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OPTIMAL AND DUAL STABILITY RESULTS FOR L^1 VISCOSITY AND L^∞ ENTROPY SOLUTIONS

NATHAËL ALIBAUD, JØRGEN ENDAL, AND ESPEN R. JAKOBSEN

ABSTRACT. This paper has three contribution: (i) nonstandard and optimal contraction results for L^1 viscosity solutions of the Hamilton-Jacobi-Bellman equation

$$\partial_t \varphi = \sup_{\xi} \{b(\xi) \cdot D\varphi + \text{tr}(a(\xi)D^2\varphi)\},$$

(ii) nonstandard and optimal contraction results for L^∞ entropy solutions of the anisotropic degenerate parabolic equation

$$\partial_t u + \text{div}F(u) = \text{div}(A(u)Du),$$

and (iii) rigorous identification of a new duality relation between these notions of generalized solutions. More precisely, we obtain a quasicontraction principle for the first equation in the weakest possible L^1 type Banach setting where stability holds in general. We rigorously identify this topology and show that our contraction estimate is optimal for model HJB equations. For the second equation, we obtain a weighted L^1 contraction principle for possibly nonintegrable L^∞ solutions. Here we identify the optimal weight in general. It is interestingly a solution of the first equation, and it is the precise formulation of this result that leads to the above mentioned duality.

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1. INTRODUCTION

In this paper we derive nonstandard stability and duality results for two central notions of weak solutions of nonlinear PDEs: entropy and viscosity solutions. These solution concepts were introduced for scalar conservation laws [39] and Hamilton-Jacobi (HJ) equations [23], and later extended to second order PDEs [38, 37, 19, 21]. Conservation laws are divergence form equations arising in continuum physics [25], while HJ equations are nondivergence form equations e.g. from differential geometry and optimal control theory [30, 5, 4]. The well-posedness of such PDEs is a challenging issue that requires entropy and viscosity solution theories in general. By now the literature is enormous and includes lots of applications. For reference books and the state-of-the-art, see [30, 26, 5, 4, 47, 25, 22].

The problem. Let us consider two model initial-value problems for the Hamilton-Jacobi-Bellman (HJB) equation

$$(1a) \quad \partial_t \varphi = \sup_{\xi \in \mathcal{E}} \{b(\xi) \cdot D\varphi + \text{tr}(a(\xi)D^2\varphi)\} \quad x \in \mathbb{R}^d, t > 0,$$

$$(1b) \quad \varphi(x, 0) = \varphi_0(x) \quad x \in \mathbb{R}^d,$$

and the anisotropic degenerate parabolic convection-diffusion equation

$$(2) \quad \begin{aligned} \partial_t u + \text{div}F(u) &= \text{div}(A(u)Du) & x \in \mathbb{R}^d, t > 0, \\ u(x, 0) &= u_0(x) & x \in \mathbb{R}^d, \end{aligned}$$

where “ D ,” “ D^2 ” and “ div ” respectively denote the gradient, the Hessian and the divergence in x , and “ tr ” is the trace. We assume that

$$(H1) \quad \begin{cases} \mathcal{E} \text{ is a nonempty set,} \\ b : \mathcal{E} \rightarrow \mathbb{R}^d \text{ a bounded function,} \\ a = \sigma^a (\sigma^a)^\text{T} \text{ for some bounded } \sigma^a : \mathcal{E} \rightarrow \mathbb{R}^{d \times K}, \end{cases}$$

$$(H2) \quad F \in W_{\text{loc}}^{1,\infty}(\mathbb{R}, \mathbb{R}^d) \quad \text{and} \quad A = \sigma^A (\sigma^A)^\text{T} \quad \text{for} \quad \sigma^A \in L_{\text{loc}}^\infty(\mathbb{R}, \mathbb{R}^{d \times K}),$$

where K is the maximal rank of $a(\xi)$ and $A(u)$. By [38, 37, 21], (1) and (2) are respectively well-posed in the sense of viscosity and entropy solutions and satisfy the following contraction principles:

$$(3) \quad \|(\varphi - \psi)(t)\|_\infty \leq \|\varphi_0 - \psi_0\|_\infty \quad \text{and} \quad \|(u - v)(t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1}.$$

Here we derive contraction estimates in the opposite settings: L^1 for (1) and L^∞ for (2). Our results are quantitative and optimal in a sense to be specified. We also connect (1) and (2) revealing new duality relations between entropy and viscosity solutions.

Optimal contraction results for (1) in L^1 . For previous results on L^1 viscosity solutions, see [17] for uniformly elliptic/parabolic PDEs and [41, 2, 1, 6, 16, 18, 27] for various other integro-PDEs. For fully nonlinear second order degenerate PDEs, we only know a technical L^1 estimate from [27] for exponentially decaying initial data. Here we develop a general L^1 theory for (1).

We first show that L^1 is too weak to get stability for model HJB equations, including pure diffusive PDEs, cf. Propositions 22–26. We then consider the smallest Banach topology stronger than L^1 for which we have stability for all equations of the form (1a). We show that it is generated by the norm

$$\|\varphi_0\|_{\text{int}} := \int \sup_{x+[-1,1]^d} |\varphi_0| \, dx$$

(cf. Theorem 28) which is the norm of L_{int}^∞ as defined in [2, 1]. Since $L_{\text{int}}^\infty \rightarrow L^1 \cap L^\infty$ is a continuous embedding, our notion of viscosity solutions for (1) is the usual one, cf. Remark 29(b).

Then we derive stability estimates for the above norm. To quantify the influence of the nonlinearities, we introduce the norms $|H|_{\text{conv}}$ and $|H|_{\text{diff}}$ on $D_p^2 H$ and $D_X^2 H$ (see Section 2.4) where

$$(4) \quad H : (p, X) \in \mathbb{R}^d \times \mathbb{S}_d \mapsto \sup_{\mathcal{E}} \{b \cdot p + \text{tr}(aX)\} \in \mathbb{R}$$

is the Hamiltonian of (1a) and $\mathbb{S}_d \subset \mathbb{R}^{d \times d}$ the symmetric matrices. Hence (1a) has the form $\partial_t \varphi = H(D\varphi, D^2\varphi)$. We show two main estimates for (1), cf. Theorems 31 and 35. First that

$$(5) \quad \|(\varphi - \psi)(t)\|_{\text{int}} \leq (1 + \omega_d(t|H|_{\text{diff}})) (1 + t|H|_{\text{conv}})^d \|\varphi_0 - \psi_0\|_{\text{int}} \quad \forall t \geq 0,$$

for some modulus of continuity ω_d . Then that there is an equivalent norm $\|\cdot\|$ on L_{int}^∞ such that

$$(6) \quad \|(\varphi - \psi)(t)\| \leq e^{(|H|_{\text{conv}} \vee |H|_{\text{diff}})t} \|\varphi_0 - \psi_0\| \quad \forall t \geq 0,$$

where $a \vee b := \max\{a, b\}$. Estimates (5) and (6) are optimal for model PDEs, cf. Proposition 98. But contrarily to (5), (6) is a true quasicontraction result as in semigroup theory [33, 28, 13] with the constant 1 in front of the exponential.

Contraction and quasicontraction semigroups are important for their relations to accretive operators, splitting methods, etc. The quasicontractivity may fail for $\|\cdot\|_{\text{int}}$ roughly speaking because ω_d in (5) is typically a square root, cf. Proposition 99. However $\|\cdot\|_{\text{int}}$ is easier to compute than $\|\cdot\|$ so both (5) and (6) have their own interest.

Optimal contraction results for (2) in L^∞ . In [39], Kruzkov showed that entropy solutions of first order equations of the form (2) are well-posed in L^∞ without assuming integrability. Kinetic and renormalized solution theories to handle unbounded solutions in L^1 were developed later in [42, 43, 12]. For second order PDEs, isotropic diffusions were treated in [19], and anisotropic diffusions in [21], in $L^1 \cap L^\infty$ and L^1 . Well-posedness of (2) in the original Kruzhkov setting of possibly nonintegrable L^∞ solutions is not standard, but results exist in [31], see also [20, 11, 3, 27, 44].

Here our contribution concerns the quantitative stability in L^∞ . There can be no L^∞ contraction since solutions can develop discontinuities in finite time. In the literature stability is therefore quantified by weighted L^1 contraction principles like the well-known finite speed of propagation estimate for first order PDEs [39, 45]:

$$(7) \quad \int_{|x-x_0|<R} |u(x,t) - v(x,t)| \, dx \leq \int_{|x-x_0|<R+Ct} |u_0(x) - v_0(x)| \, dx.$$

An example for second order PDEs (cf. [14, 20, 48, 31]) is the following estimate:

$$(8) \quad \int |u(x, t) - v(x, t)| e^{-\sqrt{1+|x|^2}} dx \leq e^{Ct} \int |u_0(x) - v_0(x)| e^{-\sqrt{1+|x|^2}} dx.$$

Note that (8) does not imply (7) when $A \equiv 0$. A finer result is given in [27] where the weight is a convolution product involving a diffusion kernel constructed via viscosity solutions. But this result still loses information since it does not imply the global L^1 contraction (3), see Remarks 2.7(b) and 2.11(e) in [27].

Our main result for (2) is a very accurate weighted L^1 contraction principle, Theorem 38, roughly speaking stating that

$$(9) \quad \int |u(x, t) - v(x, t)| \varphi_0(x) dx \leq \int |u_0(x) - v_0(x)| \varphi(x, t) dx,$$

where $\varphi_0 \geq 0$ is arbitrary and φ is the solution of (1) with $b = F'$ and $a = A$. Using stochastic control theory [30], we show that (9) has the form (Corollary 42):

$$(10) \quad \int_{|x-x_0|<R} |u(x, t) - v(x, t)| dx \leq \int |u_0(x) - v_0(x)| \sup_{\xi} \mathbb{P}(|\mathbf{X}_t^x - x_0| < R) dx,$$

where \mathbb{P} is the probability, ξ_s an adapted control, \mathbf{B}_s a Brownian motion, and \mathbf{X}_s^x an Ito process satisfying

$$d\mathbf{X}_s^x = F'(\xi_s) ds + \sqrt{2} \sigma^A(\xi_s) d\mathbf{B}_s, \quad \mathbf{X}_{s=0}^x = x.$$

Estimate (10) is a natural second order extension of (7). Its general form (9) not only implies (3), (7), (8), and [27], but the weight is also optimal in a certain class satisfying a semigroup property, see Corollary 46.

Duality between (1) and (2). Interestingly, our analysis reveals a duality relation between viscosity solutions of (1) and entropy solutions of (2) (Theorem 44). It can be stated in terms of semigroups roughly speaking as follows (Corollary 47):

If G_t and S_t are the solution semigroups of (1) and (2), $b = F'$ and $a = A$, then G_t is the smallest semigroup satisfying

$$(11) \quad \int |S_t u_0 - S_t v_0| \varphi_0 dx \leq \int |u_0 - v_0| G_t \varphi_0 dx,$$

for any $u_0, v_0 \in L^\infty$ and $0 \leq \varphi_0 \in L_{\text{int}}^\infty$.

Inequality (11) can be interpreted as a *nonlinear dual* inequality and G_t as a *dual* semigroup of S_t since the semigroup G_t is entirely determined by (11) and knowledge of S_t . The question of duality in the other direction is open, cf. Remark 49.

Outline. Preliminary results are given in Section 2, in Sections 3 and 4 we state and prove our main results, and at the end there are appendices containing complementary results and technical proofs.

Notation. \mathbb{R}^+ (resp. \mathbb{R}^-) denotes nonnegative (resp. nonpositive) real numbers and \mathbb{S}_d^+ nonnegative symmetric $d \times d$ matrices. A modulus of continuity is a function $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\omega(t) \rightarrow 0$ as $t \rightarrow 0^+$. Finally, the symbols \vee and \wedge denote max and min respectively.

2. PRELIMINARIES

In this section, we recall some basic facts on viscosity and entropy solutions; for proofs, see for instance [22, 30, 5, 4] and [21, 11, 25] respectively. We also define the space L_{int}^∞ and the semi-norms $|\cdot|_{\text{conv}}$ and $|\cdot|_{\text{diff}}$.

2.1. Viscosity solutions of (1). Let φ^* (resp. φ_*) denote the upper (resp. lower) semicontinuous envelope of φ .

Definition 1 (Viscosity solutions). *Assume (H1) and $\varphi_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ is bounded.*

(a) *A locally bounded function $\varphi : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is a viscosity subsolution (resp. supersolution) of (1) if*

i) *for every $\phi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^+)$ and local maximum $(x, t) \in \mathbb{R}^d \times (0, \infty)$ of $\varphi^* - \phi$ (resp. minimum),*

$$\partial_t \phi(x, t) \leq H(D\phi(x, t), D^2\phi(x, t)) \quad (\text{resp. } \geq),$$

ii) *and for every $x \in \mathbb{R}^d$,*

$$\varphi^*(x, 0) \leq (\varphi_0)^*(x) \quad (\text{resp. } \varphi_*(x, 0) \geq (\varphi_0)_*(x)).$$

(b) *A function φ is a viscosity solution if it is both a sub and supersolution.*

Remark 2. We say that φ is a viscosity subsolution (resp. supersolution) of (1a) if (ai) holds.

We recall the well-known comparison and the well-posedness for (1).

Theorem 3 (Comparison principle). *Assume (H1). If φ and ψ are bounded sub and supersolutions of (1a), and*

$$\varphi^*(x, 0) \leq \psi_*(x, 0) \quad \forall x \in \mathbb{R}^d,$$

then $\varphi^ \leq \psi_*$ on $\mathbb{R}^d \times \mathbb{R}^+$.*

Theorem 4 (Existence and uniqueness). *Assume (H1) and $\varphi_0 \in C_b(\mathbb{R}^d)$. Then there exists a unique viscosity solution $\varphi \in C_b(\mathbb{R}^d \times \mathbb{R}^+)$ of (1).*

Remark 5. We also have the maximum principle $\inf \varphi_0 \leq \varphi \leq \sup \varphi_0$, and for solutions φ and ψ with initial data φ_0 and ψ_0 ,

$$\|\varphi(\cdot, t) - \psi(\cdot, t)\|_\infty \leq \|\varphi_0 - \psi_0\|_\infty \quad \forall t \geq 0.$$

We may take φ_0 to be discontinuous as in (10). In that case, we loose uniqueness and we have to work with minimal and maximal solutions [24, 10, 32] (see also [4] for bilateral solutions). From now on, we denote by *BLSC* (resp. *BUSC*) bounded and lower (resp. upper) semicontinuous functions.

Theorem 6 (Minimal and maximal solutions). *Assume (H1) and $\varphi_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ is bounded. Then there exists a pair of viscosity solutions of (1),*

$$(\underline{\varphi}, \overline{\varphi}) \in \text{BLSC}(\mathbb{R}^d \times \mathbb{R}^+) \times \text{BUSC}(\mathbb{R}^d \times \mathbb{R}^+),$$

where $\underline{\varphi}$ is minimal and $\overline{\varphi}$ is maximal in the sense that

$$\underline{\varphi} \leq \varphi \leq \overline{\varphi} \quad \text{for any bounded viscosity solution } \varphi \text{ of (1).}$$

Moreover, at $t = 0$,

$$\underline{\varphi}(x, 0) = (\varphi_0)_*(x) \quad \text{and} \quad \overline{\varphi}(x, 0) = (\varphi_0)^*(x) \quad \forall x \in \mathbb{R}^d.$$

Note that $\underline{\varphi}$ and $\overline{\varphi}$ are unique by definition.

Proposition 7 (Comparison). *Assume (H1) and $\varphi_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ is bounded.*

(i) *For any bounded supersolution φ of (1) (resp. subsolution).*

$$\underline{\varphi} \leq \varphi_* \quad (\text{resp. } \overline{\varphi} \geq \varphi^*).$$

(ii) *In particular, $\underline{\varphi} \leq \underline{\psi}$ and $\overline{\varphi} \leq \overline{\psi}$ for any bounded initial data $\varphi_0 \leq \psi_0$.*

For completeness, the proofs of Theorems 6 and Proposition 7 are given in Appendix A.1 because [24, 10, 32, 4] consider slightly different problems. Let us continue with representation formulas for the solution $\underline{\varphi}$ from control theory [30, 4, 34, 35]. Throughout, “co” denote the convex hull and “Im” the image.

Proposition 8 (First order). *Assume (H1), $a \equiv 0$, and $\varphi_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ bounded. Then the minimal viscosity solution of (1) is given by*

$$\underline{\varphi}(x, t) = \sup_{x+t\mathcal{C}} (\varphi_0)_* \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+,$$

where $\mathcal{C} = \overline{\text{co}\{\text{Im}(b)\}}$.

In the second order case, we need a probabilistic framework. For simplicity, we fix for the rest of this paper

$$(12) \quad \begin{cases} \text{a complete filtered probability space } (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}), \text{ and} \\ \text{a standard } d\text{-dimensional Brownian } \mathbf{B}_t \text{ on this filtration.} \end{cases}$$

We will assume without mention that all stochastic processes in this paper are defined on this filtered probability space, and that whenever we need a Brownian motion, then we take the above Brownian motion. Let us denote the expectation by \mathbb{E} . Then:

Proposition 9 (Second order). *Assume (H1), $\varphi_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ is bounded, and*

$$(13) \quad \text{the set } \mathcal{E} \text{ is compact and the functions } b(\cdot) \text{ and } \sigma^a(\cdot) \text{ are continuous.}$$

Then the minimal viscosity solution of (1) is given by

$$\underline{\varphi}(x, t) = \sup_{\xi} \mathbb{E} \{ (\varphi_0)_*(\mathbf{X}_t^x) \},$$

where ξ_s is a progressively measurable \mathcal{E} -valued control and \mathbf{X}_s^x an Ito process satisfying the SDE

$$\begin{cases} d\mathbf{X}_s^x = b(\xi_s) ds + \sqrt{2} \sigma^a(\xi_s) d\mathbf{B}_s, & s > 0, \\ \mathbf{X}_{s=0}^x = x. \end{cases}$$

These results are standard for continuous viscosity solutions [30, 4], see also [4, 34, 35] for maximal solutions. For minimal solutions, we did not find any reference so we provide the proofs in Appendix A.2.

2.2. Entropy solutions of (2). In the nonstandard pure L^∞ setting, the well-posedness of (2) is essentially considered in [31] for smooth fluxes, see also [21, 11] for L^1 . Let us now recall these results in the form we need and provide complementary proofs in Appendix B for completeness.

Definition 10 (Entropy-entropy flux triple). *We say that (η, q, r) is an entropy-entropy flux triple if $\eta \in C^2(\mathbb{R})$ is convex, $q' = \eta' F'$ and $r' = \eta' A$.*

Given $\beta \in C(\mathbb{R})$, we also need the notation

$$\zeta_{ik}(u) := \int_0^u \sigma_{ik}^A(\xi) d\xi \quad \text{and} \quad \zeta_{ik}^\beta(u) := \int_0^u \sigma_{ik}^A(\xi) \beta(\xi) d\xi.$$

Definition 11 (Entropy solutions). *Assume (H2) and $u_0 \in L^\infty(\mathbb{R}^d)$. A function $u \in L^\infty(\mathbb{R}^d \times \mathbb{R}^+) \cap C(\mathbb{R}^+; L_{\text{loc}}^1(\mathbb{R}^d))$ is an entropy solution of (2) if*

- (a) $\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) \in L_{\text{loc}}^2(\mathbb{R}^d \times \mathbb{R}^+)$ for any $k = 1, \dots, K$,
- (b) for any $k = 1, \dots, K$ and any $\beta \in C(\mathbb{R})$

$$\sum_{i=1}^d \partial_{x_i} \zeta_{ik}^\beta(u) = \beta(u) \sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) \in L_{\text{loc}}^2(\mathbb{R}^d \times \mathbb{R}^+),$$

(c) and for all entropy-entropy flux triples (η, q, r) and $0 \leq \phi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+)$,

$$\begin{aligned} & \iint_{\mathbb{R}^d \times \mathbb{R}^+} \left(\eta(u) \partial_t \phi + \sum_{i=1}^d q_i(u) \partial_{x_i} \phi + \sum_{i,j=1}^d r_{ij}(u) \partial_{x_i x_j}^2 \phi \right) dx dt \\ & + \int_{\mathbb{R}^d} \eta(u_0(x)) \phi(x, 0) dx \geq \iint_{\mathbb{R}^d \times \mathbb{R}^+} \eta''(u) \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) \right)^2 \phi dx dt. \end{aligned}$$

Theorem 12 (Existence and uniqueness). *Assume (H2) and $u_0 \in L^\infty(\mathbb{R}^d)$. Then there exists a unique entropy solution $u \in L^\infty(\mathbb{R}^d \times \mathbb{R}^+) \cap C(\mathbb{R}^+; L^1_{\text{loc}}(\mathbb{R}^d))$ of (2).*

See [31, Theorem 1.1] or Appendix B for the proof.

Remark 13. (a) In the L^1 settings of [21, 11], the following contraction principle holds: For solutions u and v of (2) with initial data u_0 and v_0 ,

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1} \quad \forall t \geq 0.$$

- (b) In the L^∞ setting of [31], uniqueness is based on the weighted L^1 contraction principle (8), see also Lemma 94 in Appendix B.
 (c) In all cases, we have comparison and maximum principles as stated in Lemma 96 in Appendix B.

In L^∞ , uniqueness is based on a doubling of variables arguments developed in [39, 19, 11]. This argument leads to (14) below, and this inequality will be the starting point of our analysis of (2).

Lemma 14 (Kato inequality). *Assume (H2) and u, v are entropy solutions of (2) with initial data $u_0, v_0 \in L^\infty(\mathbb{R}^d)$. Then for all $T \geq 0$ and nonnegative test functions $\phi \in C_c^\infty(\mathbb{R}^d \times [0, T])$,*

$$(14) \quad \int_{\mathbb{R}^d} |u - v|(x, T) \phi(x, T) dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \phi(x, 0) dx \\ + \iint_{\mathbb{R}^d \times (0, T)} \left(|u - v| \partial_t \phi + \sum_{i=1}^d q_i(u, v) \partial_{x_i} \phi + \sum_{i,j=1}^d r_{ij}(u, v) \partial_{x_i x_j}^2 \phi \right) dx dt,$$

where

$$q_i(u, v) = \text{sign}(u - v) \int_v^u F'_i(\xi) d\xi, \quad r_{ij}(u, v) = \text{sign}(u - v) \int_v^u A_{ij}(\xi) d\xi.$$

See Appendix B for a sketch of the proof of this lemma with references to computations in [11].

2.3. The function space L^∞_{int} . Consider the following normed space

$$L^\infty_{\text{int}}(\mathbb{R}^d) := \{ \varphi_0 \in L^1 \cap L^\infty(\mathbb{R}^d) \text{ s.t. } \|\varphi_0\|_{L^\infty_{\text{int}}} < \infty \}$$

where $\|\varphi_0\|_{L^\infty_{\text{int}}} := \int \text{ess sup}_{\overline{Q}_1(x)} |\varphi_0| dx$ and $\overline{Q}_r(x) := x + [-r, r]^d$ for $r > 0$.

Theorem 15. *This is a Banach space continuously embedded into $L^1 \cap L^\infty(\mathbb{R}^d)$.*

For the proof, see [2, 1] from which we have taken the above notation. From now on, we would rather consider the pointwise sup, and to avoid confusion we then use the notation

$$\|\varphi_0\|_{\text{int}} := \int \sup_{\overline{Q}_1(x)} |\varphi_0| dx.$$

Note that $\|\varphi_0\|_{\text{int}} = \|\varphi_0\|_{L^\infty_{\text{int}}}$ if φ_0 is continuous. Here is another result of [2, 1], see for instance [1, Lemma 2.5.1].

Lemma 16. *For any $r > 0$ and $\varepsilon \geq 0$, there is a constant $C_{r,\varepsilon} \geq 0$ such that*

$$\int \sup_{\overline{Q}_{r+\varepsilon}(x)} |\varphi_0| dx \leq C_{r,\varepsilon} \int \sup_{\overline{Q}_r(x)} |\varphi_0| dx \quad \forall \varphi_0 : \mathbb{R}^d \rightarrow \mathbb{R}.$$

Remark 17. (a) This result will be used with the pointwise sup for discontinuous φ_0 , typically lower or uppersemicontinuous.

(b) A more precise and possibly new result is given in Lemma 91 in Appendix A.5.

2.4. Semi-norms $|\cdot|_{\text{conv}}$ and $|\cdot|_{\text{diff}}$. The set of Hamiltonians associated to equations of the form (1) is a convex cone which we denote by

$$\Gamma := \{H : \mathbb{R}^d \times \mathbb{S}_d \rightarrow \mathbb{R} \text{ s.t. } H \text{ satisfy (4) for some } (\mathcal{E}, b, a) \text{ satisfying (H1)}\}.$$

Note that the same $H \in \Gamma$ can be represented by different (\mathcal{E}, b, a) satisfying (H1) as long as $\text{Im}(b, a)$ is the same (see Lemma 88 or Remark 90(a)). Let us endow Γ with semi-norms.

Definition 18. *For each $H \in \Gamma$, set*

$$|H|_{\text{conv}} := \inf_{b_0 \in \mathbb{R}^d} \sup_{\xi \in \mathcal{E}} |b(\xi) - b_0| \quad \text{and} \quad |H|_{\text{diff}} := \inf_{\mathbb{S}_d^+ \ni a_0 \leq \text{Im}(a)} \sup_{\xi \in \mathcal{E}} \text{tr}(a(\xi) - a_0),$$

where (\mathcal{E}, b, a) is a representative of H , i.e. a triplet satisfying (H1) such that (4) holds.

Remark 19. (a) In the second inf, “ $a_0 \leq \text{Im}(a)$ ” means that $a_0 \leq a(\xi)$, $\forall \xi \in \mathcal{E}$.

(b) You may think that these quantities depend of the choice of the representative (\mathcal{E}, b, a) , but this is not the case: see Lemma 89 in Appendix A.4.

These semi-norms roughly speaking measure the nonlinearities of the Hamiltonians.

Proposition 20. *The above $|\cdot|_{\text{conv}}$ and $|\cdot|_{\text{diff}}$ are semi-norms. Moreover, for each Hamiltonian $H = H(p, X) \in \Gamma$, we have:*

$$(15) \quad \begin{cases} |H|_{\text{conv}} = 0 & \iff H \text{ is affine in the gradient } p, \text{ and} \\ |H|_{\text{diff}} = 0 & \iff H \text{ is affine in the Hessian } X. \end{cases}$$

See Appendix A.4 for the proof.

Remark 21. Let us give an example. Let $d = 1$ and

$$\mathcal{E} := \{c, d\} \times \{e, f\}$$

for some $c, d, e, f \in \mathbb{R}$ such that $e, f \geq 0$. Let $b(\xi) = \xi_1$ and $a(\xi) = \xi_2$ where $\xi = (\xi_1, \xi_2)$, and

$$H(p, X) := \sup_{\xi \in \mathcal{E}} \{b(\xi) \cdot p + \text{tr}(a(\xi)X)\}.$$

Then

$$|H|_{\text{conv}} = \frac{|c-d|}{2} \quad \text{and} \quad |H|_{\text{diff}} = e \vee f - e \wedge f.$$

Note also that

$$H(p, X) = \max\{cp, dp\} + \max\{eX, fX\},$$

so that for this particular example,

$$|H|_{\text{conv}} = \frac{1}{2} \int_{\mathbb{R}} |\partial_{pp}^2 H| \quad \text{and} \quad |H|_{\text{diff}} = \int_{\mathbb{R}} |\partial_{XX}^2 H|$$

in the sense of the total variations. For general dimensions and Hamiltonians, these total variations may be infinite, but $|H|_{\text{conv}}$ and $|H|_{\text{diff}}$ are always finite under our assumptions.

See Remark 90 in Appendix A.4, for connections to support functions and convex analysis.

3. MAIN RESULTS

We are ready to state our main results. The long proofs are given in Section 4.

3.1. Some L^1 instabilities for (1). Let us begin with (1) and explain why we can not have pure L^1 stability results. Let us consider the unique viscosity solution of the eikonal equation

$$(16) \quad \partial_t \varphi = \sum_{i=1}^d |\partial_{x_i} \varphi|,$$

with a given initial data $\varphi_0 \in C_b(\mathbb{R}^d)$. Under which condition is it integrable?

Proposition 22 (Integrability condition). *We have*

$$[\varphi(\cdot, t) \in L^1(\mathbb{R}^d) \quad \forall t \geq 0] \iff [\varphi_0^- \in L^1(\mathbb{R}^d) \text{ and } \varphi_0^+ \in L_{\text{int}}^\infty(\mathbb{R}^d)].$$

Proof. Use that $\varphi(x, t) = \sup_{\overline{Q}_t(x)} \varphi_0$ by Proposition 8, and then Lemma 16. \square

We can therefore not expect general L^1 stability because of the positive parts.

