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► **To cite this version:**

Nathaël Alibaud, Jørgen Endal, Espen Robstad Jakobsen. Optimal stability results and nonlinear duality for L^∞ entropy and L^1 viscosity solutions. 2023. hal-01945687v3

HAL Id: hal-01945687

<https://hal.science/hal-01945687v3>

Preprint submitted on 26 Apr 2023 (v3), last revised 12 Apr 2024 (v4)

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OPTIMAL STABILITY RESULTS AND NONLINEAR DUALITY FOR L^∞ ENTROPY AND L^1 VISCOSITY SOLUTIONS

NATHAËL ALIBAUD, JØRGEN ENDAL, AND ESPEN R. JAKOBSEN

ABSTRACT. We give a new and rigorous duality relation between two central notions of weak solutions of nonlinear PDEs: entropy and viscosity solutions. It takes the form of the *nonlinear dual inequality*:

$$(\star) \quad \int |S_t u_0 - S_t v_0| \varphi_0 \, dx \leq \int |u_0 - v_0| G_t \varphi_0 \, dx, \quad \forall \varphi_0 \geq 0, \forall u_0, \forall v_0,$$

where S_t is the entropy solution semigroup of the anisotropic degenerate parabolic equation

$$\partial_t u + \operatorname{div} F(u) = \operatorname{div}(A(u)Du),$$

and where we look for the smallest semigroup G_t satisfying (\star) . This amounts to finding an optimal weighted L^1 contraction estimate for S_t . Our main result is that G_t is the viscosity solution semigroup of the Hamilton-Jacobi-Bellman equation

$$\partial_t \varphi = \sup_{\xi} \{F'(\xi) \cdot D\varphi + \operatorname{tr}(A(\xi)D^2\varphi)\}.$$

Since weighted L^1 contraction results are mainly used for possibly nonintegrable L^∞ solutions u , the natural spaces behind this duality are L^∞ for S_t and L^1 for G_t . We therefore develop a corresponding L^1 theory for viscosity solutions φ . But L^1 itself is too large for well-posedness, and we rigorously identify the weakest L^1 type Banach setting where we can have it – a subspace of L^1 called L_{int}^∞ . A consequence of our results is a new and optimal domain of dependence estimate for second order anisotropic degenerate parabolic PDEs. It is given in terms of a stochastic target problem and extends in a natural way recent results for first order hyperbolic PDEs by [N. Pogodaev, *J. Differ. Equ.*, 2018].

1. INTRODUCTION

In this paper we study two central notions of weak solutions of nonlinear PDEs and their interplay – entropy solutions and viscosity solutions. Originally introduced for first order scalar conservation laws [37] and Hamilton-Jacobi equations [24] respectively, both solution concepts have later been extended to second order PDEs [35, 34, 18, 21]. Conservation laws are divergence form equations arising in continuum physics [26], while Hamilton-Jacobi equations are nondivergence form equations from e.g. differential geometry and optimal control theory [29, 5, 4]. The well-posedness of these equations is an important topic and requires the entropy and viscosity solution theories in general. The literature is by now very large and includes lots of applications. See [29, 27, 5, 4, 43, 26, 23] for the state-of-the-art.

Here we develop a new connection between these solution concepts. It is already well-known that viscosity solutions are integrated entropy solutions in space dimension one [19, 36, 22]. Our connection is valid in any dimension and is expressed through weighted L^1 contraction results for entropy solutions: The optimal weight is the viscosity solution of a well-determined *dual equation*. Since L^∞ is a

2010 *Mathematics Subject Classification.* 35L65, 35K65, 35B30, 35B35, 35D30, 35D40.

Key words and phrases. Scalar conservation laws, anisotropic degenerate parabolic equations, Hamilton-Jacobi-Bellman equations, L^∞ entropy solutions, L^1 viscosity solutions, weighted L^1 contraction estimates, optimal estimates, nonlinear duality, nonlinear semigroups.

natural space for such weighted estimates, we need and do develop an L^1 theory for viscosity solutions of the dual equation. Consequences are a new and optimal domain of dependence result for second order PDEs in terms of a stochastic target problem, a new rigorous form of duality between L^∞ entropy and L^1 viscosity solutions in terms of nonlinear semigroups, and a new characterization of viscosity supersolutions; see (7), (8) and (9) respectively.

The idea of using viscosity solutions to get estimates for entropy solutions was from [28]. The corresponding results were rather accurate but not optimal yet. In this paper we prove optimal estimates for entropy solutions – and – that viscosity solutions are in fact needed to prove this optimality. This is exactly what leads to rigorous duality results. Also note that we consider nonlinear anisotropic diffusions as opposed to [28]. For an early discussion and open questions about “duality between nonlinear semigroups,” see [14, pp. 28–29].

To be more precise, we consider the following two Cauchy problems: For the anisotropic degenerate parabolic convection-diffusion equation

$$(1) \quad \begin{aligned} \partial_t u + \operatorname{div} F(u) &= \operatorname{div} (A(u) Du) & x \in \mathbb{R}^d, t > 0, \\ u(x, 0) &= u_0(x) & x \in \mathbb{R}^d, \end{aligned}$$

and for the Hamilton-Jacobi-Bellman (HJB) equation

$$(2a) \quad \partial_t \varphi = \sup_{\xi \in \mathcal{E}} \{ b(\xi) \cdot D\varphi + \operatorname{tr} (a(\xi) D^2 \varphi) \} \quad x \in \mathbb{R}^d, t > 0,$$

$$(2b) \quad \varphi(x, 0) = \varphi_0(x) \quad x \in \mathbb{R}^d,$$

where “ D ,” “ D^2 ” and “ div ” respectively denote the gradient, the Hessian and the divergence in x , and “ tr ” is the trace. We assume that

$$(H1) \quad F \in W_{\operatorname{loc}}^{1,\infty}(\mathbb{R}, \mathbb{R}^d) \quad \text{and} \quad A = \sigma^A (\sigma^A)^\top \quad \text{for} \quad \sigma^A \in L_{\operatorname{loc}}^\infty(\mathbb{R}, \mathbb{R}^{d \times K}),$$

as well as

$$(H2) \quad \begin{cases} \mathcal{E} \text{ is a nonempty set,} \\ b : \mathcal{E} \rightarrow \mathbb{R}^d \text{ a bounded function,} \\ a = \sigma^a (\sigma^a)^\top \text{ for some bounded } \sigma^a : \mathcal{E} \rightarrow \mathbb{R}^{d \times K}, \end{cases}$$

where K is the maximal rank of $A(u)$ and $a(\xi)$. The entropy solution theory for first order PDEs [37] was extended in [18, 21] to show well-posedness of (1) in $L^1 \cap L^\infty$ or L^1 . Well-posedness in L^∞ is less standard for second order PDEs, but results exist in [20, 3, 28, 40]; see [30] for anisotropic diffusions. Our main objective is to derive an optimal weighted L^1 contraction result for L^∞ entropy solutions of (1). This then will require the development of a corresponding L^1 theory for a dual equation of the form (2), a nonstandard generalization of classical viscosity solution theory [24, 35, 34, 23, 29, 5, 4].

Contraction type estimates are quantitative continuous dependence results on the initial data. A simple example is the L^1 contraction principle [37, 18, 21]:

$$(3) \quad \|(u - v)(t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1}.$$

For possibly nonintegrable L^∞ solutions, we need weighted estimates. An important result is the finite speed of propagation property for first order PDEs [37]:

$$(4) \quad \int_{|x-x_0| < R} |u(x, t) - v(x, t)| \, dx \leq \int_{|x-x_0| < R+Ct} |u_0(x) - v_0(x)| \, dx;$$

see [41] for optimal estimates. For second order PDEs, a standard example is given in [13, 20, 44, 30]:

$$(5) \quad \int |u(x, t) - v(x, t)| e^{-\sqrt{1+|x|^2}} \, dx \leq e^{Ct} \int |u_0(x) - v_0(x)| e^{-\sqrt{1+|x|^2}} \, dx.$$

Note that (5) does not imply (3) and (4). A finer result that is closer to (4) is given in [28] but it still does not imply (3), see [28, Rem. 2.7(b)].

We continue with a formal presentation of our main results; see Section 3 for precise statements. We first give a new and very accurate weighted L^1 contraction estimate for (1) of the form

$$(6) \quad \int |u(x, t) - v(x, t)| \varphi_0(x) \, dx \leq \int |u_0(x) - v_0(x)| \varphi(x, t) \, dx,$$

where $\varphi_0 \geq 0$ is arbitrary and the weight φ is the viscosity solution of (2) with $b = F'$ and $a = A$. From control theory [29], we can reformulate (6) in terms of a stochastic target problem. The result is a natural extension of the optimal domain of dependence result for first order PDEs given in [41]:

$$(7) \quad \int_U |u(x, t) - v(x, t)| \, dx \leq \int |u_0(x) - v_0(x)| \sup_{\xi} \mathbb{P}(\mathbf{X}_t^x \in U) \, dx$$

where $U \subseteq \mathbb{R}^d$ is arbitrary, \mathbb{P} is the probability, ξ_s an adapted control, \mathbf{B}_s a Brownian motion, \mathbf{X}_s^x an Ito process satisfying

$$d\mathbf{X}_s^x = F'(\xi_s) \, ds + \sqrt{2} \sigma^A(\xi_s) \, d\mathbf{B}_s, \quad \mathbf{X}_{s=0}^x = x,$$

and the weight in (7) maximizes the chance to reach the target U at time t . Note that here t is fixed and \mathbf{X}_t^x depends on the process ξ_s . These two equivalent results imply (3), (4), (5), the related results in [28, 41], and as we will see, they are optimal in a rigorous sense.

To discuss the optimality of (6), we fix φ_0 and try to identify the minimal φ satisfying (6) for any u_0, v_0 . The key result is a characterization of viscosity supersolutions of (2) in terms of contraction estimates for (1):

A function φ is a viscosity supersolution of (2a) if and only if

$$(8) \quad \int |u(x, t) - v(x, t)| \varphi(x, s) \, dx \leq \int |u_0(x) - v_0(x)| \varphi(x, t + s) \, dx,$$

for all $t, s \geq 0$ and u_0, v_0 with associated entropy solutions u, v of (1) with $F' = b$ and $A = a$.

Roughly speaking this result implies that if we restrict to weights satisfying a natural semigroup property, then the best weight in (6) is the viscosity solution of (2) since by comparison solutions are always smaller than supersolutions. This then leads to our most original result:

If S_t and G_t are the solution semigroups of (1) and (2), $b = F'$ and $a = A$, then G_t is the smallest semigroup satisfying

$$(9) \quad \int |S_t u_0 - S_t v_0| \varphi_0 \, dx \leq \int |u_0 - v_0| G_t \varphi_0 \, dx, \quad \forall \varphi_0 \geq 0, \forall u_0, \forall v_0.$$

We can interpret (9) as a *nonlinear dual inequality* and G_t as a *dual semigroup* of S_t , because G_t is entirely determined by (9) and knowledge of S_t . The duality in the other direction is open (Remark 40). Since S_t is taken on L^∞ from the beginning, it remains to properly define G_t on L^1 .

Classical viscosity solution theory starting from [24, 35, 34] and summarized in e.g. [23, 29, 5, 4], typically considers bounded continuous C_b solutions. For solutions in L^1 or L^p (in space) there are fewer results, see e.g. [16] for nondegenerate PDEs and [39, 2, 1, 6, 15, 17, 28] for various other PDEs. Here we show that (2) can be ill-posed in L^1 in general. We then consider stronger norm topologies and identify the weakest one for which (2) is well-posed in general: It is generated by the norm

$$\varphi_0 \mapsto \int \sup_{x+[-1,1]^d} |\varphi_0| \, dx$$

which is the norm of the space L_{int}^∞ as defined in [2, 1]. Since $L_{\text{int}}^\infty \subset L^1 \cap L^\infty$, it follows that $C_b \cap L_{\text{int}}^\infty$ is a natural L^1 type Banach space for the dual equation (2) and its solution semigroup G_t in (9).

Our results on L^1 viscosity solutions may be of independent interest. Let us comment them further. The estimates of [2] are not in L_{int}^∞ but in its predual $L_{\text{unif}}^1 \not\subset L^1$, while [1] gives weighted L_{int}^∞ estimates for unbounded solutions with linear diffusions. In [28] there are L^1 estimates for fully nonlinear degenerate PDEs with isotropic diffusions and exponentially decaying initial data. Equation (2) is fully nonlinear, degenerate, possibly anisotropic, and we consider general L_{int}^∞ data while identifying L_{int}^∞ as the most natural L^1 viscosity solution setting.

The rest of this paper is organized as follows. We recall basic facts in Section 2, we state our main results in Section 3, and prove them in Section 4. For completeness, some results for minimal discontinuous viscosity solutions are proved in Appendix A, while a complete proof of well-posedness for L^∞ entropy solutions is given in Appendix B.

General notation. We let $\mathbb{R}^+ := [0, \infty)$ and $\mathbf{1}_U$ denote the indicator function of a set U . Given two normed spaces $X \subseteq Y$, X is continuously embedded into Y if the canonical injection is continuous. More specific notation are introduced progressively when we need them, essentially in Section 2.

2. PRELIMINARIES

This section recalls basic facts on C_b viscosity and L^∞ entropy solutions; for proofs, see e.g. [23, 29, 5, 4] and [21, 11, 26] respectively. We also define the space L_{int}^∞ .

2.1. Viscosity solutions of (2). Let φ^* (resp. φ_*) denote the upper (resp. lower) semicontinuous envelope of φ .

Definition 1 (Viscosity solutions). *Assume (H2) and $\varphi_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ is bounded.*

(a) *A locally bounded function $\varphi : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is a viscosity subsolution (resp. supersolution) of (2) if*

i) *for every $\phi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^+)$ and local maximum $(x, t) \in \mathbb{R}^d \times (0, \infty)$ of $\varphi^* - \phi$ (resp. minimum),*

$$\partial_t \phi(x, t) \leq \sup_{\mathcal{E}} \{b \cdot D\phi + \text{tr}(aD^2\phi)\}(x, t) \quad (\text{resp. } \geq),$$

ii) *and for every $x \in \mathbb{R}^d$,*

$$\varphi^*(x, 0) \leq (\varphi_0)^*(x) \quad (\text{resp. } \varphi_*(x, 0) \geq (\varphi_0)_*(x)).$$

(b) *A function φ is a viscosity solution if it is both a sub and supersolution.*

Remark 2. We say that φ is a viscosity subsolution (resp. supersolution) of (2a) if (ai) holds.

We recall the well-known comparison and the well-posedness for (2).

Theorem 3 (Comparison principle). *Assume (H2). If φ and ψ are bounded sub and supersolutions of (2a), and*

$$\varphi^*(x, 0) \leq \psi_*(x, 0) \quad \forall x \in \mathbb{R}^d,$$

then $\varphi^ \leq \psi_*$ on $\mathbb{R}^d \times \mathbb{R}^+$.*

Theorem 4 (Existence and uniqueness). *Assume (H2) and $\varphi_0 \in C_b(\mathbb{R}^d)$. Then there exists a unique viscosity solution $\varphi \in C_b(\mathbb{R}^d \times \mathbb{R}^+)$ of (2).*

Remark 5. By the comparison principle, $\inf \varphi_0 \leq \varphi \leq \sup \varphi_0$ and we have the following contraction property: $\|\varphi - \psi\|_\infty \leq \|\varphi_0 - \psi_0\|_\infty$ for every pair of solutions φ and ψ with initial data φ_0 and ψ_0 .

We may take φ_0 to be discontinuous as in (7). In that case, we loose uniqueness and we have to work with minimal and maximal solutions [25, 10, 31] (see also [4] for bilateral solutions). For our considerations, we only need minimal solutions. We denote by *BLSC* bounded lower semicontinuous functions.

Theorem 6 (Minimal solutions). *Assume (H2) and $\varphi_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ bounded. Then there exists a minimal viscosity solution $\underline{\varphi} \in \text{BLSC}(\mathbb{R}^d \times \mathbb{R}^+)$ of (2), in the sense that $\underline{\varphi} \leq \varphi$ for any bounded viscosity solution φ of (2). Moreover $\underline{\varphi}(x, t = 0) = (\varphi_0)_*(x)$ for any $x \in \mathbb{R}^d$.*

Note that $\underline{\varphi}$ is unique by definition. Actually, it is more precisely the minimal supersolution.

