i_1}/2}}{\partial \tau_{i_0} \partial \nu_j} \right] \end{aligned}$$

(iv) for $i_1 = 1, \dots, m$, we have

$$\begin{aligned} \frac{\partial^2 D_{0t}(i_1, j_1)}{\partial \tau_{i_1}^2} &= \frac{1}{\tau_{i_1}} h_{i_1, t}^{1/2} \left[-\frac{1}{\tau_{i_1}} \left(-\frac{1}{\tau_{i_1}} \log(h_{i_1, t}^{\tau_{i_1}/2}) + \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial h_{i_1, t}^{\tau_{i_1}/2}}{\partial \tau_{i_1}} \right) \right. \\ &\quad \left. + \frac{1}{\tau_{i_1}} \left(-\frac{1}{\tau_{i_1}} \log(h_{i_1, t}^{\tau_{i_1}/2}) + \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial h_{i_1, t}^{\tau_{i_1}/2}}{\partial \tau_{i_1}} \right)^2 + \frac{1}{\tau_{i_1}^2} \log(h_{i_1, t}^{\tau_{i_1}/2}) \right. \\ &\quad \left. - \frac{1}{\tau_{i_1}} \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial h_{i_1, t}^{\tau_{i_1}/2}}{\partial \tau_{i_1}} - \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial h_{i_1, t}^{\tau_{i_1}/2}}{\partial \tau_{i_1}} \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial h_{i_1, t}^{\tau_{i_1}/2}}{\partial \tau_{i_1}} + \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial^2 h_{i_1, t}^{\tau_{i_1}/2}}{\partial \tau_{i_1}^2} \right] \end{aligned}$$

(v) finally for $i_1, i_0 = 1, \dots, m$, we obtain

$$\begin{aligned} \frac{\partial^2 D_{0t}(i_1, j_1)}{\partial \tau_{i_1} \partial \tau_{i_0}} &= \frac{1}{\tau_{i_1}} h_{i_1, t}^{1/2} \left[\frac{1}{\tau_{i_0}} \left(-\frac{1}{\tau_{i_0}} \log(h_{i_1, t}^{\tau_{i_1}/2}) + \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial h_{i_1, t}^{\tau_{i_1}/2}}{\partial \tau_{i_0}} \right) \left(-\frac{1}{\tau_{i_1}} \log(h_{i_1, t}^{\tau_{i_1}/2}) + \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial h_{i_1, t}^{\tau_{i_1}/2}}{\partial \tau_{i_1}} \right) \right. \\ &\quad \left. - \frac{1}{\tau_{i_1}} \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial h_{i_1, t}^{\tau_{i_1}/2}}{\partial \tau_{i_0}} + \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial h_{i_1, t}^{\tau_{i_1}/2}}{\partial \tau_{i_0}} \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial h_{i_1, t}^{\tau_{i_1}/2}}{\partial \tau_{i_1}} + \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial^2 h_{i_1, t}^{\tau_{i_1}/2}}{\partial \tau_{i_1} \partial \tau_{i_0}} \right] \end{aligned}$$

Since the first order derivatives are already controlled, and since the estimations done in the case with known power, it remains to prove that

$$\mathbb{E} \left| \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial^2 h_{i_1, t}^{\tau_{i_1}/2}}{\partial \tau_i \partial \nu_j} (\nu_0) \right|^{r_0} < \infty, \text{ for } i = 1, \dots, m \text{ and } j = 1, \dots, s_2. \quad (\text{A.94})$$

By (A.76), we have

$$\mathbb{H}_t^{(\tau)}(i_1) = \sum_{k=0}^{\infty} \sum_{j_1=1}^d \mathbb{B}^k(i_1, j_1) \mathcal{C}_{t-k}^{(\tau)}(j_1).$$

It is easy to notice that

$$\frac{\partial^2 \mathbb{H}_t^{(\tau)}(i_1)}{\partial \tau_i \partial \omega_j} = 0,$$

and for $i \neq j$

$$\frac{\partial^2 \mathbb{H}_t^{(\tau)}(i_1)}{\partial \tau_i \partial \tau_j} = 0.$$

It remains to treat three cases to prove (A.94).

(a) For $i = 1, \dots, m$ we have

$$\frac{\partial^2 \mathbb{H}_t^{(\tau)}(i_1)}{\partial \tau_i \partial \alpha_j^{\pm}} = \sum_{k=0}^{\infty} \sum_{j_1=1}^m \sum_{l=1}^q \mathbb{B}^k(i_1, j_1) \frac{\partial A_l^{\pm}(i_1, j_1)}{\partial \alpha_j^{\pm}} \log(\pm \varepsilon_{i_1, t-l}^{\pm})(\pm \varepsilon_{i_1, t-l}^{\pm})^{\tau_{i_1}},$$

where $\partial A_l^{\pm}(i_1, j_1) / \partial \alpha_j^{\pm}$ is a matrix with only one non null element equal 1 at the place of α_j^{\pm} . It follows that

$$\begin{aligned} \alpha_j^{\pm} \frac{\partial^2 \mathbb{H}_t^{(\tau)}(i_1)}{\partial \tau_i \partial \alpha_j^{\pm}} &\leq \sum_{k=0}^{\infty} \sum_{j_1=1}^m \sum_{l=1}^q \mathbb{B}^k(i_1, j_1) \alpha_j^{\pm} \log(\pm \varepsilon_{i_1, t-l}^{\pm})(\pm \varepsilon_{i_1, t-l}^{\pm})^{\tau_{i_1}} \\ &\leq \sum_{k=0}^{\infty} \sum_{j_1=1}^m \mathbb{B}^k(i_1, j_1) \frac{\partial \mathcal{C}_{t-k}^{(\tau)}(j_1)}{\partial \tau_i}. \end{aligned}$$

Using the same techniques used to prove (A.93) with $\kappa = 1$, we obtain that

$$\mathbb{E} \left| \frac{\alpha_j^{\pm}}{\mathbb{H}_t^{(\tau)}(i_1)} \frac{\partial^2 \mathbb{H}_t^{(\tau)}(i_1)}{\partial \tau_i \partial \alpha_j^{\pm}} \right|^{r_0} < \infty.$$