Proposition 23 (L^1 instability for nonnegative solutions). *Let $\varphi_0^n(x) := (1-n|x|)^+$ for all $n \geq 1$, and φ_n be the solution of (16) with φ_0^n as initial data. Then $\varphi_0^n \in C_b \cap L^1(\mathbb{R}^d)$ and*

$$\lim_{n \rightarrow \infty} \varphi_0^n = 0 \quad \text{in } L^1(\mathbb{R}^d),$$

but

$$\lim_{n \rightarrow \infty} \varphi_n(\cdot, t) = \mathbf{1}_{\overline{Q}_t}(\cdot) \neq 0 \quad \text{in } L^1(\mathbb{R}^d), \quad \forall t > 0.$$

Proof. Use again that $\varphi_n(x, t) = \sup_{\overline{Q}_t(x)} \varphi_0^n$. \square

You might expect L^1 stability for nonpositive solutions, because the negative parts behave better in Proposition 22. But this is not the case either:

Proposition 24 (L^1 instability for nonpositive solutions). *There are nonpositive $\varphi_0, \varphi_0^n \in C_b \cap L^1(\mathbb{R}^d)$ such that*

$$\lim_{n \rightarrow \infty} \varphi_0^n = \varphi_0 \quad \text{in } L^1(\mathbb{R}^d),$$

but the solutions φ and φ_n of (16) with initial data φ_0 and φ_0^n satisfy

$$\liminf_{n \rightarrow \infty} \|\varphi_n(\cdot, t) - \varphi(\cdot, t)\|_{L^1} > 0 \quad \forall t > 0 \text{ (small enough)}.$$

See Section 4.2 for the proof. To see that L^1 is not good for pure diffusion equations of the form (1a) either, we consider an equation in one space dimension

$$(17) \quad \partial_t \varphi = (\partial_{xx}^2 \varphi)^+.$$

Here again, to have L^1 solutions, we need $\varphi_0^+ \in L_{\text{int}}^\infty$.

Proposition 25 (L_{int}^∞ and nonlinear diffusions). *Let $\varphi_0 \in C_b(\mathbb{R})$ be nonnegative and φ be the solution of (17) with φ_0 as initial data. Then,*

$$[\varphi(\cdot, t) \in L^1(\mathbb{R}) \quad \forall t \geq 0] \iff \varphi_0 \in L_{\text{int}}^\infty(\mathbb{R}).$$

See Section 4.2 for the proof. To simplify, we omit the analysis of nonpositive solutions of (17) and discuss instead the lack of a fundamental solution. Consider solutions φ_n of (17) with an approximate delta-functions as initial data:

$$(18) \quad \varphi_n(x, t = 0) = n\rho(nx),$$

where $0 \leq \rho \in C_c(\mathbb{R})$ is nontrivial. Then:

Proposition 26 (Blow-up everywhere). $\lim_{n \rightarrow \infty} \varphi_n(x, t) = \infty, \forall x \in \mathbb{R}, \forall t > 0.$

See Section 4.2 for the proof. For linear diffusion equations, φ_n would converge to the fundamental solution, but here it explodes pointwise and in all L_{loc}^p .

3.2. Optimal L^1 framework for (1). We now look for the smallest topology which is stronger than L^1 for which we have stability. To get a general theory, we restrict to normed spaces and results that hold for all equations of the form (1a). We will obtain a quasicontraction result for the corresponding semigroups. Let us recall some definitions from [33, 28, 13].

Definition 27. Let E be a normed space. We say that G_t is a semigroup on E if it is a family of maps $G_t : E \rightarrow E$, parametrized by $t \geq 0$, satisfying

$$\begin{cases} G_{t=0} = \text{id (the identity), and} \\ G_{t+s} = G_t G_s \text{ (meaning the composition) for any } t, s \geq 0. \end{cases}$$

It is a contraction semigroup if

$$\|G_t \varphi_0 - G_t \psi_0\|_E \leq \|\varphi_0 - \psi_0\|_E,$$

and a quasicontraction semigroup if there is some $\gamma \in \mathbb{R}$ such that

$$\|G_t \varphi_0 - G_t \psi_0\|_E \leq e^{\gamma t} \|\varphi_0 - \psi_0\|_E,$$

for any $\varphi_0, \psi_0 \in E$ and $t \geq 0$.

For example, let φ be the unique viscosity solution of (1) and define

$$(19) \quad G_t : \varphi_0 \in C_b(\mathbb{R}^d) \mapsto \varphi(\cdot, t) \in C_b(\mathbb{R}^d).$$

Then G_t is a contraction semigroup. Consider now spaces E such that

$$(20) \quad \begin{cases} E \text{ is a vector subspace of } C_b \cap L^1(\mathbb{R}^d), \\ E \text{ is a normed space,} \\ E \text{ is continuously embedded}^1 \text{ into } L^1(\mathbb{R}^d), \end{cases}$$

and for any triplet of data (\mathcal{E}, b, a) satisfying (H1), the associated semigroup (19) is such that for any $t \geq 0$,

$$(21) \quad G_t \text{ maps } E \text{ into itself and } G_t : E \rightarrow E \text{ is continuous.}$$

The best possible E is given below.

Theorem 28 (Optimal L^1 setting for HJB equations). *The space $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ is a Banach space satisfying the properties (20)–(21). Moreover, any other space E satisfying (20)–(21) is continuously embedded¹ into $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$.*

Remark 29. (a) We will also see that G_t is strongly continuous on $C_b \cap L_{\text{int}}^\infty$, i.e. $t \geq 0 \mapsto G_t \varphi_0$ is continuous for the strong topology, cf. Lemma 87.

(b) We have considered spaces E of continuous functions in order to avoid uniqueness issues. But this is not restrictive since the best E above is a complete space by Theorem 15.

Let us now focus on explicit estimates. We first give a rough L_{int}^∞ a priori estimate.

Theorem 30 (General L_{int}^∞ stability). *Assume (H1) and $T \geq 0$. For any bounded subsolution φ and supersolution ψ of (1a),*

$$(22) \quad \int_{\overline{Q_1(x)} \times [0, T]} \sup (\varphi^* - \psi_*)^+ dx \leq C \int_{\overline{Q_1(x)}} \sup (\varphi^* - \psi_*)^+ (\cdot, 0) dx,$$

for some constant $C = C(d, \|a\|_\infty, \|b\|_\infty, T) \geq 0$.

¹Let X, Y be normed spaces such that $X \subseteq Y$. X is continuously embedded into Y if there is $C \geq 0$ such that $\|x\|_Y \leq C\|x\|_X$ for all $x \in X$.

Note that the supremum in time is inside the integral. In the second estimate, we precisely quantify the influence of the convective and diffusive nonlinearities.

Theorem 31 (Quantitative L_{int}^∞ stability). *Assume (H1) and let $\mathcal{C} = \overline{\text{co}\{\text{Im}(b)\}}$. There exists a modulus of continuity $\omega_d(\cdot)$ only depending on the dimension d such that, for any bounded subsolution φ and supersolution ψ of (1a),*

$$(23) \quad \int \sup_{\overline{Q_1(x)}} (\varphi^* - \psi_*)^+(\cdot, t) \, dx \leq (1 + \omega_d(t|H|_{\text{diff}})) \int \sup_{\overline{Q_1(x)+t\mathcal{C}}} (\varphi^* - \psi_*)^+(\cdot, 0) \, dx \quad \forall t \geq 0.$$

Remark 32. (a) The modulus ω_d is defined in Lemma 67, see also Lemma 70.
 (b) The effect of convection is seen in the set $t\mathcal{C}$ and the diffusion in $t|H|_{\text{diff}}$.

Let us now consider the semigroup (19).

Corollary 33 (Contraction like estimate in $\|\cdot\|_{\text{int}}$). *Assume (H1) and $\omega_d(\cdot)$ is defined in Theorem 31. Then, for any $\varphi_0, \psi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ and $t \geq 0$,*

$$(24) \quad \|G_t \varphi_0 - G_t \psi_0\|_{\text{int}} \leq (1 + \omega_d(t|H|_{\text{diff}})) (1 + t|H|_{\text{conv}})^d \|\varphi_0 - \psi_0\|_{\text{int}}.$$

Remark 34. (a) The constant in (24) is optimal for small $t(|H|_{\text{conv}} \vee |H|_{\text{diff}})$, cf. the discussion in Appendix C and more precisely Proposition 98.
 (b) We deduce from Proposition 20 that G_t is quasicontractive if the diffusion is linear, and contractive if the whole equation is linear. But for fully nonlinear PDEs, G_t may not be quasicontractive roughly speaking because ω_d is typically a square root, cf. Proposition 99.

To get a quasicontraction principle for fully nonlinear PDEs, we need to change the norm. Below and throughout, “Sp” denotes the spectrum.

Theorem 35 (General quasicontraction). *For any $\varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$, define*

$$\|\varphi_0\| := \sup_{t \geq 0} e^{-t} \|G_t^{\text{mod}} \varphi_0\|_{\text{int}},$$

where G_t^{mod} is the semigroup on $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ associated to the model equation

$$\partial_t \varphi = |D\varphi| + \sup_{\lambda \in \text{Sp}(D^2 \varphi)} \lambda^+.$$

Then $\|\cdot\|$ is a norm on $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ which is equivalent to $\|\cdot\|_{\text{int}}$. Moreover, under (H1), the semigroup (19) is $\|\cdot\|$ -quasicontractive with

$$(25) \quad \|G_t \varphi_0 - G_t \psi_0\| \leq e^{(|H|_{\text{conv}} \vee |H|_{\text{diff}})t} \|\varphi_0 - \psi_0\|,$$

for any $\varphi_0, \psi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ and $t \geq 0$.

Remark 36. (a) The constant in (25) is optimal for small $t(|H|_{\text{conv}} \vee |H|_{\text{diff}})$, cf. Proposition 98.
 (b) The choice of $\|\cdot\|$ is inspired by linear semigroup theory, see Theorem 2.13 in [33]. Here, our semigroup is nonlinear, and remarkably $\|\cdot\|$ does not depend on the particular semigroup G_t that we consider as in [33]. More precisely, G_t^{mod} is a fixed model semigroup, and $\|\cdot\|$ is a fixed norm in which all semigroups of equations of the form (1a) are quasicontractive.

Remark 37. This is the L^1 counterpart of the L^∞ contraction for (1),

$$\|(\varphi - \psi)(t)\|_\infty \leq \|\varphi_0 - \psi_0\|_\infty.$$

In the L^1 setting we have

$$\|(\varphi - \psi)(t)\| \leq e^{(|H|_{\text{conv}} \vee |H|_{\text{diff}})t} \|\varphi_0 - \psi_0\|,$$

where, for some constant $M \geq 1$ and any $\phi = \phi(x)$,

$$\|\phi\|_{L^1} \leq \int \sup_{\overline{Q_1(x)}} |\phi| \, dx \leq \|\phi\| \leq M \int \sup_{\overline{Q_1(x)}} |\phi| \, dx.$$

The proofs of all results in this section can be found in Sections 4.3–4.6. Optimal examples for (24) and (25) are given in Appendix C.

3.3. Weighted L^1 contraction for (2). Let us continue with Problem (2) and state a new weighted L^1 contraction principle for L^∞ entropy solutions. The weight will be the viscosity solution of the following problem:

$$(26a) \quad \partial_t \varphi = \operatorname{ess\,sup}_{m \leq \xi \leq M} \{F'(\xi) \cdot D\varphi + \operatorname{tr}(A(\xi)D^2\varphi)\} \quad x \in \mathbb{R}^d, t > 0,$$

$$(26b) \quad \varphi(x, 0) = \varphi_0(x) \quad x \in \mathbb{R}^d,$$

for given $m < M$ and φ_0 . Here is the precise statement.

Theorem 38 (Weighted L^1 contraction). *Assume (H2), $m < M$, and take measurable $u_0 = u_0(x)$ and $v_0 = v_0(x)$ with values in $[m, M]$. Take also a nonnegative initial weight $\varphi_0 \in BLSC(\mathbb{R}^d)$. Then, the associated entropy solutions u and v of (2) and viscosity minimal solution $\underline{\varphi}$ of (26) satisfy*

$$(27) \quad \int_{\mathbb{R}^d} |u - v|(x, t) \varphi_0(x) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \underline{\varphi}(x, t) \, dx \quad \forall t \geq 0.$$

- Remark 39.* (a) Problem (26) is of the form (1) with a pointwise sup taken over the Lebesgue points of F' and A .
(b) The right-hand side of (27) can be infinite. To get finite integrals, it suffices to take $\varphi_0 \in L_{\text{int}}^\infty$ or $u_0 - v_0 \in L^1$.
(c) The same result holds when $\underline{\varphi}$ is replaced by any measurable supersolution of (26), since it is greater than $\underline{\varphi}$.

Let us be more explicit. By the representation formula for first order PDEs, we obtain the following estimate on the domain of dependence:

Corollary 40 (First order equations). *Assume (H1) with $A \equiv 0$, $m < M$, u_0 and v_0 are measurable functions with values in $[m, M]$, and u and v are entropy solutions of (2) with initial data u_0 and v_0 . Then*

$$\int_B |u - v|(x, t) \, dx \leq \int_{B-t\mathcal{C}} |u_0 - v_0|(x) \, dx$$

for any Borel set $B \subseteq \mathbb{R}^d$ and $t \geq 0$, where

$$\mathcal{C} = \overline{\operatorname{co} \left\{ \operatorname{ess\,Im} \left((F')|_{[m, M]} \right) \right\}}$$

and $\operatorname{ess\,Im}$ is the essential image.

Proof. Let $U \supseteq B$ be an open set and take $\varphi_0 = \mathbf{1}_U$, the indicator function of U . By Proposition 8, the minimal solution of (26) is $\underline{\varphi}(x, t) = \mathbf{1}_{U-t\mathcal{C}}(x)$. Apply then Theorem 38 and take the infimum over all open $U \supseteq B$. \square

Remark 41. The above result is also a consequence of [45]. In that recent paper, the author gives similar estimates for (x, t) -dependent first order PDEs using different techniques based on differential inclusions.

For second order equations, we have the following result.

Corollary 42 (Second order equations). *Assume (H1), $F'(\cdot)$ and $\sigma^A(\cdot)$ continuous, $m < M$, u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$, and u and v solutions of (2) with u_0 and v_0 as initial data. Then for any open $U \subseteq \mathbb{R}^d$ and $t \geq 0$,*

$$\int_U |u - v|(x, t) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \sup_{\xi} \mathbb{P}(\mathbf{X}_t^x \in U) \, dx,$$

where ξ_s is a progressively measurable $[m, M]$ -valued process and \mathbf{X}_s^x is an Ito process satisfying the SDE

$$(28) \quad \begin{cases} d\mathbf{X}_s^x = F'(\xi_s) \, ds + \sqrt{2} \sigma^A(\xi_s) \, dB_s, & s > 0, \\ \mathbf{X}_{s=0}^x = x. \end{cases}$$

Remark 43. \mathbf{X}_s^x is a stochastic process starting from x at time $s = 0$. The dynamics of \mathbf{X}_s^x is given by the controlled SDE (28) where the coefficients are (derivatives of) the fluxes in Equation (2). The control is determined to maximize the probability for the process to reach U at time $s = t$. Equation (26) is the dynamic programming equation for this control problem.

Proof. Take $\varphi_0 = \mathbf{1}_U$ and apply Proposition 9 to compute $\underline{\varphi}$ in Theorem 38. \square

The proof of Theorem 38 is given in Section 4.7.

3.4. Optimal weight and duality between (1) and (2). Let us discuss the optimality of Theorem 38. First we give a reformulation of the definition of viscosity supersolutions in terms of weights in L^1 contraction estimates for (2).

Theorem 44 (Weights and supersolutions). *Assume (H2), $m < M$, and $0 \leq \varphi \in BLSC(\mathbb{R}^d \times \mathbb{R}^+)$. Then the assertions below are equivalent.*

(I) *For any measurable functions u_0 and v_0 with values in $[m, M]$ and entropy solutions u and v of (2) with initial data u_0 and v_0 ,*

$$\int_{\mathbb{R}^d} |u - v|(x, t) \varphi(x, s) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x, t + s) \, dx \quad \forall t, s \geq 0.$$

(II) *The function*

$$\varphi_{\#}(x, t) := \liminf_{\substack{r \rightarrow 0^+ \\ y \rightarrow x}} \frac{1}{\text{meas}(B_r(y))} \int_{B_r(y)} \varphi(z, t) \, dz$$

is a viscosity supersolution of (26a).

We have used the notation $B_r(y) := \{z : |z - y| < r\}$.

Remark 45. The function $\varphi_{\#}$ satisfies (I) if and only if φ does since it is an a.e. representative in space of φ .

Our weight is therefore optimal in the class of weights

$$\mathscr{W}_{m, M, \varphi_0} := \{0 \leq \varphi \in BLSC(\mathbb{R}^d \times \mathbb{R}^+) \text{ satisfying (I) and } \varphi(t = 0) \geq \varphi_0\}.$$

Corollary 46 (Optimality of the weight). *Assume (H2), $m < M$, and $0 \leq \varphi_0 \in BLSC(\mathbb{R}^d)$. Then the weight $\underline{\varphi}$ from Theorem 38 belongs to the class $\mathscr{W}_{m, M, \varphi_0}$ and satisfies*

$$(\underline{\varphi})_{\#}(x, t) = \inf \{\varphi_{\#}(x, t) : \varphi \in \mathscr{W}_{m, M, \varphi_0}\} \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+.$$

Property (I) is stronger than (27) since it holds for any $s \geq 0$. This may be interpreted as a semigroup property. In that context the above result reflects some form of duality. For each $t \geq 0$, let

$$S_t : u_0 \in L^\infty(\mathbb{R}^d) \mapsto u(\cdot, t) \in L^\infty(\mathbb{R}^d)$$

where u is the entropy solution of (2), and let

$$G_t : \varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d) \mapsto \varphi(\cdot, t) \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$$

where φ is the viscosity solution of (26). Note that $G_t = G_t^{m,M}$ depends on the parameters m and M through Equation (26a). We have the following result.

Corollary 47 (A form of duality). *Assume (H2), $m < M$, and S_t, G_t defined as above. Then G_t is the smallest strongly continuous semigroup on $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ satisfying*

$$(29) \quad \int_{\mathbb{R}^d} |S_t u_0 - S_t v_0| \varphi_0 \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0| G_t \varphi_0 \, dx,$$

for every u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$, $0 \leq \varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$, and $t \geq 0$.

Remark 48. Here “smallest” means that any other semigroup H_t satisfying the same properties is such that

$$G_t \varphi_0 \leq H_t \varphi_0 \quad \forall t \geq 0, \forall \varphi_0 \geq 0.$$

The proofs of Theorem 44 and Corollaries 46 and 47 are given in Section 4.8.

Remark 49. (a) Inequality (29) can be seen as a nonlinear dual inequality between S_t and G_t , and G_t as a dual semigroup of S_t entirely determined by S_t through (29).

(b) The question of duality in the other direction is open. Let us formulate it precisely. Consider the whole family $\{G_t^{m,M} : m < M\}$ defined just before Corollary 47, and let \mathcal{S} be the set of weakly- \star continuous semigroups T_t on $L^\infty(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} |T_t u_0 - T_t v_0| \varphi_0 \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0| G_t^{m,M} \varphi_0 \, dx,$$

for every $m < M$, u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$, $0 \leq \varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$, and $t \geq 0$. Then the solution semigroup S_t of (2) belongs to \mathcal{S} , but this set contains infinitely many other semigroups. For instance, the solution semigroup associated to

$$\partial_t u + \gamma u + \operatorname{div} F(u) = \operatorname{div}(A(u)Du),$$

with $\gamma \geq 0$, also belongs to \mathcal{S} . The question of recovering S_t from the knowledge of $\{G_t^{m,M} : m < M\}$ then amounts to finding an easy criterion to select S_t amongst all the semigroups of \mathcal{S} . If for instance the PDE of (2) is linear, we claim that S_t is the unique semigroup of \mathcal{S} preserving constants. We omit the detailed verification which does not contain any particular difficulty. In the nonlinear case, we do not know if this result still holds.

4. PROOFS

Let us now prove the previous results. We begin with Problem (1).

4.1. More on viscosity solutions of (1). We need further classical results that can be found in [22, 30, 5, 4]. Let us begin with stability.

Proposition 50 (Stability by sup and inf). *Assume (H1) and $\mathcal{F} \neq \emptyset$ is a uniformly locally bounded family of viscosity subsolutions (resp. supersolutions) of (1a). Then, the function*

$$(x, t) \mapsto \sup\{\varphi(x, t) : \varphi \in \mathcal{F}\}$$

is a viscosity subsolution of (1a) (resp. supersolution but with the “inf”).

The second result concerns relaxed limits defined as follows:

$$\limsup^* \varphi_\varepsilon(x, t) := \limsup_{\substack{(y, s) \rightarrow (x, t) \\ \varepsilon \rightarrow 0^+}} \varphi_\varepsilon(y, s) \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+,$$

and $\liminf_* \varphi_\varepsilon := -\limsup^*(-\varphi_\varepsilon)$.

Proposition 51 (Stability with respect to relaxed limits). *Assume (H1) and let $(\varphi_\varepsilon)_{\varepsilon>0}$ be a family of uniformly locally bounded viscosity subsolutions (resp. supersolutions) of (1a). Then $\limsup^* \varphi_\varepsilon$ (resp. $\liminf_* \varphi_\varepsilon$) is a subsolution of (1a) (resp. supersolution).*

Remark 52. The notion of solution is thus stable under local uniform convergence (equivalent to $\limsup^* \varphi_\varepsilon = \liminf_* \varphi_\varepsilon$).

Here is the case of extremal solutions, see Appendix A.1 for the proof.

Proposition 53 (Stability of extremal solutions). *Assume (H1) and $(\varphi_0^n)_n$ is a nondecreasing (resp. nonincreasing) uniformly globally bounded sequence. If $\underline{\varphi}_n$ (resp. $\overline{\varphi}_n$) is the min (resp. max) solution of (1) with φ_0^n as initial data, then*

$$\sup_n \underline{\varphi}_n \quad (\text{resp. } \inf_n \overline{\varphi}_n)$$

is the minimal (resp. maximal) solution of (1) with initial data

$$\varphi_0 := \sup_n (\varphi_0^n)_* \quad (\text{resp. } \inf_n (\varphi_0^n)^*).$$

Let us continue with regularization procedures. Given $\varphi = \varphi(x, t)$ and $\varepsilon > 0$, we define the space supconvolution of φ as follows:

$$(30) \quad \varphi^\varepsilon(x, t) := \sup_{y \in \mathbb{R}^d} \left\{ \varphi^*(y, t) - \frac{|x - y|^2}{2\varepsilon^2} \right\}.$$

For standard properties of supconvolution, see e.g. [22, 30, 5, 4].

Lemma 54. *Assume (H1) and φ is a bounded subsolution of (1a). Then φ^ε is a subsolution of the same equation such that $\varphi^\varepsilon \in \text{BUSC}(\mathbb{R}^d \times \mathbb{R}^+)$ and is globally Lipschitz in x uniformly in t . Moreover $\varphi^*(x, t) = \lim_{\varepsilon \downarrow 0} \downarrow \varphi^\varepsilon(x, t)$ for any $x \in \mathbb{R}^d$ and $t \geq 0$.*

For supersolutions, we can regularize by convolution because of the convexity of the Hamiltonian, see [7, 8] (the ideas were introduced in [40]).

Lemma 55. *Assume (H1), $\varphi \in \text{BLSC}(\mathbb{R}^d \times \mathbb{R}^+)$ is a supersolution of (1a), and $0 \leq f \in L^1(\mathbb{R}^d \times \mathbb{R}^-)$. Then $\varphi *_{x,t} f$ is a supersolution of (1a).*

Here and throughout we extend the functions by zero to all $t \in \mathbb{R}$ to give a meaning to the convolutions in time. Below is another version that will be needed. We use the notation $M^1 = (C_0)'$ for spaces of bounded Borel measures.

Lemma 56. *Assume (H1), $\varphi \in C_b(\mathbb{R}^d \times \mathbb{R}^+)$ is a supersolution of (1a), and $0 \leq \mu \in M^1(\mathbb{R}^d)$. Then $\varphi * \mu \in C_b(\mathbb{R}^d \times \mathbb{R}^+)$ is also a supersolution (with $*$ = $*_x$ for short).*

The latter lemma is not proven in [7, 8], but can be obtained via a standard approximation procedure. Let us give it for completeness. Throughout this paper ρ_ν is a space approximate unit as $\nu \rightarrow 0^+$ of the form

$$(31) \quad \rho_\nu(x) := \frac{1}{\nu^d} \rho\left(\frac{x}{\nu}\right),$$

where $0 \leq \rho \in C_c^\infty(\mathbb{R}^d)$ and $\int \rho = 1$. Moreover θ_δ is a time approximate unit of the form

$$(32) \quad \theta_\delta(t) := \frac{1}{\delta} \theta\left(\frac{t}{\delta}\right),$$

where $0 \leq \theta \in C_c^\infty((-\infty, 0))$ and $\int \theta = 1$. Below $\delta = \nu$ but these parameters may be taken differently later.

Proof of Lemma 56. By Lemma 55, $\varphi_\nu := \varphi *_x \mu *_x \rho_\nu *_t \theta_\nu$ is a supersolution of (1a). It remains to pass to the limit as $\nu \rightarrow 0^+$. We will show that the convergence is local uniform towards $\varphi *_x \mu$, which will be sufficient by stability of the equation. With the assumed regularity on φ ,

$$\lim_{\nu \rightarrow 0^+} \varphi *_x \rho_\nu *_t \theta_\nu = \varphi \quad \text{locally uniformly,}$$

and $\|\varphi *_x \rho_\nu *_t \theta_\nu\|_\infty \leq \|\varphi\|_\infty$. Moreover, for any $x \in \mathbb{R}^d$, $t \geq 0$ and $R \geq 0$,

$$\begin{aligned} |\varphi_\nu - \varphi *_x \mu|(x, t) &\leq |\varphi *_x \rho_\nu *_t \theta_\nu - \varphi| *_x \mu(x, t) \\ &\leq \left(\sup_{|y| \leq R} |\varphi *_x \rho_\nu *_t \theta_\nu - \varphi|(x - y, t) \right) \int_{|y| \leq R} d\mu(y) \\ &\quad + 2\|\varphi\|_\infty \int_{|y| > R} d\mu(y). \end{aligned}$$

This is enough to conclude since $\lim_{R \rightarrow \infty} \int_{|y| > R} d\mu(y) = 0$. \square

4.2. L^1 instability: Proofs of Propositions 24, 25 and 26.

Proof of Proposition 24. Let us first consider the case $d = 1$ (i.e. $x \in \mathbb{R}$). The eikonal equation (16) reduces to

$$(33) \quad \partial_t \varphi = |\partial_x \varphi|.$$

Let φ_0 be a continuous approximation of $-\mathbf{1}_{(-1,1)}$, e.g. $\varphi_0(x) := -g(|x|)$ where

$$g(r) := \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 2 - r & \text{if } 1 < r \leq 2, \\ 0 & \text{if } r > 2, \end{cases}$$

and take $\varphi_0^n := \varphi_0 + \mathbf{1}_{[-1/n, 1/n]} \leq 0$ which is discontinuous. With this choice, $\varphi_0^n \rightarrow \varphi_0$ in $L^1(\mathbb{R})$ as $n \rightarrow \infty$. Let now φ and $\underline{\varphi}_n$ be the exact and minimal viscosity solutions of (33) with φ_0 and φ_0^n as initial data. By Proposition 8,

$$\begin{aligned} \varphi(x, t) &= \sup_{x+[-t, t]} \varphi_0 = -g(|x| + t) \quad \text{and} \\ \underline{\varphi}_n(x, t) &= \sup_{x+[-t, t]} (\varphi_0^n)_* = -g(|x| + t) + \mathbf{1}_{(-t-1/n, t+1/n)}(x) \end{aligned}$$

for $t \leq (1 - 1/n)/2$. Thus for $0 < t \leq 1/4$ and $n \geq 2$,

$$\int (\underline{\varphi}_n - \varphi)(x, t) dx = \int_{-t-1/n}^{t+1/n} dx \geq 2t.$$

Next we take continuous $(\tilde{\varphi}_0^n)_n$ such that

$$\tilde{\varphi}_0^n \geq \varphi_0^n \quad \text{and} \quad \lim_{n \rightarrow \infty} \|\tilde{\varphi}_0^n - \varphi_0^n\|_{L^1} = 0.$$

Clearly, we have $\tilde{\varphi}_0^n \rightarrow \varphi_0$ in $L^1(\mathbb{R})$ as $n \rightarrow \infty$, and the corresponding solutions of (33) satisfy $\tilde{\varphi}_n \geq \underline{\varphi}_n$ by Proposition 7(ii). Hence, for any $0 < t < 1/4$ and $n \geq 2$,

$$\int (\tilde{\varphi}_n - \varphi)(x, t) dx \geq \int (\underline{\varphi}_n - \varphi)(x, t) dx \geq 2t.$$

Since $\tilde{\varphi}_n \geq \varphi$ as well, $\liminf_{n \rightarrow \infty} \|\tilde{\varphi}_n(\cdot, t) - \varphi(\cdot, t)\|_{L^1} > 0$ and the proof for $d = 1$ is complete. In d dimensions, we replace $\mathbf{1}_{(-1,1)}$ by $\mathbf{1}_{(-1,1)^d}$. The argument is the same and we leave the details to the reader. \square

To prove Propositions 25 and 26, we need the lemma below.