Proposition 7. *Assume (H2) and $\varphi_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ is bounded. Then any bounded supersolution φ of (2) is such that $\underline{\varphi} \leq \varphi_*$.*

Remark 8. In particular, we have the following comparison principle: $\underline{\varphi} \leq \underline{\psi}$ for any bounded $\varphi_0 \leq \psi_0$.

For completeness, the proofs of Theorems 6 and Proposition 7 are given in Appendix A.1 because [25, 10, 31, 4] consider slightly different problems. Let us continue with representation formulas for the solution $\underline{\varphi}$ from control theory [29, 4, 32, 33]. Throughout, “co” denotes the convex hull and “Im” the image.

Proposition 9 (First order). *Assume (H2), $a \equiv 0$, and $\varphi_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ bounded. Then the minimal viscosity solution of (2) is given by*

$$\underline{\varphi}(x, t) = \sup_{x+t\mathcal{C}} (\varphi_0)_* \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+,$$

where $\mathcal{C} = \overline{\text{co}\{\text{Im}(b)\}}$.

In the second order case, we need a probabilistic framework. For simplicity, we fix for the rest of this paper

$$\begin{cases} \text{a complete filtered probability space } (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}), \text{ and} \\ \text{a standard } d\text{-dimensional Brownian } \mathbf{B}_t \text{ on this filtration.} \end{cases}$$

We will assume without mention that all stochastic processes in this paper are defined on this filtered probability space, and that whenever we need a Brownian motion, then we take the above Brownian motion. Let us denote the expectation by \mathbb{E} . Then:

Proposition 10 (Second order). *Assume (H2), $\varphi_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ is bounded,*

(10) *the set \mathcal{E} is compact and the functions $b(\cdot)$ and $\sigma^a(\cdot)$ are continuous.*

Then the minimal viscosity solution of (2) is given by

$$\underline{\varphi}(x, t) = \sup_{\xi} \mathbb{E} \{ (\varphi_0)_*(\mathbf{X}_t^x) \},$$

where ξ_s is a progressively measurable \mathcal{E} -valued control and \mathbf{X}_s^x an Ito process satisfying the SDE

$$\begin{cases} d\mathbf{X}_s^x = b(\xi_s) ds + \sqrt{2} \sigma^a(\xi_s) d\mathbf{B}_s, & s > 0, \\ \mathbf{X}_{s=0}^x = x. \end{cases}$$

These results are standard for continuous viscosity solutions [29, 4], see also [4, 32, 33] for maximal solutions. For minimal solutions, we did not find any reference so we provide the proofs in Appendix A.2.

2.2. Entropy solutions of (1). Well-posedness of (1) in L^∞ is essentially established in [30] for smooth fluxes, see [21, 11] for previous results in $L^\infty \cap L^1$ or L^1 . Let us now recall these results in the form needed here and provide complementary proofs in Appendix B for completeness.

Definition 11 (Entropy-entropy flux triple). *We say that (η, q, r) is an entropy-entropy flux triple if $\eta \in C^2(\mathbb{R})$ is convex, $q' = \eta' F'$ and $r' = \eta' A$.*

Given $\beta \in C(\mathbb{R})$, we also need the notation

$$\zeta_{ik}(u) := \int_0^u \sigma_{ik}^A(\xi) \, d\xi \quad \text{and} \quad \zeta_{ik}^\beta(u) := \int_0^u \sigma_{ik}^A(\xi) \beta(\xi) \, d\xi.$$

Definition 12 (Entropy solutions). *Assume (H1) and $u_0 \in L^\infty(\mathbb{R}^d)$. A function $u \in L^\infty(\mathbb{R}^d \times \mathbb{R}^+) \cap C(\mathbb{R}^+; L^1_{\text{loc}}(\mathbb{R}^d))$ is an entropy solution of (1) if*

- (a) $\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) \in L^2_{\text{loc}}(\mathbb{R}^d \times \mathbb{R}^+)$ for any $k = 1, \dots, K$,
- (b) for any $k = 1, \dots, K$ and any $\beta \in C(\mathbb{R})$

$$\sum_{i=1}^d \partial_{x_i} \zeta_{ik}^\beta(u) = \beta(u) \sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) \in L^2_{\text{loc}}(\mathbb{R}^d \times \mathbb{R}^+),$$

- (c) and for all entropy-entropy flux triples (η, q, r) and $0 \leq \phi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+)$,

$$\begin{aligned} & \iint_{\mathbb{R}^d \times \mathbb{R}^+} \left(\eta(u) \partial_t \phi + \sum_{i=1}^d q_i(u) \partial_{x_i} \phi + \sum_{i,j=1}^d r_{ij}(u) \partial_{x_i x_j}^2 \phi \right) dx \, dt \\ & + \int_{\mathbb{R}^d} \eta(u_0(x)) \phi(x, 0) \, dx \geq \iint_{\mathbb{R}^d \times \mathbb{R}^+} \eta''(u) \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) \right)^2 \phi \, dx \, dt. \end{aligned}$$

Theorem 13 (Existence and uniqueness). *Assume (H1) and $u_0 \in L^\infty(\mathbb{R}^d)$. Then there exists a unique entropy solution $u \in L^\infty(\mathbb{R}^d \times \mathbb{R}^+) \cap C(\mathbb{R}^+; L^1_{\text{loc}}(\mathbb{R}^d))$ of (1).*

See [30, Theorem 1.1] or Appendix B for the proof.

Remark 14. (a) In the L^1 settings of [21, 11], the following contraction principle holds: For solutions u and v of (1) with initial data u_0 and v_0 ,

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1} \quad \forall t \geq 0.$$

- (b) In the L^∞ setting of [30], uniqueness is based on the weighted L^1 contraction principle (5), see also Lemma 63 in Appendix B.
- (c) In all cases, we have comparison and maximum principles as stated in Lemma 65 in Appendix B.

In L^∞ , uniqueness is based on a doubling of variables arguments developed in [37, 18, 11]. This argument leads to (11) below, and this inequality will be the starting point of our analysis.

Lemma 15 (Kato inequality). *Assume (H1) and u, v are entropy solutions of (1) with initial data $u_0, v_0 \in L^\infty(\mathbb{R}^d)$. Then for all $T \geq 0$ and nonnegative test functions $\phi \in C_c^\infty(\mathbb{R}^d \times [0, T])$,*

$$\begin{aligned} (11) \quad & \int_{\mathbb{R}^d} |u - v|(x, T) \phi(x, T) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \phi(x, 0) \, dx \\ & + \iint_{\mathbb{R}^d \times (0, T)} \left(|u - v| \partial_t \phi + \sum_{i=1}^d q_i(u, v) \partial_{x_i} \phi + \sum_{i,j=1}^d r_{ij}(u, v) \partial_{x_i x_j}^2 \phi \right) dx \, dt, \end{aligned}$$

where

$$q_i(u, v) = \text{sign}(u - v) \int_v^u F'_i(\xi) d\xi, \quad r_{ij}(u, v) = \text{sign}(u - v) \int_v^u A_{ij}(\xi) d\xi.$$

See Appendix B for precise references to the computations in [11] on how to show this lemma in our setting.

2.3. The function space L_{int}^∞ .

$$L_{\text{int}}^\infty(\mathbb{R}^d) := \{\varphi_0 \in L_{\text{loc}}^1(\mathbb{R}^d) \text{ s.t. } \|\varphi_0\|_{L_{\text{int}}^\infty} < \infty\}$$

where $\|\varphi_0\|_{L_{\text{int}}^\infty} := \int \text{ess sup}_{\overline{Q}_1(x)} |\varphi_0| dx$ and $\overline{Q}_r(x) := x + [-r, r]^d$ for $r > 0$.

Theorem 16. *The space $L_{\text{int}}^\infty(\mathbb{R}^d)$ is a Banach space, and it is continuously embedded into $L^1 \cap L^\infty(\mathbb{R}^d)$.*

See [2, 1] for the proof and choice of the above notation. From now on we use the pointwise sup, and to avoid confusion, the notation

$$\|\varphi_0\|_{\text{int}} := \int \sup_{\overline{Q}_1(x)} |\varphi_0| dx.$$

Note that $\|\varphi_0\|_{\text{int}} = \|\varphi_0\|_{L_{\text{int}}^\infty}$ if φ_0 is continuous. We also need the following result:

Lemma 17. *For any $r > 0$ and $\varepsilon \geq 0$, there is a constant $C_{r,\varepsilon} \geq 0$ such that*

$$\int \sup_{\overline{Q}_{r+\varepsilon}(x)} |\varphi_0| dx \leq C_{r,\varepsilon} \int \sup_{\overline{Q}_r(x)} |\varphi_0| dx \quad \forall \varphi_0 : \mathbb{R}^d \rightarrow \mathbb{R}.$$

Remark 18. This result will be used with the pointwise sup for discontinuous φ_0 , typically lower or upper semicontinuous.

The proof can be found in [2, 1], see e.g. [1, Lemma 2.5.1].

3. MAIN RESULTS

We are ready to precisely state our main results. The L^1 theory for (2) is given in Section 3.1, the estimates for (1) in Section 3.2, and the duality results between (1) and (2) in Section 3.3. The long proofs are given in Section 4.

3.1. L_{int}^∞ as a natural L^1 setting for (2). We first explain why the pure L^1 setting is too weak to develop a general well-posedness theory for (2). Consider the unique viscosity solution of the eikonal equation

$$(12) \quad \partial_t \varphi = \sum_{i=1}^d |\partial_{x_i} \varphi|,$$

with a given initial data $\varphi_0 \in C_b(\mathbb{R}^d)$. Under which condition is it integrable?

Proposition 19 (Necessary and sufficient integrability condition). *We have*

$$[\varphi(\cdot, t) \in L^1(\mathbb{R}^d) \quad \forall t \geq 0] \iff [\varphi_0^- \in L^1(\mathbb{R}^d) \text{ and } \varphi_0^+ \in L_{\text{int}}^\infty(\mathbb{R}^d)].$$

Proof. Since $\varphi(x, t) = \sup_{\overline{Q}_t(x)} \varphi_0$ by Proposition 9, we conclude by Lemma 17. \square

We continue by showing that the L^1 topology is too weak to get the continuous dependence on the initial data, even for solutions which remain integrable.

Proposition 20 (Failure of the L^1 continuous dependence). *For all $n \geq 1$, let $\varphi_0^n(x) := (1 - n|x|)^+$, and φ_n be the solution of (12) with initial data φ_0^n . Then $\varphi_0^n \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ and*

$$\lim_{n \rightarrow \infty} \varphi_0^n = 0 \quad \text{in } L^1(\mathbb{R}^d),$$

but

$$\lim_{n \rightarrow \infty} \varphi_n(\cdot, t) = \mathbf{1}_{\overline{Q}_t}(\cdot) \neq 0 \quad \text{in } L^1(\mathbb{R}^d), \quad \forall t > 0.$$

Proof. Use again that $\varphi_n(x, t) = \sup_{\overline{Q}_t(x)} \varphi_0^n$. \square

Interestingly a similar analysis works also for purely diffusive PDEs. Consider e.g. an equation in one space dimension

$$(13) \quad \partial_t \varphi = (\partial_{xx}^2 \varphi)^+.$$

To have L^1 solutions, we need again that $\varphi_0^+ \in L_{\text{int}}^\infty$.

Proposition 21 (L_{int}^∞ and nonlinear diffusions). *Let $\varphi_0 \in C_b(\mathbb{R})$ be nonnegative and φ be the solution of (13) with φ_0 as initial data. Then,*

$$[\varphi(\cdot, t) \in L^1(\mathbb{R}) \quad \forall t \geq 0] \iff \varphi_0 \in L_{\text{int}}^\infty(\mathbb{R}).$$

See Section 4.5 for the proof. We now use the lack of a fundamental solution of (13) to show that there is no continuous dependence on the initial data in L^1 . Consider solutions φ_n of (13) with an approximate delta-function as initial data:

$$(14) \quad \varphi_n(x, t = 0) = n\rho(nx),$$

where $0 \leq \rho \in C_c(\mathbb{R})$ is nontrivial. Then:

Proposition 22 (Blow-up everywhere). $\lim_{n \rightarrow \infty} \varphi_n(x, t) = \infty, \forall x \in \mathbb{R}, \forall t > 0$.

See Section 4.5 for the proof. For linear diffusion equations, φ_n would converge to the fundamental solution, but here it explodes pointwise and in all L_{loc}^p . A counterexample to the L^1 continuous dependence for (13) is then given by the sequence of solutions

$$\psi_n(x, t) := \varphi_n(x, t) / \sqrt{\|\varphi_n(\cdot, t_0)\|_{L^1}} \quad \text{for a fixed } t_0 > 0,$$

since $\|\psi_n(t = 0)\|_{L^1} \rightarrow 0$ while $\|\psi_n(\cdot, t)\|_{L^1} \geq \|\psi_n(\cdot, t_0)\|_{L^1} \rightarrow \infty$ for any $t \geq t_0$.

In view of the previous results, we now look for a Banach space $X \subset L^1$ that is strong enough to get well-posedness for (2) in general. We are mainly interested in properly defining an associated semigroup; see e.g. [14, 12] for a general presentation of nonlinear semigroups.

Definition 23. *Let E be a normed space.*

(a) *A family of maps $G_t : E \rightarrow E$ parametrized by $t \geq 0$ is a semigroup on E if*

$$\begin{cases} G_{t=0} = \text{id (the identity), and} \\ G_{t+s} = G_t G_s \text{ (meaning the composition) for any } t, s \geq 0. \end{cases}$$

(b) *It is a semigroup of continuous operators if in addition $G_t : E \rightarrow E$ is continuous for each $t \geq 0$.*

(c) *And it is strongly continuous if for each $\varphi_0 \in E, t \geq 0 \mapsto G_t \varphi_0 \in E$ is strongly continuous (i.e. continuous in norm).*

Let φ be the unique viscosity solution of (2) and define

$$(15) \quad G_t : \varphi_0 \in C_b(\mathbb{R}^d) \mapsto \varphi(\cdot, t) \in C_b(\mathbb{R}^d).$$

Then G_t is a semigroup of Lipschitz continuous operators by Remark 5. A natural construction is to define X as the completion of some $E \subseteq C_b \cap L^1$, such that $X \subset L^1$ and G_t can be extended from E onto X . More precisely we require that

$$(16) \quad \begin{cases} E \text{ is a vector subspace of } C_b \cap L^1(\mathbb{R}^d), \\ E \text{ is a normed space,} \\ E \text{ is continuously embedded into } L^1(\mathbb{R}^d), \end{cases}$$

and for any data (\mathcal{E}, b, a) satisfying (H2), the semigroup (15) satisfies:

$$(17) \quad \forall t \geq 0, \quad \begin{cases} G_t(E) \subseteq X := \overline{E}^{\|\cdot\|^E}, G_t : E \rightarrow X \text{ is continuous, and} \\ G_t \text{ admits an extension onto } X \text{ as a continuous operator.} \end{cases}$$

The best E is given below.

Theorem 24 (A natural L^1 setting for (2)). *The space $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ is a Banach space satisfying the properties (16)–(17). Moreover, any other space E satisfying (16)–(17) is continuously embedded into $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$.*

Remark 25. Since the best $E = X$ is a Banach space by Theorem 16, it is a posteriori not necessary to extend G_t outside C_b . The classical notion of viscosity solutions is then already satisfactory to study L^1 solutions of fully nonlinear degenerate PDEs.

Theorem 24 relies on the following estimate:

Theorem 26 (General L_{int}^∞ stability). *Assume (H2) and $T \geq 0$. For any bounded subsolution φ and supersolution ψ of (2a),*

$$(18) \quad \int_{\overline{Q}_1(x) \times [0, T]} \sup (\varphi^* - \psi_*)^+ dx \leq C \int_{\overline{Q}_1(x)} \sup (\varphi^* - \psi_*)^+(\cdot, 0) dx,$$

for some constant $C = C(d, \|a\|_\infty, \|b\|_\infty, T) \geq 0$.