(b) For $i = 1, \dots, m$ it holds

$$\frac{\partial^2 \mathbb{H}_t^{(\tau)}(i_1)}{\partial \tau_i \partial \beta_j} = \sum_{k=1}^{\infty} \sum_{j_1=1}^m \left\{ \sum_{l=1}^k \mathbb{B}^{l-1}(i_1, j_1) \mathbb{B}^{(j)}(i_1, j_1) \mathbb{B}^{k-l}(i_1, j_1) \right\} \frac{\partial \mathcal{C}_{t-k}^{(\tau)}(i_1)}{\partial \tau_i}.$$

Consequently

$$\beta_j \frac{\partial^2 \mathbb{H}_t^{(\tau)}(i_1)}{\partial \tau_i \partial \beta_j} \leq \sum_{k=1}^{\infty} \sum_{j_1=1}^m k \mathbb{B}^k(i_1, j_1) \frac{\partial \mathcal{C}_{t-k}^{(\tau)}(i_1)}{\partial \tau_i},$$

and we proceed as in the previous case to conclude that

$$\mathbb{E} \left| \frac{\beta_j}{\mathbb{H}_t^{(\tau)}(i_1)} \frac{\partial^2 \mathbb{H}_t^{(\tau)}(i_1)}{\partial \tau_i \partial \beta_j} \right|^{r_0} < \infty.$$

(c) For $i = 1, \dots, m$

$$\frac{\partial^2 \mathbb{H}_t^{(\tau)}(i_1)}{\partial \tau_i^2} = \sum_{k=0}^{\infty} \sum_{j_1=1}^m \mathbb{B}^k(i_1, j_1) \frac{\partial^2 \mathcal{C}_{t-k}^{(\tau)}(i_1)}{\partial \tau_i^2},$$

and we use (A.93) with $\kappa = 2$ in order to obtain

$$\mathbb{E} \left| \frac{1}{\mathbb{H}_t^{(\tau)}(i_1)} \frac{\partial^2 \mathbb{H}_t^{(\tau)}(i_1)}{\partial \tau_i^2} \right|^{r_0} < \infty.$$

The above arguments can be generalized on a neighborhood of $V(\nu_0)$ of ν_0 . So we have for all $i_1 = 1, \dots, m$ and all $i, j = 1, \dots, s_1$:

$$\mathbb{E} \sup_{\nu \in V(\nu_0)} \left| \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial^2 h_{i_1, t}^{\tau_{i_1}/2}}{\partial \nu_i \partial \nu_j}(\nu) \right|^{r_0} < \infty.$$

Using the same arguments as in Subsection A.3.4 combined with the previous modification taking into account that the power is unknown (especially the estimation (A.93) with $\kappa = 3$) we obtain that for $i_1 = 1, \dots, m$ and $i, j, k = 1, \dots, s_1$, it holds

$$\mathbb{E} \sup_{\nu \in V(\nu_0)} \left| \frac{1}{h_{i_1, t}^{\tau_{i_1}/2}} \frac{\partial^3 h_{i_1, t}^{\tau_{i_1}/2}}{\partial \nu_i \partial \nu_j \partial \nu_k}(\nu) \right|^{r_0} < \infty.$$

A.5.5. Invertibility of the matrix J

Replacing θ by ν in the arguments that lead to (A.65), one obtains that the rows $1, m+1, \dots, m^2$ of the following equations

$$\mathbf{dc} = \sum_{i=1}^{s_0} c_i \frac{\partial}{\partial \nu_i} [(D_{0t} \otimes D_{0t}) \text{vec}(R_0)] = 0_{m^2} \quad (\text{A.95})$$

yield

$$\sum_{i=1}^{s_1} c_i \frac{\partial}{\partial \nu_i} h_i^{\tau/2}(\nu_0) = 0_m, \quad a.s. \quad (\text{A.96})$$

$$(D_{0t} \otimes D_{0t}) \sum_{k=s_1+1}^{s_0} c_k \frac{\partial}{\partial \nu_k} \text{vec} R_0 = 0_{m^2}, \quad a.s. \quad (\text{A.97})$$

Under **A8**, Equation (A.97) is equivalent to

$$\sum_{i=s_1+1}^{s_0} c_k \frac{\partial}{\partial \nu_k} \text{vec} R_0 = 0_{m^2}.$$

Since the vectors $\partial \text{vec} R_0 / \partial \nu_k$, $k = s_1 + 1, \dots, s_0$ are linearly independent, the vector $\mathbf{c}_3 = (c_{s_1+1}, \dots, c_{s_0})'$ is null. Consequently, Equation (A.95) becomes

$$\sum_{i=1}^{s_2} c_i \frac{\partial}{\partial \theta_i} h_i^{\tau/2}(\nu_0) + \sum_{i=s_2+1}^{s_2+m} c_i \frac{\partial}{\partial \nu_i} h_i^{\tau/2}(\nu_0) = 0_m, \quad a.s.$$

or equivalently

$$\sum_{i=1}^{s_2} c_{1,i} \frac{\partial}{\partial \theta_i} h_i^{\tau/2}(\nu_0) + \sum_{i=1}^m c_{2,i} \frac{\partial}{\partial \nu_i} h_i^{\tau/2}(\nu_0) = 0_m, \quad a.s. \quad (\text{A.98})$$

In view of (2.1), the i_1^{th} component of $\underline{h}_t^{\tau/2}(\nu_0)$ is

$$\begin{aligned} h_{i_1,t}^{\tau_{i_1}/2} &= \underline{\omega}_0(i_1) + \sum_{i_2=1}^m \sum_{i=1}^q \left(A_{0i}^+(i_1, i_2) \left(\varepsilon_{i_2,t-i}^+ \right)^{\tau_{i_2}} + A_{0i}^-(i_1, i_2) \left(-\varepsilon_{i_2,t-i}^- \right)^{\tau_{i_2}} \right) \\ &\quad + \sum_{i_2=1}^m \sum_{i=1}^p B_{0i}(i_1, i_2) h_{i_2,t-i}^{\tau_{i_2}/2} \end{aligned} \quad (\text{A.99})$$

For $i_1 = 1, \dots, m$, from (A.85) and (A.86), Equation (A.98) reduces as

$$h_{i_1,t}^{1/2} \frac{1}{\tau_{i_1}^2} \frac{1}{h_{i_1,t}^{\tau_{i_1}/2}} \left[\sum_{i=1}^{s_2} c_{1,i} \tau_{i_1} \frac{\partial h_{i_1,t}^{\tau_{i_1}/2}}{\partial \theta_i}(\nu_0) + \sum_{i=1}^m c_{2,i} \tau_{i_1} \frac{\partial h_{i_1,t}^{\tau_{i_1}/2}}{\partial \tau_i}(\nu_0) - c_{2,i_1} h_{i_1,t}^{\tau_{i_1}/2} \log \left(h_{i_1,t}^{\tau_{i_1}/2} \right) \right] = 0, \quad a.s. \quad (\text{A.100})$$