Lemma 57. *For any $(x, t) \in \mathbb{R} \times \mathbb{R}^+$, let*

$$\psi(x, t) := \begin{cases} U(|x|/\sqrt{t}) & \text{if } t > 0, \\ \mathbf{1}_{\{0\}}(x) & \text{if } t = 0, \end{cases}$$

where

$$U(r) := c_0 \int_r^\infty e^{-\frac{s^2}{4}} ds \quad \text{with} \quad c_0 := \left(\int_0^\infty e^{-\frac{s^2}{4}} ds \right)^{-1}.$$

Then $\psi \in BUSC(\mathbb{R}^d \times \mathbb{R}^+)$ and is a subsolution of (17).

Proof of Proposition 25. In addition to the above lemma, we also need Theorem 28. Theorem 28 says that $E = C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ satisfies (20)–(21). Hence, the semigroup G_t associated to (17) is such that

$$G_t : C_b \cap L_{\text{int}}^\infty(\mathbb{R}) \rightarrow C_b \cap L_{\text{int}}^\infty(\mathbb{R}) \subset L^1(\mathbb{R}) \quad \forall t \geq 0.$$

This immediately implies the if-part of Proposition 25. Let us continue with the only-if-part. It is based on the following pointwise lower bound:

$$(34) \quad \varphi(x, t) \geq U(1/\sqrt{t}) \sup_{x+[-1,1]} \varphi_0 \quad \forall x \in \mathbb{R}, \forall t > 0,$$

where U is the profile from the previous lemma, $0 \leq \varphi_0 \in C_b(\mathbb{R}^d)$ and φ is the solution of (17) with φ_0 as initial data. Let us prove (34). Fix x and t . The sup on the right-hand side is attained at some $x_0 \in x + [-1, 1]$. By the previous lemma,

$$(y, s) \mapsto \varphi_0(x_0)U(|y - x_0|/\sqrt{s})$$

is a *BUSC* subsolution of (17). At $s = 0$, it equals the function

$$y \mapsto \varphi_0(x_0)\mathbf{1}_{\{x_0\}}(y)$$

which is less or equal to $\varphi_0 = \varphi_0(y)$. By the comparison principle (Theorem 3),

$$\varphi(y, s) \geq \varphi_0(x_0)U(|y - x_0|/\sqrt{s}) \quad \forall y \in \mathbb{R}, \forall s > 0.$$

Taking $(y, s) = (x, t)$, we then get that

$$\varphi(x, t) \geq \underbrace{\varphi_0(x_0)}_{= \sup_{x+[-1,1]} \varphi_0} \underbrace{U(|x - x_0|/\sqrt{t})}_{\geq U(1/\sqrt{t})}.$$

This completes the proof of (34). From that bound the only-if-part of Proposition 25 is obvious since $U(1/\sqrt{t})$ is positive for $t > 0$. \square

Proof of Proposition 26. Let $x_0 \in \mathbb{R}$ and $c > 0$ be such that

$$\rho \geq c\mathbf{1}_{\{x_0\}},$$

where ρ is defined in (18), and define

$$\psi_n(x, t) := nc\psi(nx - x_0, n^2t),$$

where ψ is given by Lemma 57. It is easy to see that ψ_n remains a subsolution of (17). Moreover, it is *BUSC* with

$$\varphi_n(x, 0) \geq \psi_n(x, 0) \quad \forall x \in \mathbb{R},$$

by (18). Hence $\varphi_n \geq \psi_n$ by the comparison principle and it suffices to show that

$$\lim_{n \rightarrow \infty} \psi_n(x, t) = \infty \quad \forall x \in \mathbb{R}, \forall t > 0.$$

But this is quite easy because

$$\psi_n(x, t) = ncU \left(\left| x - \frac{x_0}{n} \right| / \sqrt{t} \right),$$

for any $x \in \mathbb{R}$ and $t > 0$, and both the constant c and the profile $U(\cdot)$ are positive. The proof of Proposition 26 is complete. \square

Let us now prove Lemma 57.

Proof of Lemma 57. Let us first prove that ψ , defined as in the lemma, is a subsolution of (17). In the domain $\{x \neq 0, t > 0\}$, we have

$$\partial_t \psi = \partial_{xx}^2 \psi = (\partial_{xx}^2 \psi)^+$$

in the classical sense, by direct computations based on the explicit formula for $U(\cdot)$. If now $x = 0$, we have

$$\partial_t \psi(0, \cdot) = 0 \leq (\partial_{xx}^2 \psi(0, \cdot))^+$$

since $\psi(0, \cdot)$ is constant in time.

Let us now show that ψ is *BUSC*. It is clearly continuous for positive t and it only remains to prove that

$$\mathbf{1}_{x=0} \geq \limsup_{\mathbb{R} \times \mathbb{R}^+ \ni (y, t) \rightarrow (x, 0)} U \left(|y| / \sqrt{t} \right),$$

for any $x \in \mathbb{R}$. If $x = 0$, the result follows since $U(r) \leq U(0) = 1$, for any $r \geq 0$, by the choice of c_0 in the statement of the lemma. If $x \neq 0$, then we use that

$$|y| / \sqrt{t} \rightarrow \infty \quad \text{as } (y, t) \rightarrow (x, 0^+)$$

together with the fact that $\lim_{r \rightarrow \infty} U(r) = 0$. The proof of Lemma 57 is now complete. \square

4.3. L_{int}^∞ stability: Proof of Theorem 31. The proof of Theorem 30 is much easier and done later in Section 4.4. For Theorem 31, we precisely quantify the influence of the equation's nonlinearities on the estimates.

4.3.1. Preliminary technical results. Let us give several lemmas that will be needed. They consist in constructing special supersolutions of (1a) or variants. The first one involves the successive resolution of the pure convective and diffusive PDEs.

Lemma 58. *Assume (H1) and φ is a bounded subsolution of (1a). Then, for any $x \in \mathbb{R}^d$ and $t_0 \geq 0$,*

$$\varphi^*(x, t_0) \leq \overline{\phi}(x, t_0),$$

where $\overline{\phi}$ is the maximal solution of

$$(35) \quad \partial_t \phi = \sup_{\xi \in \mathcal{E}} \text{tr} \left(a(\xi) D^2 \phi \right), \quad \phi_0(x) := \sup_{x+t_0\mathcal{C}} \varphi^*(\cdot, 0)$$

(with $\mathcal{C} = \overline{\text{co}\{\text{Im}(b)\}}$).

The proof uses another version of Proposition 9 which holds without assumption (13). It involves the new control set

$$(36) \quad \mathcal{U} := \{\text{progressively measurable } \mathcal{U}\text{-valued processes } (\mathbf{q}_t, \boldsymbol{\sigma}_t)\},$$

where $\mathcal{U} := \overline{\{(q, \sigma) \in \mathbb{R}^d \times \mathbb{S}_d : (q, \sigma^2) \in \text{co}\{\text{Im}(b, a)\}\}}$. Here is this version:

Proposition 59. *Assume (H1) and $\varphi_0 \in C_b(\mathbb{R}^d)$. Then the viscosity solution of (1) satisfies*

$$\varphi(x, t) = \sup_{(q, \cdot, \cdot) \in \mathcal{U}} \mathbb{E} \{ \varphi_0(\mathbf{X}_t^x) \} \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+,$$

where the Ito process \mathbf{X}_s^x satisfies the Ito SDE

$$\begin{cases} d\mathbf{X}_s^x = \mathbf{q}_s ds + \sqrt{2} \boldsymbol{\sigma}_s d\mathbf{B}_s, & s > 0, \\ \mathbf{X}_{s=0}^x = x. \end{cases}$$

Proof. By Lemma 88 in Appendix A.4, we can rewrite (1a) as follows:

$$\partial_t \varphi = \sup_{(q, \sigma) \in \mathcal{U}} \{q \cdot D\varphi + \text{tr}(\sigma^2 D^2 \varphi)\},$$

where \mathcal{U} is defined below (36). Since \mathcal{U} is compact and $(q, \sigma) \in \mathcal{U} \mapsto q \in \mathbb{R}^d$ and $(q, \sigma) \in \mathcal{U} \mapsto \sigma \in \mathbb{S}_d$ are continuous, we can apply results from stochastic control theory [30], see also Proposition 9. \square

Proof of Lemma 58. Take the supconvolution in space, φ^ε for $\varepsilon > 0$, see (30). By Lemma 54, it is a subsolution of (1a), Lipschitz continuous in x , and *BUSC* in (x, t) . Let us first derive an inequality for φ^ε . By the representation formula Proposition 59 and the comparison principle, we have

$$\varphi^\varepsilon(x, t_0) \leq \sup_{(q, \sigma) \in \mathcal{U}} \mathbb{E} \{ \varphi^\varepsilon(\mathbf{X}_{t_0}^x, 0) \}$$

where $\mathbf{X}_{t_0}^x = x + \int_0^{t_0} \mathbf{q}_t dt + \sqrt{2} \int_0^{t_0} \boldsymbol{\sigma}_t d\mathbf{B}_t$; indeed, the right-hand side is the unique solution with initial data $\varphi^\varepsilon(\cdot, 0)$. By the definition of \mathcal{U} , see (36), we infer that

$$\int_0^{t_0} \mathbf{q}_t(\omega) dt \in t_0 \mathcal{C} \quad \forall (\mathbf{q}_t, \boldsymbol{\sigma}_t) \in \mathcal{U}, \forall \omega \in \Omega,$$

where Ω is the sample space of (12). Hence, for such given controls,

$$\varphi^\varepsilon(\mathbf{X}_{t_0}^x, 0) \leq \sup_{\mathbf{Y}_{t_0}^x + t_0 \mathcal{C}} \varphi^\varepsilon(\cdot, 0)$$

where $\mathbf{Y}_{t_0}^x := x + \sqrt{2} \int_0^{t_0} \boldsymbol{\sigma}_t d\mathbf{B}_t$. Let ϕ_0^ε be the supconvolution of ϕ_0 , then

$$\begin{aligned} \varphi^\varepsilon(\mathbf{X}_{t_0}^x, 0) &\leq \sup_{y \in \mathbf{Y}_{t_0}^x + t_0 \mathcal{C}} \sup_{z \in \mathbb{R}^d} \left\{ \varphi^*(y + z, 0) - \frac{|z|^2}{2\varepsilon^2} \right\} \\ &= \sup_{z \in \mathbb{R}^d} \left[\left\{ \sup_{y \in \mathbf{Y}_{t_0}^x + t_0 \mathcal{C}} \varphi^*(y + z, 0) \right\} - \frac{|z|^2}{2\varepsilon^2} \right] \\ &= \sup_{z \in \mathbb{R}^d} \left\{ \phi_0(\mathbf{Y}_{t_0}^x + z) - \frac{|z|^2}{2\varepsilon^2} \right\} \\ &= \phi_0^\varepsilon(\mathbf{Y}_{t_0}^x). \end{aligned}$$

Taking expectation and sup over the controls, we get

$$(37) \quad \varphi^\varepsilon(x, t_0) \leq \sup_{\boldsymbol{\sigma} \in \sqrt{\mathcal{D}}} \mathbb{E} \{ \phi_0^\varepsilon(\mathbf{Y}_{t_0}^x) \}$$

for $\mathbf{Y}_{t_0}^x = x + \sqrt{2} \int_0^{t_0} \boldsymbol{\sigma}_t d\mathbf{B}_t$,

$$\sqrt{\mathcal{D}} := \left\{ \text{progressively measurable } \boldsymbol{\sigma}_t \text{ with values in } \sqrt{\mathcal{D}} \right\}$$

and

$$\sqrt{\mathcal{D}} := \overline{\{ \sigma \in \mathbb{S}_d : \sigma^2 \in \text{co} \{ \text{Im}(a) \} \}}.$$

Here we have used that if $(\mathbf{q}_t, \boldsymbol{\sigma}_t) \in \mathcal{U}$ then necessarily $\boldsymbol{\sigma}_t \in \sqrt{\mathcal{D}}$. But using again the representation of Proposition 59, the right-hand side of (37) is the solution ϕ_ε of (35) with initial data ϕ_0^ε . Hence,

$$\varphi^\varepsilon(x, t_0) \leq \phi_\varepsilon(x, t_0) \quad \forall x \in \mathbb{R}^d,$$

and we deduce the result as $\varepsilon \downarrow 0$. Indeed, since $\phi_0^\varepsilon \downarrow \phi_0$ (pointwise) by standard properties, $\phi_\varepsilon \downarrow \bar{\phi}$ by the stability result for extremal solutions in Proposition 53. The proof is complete (since $\varphi^\varepsilon \downarrow \varphi^*$ too). \square

The next lemma involves the convolution semigroup of the equation

$$(38) \quad \partial_t \varphi = \text{tr} (a_0 D^2 \varphi),$$

with a fixed a_0 . Let us recall its definition, see [15] and the references therein.

Proposition 60 (Convolution semigroups). *For any $0 \leq a_0 \in \mathbb{S}_d$, there exists a family of measures $\mu_t \in M^1(\mathbb{R}^d)$, parametrized by $t \geq 0$, such that*

- (i) $\mu_{t=0}$ is the Dirac delta at $x = 0$,
- (ii) $\mu_{t+s} = \mu_t * \mu_s$ for any $t, s \geq 0$,
- (iii) $t \geq 0 \mapsto \mu_t$ is weakly- \star continuous, i.e.

$$\lim_{t \rightarrow t_0} \int \varphi_0 d\mu_t = \int \varphi_0 d\mu_{t_0} \quad \forall t_0 \in \mathbb{R}^+, \forall \varphi_0 \in C_0(\mathbb{R}^d),$$

- (iv) $\mu_t \geq 0$ and $\mu_t(\mathbb{R}^d) \leq 1$ for any $t \geq 0$, and
- (v) for any $\varphi_0 \in C_b^\infty(\mathbb{R}^d)$,

$$\varphi(x, t) := \mu_t * \varphi_0(x) \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^+)$$

solves (38) and satisfies $\varphi(\cdot, t = 0) = \varphi_0(\cdot)$.

Lemma 61. *Assume $\tilde{b} = \tilde{b}(\xi)$ and $\tilde{a} = \tilde{a}(\xi)$ satisfy (H1), and that $a_0 \in \mathbb{S}_d$ is a nonnegative matrix with convolution semigroup μ_t . Then for any bounded and nonnegative subsolution ϕ of the equation*

$$(39) \quad \partial_t \phi = \sup_{\xi \in \mathcal{E}} \left\{ \tilde{b}(\xi) \cdot D\phi + \text{tr} (\tilde{a}(\xi) D^2 \phi) \right\} + \text{tr} (a_0 D^2 \phi)$$

and any time $t \geq 0$,

$$(40) \quad \phi^*(\cdot, t) \leq \mu_t * \bar{\varphi}(\cdot, t),$$

where $\bar{\varphi}$ is the maximal solution of

$$(41) \quad \partial_t \varphi = \sup_{\xi \in \mathcal{E}} \left\{ \tilde{b}(\xi) \cdot D\varphi + \text{tr} (\tilde{a}(\xi) D^2 \varphi) \right\}, \quad \varphi(\cdot, 0) = \phi^*(\cdot, 0).$$

For the proof, we need a classical result for viscosity solutions (see e.g. [22]). Let BUC be the spaces of bounded uniformly continuous functions.

Lemma 62. *Assume (H1) and $\varphi_0 \in BUC(\mathbb{R}^d)$. Then the viscosity solution φ of (1) satisfies $\varphi \in BUC(\mathbb{R}^d \times \mathbb{R}^+)$.*

Let us prove the previous lemma.

Proof of Lemma 61. Take the space approximate unit (31) and consider

$$(x, t) \mapsto \mu_t * \rho_\nu(x).$$

By Proposition 60, it is nonnegative, C_b^∞ in space-time, and solves:

$$\partial_t (\mu_t * \rho_\nu) = \text{tr} (a_0 D^2 (\mu_t * \rho_\nu)), \quad \mu_{t=0} * \rho_\nu = \rho_\nu.$$

Then take the supconvolution ϕ^ε in space of ϕ , as in (30). By Lemma 54, it is a $BUSC$ subsolution of the same equation as ϕ , that is (39). It is also Lipschitz continuous in space, and hence by Lemma 62 there exists a unique solution $\varphi_\varepsilon \in BUC(\mathbb{R}^d \times \mathbb{R}^+)$ of the PDE part of (41) with initial data

$$\varphi_\varepsilon(\cdot, 0) = \phi^\varepsilon(\cdot, 0).$$

By the comparison principle, and since $\phi \geq 0$, we know that φ_ε is nonnegative. Now let θ_ν be the time approximate unit (32) with $\delta = \nu$, and define

$$\varphi_{\varepsilon,\nu} := \varphi_\varepsilon *_{x,t} (\rho_\nu \theta_\nu).$$

This function is nonnegative, C_b^∞ in space-time, and by Lemma 55 a supersolution of the PDE part of (41). Now, we claim that:

(42) The function $\phi_{\varepsilon,\nu}(x, t) := \mu_t * \rho_\nu * \varphi_{\varepsilon,\nu}(\cdot, t)(x)$ is a supersolution of (39).

This follows from the computations

$$\begin{aligned} \partial_t \phi_{\varepsilon,\nu} &= \mu_t * \rho_\nu * \partial_t \varphi_{\varepsilon,\nu} + \varphi_{\varepsilon,\nu} * \partial_t (\mu_t * \rho_\nu) \\ &\geq \mu_t * \rho_\nu * \sup_{\mathcal{E}} \left\{ \tilde{b} \cdot D\varphi_{\varepsilon,\nu} + \text{tr}(\tilde{a} D^2 \varphi_{\varepsilon,\nu}) \right\} + \varphi_{\varepsilon,\nu} * \text{tr}(a_0 D^2 (\mu_t * \rho_\nu)), \end{aligned}$$

by the equations satisfied by $\varphi_{\varepsilon,\nu}$ and $\mu_t * \rho_\nu$. It follows that

$$\begin{aligned} \partial_t \phi_{\varepsilon,\nu} &\geq \sup_{\mathcal{E}} \left\{ \tilde{b} \cdot (\mu_t * \rho_\nu * D\varphi_{\varepsilon,\nu}) + \text{tr}(\tilde{a} (\mu_t * \rho_\nu * D^2 \varphi_{\varepsilon,\nu})) \right\} \\ &\quad + \text{tr}(a_0 (\varphi_{\varepsilon,\nu} * D^2 (\mu_t * \rho_\nu))) \\ &= \sup_{\mathcal{E}} \left\{ \tilde{b} \cdot D\phi_{\varepsilon,\nu} + \text{tr}(\tilde{a} D^2 \phi_{\varepsilon,\nu}) \right\} + \text{tr}(a_0 D^2 \phi_{\varepsilon,\nu}), \end{aligned}$$

which completes the proof of (42).

It remains to pass to the limits to obtain (40). Since $\varphi_\varepsilon \in BUC(\mathbb{R}^d \times \mathbb{R}^+)$, it is standard that $\varphi_{\varepsilon,\nu} = \varphi_\varepsilon *_{x,t} (\rho_\nu \theta_\nu)$ converges uniformly towards φ_ε as $\nu \rightarrow 0^+$. Then $\phi_{\varepsilon,\nu}(x, t)$ also converges uniformly towards $\mu_t * \varphi_\varepsilon(\cdot, t)(x)$ as $\nu \rightarrow 0^+$. Indeed,

$$\begin{aligned} \|\phi_{\varepsilon,\nu} - \mu_t * \varphi_\varepsilon\|_\infty &= \|\mu_t * \rho_\nu * \varphi_{\varepsilon,\nu} - \mu_t * \varphi_\varepsilon\|_\infty \\ &= \|\mu_t * (\rho_\nu * \varphi_{\varepsilon,\nu} - \varphi_\varepsilon)\|_\infty \\ &\leq \|\rho_\nu * \varphi_{\varepsilon,\nu} - \varphi_\varepsilon\|_\infty, \end{aligned}$$

where the infinity norm is in space-time and we used that $\mu_t(\mathbb{R}^d) \leq 1$, see Proposition 60. By stability, see e.g. Proposition 51, we infer that $\mu_t * \varphi_\varepsilon$ is a *BUC* viscosity supersolution of (39) since $\phi_{\varepsilon,\nu}$ was by (42). Moreover, at the initial time,

$$\mu_{t=0} * \varphi_\varepsilon(\cdot, 0) = \varphi_\varepsilon(\cdot, 0) = \phi^\varepsilon(\cdot, 0) \geq \phi^*(\cdot, 0).$$

Comparing the sub and supersolutions ϕ and $\mu_t * \varphi_\varepsilon$ of (39) implies that

$$(43) \quad \phi^*(\cdot, t) \leq \mu_t * \varphi_\varepsilon(\cdot, t) \quad \forall t \geq 0.$$

We now conclude by the stability of extremal solutions, see Proposition 53. Indeed, recall that $\varphi_\varepsilon(\cdot, 0) = \phi^\varepsilon(\cdot, 0) \downarrow \phi^*(\cdot, 0)$ as $\varepsilon \downarrow 0$, so that the corresponding solutions of (41) satisfy $\varphi_\varepsilon \downarrow \bar{\varphi}$. We then get (40) by passing to the limit in (43) using the dominated convergence theorem. Let us justify this. Since the measure μ_t is bounded, it suffices to derive an upper bound on φ_ε uniform in ε . To do so, use the maximum principle $\varphi_\varepsilon \leq \sup \varphi_\varepsilon(\cdot, 0)$ and recall that $\varphi_\varepsilon(\cdot, 0) = \phi^\varepsilon(\cdot, 0)$ to find that $\varphi_\varepsilon \leq \sup \phi$. The proof is complete. \square

4.3.2. *Proof of Theorem 31.* We are now ready to focus on the proof of Theorem 31, that is to say of Estimate (23). First we will roughly speaking reduce the proof to considering the pure diffusive equation

$$(44) \quad \partial_t \phi = \sup_{\xi \in \mathcal{E}} \text{tr}(a(\xi) D^2 \phi).$$

Define $c_a(t)$ to be the smallest $c \in [1, \infty]$ such that for any bounded and nonnegative subsolution ϕ of (44),

$$(45) \quad \int \sup_{\bar{Q}_1(x)} \phi^*(\cdot, t) \, dx \leq c \int \sup_{\bar{Q}_1(x)} \phi^*(\cdot, 0) \, dx.$$

Henceforth it is agreed that (45) holds whenever the left-hand side is zero or the right-hand side is not finite. Note then that $c = \infty$ satisfies (45) because $\phi^* \equiv 0$ whenever $\phi^*(\cdot, 0) \equiv 0$, by the comparison principle. Actually, we shall see that $c_a(t)$ is always finite; but, let us first explain how to reduce the proof of (23) to the qualitative analysis of this constant.

Lemma 63. *Assume (H1), and φ and ψ are bounded sub and supersolutions of (1a). Then for all $t \geq 0$,*

$$(46) \quad \int \sup_{\overline{Q}_1(x)} (\varphi^* - \psi_*)^+(\cdot, t) \, dx \leq c_a(t) \int \sup_{\overline{Q}_1(x)+t\mathcal{C}} (\varphi^* - \psi_*)^+(\cdot, 0) \, dx.$$

The proof uses the elementary result:

Lemma 64. *Assume (H1), and φ and ψ are bounded sub and supersolutions of (1a). Then $\varphi^* - \psi_*$ is a subsolution.*

Skech of the proof. Let

$$H_\xi(\varphi) := b(\xi) \cdot D\varphi + \text{tr}(a(\xi)D^2\varphi)$$

and note that then

$$\partial_t(\varphi - \psi) \leq \sup_{\xi \in \mathcal{E}} H_\xi(\varphi) - \sup_{\xi \in \mathcal{E}} H_\xi(\psi) \leq \sup_{\xi \in \mathcal{E}} (H_\xi(\varphi) - H_\xi(\psi)),$$

which gives us the result. \square

Rigourously, we need to argue with a test function. This can be done with the help of the Ishii lemma and semijets, see [22, Theorem 8.3]. Such a justification is standard and left to the reader.

Proof of Lemma 63. Lemma 64 implies that $\varphi^* - \psi_*$ is a subsolution of (1a); hence, Lemma 58 implies that

$$(\varphi^* - \psi_*)(x, t_0) \leq \overline{\phi}(x, t_0),$$

for any $x \in \mathbb{R}^d$ and $t_0 \geq 0$, where $\overline{\phi}$ is the maximal solution of (44) with initial data $\phi_0(x) = \sup_{x+t_0\mathcal{C}} (\varphi^* - \psi_*)(\cdot, 0)$. Taking the sup in x over cubes, we get

$$\sup_{\overline{Q}_1(x)} (\varphi^* - \psi_*)^+(\cdot, t_0) \leq \sup_{\overline{Q}_1(x)} (\overline{\phi})^+(\cdot, t_0).$$

Note that $(\overline{\phi})^+$ is still a subsolution of (44), by the stability by supremum of viscosity subsolutions, see Proposition 50. In particular,

$$\int \sup_{\overline{Q}_1(x)} (\overline{\phi})^+(\cdot, t_0) \, dx \leq c_a(t_0) \int \sup_{\overline{Q}_1(x)} (\phi_0)^+ \, dx$$

by the definition of $c_a(t_0)$. We then readily get the desired result from the previous inequalities and using the definition of ϕ_0 . \square

We now estimate $c_a(t)$ in terms of $|H|_{\text{diff}}$ from Definition 18. We will need the lemmas below.

Lemma 65. *Let $a = a(\xi)$ satisfy (H1), $a_0 \in \mathbb{S}_d$, $0 \leq a_0 \leq \text{Im}(a)$, and ϕ be a bounded and nonnegative subsolution of (44). Then for any $t \geq 0$,*

$$(47) \quad \int \sup_{\overline{Q}_1(x)} \phi^*(\cdot, t) \, dx \leq \int \sup_{\overline{Q}_1(x)} \overline{\varphi}(\cdot, t) \, dx,$$

where $\overline{\varphi}$ is the maximal solution of

$$\partial_t \varphi = \sup_{\xi \in \mathcal{E}} \text{tr}((a(\xi) - a_0)D^2\varphi), \quad \varphi(\cdot, 0) = \phi^*(\cdot, 0).$$

Proof. We can rewrite (44) as follows:

$$\partial_t \phi = \sup_{\xi} \{ \operatorname{tr}((a - a_0)D^2 \phi) \} + \operatorname{tr}(a_0 D^2 \phi).$$

We recognize Equation (39) with $\tilde{b} \equiv 0$ and $\tilde{a}(\xi) = a(\xi) - a_0$. Applying Lemma 61, we obtain that

$$\phi^*(\cdot, t) \leq \mu_t * \bar{\varphi}(\cdot, t),$$

for any $t \geq 0$ and some nonnegative measure μ_t with mass $\mu_t(\mathbb{R}^d) \leq 1$. The desired inequality (47) then readily follows by a simple Young type inequality for convolutions. \square

Corollary 66. *Assume $a = a(\xi)$ satisfies (H1) and $c_a(t)$ is defined in (45). Then for any nonnegative $a_0 \in \mathbb{S}_d$ such that $a_0 \leq \operatorname{Im}(a)$ and any $t \geq 0$,*

$$c_a(t) \leq c_{a-a_0}(t).$$

Proof. This is immediate from Lemma 65. \square

Here is the estimate of $c_a(t)$ which involves the modulus below.

Lemma 67. *For any $r \geq 0$, let $\omega_d(r)$ be the smallest $\omega \in [0, \infty]$ satisfying*

$$(48) \quad \int \sup_{\bar{Q}_1(x)} \varphi^*(\cdot, r) \, dx \leq (1 + \omega) \int \sup_{\bar{Q}_1(x)} \varphi^*(\cdot, 0) \, dx,$$

for any nonnegative and bounded subsolution of

$$(49) \quad \partial_t \varphi = \sup_{\lambda \in \operatorname{Sp}(D^2 \varphi)} \lambda^+.$$

Then, for any $a = a(\xi)$ satisfying (H1) and $t \geq 0$,

$$(50) \quad c_a(t) \leq 1 + \omega_d(t|H|_{\text{diff}}).$$

The proof is based on the Ky Fan type inequality (see e.g. [46]),

$$(51) \quad \operatorname{tr}(XY) \leq \sum_{i=1}^d \lambda_i(X) \lambda_i(Y) \quad \forall X, Y \in \mathbb{S}_d,$$

where $\lambda_1 \leq \dots \leq \lambda_d$ denote the ordered eigenvalues.

Proof of Lemma 67. Let ϕ be a nonnegative bounded subsolution of (44). By (51),

$$\partial_t \phi \leq \|\operatorname{tr}(a)\|_{\infty} \sup_{\lambda \in \operatorname{Sp}(D^2 \phi)} \lambda^+$$

in the viscosity sense. We recognize Equation (49) up to a rescaling. Indeed, set

$$\varphi(x, t) := \phi \left(x, \frac{t}{\|\operatorname{tr}(a)\|_{\infty}} \right)$$

and observe that φ is a subsolution of (49). By the definition of $\omega_d(\cdot)$ in (48), we deduce that

$$\begin{aligned} \int \sup_{\bar{Q}_1(x)} \phi^*(\cdot, t) \, dx &= \int \sup_{\bar{Q}_1(x)} \varphi^*(\cdot, t \|\operatorname{tr}(a)\|_{\infty}) \, dx \\ &\leq (1 + \omega_d(t \|\operatorname{tr}(a)\|_{\infty})) \int \sup_{\bar{Q}_1(x)} \varphi^*(\cdot, 0) \, dx \\ &= (1 + \omega_d(t \|\operatorname{tr}(a)\|_{\infty})) \int \sup_{\bar{Q}_1(x)} \phi^*(\cdot, 0) \, dx. \end{aligned}$$

Recalling that ϕ is an arbitrary nonnegative and bounded subsolution of (44), it follows that

$$c_a(t) \leq 1 + \omega_d(t \|\operatorname{tr}(a)\|_{\infty})$$

(this constant being the smallest one satisfying (45)). Taking now $a_0 \in \mathbb{S}_d^+$ such that $a_0 \leq \text{Im}(a)$ and repeating the above arguments, we also have

$$c_{a-a_0}(t) \leq 1 + \omega_d(t \|\text{tr}(a - a_0)\|_\infty).$$

Using in addition Corollary 66 and taking the infimum in a_0 , we infer that

$$c_a(t) \leq 1 + \omega_d \left(t \inf_{a_0 \text{ such as above}} \|\text{tr}(a - a_0)\|_\infty \right),$$

by the monotonicity of $\omega_d(\cdot)$. This function is nondecreasing because the right-hand side of (49) is nonnegative such that the solutions of (49) are nondecreasing in time. We recognize the above inf as the quantity $|H|_{\text{diff}}$ from Definition 18 and this gives us the desired inequality (50).