As a consequence we have the following result:

Corollary 27 (L_{int}^∞ well-posedness of (2)). *Assume (H2) and G_t is the solution semigroup defined in (15). Then its restriction to $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ is a strongly continuous semigroup of Lipschitz continuous operators.*

The proofs of Theorem 26 and Corollary 27 are given in Section 4.2, while Theorem 24 is proved in Section 4.5.

3.2. Weighted L^1 contraction for (1). We state our new and improved weighted L^1 contraction principle for L^∞ entropy solutions of (1). The weight will be the viscosity solution of the following problem:

$$(19a) \quad \partial_t \varphi = \operatorname{ess\,sup}_{m \leq \xi \leq M} \{F'(\xi) \cdot D\varphi + \operatorname{tr}(A(\xi)D^2\varphi)\} \quad x \in \mathbb{R}^d, t > 0,$$

$$(19b) \quad \varphi(x, 0) = \varphi_0(x) \quad x \in \mathbb{R}^d,$$

for given $m < M$ and φ_0 .

Theorem 28 (Weighted L^1 contraction). *Assume (H1), $m < M$, $u_0 = u_0(x)$ and $v_0 = v_0(x)$ are measurable with values in $[m, M]$, and $0 \leq \varphi_0 \in BLSC(\mathbb{R}^d)$. Then the corresponding entropy solutions u and v of (1) and minimal viscosity solution $\underline{\varphi}$ of (19) satisfy*

$$(20) \quad \int_{\mathbb{R}^d} |u - v|(x, t) \varphi_0(x) dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \underline{\varphi}(x, t) dx \quad \forall t \geq 0.$$

Remark 29. (a) Problem (19) is of the form (2) with a pointwise sup taken over the Lebesgue points of F' and A .

- (b) The right-hand side of (20) can be infinite. To get finite integrals, it suffices to take $\varphi_0 \in L^\infty_{\text{int}}$ or $u_0 - v_0 \in L^1$.
- (c) The same result holds when $\underline{\varphi}$ is replaced by any measurable supersolution of (19), since it is greater than $\underline{\varphi}$.

As a consequence of this result and the representation formula for first order PDEs, we obtain the following estimate on the domain of dependence:

Corollary 30 (First order equations). *Assume (H2) with $A \equiv 0$, $m < M$, u_0 and v_0 are measurable functions with values in $[m, M]$, and u and v are entropy solutions of (1) with initial data u_0 and v_0 . Then*

$$\int_B |u - v|(x, t) \, dx \leq \int_{B-t\mathcal{C}} |u_0 - v_0|(x) \, dx$$

for any Borel set $B \subseteq \mathbb{R}^d$ and $t \geq 0$, where

$$\mathcal{C} = \overline{\text{co} \left\{ \text{ess Im} \left((F')|_{[m, M]} \right) \right\}}$$

and ess Im is the essential image.

Proof. Let $U \supseteq B$ be an open set and take $\varphi_0 = \mathbf{1}_U$. By Proposition 9, the minimal solution of (19) is $\underline{\varphi}(x, t) = \mathbf{1}_{U-t\mathcal{C}}(x)$. Apply then Theorem 28 and take the infimum over all open $U \supseteq B$. \square

Remark 31. This result also follows from [41] where similar estimates are given for (x, t) -dependent first order PDEs using techniques based on differential inclusions.

For second order equations, we have the following extension of the domain of dependence result:

Corollary 32 (Second order equations). *Assume (H2), $F'(\cdot)$ and $\sigma^A(\cdot)$ continuous, $m < M$, u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$, and u and v solutions of (1) with u_0 and v_0 as initial data. Then for any open $U \subseteq \mathbb{R}^d$ and $t \geq 0$,*

$$\int_U |u - v|(x, t) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \sup_{\xi} \mathbb{P}(\mathbf{X}_t^x \in U) \, dx,$$

where ξ_s is a progressively measurable $[m, M]$ -valued process and \mathbf{X}_s^x is an Ito process satisfying the SDE

$$(21) \quad \begin{cases} d\mathbf{X}_s^x = F'(\xi_s) \, ds + \sqrt{2} \sigma^A(\xi_s) \, d\mathbf{B}_s, & s > 0, \\ \mathbf{X}_{s=0}^x = x. \end{cases}$$

Remark 33. (1) \mathbf{X}_s^x is a stochastic process starting from x at time $s = 0$. The dynamics of \mathbf{X}_s^x is given by the controlled SDE (21) where the coefficients are (derivatives of) the fluxes in Equation (1). The control is determined to maximize the probability for the process to reach U at time $s = t$. Equation (19) is the dynamic programming equation for this control problem.

(2) If $A \equiv 0$, the problem is deterministic and we recover Corollary 30.

Proof. Take $\varphi_0 = \mathbf{1}_U$ and apply Proposition 10 to compute $\underline{\varphi}$ in Theorem 28. \square

The proof of Theorem 28 is given in Section 4.3.

3.3. Optimal weight and duality between (1) and (2). We discuss the optimality of Theorem 28 and its rigorous interpretation in terms of duality relations. The first step is a reformulation of the definition of viscosity supersolutions in terms of weights in L^1 contraction estimates for (1).

Theorem 34 (Weights and supersolutions). *Assume (H1), $m < M$, and $0 \leq \varphi \in \text{BLSC}(\mathbb{R}^d \times \mathbb{R}^+)$. Then the statements below are equivalent.*

(I) For any measurable functions u_0 and v_0 with values in $[m, M]$ and entropy solutions u and v of (1) with initial data u_0 and v_0 ,

$$\int_{\mathbb{R}^d} |u - v|(x, t) \varphi(x, s) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x, t + s) \, dx \quad \forall t, s \geq 0.$$

(II) The function

$$\varphi_{\#}(x, t) := \liminf_{\substack{r \rightarrow 0^+ \\ y \rightarrow x}} \frac{1}{\text{meas}(B_r(y))} \int_{B_r(y)} \varphi(z, t) \, dz$$

is a viscosity supersolution of (19a).

We have used the notation $B_r(y) := \{z : |z - y| < r\}$.

Remark 35. The function $\varphi_{\#}$ satisfies (I) if and only if φ does since it is an a.e. representative in space of φ .

We will therefore roughly speaking deduce from the comparison principle that our weight is optimal in the class of weights

$$\mathcal{W}_{m, M, \varphi_0} := \{0 \leq \varphi \in BLSC(\mathbb{R}^d \times \mathbb{R}^+) \text{ satisfying (I) and } \varphi(t=0) \geq \varphi_0\}.$$

Corollary 36 (Optimality of the weight). *Assume (H1), $m < M$, and $0 \leq \varphi_0 \in BLSC(\mathbb{R}^d)$. Then the weight $\underline{\varphi}$ from Theorem 28 belongs to the class $\mathcal{W}_{m, M, \varphi_0}$ and satisfies*

$$(\underline{\varphi})_{\#}(x, t) = \inf \{\varphi_{\#}(x, t) : \varphi \in \mathcal{W}_{m, M, \varphi_0}\} \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+.$$

Remark 37. (a) Property (I) is stronger than (20) since it holds for any $s \geq 0$. This may be interpreted as a semigroup property.

(b) Property (I) is satisfied by most of the weights from the literature, as e.g. for

$$\varphi \equiv 1, \quad \varphi(x, t) = \mathbf{1}_{|x-x_0| < R+Ct} \quad \text{and} \quad \varphi(x, t) = e^{Ct} e^{-\sqrt{1+|x|^2}},$$

in respectively (3), (4) and (5); see also the stability results from [37, 13, 20, 44, 21, 28, 30, 41].

In the context of semigroups, the above result reflects some form of duality. For each $t \geq 0$, let

$$S_t : u_0 \in L^\infty(\mathbb{R}^d) \mapsto u(\cdot, t) \in L^\infty(\mathbb{R}^d)$$

where u is the entropy solution of (1), and let

$$G_t : \varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d) \mapsto \varphi(\cdot, t) \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$$

where φ is the viscosity solution of (19). Note that $G_t = G_t^{m, M}$ depends on the parameters m and M through Equation (19a).

Corollary 38 (A form of duality). *Assume (H1), $m < M$, and consider the semigroups S_t and G_t defined as above. Then G_t is the smallest strongly continuous semigroup of continuous operators on $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ satisfying*

$$(22) \quad \int_{\mathbb{R}^d} |S_t u_0 - S_t v_0| \varphi_0 \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0| G_t \varphi_0 \, dx,$$

for every u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$, $0 \leq \varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$, and $t \geq 0$.

Remark 39. Here “smallest” means that any other semigroup H_t satisfying the same properties is such that

$$G_t \varphi_0 \leq H_t \varphi_0 \quad \forall \varphi_0 \geq 0, \forall t \geq 0.$$

The proofs of Theorem 34, and Corollaries 36 and 38 are given in Section 4.4.

- Remark 40.* (a) Inequality (22) can be seen as a nonlinear dual inequality between S_t and G_t , and G_t as a dual semigroup of S_t whose restriction over the cone $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d, \mathbb{R}^+)$ is entirely determined by S_t through (22).
- (b) The question of duality in the other direction is open. Let us formulate it precisely. Consider S_t and the whole family $\{G_t^{m,M} : m < M\}$ defined just before Corollary 38.

Open question. *Is S_t the unique weakly- \star continuous semigroup on $L^\infty(\mathbb{R}^d)$ such that for all $m < M$, $G_t^{m,M}$ is the smallest strongly continuous semigroup of continuous operators on $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ satisfying*

$$(23) \quad \int_{\mathbb{R}^d} |S_t u_0 - S_t v_0| \varphi_0 \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0| G_t^{m,M} \varphi_0 \, dx,$$

for all u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$, $0 \leq \varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$, and $t \geq 0$?

A positive answer would mean that S_t is conversely entirely determined by $\{G_t^{m,M} : m < M\}$ through (23).

4. PROOFS

This section is devoted to the proofs of the results of Section 3. We progressively establish what we need to arrive at our concluding duality results (Corollaries 36 and 38). The proofs of Propositions 21 and 22 and Theorem 24 are independent of this development and given at the end of the section.

4.1. More on viscosity solutions of (2). We need further classical results that can be found in [23, 29, 5, 4].

Proposition 41 (Stability w.r.t. sup). *Assume (H2) and $\mathcal{F} \neq \emptyset$ is a uniformly locally bounded family of viscosity subsolutions of (2a). Then, the function*

$$(x, t) \mapsto \sup\{\varphi(x, t) : \varphi \in \mathcal{F}\}$$

is a viscosity subsolution of (2a).

The next results concern relaxed limits. Consider the upper limit

$$\limsup^* \varphi_\varepsilon(x, t) := \limsup_{\substack{(y, s) \rightarrow (x, t) \\ \varepsilon \rightarrow 0^+}} \varphi_\varepsilon(y, s) \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+,$$

as well as the lower limit $\liminf_* \varphi_\varepsilon := -\limsup^*(-\varphi_\varepsilon)$. Then:

Proposition 42 (Stability w.r.t. relaxed limits). *Assume (H2) and let $(\varphi_\varepsilon)_{\varepsilon > 0}$ be a family of uniformly locally bounded viscosity subsolutions (resp. supersolutions) of (2a). Then $\limsup^* \varphi_\varepsilon$ (resp. $\liminf_* \varphi_\varepsilon$) is a subsolution of (2a) (resp. supersolution).*

Remark 43. The notion of solution (or semisolution) is thus stable under local uniform convergence (equivalent to $\limsup^* \varphi_\varepsilon = \liminf_* \varphi_\varepsilon$).

Proposition 44 (Limiting initial data). *Assume (H2) and $(\varphi_\varepsilon)_{\varepsilon > 0}$ is a uniformly locally bounded family of viscosity subsolutions (resp. supersolutions) of (2a). Then $\limsup^* \varphi_\varepsilon$ (resp. $\liminf_* \varphi_\varepsilon$) satisfies*

$$\{\limsup^* \varphi_\varepsilon\}(x, 0) = \{\limsup^*(\varphi_\varepsilon)^*(\cdot, 0)\}(x) \quad \forall x \in \mathbb{R}^d$$

(resp. $\{\liminf_* \varphi_\varepsilon\}(x, 0) = \{\liminf_*(\varphi_\varepsilon)_*(\cdot, 0)\}(x)$).

Remark 45. For subsolutions this means that

$$\limsup_{\substack{\mathbb{R}^d \times \mathbb{R}^+ \ni (y,s) \rightarrow (x,0) \\ \varepsilon \rightarrow 0^+}} \varphi_\varepsilon(y,s) = \limsup_{\substack{\mathbb{R}^d \ni y \rightarrow x \\ \varepsilon \rightarrow 0^+}} (\varphi_\varepsilon)^*(y,0),$$

where $(\varphi_\varepsilon)^*$ is the upper semicontinuous envelope computed in (x,t) . The proof can be found in [9] and [5, Theorem 4.7].¹

Here is the stability for minimal solutions, see Appendix A.1 for the proof.

Proposition 46 (Stability of minimal solutions). *Assume (H2) and $(\varphi_0^n)_n$ is a nondecreasing uniformly globally bounded sequence. If $\underline{\varphi}_n$ is the minimal solution of (2) with φ_0^n as initial data, then $\sup_n \underline{\varphi}_n$ is the minimal solution of (2) with initial data $\sup_n (\varphi_0^n)_*$.*

Let us continue with regularization procedures. Usually we consider inf and supconvolutions, but for convex Hamiltonians we can use the classical convolution for supersolutions, see [7, 8] (the ideas were introduced in [38]).

Lemma 47. *Assume (H2), $\varphi \in BLSC(\mathbb{R}^d \times (0, \infty))$ is a supersolution of (2a), and $0 \leq f \in L^1(\mathbb{R}^d \times (-\infty, 0))$. Then $\varphi *_x *_t f$ is a supersolution of (2a).*

Here and throughout we extend the functions by zero to all $t \in \mathbb{R}$ to give a meaning to the convolutions in time. Below is another version that will be needed.

Lemma 48. *Assume (H2), $\varphi \in C_b(\mathbb{R}^d \times (0, \infty))$ is a supersolution of (2a), and $0 \leq g \in L^1(\mathbb{R}^d)$. Then $\varphi *_x g$ remains a supersolution.*

The latter lemma is not proven in [7, 8], but can be obtained via a standard approximation procedure. Let us give it for completeness. Throughout this paper ρ_ν is a space approximate unit as $\nu \rightarrow 0^+$ of the form

$$(24) \quad \rho_\nu(x) := \frac{1}{\nu^d} \rho\left(\frac{x}{\nu}\right),$$

where $0 \leq \rho \in C_c^\infty(\mathbb{R}^d)$ and $\int \rho = 1$. Moreover θ_ν is a time approximate unit of the form

$$(25) \quad \theta_\nu(t) := \frac{1}{\nu} \theta\left(\frac{t}{\nu}\right),$$

where $0 \leq \theta \in C_c^\infty((-\infty, 0))$ and $\int \theta = 1$.

Proof of Lemma 48. By Lemma 47, $\varphi_\nu := \varphi *_x g *_x \rho_\nu *_t \theta_\nu$ is a supersolution of (2a). It remains to pass to the limit as $\nu \rightarrow 0^+$. We will show that the convergence is local uniform towards $\varphi *_x g$, which will be sufficient by stability of the equation. We only need a local uniform convergence for $t > 0$ because the conclusion concerns the PDE only. With the assumed regularity on φ ,

$$\lim_{\nu \rightarrow 0^+} \varphi *_x \rho_\nu *_t \theta_\nu = \varphi \quad \text{locally uniformly,}$$

¹Let us briefly recall the ideas for the reader's convenience. First consider $\varphi := \limsup^* \varphi_\varepsilon$, $\varphi_0(x) := \{\limsup^*(\varphi_\varepsilon)^*(\cdot, 0)\}(x)$, and show that $\min\{\partial_t \varphi - H(D\varphi, D^2\varphi), \varphi - \varphi_0\} \leq 0$ at $t = 0$ in the viscosity sense. Fix then some x and use the viscosity inequalities at a $\max(\bar{y}, \bar{t})$ of the function $\varphi(y, t) - |y - x|^2/\bar{\varepsilon} - Ct$ with C large enough such that $\bar{t} = 0$. We get $\varphi(x, 0) \leq \varphi_0(\bar{y})$ and conclude as $\bar{\varepsilon} \rightarrow 0^+$.

and $\|\varphi *_x \rho_\nu *_t \theta_\nu\|_\infty \leq \|\varphi\|_\infty$. Moreover, for any $x \in \mathbb{R}^d$, $t > 0$ and $R \geq 0$,

$$\begin{aligned} |\varphi_\nu - \varphi *_x g|(x, t) &\leq |\varphi *_x \rho_\nu *_t \theta_\nu - \varphi *_x g|(x, t) \\ &\leq \left(\sup_{|y| \leq R} |\varphi *_x \rho_\nu *_t \theta_\nu - \varphi|(x - y, t) \right) \int_{|y| \leq R} g(y) \, dy \\ &\quad + 2\|\varphi\|_\infty \int_{|y| > R} g(y) \, dy. \end{aligned}$$

This is enough to conclude since $\lim_{R \rightarrow \infty} \int_{|y| > R} g(y) \, dy = 0$. \square

4.2. L^∞_{int} well-posedness: Proofs of Theorem 26 and Corollary 27. Let us now show that (2) is well-posed in L^∞_{int} as stated in Corollary 27. We first need to prove Theorem 26 for which we will use the lemmas below.