From (A.99), the derivatives are defined recursively by

$$\begin{aligned} \frac{\partial h_{i_1,t}^{\tau_{i_1}/2}}{\partial \theta}(\nu) &= \tilde{c}_t(\nu) + \sum_{i_2=1}^m \sum_{i=1}^p B_i(i_1, i_2) \frac{\partial h_{i_2,t-i}^{\tau_{i_2}/2}}{\partial \theta}(\nu), \\ \frac{\partial h_{i_1,t}^{\tau_{i_1}/2}}{\partial \tau_j}(\nu) &= \sum_{i=1}^q \left(A_i^+(i_1, j) \log \left(\varepsilon_{j,t-i}^+ \right) \left(\varepsilon_{j,t-i}^+ \right)^{\tau_j} + A_i^-(i_1, j) \log \left(-\varepsilon_{j,t-i}^- \right) \left(-\varepsilon_{j,t-i}^- \right)^{\tau_j} \right) \\ &\quad + \sum_{i_2=1}^m \sum_{i=1}^p B_i(i_1, i_2) \frac{\partial h_{i_2,t-i}^{\tau_{i_2}/2}}{\partial \tau_j}(\nu) \end{aligned}$$

for $j = 1, \dots, m$, where $\tilde{c}_t(\nu)$ is defined by

$$\begin{aligned} \tilde{c}_t(\nu) &= \left(0, \dots, 1, 0, \dots, \left(\varepsilon_{i_1,t-1}^+ \right)^{\tau_{i_1}}, 0, \dots, \left(\varepsilon_{i_m,t-1}^+ \right)^{\tau_{i_m}}, 0, \dots, \left(\varepsilon_{i_1,t-q}^+ \right)^{\tau_{i_1}}, 0, \dots, \left(\varepsilon_{i_m,t-q}^+ \right)^{\tau_{i_m}}, \right. \\ &\quad \left. 0, \dots, \left(-\varepsilon_{i_1,t-1}^- \right)^{\tau_{i_1}}, 0, \dots, \left(-\varepsilon_{i_m,t-1}^- \right)^{\tau_{i_m}}, 0, \dots, \left(-\varepsilon_{i_1,t-q}^- \right)^{\tau_{i_1}}, 0, \dots, \left(-\varepsilon_{i_m,t-q}^- \right)^{\tau_{i_m}}, \right. \\ &\quad \left. 0, \dots, h_{i_1,t-1}^{\tau_{i_1}/2}, 0, \dots, h_{i_m,t-1}^{\tau_{i_m}/2}, 0, \dots, h_{i_1,t-p}^{\tau_{i_1}/2}, 0, \dots, h_{i_m,t-p}^{\tau_{i_m}/2}, \dots, 0 \right)'. \end{aligned} \quad (\text{A.101})$$

Let R_t a random variable measurable with respect to $\sigma\{\eta_u, u \leq t\}$.

We restrict ourselves to the particular case of a CCC-APGARCH($p, 1$) (see (2.1)). The general case can be easily deduced from the following arguments. We thus have

$$\begin{aligned} \frac{\partial h_{i_1,t}^{\tau_{i_1}/2}}{\partial \tau_j}(\nu) &= A_1^+(i_1, j) \log \left(\varepsilon_{j,t-1}^+ \right) \left(\varepsilon_{j,t-1}^+ \right)^{\tau_j} + A_1^-(i_1, j) \log \left(-\varepsilon_{j,t-1}^- \right) \left(-\varepsilon_{j,t-1}^- \right)^{\tau_j} + R_{t-2}, \\ h_{i_1,t}^{\tau_{i_1}/2}(\nu) &= \sum_{i_2=1}^m \left(A_1^+(i_1, i_2) \left(\varepsilon_{i_2,t-1}^+ \right)^{\tau_{i_2}} + A_1^-(i_1, i_2) \left(-\varepsilon_{i_2,t-1}^- \right)^{\tau_{i_2}} \right) + R_{t-2}. \end{aligned}$$

Combining the above expressions and (A.101), under **A8**, Equation (A.100) becomes

$$\begin{aligned}
0 &= \sum_{i=1}^m \tau_{i_1} \left(c_{1,i_1+i \cdot m} \left(\varepsilon_{i,t-1}^+ \right)^{\tau_i} + c_{1,i_1+i \cdot m^2} \left(-\varepsilon_{i,t-1}^- \right)^{\tau_i} \right) + R_{t-2}^1 \\
&\quad + \sum_{i=1}^m c_{2,i} \tau_{i_1} \left(A_{01}^+(i_1, i) \log \left(\varepsilon_{i,t-1}^+ \right) \left(\varepsilon_{i,t-1}^+ \right)^{\tau_i} + A_{01}^-(i_1, i) \log \left(-\varepsilon_{i,t-1}^- \right) \left(-\varepsilon_{i,t-1}^- \right)^{\tau_i} + R_{t-2}^2 \right) \\
&\quad - c_{2,i_1} \left(\sum_{i_2=1}^m \left(A_{01}^+(i_1, i_2) \left(\varepsilon_{i_2,t-1}^+ \right)^{\tau_{i_2}} + A_{01}^-(i_1, i_2) \left(-\varepsilon_{i_2,t-1}^- \right)^{\tau_{i_2}} \right) + R_{t-2}^3 \right) \\
&\quad \times \log \left[\sum_{i_2=1}^m \left(A_{01}^+(i_1, i_2) \left(\varepsilon_{i_2,t-1}^+ \right)^{\tau_{i_2}} + A_{01}^-(i_1, i_2) \left(-\varepsilon_{i_2,t-1}^- \right)^{\tau_{i_2}} \right) + R_{t-2}^3 \right], \quad a.s.
\end{aligned}$$