Note finally that we have assumed $\|\text{tr}(a - a_0)\|_\infty$ to be positive when dividing by it. If this is not the case then, $a \equiv a_0$ for some a_0 and Equation (44) becomes linear. Its solutions are then given by Proposition 60, and it is readily seen that $c_a(t) = 1$. This completes the proof. \square

In order to establish Theorem 31, it remains to show that $\omega_d(\cdot)$ is a modulus of continuity. Here is a technical lemma. It involves again the profile

$$U(r) = c_0 \int_r^\infty e^{-\frac{s^2}{4}} ds \quad \text{with} \quad c_0 = \left(\int_0^\infty e^{-\frac{s^2}{4}} ds \right)^{-1}.$$

Lemma 68. *For any $(x, t) \in \mathbb{R}^d \times \mathbb{R}^+$, define*

$$\Phi(x, t) := \begin{cases} U \left((|x| - 1)^+ / \sqrt{t} \right) & \text{if } t > 0, \\ \mathbf{1}_{|x| < 1} & \text{if } t = 0. \end{cases}$$

Then Φ is a supersolution of (49) which is BLSC on $\mathbb{R}^d \times \mathbb{R}^+$.

Proof. The lower semicontinuity up to $t = 0$ is immediate. Arguing as in Lemma 57, we observe that (49) is satisfied for $|x| < 1$, since Φ is constant in that region. It is satisfied if $|x| = 1$, because the subsets are empty. Now if $|x| > 1$,

$$\Phi(x, t) = U \left((|x| - 1) / \sqrt{t} \right),$$

so that

$$\partial_t \Phi = -\frac{|x| - 1}{2t^{\frac{3}{2}}} U' \quad \text{and} \quad \partial_{x_i x_j}^2 \Phi = \left(\frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3} \right) \frac{U'}{\sqrt{t}} + \frac{x_i x_j}{|x|^2} \frac{U''}{t}.$$

Since $U' \leq 0$ and $U'' \geq 0$, we have $\sum_{i,j=1}^d \partial_{x_i x_j}^2 \Phi h_i h_j \leq \frac{U''}{t}$ for any $h = (h_i)$ with $|h| = 1$ and we have equality if $h = x/|x|$. Hence

$$(52) \quad \sup_{\lambda \in \text{Sp}(D^2 \Phi)} \lambda^+ = \frac{U''}{t}$$

and $\partial_t \Phi - \sup_{\lambda \in \text{Sp}(D^2 \Phi)} \lambda^+ = -(rU'(r)/2 + U''(r))/t$ with $r := (|x| - 1)/\sqrt{t}$. But the right-hand side is zero because of the definition of U . \square

Here are some properties on Φ .

Lemma 69. *Let Φ be defined in Lemma 68. Then Φ is nonnegative, nondecreasing with respect to t , and integrable with respect to x .*

Proof. The nonnegativity and monotonicity are clear from the definition of Φ . Moreover, using polar coordinates, we see that

$$(53) \quad \begin{aligned} \int_{\mathbb{R}^d} \Phi(x, t) \, dx &= U(0) \int_{|x| \leq 1} dx + \int_{|x| > 1} U\left(\frac{|x| - 1}{\sqrt{t}}\right) dx \\ &\leq c_d \left(1 + \sqrt{t} \int_0^\infty U(s) \, ds + \sqrt{t}^d \int_0^\infty s^{d-1} U(s) \, ds\right) < \infty. \end{aligned}$$

The proof is complete. For later use, note that the above constant c_d only depends on the dimension d . \square

We are now in position to prove the result below.

Lemma 70. *Let $\omega_d(\cdot)$ be defined as in Lemma 67. Then $\omega_d(\cdot)$ is a modulus of continuity in the sense that it is everywhere finite and has limit zero at zero.*

Proof. We would like to derive an upper bound on $\omega_d(r)$ for any $r \geq 0$. Take an arbitrary bounded and nonnegative subsolution φ of (49) and let us try to get an estimate of the form (48) with a finite ω independent of φ . We have to be precise enough to conclude in the end that $\omega_d(r) \rightarrow 0$ as $r \rightarrow 0^+$. If $\int \sup_{\overline{Q}_1(x)} \varphi^*(\cdot, 0) \, dx = \infty$, then (48) holds for any $\omega \geq 0$. The interesting case is therefore when

$$(54) \quad \int \sup_{\overline{Q}_1(x)} \varphi^*(\cdot, 0) \, dx < \infty,$$

which we assume from now on. The idea to get (48) is to construct an integrable supersolution of (49), see also [27]. To do so, we need mollifiers ρ_ν in space, as in (31), and it will be important to choose them with supports

$$(55) \quad \text{supp}(\rho_\nu) \subset B_\nu(0).$$

For any $\nu > 0$, let Ψ_ν be the solution of (49) with initial data

$$\Psi_\nu(\cdot, 0) = \rho_\nu(\cdot).$$

Next, for any $\delta > 0$, we introduce

$$\psi_{\nu, \delta} := \Psi_\nu *_{x, t} \left(\theta_\delta \sup_{\overline{Q}_\nu(\cdot)} \varphi^*(\cdot, 0) \right)$$

with the time approximate unit θ_δ from (32). This notation means that

$$\psi_{\nu, \delta}(x, t) := \iint_{\mathbb{R}^d \times \mathbb{R}^-} \Psi_\nu(x - y, t - s) \theta_\delta(s) \sup_{\overline{Q}_\nu(y)} \varphi^*(\cdot, 0) \, dy \, ds.$$

Note that $\psi_{\nu, \delta} \in BUC(\mathbb{R}^d \times \mathbb{R}^+)$ since $\Psi_\nu \in BUC(\mathbb{R}^d \times \mathbb{R}^+)$ and

$$(x, t) \mapsto \theta_\delta(t) \sup_{\overline{Q}_\nu(x)} \varphi^*(\cdot, 0) \in L^1(\mathbb{R}^d \times \mathbb{R}^-),$$

by Lemma 62 and (54). It is also standard that

$$\lim_{\delta \rightarrow 0^+} \psi_{\nu, \delta} = \psi_\nu := \Psi_\nu * \sup_{\overline{Q}_\nu(\cdot)} \varphi^*(\cdot, 0) \quad \text{uniformly on } \mathbb{R}^d \times \mathbb{R}^+$$

(with $* = *_x$). The latter limit will be the supersolution of (49) that we will use to derive an L_{int}^∞ a priori estimate of the form (48).

Let us first compare φ with ψ_ν . Note that ψ_ν is indeed a supersolution of (49) because so were all the $\psi_{\nu, \delta}$ by Lemma 55. It is also BUC up to $t = 0$ as a uniform limit of BUC functions. Since, by (55),

$$\psi_\nu(x, 0) = \int \rho_\nu(y) \sup_{\overline{Q}_\nu(x-y)} \varphi^*(\cdot, 0) \, dy \geq \varphi^*(x, 0) \quad \forall x \in \mathbb{R}^d,$$

the comparison principle implies that

$$\varphi^* \leq \psi_\nu.$$

It follows that for any $r \geq 0$ and $x \in \mathbb{R}^d$,

$$\begin{aligned} \frac{\sup}{\overline{Q}_1(x)} \varphi^*(\cdot, r) &\leq \sup_{z \in [-1, 1]^d} \int \Psi_\nu(y, r) \frac{\sup}{\overline{Q}_\nu(x+z-y)} \varphi^*(\cdot, 0) dy \\ &\leq \int \Psi_\nu(y, r) \frac{\sup}{\overline{Q}_{1+\nu}(x-y)} \varphi^*(\cdot, 0) dy, \end{aligned}$$

so that

$$\int \frac{\sup}{\overline{Q}_1(x)} \varphi^*(\cdot, r) dx \leq \int_{\mathbb{R}^d} \Psi_\nu(y, r) dy \int \frac{\sup}{\overline{Q}_{1+\nu}(x)} \varphi^*(\cdot, 0) dx.$$

Applying Lemma 91 of the appendix, we get that

$$\int \frac{\sup}{\overline{Q}_1(x)} \varphi^*(\cdot, r) dx \leq (1 + \omega) \int \frac{\sup}{\overline{Q}_1(x)} \varphi^*(\cdot, 0) dx,$$

where

$$\omega := (1 + \nu)^d \int_{\mathbb{R}^d} \Psi_\nu(y, r) dy - 1.$$

This ω is nonnegative since Ψ_ν is nondecreasing in time as a solution of (49). It is also independent of φ so we can conclude that

$$(56) \quad \omega_d(r) \leq \inf_{\nu > 0} (1 + \nu)^d \int_{\mathbb{R}^d} \Psi_\nu(y, r) dy - 1 \quad \forall r \geq 0,$$

by the definition of $\omega_d(r)$ given in Lemma 67. From this inequality, it is sufficient to prove that $\Psi_\nu \in C(\mathbb{R}^+; L^1(\mathbb{R}^d))$ for any $\nu > 0$, to conclude that $\omega_d(\cdot)$ is a modulus of continuity. Indeed, $\omega_d(\cdot)$ will then be everywhere finite and have limit zero at zero since $\int \Psi_\nu(y, 0) dy = \int \rho_\nu = 1$ and ν can be taken arbitrary small.

Let us check that $\Psi_\nu \in C(\mathbb{R}^+; L^1(\mathbb{R}^d))$ to finish the proof. We need the supersolution Φ from Lemma 68. Let

$$(57) \quad \Phi_\nu(x, t) := \frac{c}{\nu^d} \Phi\left(\frac{x}{\nu}, \frac{t}{\nu^2}\right),$$

for some constant $c > 0$ and note that this is a supersolution of (49) since so was Φ . Take c so large that $c \mathbf{1}_{|\cdot| < 1} \geq \rho(\cdot)$ which implies that

$$\Phi_\nu(x, 0) = \frac{c}{\nu^d} \mathbf{1}_{|x| < \nu} \geq \frac{1}{\nu^d} \rho\left(\frac{x}{\nu}\right) = \Psi_\nu(x, 0),$$

for any $x \in \mathbb{R}^d$. It follows that $\Psi_\nu \leq \Phi_\nu$ by the comparison principle. The continuity of Ψ_ν in time with values in $L^1(\mathbb{R}^d)$ is then an immediate consequence of the properties of Φ stated in Lemma 69. Indeed, fixing for instance $T > 0$, we have

$$(58) \quad \Psi_\nu(x, t) \leq \Phi_\nu(x, T) \quad \forall x \in \mathbb{R}^d, \forall t \in [0, T],$$

since Φ is nondecreasing in time. The fixed integrable function $\Phi_\nu(\cdot, T)$ can then serve as a dominating function and continuity follows by the dominated convergence theorem. The proof is complete. \square

End of the proof of Theorem 31. Estimates (46) and (50) imply the desired inequality (23), with an $\omega_d(\cdot)$ being indeed a modulus of continuity by Lemma 70. \square

4.4. L_{int}^∞ stability: Proof of Theorem 30.

Proof of Theorem 30. Let $L_b := \|b\|_\infty$ and $L_a := \|\text{tr}(a)\|_\infty$ and assume $L_a > 0$. By the Ky Fan inequality (51), the Hamiltonian (4) of Equation (1a) satisfies

$$H(p, X) \leq L_b|p| + L_a \sup_{\lambda \in \text{Sp}(X)} \lambda^+,$$

and hence, any subsolution of (1a) is a subsolution of the equation

$$(59) \quad \partial_t \varphi = L_b|D\varphi| + L_a \sup_{\lambda \in \text{Sp}(D^2\varphi)} \lambda^+.$$

By Lemma 64, if φ and ψ are bounded sub and supersolutions of (1a), then $\varphi^* - \psi_*$ is a subsolution of the same equation. In particular, $\varphi^* - \psi_*$ is a subsolution of (59). To prove Estimate (22), we will construct an integrable supersolution of (59).

Consider again

$$U : r \geq 0 \mapsto c_0 \int_r^\infty e^{-\frac{s^2}{4}} ds$$

where $c_0 > 0$ is such that $U(0) = 1$, and

$$(60) \quad \Psi(x, t) := U\left(\frac{(|x| - 1 - L_b t)^+}{\sqrt{L_a t}}\right) \quad \text{with} \quad \Psi(x, t = 0) := \mathbf{1}_{|x| < 1}.$$

Arguing as in the proof of Lemma 68, we see that Ψ is a supersolution of (59). Define now

$$\psi := \Psi *_x \sup_{\overline{Q}_1(\cdot)} \phi_0,$$

where $\phi_0(x) := (\varphi^* - \psi_*)^+(x, t = 0)$. We will use Lemma 56 to show that ψ is a supersolution of (59). To do this, Ψ should be bounded and continuous, and $\sup_{\overline{Q}_1(\cdot)} \phi_0$ integrable. The latter condition can be assumed since (22) trivially holds if not. Since Ψ is bounded continuous only for $t > 0$, we apply Lemma 55 for $t > \varepsilon > 0$. Since ε is arbitrary, we conclude that ψ is a supersolution of (59). Since $\Psi \in C(\mathbb{R}^+, L^1)$ and $\sup_{\overline{Q}_1(\cdot)} \phi_0 \in L^\infty(\mathbb{R}^d)$, this supersolution is continuous up to $t = 0$ and satisfies

$$\psi(x, 0) = \int \mathbf{1}_{|y| < 1} \sup_{\overline{Q}_1(x-y)} \underbrace{\phi_0}_{=(\varphi^* - \psi_*)^+(t=0)} dy \geq (\varphi^* - \psi_*)(x, 0).$$

Since $\varphi^* - \psi_*$ is a subsolution of (59), $(\varphi^* - \psi_*)^+ \leq \psi$ everywhere (where we can take the positive part because $\psi \geq 0$). Hence

$$\begin{aligned} \int \sup_{\overline{Q}_1(x) \times [0, T]} (\varphi^* - \psi_*)^+ dx &\leq \int \sup_{\overline{Q}_1(x) \times [0, T]} \psi dx \\ &\leq \int \sup_{t \in [0, T]} \Psi(y, t) dy \int \sup_{\overline{Q}_2(x)} \phi_0 dx, \end{aligned}$$

by the Fubini theorem, etc. The first integral satisfies

$$\int \sup_{t \in [0, T]} \Psi(y, t) dy \leq \int U\left(\frac{(|y| - 1 - L_b T)^+}{\sqrt{L_a T}}\right) dy < \infty,$$

by (60) and since U is nondecreasing and integrable. For the second integral, Lemma 16 implies that

$$\int \sup_{\overline{Q}_2(x)} \phi_0 dx \leq C \int \sup_{\overline{Q}_1(x)} (\varphi^* - \psi_*)^+(\cdot, 0) dx,$$

for a constant C which only depends on d . Combining the three inequalities above completes the proof of (22) when $L_a = \|\text{tr}(a)\|_\infty > 0$. If $L_a = 0$, there is no diffusive part in (1a), and (22) follows from Proposition 8 and Lemma 16. \square

4.5. Consequences: Proofs of Theorem 28 and Corollary 33.

Lemma 71. *Assume (H1) and φ and ψ are continuous viscosity solutions of (1a). Then $|\varphi - \psi|$ is a subsolution of the same PDE.*

Proof. By Lemma 64, both $\varphi - \psi$ and $\psi - \varphi$ are subsolutions. Thus, by the stability by sup, $(\varphi - \psi)^+ = \max\{0, \varphi - \psi\}$ and $(\psi - \varphi)^+$ are subsolutions and then also $|\varphi - \psi| = \max\{(\varphi - \psi)^+, (\psi - \varphi)^+\}$. \square

In the sequel, G_t is the semigroup on $C_b(\mathbb{R}^d)$ of (1) defined in (19).

Proof of Theorem 28. Let us first prove that $E = C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ satisfies (20)–(21). Property (20) follows from Theorem 15. The fact that G_t maps $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ into itself follows from Theorem 31. Indeed, if $\varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$, then the function $(x, t) \mapsto |G_t \varphi_0(x)|$ is a bounded subsolution of (1a), by Lemma 71 with $\psi \equiv 0$. Estimate (23) then implies that for any $t \geq 0$,

$$\begin{aligned} \|G_t \varphi_0\|_{\text{int}} &= \int \sup_{\overline{Q}_1(x)} |G_t \varphi_0| \, dx \\ &\leq (1 + \omega_d(t|H|_{\text{diff}})) \int \sup_{\overline{Q}_1(x)+t\mathcal{C}} |\varphi_0| \, dx. \end{aligned}$$

By the boundedness of $\mathcal{C} = \overline{\text{co}\{\text{Im}(b)\}}$ and Lemma 16, these integrals are finite whenever $\varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$. It remains to prove that

$$G_t : C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d) \rightarrow C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$$

is continuous for any $t \geq 0$. Let us apply again (23) to

$$(x, t) \mapsto |G_t \varphi_0(x) - G_t \psi_0(x)|,$$

which is a subsolution of (1a) by Lemma 71. As above we get that

$$(61) \quad \|G_t \varphi_0 - G_t \psi_0\|_{\text{int}} \leq (1 + \omega_d(t|H|_{\text{diff}})) \int \sup_{\overline{Q}_1(x)+t\mathcal{C}} |\varphi_0 - \psi_0| \, dx.$$

Arguing as above we deduce the desired continuity. This completes the proof of the fact that $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ satisfies (20)–(21).

Let now E be another normed space satisfying such properties and let us prove that it is continuously embedded into $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$. We will need the following lemma (whose proof is elementary).

Lemma 72. *For any $\varphi_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\sup |\varphi_0| \leq |\sup \varphi_0| + |\inf \varphi_0|$.*

Recall then that (21) is required to hold for any data $b = b(\xi)$ and $a = a(\xi)$ satisfying (H1). Choose for instance the eikonal equation

$$\partial_t \varphi = \sum_{i=1}^d |\partial_{x_i} \varphi|$$

and denote by G_t^e its semigroup. By the representation Proposition 8,

$$G_t^e \varphi_0(x) = \sup_{x+t[-1,1]^d} \varphi_0.$$

Since $G_{t=1}^e$ maps $E \subset L^1(\mathbb{R}^d)$ into itself by assumption, the function

$$x \mapsto \sup_{x+[-1,1]^d} \varphi_0$$

belongs to $L^1(\mathbb{R}^d)$ for any $\varphi_0 \in E$. Using that E is a vector space, $-\varphi_0 \in E$, and the function

$$x \mapsto \inf_{x+[-1,1]^d} \varphi_0$$

also belongs to $L^1(\mathbb{R}^d)$. By Lemma 72, we conclude that $E \subseteq C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$. Finally we use that

$$G_{t=1}^e : E \rightarrow E$$

is continuous at $\varphi_0 \equiv 0$ to obtain that for any $\|\varphi_0^n\|_E \rightarrow 0$,

$$\|G_{t=1}^e \varphi_0^n\|_E \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Combining this with the continuity of the embedding $E \subset L^1(\mathbb{R}^d)$, we obtain that

$$\left\| \sup_{x+[-1,1]^d} \varphi_0^n \right\|_{L_x^1} \rightarrow 0.$$

Using once again that E is a normed space, the same holds with $-\varphi_0$, that is

$$\left\| \inf_{x+[-1,1]^d} \varphi_0^n \right\|_{L_x^1} \rightarrow 0.$$

By Lemma 72, we conclude that $\|\varphi_0^n\|_{\text{int}} \rightarrow 0$ which completes the proof. \square

Before proving Corollary 33, we need a technical result.

Lemma 73. *Assume (H1) and let $|H|_{\text{conv}}$ be defined in Definition 18. Then there exists $b_0 \in \mathbb{R}^d$ such that $\mathcal{C} = \overline{\text{co}\{\text{Im}(b)\}}$ is included in the cube $\overline{Q}_{|H|_{\text{conv}}}(b_0)$.*

Proof. Let $(b_0^n)_n$ be a minimizing sequence in \mathbb{R}^d for the infimum

$$|H|_{\text{conv}} = \inf_{b_0 \in \mathbb{R}^d} \underbrace{\|b - b_0\|_\infty}_{= \sup_{\xi \in \mathcal{E}} |b(\xi) - b_0|}.$$

The sequence is bounded since b is bounded, and (a subsequence of) it converges to some b_0 . But then $|H|_{\text{conv}} = \lim_{n \rightarrow \infty} \|b - b_0^n\|_\infty = \|b - b_0\|_\infty$. Hence,

$$\text{Im}(b) \subseteq \overline{Q}_{\|b - b_0\|_\infty}(b_0) = \overline{Q}_{|H|_{\text{conv}}}(b_0)$$

because for any $\xi \in \mathcal{E}$ and $i = 1, \dots, d$,

$$|(b(\xi))_i - (b_0)_i| \leq \sup_{\mathcal{E}} \sqrt{\sum_{j=1}^d (b_j - (b_0)_j)^2} = \|b - b_0\|_\infty.$$

The convex envelope \mathcal{C} is therefore also included in this cube. \square

Proof of Corollary 33. Let us continue the computations from (61). We have, for any φ_0 and ψ_0 in $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$, and any $t \geq 0$,

$$\begin{aligned} \|G_t \varphi_0 - G_t \psi_0\|_{\text{int}} &\leq (1 + \omega_d(t|H|_{\text{diff}})) \int \sup_{\overline{Q}_1(x) + t\mathcal{C}} |\varphi_0 - \psi_0| dx \\ &\leq (1 + \omega_d(t|H|_{\text{diff}})) \int \sup_{\overline{Q}_1(x) + t\overline{Q}_{|H|_{\text{conv}}}(b_0)} |\varphi_0 - \psi_0| dx, \end{aligned}$$

for some center $b_0 \in \mathbb{R}^d$ given by Lemma 73. We change the variables $x + tb_0 \mapsto x$ to find that

$$\|G_t \varphi_0 - G_t \psi_0\|_{\text{int}} \leq (1 + \omega_d(t|H|_{\text{diff}})) \int \sup_{\overline{Q}_{1+t|H|_{\text{conv}}}(x)} |\varphi_0 - \psi_0| dx.$$

Applying the inequality given by Lemma 91 in the appendix, then leads to

$$\|G_t \varphi_0 - G_t \psi_0\|_{\text{int}} \leq (1 + \omega_d(t|H|_{\text{diff}})) (1 + t|H|_{\text{conv}})^d \|\varphi_0 - \psi_0\|_{\text{int}}. \quad \square$$

4.6. L_{int}^∞ **quasicontraction: Proof of Theorem 35.** Recall that G_t^{mod} is the semigroup associated to the model HJB equation

$$(62) \quad \partial_t \varphi = |D\varphi| + \sup_{\lambda \in \text{Sp}(D^2\varphi)} \lambda^+.$$

It is a semigroup of Lipschitz operators in $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ by the previous analysis. Recall also that

$$(63) \quad \|\varphi_0\| = \sup_{t \geq 0} e^{-t} \|G_t^{\text{mod}}|\varphi_0|\|_{\text{int}}.$$

To show that it is a norm, we need to know that the modulus from Theorem 31 does not increase faster than an exponential at infinity.

Lemma 74. *The modulus from Theorem 31 satisfies*

$$\omega_d(r) = O(r^{\frac{d}{2}}) \quad \text{as } r \rightarrow \infty.$$

Proof. We have established in (56) that

$$\omega_d(r) \leq \inf_{\nu > 0} (1 + \nu)^d \int_{\mathbb{R}^d} \Psi_\nu(y, r) \, dy - 1,$$

where, by (57) and (58), $\Psi_\nu(y, r) \leq c\nu^{-d}\Phi(y\nu^{-1}, r\nu^{-2})$ for a $c > 0$ independent of ν, y, r . Taking $\nu = 1$,

$$\omega_d(r) \leq 2^d c \int_{\mathbb{R}^d} \Phi(y, r) \, dy,$$

and the result easily follows from Estimate (53). \square

Let us now establish that:

Lemma 75. *The function $\|\cdot\|$ defined in (63) is a norm on $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ which is equivalent to the usual one $\|\cdot\|_{\text{int}}$.*

Proof. Assume first that it is a norm and let us prove the equivalence. Simple computations show that $|H|_{\text{conv}} = |H|_{\text{diff}} = 1$ for the Hamiltonian of (62).² Applying Corollary 33 to that equation, we deduce that

$$\|G_t^{\text{mod}}|\varphi_0|\|_{\text{int}} \leq (1 + \omega_d(t)) (1 + t)^d \|\varphi_0\|_{\text{int}},$$

for any $\varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ and $t \geq 0$. Hence, by Lemma 74,

$$\|G_t^{\text{mod}}|\varphi_0|\|_{\text{int}} \leq M e^t \|\varphi_0\|_{\text{int}}$$

for some constant $M = M(d) \geq 1$. From that, the equivalence between the norms is clear.

Let us now prove that $\|\cdot\|$ is a norm on $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$. Using again the previous inequality, we deduce that the sup in (63) is always finite. Moreover, given any initial data φ_0 and ψ_0 in $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$, the sum of the solutions

$$\varphi(x, t) := G_t^{\text{mod}}|\varphi_0|(x) \quad \text{and} \quad \psi(x, t) := G_t^{\text{mod}}|\psi_0|(x)$$

is a supersolution of the same equation; indeed,

$$\partial_t(\varphi + \psi) = \sup\{\dots\} + \sup\{\dots\} \geq \sup\{\dots + \dots\}.$$

Hence, the comparison principle implies that, for any $t \geq 0$,

$$G_t^{\text{mod}}|\varphi_0 + \psi_0| \leq G_t^{\text{mod}}(|\varphi_0| + |\psi_0|) \leq G_t^{\text{mod}}|\varphi_0| + G_t^{\text{mod}}|\psi_0|.$$

It is thus clear that $\|\varphi_0 + \psi_0\| \leq \|\varphi_0\| + \|\psi_0\|$. To obtain that $\|\alpha\varphi_0\| = |\alpha|\|\varphi_0\|$, for any φ_0 and scalar $\alpha \in \mathbb{R}$, we use that

$$G_t^{\text{mod}}|\alpha\varphi_0| = |\alpha|G_t^{\text{mod}}|\varphi_0|.$$

²Use that the Hamiltonian of (62) is $H(p, X) = \sup_{\mathcal{E}} \{q \cdot p + \text{tr}((\bar{q} \otimes \bar{q})X)\}$, where the sup is taken over the set $\mathcal{E} = \{(q, \bar{q}) \in \mathbb{R}^d \times \mathbb{R}^d \text{ s.t. } |q|, |\bar{q}| \leq 1\}$.

This follows from the invariance of Equation (62) with respect to multiplication of $|\alpha| \geq 0$. Finally, if $\|\varphi_0\| = 0$, then choosing $t = 0$ in its definition implies that $\|\varphi_0\| \geq \|\varphi_0\|_{\text{int}} = 0$ and so $\varphi_0 \equiv 0$. The proof is complete. \square

At this stage, it only remains to obtain (25). Here is a first inequality.

Lemma 76. *Assume (H1). Then for any $\varphi_0, \psi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ and $t, s \geq 0$,*

$$G_s^{\text{mod}} |G_t \varphi_0 - G_t \psi_0| \leq G_s^{\text{mod}} G_t |\varphi_0 - \psi_0|,$$

where G_t and G_s^{mod} are the semigroups associated to (1) and (62).

Proof. By Lemma 71, the difference $|G_t \varphi_0 - G_t \psi_0|$ is a subsolution of (1a). By comparison, we obtain that

$$|G_t \varphi_0 - G_t \psi_0| \leq G_t |\varphi_0 - \psi_0|,$$

for any $t \geq 0$. We conclude by next using the comparison for Equation (62). \square

In the next lemma we use the translation operator \mathcal{T}_h defined by $\mathcal{T}_h \varphi_0(\cdot) := \varphi_0(\cdot - h)$ for $h \in \mathbb{R}^d$.

Lemma 77. *Assume (H1), $b_0 \in \mathbb{R}^d$, $a_0 \in \mathbb{S}_d$, $0 \leq a_0 \leq \text{Im}(a)$, and μ_t is the convolution semigroup corresponding to a_0 given by Proposition 60. For any nonnegative $\phi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$,*

$$\mathcal{T}_{tb_0} G_s^{\text{mod}} G_t \phi_0 \leq \mu_t * \{G_{s+\gamma t}^{\text{mod}} \phi_0\} \quad \forall t, s \geq 0,$$

where $\gamma = \|b - b_0\|_\infty \vee \|\text{tr}(a - a_0)\|_\infty$.