Lemma 49. *Assume (H2), and φ and ψ are sub and supersolutions of (2a). Then $(\varphi^* - \psi_*)^+$ remains a subsolution.*

Sketch of the proof. Let $H_\xi(\varphi) := b(\xi) \cdot D\varphi + \text{tr}(a(\xi)D^2\varphi)$ and note that

$$\partial_t(\varphi - \psi) \leq \sup_{\xi \in \mathcal{E}} H_\xi(\varphi) - \sup_{\xi \in \mathcal{E}} H_\xi(\psi) \leq \sup_{\xi \in \mathcal{E}} (H_\xi(\varphi) - H_\xi(\psi)).$$

By stability, $(\varphi^* - \psi_*)^+ = \max\{\varphi^* - \psi_*, 0\}$ is thus indeed a subsolution. \square

The rigorous justification of the above computations can be done by using a test function, Ishii lemma, and semijets [23, Theorem 8.3]. The details are standard and left to the reader. Here is a second lemma involving the profile

$$U : r \geq 0 \mapsto c_0 \int_r^\infty e^{-\frac{s^2}{4}} \, ds,$$

where $c_0 > 0$ is chosen such that $U(0) = 1$. Below and after ‘‘Sp’’ is the spectrum.

Lemma 50. *Let $L_b \geq 0$ and $L_a > 0$. For any $(x, t) \in \mathbb{R}^d \times \mathbb{R}^+$, define*

$$(26) \quad \Psi(x, t) := \begin{cases} U\left(\frac{(|x| - 1 - L_b t)^+}{\sqrt{L_a t}}\right) & \text{if } t > 0, \\ \mathbf{1}_{|x| < 1} & \text{if } t = 0. \end{cases}$$

Then in the viscosity sense,

$$\partial_t \Psi \geq L_b |D\Psi| + L_a \sup_{\lambda \in \text{Sp}(D^2\Psi)} \lambda^+ \quad \text{in } \mathbb{R}^d \times (0, \infty)$$

(it is in fact an equality). Moreover $\Psi \in C_b(\mathbb{R}^d \times (0, \infty)) \cap C(\mathbb{R}^+; L^1(\mathbb{R}^d))$ where the latter time continuity holds up to $t = 0$.

Remark 51. Roughly speaking, we will use Ψ as a fundamental solution to construct L^1 supersolutions of (2), but we cannot take it as a Dirac mass at $t = 0$ because of Proposition 22.

Proof. The desired PDE holds if $|x| < 1 + L_b t$ since Ψ is constant in that region. It is also satisfied if $|x| = 1 + L_b t$ because the subsets are empty. Now if $|x| > 1 + L_b t$, then

$$\partial_t \Psi = -L_a \frac{|x| - 1 - L_b t}{2(L_a t)^{\frac{3}{2}}} U' - \frac{L_b}{\sqrt{L_a t}} U', \quad D\Psi = \frac{x}{|x|} \frac{U'}{\sqrt{L_a t}},$$

and

$$\partial_{x_i x_j}^2 \Psi = \left(\frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3} \right) \frac{U'}{\sqrt{L_a t}} + \frac{x_i x_j}{|x|^2} \frac{U''}{L_a t}.$$

Since $U' \leq 0$ and $U'' \geq 0$, we have $\sum_{i,j=1}^d \partial_{x_i x_j}^2 \Psi h_i h_j \leq \frac{U''}{L_a t}$ for any $h = (h_i)$ with $|h| = 1$. Hence $\sup_{\lambda \in \text{Sp}(D^2 \Psi)} \lambda^+ \leq \frac{U''}{L_a t}$ and

$$\partial_t \Psi - L_b |D\Psi| - L_a \sup_{\lambda \in \text{Sp}(D^2 \Psi)} \lambda^+ \geq -\frac{rU'(r)/2 + U''(r)}{t}$$

with $r = (|x| - 1 - L_b t)/\sqrt{L_a t}$. The right-hand side is zero by definition of U , and we obtain the desired equation for positive times. Now the detailed verification that $\Psi \in C_b(\mathbb{R}^d \times (0, \infty)) \cap C(\mathbb{R}^+; L^1(\mathbb{R}^d))$ does not contain any particular difficulty and is left to the reader. The proof is complete. \square

Proof of Theorem 26. Let $L_b := \|b\|_\infty$ and $L_a := \|\text{tr}(a)\|_\infty$ and assume $L_a > 0$. We will use the following Ky Fan inequality [42]:

$$(27) \quad \text{tr}(XY) \leq \sum_{i=1}^d \lambda_i(X) \lambda_i(Y) \quad \forall X, Y \text{ real } d \times d \text{ symmetric matrices,}$$

with the ordered eigenvalues $\lambda_1 \leq \dots \leq \lambda_d$. It implies that any subsolution of (2a) is a subsolution of the equation

$$(28) \quad \partial_t \varphi = L_b |D\varphi| + L_a \sup_{\lambda \in \text{Sp}(D^2 \varphi)} \lambda^+.$$

Consider now arbitrary bounded sub and supersolutions φ and ψ of (2a). By Lemma 49, $(\varphi^* - \psi_*)^+$ is a subsolution of (2a) thus of (28). To prove Estimate (18), we will construct an integrable supersolution of (28). We will take it of the form

$$\psi := \Psi *_x \sup_{\overline{Q}_1(\cdot)} \phi_0,$$

where $\phi_0(x) := (\varphi^* - \psi_*)^+(x, t = 0)$ and Ψ is defined in Lemma 50. Let us use Lemma 48 to show that ψ is a supersolution of (28). We need $\sup_{\overline{Q}_1(\cdot)} \phi_0$ to be integrable, and this can be assumed without loss of generality since (18) trivially holds if not. Now recalling that $\Psi \in C_b(\mathbb{R}^d \times (0, \infty))$ is a supersolution of (28), Lemma 48 applies and ψ remains a supersolution. Since moreover $\Psi \in C(\mathbb{R}^+; L^1(\mathbb{R}^d))$ and $\sup_{\overline{Q}_1(\cdot)} \phi_0 \in L^\infty(\mathbb{R}^d)$, this supersolution is continuous up to $t = 0$ and satisfies

$$\psi(x, 0) = \int \mathbf{1}_{|y| < 1} \sup_{\overline{Q}_1(x-y)} \underbrace{\phi_0}_{=(\varphi^* - \psi_*)^+(t=0)} dy \geq (\varphi^* - \psi_*)^+(x, 0).$$

Hence $(\varphi^* - \psi_*)^+ \leq \psi$ everywhere by the comparison principle, and

$$\begin{aligned} \int \sup_{\overline{Q}_1(x) \times [0, T]} (\varphi^* - \psi_*)^+ dx &\leq \int \sup_{\overline{Q}_1(x) \times [0, T]} \psi dx \\ &\leq \int \sup_{t \in [0, T]} \Psi(y, t) dy \int \sup_{\overline{Q}_2(x)} \phi_0 dx, \end{aligned}$$

by the Fubini theorem, etc. The first integral satisfies

$$\int \sup_{t \in [0, T]} \Psi(y, t) dy \leq \int U \left((|y| - 1 - L_b T)^+ / \sqrt{L_a T} \right) dy < \infty,$$

by (26) and since U is nondecreasing and integrable. For the second integral, Lemma 17 implies that

$$\int \sup_{\overline{Q}_2(x)} \phi_0 dx \leq C \int \sup_{\overline{Q}_1(x)} (\varphi^* - \psi_*)^+(\cdot, 0) dx,$$

for a constant C which only depends on d . Combining the three inequalities above completes the proof of (18) when $L_a = \|\operatorname{tr}(a)\|_\infty > 0$. If $L_a = 0$, there is no diffusive part in (2a), and (18) follows from Proposition 9 and Lemma 17. \square

We are ready to prove Corollary 27. We need the result below.

Lemma 52. *Assume (H2) and φ and ψ are continuous viscosity solutions of (2a). Then $|\varphi - \psi|$ is a subsolution of the same PDE.*

Proof. Use that $|\varphi - \psi| = \max\{(\varphi - \psi)^+, (\psi - \varphi)^+\}$ and Lemma 49. \square

Proof of Corollary 27. The fact that G_t maps $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ into itself follows from Theorem 26. Indeed, if $\varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$, then the function $(x, t) \mapsto |G_t \varphi_0(x)|$ is a bounded subsolution of (2a), by Lemma 52 with $\psi \equiv 0$. Estimate (18) then implies that for any $t \geq 0$,

$$\|G_t \varphi_0\|_{\text{int}} = \int \sup_{\overline{Q_1(x)}} |G_t \varphi_0| \, dx \leq C \int \sup_{\overline{Q_1(x)}} |\varphi_0| \, dx,$$

for some constant $C = C(d, \|a\|_\infty, \|b\|_\infty, t)$. Let us now prove that

$$G_t : C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d) \rightarrow C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$$

is Lipschitz continuous for any $t \geq 0$. Let us apply again (18) to

$$(x, t) \mapsto |G_t \varphi_0(x) - G_t \psi_0(x)|,$$

which is a subsolution of (2a) by Lemma 52. As above we get that

$$\|G_t \varphi_0 - G_t \psi_0\|_{\text{int}} \leq C \int \sup_{\overline{Q_1(x)}} |\varphi_0 - \psi_0| \, dx,$$

and deduce the desired continuity because C does not depend on the initial data. Hence G_t is a semigroup of Lipschitz continuous operators on $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ and it remains to prove the time continuity. Fix $t_0 \geq 0$ and let us show that

$$\int \sup_{\overline{Q_1(x)}} |G_t \varphi_0 - G_{t_0} \varphi_0| \, dx \rightarrow 0 \quad \text{as } t \rightarrow t_0.$$

The pointwise convergence follows from the continuity of $(x, t) \mapsto G_t \varphi_0(x)$ (as continuous solution of (2)), and a dominating function is given by

$$x \mapsto \sup_{(y, s) \in \overline{Q_1(x)} \times [0, t_0 + 1]} 2|G_s \varphi_0(y)|$$

which is integrable by Theorem 26. \square

4.3. Weighted L^1 contraction: Proof of Theorem 28. We continue with the general weighted L^1 contraction principle for (1).

Proof of Theorem 28. We have to show that

$$(29) \quad \int_{\mathbb{R}^d} |u - v|(x, T) \varphi_0(x) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x, T) \, dx \quad \forall T \geq 0.$$

Let us use the Kato inequality (11). For a.e. $x \in \mathbb{R}^d$ and $t \geq 0$,

$$\begin{aligned} & \left\{ \sum_{i=1}^d q_i(u, v) \partial_{x_i} \phi + \sum_{i,j=1}^d r_{ij}(u, v) \partial_{x_i x_j}^2 \phi \right\} (x, t) \\ &= \operatorname{sign}(u(x, t) - v(x, t)) \int_{v(x, t)}^{u(x, t)} \{F'(\xi) \cdot D\phi(x, t) + \operatorname{tr}(A(\xi) D^2 \phi(x, t))\} \, d\xi \\ &\leq |u(x, t) - v(x, t)| \operatorname{ess\,sup}_{m \leq \xi \leq M} \{F'(\xi) \cdot D\phi(x, t) + \operatorname{tr}(A(\xi) D^2 \phi(x, t))\}, \end{aligned}$$

where we have taken the sup over $[m, M]$ because of the maximum principle Lemma 65. Injecting into (11), we get that

$$(30) \quad \int_{\mathbb{R}^d} |u - v|(x, T)\phi(x, T) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x)\phi(x, 0) \, dx \\ + \iint_{\mathbb{R}^d \times (0, T)} |u - v| \left(\partial_t \phi + \operatorname{ess\,sup}_{m \leq \xi \leq M} \{F'(\xi) \cdot D\phi + \operatorname{tr}(A(\xi)D^2\phi)\} \right) \, dx \, dt.$$

In the third integral, we recognize the backward in time version of (19a). The proof of (29) then consists in taking $\phi(x, t) = \underline{\varphi}(x, T - t)$.

Simplified case: $0 \leq \varphi_0 \in C_c(\mathbb{R}^d)$.

Now (19) has a unique viscosity solution φ which coincides with $\underline{\varphi}$. It belongs to $C_b(\mathbb{R}^d \times \mathbb{R}^+) \cap C(\mathbb{R}^+; L^1(\mathbb{R}^d))$ by Corollary 27 and Theorem 16. Let us regularize it by convolution

$$\varphi_\nu := \varphi *_{x,t} (\rho_\nu \theta_\nu),$$

with the mollifiers (24) and (25). It follows that

$$\varphi_\nu \in C^\infty(\mathbb{R}^d \times \mathbb{R}^+) \cap C(\mathbb{R}^+; L^1(\mathbb{R}^d))$$

along with all its derivatives. This is enough to take $\phi_\nu(x, t) := \varphi_\nu(x, T - t)$ as a test function in (30) by approximation. Note that ϕ_ν is a supersolution of the backward version of (19a) by Lemma 47, i.e.

$$\partial_t \phi_\nu + \operatorname{ess\,sup}_{m \leq \xi \leq M} \{F'(\xi) \cdot D\phi_\nu + \operatorname{tr}(A(\xi)D^2\phi_\nu)\} \leq 0 \quad \text{for any } t < T.$$

Inequality (30) with the test function ϕ_ν then implies that

$$\int_{\mathbb{R}^d} |u - v|(x, T)\varphi_\nu(x, 0) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x)\varphi_\nu(x, T) \, dx,$$

for any $T \geq 0$ and $\nu > 0$. By the $C(\mathbb{R}^+; L^1(\mathbb{R}^d))$ regularity of φ , the convolution $\varphi_\nu = \varphi *_{x,t} (\rho_\nu \theta_\nu)$ converges to φ in $C([0, T]; L^1(\mathbb{R}^d))$ as $\nu \rightarrow 0^+$. Passing to the limit as $\nu \rightarrow 0^+$ then yields (29).

General case: $0 \leq \varphi_0 \in BLSC(\mathbb{R}^d)$.

We would like to pointwise approximate φ_0 by a monotone sequence $\varphi_0^n \uparrow \varphi_0$ such that $0 \leq \varphi_0^n \in C_c(\mathbb{R}^d)$. Take

$$\varphi_0^n(x) := \inf_{y \in \mathbb{R}^d} \{ \varphi_0(y) \mathbf{1}_{|y| < n} + n|x - y|^2 \} \geq 0.$$

Then φ_0^n is continuous as an infconvolution, see e.g. [23, 29, 5, 4]. Also,

$$\varphi_0^n(x) \leq \varphi_0(x) \mathbf{1}_{|x| < n} \quad \forall x \in \mathbb{R}^d,$$

which implies that $\varphi_0^n \in C_c(\mathbb{R}^d)$. In the limit $n \rightarrow \infty$, we have $\varphi_0^n \uparrow (\varphi_0)_* = \varphi_0$. Let φ_n be the solution of (19) with initial data φ_0^n , then by the previous step,

$$\int_{\mathbb{R}^d} |u - v|(x, T)\varphi_0^n(x) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x)\varphi_n(x, T) \, dx,$$

for any $T \geq 0$ and n . By the stability of minimal solutions (see Proposition 46), these solutions satisfy $\varphi_n \uparrow \underline{\varphi}$ pointwise. So we conclude the proof of (29) by passing to the limit as $n \rightarrow \infty$ using the monotone convergence theorem. \square

4.4. Duality: Proofs of Theorem 34 and Corollaries 36 and 38. Let us now establish the new characterization of viscosity supersolutions (Theorem 34). We need several technical lemmas.