which is equivalent, almost surely, to the following two equations

$$\begin{aligned}
&\sum_{i=1}^m \tau_{i_1} c_{1,i_1+i \cdot m} \left(\varepsilon_{i,t-1}^+ \right)^{\tau_i} + R_{t-2}^1 + \sum_{i=1}^m c_{2,i} \tau_{i_1} \left(A_{01}^+(i_1, i) \log \left(\varepsilon_{i,t-1}^+ \right) \left(\varepsilon_{i,t-1}^+ \right)^{\tau_i} + R_{t-2}^2 \right) \\
&- c_{2,i_1} \left(\sum_{i_2=1}^m A_{01}^+(i_1, i_2) \left(\varepsilon_{i_2,t-1}^+ \right)^{\tau_{i_2}} + R_{t-2}^3 \right) \log \left[\sum_{i_2=1}^m A_{01}^+(i_1, i_2) \left(\varepsilon_{i_2,t-1}^+ \right)^{\tau_{i_2}} + R_{t-2}^3 \right] = 0
\end{aligned} \tag{A.102}$$

$$\begin{aligned}
&\sum_{i=1}^m \tau_{i_1} c_{1,i_1+i \cdot m^2} \left(-\varepsilon_{i,t-1}^- \right)^{\tau_i} + R_{t-2}^1 + \sum_{i=1}^m c_{2,i} \tau_{i_1} \left(A_{01}^-(i_1, i) \log \left(-\varepsilon_{i,t-1}^- \right) \left(-\varepsilon_{i,t-1}^- \right)^{\tau_i} + R_{t-2}^2 \right) \\
&- c_{2,i_1} \left(\sum_{i_2=1}^m A_{01}^-(i_1, i_2) \left(-\varepsilon_{i_2,t-1}^- \right)^{\tau_{i_2}} + R_{t-2}^3 \right) \log \left[\sum_{i_2=1}^m A_{01}^-(i_1, i_2) \left(-\varepsilon_{i_2,t-1}^- \right)^{\tau_{i_2}} + R_{t-2}^3 \right] = 0.
\end{aligned} \tag{A.103}$$

It follows from **A8** that, for some $\prod_{i=1}^m [a_i, b_i] \subset [0, +\infty[^m$,

$$\begin{aligned}
&\sum_{i=1}^m \tau_{i_1} c_{1,i_1+i \cdot m} x_i + R_{t-2}^1 + \sum_{i=1}^m c_{2,i} \left(A_{01}^+(i_1, i) x_i \log(x_i) + R_{t-2}^2 \right) \\
&- c_{2,i_1} \left(\sum_{i_2=1}^m A_{01}^+(i_1, i_2) x_{i_2} + R_{t-2}^3 \right) \log \left[\sum_{i_2=1}^m A_{01}^+(i_1, i_2) x_{i_2} + R_{t-2}^3 \right] = 0, \quad a.s.
\end{aligned} \tag{A.104}$$

$$\begin{aligned}
&\sum_{i=1}^m \tau_{i_1} c_{1,i_1+i \cdot m^2} x_i + R_{t-2}^1 + \sum_{i=1}^m c_{2,i} \left(A_{01}^-(i_1, i) x_i \log(x_i) + R_{t-2}^2 \right) \\
&- c_{2,i_1} \left(\sum_{i_2=1}^m A_{01}^-(i_1, i_2) x_{i_2} + R_{t-2}^3 \right) \log \left[\sum_{i_2=1}^m A_{01}^-(i_1, i_2) x_{i_2} + R_{t-2}^3 \right] = 0, \quad a.s.
\end{aligned} \tag{A.105}$$

for any $(x_1, \dots, x_m) \in \prod_{i=1}^m [a_i, b_i]$. Differentiating three times Equations (A.104) and (A.105) with respect to x_i , for $i = 1, \dots, m$, we obtain that

$$(c_{2,i} - c_{2,i_1}) \left[\left(A_{01}^+(i_1, i) \right)^2 + \left(A_{01}^-(i_1, i) \right)^2 \right] = 0.$$

Under Assumption **A4** : if $p > 0$, $\mathcal{A}_0^+(1) + \mathcal{A}_0^-(1) \neq 0$, it is impossible to have $A_{01}^+(i_1, i) = A_{01}^-(i_1, i) = 0$, for all $i = 1, \dots, m$. Then there exists i_0 such that $\left(A_{01}^+(i_1, i_0) \right)^2 + \left(A_{01}^-(i_1, i_0) \right)^2 \neq 0$ and then we have

$c_{2,i_0} - c_{2,i_1} = 0$. Equations (A.104) and (A.105) become

$$\begin{aligned} & \sum_{i=1}^m \tau_{i_1} c_{1,i_1+i \cdot m} x_i + R_{t-2}^1 + \sum_{i=1}^m c_{2,i} (A_{01}^+(i_1, i) x_i \log(x_i) + R_{t-2}^2) \\ & - \frac{1}{2} c_{2,i_0} \left(\sum_{i_2=1}^m A_{01}^+(i_1, i_2) x_{i_2} + R_{t-2}^3 \right) \log \left[\sum_{i_2=1}^m A_{01}^+(i_1, i_2) x_{i_2} + R_{t-2}^3 \right] = 0, \quad a.s. \end{aligned} \quad (\text{A.106})$$

$$\begin{aligned} & \sum_{i=1}^m \tau_{i_1} c_{1,i_1+i \cdot m^2} x_i + R_{t-2}^1 + \sum_{i=1}^m c_{2,i} (A_{01}^-(i_1, i) x_i \log(x_i) + R_{t-2}^2) \\ & - \frac{1}{2} c_{2,i_0} \left(\sum_{i_2=1}^m A_{01}^-(i_1, i_2) x_{i_2} + R_{t-2}^3 \right) \log \left[\sum_{i_2=1}^m A_{01}^-(i_1, i_2) x_{i_2} + R_{t-2}^3 \right] = 0, \quad a.s. \end{aligned} \quad (\text{A.107})$$

Differentiating twice again Equations (A.106) and (A.107) with respect to x_{i_0} , we find that

$$c_{2,i_0} (A_{01}^+(i_1, i_0) + A_{01}^-(i_1, i_0)) \left[\sum_{\substack{i_2=1 \\ i_2 \neq i_0}}^m \left(A_{01}^+(i_1, i_2) (\varepsilon_{i_2, t-1}^+)^{\tau_{i_2}} + A_{01}^-(i_1, i_2) (-\varepsilon_{i_2, t-1}^-)^{\tau_{i_2}} \right) + R_{t-2}^3 \right] = 0.$$

Since the law of η_t is non degenerated (see Assumption **A3**), we deduce that

$$\sum_{\substack{i_2=1 \\ i_2 \neq i_0}}^m \left(A_{01}^+(i_1, i_2) (\varepsilon_{i_2, t-1}^+)^{\tau_{i_2}} + A_{01}^-(i_1, i_2) (-\varepsilon_{i_2, t-1}^-)^{\tau_{i_2}} \right) + R_{t-2}^3 \neq 0.$$

So $c_{2,i_0} = c_{2,i_1} = 0$. Since i_1 is arbitrary, we deduce that $c_{2,i} = 0$ for any $i = 1, \dots, m$, or equivalently that $c_{1,i_1+i_0 \cdot m} = c_{1,i_1+i_0 \cdot m^2} = 0$.