Proof. We will show that $\phi(x, t) := G_t \phi_0(x)$ is a subsolution of an equation related to (62). Starting from its equation (1a), we have

$$\begin{aligned} \partial_t \phi &= \sup_{\mathcal{E}} \{ (b - b_0) \cdot D\phi + \text{tr}((a - a_0)D^2\phi) \} + b_0 \cdot D\phi + \text{tr}(a_0 D^2\phi) \\ &\leq \sup_{\mathcal{E}} \{ (b - b_0) \cdot D\phi \} + \sup_{\mathcal{E}} \{ \text{tr}((a - a_0)D^2\phi) \} + b_0 \cdot D\phi + \text{tr}(a_0 D^2\phi). \end{aligned}$$

For the first term, we have

$$\sup_{\mathcal{E}} \{ (b - b_0) \cdot D\phi \} \leq \|b - b_0\|_\infty |D\phi|,$$

and for the second one,

$$\sup_{\mathcal{E}} \{ \text{tr}((a - a_0)D^2\phi) \} \leq \|\text{tr}(a - a_0)\|_\infty \sup_{\lambda \in \text{Sp}(D^2\phi)} \lambda^+,$$

by the Ky Fan inequality (51). Hence

$$(64) \quad \partial_t \phi \leq \|b - b_0\|_\infty |D\phi| + \|\text{tr}(a - a_0)\|_\infty \sup_{\lambda \in \text{Sp}(D^2\phi)} \lambda^+ + b_0 \cdot D\phi + \text{tr}(a_0 D^2\phi).$$

We almost recognize (62). To get rid of the linear convection term, we use the change of variables:

$$\tilde{\phi}(x, t) := \phi(x - tb_0, t) = \mathcal{T}_{tb_0} \phi(x, t),$$

and we rescale to normalize the coefficients:

$$\tilde{\tilde{\phi}}(x, t) := \tilde{\phi}(x, \gamma^{-1}t) \quad \text{with} \quad \gamma := \|b - b_0\|_\infty \vee \|\text{tr}(a - a_0)\|_\infty.$$

Then $\tilde{\tilde{\phi}}$ satisfies the PDE

$$\partial_t \tilde{\tilde{\phi}} \leq |D\tilde{\tilde{\phi}}| + \sup_{\lambda \in \text{Sp}(D^2\tilde{\tilde{\phi}})} \lambda^+ + \gamma^{-1} \text{tr}(a_0 D^2\tilde{\tilde{\phi}}),$$

in the viscosity sense, i.e. (62) plus a linear diffusion term. We have a PDE of the form (39). By Lemma 61, we deduce that for any $t \geq 0$,

$$\tilde{\phi}(\cdot, t) \leq \mu_{\gamma^{-1}t} * \{G_t^{\text{mod}} \tilde{\phi}(\cdot, 0)\} = \mu_{\gamma^{-1}t} * \{G_t^{\text{mod}} \phi(\cdot, 0)\}$$

for the convolution semigroup μ_t associated to the diffusion matrix a_0 . Note that by scaling, the semigroup associated to $\gamma^{-1}a_0$ is $\mu_{\gamma^{-1}t}$. Since

$$\tilde{\phi}(\cdot, t) = \tilde{\phi}(\cdot, \gamma^{-1}t) = \mathcal{T}_{\gamma^{-1}tb_0} \phi(\cdot, \gamma^{-1}t) = \mathcal{T}_{\gamma^{-1}tb_0} G_{\gamma^{-1}t} \phi_0,$$

we conclude that

$$\mathcal{T}_{\gamma^{-1}tb_0} G_{\gamma^{-1}t} \phi_0 \leq \mu_{\gamma^{-1}t} * \{G_t^{\text{mod}} \phi_0\},$$

or equivalently

$$\mathcal{T}_{tb_0} G_t \phi_0 \leq \mu_t * \{G_{\gamma t}^{\text{mod}} \phi_0\}.$$

At this stage, we fix t and let only s move. If we apply the semigroup G_s^{mod} to the previous inequality, we get that

$$G_s^{\text{mod}} \mathcal{T}_{tb_0} G_t \phi_0 \leq G_s^{\text{mod}} \{\mu_t * G_{\gamma t}^{\text{mod}} \phi_0\}$$

for any $s \geq 0$. This is a consequence of the comparison principle for (62). Using also the invariance of (62) with respect to the fixed translation $-tb_0$, we can commute the operators G_s^{mod} and \mathcal{T}_{tb_0} ,

$$(65) \quad \mathcal{T}_{tb_0} G_s^{\text{mod}} G_t \phi_0 \leq G_s^{\text{mod}} \{\mu_t * G_{\gamma t}^{\text{mod}} \phi_0\}.$$

Moreover, by Lemma 56,

$$(x, s) \mapsto \mu_t * \{G_s^{\text{mod}} G_{\gamma t}^{\text{mod}} \phi_0\}(x)$$

is a supersolution of (62). At $s = 0$ we have

$$G_{s=0}^{\text{mod}} \{\mu_t * G_{\gamma t}^{\text{mod}} \phi_0\} = \mu_t * G_{\gamma t}^{\text{mod}} \phi_0 = \mu_t * \{G_{s=0}^{\text{mod}} G_{\gamma t}^{\text{mod}} \phi_0\},$$

so we can use again the comparison principle for (62) to deduce that

$$(66) \quad G_s^{\text{mod}} \{\mu_t * G_{\gamma t}^{\text{mod}} \phi_0\} \leq \mu_t * \underbrace{\{G_s^{\text{mod}} G_{\gamma t}^{\text{mod}} \phi_0\}}_{=G_{s+\gamma t}^{\text{mod}} \phi_0},$$

for any $s \geq 0$. By (65) and (66) we can conclude the proof when $\gamma = \|b - b_0\|_\infty \vee \|\text{tr}(a - a_0)\|_\infty > 0$ (we divided by it when we rescaled). If $\gamma = 0$, Equation (64) is linear and the same reasoning applies without rescaling the time. \square

We are now ready to prove the theorem.

Proof of Theorem 35. The norms are equivalent by Lemma 75, so it remains to show Estimate (25):

$$(67) \quad e^{-s} \|G_s^{\text{mod}} |G_t \varphi_0 - G_t \psi_0|\|_{\text{int}} \leq e^{(|H|_{\text{conv}} \vee |H|_{\text{diff}})t} \sup_{\tau \geq 0} e^{-\tau} \|G_\tau^{\text{mod}} |\varphi_0 - \psi_0|\|_{\text{int}},$$

for arbitrary $\varphi_0, \psi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ and times t and s . By Lemma 76 with $\phi_0 := |\varphi_0 - \psi_0|$, it suffices to show that

$$(68) \quad e^{-s} \|G_s^{\text{mod}} G_t \phi_0\|_{\text{int}} \leq e^{(|H|_{\text{conv}} \vee |H|_{\text{diff}})t} \sup_{\tau \geq 0} e^{-\tau} \|G_\tau^{\text{mod}} \phi_0\|_{\text{int}},$$

for any t and s . Indeed that lemma shows that

$$G_s^{\text{mod}} |G_t \varphi_0 - G_t \psi_0| \leq G_s^{\text{mod}} G_t \phi_0,$$

where the left-hand side is nonnegative by the comparison principle for (62). Hence

$$\|G_s^{\text{mod}} |G_t \varphi_0 - G_t \psi_0|\|_{\text{int}} \leq \|G_s^{\text{mod}} G_t \phi_0\|_{\text{int}}$$

and (68) will imply (67). To show (68) we use Lemma 77. It implies that for any $b_0 \in \mathbb{R}^d$ and $a_0 \in \mathbb{S}_d^+$ such that $a_0 \leq \text{Im}(a)$,

$$\mathcal{T}_{tb_0} G_s^{\text{mod}} G_t \phi_0 \leq \mu_t * \{G_{s+\gamma t}^{\text{mod}} \phi_0\},$$

for some measure $\mu_t \geq 0$, $\mu_t(\mathbb{R}^d) \leq 1$, and

$$\gamma = \|b - b_0\|_\infty \vee \|\text{tr}(a - a_0)\|_\infty.$$

Since these functions are nonnegative by the comparison principle, Young's inequality implies that

$$\|\mathcal{T}_{tb_0} G_s^{\text{mod}} G_t \phi_0\|_{\text{int}} \leq \|G_{s+\gamma t}^{\text{mod}} \phi_0\|_{\text{int}},$$

for any t and s . Changing variables as before, we see that

$$\int \sup_{y \in [-1,1]^d} G_s^{\text{mod}} G_t \phi_0(x + y - tb_0) dx = \int \sup_{y \in [-1,1]^d} G_s^{\text{mod}} G_t \phi_0(x - y) dx,$$

which means that

$$\|G_s^{\text{mod}} G_t \phi_0\|_{\text{int}} = \|\mathcal{T}_{tb_0} G_s^{\text{mod}} G_t \phi_0\|_{\text{int}} \leq \|G_{s+\gamma t}^{\text{mod}} \phi_0\|_{\text{int}},$$

and multiplying by e^{-s} it easily follows that

$$e^{-s} \|G_s^{\text{mod}} G_t \phi_0\|_{\text{int}} \leq e^{(\|b-b_0\|_\infty \vee \|\text{tr}(a-a_0)\|_\infty)t} \sup_{\tau \geq 0} e^{-\tau} \|G_\tau^{\text{mod}} \phi_0\|_{\text{int}},$$

for any t, s, b_0 , and a_0 . Taking the infimum in b_0 and a_0 implies (68) and thus the desired estimate (25). \square

The proofs of the results of Section 3.2 are complete.

4.7. Weighted L^1 contraction: Proof of Theorem 38. We recall that we will use the *dual* problem (26), which can be written with a pointwise supremum taken over the Lebesgue points of F' and A . It is therefore of the form (1) and the viscosity solution theory applies.

Proof of Theorem 38. Recall that $m < M$, u and v are the solutions of (2) with initial data u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$, $0 \leq \varphi_0 \in BLSC(\mathbb{R}^d)$, and $\underline{\varphi}$ is the minimal solution of (26) with φ_0 as initial data.

We have to show that

$$(69) \quad \int_{\mathbb{R}^d} |u - v|(x, T) \varphi_0(x) dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \underline{\varphi}(x, T) dx \quad \forall T \geq 0.$$

Let us use the Kato inequality (14). For almost every $x \in \mathbb{R}^d$ and $t \geq 0$,

$$\begin{aligned} & \left\{ \sum_{i=1}^d q_i(u, v) \partial_{x_i} \phi + \sum_{i,j=1}^d r_{ij}(u, v) \partial_{x_i x_j}^2 \phi \right\} (x, t) \\ &= \text{sign}(u(x, t) - v(x, t)) \int_{v(x, t)}^{u(x, t)} \{F'(\xi) \cdot D\phi(x, t) + \text{tr}(A(\xi) D^2 \phi(x, t))\} d\xi \\ &\leq |u(x, t) - v(x, t)| \text{ess sup}_{m \leq \xi \leq M} \{F'(\xi) \cdot D\phi(x, t) + \text{tr}(A(\xi) D^2 \phi(x, t))\}, \end{aligned}$$

where we have taken the sup over $[m, M]$ because of the maximum principle Lemma 96. Injecting into (14), we get that

$$(70) \quad \begin{aligned} & \int_{\mathbb{R}^d} |u - v|(x, T) \phi(x, T) dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \phi(x, 0) dx \\ & + \iint_{\mathbb{R}^d \times (0, T)} |u - v| \left(\partial_t \phi + \text{ess sup}_{m \leq \xi \leq M} \{F'(\xi) \cdot D\phi + \text{tr}(A(\xi) D^2 \phi)\} \right) dx dt. \end{aligned}$$

In the third integral, we recognize the backward in time version of (26a). The proof of (69) then consists in taking $\phi(x, t) := \underline{\varphi}(x, T - t)$.

Simplified case: $0 \leq \varphi_0 \in C_c(\mathbb{R}^d)$.

Now (26) has a unique viscosity solution φ which coincides with $\underline{\varphi}$. It belongs to $BUC(\mathbb{R}^d \times \mathbb{R}^+)$ by Lemma 62, and it is continuous in time with values in $L^1(\mathbb{R}^d)$ by Lemma 87. Let us regularize it by convolution

$$\varphi_\nu := \varphi *_{x,t} (\rho_\nu \theta_\nu),$$

with the mollifiers (31) and (32). It follows that

$$\varphi_\nu \in C^\infty(\mathbb{R}^d \times \mathbb{R}^+) \cap C(\mathbb{R}^+; L^1(\mathbb{R}^d))$$

along with all its derivatives. This is enough to take $\phi_\nu(x, t) := \varphi_\nu(x, T - t)$ as a test function in (70) by approximation. Note that ϕ_ν is a supersolution of the backward version of (26a) by Lemma 55, i.e.

$$\partial_t \phi_\nu + \operatorname{ess\,sup}_{m \leq \xi \leq M} \{F'(\xi) \cdot D\phi_\nu + \operatorname{tr}(A(\xi)D^2\phi_\nu)\} \leq 0 \quad \text{for any } t < T.$$

Inequality (70) with the test function ϕ_ν then implies that

$$\int_{\mathbb{R}^d} |u - v|(x, T) \varphi_\nu(x, 0) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi_\nu(x, T) \, dx,$$

for any $T \geq 0$ and $\nu > 0$. By the $C(\mathbb{R}^+; L^1(\mathbb{R}^d))$ regularity of φ , the convolution $\varphi_\nu = \varphi *_{x,t} (\rho_\nu \theta_\nu)$ converges to φ in $C([0, T]; L^1(\mathbb{R}^d))$ as $\nu \rightarrow 0^+$. Passing to the limit as $\nu \rightarrow 0^+$ then yields (69).

General case: $0 \leq \varphi_0 \in BLSC(\mathbb{R}^d)$.

We would like to pointwise approximate φ_0 by a monotone sequence $\varphi_0^n \uparrow \varphi_0$ such that $0 \leq \varphi_0^n \in C_c(\mathbb{R}^d)$. Take

$$\varphi_0^n(x) := \inf_{y \in \mathbb{R}^d} \{ \varphi_0(y) \mathbf{1}_{|y| < n} + n|x - y|^2 \} \geq 0.$$

Then φ_0^n is continuous as an infconvolution. Also,

$$\varphi_0^n(x) \leq \varphi_0(x) \mathbf{1}_{|x| < n} \quad \forall x \in \mathbb{R}^d,$$

which implies that $\varphi_0^n \in C_c(\mathbb{R}^d)$. In the limit $n \rightarrow \infty$, we have $\varphi_0^n \uparrow (\varphi_0)_* = \varphi_0$. Let φ_n be the solution of (26) with initial data φ_0^n , then by the previous step,

$$\int_{\mathbb{R}^d} |u - v|(x, T) \varphi_0^n(x) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi_n(x, T) \, dx,$$

for any $T \geq 0$ and n . By the stability of extremal solutions (see Proposition 53), these solutions satisfy $\varphi_n \uparrow \underline{\varphi}$ pointwise. So we conclude the proof of (69) by passing to the limit as $n \rightarrow \infty$ using the monotone convergence theorem. \square

4.8. Duality: Proofs of Theorem 44 and Corollaries 46 and 47. We need some auxiliary lemmas. Here is a first classical result on entropy solutions.

Lemma 78. *Assume (H2) and $u_0 \in L^\infty(\mathbb{R}^d)$. Then, the entropy solution of (2) is a distributional solution of (2),*

$$\begin{aligned} \iint_{\mathbb{R}^d \times \mathbb{R}^+} \left(u \partial_t \phi + \sum_{i=1}^d F_i(u) \partial_{x_i} \phi + \sum_{i,j=1}^d \mathcal{A}_{ij}(u) \partial_{x_i x_j}^2 \phi \right) dx dt \\ + \int_{\mathbb{R}^d} u_0(x) \phi(x, 0) \, dx = 0 \quad \forall \phi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+), \end{aligned}$$

where $\mathcal{A}_{ij}(u) := \int_0^u A_{ij}(\xi) \, d\xi$.

Proof. Take $\eta(u) = \pm u$ successively in the entropy inequalities, Definition 11(c). \square

Here is another result on the continuity in time.

Lemma 79. *Assume (H2), $u_0, v_0 \in L^\infty(\mathbb{R}^d)$ with $u_0 - v_0 \in L^1(\mathbb{R}^d)$, u and v entropy solutions of (2) with initial data u_0 and v_0 . Then $u - v \in C(\mathbb{R}^+; L^1(\mathbb{R}^d))$.*

To prove it, we need the following result on viscosity solutions.

Proposition 80 (Limiting initial data). *Assume (H1) and $(\varphi_\varepsilon)_{\varepsilon>0}$ is a uniformly locally bounded family of viscosity subsolutions (resp. supersolutions) of (1a). Then $\limsup^* \varphi_\varepsilon$ (resp. $\liminf_* \varphi_\varepsilon$) satisfies*

$$\begin{aligned} \{\limsup^* \varphi_\varepsilon\}(x, 0) &= \{\limsup^*(\varphi_\varepsilon)^*(\cdot, 0)\}(x) \quad \forall x \in \mathbb{R}^d \\ (\text{resp. } \{\liminf_* \varphi_\varepsilon\}(x, 0) &= \{\liminf_*(\varphi_\varepsilon)_*(\cdot, 0)\}(x)). \end{aligned}$$

Remark 81. For subsolutions this means that

$$\limsup_{\substack{\mathbb{R}^d \times \mathbb{R}^+ \ni (y, s) \rightarrow (x, 0) \\ \varepsilon \rightarrow 0^+}} \varphi_\varepsilon(y, s) = \limsup_{\substack{\mathbb{R}^d \ni y \rightarrow x \\ \varepsilon \rightarrow 0^+}} (\varphi_\varepsilon)^*(y, 0),$$

where $(\varphi_\varepsilon)^*$ is the upper semicontinuous envelope computed in (x, t) . The proof can be found in [9] and [5, Theorem 4.7].³

Proof of Lemma 79. By Theorem 38 with $\varphi_0 \equiv 1$, we have

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1} \quad \forall t \geq 0.$$

Since the left-hand side is finite, $u - v \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^d))$. By the continuity in time with values in $L^1_{\text{loc}}(\mathbb{R}^d)$ of these functions, it remains to prove that

$$(71) \quad \lim_{R \rightarrow \infty} \sup_{t \in [0, T]} \int_{|x| \geq R} |u(x, t) - v(x, t)| dx = 0 \quad \forall T \geq 0.$$

To do so, we will use again Theorem 38.

Fix $m < M$ such that u_0 and v_0 take their values in $[m, M]$, and consider

$$\varphi_0^R(x) := \varphi_0\left(\frac{x}{R}\right), \quad R > 0,$$

where $\varphi_0 = \varphi_0(x)$ is some kernel such that

$$\begin{cases} 0 \leq \varphi_0 \in C_b(\mathbb{R}^d), \\ \varphi_0(x) = 0 \text{ for } |x| \leq 1/2, \\ \text{and } \varphi_0(x) = 1 \text{ for } |x| \geq 1. \end{cases}$$

With that choice, $\varphi_0^R \rightarrow 0$ as $R \rightarrow \infty$ locally uniformly. We then claim that the solutions φ_R of (26) with initial data φ_0^R converge locally uniformly to zero too. This is a consequence of the method of relaxed semilimits [9]. Let us give details for completeness. By the maximum principle,

$$\|\varphi_R\|_\infty \leq \|\varphi_0^R\|_\infty = \|\varphi_0\|_\infty \quad \forall R > 0.$$

We can then apply Propositions 51 and 80 to $\limsup^* \varphi_R$ as $R \rightarrow \infty$ and get that it is a subsolution of (26a) satisfying

$$\limsup^* \varphi_R(x, 0) = \limsup^* \varphi_0^R(x) = 0 \quad \forall x \in \mathbb{R}^d.$$

³Let us briefly recall the ideas for the reader's convenience. First consider $\varphi = \limsup^* \varphi_\varepsilon$, $\varphi_0(x) = \{\limsup^*(\varphi_\varepsilon)^*(\cdot, 0)\}(x)$, and show that $\min\{\partial_t \varphi - H(D\varphi, D^2\varphi), \varphi - \varphi_0\} \leq 0$ at $t = 0$ in the viscosity sense, fix then some x and use the viscosity inequalities at a $\max(\bar{y}, \bar{t})$ of the function $\varphi(y, t) - |y - x|^2/\varepsilon - Ct$ with C large enough such that $\bar{t} = 0$. We get $\varphi(x, 0) \leq \varphi_0(\bar{y})$ and conclude as $\varepsilon \rightarrow 0^+$.

Similarly, $\liminf_* \varphi_R$ is a supersolution of (1) with zero as initial data. The comparison principle then implies that

$$\limsup^* \varphi_R \leq \liminf_* \varphi_R.$$

Hence φ_R converges locally uniformly to the unique solution of (26) with zero initial data, that is zero itself.

Now we can show (71). By Theorem 38 with the previous m , M , and φ_0^R ,

$$\begin{aligned} \int_{|x| \geq R} |u(x, t) - v(x, t)| dx &\leq \int_{\mathbb{R}^d} |u(x, t) - v(x, t)| \varphi_0^R(x) dx \\ &\leq \int_{\mathbb{R}^d} |u_0(x) - v_0(x)| \varphi_R(x, t) dx \leq \int_{\mathbb{R}^d} |u_0(x) - v_0(x)| \sup_{s \in [0, T]} \varphi_R(x, s) dx, \end{aligned}$$

for any $T \geq t \geq 0$. The right-hand side vanishes as $R \rightarrow \infty$ by the discussion above and the dominated convergence theorem. The proof of (71) is complete. \square

Let us continue by introducing a regularization procedure for the weights.

Lemma 82. *Assume (H2), $m < M$, ρ_ν and θ_ν are defined in (31) and (32), and $0 \leq \varphi \in BLSC(\mathbb{R}^d \times \mathbb{R}^+)$ satisfies (I) in Theorem 44. Then for any $\nu > 0$, the convolution*

$$\varphi_\nu := \varphi *_{x,t} (\rho_\nu \theta_\nu) \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^+)$$

also satisfies (I) in Theorem 44.

Proof. By assumption,

$$(72) \quad \int_{\mathbb{R}^d} |u - v|(x, t) \varphi(x, s) dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x, t + s) dx,$$

for any $t, s \geq 0$, u_0 and v_0 with values in $[m, M]$, and entropy solutions u and v of (2) with u_0 and v_0 as initial data. Our aim is to get the same result for φ_ν . Let us use (72) not for u_0 and v_0 , but their translations $u_0(\cdot + y)$ and $v_0(\cdot + y)$ for some fixed $y \in \mathbb{R}^d$. Since the PDE part of (2) is invariant with respect to translation, the corresponding solutions are $u(x + y, t)$ and $v(x + y, t)$. Hence,

$$\int_{\mathbb{R}^d} |u - v|(x + y, t) \varphi(x, s) dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x + y) \varphi(x, t + s) dx$$

for any $t, s \geq 0$. By changing the variable of integration, we obtain that

$$\int_{\mathbb{R}^d} |u - v|(x, t) \varphi(x - y, s) dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x - y, t + s) dx.$$

Now we fix $\tau \leq 0$ and apply this formula, not for s but $s - \tau$. We deduce that

$$\int_{\mathbb{R}^d} |u - v|(x, t) \varphi(x - y, s - \tau) dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x - y, t + s - \tau) dx.$$

Multiply then by $\rho_\nu(y) \theta_\nu(\tau)$ and integrate over $(y, \tau) \in \mathbb{R}^d \times \mathbb{R}^-$ to conclude. \square

We want to pass to the limit and compare the functions

$$\varphi_\flat := \liminf_* \varphi_\nu \quad \text{as } \nu \rightarrow 0^+$$

and

$$(73) \quad \varphi_\#(x, t) = \liminf_{\substack{r \rightarrow 0^+ \\ y \rightarrow x}} \frac{1}{\text{meas}(B_r(y))} \int_{B_r(y)} \varphi(z, t) dz.$$

To do so, we assume in addition that

$$(74) \quad \text{supp}(\rho_\nu) \subset B_\nu(0) \quad \text{and} \quad \text{supp}(\theta_\nu) \subset (-\nu, 0).$$

Let us first give fundamental properties on φ_\flat and $\varphi_\#$.

Lemma 83. Assume $\varphi \in BLSC(\mathbb{R}^d \times \mathbb{R}^+)$, φ_b and $\varphi_{\#}$ are as above, and (74) holds. Then:

- (i) The limit φ_b is the pointwise largest function in $BLSC(\mathbb{R}^d \times \mathbb{R}^+)$ less than or equal φ a.e. Moreover $\varphi_b = \varphi$ a.e.
- (ii) For any $t \geq 0$, $\varphi_{\#}(\cdot, t)$ is the pointwise largest function in $BLSC(\mathbb{R}^d)$ less than or equal $\varphi(\cdot, t)$ a.e. Moreover $\varphi_{\#} = \varphi(\cdot, t)$ a.e.

Remark 84. It is understood that ‘‘a.e.’’ holds in (x, t) in (i) and x in (ii).

Proof. Let us prove (i). Note first that φ_b is lower semicontinuous as a lower relaxed limit. To prove that it is not greater than φ a.e., it suffices to do it for the Lebesgue points of φ . Such points $(x, t) \in \mathbb{R}^d \times (0, \infty)$ satisfy

$$\lim_{\nu \rightarrow 0^+} \frac{1}{\nu^{d+1}} \iint_{B_\nu(x) \times (t-\nu, t+\nu)} |\varphi(y, s) - \varphi(x, t)| \, dy \, ds = 0,$$

so by the assumptions on the mollifiers, see (31), (32) and (74), we find that

$$\begin{aligned} |\varphi_\nu(x, t) - \varphi(x, t)| &\leq \frac{1}{\nu^{d+1}} \iint_{B_\nu(x) \times (t, t+\nu)} |\varphi(y, s) - \varphi(x, t)| \\ &\quad \cdot \rho\left(\frac{x-y}{\nu}\right) \theta\left(\frac{t-s}{\nu}\right) \, dy \, ds \rightarrow 0 \quad \text{as } \nu \rightarrow 0^+. \end{aligned}$$

It follows that

$$\varphi_b(x, t) \leq \lim_{\nu \rightarrow 0^+} \varphi_\nu(x, t) = \varphi(x, t),$$

at any Lebesgue point. Moreover, for any fixed (x, t) , lower semicontinuity of φ implies that

$$\varphi_\nu(y, s) = \iint_{B_\nu(y) \times (s, s+\nu)} \underbrace{\varphi(z, \tau)}_{\geq \varphi(x, t) + o(1)} \rho\left(\frac{y-z}{\nu}\right) \theta\left(\frac{s-\tau}{\nu}\right) \, dz \, d\tau \geq \varphi(x, t) + o(1)$$

as $(y, s, \nu) \rightarrow (x, t, 0^+)$, and we conclude that

$$\varphi_b(x, t) = \liminf_* \varphi_\nu(x, t) \geq \varphi(x, t).$$

We conclude that $\varphi_b = \varphi$ a.e.

Now, to conclude the proof of (i), it remains to prove that $\varphi_b \geq \psi$ pointwise for any other $\psi \in BLSC(\mathbb{R}^d \times \mathbb{R}^+)$ such that $\psi \leq \varphi$ a.e. Given such a function, let

$$\psi_b := \liminf_* \psi *_{x,t} (\rho_\nu \theta_\nu).$$

As above, $\psi \leq \psi_b$ pointwise; but also $\psi_b \leq \varphi_b$ pointwise since

$$\psi *_{x,t} (\rho_\nu \theta_\nu) \leq \varphi *_{x,t} (\rho_\nu \theta_\nu).$$

This completes the proof of (i). The arguments for (ii) are similar. \square

Here is now a general inequality between φ_b and $\varphi_{\#}$.

Lemma 85. Under the hypotheses of the previous lemma, $(\varphi_{\#})_* \leq \varphi_b$ pointwise.

Proof. Let us first prove that $\varphi_{\#}$ is measurable in (x, t) . We have

$$\varphi_{\#}(x, t) = \sup_{n \geq 1} \underbrace{\inf_{m \geq n} \inf_{\substack{\frac{1}{m} \leq r \leq \frac{1}{n} \\ |y| \leq \frac{1}{n}}} \frac{1}{\text{meas}(B_r(y))} \int_{B_r(y)} \varphi(x+z, t) \, dz}_{=:\varphi_{n,m}(x,t)} \quad \underbrace{\quad}_{=:\varphi_n(x,t)}$$

where n and m are integers. For each $\frac{1}{m} \leq r \leq \frac{1}{n}$ and $|y| \leq \frac{1}{n}$, the function

$$(x, t) \mapsto \frac{1}{\text{meas}(B_r(y))} \int_{B_r(y)} \varphi(x+z, t) dz$$

is lower semicontinuous by Fatou's lemma and $\varphi \in BLSC$ (assumption in the previous lemma). The infimum $\varphi_{n,m}$ remains lower semicontinuous, because r and y live in compact sets. Hence, $\varphi_n = \inf_{m \geq n} \varphi_{n,m}$ is measurable in (x, t) and so is $\varphi_{\#} = \sup_{n \geq 1} \varphi_n$.