Here is a first classical result on entropy solutions.

Lemma 53. *Assume (H1) and $u_0 \in L^\infty(\mathbb{R}^d)$. Then, the entropy solution of (1) is a distributional solution of (1),*

$$\iint_{\mathbb{R}^d \times \mathbb{R}^+} \left(u \partial_t \phi + \sum_{i=1}^d F_i(u) \partial_{x_i} \phi + \sum_{i,j=1}^d \mathcal{A}_{ij}(u) \partial_{x_i x_j}^2 \phi \right) dx dt + \int_{\mathbb{R}^d} u_0(x) \phi(x, 0) dx = 0 \quad \forall \phi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+),$$

where $\mathcal{A}_{ij}(u) = \int_0^u A_{ij}(\xi) d\xi$.

Proof. Take $\eta(u) = \pm u$ successively in the entropy inequalities, Definition 12(c). \square

Here is another result on the continuity in time.

Lemma 54. *Assume (H1), $u_0, v_0 \in L^\infty(\mathbb{R}^d)$ with $u_0 - v_0 \in L^1(\mathbb{R}^d)$, u and v entropy solutions of (1) with initial data u_0 and v_0 . Then $u - v \in C(\mathbb{R}^+; L^1(\mathbb{R}^d))$.*

Proof of Lemma 54. By Theorem 28 with $\varphi_0 \equiv 1$, we have

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1} \quad \forall t \geq 0.$$

Since the left-hand side is finite, $u - v \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^d))$. By the continuity in time with values in $L^1_{\text{loc}}(\mathbb{R}^d)$ of these functions, it remains to prove that

$$(31) \quad \lim_{R \rightarrow \infty} \sup_{t \in [0, T]} \int_{|x| \geq R} |u(x, t) - v(x, t)| dx = 0 \quad \forall T \geq 0.$$

To do so, we will use again Theorem 28.

Fix $m < M$ such that u_0 and v_0 take their values in $[m, M]$, and consider

$$\varphi_0^R(x) := \varphi_0\left(\frac{x}{R}\right), \quad R > 0,$$

where $\varphi_0 = \varphi_0(x)$ is some kernel such that

$$\begin{cases} 0 \leq \varphi_0 \in C_b(\mathbb{R}^d), \\ \varphi_0(x) = 0 \text{ for } |x| \leq 1/2, \\ \text{and } \varphi_0(x) = 1 \text{ for } |x| \geq 1. \end{cases}$$

With that choice, $\varphi_0^R \rightarrow 0$ as $R \rightarrow \infty$ locally uniformly. We then claim that the solutions φ_R of (19) with initial data φ_0^R converge locally uniformly to zero too. This is a consequence of the method of relaxed semilimits [9]. Let us give details for completeness. By the maximum principle,

$$\|\varphi_R\|_\infty \leq \|\varphi_0^R\|_\infty = \|\varphi_0\|_\infty \quad \forall R > 0.$$

We can then apply Propositions 42 and 44 to $\limsup^* \varphi_R$ as $R \rightarrow \infty$ and get that it is a subsolution of (19a) satisfying

$$\limsup^* \varphi_R(x, 0) = \limsup^* \varphi_0^R(x) = 0 \quad \forall x \in \mathbb{R}^d.$$

Similarly, $\liminf_* \varphi_R$ is a supersolution of (2) with zero as initial data. The comparison principle then implies that

$$\limsup^* \varphi_R \leq \liminf_* \varphi_R.$$

Hence φ_R converges locally uniformly to the unique solution of (19) with zero initial data, that is zero itself.

Now we can show (31). By Theorem 28 with the previous m , M , and φ_0^R ,

$$\begin{aligned} \int_{|x| \geq R} |u(x, t) - v(x, t)| dx &\leq \int_{\mathbb{R}^d} |u(x, t) - v(x, t)| \varphi_0^R(x) dx \\ &\leq \int_{\mathbb{R}^d} |u_0(x) - v_0(x)| \varphi_R(x, t) dx \leq \int_{\mathbb{R}^d} |u_0(x) - v_0(x)| \sup_{s \in [0, T]} \varphi_R(x, s) dx, \end{aligned}$$

for any $T \geq t \geq 0$. The right-hand side vanishes as $R \rightarrow \infty$ by the discussion above and the dominated convergence theorem. The proof of (31) is complete. \square

Here is a regularization procedure for the weights.

Lemma 55. *Assume (H1), $m < M$, ρ_ν and θ_ν are defined in (24) and (25), and $0 \leq \varphi \in BLSC(\mathbb{R}^d \times \mathbb{R}^+)$ satisfies (I) in Theorem 34. Then for any $\nu > 0$, the convolution*

$$\varphi_\nu := \varphi *_{x,t} (\rho_\nu \theta_\nu) \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^+)$$

also satisfies (I) in Theorem 34.

Proof. By assumption,

$$(32) \quad \int_{\mathbb{R}^d} |u - v|(x, t) \varphi(x, s) dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x, t + s) dx,$$

for any $t, s \geq 0$, u_0 and v_0 with values in $[m, M]$, and entropy solutions u and v of (1) with u_0 and v_0 as initial data. Our aim is to get the same result for φ_ν . Let us use (32) not for u_0 and v_0 , but their translations $u_0(\cdot + y)$ and $v_0(\cdot + y)$ for some fixed $y \in \mathbb{R}^d$. Since the PDE part of (1) is invariant w.r.t. translation, the corresponding solutions are $u(x + y, t)$ and $v(x + y, t)$. Hence,

$$\int_{\mathbb{R}^d} |u - v|(x + y, t) \varphi(x, s) dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x + y) \varphi(x, t + s) dx$$

for any $t, s \geq 0$. By changing the variable of integration, we obtain that

$$\int_{\mathbb{R}^d} |u - v|(x, t) \varphi(x - y, s) dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x - y, t + s) dx.$$

Now we fix $\tau \leq 0$ and apply this formula, not for s but $s - \tau$. We deduce that

$$\int_{\mathbb{R}^d} |u - v|(x, t) \varphi(x - y, s - \tau) dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x - y, t + s - \tau) dx.$$

Multiply then by $\rho_\nu(y) \theta_\nu(\tau)$ and integrate over $(y, \tau) \in \mathbb{R}^d \times \mathbb{R}^-$ to conclude. \square

Later we will pass to the limit

$$\varphi_b := \liminf_* \varphi_\nu \quad \text{as } \nu \rightarrow 0^+,$$

and compare φ_b with the function

$$(33) \quad \varphi_\#(x, t) = \liminf_{\substack{r \rightarrow 0^+ \\ y \rightarrow x}} \frac{1}{\text{meas}(B_r(y))} \int_{B_r(y)} \varphi(z, t) dz.$$

To do so, we will assume in addition that

$$(34) \quad \text{supp}(\rho_\nu) \subset B_\nu(0) \quad \text{and} \quad \text{supp}(\theta_\nu) \subset (-\nu, 0).$$

Here are fundamental properties on φ_b and $\varphi_\#$ that will be needed.

Lemma 56. *Assume $\varphi \in BLSC(\mathbb{R}^d \times \mathbb{R}^+)$, φ_b and $\varphi_\#$ are as above, and (34) holds. Then:*

- (i) *The limit φ_b is the pointwise largest function in $BLSC(\mathbb{R}^d \times \mathbb{R}^+)$ that is less than or equal φ a.e. Moreover $\varphi_b = \varphi$ a.e.*

- (ii) For any $t \geq 0$, $\varphi_{\#}(\cdot, t)$ is the pointwise largest function in $BLSC(\mathbb{R}^d)$ less than or equal $\varphi(\cdot, t)$ a.e. Moreover $\varphi_{\#} = \varphi(\cdot, t)$ a.e.

Remark 57. It is understood that ‘‘a.e.’’ holds in (x, t) in (i) and x in (ii).

Proof. Let us prove (i). Note first that φ_b is lower semicontinuous as a lower relaxed limit. To prove that $\varphi_b \leq \varphi$ a.e., it suffices to do it for the Lebesgue points of φ . Such points $(x, t) \in \mathbb{R}^d \times (0, \infty)$ satisfy

$$\lim_{\nu \rightarrow 0^+} \frac{1}{\nu^{d+1}} \iint_{B_\nu(x) \times (t-\nu, t+\nu)} |\varphi(y, s) - \varphi(x, t)| \, dy \, ds = 0,$$

so by the assumptions on the mollifiers, see (24), (25) and (34), we find that

$$\begin{aligned} |\varphi_\nu(x, t) - \varphi(x, t)| &\leq \frac{1}{\nu^{d+1}} \iint_{B_\nu(x) \times (t, t+\nu)} |\varphi(y, s) - \varphi(x, t)| \\ &\quad \cdot \rho\left(\frac{x-y}{\nu}\right) \theta\left(\frac{t-s}{\nu}\right) \, dy \, ds \rightarrow 0 \quad \text{as } \nu \rightarrow 0^+. \end{aligned}$$

It follows that

$$\varphi_b(x, t) \leq \lim_{\nu \rightarrow 0^+} \varphi_\nu(x, t) = \varphi(x, t),$$

at any Lebesgue point. Moreover, for any fixed (x, t) , lower semicontinuity of φ implies that

$$\varphi_\nu(y, s) = \iint_{B_\nu(y) \times (s, s+\nu)} \underbrace{\varphi(z, \tau)}_{\geq \varphi(x, t) + o(1)} \rho\left(\frac{y-z}{\nu}\right) \theta\left(\frac{s-\tau}{\nu}\right) \, dz \, d\tau \geq \varphi(x, t) + o(1)$$

as $(y, s, \nu) \rightarrow (x, t, 0^+)$, and we get that

$$\varphi_b(x, t) = \liminf_* \varphi_\nu(x, t) \geq \varphi(x, t).$$

We conclude that $\varphi_b = \varphi$ a.e.

Now, to complete the proof of (i), it remains to prove that $\varphi_b \geq \psi$ pointwise for any other $\psi \in BLSC(\mathbb{R}^d \times \mathbb{R}^+)$ such that $\psi \leq \varphi$ a.e. Given such a function, let

$$\psi_b := \liminf_* \psi *_{x,t} (\rho_\nu \theta_\nu).$$

As above, $\psi \leq \psi_b$ pointwise; but also $\psi_b \leq \varphi_b$ pointwise since

$$\psi *_{x,t} (\rho_\nu \theta_\nu) \leq \varphi *_{x,t} (\rho_\nu \theta_\nu).$$

This proves (i) and the arguments for (ii) are similar. \square

Here is also a general inequality between φ_b and $\varphi_{\#}$ that will be needed.

Lemma 58. *Under the hypotheses of the previous lemma, $(\varphi_{\#})_* \leq \varphi_b$ pointwise.*

Proof. Let us first prove that $\varphi_{\#}$ is measurable in (x, t) . We have

$$\varphi_{\#}(x, t) = \sup_{n \geq 1} \inf_{m \geq n} \underbrace{\inf_{\substack{\frac{1}{m} \leq r \leq \frac{1}{n} \\ |y| \leq \frac{1}{n}}} \frac{1}{\text{meas}(B_r(y))} \int_{B_r(y)} \varphi(x+z, t) \, dz}_{=:\varphi_{n,m}(x,t)} \quad \overbrace{\hspace{10em}}^{=:\varphi_n(x,t)},$$

where n and m are integers. For each $\frac{1}{m} \leq r \leq \frac{1}{n}$ and $|y| \leq \frac{1}{n}$, the function

$$(x, t) \mapsto \frac{1}{\text{meas}(B_r(y))} \int_{B_r(y)} \varphi(x+z, t) \, dz$$

is lower semicontinuous by Fatou’s lemma and $\varphi \in BLSC$ (assumption in the previous lemma). The infimum $\varphi_{n,m}$ remains lower semicontinuous, because r and

y live in compact sets. Hence, $\varphi_n = \inf_{m \geq n} \varphi_{n,m}$ is measurable in (x, t) and so is $\varphi_{\#} = \sup_{n \geq 1} \varphi_n$.

We can now prove the lemma. For any $t \geq 0$, the measurable functions $\varphi, \varphi_{\#}$ satisfy $\varphi_{\#}(\cdot, t) = \varphi(\cdot, t)$ a.e., hence we may use the Fubini theorem to conclude that

$$\iint_{\mathbb{R}^d \times \mathbb{R}^+} \mathbf{1}_{\{\varphi_{\#} = \varphi\}} dx dt = \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^d} \mathbf{1}_{\{\varphi_{\#}(x,t) = \varphi(x,t)\}} dx \right) dt = 0.$$

This proves that $\varphi_{\#} = \varphi$ a.e. in (x, t) , so that $(\varphi_{\#})_* \leq \varphi$ a.e. in (x, t) . Hence $(\varphi_{\#})_* \leq \varphi_b$ pointwise by Lemma 56(i). \square

Here are further properties that we will need.

Lemma 59. *Let $\varphi, \psi \in BLSC(\mathbb{R}^d \times \mathbb{R}^+)$ and $\varphi_{\#}, \psi_{\#}$ as in (33). Then*

- (i) $\varphi \leq (\varphi_{\#})_*$ pointwise, and
- (ii) if $\varphi \leq \psi_{\#}$ pointwise, then $\varphi_{\#} \leq \psi_{\#}$ pointwise.

Proof. We can show that $\varphi \leq \varphi_{\#}$ from the definition of $\varphi_{\#}$ and the lower semi-continuity of φ , exactly as we showed that $\varphi \leq \varphi_b$ in the proof of Lemma 56. In particular, $\varphi \leq (\varphi_{\#})_*$ which is part (i). For part (ii), use Lemma 56(ii). It says that $\psi_{\#}(\cdot, t) = \psi(\cdot, t)$ a.e. in x , for each fixed $t \geq 0$. Hence, $\varphi(\cdot, t) \leq \psi(\cdot, t)$ a.e. and the desired inequality follows again from the definitions of $\varphi_{\#}$ and $\psi_{\#}$. \square

We are now in position to prove Theorem 34.

Proof of Theorem 34. Let us proceed in several steps.

1) (II) \implies (I).

By (II), $(\varphi_{\#})_*$ is a *BLSC* supersolution of (19a). In particular, for any fixed $s \geq 0$, the function

$$(x, t) \mapsto (\varphi_{\#})_*(x, t + s)$$

is also a supersolution of (19a). By Remark 29(c), we can apply Theorem 28 to this supersolution with the *BLSC* initial weight $(\varphi_{\#})_*(\cdot, s)$. The result is that

$$\int_{\mathbb{R}^d} |u - v|(x, t)(\varphi_{\#})_*(x, s) dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x)(\varphi_{\#})_*(x, t + s) dx,$$

for any $u_0 = u_0(x)$ and $v_0 = v_0(x)$ with values in $[m, M]$, u and v entropy solutions of (1) with u_0 and v_0 as initial data, and $t, s \geq 0$. This is exactly (I) but with $(\varphi_{\#})_*$ instead of φ . To replace $(\varphi_{\#})_*$ by φ , we use Lemma 59(i) for the left-hand side. For the right-hand side, we use that $(\varphi_{\#})_* \leq \varphi_{\#}$ pointwise and the fact that $\varphi_{\#}(x, t + s) = \varphi(x, t + s)$ for a.e. x , see Lemma 56(ii). This implies (I) with φ , as desired.

2) (I) \implies (II) for smooth weights φ .