Thus the vector $\mathbf{c}_2 = (c_{2,1}, \dots, c_{2,m})' = (c_{s_2+1}, \dots, c_{s_1})'$ is null. We recall that $\mathbf{c}_3 = (c_{s_1+1}, \dots, c_{s_0})'$ is null, the invertibility of the matrix J is thus shown in this case of a CCC-APGARCH($p, 1$).

In the general case of a CCC-APGARCH(p, q), we show by induction that (A.98) entails necessarily

$$A_{01}^+(i_1, i_0) + A_{01}^-(i_1, i_0) = \dots = A_{0q}^+(i_1, i_0) + A_{0q}^-(i_1, i_0) = 0, \quad \forall, i_0, i_1 = 1, \dots, m,$$

which is impossible under Assumption **A4** and thus $\mathbf{c} = 0$.

This is in contradiction with $\mathbf{c}'\mathbf{h}'\mathbf{h}\mathbf{c} = \mathbf{c}'\mathbf{d}'\mathbf{H}^2\mathbf{d}\mathbf{c} = 0$ almost-surely. Therefore the assumption that J is not singular is absurd.

A.5.6. Forgetting of the initial values

It suffices to adapt the arguments used in Subsection A.3.6 when the power is known.

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B. Details on the proof of Theorem 2

B.1. Minimization of the quasi-likelihood on the true value

The criterion is not integrable on all point, but we first prove that $\mathbb{E}_{\theta_0} [l_t(\theta)]$ is well defined on $\mathbb{R} \cup \{+\infty\}$ for all θ . Indeed we have

$$\mathbb{E}_{\theta_0} [l_t^-(\theta)] \leq \mathbb{E}_{\theta_0} [\log^- |H_t|] = \mathbb{E}_{\theta_0} [\log^- |D_t R D_t|] \leq \max\{0, \log(|R| \min_i \underline{\omega}(i)^m)\} < \infty.$$

Now we show that $\mathbb{E}_{\theta_0} [l_t(\theta_0)]$ is well defined on \mathbb{R} . We use $|\det(A)| \leq \rho(A)^m \leq \|A\|^m$, the Jensen inequality and the Corollary 2

$$\begin{aligned} \mathbb{E}_{\theta_0} [\log |H_t(\theta_0)|] &= \mathbb{E}_{\theta_0} \left[\frac{m}{s} \log |H_t(\theta_0)|^{\frac{s}{m}} \right] \\ &\leq \frac{m}{s} \log \left(\mathbb{E}_{\theta_0} \|H_t(\theta_0)\|^{\frac{s}{m}} \right) \\ &\leq \frac{m}{s} \log \left(\mathbb{E}_{\theta_0} \|R\|^s \|D_{0t}\|^{2s} \right) \\ &\leq K + \frac{m}{s} \log \left(\mathbb{E}_{\theta_0} \|D_{0t}\|^{2s} \right) \\ &= K + \frac{m}{s} \log \left(\mathbb{E}_{\theta_0} \left[\max_i (h_{it}(\theta_0))^s \right] \right) \\ &\leq K + \frac{m}{s} \log \left(\mathbb{E}_{\theta_0} \|\underline{h}_t(\theta_0)\|^s \right) < \infty . \end{aligned}$$

Therefore we have

$$\begin{aligned} \mathbb{E}_{\theta_0} [l_t(\theta_0)] &= \mathbb{E}_{\theta_0} [\varepsilon_t' H_t^{-1}(\theta_0) \varepsilon_t + \log |H_t(\theta_0)|] \\ &= \mathbb{E}_{\theta_0} \left[\eta_t' (H_t^{1/2})'(\theta_0) H_t^{-1}(\theta_0) H_t^{1/2}(\theta_0) \eta_t + \log |H_t(\theta_0)| \right] \\ &= \mathbb{E}_{\theta_0} \left[\text{Tr}(\eta_t' (H_t^{1/2})'(\theta_0) H_t^{-1}(\theta_0) H_t^{1/2}(\theta_0) \eta_t) + \log |H_t(\theta_0)| \right] \\ &= \mathbb{E}_{\theta_0} [\text{Tr}(\eta_t' \eta_t) + \log |H_t(\theta_0)|] \\ &= m + \mathbb{E}_{\theta_0} [\log |H_t(\theta_0)|] < +\infty. \end{aligned}$$

Since $\mathbb{E}_{\theta_0} [l_t^-(\theta)] < \infty$ for any θ , $\mathbb{E}_{\theta_0} [l_t^-(\theta_0)] < \infty$ and we deduce that $\mathbb{E}_{\theta_0} [l_t(\theta_0)]$ is well defined in \mathbb{R} . So, when one studies the function $\theta \mapsto \mathbb{E}_{\theta} [l_t(\theta)]$, we can restrict our study to the values of θ such that $\mathbb{E}_{\theta_0} |l_t(\theta)| < \infty$.

We denote $\lambda_{i,t}$ the positive eigenvalues of $H_t(\theta_0)H_t^{-1}(\theta)$. We have