We can now prove the lemma. For any $t \geq 0$, the measurable functions $\varphi, \varphi_{\#}$ satisfy $\varphi_{\#}(\cdot, t) = \varphi(\cdot, t)$ a.e., hence we may use the Fubini theorem to conclude that

$$\iint_{\mathbb{R}^d \times \mathbb{R}^+} \mathbf{1}_{\{\varphi_{\#} = \varphi\}} dx dt = \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^d} \mathbf{1}_{\{\varphi_{\#}(x,t) = \varphi(x,t)\}} dx \right) dt = 0.$$

This proves that $\varphi_{\#} = \varphi$ a.e. in (x, t) , so that $(\varphi_{\#})_* \leq \varphi$ a.e. in (x, t) . Hence $(\varphi_{\#})_* \leq \varphi_b$ pointwise by Lemma 83(i). \square

Here are further properties that we will need.

Lemma 86. *Let $\varphi, \psi \in BLSC(\mathbb{R}^d \times \mathbb{R}^+)$ and $\varphi_{\#}, \psi_{\#}$ as in (73). Then*

- (i) $\varphi \leq (\varphi_{\#})_*$ pointwise, and
- (ii) if $\varphi \leq \psi_{\#}$ pointwise, then $\varphi_{\#} \leq \psi_{\#}$ pointwise.

Proof. We can show that $\varphi \leq \varphi_{\#}$ from the definition of $\varphi_{\#}$ and the lower semicontinuity of φ , exactly as we showed that $\varphi \leq \varphi_b$ in the proof of Lemma 83. In particular, $\varphi \leq (\varphi_{\#})_*$ which is part (i). For part (ii), use Lemma 83(ii). It says that $\psi_{\#}(\cdot, t) = \psi(\cdot, t)$ a.e. in x , for each fixed $t \geq 0$. Hence, $\varphi(\cdot, t) \leq \psi(\cdot, t)$ a.e. and the desired inequality follows again from the definitions of $\varphi_{\#}$ and $\psi_{\#}$. \square

Proof of Theorem 44. Let us proceed in several steps.

1) (II) \implies (I).

By (II), $(\varphi_{\#})_*$ is a *BLSC* supersolution of (26a). In particular, for any fixed $s \geq 0$, the function

$$(x, t) \mapsto (\varphi_{\#})_*(x, t + s)$$

is also a supersolution of (26a). By Remark 39(c), we can apply Theorem 38 to this supersolution with the *BLSC* initial weight $(\varphi_{\#})_*(\cdot, s)$. The result is that

$$\int_{\mathbb{R}^d} |u - v|(x, t)(\varphi_{\#})_*(x, s) dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x)(\varphi_{\#})_*(x, t + s) dx,$$

for any $u_0 = u_0(x)$ and $v_0 = v_0(x)$ with values in $[m, M]$, u and v entropy solutions of (2) with u_0 and v_0 as initial data, and $t, s \geq 0$. This is exactly (I) but with $(\varphi_{\#})_*$ instead of φ . To replace $(\varphi_{\#})_*$ by φ , we use Lemma 86(i) for the left-hand side. For the right-hand side, we use that $(\varphi_{\#})_* \leq \varphi_{\#}$ pointwise and the fact that $\varphi_{\#}(x, t + s) = \varphi(x, t + s)$ for almost each x , see Lemma 83(ii). This implies (I) with φ , as desired.

2) (I) \implies (II) for smooth weights φ .

Let us prove the reverse implication when $0 \leq \varphi \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^+)$. We will appropriately choose u_0 and v_0 later. For the moment, we assume that

$$m \leq v_0 \leq u_0 \leq M \quad \text{and} \quad u_0 - v_0 \in L^1(\mathbb{R}^d).$$

By Lemmas 96 and 79, $0 \leq u - v \in C(\mathbb{R}^+; L^1(\mathbb{R}^d))$, and then we can use (I) to get

$$(75) \quad \int_{\mathbb{R}^d} (u - v)(x, T)\varphi(x, s) dx \leq \int_{\mathbb{R}^d} (u_0 - v_0)(x)\varphi(x, T + s) dx,$$

for any $T, s \geq 0$. Let us fix $s > 0$ and determine what PDE φ satisfies. This will be done by injecting the weak formulation of (2) into (75) and then pass to the limit as $T \rightarrow 0^+$. By Lemma 78,

$$\begin{aligned} \int_{\mathbb{R}^d} (u - v)(x, T) \phi(x, T) \, dx &= \iint_{\mathbb{R}^d \times (0, T)} \left((u - v) \partial_t \phi + \sum_{i=1}^d (F_i(u) - F_i(v)) \partial_{x_i} \phi \right. \\ &\quad \left. + \sum_{i,j=1}^d (\mathcal{A}_{ij}(u) - \mathcal{A}_{ij}(v)) \partial_{x_i x_j}^2 \phi \right) \, dx \, dt \\ &\quad + \int_{\mathbb{R}^d} (u_0 - v_0)(x) \phi(x, 0) \, dx, \end{aligned}$$

for any $\phi \in C_c^\infty(\mathbb{R}^d \times [0, T])$ and $\mathcal{A}'_{ij} = \mathcal{A}_{ij}$. Note that we have rewritten the equation given by Lemma 78 with integrals in $t < T$ and an additional final term at $t = T$. This follows from standard arguments using the L^1_{loc} continuity in time of u and v . Since $\varphi \in C_b^\infty$, $u - v \in C_t(L^1_x)$ and $u, v \in L^\infty$, a standard approximation argument shows that we can take ϕ to be

$$\phi(x, t) = \varphi(x, t + s - T),$$

and get that

$$(76) \quad \begin{aligned} \int_{\mathbb{R}^d} (u - v)(x, T) \varphi(x, s) \, dx &= \iint_{\mathbb{R}^d \times (0, T)} \left((u - v) \partial_t \varphi(t + s - T) \right. \\ &\quad \left. + \sum_{i=1}^d (F_i(u) - F_i(v)) \partial_{x_i} \varphi(t + s - T) \right. \\ &\quad \left. + \sum_{i,j=1}^d (\mathcal{A}_{ij}(u) - \mathcal{A}_{ij}(v)) \partial_{x_i x_j}^2 \varphi(t + s - T) \right) \, dx \, dt \\ &\quad + \int_{\mathbb{R}^d} (u_0 - v_0)(x) \varphi(x, s - T) \, dx. \end{aligned}$$

Here we assume that $s > 0$ and T is so small that $s - T > 0$. Inserting (76) into (75), we get

$$\begin{aligned} \int_{\mathbb{R}^d} (u_0 - v_0)(x) \varphi(x, s + T) \, dx - \int_{\mathbb{R}^d} (u_0 - v_0)(x) \varphi(x, s - T) \, dx \\ \geq \iint_{\mathbb{R}^d \times (0, T)} (\dots) \, dx \, dt. \end{aligned}$$

We now would like to divide by $2T$ and pass to the limit as $T \rightarrow 0^+$. All the computations are justified, again because $\varphi \in C_b^\infty$, the solutions u and v are bounded, and $u - v \in C(\mathbb{R}^+; L^1(\mathbb{R}^d))$. We get that

$$\begin{aligned} &\int_{\mathbb{R}^d} (u_0(x) - v_0(x)) \partial_s \varphi(x, s) \, dx \\ &\geq \frac{1}{2} \int_{\mathbb{R}^d} \left((u_0 - v_0) \partial_s \varphi(s) + \sum_{i=1}^d (F_i(u_0) - F_i(v_0)) \partial_{x_i} \varphi(s) \right. \\ &\quad \left. + \sum_{i,j=1}^d (\mathcal{A}_{ij}(u_0) - \mathcal{A}_{ij}(v_0)) \partial_{x_i x_j}^2 \varphi(s) \right) \, dx. \end{aligned}$$

Subtracting the term $\int (u_0 - v_0) \partial_s \varphi(s) dx/2$ of the right-hand side implies that

$$\begin{aligned}
(77) \quad & \int_{\mathbb{R}^d} (u_0(x) - v_0(x)) \partial_s \varphi(x, s) dx \\
& \geq \int_{\mathbb{R}^d} \left(\sum_{i=1}^d (F_i(u_0) - F_i(v_0)) \partial_{x_i} \varphi(s) + \sum_{i,j=1}^d (\mathcal{A}_{ij}(u_0) - \mathcal{A}_{ij}(v_0)) \partial_{x_i x_j}^2 \varphi(s) \right) dx \\
& = \int_{\mathbb{R}^d} \int_{v_0(x)}^{u_0(x)} \{ F'(\xi) \cdot D\varphi(x, s) + \text{tr} (A_{ij}(\xi) D^2 \varphi(x, s)) \} d\xi dx,
\end{aligned}$$

for any $s > 0$ and $0 \leq u_0 - v_0 \in L^1(\mathbb{R}^d)$ such that both u_0 and v_0 take their values in the interval $[m, M]$. It remains to choose $u_0 - v_0$ as an approximate unit, up to some multiplicative constant.

Let us introduce new parameters: $x_0 \in \mathbb{R}^d$, $\varepsilon > 0$ and $m \leq a < b \leq M$. We would like to choose

$$u_0 - v_0 = (b - a) \mathbf{1}_{x_0 + (-\varepsilon, \varepsilon)^d},$$

with the constraint that both u_0 and v_0 only take the two values a and b . Writing $x = (x_i)$, take for instance

$$u_0(x) := \begin{cases} a & \text{if } x_1 > (x_0)_1 + \varepsilon, \\ b & \text{if not,} \end{cases}$$

and

$$v_0(x) := \begin{cases} a & \text{if } x_1 > (x_0)_1 + \varepsilon \text{ or } x \in x_0 + (-\varepsilon, \varepsilon)^d, \\ b & \text{if not.} \end{cases}$$

Then $m \leq v_0 \leq u_0 \leq M$ and $u_0 - v_0 \in L^1(\mathbb{R}^d)$ as required. Inserting our choice into (77) and dividing by $(b - a)\varepsilon^d$, we deduce that

$$\begin{aligned}
& \frac{1}{\varepsilon^d} \int_{x_0 + (-\varepsilon, \varepsilon)^d} \partial_s \varphi(x, s) dx \\
& \geq \frac{1}{\varepsilon^d} \int_{x_0 + (-\varepsilon, \varepsilon)^d} \frac{1}{b - a} \int_a^b \{ F'(\xi) \cdot D\varphi(x, s) + \text{tr} (A_{ij}(\xi) D^2 \varphi(x, s)) \} d\xi dx.
\end{aligned}$$

Let now $\xi \in (m, M)$ be any Lebesgue point of any arbitrarily chosen a.e. representative of (F', A) . Take first the limit as $a, b \rightarrow \xi$ such that ξ is the center of each $[a, b]$ in order to use the Lebesgue point property; take next the limit as $\varepsilon \rightarrow 0^+$. This gives us that

$$\partial_s \varphi(x_0, s) \geq F'(\xi) \cdot D\varphi(x_0, s) + \text{tr} (A_{ij}(\xi) D^2 \varphi(x_0, s)),$$

for any $x_0 \in \mathbb{R}^d$, $s > 0$, and Lebesgue point ξ . That is φ is a supersolution of (26). This completes the proof of the remaining implication in the case where φ is C_b^∞ (and then $\varphi_\# = \varphi$).

3) (I) \implies (II) for nonnegative BLSC weights φ .

In this case we use the regularization procedure of Lemma 82. By this lemma

$$\varphi_\nu = \varphi *_{x,t} (\rho_\nu \theta_\nu)$$

satisfies (I) since φ does by assumption. By the previous step we deduce that φ_ν is a supersolution of (26a). Hence

$$\varphi_b = \liminf_* \varphi_\nu$$

is also a supersolution by stability (cf. Proposition 51). But to prove (II), we need to show that $\varphi_\#$ is a supersolution. We will do this by showing that $\varphi_b = (\varphi_\#)_*$

pointwise (at least for positive times). We already have $(\varphi_{\#})_* \leq \varphi_b$ by Lemma 85. To prove that $\varphi_b \leq (\varphi_{\#})_*$, we need to use (I). By (I),

$$\int_{\mathbb{R}^d} |u - v|(x, t) \varphi(x, s) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x, t + s) \, dx,$$

for any u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$ and corresponding solutions u and v of (2) and $t, s \geq 0$. By Lemma 83(i), we also have that $\varphi_b = \varphi$ a.e. In particular, there is a null set $N \subset \mathbb{R}^+$ such that $\varphi(\cdot, s) = \varphi_b(\cdot, s)$ a.e., for any $s \notin N$.⁴ Fixing $T > 0$, there thus exists a sequence $s_n \rightarrow T^-$ such that $s_n \notin N$, for any n . Choosing moreover $t_n := T - s_n$, we deduce that

$$\int_{\mathbb{R}^d} |u - v|(x, t_n) \varphi_b(x, s_n) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x, T) \, dx.$$

Let us pass to the limit as $n \rightarrow \infty$ in the left-hand side. To do so, we use Fatou's lemma, which is possible because of the lower semicontinuity of φ_b and the continuity of entropy solutions with values in $L^1_{\text{loc}}(\mathbb{R}^d)$ which implies that

$$|u - v|(x, t_n) \rightarrow |u_0 - v_0|(x) \quad \text{for a.e. } x$$

(along a subsequence). In the limit, it then follows that

$$\int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi_b(x, T) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x, T) \, dx$$

for any u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$ and $T > 0$. To continue, we argue as in the previous step where we chose $0 \leq u_0 - v_0 \in L^1(\mathbb{R}^d)$ to be an approximate unit up to a multiplicative constant. The same arguments imply that for any $T > 0$,

$$\varphi_b(\cdot, T) \leq \varphi(\cdot, T) \quad \text{a.e.}$$

By Lemma 83(ii), we conclude that $\varphi_b \leq \varphi_{\#}$ pointwise (for positive times). Hence, $\varphi_b \leq (\varphi_{\#})_*$ and then $\varphi_b = (\varphi_{\#})_*$ (for positive times). This implies that $(\varphi_{\#})_* = \varphi_b$ is a supersolution of (26a). The proof of Theorem 44 is complete. \square

Let us finally prove Corollaries 46 and 47.

Proof of Corollary 46. We already know that $\underline{\varphi} \in \mathscr{W}_{m, M, \varphi_0}$ by Theorem 38. Let us prove the formula with the inf. Take $\varphi \in \mathscr{W}_{m, M, \varphi_0}$, which means that $\varphi \in BLSC$ and satisfies Theorem 44(I) with $\varphi(t = 0) \geq \varphi_0$. By this theorem, φ satisfies (II) as well, that is $\varphi_{\#}$ is a supersolution of (26a). Recall that $\varphi \leq (\varphi_{\#})_*$ pointwise by Lemma 86(i). In particular

$$(\varphi_{\#})_*(t = 0) \geq \varphi(t = 0) \geq \varphi_0.$$

Thus $\varphi_{\#}$ is a supersolution of the Cauchy problem (26), and $\underline{\varphi} \leq \varphi_{\#}$ by Proposition 7(i). Then Lemma 86(ii) implies that $(\underline{\varphi})_{\#} \leq \varphi_{\#}$, and we conclude that

$$(\underline{\varphi})_{\#}(x, t) = \inf \{ \varphi_{\#}(x, t) : \varphi \in \mathscr{W}_{m, M, \varphi_0} \} \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+$$

(with an equality because $\underline{\varphi} \in \mathscr{W}_{m, M, \varphi_0}$). The proof is complete. \square

Proof of Corollary 47. Let H_t be an arbitrary strongly continuous semigroup on $C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$ satisfying

$$\int_{\mathbb{R}^d} |S_t u_0 - S_t v_0| \varphi_0 \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0| H_t \varphi_0 \, dx,$$

for any u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$, $0 \leq \varphi_0 \in C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$, and $t \geq 0$. We have to prove that for any such φ_0 and t ,

$$G_t \varphi_0 \leq H_t \varphi_0,$$

⁴To find N use that $\iint_{\mathbb{R}^d \times \mathbb{R}^+} \mathbf{1}_{\{\varphi_b = \varphi\}} \, dx \, ds = 0 = \int_{\mathbb{R}^+} \text{meas}\{\varphi(\cdot, s) = \varphi_b(\cdot, s)\} \, ds$ by Fubini.

where G_t is the semigroup corresponding to (26). For any such φ_0 , the minimal solution of (26) is the unique continuous solution, that is

$$\underline{\varphi}(x, t) = G_t \varphi_0(x) \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+.$$

Moreover, the above assumption on H_t implies that

$$H_t \varphi_0(x) \in \mathcal{W}_{m, M, \varphi_0}.$$

By Corollary 46 we deduce that for any $x \in \mathbb{R}^d$ and $t \geq 0$,

$$(G_t \varphi_0)_\#(x) \leq (H_t \varphi_0)_\#(x),$$

where we recall that

$$(G_t \varphi_0)_\#(x) = \liminf_{\substack{r \rightarrow 0^+ \\ y \rightarrow x}} \frac{1}{\text{meas}(B_r(y))} \int_{B_r(y)} G_t \varphi_0(z) \, dz$$

(and similarly for H). Since both $G_t \varphi_0(x)$ and $H_t \varphi_0(x)$ are continuous in x , we have $(G_t \varphi_0)_\# = G_t \varphi_0$ and $(H_t \varphi_0)_\# = H_t \varphi_0$ pointwise and the proof is complete. \square

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APPENDIX A. TECHNICAL PROOFS

A.1. Min and max viscosity solutions. Here are the proofs of Theorem 6 and Propositions 7 and 53; the ideas are inspired by [24, 10, 32] and the details are given for completeness.

Proof of Theorem 6. Consider the inf and supconvolutions of φ_0 ,

$$(\varphi_0)_\varepsilon(x) := \inf_{y \in \mathbb{R}^d} \left\{ (\varphi_0)_*(y) + \frac{|x - y|^2}{2\varepsilon^2} \right\}$$

and

$$(\varphi_0)^\varepsilon(x) := \sup_{y \in \mathbb{R}^d} \left\{ (\varphi_0)^*(y) - \frac{|x - y|^2}{2\varepsilon^2} \right\}.$$

Recall that they are at least C_b with $\inf \varphi_0 \leq (\varphi_0)_\varepsilon \leq \varphi_0 \leq (\varphi_0)^\varepsilon \leq \sup \varphi_0$,

$$\lim_{\varepsilon \downarrow 0} \uparrow (\varphi_0)_\varepsilon = \sup_{\varepsilon > 0} (\varphi_0)_\varepsilon = (\varphi_0)_* \quad \text{and} \quad \lim_{\varepsilon \downarrow 0} \downarrow (\varphi_0)^\varepsilon = \inf_{\varepsilon > 0} (\varphi_0)^\varepsilon = (\varphi_0)^*,$$

see e.g. [22, 30, 5, 4]. Let φ_ε and φ^ε be the viscosity solutions of (1a) with initial data $(\varphi_0)_\varepsilon$ and $(\varphi_0)^\varepsilon$, whose well-posedness is ensured by Theorem 4. By the comparison and the maximum principles, see Theorem 3 and Remark 5, we have the bounds

$$\inf \varphi_0 \leq \varphi_\varepsilon \leq \varphi^\varepsilon \leq \sup \varphi_0.$$

We can then define real-valued and bounded functions by

$$\underline{\varphi} := \sup_{\varepsilon > 0} \varphi_\varepsilon \quad \text{and} \quad \overline{\varphi} := \inf_{\varepsilon > 0} \varphi^\varepsilon.$$

We will see that these are our desired extremal solutions. First note that $\underline{\varphi}$ and $\overline{\varphi}$ are respectively lower and upper semicontinuous, as sup and inf of continuous functions. Moreover, $\underline{\varphi} \leq \overline{\varphi}$, so that at the initial time we immediately have

$$\underline{\varphi}(t=0) = (\varphi_0)_* \quad \text{and} \quad (\underline{\varphi})^*(t=0) \leq \overline{\varphi}(t=0) = (\varphi_0)^*.$$

This means that (the lower semicontinuous function) $\underline{\varphi}$ satisfies the desired discontinuous conditions of Definition 1(aii). Also, we have the exact pointwise initial data $\underline{\varphi}(t=0) = (\varphi_0)_*$ as claimed in the theorem. We argue the same way to get the desired initial data for $\overline{\varphi}$. Let us now show that these functions solve (1a). By the stability of sup and inf, see Proposition 50, we already know that $\underline{\varphi}$ and $\overline{\varphi}$ are sub and supersolutions, respectively. To get the other viscosity inequalities, we need to use that

$$(78) \quad \underline{\varphi} = \sup_{\varepsilon>0} \varphi_\varepsilon = \liminf_* \varphi_\varepsilon \quad \text{and} \quad \overline{\varphi} = \inf_{\varepsilon>0} \varphi^\varepsilon = \limsup^* \varphi^\varepsilon$$

where the relaxed limits are taken as $\varepsilon \rightarrow 0^+$. This follows by elementary arguments (see e.g. [5, 4]) since φ^ε (resp. φ_ε) is at least lower semicontinuous and increases (resp. upper semicontinuous and decreases) as $\varepsilon \downarrow 0$, which again follows by comparison since $(\varphi_0)^\varepsilon$ (resp. $(\varphi_0)_\varepsilon$) increases (resp. decreases) as $\varepsilon \downarrow 0$. For the reader's convenience, we do it for $\underline{\varphi}$. For any fixed (x, t) ,

$$\liminf_* \varphi_\varepsilon(x, t) \leq \lim_{\varepsilon \rightarrow 0^+} \varphi_\varepsilon(x, t) = \underline{\varphi}(x, t).$$

Moreover, for any sequence $(x_n, t_n, \varepsilon_n) \rightarrow (x, t, 0^+)$ such that $\varepsilon_n \leq \varepsilon_m$ for any $n \geq m$, we have $\varphi_{\varepsilon_n}(x_n, t_n) \geq \varphi_{\varepsilon_m}(x_n, t_n)$. Fixing m and taking the limit in n ,

$$\liminf_{n \rightarrow \infty} \varphi_{\varepsilon_n}(x_n, t_n) \geq \liminf_{n \rightarrow \infty} \varphi_{\varepsilon_m}(x_n, t_n) \geq \varphi_{\varepsilon_m}(x, t)$$

by lower semicontinuity of φ_{ε_m} . Taking the limit in m ,

$$\liminf_{n \rightarrow \infty} \varphi_{\varepsilon_n}(x_n, t_n) \geq \lim_{m \rightarrow \infty} \varphi_{\varepsilon_m}(x, t) = \underline{\varphi}(x, t).$$

This proves the first part of (78) and the second part can be obtained similarly. By the stability with respect to lower and upper relaxed limits, see Proposition 51, we conclude that $\underline{\varphi}$ and $\overline{\varphi}$ are viscosity solutions of (1a), and thus of (1) since we already established (1b) in the sense of Definition 1(aii).

At this stage, it only remains to prove that these solutions are extremal. Let φ be another bounded discontinuous solution. Noting that

$$(\varphi_0)_\varepsilon \leq (\varphi_0)_* \leq \varphi_*(t=0) \leq \varphi^*(t=0) \leq (\varphi_0)^* \leq (\varphi_0)^\varepsilon,$$

we use once more the comparison principle to deduce that

$$\varphi_\varepsilon \leq \varphi \leq \varphi^\varepsilon,$$

for any $\varepsilon > 0$. In the limit as $\varepsilon \rightarrow 0^+$, we conclude that $\underline{\varphi} \leq \varphi \leq \overline{\varphi}$ and the proof is complete. \square

Proof of Proposition 7. To prove (i), we simply argue as in the end of the proof of Theorem 6. For instance, assume that φ is a bounded supersolution of (1). Then,

$$(\varphi_0)_\varepsilon \leq (\varphi_0)_* \leq \varphi_*(t=0)$$

and, by comparison, $\varphi_\varepsilon \leq \varphi$ for any $\varepsilon > 0$, etc. Part (ii) follows from part (i). \square

Proof of Proposition 53. We only give the proof for the minimal solutions, the other case is similar. Let $\underline{\varphi}$ denote the minimal solution of (1) with initial data $\varphi_0 := \sup_n (\varphi_0^n)_*$. We have to prove that $\underline{\varphi} = \sup_n \underline{\varphi}_n$, where $\underline{\varphi}_n$ is the minimal solution of (1) with initial data φ_0^n . By Proposition 7(i), we have $\underline{\varphi}_n \leq \underline{\varphi}$ for any integer n . We thus already know that $\underline{\varphi} \geq \sup_n \underline{\varphi}_n$ and it only remains to prove the other inequality. To do so, it suffices to show that $\sup_n \underline{\varphi}_n$ is a supersolution of (1) (with

initial data φ_0). Indeed, by Proposition 7(i), we then get $\underline{\varphi} \leq \sup_n \underline{\varphi}_n$. It is at this stage that we need to use monotonicity. Recall that $n \mapsto \varphi_0^n(x)$ is nondecreasing for any x . By the comparison principle, cf. Proposition 7(ii), the same monotonicity holds for the minimal solutions which means that $n \mapsto \underline{\varphi}_n(x, t)$ is nondecreasing for any fixed x and t . Since $\underline{\varphi}_n$ is lower semicontinuous, we can argue as for (78) and get that

$$\sup_n \underline{\varphi}_n = \liminf_* \varphi_n \quad \text{as } n \rightarrow \infty.$$

By stability, see Propositions 51 and 80, we deduce that $\liminf_* \varphi_n$ is a supersolution of (1a) with initial data

$$\liminf_* \varphi_n(t=0) = \liminf_*(\varphi_0^n)_*$$

(the first liminf is taken in (x, t) and the second only in x). But this initial data is precisely

$$\liminf_*(\varphi_0^n)_* = \sup_n(\varphi_0^n)_* = \varphi_0,$$

again by similar arguments than for (78). This completes the proof. \square

A.2. Representation formulas. Let us prove Propositions 8 and 9. These results are classical in the control theory, but usually written for continuous or maximal solutions, see [30, 4, 34, 35]. Here we give the proofs for minimal solutions.

Proof of Proposition 8. By the assumption that $a \equiv 0$, (1a) is now

$$\partial_t \varphi = \sup_{\xi \in \mathcal{E}} \{b(\xi) \cdot D\varphi\} = \sup_{q \in \mathcal{C}} \{q \cdot D\varphi\},$$

where $\mathcal{C} = \overline{\text{co}\{\text{Im}(b)\}}$ is compact. By control theory [5, 4] the viscosity solutions of (1) is given by

$$\varphi(x, t) = \sup_{x+t\mathcal{C}} \varphi_0$$

if $\varphi_0 \in BUC$. In the general case, consider the infconvolution

$$(\varphi_0)_\varepsilon(x) := \inf_{y \in \mathbb{R}^d} \left\{ \varphi_0(y) + \frac{|x-y|^2}{2\varepsilon^2} \right\}.$$

Recall that $(\varphi_0)_\varepsilon$ is at least BUC and $(\varphi_0)_\varepsilon \uparrow (\varphi_0)_*$ pointwise as $\varepsilon \downarrow 0$. It follows that the solution of (1a) with $(\varphi_0)_\varepsilon$ as initial data is

$$\varphi_\varepsilon(x, t) = \sup_{x+t\mathcal{C}} (\varphi_0)_\varepsilon.$$

By Proposition 53, the minimal solution of (1) is thus

$$\underline{\varphi}(x, t) = \sup_{\varepsilon > 0} \varphi_\varepsilon(x, t) = \sup_{\varepsilon > 0} \sup_{x+t\mathcal{C}} (\varphi_0)_\varepsilon = \sup_{x+t\mathcal{C}} \sup_{\varepsilon > 0} (\varphi_0)_\varepsilon = \sup_{x+t\mathcal{C}} (\varphi_0)_*.$$

Rigorously speaking, Proposition 53 implies that this is the minimal solution with initial data $(\varphi_0)_*$, but it coincides with the minimal solution associated to φ_0 by Proposition 7(i). \square

Proof of Proposition 9. Equation (1a) is given by

$$\partial_t \varphi = \sup_{\xi \in \mathcal{E}} \{b(\xi) \cdot D\varphi + \text{tr}(\sigma^a(\xi)(\sigma^a)^\top(\xi)D^2\varphi)\},$$

where \mathcal{E} is compact and the coefficients b and σ^a are continuous by (13). By stochastic control theory [30], the viscosity solution of (1) is given by

$$\varphi(x, t) = \sup_{\xi} \mathbb{E} \{ \varphi_0(\mathbf{X}_t^x) \},$$

if $\varphi_0 \in BUC$ and where ξ_s and \mathbf{X}_s^x are defined as in Proposition 9. Let us now repeat the argument of the proof of Proposition 8 considering the infconvolution $(\varphi_0)_\varepsilon$ and the corresponding solution of (1a)

$$\varphi_\varepsilon(x, t) = \sup_{\xi} \mathbb{E} \{ (\varphi_0)_\varepsilon(\mathbf{X}_t^x) \}.$$

We find that the minimal solution of (1) is

$$\underline{\varphi}(x, t) = \sup_{\varepsilon > 0} \varphi_\varepsilon(x, t) = \sup_{\xi} \sup_{\varepsilon > 0} \mathbb{E} \{ (\varphi_0)_\varepsilon(\mathbf{X}_t^x) \}.$$

Since $(\varphi_0)_\varepsilon \uparrow (\varphi_0)_*$ as $\varepsilon \downarrow 0$, we conclude the proof using the monotone convergence theorem:

$$\begin{aligned} & \sup_{\varepsilon > 0} \mathbb{E} \{ (\varphi_0)_\varepsilon(\mathbf{X}_t^x) \} \\ &= \lim_{\varepsilon \downarrow 0} \uparrow \mathbb{E} \{ (\varphi_0)_\varepsilon(\mathbf{X}_t^x) \} = \mathbb{E} \left\{ \lim_{\varepsilon \downarrow 0} \uparrow (\varphi_0)_\varepsilon(\mathbf{X}_t^x) \right\} = \mathbb{E} \{ (\varphi_0)_*(\mathbf{X}_t^x) \}. \quad \square \end{aligned}$$

A.3. L_{int}^∞ strong continuity in time. During the proofs, we have assumed that the solution of (1) is continuous in time with values in L_{int}^∞ . Let us prove it here.