Let us prove the reverse implication when $0 \leq \varphi \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^+)$. We will appropriately choose u_0 and v_0 later. For the moment, we assume that

$$m \leq v_0 \leq u_0 \leq M \quad \text{and} \quad u_0 - v_0 \in L^1(\mathbb{R}^d).$$

By Lemmas 65 and 54, $0 \leq u - v \in C(\mathbb{R}^+; L^1(\mathbb{R}^d))$, and then we can use (I) to get

$$(35) \quad \int_{\mathbb{R}^d} (u - v)(x, T)\varphi(x, s) dx \leq \int_{\mathbb{R}^d} (u_0 - v_0)(x)\varphi(x, T + s) dx,$$

for any $T, s \geq 0$. Let us fix $s > 0$ and determine what PDE φ satisfies. This will be done by injecting the weak formulation of (1) into (35) and then pass to the limit

as $T \rightarrow 0^+$. By Lemma 53,

$$\begin{aligned} \int_{\mathbb{R}^d} (u - v)(x, T) \phi(x, T) \, dx &= \iint_{\mathbb{R}^d \times (0, T)} \left((u - v) \partial_t \phi + \sum_{i=1}^d (F_i(u) - F_i(v)) \partial_{x_i} \phi \right. \\ &\quad \left. + \sum_{i,j=1}^d (\mathcal{A}_{ij}(u) - \mathcal{A}_{ij}(v)) \partial_{x_i x_j}^2 \phi \right) \, dx \, dt \\ &\quad + \int_{\mathbb{R}^d} (u_0 - v_0)(x) \phi(x, 0) \, dx, \end{aligned}$$

for any $\phi \in C_c^\infty(\mathbb{R}^d \times [0, T])$ and $\mathcal{A}'_{ij} = \mathcal{A}_{ij}$. Note that we have rewritten the equation given by Lemma 53 with integrals in $t < T$ and an additional final term at $t = T$. This follows from standard arguments using the L^1_{loc} continuity in time of u and v . Since $\varphi \in C_b^\infty$, $u - v \in C_t(L^1_x)$ and $u, v \in L^\infty$, a standard approximation argument shows that we can take ϕ to be

$$\phi(x, t) = \varphi(x, t + s - T),$$

and get that

$$(36) \quad \begin{aligned} \int_{\mathbb{R}^d} (u - v)(x, T) \varphi(x, s) \, dx &= \iint_{\mathbb{R}^d \times (0, T)} \left((u - v) \partial_t \varphi(t + s - T) \right. \\ &\quad \left. + \sum_{i=1}^d (F_i(u) - F_i(v)) \partial_{x_i} \varphi(t + s - T) \right. \\ &\quad \left. + \sum_{i,j=1}^d (\mathcal{A}_{ij}(u) - \mathcal{A}_{ij}(v)) \partial_{x_i x_j}^2 \varphi(t + s - T) \right) \, dx \, dt \\ &\quad + \int_{\mathbb{R}^d} (u_0 - v_0)(x) \varphi(x, s - T) \, dx. \end{aligned}$$

Here we assume that $s > 0$ and T is so small that $s - T > 0$. Inserting (36) into (35), we get

$$\begin{aligned} \int_{\mathbb{R}^d} (u_0 - v_0)(x) \varphi(x, s + T) \, dx - \int_{\mathbb{R}^d} (u_0 - v_0)(x) \varphi(x, s - T) \, dx \\ \geq \iint_{\mathbb{R}^d \times (0, T)} (\dots) \, dx \, dt. \end{aligned}$$

We now would like to divide by $2T$ and pass to the limit as $T \rightarrow 0^+$. All the computations are justified, again because $\varphi \in C_b^\infty$, the solutions u and v are bounded, and $u - v \in C_t(L^1_x)$. We get that

$$\begin{aligned} &\int_{\mathbb{R}^d} (u_0(x) - v_0(x)) \partial_s \varphi(x, s) \, dx \\ &\geq \frac{1}{2} \int_{\mathbb{R}^d} \left((u_0 - v_0) \partial_s \varphi(s) + \sum_{i=1}^d (F_i(u_0) - F_i(v_0)) \partial_{x_i} \varphi(s) \right. \\ &\quad \left. + \sum_{i,j=1}^d (\mathcal{A}_{ij}(u_0) - \mathcal{A}_{ij}(v_0)) \partial_{x_i x_j}^2 \varphi(s) \right) \, dx. \end{aligned}$$

Subtracting the term $\int (u_0 - v_0) \partial_s \varphi(s) dx/2$ of the right-hand side implies that

$$\begin{aligned}
(37) \quad & \int_{\mathbb{R}^d} (u_0(x) - v_0(x)) \partial_s \varphi(x, s) dx \\
& \geq \int_{\mathbb{R}^d} \left(\sum_{i=1}^d (F_i(u_0) - F_i(v_0)) \partial_{x_i} \varphi(s) + \sum_{i,j=1}^d (\mathcal{A}_{ij}(u_0) - \mathcal{A}_{ij}(v_0)) \partial_{x_i x_j}^2 \varphi(s) \right) dx \\
& = \int_{\mathbb{R}^d} \int_{v_0(x)}^{u_0(x)} \{ F'(\xi) \cdot D\varphi(x, s) + \text{tr} (A_{ij}(\xi) D^2 \varphi(x, s)) \} d\xi dx,
\end{aligned}$$

for any $s > 0$ and $0 \leq u_0 - v_0 \in L^1(\mathbb{R}^d)$ such that both u_0 and v_0 take their values in the interval $[m, M]$. It remains to choose $u_0 - v_0$ as an approximate unit, up to some multiplicative constant.

Let us introduce new parameters: $x_0 \in \mathbb{R}^d$, $\varepsilon > 0$ and $m \leq a < b \leq M$. We would like to choose

$$u_0 - v_0 = (b - a) \mathbf{1}_{x_0 + (-\varepsilon, \varepsilon)^d},$$

with the constraint that both u_0 and v_0 only take the two values a and b . Writing $x = (x_i)$, take e.g.

$$u_0(x) := \begin{cases} a & \text{if } x_1 > (x_0)_1 + \varepsilon, \\ b & \text{if not,} \end{cases}$$

and

$$v_0(x) := \begin{cases} a & \text{if } x_1 > (x_0)_1 + \varepsilon \text{ or } x \in x_0 + (-\varepsilon, \varepsilon)^d, \\ b & \text{if not.} \end{cases}$$

Then $m \leq v_0 \leq u_0 \leq M$ and $u_0 - v_0 \in L^1(\mathbb{R}^d)$ as required. Inserting our choice into (37) and dividing by $(b - a)\varepsilon^d$, we deduce that

$$\begin{aligned}
& \frac{1}{\varepsilon^d} \int_{x_0 + (-\varepsilon, \varepsilon)^d} \partial_s \varphi(x, s) dx \\
& \geq \frac{1}{\varepsilon^d} \int_{x_0 + (-\varepsilon, \varepsilon)^d} \frac{1}{b - a} \int_a^b \{ F'(\xi) \cdot D\varphi(x, s) + \text{tr} (A_{ij}(\xi) D^2 \varphi(x, s)) \} d\xi dx.
\end{aligned}$$

Let now $\xi \in (m, M)$ be any Lebesgue point of any arbitrarily chosen a.e. representative of (F', A) . Take first the limit as $a, b \rightarrow \xi$ such that ξ is the center of each $[a, b]$ in order to use the Lebesgue point property; take next the limit as $\varepsilon \rightarrow 0^+$. This gives us that

$$\partial_s \varphi(x_0, s) \geq F'(\xi) \cdot D\varphi(x_0, s) + \text{tr} (A_{ij}(\xi) D^2 \varphi(x_0, s)),$$

for any $x_0 \in \mathbb{R}^d$, $s > 0$, and Lebesgue point ξ . That is φ is a supersolution of (19). This completes the proof of the remaining implication in the case where φ is C_b^∞ (and then $\varphi_\# = \varphi$).

3) (I) \implies (II) for nonnegative BLSC weights φ .

In this case we use the regularization procedure of Lemma 55. By this lemma

$$\varphi_\nu = \varphi *_{x,t} (\rho_\nu \theta_\nu)$$

satisfies (I) since φ does by assumption. By the previous step we deduce that φ_ν is a supersolution of (19a). Hence

$$\varphi_b = \liminf_* \varphi_\nu$$

is also a supersolution by stability (cf. Proposition 42). But to prove (II), we need to show that $\varphi_\#$ is a supersolution. We will do this by showing that $\varphi_b = (\varphi_\#)_*$

pointwise (at least for positive times). We already have $(\varphi\#)_* \leq \varphi_b$ by Lemma 58. To prove that $\varphi_b \leq (\varphi\#)_*$, we need to use (I). By (I),

$$\int_{\mathbb{R}^d} |u - v|(x, t) \varphi(x, s) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x, t + s) \, dx,$$

for any u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$ and corresponding solutions u and v of (1) and $t, s \geq 0$. By Lemma 56(i), we also have that $\varphi_b = \varphi$ a.e. In particular, there is a null set $N \subset \mathbb{R}^+$ such that $\varphi(\cdot, s) = \varphi_b(\cdot, s)$ a.e., for any $s \notin N$.² Fixing $T > 0$, there thus exists a sequence $s_n \rightarrow T^-$ such that $s_n \notin N$, for any n . Choosing moreover $t_n := T - s_n$, we deduce that

$$\int_{\mathbb{R}^d} |u - v|(x, t_n) \varphi_b(x, s_n) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x, T) \, dx.$$

Let us pass to the limit as $n \rightarrow \infty$ in the left-hand side. To do so, we use Fatou's lemma, which is possible because of the lower semicontinuity of φ_b and the continuity of entropy solutions with values in $L^1_{\text{loc}}(\mathbb{R}^d)$ which implies that

$$|u - v|(x, t_n) \rightarrow |u_0 - v_0|(x) \quad \text{for a.e. } x$$

(along a subsequence). In the limit, it then follows that

$$\int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi_b(x, T) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x, T) \, dx$$

for any u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$ and $T > 0$. To continue, we argue as in the previous step where we chose $0 \leq u_0 - v_0 \in L^1(\mathbb{R}^d)$ to be an approximate unit up to a multiplicative constant. The same arguments imply that for any $T > 0$,

$$\varphi_b(\cdot, T) \leq \varphi(\cdot, T) \quad \text{a.e.}$$

By Lemma 56(ii), we conclude that $\varphi_b \leq \varphi\#$ pointwise (for positive times). Hence, $\varphi_b \leq (\varphi\#)_*$ and then $\varphi_b = (\varphi\#)_*$ (for positive times). This implies that $(\varphi\#)_* = \varphi_b$ is a supersolution of (19a). The proof of Theorem 34 is complete. \square

We have now established all preliminary results and are ready to prove our duality results (Corollaries 36 and 38).

Proof of Corollary 36. We already know that $\underline{\varphi} \in \mathscr{W}_{m, M, \varphi_0}$ by Theorem 28. Let us prove the formula with the inf. Take $\varphi \in \mathscr{W}_{m, M, \varphi_0}$, which means that $\varphi \in BLSC$ and satisfies Theorem 34(I) with $\varphi(t = 0) \geq \varphi_0$. By this theorem, φ satisfies (II) as well, that is $\varphi\#$ is a supersolution of (19a). Recall that $\varphi \leq (\varphi\#)_*$ pointwise by Lemma 59(i). In particular

$$(\varphi\#)_*(t = 0) \geq \varphi(t = 0) \geq \varphi_0.$$

Thus $\varphi\#$ is a supersolution of the Cauchy problem (19), and $\underline{\varphi} \leq \varphi\#$ by Proposition 7. Then Lemma 59(ii) implies that $(\underline{\varphi})\# \leq \varphi\#$, and we conclude that

$$(\underline{\varphi})\#(x, t) = \inf \{ \varphi\#(x, t) : \varphi \in \mathscr{W}_{m, M, \varphi_0} \} \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+$$

(with an equality because $\underline{\varphi} \in \mathscr{W}_{m, M, \varphi_0}$). The proof is complete. \square

Proof of Corollary 38. Fix $m < M$. By what precedes, the solution semigroup G_t of (19) is a strongly continuous semigroup of continuous operators on $C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$ and satisfies (22). Let now H_t be another arbitrary such semigroup satisfying (22), i.e. such that

$$\int_{\mathbb{R}^d} |S_t u_0 - S_t v_0| \varphi_0 \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0| H_t \varphi_0 \, dx,$$

²To find N use that $\iint_{\mathbb{R}^d \times \mathbb{R}^+} \mathbf{1}_{\{\varphi_b = \varphi\}} \, dx \, ds = 0 = \int_{\mathbb{R}^+} \text{meas}\{\varphi(\cdot, s) = \varphi_b(\cdot, s)\} \, ds$ by Fubini.

for any u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$, $0 \leq \varphi_0 \in C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$, and $t \geq 0$. We have to prove that for any such φ_0 and t ,

$$G_t \varphi_0 \leq H_t \varphi_0.$$

First the minimal solution of (19) is the unique continuous solution, that is

$$\underline{\varphi}(x, t) = G_t \varphi_0(x) \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+.$$

Moreover, the above assumption on H_t implies that

$$H_t \varphi_0(x) \in \mathcal{W}_{m, M, \varphi_0}.$$

By Corollary 36 we deduce that for any $x \in \mathbb{R}^d$ and $t \geq 0$,

$$(G_t \varphi_0)_\#(x) \leq (H_t \varphi_0)_\#(x),$$

where we recall that

$$(G_t \varphi_0)_\#(x) = \liminf_{\substack{r \rightarrow 0^+ \\ y \rightarrow x}} \frac{1}{\text{meas}(B_r(y))} \int_{B_r(y)} G_t \varphi_0(z) \, dz$$

(and similarly for H). Since both $G_t \varphi_0(x)$ and $H_t \varphi_0(x)$ are continuous in x , we have $(G_t \varphi_0)_\# = G_t \varphi_0$ and $(H_t \varphi_0)_\# = H_t \varphi_0$ pointwise and the proof is complete. \square

4.5. L^∞_{int} versus L^1 : Proofs of Propositions 21, 22, and Theorem 24. Recall that these results justify the use of L^∞_{int} for (2), instead of L^1 . We need a result on the profile $U(r) = c_0 \int_r^\infty e^{-\frac{s^2}{4}} \, ds$ with c_0 such that $U(0) = 1$.

Lemma 60. *For any $(x, t) \in \mathbb{R} \times \mathbb{R}^+$, let*

$$\psi(x, t) := \begin{cases} U(|x|/\sqrt{t}) & \text{if } t > 0, \\ \mathbf{1}_{\{0\}}(x) & \text{if } t = 0. \end{cases}$$

Then $\psi \in \text{BUSC}(\mathbb{R} \times \mathbb{R}^+)$ and is a subsolution of (13).