$$\begin{aligned}
\mathbb{E}_{\theta_0} [l_t(\theta)] - \mathbb{E}_{\theta_0} [l_t(\theta_0)] &= \mathbb{E}_{\theta_0} [\varepsilon_t' H_t^{-1}(\theta) \varepsilon_t + \log |H_t(\theta)|] - \mathbb{E}_{\theta_0} [\varepsilon_t' H_t^{-1}(\theta_0) \varepsilon_t + \log |H_t(\theta_0)|] \\
&= \mathbb{E}_{\theta_0} [\varepsilon_t' H_t^{-1}(\theta) \varepsilon_t - \varepsilon_t' H_t^{-1}(\theta_0) \varepsilon_t] + \mathbb{E}_{\theta_0} [\log |H_t(\theta)| - \log |H_t(\theta_0)|] \\
&= \mathbb{E}_{\theta_0} [\varepsilon_t' \{H_t^{-1}(\theta) - H_t^{-1}(\theta_0)\} \varepsilon_t] + \mathbb{E}_{\theta_0} [\log |H_t(\theta) H_t^{-1}(\theta_0)|] \\
&= \mathbb{E}_{\theta_0} \left[\eta_t' (H_t^{1/2})'(\theta_0) \{H_t^{-1}(\theta) - H_t^{-1}(\theta_0)\} H_t^{1/2}(\theta_0) \eta_t \right] \\
&\quad + \mathbb{E}_{\theta_0} [\log |H_t(\theta) H_t^{-1}(\theta_0)|] \\
&= \mathbb{E}_{\theta_0} \left[\eta_t' \{ (H_t^{1/2})'(\theta_0) H_t^{-1}(\theta) H_t^{1/2}(\theta_0) - (H_t^{1/2})'(\theta_0) H_t^{-1}(\theta_0) H_t^{1/2}(\theta_0) \} \eta_t \right] \\
&\quad + \mathbb{E}_{\theta_0} [\log |H_t(\theta) H_t^{-1}(\theta_0)|] \\
&= \mathbb{E}_{\theta_0} \left[\eta_t' \{ (H_t^{1/2})'(\theta_0) H_t^{-1}(\theta) H_t^{1/2}(\theta_0) - I_m \} \eta_t \right] + \mathbb{E}_{\theta_0} [\log |H_t(\theta) H_t^{-1}(\theta_0)|] \\
&= \text{Tr} \left\{ \mathbb{E}_{\theta_0} \left[(H_t^{1/2})'(\theta_0) H_t^{-1}(\theta) H_t^{1/2}(\theta_0) - I_m \right] \mathbb{E} [\eta_t \eta_t'] \right\} \\
&\quad + \mathbb{E}_{\theta_0} [\log |H_t(\theta) H_t^{-1}(\theta_0)|] \\
&= \mathbb{E}_{\theta_0} \left[\text{Tr} \left((H_t^{1/2})'(\theta_0) H_t^{-1}(\theta) H_t^{1/2}(\theta_0) - I_m \right) \right] + \mathbb{E}_{\theta_0} [\log |H_t(\theta) H_t^{-1}(\theta_0)|] \\
&= \mathbb{E}_{\theta_0} \left[\sum_{i=1}^m (\lambda_{i,t} - 1 - \log(\lambda_{i,t})) \right] \geq 0,
\end{aligned}$$

where we have used the inequality $\log(x) \leq x - 1$, for all $x > 0$. But, if $x = 1$, $\log(x) = x - 1$, then the inequality is strict, except if for all i , $\lambda_{i,t} = 1$ \mathbb{P}_{θ_0} -a.s. This condition means that $H_t(\theta) = H_t(\theta_0)$ \mathbb{P}_{θ_0} -a.s. It follows $\underline{h}_t(\theta) = \underline{h}_t(\theta_0)$ \mathbb{P}_{θ_0} -a.s. and $R(\theta) = R(\theta_0)$. By the identifiability proved in subsection A.2.2, we deduce that $\theta = \theta_0$.

B.2. Proof of (A.10)

We recall that

$$\mathcal{L}_n(\theta) = \frac{1}{n} \sum_{t=1}^n l_t, \quad l_t = l_t(\theta) = \varepsilon_t' H_t^{-1} \varepsilon_t + \log |H_t|.$$

For $\theta \in \Theta$ and an integer $k > 0$, we denote $V_k(\theta)$ the open ball of radius $1/k$ centered on θ . We have

$$\begin{aligned}
\inf_{\theta^* \in V_k(\theta) \cap \Theta} \tilde{\mathcal{L}}_n(\theta^*) &= \inf_{\theta^* \in V_k(\theta) \cap \Theta} \left(\mathcal{L}_n(\theta^*) + \tilde{\mathcal{L}}_n(\theta^*) - \mathcal{L}_n(\theta^*) \right) \\
&\geq \inf_{\theta^* \in V_k(\theta) \cap \Theta} \mathcal{L}_n(\theta^*) + \inf_{\theta^* \in V_k(\theta) \cap \Theta} \left(\tilde{\mathcal{L}}_n(\theta^*) - \mathcal{L}_n(\theta^*) \right) \\
&\geq \inf_{\theta^* \in V_k(\theta) \cap \Theta} \mathcal{L}_n(\theta^*) - \sup_{\theta^* \in V_k(\theta) \cap \Theta} \left| \tilde{\mathcal{L}}_n(\theta^*) - \mathcal{L}_n(\theta^*) \right|
\end{aligned}$$

Then

$$\begin{aligned}
\liminf_{n \rightarrow +\infty} \inf_{\theta^* \in V_k(\theta) \cap \Theta} \tilde{\mathcal{L}}_n(\theta^*) &\geq \liminf_{n \rightarrow +\infty} \inf_{\theta^* \in V_k(\theta) \cap \Theta} \mathcal{L}_n(\theta^*) + \liminf_{n \rightarrow +\infty} \left(- \sup_{\theta^* \in V_k(\theta) \cap \Theta} \left| \tilde{\mathcal{L}}_n(\theta^*) - \mathcal{L}_n(\theta^*) \right| \right) \\
&\geq \liminf_{n \rightarrow +\infty} \inf_{\theta^* \in V_k(\theta) \cap \Theta} \mathcal{L}_n(\theta^*) - \limsup_{n \rightarrow +\infty} \sup_{\theta \in \Theta} \left| \tilde{\mathcal{L}}_n(\theta) - \mathcal{L}_n(\theta) \right|,
\end{aligned}$$

and by subsection A.2.1 we deduce that

$$\begin{aligned}
\liminf_{n \rightarrow +\infty} \inf_{\theta^* \in V_k(\theta) \cap \Theta} \tilde{\mathcal{L}}_n(\theta^*) &\geq \liminf_{n \rightarrow +\infty} \inf_{\theta^* \in V_k(\theta) \cap \Theta} \mathcal{L}_n(\theta^*) = \liminf_{n \rightarrow +\infty} \inf_{\theta^* \in V_k(\theta) \cap \Theta} \frac{1}{n} \sum_{t=1}^n l_t(\theta^*) \\
&\geq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n \inf_{\theta^* \in V_k(\theta) \cap \Theta} l_t(\theta^*). \tag{B.1}
\end{aligned}$$

We can not apply the classical ergodic theorem to $\{\inf_{\theta^* \in V_k(\theta) \cap \Theta} l_t(\theta^*)\}_t$ because of the lack of integrability mentioned in Subsection B.1. So we use an extension of the classical ergodic theorem (see Billingsley (1995) pages 284 and 495) and we have