Lemma 87. *Assume (H1), G_t is the semigroup associated to (1) defined in (19), and $\varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$. Then the function $t \geq 0 \mapsto G_t \varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ is strongly continuous, i.e.*

$$\lim_{t \rightarrow t_0} \|G_t \varphi_0 - G_{t_0} \varphi_0\|_{\text{int}} = 0 \quad \forall t_0 \geq 0.$$

Proof. Fix $\varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ and let us prove that

$$t \geq 0 \mapsto G_t \varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$$

is continuous. That is to say, fix also $t_0 \in \mathbb{R}^+$ and let us show that

$$\int \sup_{\overline{Q}_1(x)} |G_t \varphi_0 - G_{t_0} \varphi_0| \, dx \rightarrow 0 \quad \text{as } t \rightarrow t_0.$$

The pointwise convergence follows from the continuity of $(x, t) \mapsto G_t \varphi_0(x)$ (this function being the continuous solution of (1)). Moreover, a dominating function is given by

$$x \mapsto \sup_{(y, s) \in \overline{Q}_1(x) \times [0, t_0 + 1]} |G_s \varphi_0(y)|.$$

It is indeed integrable by the uniform in time estimate of Theorem 30. \square

A.4. Main properties of $|\cdot|_{\text{conv}}$ and $|\cdot|_{\text{diff}}$. Let us prove what we have claimed in Section 2.4. We need support functions h_E of sets E (cf. e.g. [36] for their standard properties). For any $\emptyset \neq E \subseteq \mathbb{R}^d \times \mathbb{S}_d$, h_E is the function

$$h_E : (p, X) \in \mathbb{R}^d \times \mathbb{S}_d \mapsto \sup_{(q, Y) \in E} (q, Y) \cdot (p, X),$$

where $(q, Y) \cdot (p, X) := q \cdot p + \text{tr}(YX)$ is the inner product of $\mathbb{R}^d \times \mathbb{S}_d$.

Lemma 88. *Under (H1), the Hamiltonian of (1a) defined by (4) satisfies $H = h_{\mathcal{K}}$ where $\mathcal{K} = \text{co}\{\text{Im}(b, a)\}$.*

Proof. Since $H = h_{\text{Im}(b, a)}$ by definition, it suffices to use that $h_E = h_{\overline{\text{co}(E)}}$ for any set E . \square

Let us now prove the assertion of Remark 19(b), that is the result below.

Lemma 89. *Let $H : \mathbb{R}^d \times \mathbb{S}_d \rightarrow \mathbb{R}$ be such that there are two different triplets (\mathcal{E}, b, a) and $(\tilde{\mathcal{E}}, \tilde{b}, \tilde{a})$ satisfying (H1) for which*

$$H(p, X) = \sup_{\mathcal{E}} \{b \cdot p + \text{tr}(aX)\} = \sup_{\tilde{\mathcal{E}}} \{\tilde{b} \cdot p + \text{tr}(\tilde{a}X)\}.$$

Then

$$|H|_{\text{conv}} = \inf_{b_0 \in \mathbb{R}^d} \sup_{\xi \in \mathcal{E}} |b(\xi) - b_0| = \inf_{b_0 \in \mathbb{R}^d} \sup_{\tilde{\xi} \in \tilde{\mathcal{E}}} |\tilde{b}(\tilde{\xi}) - b_0|$$

and

$$|H|_{\text{diff}} = \inf_{\mathbb{S}_d^+ \ni a_0 \leq \text{Im}(a)} \sup_{\xi \in \mathcal{E}} \text{tr}(a(\xi) - a_0) = \inf_{\mathbb{S}_d^+ \ni a_0 \leq \text{Im}(\tilde{a})} \sup_{\tilde{\xi} \in \tilde{\mathcal{E}}} \text{tr}(\tilde{a}(\tilde{\xi}) - a_0).$$

Proof. By Lemma 88, we have $H = h_{\mathcal{K}} = h_{\tilde{\mathcal{K}}}$ where $\mathcal{K} = \overline{\text{co}\{\text{Im}(b, a)\}}$ and $\tilde{\mathcal{K}} = \overline{\text{co}\{\text{Im}(\tilde{b}, \tilde{a})\}}$. It follows that $\mathcal{K} = \tilde{\mathcal{K}}$ because any closed convex set is entirely determined by its support function. It thus suffices to prove that $|H|_{\text{conv}}$ and $|H|_{\text{diff}}$ only depend on \mathcal{K} . Let us do it for $|H|_{\text{diff}}$. Define the projection

$$\mathcal{D} := \text{proj}_{\mathbb{S}_d}(\mathcal{K}) = \overline{\text{co}\{\text{Im}(a)\}}.$$

Since $a_0 \leq \text{Im}(a)$ if and only if $a_0 \leq \mathcal{D}$,

$$|H|_{\text{diff}} = \inf_{\mathbb{S}_d^+ \ni a_0 \leq \mathcal{D}} \sup_{\mathcal{E}} \text{tr}(a - a_0),$$

where

$$\sup_{\mathcal{E}} \text{tr}(a - a_0) = \sup_{Y \in \text{Im}(a)} \text{tr}(Y - a_0) = \sup_{Y \in \overline{\text{co}\{\text{Im}(a)\}}} \text{tr}(Y - a_0).$$

It follows that

$$|H|_{\text{diff}} = \inf_{\mathbb{S}_d^+ \ni a_0 \leq \mathcal{D}} \sup_{Y \in \mathcal{D}} \text{tr}(Y - a_0) \quad \text{for } \mathcal{D} = \text{proj}_{\mathbb{S}_d}(\mathcal{K}).$$

We show in the same way that

$$|H|_{\text{conv}} = \inf_{b_0 \in \mathbb{R}^d} \sup_{q \in \mathcal{C}} |q - b_0| \quad \text{for } \mathcal{C} = \text{proj}_{\mathbb{R}^d}(\mathcal{K}).$$

This completes the proof since these formula only depend on \mathcal{K} . \square

Let us now prove the last result of Section 2.4.

Proof of Proposition 20. We only do the proof for $|\cdot|_{\text{diff}}$ since the argument is similar for $|\cdot|_{\text{conv}}$. Let us first prove that it is a semi-norm. It is clearly finitely valued because $a = a(\xi)$ is bounded. Given $H, \tilde{H} \in \Gamma$ with some respective triplets of data (\mathcal{E}, b, a) and $(\tilde{\mathcal{E}}, \tilde{b}, \tilde{a})$ satisfying (H1), we have

$$\begin{aligned} H(p, X) + \tilde{H}(p, X) &= \sup_{\mathcal{E}} \{b \cdot p + \text{tr}(aX)\} + \sup_{\tilde{\mathcal{E}}} \{\tilde{b} \cdot p + \text{tr}(\tilde{a}X)\} \\ &= \sup_{\mathcal{E} \times \tilde{\mathcal{E}}} \left\{ (b(\xi) + \tilde{b}(\tilde{\xi})) \cdot p + \text{tr} \left((a(\xi) + \tilde{a}(\tilde{\xi}))X \right) \right\}, \end{aligned}$$

for any $(p, X) \in \mathbb{R}^d \times \mathbb{S}_d$. This identifies some data which we can associate to the Hamiltonian $H + \tilde{H} \in \Gamma$. Let us denote the diffusion matrix by $a \oplus \tilde{a}$,

$$a \oplus \tilde{a} : (\xi, \tilde{\xi}) \mapsto a(\xi) + \tilde{a}(\tilde{\xi}),$$

and fix a_0 and \tilde{a}_0 in \mathbb{S}_d^+ such that $a_0 \leq \text{Im}(a)$ and $\tilde{a}_0 \leq \text{Im}(\tilde{a})$. It follows that $a_0 + \tilde{a}_0 \leq \text{Im}(a \oplus \tilde{a})$ and

$$\begin{aligned} |H + \tilde{H}|_{\text{diff}} &\leq \sup_{\mathcal{E} \times \tilde{\mathcal{E}}} \text{tr} \left((a(\xi) + \tilde{a}(\tilde{\xi})) - (a_0 + \tilde{a}_0) \right) \\ &= \sup_{\mathcal{E}} \text{tr}(a - a_0) + \sup_{\tilde{\mathcal{E}}} \text{tr}(\tilde{a} - \tilde{a}_0). \end{aligned}$$

Taking the infimum in a_0 and \tilde{a}_0 , we deduce that

$$|H + \tilde{H}|_{\text{diff}} \leq |H|_{\text{diff}} + |\tilde{H}|_{\text{diff}}.$$

We argue similarly to show that $|\alpha H|_{\text{diff}} = \alpha |H|_{\text{diff}}$, for any $\alpha \geq 0$.

Let us now prove (15). Take a minimizing sequence $(a_0^n)_n$ for the infimum

$$|H|_{\text{diff}} = \inf_{\mathbb{S}_d^+ \ni a_0 \leq \text{Im}(a)} \|\text{tr}(a - a_0)\|_\infty.$$

It is bounded since $a = a(\xi)$ is bounded and hence $a_0^n \rightarrow a_0$ for some matrix $a_0 \in \mathbb{S}_d^+$ up to a subsequence. In the limit, we get that

$$|H|_{\text{diff}} = \lim_{n \rightarrow \infty} \|\text{tr}(a - a_0^n)\|_\infty = \|\text{tr}(a - a_0)\|_\infty.$$

Hence if this semi-norm is zero, then $a \equiv a_0$ and the diffusion is linear. Conversely, assume that $H(p, X)$ is affine in X , for any p , and let us prove that $|H|_{\text{diff}} = 0$. Since

$$X \in \mathbb{S}_d \mapsto H(0, X) \in \mathbb{R}$$

is linear by assumption, Riesz representation theorem implies that there exists $a_0 \in \mathbb{S}_d$ such that $H(0, X) = \text{tr}(a_0 X)$ for any $X \in \mathbb{S}_d$. But we also have

$$H(0, X) = \sup_{\mathcal{E}} \text{tr}(aX) = \sup_{Y \in \overline{\text{co}\{\text{Im}(a)\}}} \text{tr}(YX)$$

and therefore $\overline{\text{co}\{\text{Im}(a)\}} = \{a_0\}$ (again because a closed convex set is entirely determined by its support function). In particular,

$$|H|_{\text{diff}} \leq \|\text{tr}(a - a_0)\|_\infty = 0$$

and this completes the proof. \square

During these proofs, we have seen some interesting connections with support functions. Let us summarize them for completeness.

Remark 90. (a) Recall the cone defined in Section 2.4:

$$\Gamma = \{H : \mathbb{R}^d \times \mathbb{S}_d \rightarrow \mathbb{R} \text{ s.t. } H \text{ satisfy (4) for some } (\mathcal{E}, b, a) \text{ satisfying (H1)}\}.$$

It satisfies

$$\Gamma = \{\text{support functions of compact convex sets } \mathcal{K} \subset \mathbb{R}^d \times \mathbb{S}_d^+\},$$

where the support function of \mathcal{K} is

$$h_{\mathcal{K}}(p, X) = \sup_{(q, Y) \in \mathcal{K}} (q, Y) \cdot (p, X)$$

for the inner product $(q, Y) \cdot (p, X) = q \cdot p + \text{tr}(YX)$.

(b) If $\mathcal{C} = \text{proj}_{\mathbb{R}^d}(\mathcal{K})$ and $\mathcal{D} = \text{proj}_{\mathbb{S}_d}(\mathcal{K})$ are the projected sets, then

i) $|h_{\mathcal{K}}|_{\text{conv}}$ is the minimal radius of balls containing \mathcal{C} , and

ii) $|h_{\mathcal{K}}|_{\text{diff}} = \inf_{\mathbb{S}_d^+ \ni Y_0 \leq \mathcal{D}} \sup_{Y \in \mathcal{D}} \text{tr}(Y - Y_0)$.

(c) For any triplet (\mathcal{E}, a, b) representing $H = h_{\mathcal{K}}$,

$$\mathcal{K} = \overline{\text{co}\{\text{Im}(b, a)\}}, \quad \mathcal{C} = \overline{\text{co}\{\text{Im}(b)\}} \quad \text{and} \quad \mathcal{D} = \overline{\text{co}\{\text{Im}(a)\}}.$$

A.5. An optimal L_{int}^∞ inequality.

Lemma 91. *For any nonnegative $\varphi_0 \in USC(\mathbb{R}^d)$, $r > 0$ and $\varepsilon \geq 0$,*

$$\int \sup_{\overline{Q}_{r+\varepsilon}(x)} \varphi_0 \, dx \leq \left(\frac{r+\varepsilon}{r} \right)^d \int \sup_{\overline{Q}_r(x)} \varphi_0 \, dx.$$

Remark 92. This is Lemma 16 with an explicit estimate of the constant. The constant is optimal, for instance for $\varphi_0 = \mathbf{1}_{\{x_0\}}$ and any fixed $x_0 \in \mathbb{R}^d$.

Proof. Assume the result holds for $d = 1$, and let $x = (x_1, \hat{x}) \in \mathbb{R} \times \mathbb{R}^{d-1}$. Then by assumption

$$\begin{aligned} \int \sup_{\overline{Q}_{r+\varepsilon}(x_1, \hat{x})} \varphi_0 \, dx_1 &= \int \sup_{y_1 \in [-r-\varepsilon, r+\varepsilon]} \overbrace{\sup_{\hat{y} \in [-r-\varepsilon, r+\varepsilon]^{d-1}} \varphi_0(x_1 + y_1, \hat{x} + \hat{y})}^{\text{fixed } USC \text{ and nonnegative function of } x_1} \, dx_1 \\ &\leq \frac{r+\varepsilon}{r} \int \sup_{y_1 \in [-r, r]} \sup_{\hat{y} \in [-r-\varepsilon, r+\varepsilon]^{d-1}} \varphi_0(x_1 + y_1, \hat{x} + \hat{y}) \, dx_1, \end{aligned}$$

for any $\hat{x} \in \mathbb{R}^{d-1}$. Integrating in \hat{x} , we get

$$\int \sup_{\overline{Q}_{r+\varepsilon}(x)} \varphi_0 \, dx \leq \frac{r+\varepsilon}{r} \int \sup_{y \in [-r, r] \times [-r-\varepsilon, r+\varepsilon]^{d-1}} \varphi_0(x+y) \, dx.$$

Starting from the integral to the right, we interchange the roles of x_1 and x_2 and repeat the arguments. Continuing in this way leads to the result.

Let us thus focus on the case $d = 1$. An approximation procedure allows to reduce the proof to the case where φ_0 is a step function,

$$\varphi_0(x) = \sum_{i=1}^n \alpha_i \mathbf{1}_{I_i}(x),$$

for some $\alpha_i \geq 0$ and bounded disjoint intervals I_i . To see this, it suffices to take the supconvolution followed by upper Riemann sum approximations of integrals. The details are rather standard and left to the reader. Let us focus on the step functions above. Let $0 < \tilde{\alpha}_1 < \dots < \tilde{\alpha}_m$ be all the possible different values taken by α_i , and

$$\varphi_0(x) = \sum_{i=1}^m \beta_i \mathbf{1}_{J_i}(x),$$

with nonnegative $\beta_1 := \tilde{\alpha}_1$, $\beta_2 := \tilde{\alpha}_2 - \tilde{\alpha}_1$, etc., and domains

$$J_i := \{\varphi_0 \geq \tilde{\alpha}_i\}$$

which are finite unions of intervals. For any $r > 0$, we have a similar representation for

$$\psi_0(x) := \sup_{x+[-r, r]} \varphi_0 = \sum_{i=1}^m \beta_i \mathbf{1}_{\{\psi_0 \geq \tilde{\alpha}_i\}}(x),$$

where by definition of ψ_0 ,

$$\{\psi_0 \geq \tilde{\alpha}_i\} = J_i + [-r, r].$$

In other words,

$$\sup_{x+[-r, r]} \varphi_0 = \sum_{i=1}^m \beta_i \mathbf{1}_{J_i+[-r, r]}(x).$$

By construction, it follows that

$$\int \sup_{x+[-r, r]} \varphi_0 \, dx = \sum_{i=1}^m \beta_i \text{meas}(J_i + [-r, r]).$$

The same formula holds for $r + \varepsilon$ and it thus suffices to prove that

$$\frac{\text{meas}(J_i + [-r - \varepsilon, r + \varepsilon])}{\text{meas}(J_i + [-r, r])} \leq \frac{r + \varepsilon}{r} \quad \forall i = 1, \dots, m.$$

To do so, rewrite

$$J_i + [-r, r] = \cup_{j=1}^k K_j$$

as a disjoint union of intervals K_j . In particular,

$$\text{meas}(J_i + [-r, r]) = \sum_{j=1}^k \text{meas}(K_j).$$

Observe also that

$$J_i + [-r - \varepsilon, r + \varepsilon] \subseteq \cup_{j=1}^k (K_j + [-\varepsilon, \varepsilon]),$$

where the union to the right no longer needs to be disjoint. Nevertheless, we have the estimate

$$\text{meas}(J_i + [-r - \varepsilon, r + \varepsilon]) \leq \sum_{j=1}^k \text{meas}(K_j + [-\varepsilon, \varepsilon])$$

and the proof further reduces to proving

$$(79) \quad \frac{\text{meas}(K_j + [-\varepsilon, \varepsilon])}{\text{meas}(K_j)} \leq \frac{r + \varepsilon}{r} \quad \forall j = 1, \dots, k,$$

since then

$$\frac{\text{meas}(J_i + [-r - \varepsilon, r + \varepsilon])}{\text{meas}(J_i + [-r, r])} \leq \frac{\sum_{j=1}^k \text{meas}(K_j + [-\varepsilon, \varepsilon])}{\sum_{j=1}^k \text{meas}(K_j)} \leq \frac{r + \varepsilon}{r}.$$

To prove (79), remember that $J_i + [-r, r] = \cup_{j=1}^k K_j$ is a disjoint union of intervals. Because of the addition of $[-r, r]$, each intervals K_j has a length greater than $2r$. To fix the ideas, let us say $K_j = [a, b]$. Then $b - a \geq 2r$ and

$$\frac{\text{meas}(K_j + [-\varepsilon, \varepsilon])}{\text{meas}(K_j)} = \frac{b - a + 2\varepsilon}{b - a} \leq \frac{2r + 2\varepsilon}{2r} = \frac{r + \varepsilon}{r},$$

because the function $\tau > 0 \mapsto \frac{\tau + \varepsilon}{\tau}$ is nonincreasing. The proof is complete. \square

APPENDIX B. COMPLEMENTARY PROOFS FOR ENTROPY SOLUTIONS

For completeness, we recall the proof of Theorem 12 which is Theorem 1.1 in [31] under (H2). We will take the opportunity to give details, but we will not perform the doubling of variables to show Lemma 14 for which we will refer to [11].

Recall that [21, 11] proved the well-posedness of L^1 kinetic or renormalized solutions which are equivalent to entropy solutions in $L^1 \cap L^\infty$. The definition of entropy solutions in $L^1 \cap L^\infty$ uses the energy estimate (2.8) of [21],

$$\iint_{\mathbb{R}^d \times \mathbb{R}^+} \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) \right)^2 dx dt \leq \frac{1}{2} \|u_0\|_{L^2} < \infty \quad \text{if } u_0 \in L^1 \cap L^\infty,$$

where

$$\zeta_{ik}(u) = \int_0^u \sigma_{ik}^A(\xi) d\xi.$$

As a consequence “ L^2 ” was used e.g. in [11, Definition 2.2] instead of “ L_{loc}^2 ” in Definition 11. But we have the following result:

Lemma 93 (Local energy estimate). *Assume (H2), $u_0 \in L^\infty(\mathbb{R}^d)$, $0 \leq \phi \in C_c^\infty(\mathbb{R}^d)$, and $T \geq 0$. If u is an entropy solution of (2) in the sense of Definition 11 and*

$$\|u_0\|_{L^\infty} + \|u\|_{L^\infty} + \|\phi\|_{W^{2,1}} \leq M,$$

then there is a constant C only depending on T , M , F and A such that

$$\iint_{\mathbb{R}^d \times (0, T)} \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u(x, t)) \right)^2 \phi(x) \, dx \, dt \leq C.$$

Proof. We use Definition 11(c) with the entropy $\eta(u) = |u|^2$ and the corresponding fluxes

$$q(u) = 2 \int_0^u \xi F'(\xi) \, d\xi \quad \text{and} \quad r(u) = 2 \int_0^u \xi A(\xi) \, d\xi.$$

We also take a test function $\phi(x) \mathbf{1}_{[0, T]}(t)$ where $0 \leq \phi \in C_c^\infty(\mathbb{R}^d)$. It is not smooth in time but a standard approximation argument shows that it can be used in Definition 11(c) if we add also a final value term at $t = T$. Here we need the L^1_{loc} continuity in time of entropy solutions. The result is

$$\begin{aligned} & \overbrace{\int_{\mathbb{R}^d} u^2(x, T) \phi(x) \, dx}^{\geq 0} + 2 \iint_{\mathbb{R}^d \times (0, T)} \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) \right)^2 \phi \, dx \, dt \\ & \leq \int_{\mathbb{R}^d} u_0^2(x) \phi(x) \, dx + \iint_{\mathbb{R}^d \times (0, T)} \left(\sum_{i=1}^d q_i(u) \partial_{x_i} \phi + \sum_{i, j=1}^d r_{ij}(u) \partial_{x_i x_j}^2 \phi \right) \, dx \, dt. \end{aligned}$$

By assumption $\|u_0\|_{L^\infty} + \|u\|_{L^\infty} + \|\phi\|_{W^{2,1}} \leq M$, so it follows that

$$\begin{cases} \|q(u)\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^+, \mathbb{R}^d)} \leq 2M^2 \operatorname{ess\,sup}_{-M \leq \xi \leq M} |F'(\xi)|, \text{ and} \\ \|r(u)\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^+, \mathbb{R}^d \times d)} \leq 2M^2 \operatorname{ess\,sup}_{-M \leq \xi \leq M} |A(\xi)|. \end{cases}$$

With all these estimates, the conclusion readily follows. \square

Let us now show the Kato inequality.

Sketch of the proof of Lemma 14. Copy the proof of Theorem 3.1 of [11] with $l = \infty$ and zero renormalization measures $\mu_i^u \equiv 0 \equiv \mu_i^v$. With the aid of the previous local energy estimate, check that every computation holds until (3.19)—even if u and v satisfy (a)–(b) of Definition 11 with L^2_{loc} and not L^2 as in [11]. This gives (14) with $\phi \in C_c^\infty(\mathbb{R}^d \times (0, \infty))$. Use an approximation argument for $\phi(x, t) \mathbf{1}_{[0, T]}(t)$ and the continuity in time with values in L^1_{loc} to get initial and final terms. \square

To show the uniqueness of entropy solutions, it suffices to find a *good* ϕ in (14), for instance an exponential as in [20, 31]. This gives the result below.

Lemma 94. *Assume (H2) and u, v are L^∞ entropy solutions of (2) with initial data $u_0, v_0 \in L^\infty(\mathbb{R}^d)$. Then for any $t \geq 0$ and $m < M$ such that $m \leq u, v \leq M$,*

$$\int_{\mathbb{R}^d} |u - v|(x, t) e^{-|x|} \, dx \leq e^{(L_F + L_A)t} \int_{\mathbb{R}^d} |u_0 - v_0|(x) e^{-|x|} \, dx,$$

where $L_F := \operatorname{ess\,sup}_{[-m, M]} |F'|$ and $L_A := \operatorname{ess\,sup}_{[-m, M]} \operatorname{tr}(A)$.

Remark 95. We can take $m < M$ such that u_0 and v_0 take their values in $[m, M]$ by the maximum principle. At this stage of this appendix, this principle is only known in $L^1 \cap L^\infty$ (or L^1) by [21, 11] and it will follow later in L^∞ .

Sketch of the proof. The proof is inspired by [20, 31]. Consider

$$\phi_\varepsilon(x, t) := e^{(L_F + L_A)(T-t) - \sqrt{\varepsilon^2 + |x|^2}},$$

for some arbitrary $\varepsilon > 0$, and check that

$$\begin{aligned} |u - v| \partial_t \phi_\varepsilon + \sum_{i=1}^d q_i(u, v) \partial_{x_i} \phi_\varepsilon + \sum_{i,j=1}^d r_{ij}(u, v) \partial_{x_i x_j}^2 \phi_\varepsilon \\ \leq |u - v| \left\{ \partial_t \phi_\varepsilon + L_F |D\phi_\varepsilon| + L_A \sup_{\lambda \in \text{Sp}(D^2 \phi_\varepsilon)} \lambda^+ \right\} \leq 0 \end{aligned}$$

by the Ky Fan inequality (51). Then by the Kato inequality (14) with ϕ_ε ,

$$\int_{\mathbb{R}^d} |u - v|(x, T) e^{-\sqrt{\varepsilon^2 + |x|^2}} dx \leq e^{(L_F + L_A)T} \int_{\mathbb{R}^d} |u_0 - v_0|(x) e^{-\sqrt{\varepsilon^2 + |x|^2}} dx$$

and the result follows in the limit $\varepsilon \rightarrow 0^+$. \square

Proof of Theorem 12. By Lemma 94, it remains to show the existence. The proof is inspired by [21, 11]. Given $u_0 \in L^\infty(\mathbb{R}^d)$, take $(u_0^n)_n$ in $L^1 \cap L^\infty(\mathbb{R}^d)$ such that

$$(80) \quad -\text{ess sup } u_0^- \leq u_0^n \leq \text{ess sup } u_0^+ \quad \text{and} \quad u_0^n \rightarrow u_0 \text{ in } L^1_{\text{loc}}(\mathbb{R}^d).$$

Let u_n be the entropy solution of (2) with initial data u_0^n . By the maximum principle (in $L^1 \cap L^\infty$), we know that

$$(81) \quad -\text{ess sup } u_0^- \leq u_n \leq \text{ess sup } u_0^+$$

Moreover, by Lemma 94, we have for any $R \geq 0$, $T \geq 0$, and integers n, m ,

$$\begin{aligned} \|u_m - u_n\|_{C([0, T]; L^1(\{|x| < R\}))} \\ = \sup_{t \in [0, T]} \int_{|x| < R} |u_m(x, t) - u_n(x, t)| dx \\ \leq e^R \sup_{t \in [0, T]} \int_{\mathbb{R}^d} |u_m(x, t) - u_n(x, t)| e^{-|x|} dx \\ \leq e^R e^{(L_F + L_A)T} \int_{\mathbb{R}^d} |u_0^m(x) - u_0^n(x)| e^{-|x|} dx, \end{aligned}$$

where the latter integral tends to zero as $n, m \rightarrow \infty$ by (80). Hence there exists some $u \in L^\infty(\mathbb{R}^d \times \mathbb{R}^+) \cap C(\mathbb{R}^+; L^1_{\text{loc}}(\mathbb{R}^d))$ such that

$$(82) \quad \lim_{n \rightarrow \infty} u_n = u \quad \text{in } C([0, T]; L^1_{\text{loc}}(\mathbb{R}^d)), \quad \forall T \geq 0.$$

It remains to show that u is an entropy solution with initial data u_0 .

We begin with the L^2_{loc} energy estimate of Definition 11(a). By Lemma 93 and the L^∞ bounds in (81), the sequence

$$\left\{ \sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u_n) \right\} \subset L^2(\mathbb{R}^d \times \mathbb{R}^+)$$

is uniformly bounded in $L^2(\mathcal{K})$, for any $k = 1, \dots, K$, and compact $\mathcal{K} \subset \mathbb{R}^d \times \mathbb{R}^+$. As a consequence, it weakly converges in $L^2(\mathcal{K})$ (up to some subsequence) to some function

$$f(x, t) \in L^2_{\text{loc}}(\mathbb{R}^d \times \mathbb{R}^+).$$

For any $\phi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+)$ and $k = 1, \dots, K$,

$$\iint_{\mathbb{R}^d \times \mathbb{R}^+} \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u_n) \right) \phi dx dt = - \iint_{\mathbb{R}^d \times \mathbb{R}^+} \left(\sum_{i=1}^d \zeta_{ik}(u_n) \partial_{x_i} \phi \right) dx dt.$$

Since

$$\zeta_{ik}(\cdot) = \int_0^\cdot \sigma_{ik}^A(\xi) \, d\xi \quad \text{for } \sigma_{ik}^A \in L_{\text{loc}}^\infty(\mathbb{R}, \mathbb{R}^{d \times K}),$$

the function $\zeta_{ik}(\cdot)$ is locally Lipschitz continuous. By the strong convergence stated in (82) and the L^∞ bounds in (81), we infer that $\zeta_{ik}(u_n)$ converges towards $\zeta_{ik}(u)$ in $C([0, T]; L_{\text{loc}}^1(\mathbb{R}^d))$ for all $T \geq 0$. Hence, at the limit, we deduce that

$$\iint_{\mathbb{R}^d \times \mathbb{R}^+} f \phi \, dx \, dt = - \iint_{\mathbb{R}^d \times \mathbb{R}^+} \left(\sum_{i=1}^d \zeta_{ik}(u) \partial_{x_i} \phi \right) \, dx \, dt.$$

By the definition of a weak derivatives, this means that

$$\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) = f \in L_{\text{loc}}^2(\mathbb{R}^d \times \mathbb{R}^+),$$

and the proof of part (a) in Definition 11 is complete. Moreover we have found that

$$\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u_n) \rightharpoonup \sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) \quad \text{in } L^2(\mathcal{K}),$$

for any $k = 1, \dots, K$ and compact $\mathcal{K} \subset \mathbb{R}^d \times \mathbb{R}^+$.