Proof. Let us prove that ψ is a subsolution of (13). In the domain $\{x \neq 0, t > 0\}$, we find as in the proof of Lemma 50 that

$$\partial_t \psi = \partial_{xx}^2 \psi = (\partial_{xx}^2 \psi)^+$$

in the classical sense. If now $x = 0$, we have

$$\partial_t \psi(0, \cdot) = 0 \leq (\partial_{xx}^2 \psi(0, \cdot))^+$$

since $\psi(0, \cdot)$ is constant in time. Let us now show that ψ is *BUSC*. It is clearly continuous for positive t and it only remains to prove that

$$\mathbf{1}_{x=0} \geq \limsup_{\mathbb{R} \times \mathbb{R}^+ \ni (y, t) \rightarrow (x, 0)} U(|y|/\sqrt{t}),$$

for any $x \in \mathbb{R}$. If $x = 0$, the result follows since $U(r) \leq U(0) = 1$ for any $r \geq 0$. If $x \neq 0$, then we use that

$$|y|/\sqrt{t} \rightarrow \infty \quad \text{as } (y, t) \rightarrow (x, 0^+)$$

together with the fact that $\lim_{r \rightarrow \infty} U(r) = 0$. The proof of Lemma 60 is now complete. \square

Proof of Proposition 21. Theorem 26 implies the if-part. Let us prove the only-if-part. It is based on the following pointwise lower bound:

$$(38) \quad \varphi(x, t) \geq U\left(\frac{1}{\sqrt{t}}\right) \sup_{x+[-1, 1]} \varphi_0 \quad \forall x \in \mathbb{R}, \forall t > 0,$$

where U is the profile from the previous lemma, $0 \leq \varphi_0 \in C_b(\mathbb{R})$ and φ is the solution of (13) with φ_0 as initial data. Let us prove (38). Fix x and t . The sup on the right-hand side is attained at some $x_0 \in x + [-1, 1]$. By the previous lemma,

$$(y, s) \mapsto \varphi_0(x_0)U(|y - x_0|/\sqrt{s})$$

is a *BUSC* subsolution of (13). At $s = 0$, it equals the function

$$y \mapsto \varphi_0(x_0)\mathbf{1}_{\{x_0\}}(y)$$

which is less or equal to $\varphi_0 = \varphi_0(y)$. By the comparison principle (Theorem 3),

$$\varphi(y, s) \geq \varphi_0(x_0)U(|y - x_0|/\sqrt{s}) \quad \forall y \in \mathbb{R}, \forall s > 0.$$

Taking $(y, s) = (x, t)$, we then get that

$$\varphi(x, t) \geq \underbrace{\varphi_0(x_0)}_{= \sup_{x+[-1,1]} \varphi_0} \underbrace{U(|x - x_0|/\sqrt{t})}_{\geq U(1/\sqrt{t})}.$$

This completes the proof of (38). From that bound the only-if-part of Proposition 21 is obvious since $U(1/\sqrt{t})$ is positive for $t > 0$. \square

Proof of Proposition 22. Let $x_0 \in \mathbb{R}$ and $c > 0$ be such that

$$\rho \geq c\mathbf{1}_{\{x_0\}},$$

where ρ is defined in (14), and define

$$\psi_n(x, t) := nc\psi(nx - x_0, n^2t),$$

where ψ is given by Lemma 60. It is easy to see that ψ_n remains a subsolution of (13). Moreover, it is *BUSC* with

$$\varphi_n(x, 0) \geq \psi_n(x, 0) \quad \forall x \in \mathbb{R},$$

by (14). Hence $\varphi_n \geq \psi_n$ by the comparison principle and it suffices to show that

$$\lim_{n \rightarrow \infty} \psi_n(x, t) = \infty \quad \forall x \in \mathbb{R}, \forall t > 0.$$

But this is quite easy because

$$\psi_n(x, t) = ncU\left(\left|x - \frac{x_0}{n}\right|/\sqrt{t}\right),$$

for any $x \in \mathbb{R}$ and $t > 0$, and both the constant c and the profile $U(\cdot)$ are positive. The proof of Proposition 22 is complete. \square

To show Theorem 24, we need the following lemma whose proof is elementary and left to the reader.

Lemma 61. *For any $\varphi_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\sup|\varphi_0| \leq |\sup \varphi_0| + |\inf \varphi_0|$.*

Proof of Theorem 24. The fact that $E = C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ satisfies (16)–(17) follows from Theorem 16 and Corollary 27. Let now E be another normed space satisfying such properties and let us prove that it is continuously embedded into $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$. Recall that (17) is required to hold for any data $b = b(\xi)$ and $a = a(\xi)$ satisfying (H2). Choose e.g. the eikonal equation

$$\partial_t \varphi = \sum_{i=1}^d |\partial_{x_i} \varphi|$$

and denote by G_t^e its semigroup. By the representation Proposition 9,

$$G_t^e \varphi_0(x) = \sup_{x+t[-1,1]^d} \varphi_0.$$

Since $G_{t=1}^e$ maps $E \subseteq C_b \cap L^1(\mathbb{R}^d)$ into $X = \overline{E}^{\|\cdot\|_E} \subset L^1(\mathbb{R}^d)$ by assumption, the function

$$x \mapsto \sup_{x+[-1,1]^d} \varphi_0$$

belongs to $L^1(\mathbb{R}^d)$ for any $\varphi_0 \in E$. Using that E is a vector space, $-\varphi_0 \in E$, and the function

$$x \mapsto \inf_{x+[-1,1]^d} \varphi_0$$

also belongs to $L^1(\mathbb{R}^d)$. By Lemma 61, we conclude that $E \subseteq C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$. Finally we use that $G_{t=1}^e : E \rightarrow X$ is continuous at $\varphi_0 \equiv 0$ to obtain that for any $\|\varphi_0^n\|_E \rightarrow 0$, as $n \rightarrow \infty$, we have $G_{t=1}^e \varphi_0^n \rightarrow 0$ in X . Combining this with the continuity of the inclusion $X \subset L^1(\mathbb{R}^d)$, we obtain that

$$\left\| \sup_{x+[-1,1]^d} \varphi_0^n \right\|_{L_x^1} \rightarrow 0.$$

Using once again that E is a normed space, the same holds with $-\varphi_0$, that is

$$\left\| \inf_{x+[-1,1]^d} \varphi_0^n \right\|_{L_x^1} \rightarrow 0.$$

By Lemma 61, we conclude that $\|\varphi_0^n\|_{\text{int}} \rightarrow 0$ which completes the proof. \square

ACKNOWLEDGEMENTS

E. R. Jakobsen is supported by the Research Council of Norway under grant agreement no. 325114 “IMod. Partial differential equations, statistics and data: An interdisciplinary approach to data-based modelling.” J. Endal was supported by the Research Council of Norway under the Toppforsk (research excellence) grant agreement no. 250070 “Waves and Nonlinear Phenomena (WaNP).” J. Endal received funding from the Research Council of Norway under the Toppforsk (research excellence) grant agreement no. 250070 “Waves and Nonlinear Phenomena (WaNP),” the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement no. 839749 “Novel techniques for quantitative behavior of convection-diffusion equations (techFRONT),” and from the Research Council of Norway under the MSCA-TOPP-UT grant agreement no. 312021. N. Alibaud was supported by the French ANR project CoToCoLa (no. ANR-11-JS01-006-01). Part of this paper was completed the first half of 2016 during J. Endal’s stay at Laboratory of Mathematics of Besançon (LMB), University of Bourgogne Franche-Comté (UBFC), and National Engineering Institute in Mechanics and Microtechnologies (ENSMM). We also thank Boris Andreianov for pointing out [41] to us and for other useful comments.

APPENDIX A. COMPLEMENTARY PROOFS FOR VISCOSITY SOLUTIONS

A.1. Minimal viscosity solutions. Here are the proofs of Theorem 6 and Propositions 7 and 46; the ideas are inspired by [25, 10, 31] and the details are given for completeness.

Proof of Theorem 6. Consider the infconvolution

$$(39) \quad (\varphi_0)_\varepsilon(x) := \inf_{y \in \mathbb{R}^d} \left\{ \varphi_0(y) + \frac{|x-y|^2}{2\varepsilon^2} \right\},$$

which is at least C_b with $\inf \varphi_0 \leq (\varphi_0)_\varepsilon \leq (\varphi_0)_* \leq \sup \varphi_0$, and

$$\lim_{\varepsilon \downarrow 0} \uparrow (\varphi_0)_\varepsilon = \sup_{\varepsilon > 0} (\varphi_0)_\varepsilon = (\varphi_0)_*,$$

see e.g. [23, 29, 5, 4]. Let φ_ε be the viscosity solution of (2a) with initial data $(\varphi_0)_\varepsilon$, whose well-posedness is ensured by Theorem 4. By the maximum principle, see Remark 5, we have the bounds

$$\inf \varphi_0 \leq \varphi_\varepsilon \leq \sup \varphi_0.$$

We can then define the real-valued and bounded function

$$\underline{\varphi} := \sup_{\varepsilon > 0} \varphi_\varepsilon.$$

We will see that this is our desired minimal solution.

The key step is to prove that

$$(40) \quad \underline{\varphi} = \sup_{\varepsilon > 0} \varphi_\varepsilon = \liminf_* \varphi_\varepsilon$$

where the relaxed limit is taken as $\varepsilon \rightarrow 0^+$. This follows by elementary arguments (see e.g. [5, 4]) since φ^ε is at least lower semicontinuous and nondecreases as $\varepsilon \downarrow 0$, which follows by comparison since $(\varphi_0)^\varepsilon$ nondecreases as $\varepsilon \downarrow 0$. Let us give details for the reader's convenience. For any fixed (x, t) ,

$$\liminf_* \varphi_\varepsilon(x, t) \leq \lim_{\varepsilon \rightarrow 0^+} \varphi_\varepsilon(x, t) = \underline{\varphi}(x, t).$$

Moreover, for any sequence $(x_n, t_n, \varepsilon_n) \rightarrow (x, t, 0^+)$ such that $\varepsilon_n \leq \varepsilon_m$ for any $n \geq m$, we have $\varphi_{\varepsilon_n}(x_n, t_n) \geq \varphi_{\varepsilon_m}(x_n, t_n)$. Fixing m and taking the limit in n ,

$$\liminf_{n \rightarrow \infty} \varphi_{\varepsilon_n}(x_n, t_n) \geq \liminf_{n \rightarrow \infty} \varphi_{\varepsilon_m}(x_n, t_n) \geq \varphi_{\varepsilon_m}(x, t)$$

by lower semicontinuity of φ_{ε_m} . Taking the limit in m ,

$$\liminf_{n \rightarrow \infty} \varphi_{\varepsilon_n}(x_n, t_n) \geq \lim_{m \rightarrow \infty} \varphi_{\varepsilon_m}(x, t) = \underline{\varphi}(x, t).$$

This proves (40).

By stability by sup (Proposition 41), $\underline{\varphi}$ is a subsolution of (2a), and by stability by relaxed limit (Proposition 42), $\underline{\varphi}$ is a supersolution of (2a). To pass to the limit in the initial data, use Proposition 44 to infer that

$$(\underline{\varphi})^*(x, t = 0) \leq \{\limsup^* \varphi_\varepsilon\}(x, 0) = \{\limsup^* \varphi_\varepsilon(\cdot, 0)\}(x) \leq (\varphi_0)^*(x)$$

(the first limsup being taken in (x, t) and the second in x). This gives the inequality of subsolution as in Definition 1(aii). For the other inequality, use that $\underline{\varphi}$ is lower semicontinuous, as a sup of continuous functions, with

$$\underline{\varphi}(x, t = 0) = \sup_{\varepsilon > 0} \varphi_\varepsilon(x, 0) = (\varphi_0)_*(x).$$

This proves that $\underline{\varphi}$ is a solution of (2). It only remains to prove that it is minimal. Let φ be another bounded discontinuous solution. Noting that

$$(\varphi_0)_\varepsilon \leq (\varphi_0)_* \leq \varphi_*(t = 0),$$

we use once more the comparison principle to deduce that $\varphi_\varepsilon \leq \varphi$, for any $\varepsilon > 0$, so $\underline{\varphi} \leq \varphi$ as $\varepsilon \rightarrow 0^+$. \square

Proof of Proposition 7. We argue as in the end of the proof of Theorem 6: Assume φ is a bounded supersolution of (2), then $(\varphi_0)_\varepsilon \leq (\varphi_0)_* \leq \varphi_*(t = 0)$ and, by comparison, $\varphi_\varepsilon \leq \varphi$, etc. \square

Proof of Proposition 46. Let $\underline{\varphi}$ denote the minimal solution of (2) with initial data $\varphi_0 := \sup_n (\varphi_0^n)_*$. We have to prove that $\underline{\varphi} = \sup_n \underline{\varphi}_n$, where $\underline{\varphi}_n$ is the minimal solution of (2) with initial data φ_0^n . By Proposition 7, we have $\underline{\varphi}_n \leq \underline{\varphi}$ for any integer n . We thus already know that $\underline{\varphi} \geq \sup_n \underline{\varphi}_n$ and it only remains to prove the other inequality. To do so, it suffices to show that $\sup_n \underline{\varphi}_n$ is a supersolution of (2) (with initial data φ_0). Indeed, by Proposition 7, we then get $\underline{\varphi} \leq \sup_n \underline{\varphi}_n$.

It is at this stage that we need to use monotonicity. Recall that $n \mapsto \varphi_0^n(x)$ is nondecreasing for any x . By the comparison principle, cf. Remark 8, the same monotonicity holds for the minimal solutions which means that $n \mapsto \underline{\varphi}_n(x, t)$ is nondecreasing for any fixed x and t . Since $\underline{\varphi}_n$ is lower semicontinuous, we can argue as for (40) and get that

$$\sup_n \underline{\varphi}_n = \liminf_* \underline{\varphi}_n \quad \text{as } n \rightarrow \infty.$$

By stability, see Propositions 42 and 44, we deduce that $\liminf_* \underline{\varphi}_n$ is a supersolution of (2a) with initial data

$$\liminf_* \underline{\varphi}_n(t=0) = \liminf_*(\varphi_0^n)_*.$$

But this initial data is precisely

$$\liminf_*(\varphi_0^n)_* = \sup_n (\varphi_0^n)_* = \varphi_0,$$

again by similar arguments than for (40). This completes the proof. \square

A.2. Representation formulas. Let us prove Propositions 9 and 10. These results are classical in control theory, but usually written for continuous or maximal solutions, see [29, 4, 32, 33]. Here we give the proofs for minimal solutions.

Proof of Proposition 9. By the assumption that $a \equiv 0$, (2a) is now

$$\partial_t \varphi = \sup_{\xi \in \mathcal{E}} \{b(\xi) \cdot D\varphi\} = \sup_{q \in \mathcal{C}} \{q \cdot D\varphi\},$$

where $\mathcal{C} = \overline{\text{co}\{\text{Im}(b)\}}$ is compact. By control theory [5, 4] the viscosity solutions of (2) is given by

$$\varphi(x, t) = \sup_{x+t\mathcal{C}} \varphi_0$$

if φ_0 is bounded and uniformly continuous. In the general case, consider the infconvolution (39). Recall that $(\varphi_0)_\varepsilon$ is at least bounded and uniformly continuous, and $(\varphi_0)_\varepsilon \uparrow (\varphi_0)_*$ pointwise as $\varepsilon \downarrow 0$. It follows that the solution of (2a) with $(\varphi_0)_\varepsilon$ as initial data is

$$\varphi_\varepsilon(x, t) = \sup_{x+t\mathcal{C}} (\varphi_0)_\varepsilon.$$

By Proposition 46, the minimal solution of (2) is thus

$$\underline{\varphi}(x, t) = \sup_{\varepsilon > 0} \varphi_\varepsilon(x, t) = \sup_{\varepsilon > 0} \sup_{x+t\mathcal{C}} (\varphi_0)_\varepsilon = \sup_{x+t\mathcal{C}} \sup_{\varepsilon > 0} (\varphi_0)_\varepsilon = \sup_{x+t\mathcal{C}} (\varphi_0)_*.$$

Rigorously speaking, Proposition 46 implies that this is the minimal solution with initial data $(\varphi_0)_*$, but it coincides with the minimal solution associated to φ_0 by Proposition 7. \square

Proof of Proposition 10. Equation (2a) is given by

$$\partial_t \varphi = \sup_{\xi \in \mathcal{E}} \{b(\xi) \cdot D\varphi + \text{tr}(\sigma^a(\xi)(\sigma^a)^\top(\xi)D^2\varphi)\},$$

where \mathcal{E} is compact and the coefficients b and σ^a are continuous by (10). By stochastic control theory [29], the viscosity solution of (2) is given by

$$\varphi(x, t) = \sup_{\xi} \mathbb{E} \{\varphi_0(\mathbf{X}_t^x)\}$$

if φ_0 is bounded and uniformly continuous, where ξ_s and \mathbf{X}_s^x are defined as in Proposition 10. Let us now repeat the argument of the proof of Proposition 9 considering the infconvolution $(\varphi_0)_\varepsilon$ and the corresponding solution of (2a)

$$\varphi_\varepsilon(x, t) = \sup_{\xi} \mathbb{E} \{(\varphi_0)_\varepsilon(\mathbf{X}_t^x)\}.$$

We find that the minimal solution of (2) is

$$\underline{\varphi}(x, t) = \sup_{\varepsilon > 0} \varphi_\varepsilon(x, t) = \sup_{\xi} \sup_{\varepsilon > 0} \mathbb{E} \{ (\varphi_0)_\varepsilon(\mathbf{X}_t^x) \}.$$

Since $(\varphi_0)_\varepsilon \uparrow (\varphi_0)_*$ as $\varepsilon \downarrow 0$, we conclude the proof using the monotone convergence theorem:

$$\begin{aligned} & \sup_{\varepsilon > 0} \mathbb{E} \{ (\varphi_0)_\varepsilon(\mathbf{X}_t^x) \} \\ &= \lim_{\varepsilon \downarrow 0} \uparrow \mathbb{E} \{ (\varphi_0)_\varepsilon(\mathbf{X}_t^x) \} = \mathbb{E} \left\{ \lim_{\varepsilon \downarrow 0} \uparrow (\varphi_0)_\varepsilon(\mathbf{X}_t^x) \right\} = \mathbb{E} \{ (\varphi_0)_*(\mathbf{X}_t^x) \}. \quad \square \end{aligned}$$

APPENDIX B. COMPLEMENTARY PROOFS FOR ENTROPY SOLUTIONS

For completeness, we recall the proof of Theorem 13 which is Theorem 1.1 in [30] under (H1). We will take the opportunity to give details, but we will not perform the doubling of variables to show Lemma 15 for which we will refer to [11].