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{t=1}^n \inf_{\theta^* \in V_k(\theta) \cap \Theta} l_t(\theta^*) \longrightarrow \mathbb{E} \left(\inf_{\theta^* \in V_k(\theta) \cap \Theta} l_1(\theta^*) \right)$$

By Beppo-Levi's theorem we have, when k goes to infinity

$$\liminf_{k \rightarrow +\infty} \mathbb{E} \left(\inf_{\theta^* \in V_k(\theta) \cap \Theta} l_1(\theta^*) \right) = \mathbb{E} \left(\liminf_{k \rightarrow +\infty} \inf_{\theta^* \in V_k(\theta) \cap \Theta} l_1(\theta^*) \right) \longrightarrow \mathbb{E} [l_1(\theta)].$$

Hence (B.1) implies that

$$\liminf_{n \rightarrow +\infty} \inf_{\theta^* \in V_k(\theta) \cap \Theta} \tilde{\mathcal{L}}_n(\theta^*) \geq \mathbb{E} [l_1(\theta)]$$

and (A.10) follows from the result stated in Subsection B.1.

B.3. Conclusion: proof of Theorem 2

By Subsection A.2.1, $\lim_{n \rightarrow +\infty} \tilde{\mathcal{L}}_n(\theta_0) = \lim_{n \rightarrow +\infty} \mathcal{L}_n(\theta_0)$ and by the ergodic theorem, we have $\lim_{n \rightarrow +\infty} \mathcal{L}_n(\theta_0) = \mathbb{E}_{\theta_0} [l_1(\theta_0)]$. Consequently, $\lim_{n \rightarrow +\infty} \tilde{\mathcal{L}}_n(\theta_0)$ exists and is $\mathbb{E}_{\theta_0} [l_1(\theta_0)]$. Since $\inf_{\theta^* \in V_k(\theta_0) \cap \Theta} \tilde{\mathcal{L}}_n(\theta^*) \leq \tilde{\mathcal{L}}_n(\theta_0)$, we have

$$\limsup_{n \rightarrow +\infty} \inf_{\theta^* \in V_k(\theta_0) \cap \Theta} \tilde{\mathcal{L}}_n(\theta^*) \leq \limsup_{n \rightarrow +\infty} \tilde{\mathcal{L}}_n(\theta_0) = \lim_{n \rightarrow +\infty} \tilde{\mathcal{L}}_n(\theta_0) = \lim_{n \rightarrow +\infty} \mathcal{L}_n(\theta_0) = \mathbb{E}_{\theta_0} [l_1(\theta_0)]$$

Then for a small neighborhood $V(\theta_0)$ of θ_0

$$\limsup_{n \rightarrow +\infty} \inf_{\theta^* \in V(\theta_0) \cap \Theta} \tilde{\mathcal{L}}_n(\theta^*) \leq \limsup_{n \rightarrow +\infty} \tilde{\mathcal{L}}_n(\theta_0) = \mathbb{E}_{\theta_0} [l_1(\theta_0)]. \quad (\text{B.2})$$

The parameter space Θ can be covered as

$$\Theta \subseteq V(\theta_0) + \bigcup_{\theta \in \Theta} V(\theta)$$

with $V(\theta)$ is a neighborhood of θ verifying (iv). By the compactness of Θ , there exists a finite covering of Θ

$$\Theta \subseteq V(\theta_0) \cup V(\theta_1) \cup \dots \cup V(\theta_k)$$

and then

$$\inf_{\theta \in \Theta} \tilde{\mathcal{L}}_n(\theta) = \min_{i=0,1,\dots,k} \inf_{\theta \in \Theta \cap V(\theta_i)} \tilde{\mathcal{L}}_n(\theta). \quad (\text{B.3})$$

Suppose that for all N , there exist $n \geq N$ such that $\hat{\theta}_n \in V(\theta_{i_0})$ with $i_0 = 1, \dots, k$. Let $\varepsilon > 0$, we have by (B.2) that there exists N_1 such that for all $n \geq N_1$,

$$\inf_{\theta^* \in \mathcal{V}(\theta_0)} \tilde{\mathcal{L}}_n(\theta^*) < \left[\limsup_{n \rightarrow +\infty} \inf_{\theta^* \in V(\theta_0)} \tilde{\mathcal{L}}_n(\theta^*) \right] + \varepsilon \leq \mathbb{E}_{\theta_0} [l_1(\theta_0)] + \varepsilon,$$

and by (A.10), for $i_0 \neq 0$, we obtain the existence of N_2 such that for all $n \geq N_2$,

$$\inf_{\theta^* \in \mathcal{V}(\theta_{i_0})} \tilde{\mathcal{L}}_n(\theta^*) > \left[\liminf_{n \rightarrow +\infty} \inf_{\theta^* \in V(\theta_{i_0})} \tilde{\mathcal{L}}_n(\theta^*) \right] - \varepsilon.$$

We can suppose that $N \geq N_1 \cup N_2$ and by the two latter relations and (B.3), we have for $i_0 \neq 0$

$$\liminf_{n \rightarrow +\infty} \inf_{\theta^* \in V(\theta_{i_0})} \tilde{\mathcal{L}}_n(\theta^*) - \varepsilon < \inf_{\theta \in \mathcal{V}(\theta_{i_0})} \tilde{\mathcal{L}}_n(\theta) = \inf_{\theta \in \Theta} \tilde{\mathcal{L}}_n(\theta) \leq \inf_{\theta \in \mathcal{V}(\theta_0)} \tilde{\mathcal{L}}_n(\theta) < \mathbb{E}_{\theta_0} [l_1(\theta_0)] + \varepsilon.$$

Then we will have

$$\liminf_{n \rightarrow +\infty} \inf_{\theta^* \in V(\theta_{i_0})} \tilde{\mathcal{L}}_n(\theta^*) - \varepsilon < \mathbb{E}_{\theta_0} [l_1(\theta_0)] + \varepsilon,$$

then

$$\liminf_{n \rightarrow +\infty} \inf_{\theta^* \in V(\theta_{i_0})} \tilde{\mathcal{L}}_n(\theta^*) \leq \mathbb{E}_{\theta_0} [l_1(\theta_0)]. \quad (\text{B.4})$$

But by (A.10) we have $\liminf_{n \rightarrow +\infty} \inf_{\theta^* \in V(\theta_{i_0})} \tilde{\mathcal{L}}_n(\theta^*) > \mathbb{E}_{\theta_0} [l_1(\theta_0)]$ and this is in contradiction to (B.4).

For n large enough, we conclude that $\hat{\theta}_n$ belongs to $V(\theta_0)$. \square

C. Details on the proof of Theorem 3

C.1. First derivative of log-likelihood

- Proof of (A.22) and (A.23).

We differentiate with respect to θ_i for $i = 1, \dots, s_1$ (that is with respect to $\underline{\omega}', \alpha_1^{+'}, \dots, \alpha_q^{+'}, \alpha_1^{-'}, \dots, \alpha_q^{-'}, \beta_1', \dots, \beta_p'$). Indeed,

$$\begin{aligned} \frac{\partial \underline{\varepsilon}_t' D_t^{-1} R^{-1} D_t^{-1} \underline{\varepsilon}_t}{\partial \theta_i} &= Tr \left(\frac{\partial \underline{\varepsilon}_t' D_t^{-1} R^{-1} D_t^{-1} \underline{\varepsilon}_t}{D_t'} \frac{\partial D_t}{\partial \theta_i} \right) = Tr \left(\frac{\partial Tr(\underline{\varepsilon}_t \underline{\varepsilon}_t' D_t^{-1} R^{-1} D_t^{-1})}{D_t'} \frac{\partial D_t}{\partial \theta_i} \right) \\ &= -Tr \left((\underline{\varepsilon}_t \underline{\varepsilon}_t' D_t^{-1} R^{-1} + R^{-1} D_t^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t') D_t^{-1} \frac{\partial D_t}{\partial \theta_i} D_t^{-1} \right), \\ 2 \frac{\partial \log(\det(D_t)) + \log(\det(R))}{\partial \theta_i} &= 2Tr \left(\frac{\partial \log(\det(D_t))}{\partial D_t'} \frac{\partial D_t}{\partial \theta_i} + \underbrace{\frac{\partial \log(\det(R))}{\partial D_t'}}_{=0} \frac{\partial D_t}{\partial \theta_i} \right) \\ &= 2Tr \left(D_t^{-1} \frac{\partial D_t}{\partial \theta_i} \right), \end{aligned}$$

and we obtain (A.22). Indeed, using the property $Tr(AB) = Tr(BA)$ in (A.22) yields

$$\begin{aligned} \frac{\partial l_t(\theta_0)}{\partial \theta_i} &= -Tr \left((\underline{\varepsilon}_t \underline{\varepsilon}_t' D_{0t}^{-1} R_0^{-1} + R_0^{-1} D_{0t}^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t') D_{0t}^{-1} D_{0t}^{(i)} D_t^{-1} - 2D_t^{-1} D_{0t}^{(i)} \right) \\ &= Tr \left(-D_{0t} \tilde{\eta}_t \tilde{\eta}_t' R_0^{-1} D_{0t}^{-1} D_{0t}^{(i)} D_t^{-1} - R_0^{-1} \tilde{\eta}_t \tilde{\eta}_t' D_{0t}^{(i)} D_t^{-1} + D_{0t}^{-1} D_{0t}^{(i)} + D_{0t}^{(i)} D_{0t}^{-1} \right) \\ &= Tr \left(-D_{0t}^{-1} D_{0t} \tilde{\eta}_t \tilde{\eta}_t' R_0^{-1} D_{0t}^{-1} D_{0t}^{(i)} + (I_m - R_0^{-1} \tilde{\eta}_t \tilde{\eta}_t') D_{0t}^{(i)} D_t^{-1} + D_{0t}^{-1} D_{0t}^{(i)} \right) \\ &= Tr \left((I_m - \tilde{\eta}_t \tilde{\eta}_t' R_0^{-1}) D_{0t}^{-1} D_{0t}^{(i)} + (I_m - R_0^{-1} \tilde{\eta}_t \tilde{\eta}_t') D_{0t}^{(i)} D_t^{-1} \right) \end{aligned}$$

and (A.23) is proved.

- Proof of (A.24) and (A.25).

We differentiate with respect to θ_i for $i = s_1 + 1, \dots, s_0$ (that is with respect to ρ'). Indeed we have

$$\begin{aligned} \frac{\partial \underline{\varepsilon}_t' D_t^{-1} R^{-1} D_t^{-1} \underline{\varepsilon}_t}{\partial \theta_i} &= Tr \left(\frac{\partial \underline{\varepsilon}_t' D_t^{-1} R^{-1} D_t^{-1} \underline{\varepsilon}_t}{R'} \frac{\partial R}{\partial \theta_i} \right) = Tr \left(\frac{\partial Tr(\underline{\varepsilon}_t \underline{\varepsilon}_t' D_t^{-1} R^{-1} D_t^{-1})}{R'} \frac{\partial R}{\partial \theta_i} \right) \\ &= -Tr \left(R^{-1} D_t^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' D_t^{-1} R^{-1} \frac{\partial R}{\partial \theta_i} \right), \end{aligned}$$

and

$$\begin{aligned}
2 \frac{\partial \log(\det(D_t)) + \log(\det(R))}{\partial \theta_i} &= Tr \left(\underbrace{\frac{\partial 2 \log(\det(D_t))}{\partial R'} \frac{\partial R}{\partial \theta_i}}_{=0} + \frac{\partial \log(\det(R))}{\partial R'} \frac{\partial R}{\partial \theta_i} \right) \\
&= Tr \left(R^{-1} \frac{\partial R}{\partial \theta_i} \right).
\end{aligned}$$

Hence we obtain (A.24). Now, we resume the above computations:

$$\begin{aligned}
\frac{\partial l_t(\theta_0)}{\partial \theta_i} &= -Tr \left(R_0^{-1} D_{0t}^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' D_{0t}^{-1} R_0^{-1} R_0^{(i)} - R_0^{-1} R_0^{(i)} \right) \\
&= -Tr \left((-I_m + R_0^{-1} D_{0t}^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' D_{0t}^{-1}) R_0^{-1} R_0^{(i)} \right) \\
&= Tr \left((I_m - R_0^{-1} D_{0t}^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' D_{0t}^{-1}) R_0^{-1} R_0^{(i)} \right).
\end{aligned}$$

We remark that $D_{0t}^{-1} \underline{\varepsilon}_t \underline{\varepsilon}_t' D_{0t}^{-1} = D_{0t}^{-1} D_{0t} R_0^{1/2} \eta_t \eta_t' R_0^{1/2} D_{0t} D_{0t}^{-1} = \tilde{\eta}_t \tilde{\eta}_t'$ and thus (A.25) is proved.

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