Recall that [21, 11] proved the well-posedness of L^1 kinetic or renormalized solutions which are equivalent to entropy solutions in $L^1 \cap L^\infty$. The definition of entropy solutions in $L^1 \cap L^\infty$ uses the energy estimate (2.8) of [21],

$$\iint_{\mathbb{R}^d \times \mathbb{R}^+} \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) \right)^2 dx dt \leq \frac{1}{2} \|u_0\|_{L^2} < \infty \quad \text{if } u_0 \in L^1 \cap L^\infty,$$

where $\zeta_{ik}(u) = \int_0^u \sigma_{ik}^A(\xi) d\xi$. As a consequence “ L^2 ” was used e.g. in [11, Definition 2.2] instead of “ L_{loc}^2 ” in Definition 12. But we have the following result:

Lemma 62 (Local energy estimate). *Assume (H1), $u_0 \in L^\infty(\mathbb{R}^d)$, $0 \leq \phi \in C_c^\infty(\mathbb{R}^d)$, and $T \geq 0$. If u is an entropy solution of (1) in the sense of Definition 12 and*

$$\|u_0\|_{L^\infty} + \|u\|_{L^\infty} + \|\phi\|_{W^{2,1}} \leq M,$$

then there is a constant C only depending on T, M, F and A such that

$$\iint_{\mathbb{R}^d \times (0, T)} \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u(x, t)) \right)^2 \phi(x) dx dt \leq C.$$

Proof. We use Definition 12(c) with the entropy $\eta(u) = |u|^2$ and the corresponding fluxes

$$q(u) = 2 \int_0^u \xi F'(\xi) d\xi \quad \text{and} \quad r(u) = 2 \int_0^u \xi A(\xi) d\xi.$$

We also take a test function $\phi(x) \mathbf{1}_{[0, T]}(t)$ where $0 \leq \phi \in C_c^\infty(\mathbb{R}^d)$. It is not smooth in time but a standard approximation argument shows that it can be used in Definition 12(c) if we add also a final value term at $t = T$. Here we need the L_{loc}^1 continuity in time of entropy solutions. The result is

$$\begin{aligned} & \overbrace{\int_{\mathbb{R}^d} u^2(x, T) \phi(x) dx}^{\geq 0} + 2 \iint_{\mathbb{R}^d \times (0, T)} \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) \right)^2 \phi dx dt \\ & \leq \int_{\mathbb{R}^d} u_0^2(x) \phi(x) dx + \iint_{\mathbb{R}^d \times (0, T)} \left(\sum_{i=1}^d q_i(u) \partial_{x_i} \phi + \sum_{i,j=1}^d r_{ij}(u) \partial_{x_i x_j}^2 \phi \right) dx dt. \end{aligned}$$

By assumption $\|u_0\|_{L^\infty} + \|u\|_{L^\infty} + \|\phi\|_{W^{2,1}} \leq M$, so it follows that

$$\begin{cases} \|q(u)\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^+, \mathbb{R}^d)} \leq 2M^2 \text{ess sup}_{-M \leq \xi \leq M} |F'(\xi)|, \text{ and} \\ \|r(u)\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^+, \mathbb{R}^d \times \mathbb{R}^d)} \leq 2M^2 \text{ess sup}_{-M \leq \xi \leq M} |A(\xi)|. \end{cases}$$

With all these estimates, the conclusion readily follows. \square

Let us now give precise references on how to show the Kato inequality.

Sketch of the proof of Lemma 15. Copy the proof of Theorem 3.1 of [11] with $l = \infty$ and zero renormalization measures $\mu_l^u \equiv 0 \equiv \mu_l^v$. With the aid of the previous local energy estimate, check that every computation holds until (3.19) – even if u and v satisfy (a)–(b) of Definition 12 with L_{loc}^2 and not L^2 as in [11]. This gives (11) with $\phi \in C_c^\infty(\mathbb{R}^d \times (0, \infty))$. Use an approximation argument for $\phi(x, t)\mathbf{1}_{[0, T]}(t)$ and the continuity in time with values in L_{loc}^1 to get initial and final terms. \square

To show the uniqueness of entropy solutions, it suffices to find a *good* ϕ in (11), e.g. an exponential as in [20, 30]. This gives the result below.

Lemma 63. *Assume (H1) and u, v are L^∞ entropy solutions of (1) with initial data $u_0, v_0 \in L^\infty(\mathbb{R}^d)$. Then for any $t \geq 0$ and $m < M$ such that u and v take their values in $[m, M]$,*

$$\int_{\mathbb{R}^d} |u - v|(x, t) e^{-|x|} dx \leq e^{(L_F + L_A)t} \int_{\mathbb{R}^d} |u_0 - v_0|(x) e^{-|x|} dx,$$

where $L_F = \text{ess sup}_{[m, M]} |F'|$ and $L_A = \text{ess sup}_{[m, M]} \text{tr}(A)$.

Remark 64. By the maximum principle, the result remains true for any $[m, M]$ containing the values u_0 and v_0 . But at this stage of this appendix, this principle is only known in $L^1 \cap L^\infty$ (or L^1) by [21, 11] and it will follow later in L^∞ .

Sketch of the proof. The proof is inspired by [20, 30]. Consider

$$\phi_\varepsilon(x, t) := e^{(L_F + L_A)(T-t) - \sqrt{\varepsilon^2 + |x|^2}},$$

for some arbitrary $\varepsilon > 0$, and check that

$$\begin{aligned} |u - v| \partial_t \phi_\varepsilon + \sum_{i=1}^d q_i(u, v) \partial_{x_i} \phi_\varepsilon + \sum_{i,j=1}^d r_{ij}(u, v) \partial_{x_i x_j}^2 \phi_\varepsilon \\ \leq |u - v| \left\{ \partial_t \phi_\varepsilon + L_F |D\phi_\varepsilon| + L_A \sup_{\lambda \in \text{Sp}(D^2 \phi_\varepsilon)} \lambda^+ \right\} \leq 0 \end{aligned}$$

by the Ky Fan inequality (27). Then by the Kato inequality (11) with ϕ_ε ,

$$\int_{\mathbb{R}^d} |u - v|(x, T) e^{-\sqrt{\varepsilon^2 + |x|^2}} dx \leq e^{(L_F + L_A)T} \int_{\mathbb{R}^d} |u_0 - v_0|(x) e^{-\sqrt{\varepsilon^2 + |x|^2}} dx$$

and the result follows in the limit $\varepsilon \rightarrow 0^+$. \square

Proof of Theorem 13. By Lemma 63, it remains to show the existence. The proof is inspired by [21, 11]. Given $u_0 \in L^\infty(\mathbb{R}^d)$, take $(u_0^n)_n$ in $L^1 \cap L^\infty(\mathbb{R}^d)$ such that

$$(41) \quad -\text{ess sup } u_0^- \leq u_0^n \leq \text{ess sup } u_0^+ \quad \text{and} \quad u_0^n \rightarrow u_0 \quad \text{in } L_{\text{loc}}^1(\mathbb{R}^d).$$

Let u_n be the entropy solution of (1) with initial data u_0^n . By the maximum principle (in $L^1 \cap L^\infty$), we know that

$$(42) \quad -\text{ess sup } u_0^- \leq u_n \leq \text{ess sup } u_0^+.$$

Moreover, by Lemma 63, we have for any $R \geq 0$, $T \geq 0$, and integers n, m ,

$$\begin{aligned} & \|u_m - u_n\|_{C([0,T];L^1(\{|x|<R\}))} \\ &= \sup_{t \in [0,T]} \int_{|x|<R} |u_m(x,t) - u_n(x,t)| \, dx \\ &\leq e^R \sup_{t \in [0,T]} \int_{\mathbb{R}^d} |u_m(x,t) - u_n(x,t)| e^{-|x|} \, dx \\ &\leq e^R e^{(L_F+L_A)T} \int_{\mathbb{R}^d} |u_0^m(x) - u_0^n(x)| e^{-|x|} \, dx, \end{aligned}$$

where the latter integral tends to zero as $n, m \rightarrow \infty$ by (41). Hence there exists some $u \in L^\infty(\mathbb{R}^d \times \mathbb{R}^+) \cap C(\mathbb{R}^+; L^1_{\text{loc}}(\mathbb{R}^d))$ such that

$$(43) \quad \lim_{n \rightarrow \infty} u_n = u \quad \text{in } C([0, T]; L^1_{\text{loc}}(\mathbb{R}^d)), \quad \forall T \geq 0.$$

It remains to show that u is an entropy solution with initial data u_0 .

We have to derive the L^2_{loc} energy estimate of Definition 12(a), and check that it is enough to pass to the limit in the equation as in [21, 11]. By Lemma 62 and the L^∞ bounds in (42), the sequence

$$\left\{ \sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u_n) \right\} \subset L^2(\mathbb{R}^d \times \mathbb{R}^+)$$

is uniformly bounded in $L^2(\mathcal{K})$, for any $k = 1, \dots, K$, and compact $\mathcal{K} \subset \mathbb{R}^d \times \mathbb{R}^+$. It then weakly converges in $L^2(\mathcal{K})$ to $\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u)$. We can identify the limit because $\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u_n)$ also converges to $\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u)$ in the distribution sense. Indeed

$$\zeta_{ik}(\cdot) = \int_0^\cdot \sigma_{ik}^A(\xi) \, d\xi$$

is locally Lipschitz continuous since $\sigma_{ik}^A(\cdot)$ is locally bounded, and (42) and (43) imply that $\zeta_{ik}(u_n) \rightarrow \zeta_{ik}(u)$ in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^d))$ for all $T \geq 0$. And as claimed previously, all corresponding derivatives necessarily converge in the distribution sense. The proof of part (a) in Definition 12 is complete. Moreover we have found that

$$\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u_n) \rightharpoonup \sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) \quad \text{in } L^2(\mathcal{K}),$$

for any $k = 1, \dots, K$ and compact $\mathcal{K} \subset \mathbb{R}^d \times \mathbb{R}^+$.

To show the chain rule in part (b) of Definition 12, we start from the chain rule for u_n ,

$$(44) \quad \sum_{i=1}^d \partial_{x_i} \zeta_{ik}^\beta(u_n) = \beta(u_n) \sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u_n) \in L^2(\mathbb{R}^d \times \mathbb{R}^+),$$

valid for any $\beta \in C(\mathbb{R})$, $k = 1, \dots, K$, and integer n . Recall also that

$$\zeta_{ik}^\beta(u_n) = \int_0^{u_n} \sigma_{ik}^A(\xi) \beta(\xi) \, d\xi.$$

By the previous convergence results and bounds, the right-hand side of (44) converges weakly to $\beta(u) \sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u)$ in $L^2(\mathcal{K})$. We can argue as before to show that the left-hand side converges weakly in $L^2(\mathcal{K})$ to $\sum_{i=1}^d \partial_{x_i} \zeta_{ik}^\beta(u)$. We thus get part (b) of Definition 12 in the limit. Moreover,

$$(45) \quad \sum_{i=1}^d \partial_{x_i} \zeta_{ik}^\beta(u_n) \rightharpoonup \sum_{i=1}^d \partial_{x_i} \zeta_{ik}^\beta(u) \quad \text{in } L^2(\mathcal{K}),$$

for any $\beta \in C(\mathbb{R})$, $k = 1, \dots, K$, and compact $\mathcal{K} \subset \mathbb{R}^d \times \mathbb{R}^+$.

Now, it remains to prove part (c) of Definition 12. The only difference with [21, 11] is that the previous convergences hold locally in L^2 and not globally. But since we use test functions, the reasoning is the same. Let us recall it for completeness. We focus on the quadratic term. Take $\beta = \sqrt{\eta''}$ and apply the chain rule Definition 12(b),

$$\begin{aligned}
& \iint_{\mathbb{R}^d \times \mathbb{R}^+} \eta''(u_n) \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u_n) \right)^2 \phi \, dx \, dt \\
&= \iint_{\mathbb{R}^d \times \mathbb{R}^+} \eta''(u_n) \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u_n) \right) \left(\sum_{j=1}^d \partial_{x_j} \zeta_{jk}(u_n) \right) \phi \, dx \, dt \\
&= \iint_{\mathbb{R}^d \times \mathbb{R}^+} \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}^{\sqrt{\eta''}}(u_n) \right) \left(\sum_{j=1}^d \partial_{x_j} \zeta_{jk}^{\sqrt{\eta''}}(u_n) \right) \phi \, dx \, dt \\
&= \iint_{\mathbb{R}^d \times \mathbb{R}^+} \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}^{\sqrt{\eta''}}(u_n) \sqrt{\phi} \right)^2 \, dx \, dt \\
&= \sum_{k=1}^K \left\| \sum_{i=1}^d \partial_{x_i} \zeta_{ik}^{\sqrt{\eta''}}(u_n) \sqrt{\phi} \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}^+)}^2.
\end{aligned}$$

But, by (45), we have for any $k = 1, \dots, K$,

$$\sum_{i=1}^d \partial_{x_i} \zeta_{ik}^{\sqrt{\eta''}}(u_n) \sqrt{\phi} \rightharpoonup \sum_{i=1}^d \partial_{x_i} \zeta_{ik}^{\sqrt{\eta''}}(u) \sqrt{\phi} \quad \text{in } L^2(\mathbb{R}^d \times \mathbb{R}^+).$$

It follows that

$$\left\| \sum_{i=1}^d \partial_{x_i} \zeta_{ik}^{\sqrt{\eta''}}(u) \sqrt{\phi} \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}^+)} \leq \liminf_{n \rightarrow \infty} \left\| \sum_{i=1}^d \partial_{x_i} \zeta_{ik}^{\sqrt{\eta''}}(u_n) \sqrt{\phi} \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}^+)},$$

that is

$$\begin{aligned}
& \liminf_{n \rightarrow \infty} \iint_{\mathbb{R}^d \times \mathbb{R}^+} \eta''(u_n) \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u_n) \right)^2 \phi \, dx \, dt \\
& \geq \sum_{k=1}^K \left\| \sum_{i=1}^d \partial_{x_i} \zeta_{ik}^{\sqrt{\eta''}}(u) \sqrt{\phi} \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}^+)}^2 \\
& = \iint_{\mathbb{R}^d \times \mathbb{R}^+} \eta''(u) \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) \right)^2 \phi \, dx \, dt,
\end{aligned}$$

where similar chain rule computations have been used for u . This is enough to pass to the limit in the entropy inequalities of Definition 12(b) and the proof is complete. \square

As a byproduct of the previous proof, we get the lemma below.

Lemma 65. *Assume (H1), $u_0 \in L^\infty(\mathbb{R}^d)$, and u is the entropy solution of (1). Then $\text{ess inf } u_0 \leq u \leq \text{ess sup } u_0$. Moreover, if v is the entropy solution with initial data v_0 , then $u_0 \geq v_0$ implies $u \geq v$.*

Proof. For the comparison principle, define $u_0^n(x) := u_0(x) \mathbf{1}_{|x| < n}$ and v_0^n similarly. As previously, the associated entropy solutions u_n and v_n respectively converge

towards u and v in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^d))$, $T \geq 0$, and thus a.e. up to taking a (common) subsequence. If $u_0 \geq v_0$, then $u_0^n \geq v_0^n$ for all n , so $u_n \geq v_n$ by the comparison principle in $L^1 \cap L^\infty$, and $u \geq v$ at the limit. For the maximum principle, apply the comparison principle to $v_0 := \text{ess inf } u_0$ and $\text{ess sup } u_0$ successively. \square

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