To show the chain rule in part (b) of Definition 11, we start from the chain rule for u_n ,

$$(83) \quad \sum_{i=1}^d \partial_{x_i} \zeta_{ik}^\beta(u_n) = \beta(u_n) \sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u_n) \in L^2(\mathbb{R}^d \times \mathbb{R}^+),$$

valid for any $\beta \in C(\mathbb{R})$, $k = 1, \dots, K$, and integer n . Recall also that

$$\zeta_{ik}^\beta(u_n) = \int_0^{u_n} \sigma_{ik}^A(\xi) \beta(\xi) \, d\xi.$$

By the previous convergence results and bounds, the right-hand side of (83) converges weakly to $\beta(u) \sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u)$ in $L^2(\mathcal{K})$. We can argue as before to show that the left-hand side converges weakly in $L^2(\mathcal{K})$ to $\sum_{i=1}^d \partial_{x_i} \zeta_{ik}^\beta(u)$. We thus get part (b) of Definition 11 in the limit. Moreover,

$$(84) \quad \sum_{i=1}^d \partial_{x_i} \zeta_{ik}^\beta(u_n) \rightharpoonup \sum_{i=1}^d \partial_{x_i} \zeta_{ik}^\beta(u) \quad \text{in } L^2(\mathcal{K}),$$

for any $\beta \in C(\mathbb{R})$, $k = 1, \dots, K$, and compact $\mathcal{K} \subset \mathbb{R}^d \times \mathbb{R}^+$.

Now, it remains to prove the part (c) of Definition 11. The only difficulty concerns the passage to the limit in the right-hand side, because we have a quadratic term and mere weak convergence. Let us therefore only focus on this term. We

take $\beta = \sqrt{\eta''}$ and apply the chain rule Definition 11(b),

$$\begin{aligned}
 & \iint_{\mathbb{R}^d \times \mathbb{R}^+} \eta''(u_n) \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u_n) \right)^2 \phi \, dx \, dt \\
 &= \iint_{\mathbb{R}^d \times \mathbb{R}^+} \eta''(u_n) \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u_n) \right) \left(\sum_{j=1}^d \partial_{x_j} \zeta_{jk}(u_n) \right) \phi \, dx \, dt \\
 &= \iint_{\mathbb{R}^d \times \mathbb{R}^+} \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}^{\sqrt{\eta''}}(u_n) \right) \left(\sum_{j=1}^d \partial_{x_j} \zeta_{jk}^{\sqrt{\eta''}}(u_n) \right) \phi \, dx \, dt \\
 &= \iint_{\mathbb{R}^d \times \mathbb{R}^+} \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}^{\sqrt{\eta''}}(u_n) \sqrt{\phi} \right)^2 \, dx \, dt \\
 &= \sum_{k=1}^K \left\| \sum_{i=1}^d \partial_{x_i} \zeta_{ik}^{\sqrt{\eta''}}(u_n) \sqrt{\phi} \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}^+)}^2.
 \end{aligned}$$

But, by (84), we have for any $k = 1, \dots, K$,

$$\sum_{i=1}^d \partial_{x_i} \zeta_{ik}^{\sqrt{\eta''}}(u_n) \sqrt{\phi} \rightharpoonup \sum_{i=1}^d \partial_{x_i} \zeta_{ik}^{\sqrt{\eta''}}(u) \sqrt{\phi} \quad \text{in } L^2(\mathbb{R}^d \times \mathbb{R}^+).$$

It follows that

$$\left\| \sum_{i=1}^d \partial_{x_i} \zeta_{ik}^{\sqrt{\eta''}}(u) \sqrt{\phi} \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}^+)} \leq \liminf_{n \rightarrow \infty} \left\| \sum_{i=1}^d \partial_{x_i} \zeta_{ik}^{\sqrt{\eta''}}(u_n) \sqrt{\phi} \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}^+)},$$

that is

$$\begin{aligned}
 & \liminf_{n \rightarrow \infty} \iint_{\mathbb{R}^d \times \mathbb{R}^+} \eta''(u_n) \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u_n) \right)^2 \phi \, dx \, dt \\
 & \geq \sum_{k=1}^K \left\| \sum_{i=1}^d \partial_{x_i} \zeta_{ik}^{\sqrt{\eta''}}(u) \sqrt{\phi} \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}^+)}^2 \\
 & = \iint_{\mathbb{R}^d \times \mathbb{R}^+} \eta''(u) \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) \right)^2 \phi \, dx \, dt,
 \end{aligned}$$

where similar chain rule computations have been used for u . This is enough to pass to the limit in the entropy inequalities of Definition 11(b) and the proof is complete. \square

As a byproduct of the previous proof, we get the lemma below.

Lemma 96. *Assume (H2), $u_0 \in L^\infty(\mathbb{R}^d)$, and u is the entropy solution of (2). Then $\text{ess inf } u_0 \leq u \leq \text{ess sup } u_0$. Moreover, if v is the entropy solution with initial data v_0 , then $u_0 \geq v_0$ implies $u \geq v$.*

Proof. For the comparison principle, define $u_0^n(x) := u_0(x) \mathbf{1}_{|x| < n}$ and v_0^n similarly. As previously, the associated entropy solutions u_n and v_n respectively converge towards u and v in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^d))$, $T \geq 0$, and thus almost everywhere up to taking a (common) subsequence. If $u_0 \geq v_0$, then $u_0^n \geq v_0^n$ for all n , so $u_n \geq v_n$ by the comparison principle in $L^1 \cap L^\infty$, and $u \geq v$ at the limit. For the maximum principle, apply the comparison principle to $v_0 := \text{ess inf } u_0$ and $\text{ess sup } u_0$ successively. \square

APPENDIX C. COMPLEMENTARY PROOFS FOR VISCOSITY SOLUTIONS

In Section 3.2, we claimed that (24) and (25) are optimal. Let us precise in which sense.

Lipschitz semi-norms for G_t . Let $G_t : C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d) \rightarrow C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ be the solution semigroup of (1) and define

$$\text{Lip}(G_t, \|\cdot\|) := \sup_{\varphi_0, \psi_0 \in C_b \cap L_{\text{int}}^\infty, \varphi_0 \neq \psi_0} \frac{\|G_t \varphi_0 - G_t \psi_0\|}{\|\varphi_0 - \psi_0\|}$$

with respect to a given norm $\|\cdot\|$. The contraction properties of G_t are related to the behavior of this smallest Lipschitz constant as $t \rightarrow 0^+$, for instance:

$$(85) \quad G_t \text{ is } \|\cdot\| \text{-quasicontractive} \iff \text{Lip}(G_t, \|\cdot\|) \leq 1 + O(t),$$

cf. Definition 27 and [33, 28, 13]. If we rewrite (24) and (25) in these terms, we obtain that

$$(86) \quad \begin{aligned} \text{Lip}(G_t, \|\cdot\|_{\text{int}}) &\leq 1 + O(t|H|_{\text{conv}} + \omega_d(t|H|_{\text{diff}})) \\ &\text{and } \text{Lip}(G_t, \|\cdot\|) \leq 1 + O(t(|H|_{\text{conv}} \vee |H|_{\text{diff}})) \end{aligned}$$

as $t(|H|_{\text{conv}} \vee |H|_{\text{diff}}) \rightarrow 0^+$. Let us give an example of a PDE for which (86) is optimal.

Optimal example for (24) and (25). Given $\eta, \nu \geq 0$, consider the equation

$$(87) \quad \partial_t \varphi = \eta |D\varphi| + \nu \sup_{\lambda \in \text{Sp}(D^2\varphi)} \lambda^+$$

for which $|H|_{\text{conv}} = \eta$ and $|H|_{\text{diff}} = \nu$.⁵ For the solution semigroup G_t of (87), the properties in (86) read as follows:

$$(88) \quad \text{Lip}(G_t, \|\cdot\|_{\text{int}}) \leq 1 + O(t\eta + \omega_d(t\nu)) \quad \text{and} \quad \text{Lip}(G_t, \|\cdot\|) \leq 1 + O(t(\eta \vee \nu))$$

as $t(\eta \vee \nu) \rightarrow 0^+$.

Remark 97. From now it is understood that $G_t = G_t^{\eta, \nu}$ depends on η and ν .

The result below shows that (88) is optimal.

Proposition 98. *The solution semigroup G_t of (87) satisfies*

$$\liminf_{t(\eta \vee \nu) \rightarrow 0^+} \frac{\text{Lip}(G_t, \|\cdot\|_{\text{int}}) - 1}{t\eta + \omega_d(t\nu)} > 0 \quad \text{and} \quad \liminf_{t(\eta \vee \nu) \rightarrow 0^+} \frac{\text{Lip}(G_t, \|\cdot\|) - 1}{t(\eta \vee \nu)} > 0,$$

for the modulus $\omega_d(\cdot)$ defined in Lemma 67 (see also Lemma 70).

Here is another result which justifies the necessity to introduce the new norm $\|\cdot\|$ in Theorem 35.

Proposition 99. *The above modulus satisfies $\liminf_{r \rightarrow 0^+} \omega_d(r)/\sqrt{r} > 0$. In particular, the solution semigroup G_t of (87) is not $\|\cdot\|_{\text{int}}$ -quasicontractive if $\nu > 0$.*

Proof of the second part of Proposition 99. By Proposition 98 and the first part, $\liminf_{r \rightarrow 0^+} \omega_d(r)/\sqrt{r} > 0$, the right-hand side of (85) fails and G_t is not a quasicontraction semigroup. \square

The rest of this appendix is devoted to the proofs of Proposition 98 and the liminf in Proposition 99. We need some lemmas. Set

$$USC_{\text{int}}(\mathbb{R}^d) := \{\varphi_0 \in USC(\mathbb{R}^d) : \|\varphi_0\|_{\text{int}} < \infty\}$$

with the pointwise sup in the definition of $\|\cdot\|_{\text{int}}$ according to our notation.

⁵Recall that $H(p, X) = \sup_{|q|, |\bar{q}| \leq 1} \{\eta q \cdot p + \nu \text{tr}((\bar{q} \otimes \bar{q})X)\}$.

Lemma 100. *Given $0 \leq \varphi_0 \in USC_{\text{int}}(\mathbb{R}^d)$, there is $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d) \ni \varphi_0^n \downarrow \varphi_0$ pointwise.*

Proof. Just take the supconvolution $\varphi_0^n(x) := \sup_y \{\varphi_0(y) - n|x - y|^2\}$. It is indeed in L_{int}^∞ because $\varphi_0^n(x) = \varphi_0(\bar{y}) - n|x - \bar{y}|^2$ for some $|x - \bar{y}| \leq \sqrt{\|\varphi_0\|_\infty/n} =: C$ and thus $\int \sup_{\bar{Q}_1(x)} \varphi_0^n dx \leq \int \sup_{\bar{Q}_{1+C}(x)} \varphi_0 dx < \infty$ by Lemma 16. \square

The next result is Proposition 8 for max solutions.

Lemma 101. *For any compact convex $\mathcal{C} \subset \mathbb{R}^d$, the maximal solution of $\partial_t \varphi = \sup_{q \in \mathcal{C}} q \cdot D\varphi$ with bounded φ_0 is $\bar{\varphi}(x, t) = \sup_{x+t\mathcal{C}}(\varphi_0)^*$.*

Proof. Take the supconvolution $\varphi_0^\varepsilon \downarrow (\varphi_0)^*$ as $\varepsilon \downarrow 0$. By stability, cf. Proposition 53, the solution of $\partial_t \varphi_\varepsilon = \sup_{q \in \mathcal{C}} q \cdot D\varphi_\varepsilon$ with data φ_0^ε satisfies $\varphi_\varepsilon \downarrow \bar{\varphi}$.⁶ It moreover satisfies the desired formula $\varphi_\varepsilon(x, t) = \sup_{y \in t\mathcal{C}} \varphi_0^\varepsilon(x + y)$ because φ_0^ε is regular. Writing that $\varphi_0^\varepsilon(x + y) = \sup_z \{(\varphi_0)^*(x + y + z) - |z|^2/(2\varepsilon^2)\}$ and exchanging the supremums in y and z then reveals that φ_ε is the supconvolution in space of $\sup_{x+t\mathcal{C}}(\varphi_0)^*$. Hence $\varphi_\varepsilon(x, t) \downarrow \sup_{x+t\mathcal{C}}(\varphi_0)^*$ and the proof is complete. \square

Lemma 102. *Let $\|\cdot\| = \|\cdot\|_{\text{int}}$ or $\|\cdot\|$. Then the semigroup associated to (87) satisfies for any $t \geq 0$,*

$$\text{Lip}(G_t, \|\cdot\|) \geq \sup_{0 \leq \varphi_0 \in USC_{\text{int}}, \varphi_0 \neq 0} \|\bar{G}_t \varphi_0\| / \|\varphi_0\| =: M_t.$$

Hereafter \bar{G}_t denotes the maximal solution operator associated to (87). We use the same notation for \bar{G}_t^{mod} associated to (62). The semigroup G_t^{mod} is a part of the definition (63) of $\|\cdot\|$. In Lemma 102, it is understood that this norm is defined for $0 \leq \varphi_0 \in USC_{\text{int}}(\mathbb{R}^d)$ by

$$\|\varphi_0\| := \sup_{s \geq 0} e^{-s} \|\bar{G}_s^{\text{mod}} \varphi_0\|_{\text{int}}.$$

Proof of Lemma 102. Let $\|\cdot\| = \|\cdot\|$. Given $0 \leq \varphi_0 \in USC_{\text{int}}$, $\varphi_0 \neq 0$, use Lemma 100 to get a sequence $C_b \cap L_{\text{int}}^\infty \ni \varphi_0^n \downarrow \varphi_0$ pointwise. The main step of the proof consists in showing that $\|G_t \varphi_0^n\| \rightarrow \|\bar{G}_t \varphi_0\|$ as $n \rightarrow \infty$, for any $t \geq 0$.

Take the semigroup G_t^e associated to

$$(89) \quad \partial_t \varphi = \sum_{i=1}^d |\partial_{x_i} \varphi|.$$

By Proposition 8 and Lemma 101,

$$\sup_{\bar{Q}_1(x)} G_s^{\text{mod}} G_t \varphi_0^n = G_1^e G_s^{\text{mod}} G_t \varphi_0^n(x) \quad \text{and} \quad \sup_{\bar{Q}_1(x)} \bar{G}_s^{\text{mod}} \bar{G}_t \varphi_0 = \bar{G}_1^e \bar{G}_s^{\text{mod}} \bar{G}_t \varphi_0(x)$$

for any $t, s \geq 0$ and $x \in \mathbb{R}^d$. These functions are moreover nonnegative by the comparison principle, cf. Proposition 7(ii). Hence

$$\begin{aligned} \|G_s^{\text{mod}} G_t \varphi_0^n\|_{\text{int}} &= \int_{\mathbb{R}^d} G_1^e G_s^{\text{mod}} G_t \varphi_0^n(x) dx \\ &\rightarrow \int_{\mathbb{R}^d} \bar{G}_1^e \bar{G}_s^{\text{mod}} \bar{G}_t \varphi_0 dx = \|\bar{G}_s^{\text{mod}} \bar{G}_t \varphi_0\|_{\text{int}} \quad \text{as } n \rightarrow \infty, \end{aligned}$$

because $G_1^e G_s^{\text{mod}} G_t \varphi_0^n \downarrow \bar{G}_1^e \bar{G}_s^{\text{mod}} \bar{G}_t \varphi_0$ pointwise by stability (Proposition 53) and since these functions are in $L_{\text{int}}^\infty \subset L^1$ by Section 3.2. Note now that $\|G_t \varphi_0^n\| = e^{-s_n} \|G_{s_n}^{\text{mod}} G_t \varphi_0^n\|_{\text{int}}$ for some $s_n \in [0, R_t]$ with $R_t > 0$ independent of n , because

⁶Recall that the max solutions with data φ_0 or $(\varphi_0)^*$ coincide by Proposition 7(i).

the function $s \mapsto \|G_s^{\text{mod}} G_t \varphi_0^n\|_{\text{int}}$ is continuous by Lemma 87, grows polynomially by Corollary 33 and Lemma 74, and at $s = 0$ we have

$$\|G_t \varphi_0^n\| \geq \|G_t \varphi_0^n\|_{\text{int}} \geq \|\varphi_0^n\|_{\text{int}} \geq \|\varphi_0\|_{\text{int}} > 0.$$

To get the second inequality, we have used that $G_t \varphi_0^n \geq \varphi_0^n$ pointwise because the solution of (87) is nondecreasing in t . Let $\bar{s} \in [0, R_t]$ be the limit of s_n (t being fixed from the beginning); it exists up to reasoning with subsequences if necessary. For any $m \geq n$,

$$\|G_t \varphi_0^m\| = e^{-s_m} \|G_{s_m}^{\text{mod}} G_t \varphi_0^m\|_{\text{int}} \leq e^{-s_m} \|G_{s_m}^{\text{mod}} G_t \varphi_0^n\|_{\text{int}} \rightarrow e^{-\bar{s}} \|G_{\bar{s}}^{\text{mod}} G_t \varphi_0^n\|_{\text{int}}$$

as $m \rightarrow \infty$. Taking the limit as $n \rightarrow \infty$ by what precedes, we conclude that

$$\limsup_m \|G_t \varphi_0^m\| \leq e^{-\bar{s}} \|\overline{G_{\bar{s}}^{\text{mod}}} \overline{G_t \varphi_0}\|_{\text{int}} \leq \|\overline{G_t \varphi_0}\|.$$

But for any $s \geq 0$,

$$\|G_t \varphi_0^n\| \geq e^{-s} \|G_s^{\text{mod}} G_t \varphi_0^n\|_{\text{int}} \rightarrow e^{-s} \|\overline{G_s^{\text{mod}}} \overline{G_t \varphi_0}\|_{\text{int}} \quad \text{as } n \rightarrow \infty,$$

so taking the sup in s implies that $\liminf_n \|G_t \varphi_0^n\| \geq \|\overline{G_t \varphi_0}\|$.

This proves our claim that $\|G_t \varphi_0^n\| \rightarrow \|\overline{G_t \varphi_0}\|$ for any $t \geq 0$ and in particular $\|\varphi_0^n\| \rightarrow \|\varphi_0\|$ for $t = 0$. It follows that

$$\text{Lip}(G_t, \|\cdot\|) \geq \|G_t \varphi_0^n\| / \|\varphi_0^n\| \rightarrow \|\overline{G_t \varphi_0}\| / \|\varphi_0\| \quad \text{as } n \rightarrow \infty.$$

This implies that $\text{Lip}(G_t, \|\cdot\|) \geq M_t$ by taking the sup in φ_0 . The proof is then complete for the norm $\|\cdot\|$. For $\|\cdot\|_{\text{int}}$, use that $\|G_s^{\text{mod}} G_t \varphi_0^n\|_{\text{int}} \rightarrow \|\overline{G_s^{\text{mod}}} \overline{G_t \varphi_0}\|_{\text{int}}$ with $s = 0$ to get that $\|G_t \varphi_0^n\|_{\text{int}} \rightarrow \|\overline{G_t \varphi_0}\|_{\text{int}}$ and conclude similarly. \square

At this stage, we can prove the first part of Proposition 98.

Proof of the first liminf in Proposition 98. Let us derive lower bounds on M_t from Lemma 102 with $\|\cdot\| = \|\cdot\|_{\text{int}}$. We first choose $\varphi_0 = \mathbf{1}_{\{0\}}$ and focus on the convection part of (87). Its Hamiltonian satisfies

$$\eta|p| + \nu \sup_{\lambda \in \text{Sp}(X)} \lambda^+ \geq \eta|p| \geq c_d \eta \sum_{i=1}^d |p_i|$$

with $c_d = d^{-1/2}$. Thus $t \mapsto \overline{G_{c_d t \eta}^e} \mathbf{1}_{\{0\}}$ is a subsolution of (87) since $t \mapsto \overline{G_t^e} \mathbf{1}_{\{0\}}$ solves (89). It follows that $\overline{G_t} \mathbf{1}_{\{0\}} \geq \overline{G_{c_d t \eta}^e} \mathbf{1}_{\{0\}}$, and since $\overline{G_{c_d t \eta}^e} \mathbf{1}_{\{0\}} = \mathbf{1}_{\overline{Q_{c_d t \eta}}}$ by Lemma 101,

$$\|\overline{G_t} \mathbf{1}_{\{0\}}\|_{\text{int}} \geq \int \sup_{\overline{Q_1}(x)} \mathbf{1}_{\overline{Q_{c_d t \eta}}} dx = \text{meas}(\overline{Q_{1+c_d t \eta}}).$$

Using that $\|\mathbf{1}_{\{0\}}\|_{\text{int}} = \text{meas}(\overline{Q_1})$,

$$(90) \quad M_t \geq \frac{\text{meas}(\overline{Q_{1+c_d t \eta}})}{\text{meas}(\overline{Q_1})} = (1 + c_d t \eta)^d \geq 1 + c_d t \eta.$$

Let us now treat the diffusion in (87). Given the solution semigroup G_t^{diff} of (49), we can reformulate the definition of $\omega_d(\cdot)$ in Lemma 67 as follows: For any $r \geq 0$,

$$(91) \quad \omega_d(r) = \sup_{0 \leq \varphi_0 \in USC_{\text{int}}, \varphi_0 \neq 0} \frac{\|\overline{G_r^{\text{diff}}} \varphi_0\|_{\text{int}}}{\|\varphi_0\|_{\text{int}}} - 1.$$

In particular $1 + \omega_d(t\nu) = \lim_n \|\overline{G_{t\nu}^{\text{diff}}} \varphi_0^n\|_{\text{int}} / \|\varphi_0^n\|_{\text{int}}$ for some $0 \leq \varphi_0^n \in USC_{\text{int}}$, $\varphi_0^n \neq 0$. Using this time that

$$\eta|p| + \nu \sup_{\lambda \in \text{Sp}(X)} \lambda^+ \geq \nu \sup_{\lambda \in \text{Sp}(X)} \lambda^+,$$

the functions $t \mapsto \overline{G}_{t\nu}^{\text{diff}} \varphi_0^n$ are subsolutions of (87) because $t \mapsto \overline{G}_t^{\text{diff}} \varphi_0^n$ solves (49). Then $\overline{G}_t \varphi_0^n \geq \overline{G}_{t\nu}^{\text{diff}} \varphi_0^n$ and $\|\overline{G}_t \varphi_0^n\|_{\text{int}} \geq \|\overline{G}_{t\nu}^{\text{diff}} \varphi_0^n\|_{\text{int}}$ since these functions are nonnegative. We obtain that

$$M_t \geq \|\overline{G}_t \varphi_0^n\|_{\text{int}} / \|\varphi_0^n\|_{\text{int}} \geq \|\overline{G}_{t\nu}^{\text{diff}} \varphi_0^n\|_{\text{int}} / \|\varphi_0^n\|_{\text{int}} \rightarrow 1 + \omega_d(t\nu) \quad \text{as } n \rightarrow \infty.$$

Taking the maximum with (90), we deduce that $M_t \geq 1 + (c_d t \eta \vee \omega_d(t\nu))$ with $c_d = d^{-1/2} > 0$ and the first liminf in Proposition 98 follows from Lemma 102. \square

For the second liminf, we need lemmas to compute $\|\overline{G}_t \mathbf{1}_{\{0\}}\|$. Recall that \overline{G}_t and $\overline{G}_t^{\text{mod}}$ are the maximal solution operators associated to (87) and (62).

Lemma 103. *We have $\overline{G}_s^{\text{mod}} \overline{G}_t \mathbf{1}_{\{0\}}(x) = U((|x| - \eta t - s)^+ / \sqrt{\nu t + s})$ for any $t, s \geq 0$ and $x \in \mathbb{R}^d$ with the profile $U(\cdot)$ from Lemma 68.*

Hereafter we use the convention $U(a/0) := 0$ if $a > 0$ and $U(0/0) := 1$ which means that $U((|x| - \eta t - s)^+ / 0) = \mathbf{1}_{\overline{B}_{\eta t + s}}(x)$ with the closed ball $\overline{B}_r := \{|x| \leq r\}$.

Proof. Let us take $\nu > 0$ and first show that

$$\varphi(x, t) := U((|x| - \eta t)^+ / \sqrt{\nu t}) = \overline{G}_t \mathbf{1}_{\{0\}}(x).$$

Recall that $U(\cdot)$ is smooth, decreasing, and $U''(r) + rU'(r)/2 = 0$ with $U(0) = 1$ and $U(+\infty) = 0$.

Let us verify that φ solves (87). We argue as in Lemma 68. If $|x| < \eta t$, then φ is constant and solves (87). If $|x| > \eta t$, then $\varphi(x, t) = U((|x| - \eta t) / \sqrt{\nu t})$,

$$\partial_t \varphi = \nu \frac{-|x| - \eta t}{2(\nu t)^{\frac{3}{2}}} U', \quad D\varphi = \frac{x}{|x|} \frac{U'}{\sqrt{\nu t}}, \quad \text{and} \quad \sup_{\lambda \in \text{Sp}(D^2 \varphi)} \lambda^+ = \frac{1}{\nu t} U'',$$

where the last term is as in (52). Since $U' \leq 0$, $|D\varphi| = -U' / \sqrt{\nu t}$ and taking $r := (|x| - \eta t) / \sqrt{\nu t}$,

$$\partial_t \varphi - \eta |D\varphi| - \nu \sup_{\lambda \in \text{Sp}(D^2 \varphi)} \lambda^+ = -\frac{rU'(r)/2 + U''(r)}{t} = 0.$$

The function φ thus solves (87) if $|x| \neq \eta t$. For the last case, consider the function $\psi(x, t) := \mathbf{1}_{\overline{B}_{\eta t}}(x)$. It is less or equal φ , satisfies $\partial_t \psi = \eta |D\psi|$, and is such that $\varphi(x, t) = \psi(x, t)$ if $|x| = \eta t$. Given any such (x, t) and $\phi \in C^\infty$ such that (x, t) is a max of $\varphi - \phi$, $\psi - \phi$ achieves also a max at (x, t) and

$$\partial_t \phi(x, t) \leq \eta |D\phi(x, t)| \leq \eta |D\phi(x, t)| + \nu \sup_{\lambda \in \text{Sp}(D^2 \phi(x, t))} \lambda^+.$$

This shows that φ is subsolution of (87) at (x, t) and it is also a supersolution because its subset is empty. We conclude that φ satisfies the equation associated to G_t everywhere. Now given the continuous initial data $\varphi_0^n(x) := \varphi(x, 1/n)$, we deduce that $G_t \varphi_0^n(x) = \varphi(x, t + 1/n)$. We will show that $\overline{G}_t \mathbf{1}_{\{0\}}(x) = \varphi(x, t)$ by passing to the limit as $n \rightarrow \infty$. Indeed, by monotonicity of the profile U , we have $\varphi_0^n \downarrow \mathbf{1}_{\{0\}}$ and $\varphi(\cdot, \cdot + 1/n) \downarrow \varphi(\cdot, \cdot)$ pointwise. Hence $G_t \varphi_0^n \downarrow \overline{G}_t \mathbf{1}_{\{0\}}$ by stability and $\overline{G}_t \mathbf{1}_{\{0\}}(x) = \varphi(x, t)$.

Fixing t and arguing similarly with $\tilde{\varphi}(x, s) := U((|x| - \eta t - s)^+ / \sqrt{\nu t + s})$, we find that $\overline{G}_s^{\text{mod}} \overline{G}_t \mathbf{1}_{\{0\}}(x) = \tilde{\varphi}(x, s)$ and complete the proof for $\nu > 0$. For $\nu = 0$ we know that $\overline{G}_t \mathbf{1}_{\{0\}}(x) = \varphi(x, t)$ by Lemma 101, according to our notation, and the rest of the proof is the same. \square

Lemma 104. *For any $x \in \mathbb{R}^d$, define $f(x) := \text{dist}(x, \overline{Q}_1)$. Then for any $t, s \geq 0$,*

$$\|\overline{G}_s^{\text{mod}} \overline{G}_t \mathbf{1}_{\{0\}}\|_{\text{int}} = \int_{\mathbb{R}^d} U((f(x) - \eta t - s)^+ / \sqrt{\nu t + s}) \, dx.$$

Proof. Using that $f(x) = \min_{y \in \overline{Q_1}(x)} |y|$, the monotonicity of the profile $U(\cdot)$ implies that $\sup_{y \in \overline{Q_1}(x)} U((|y| - \eta t - s)^+ / \sqrt{\nu t + s}) = U((f(x) - \eta t - s)^+ / \sqrt{\nu t + s})$. We conclude by Lemma 103. \square

The integral in Lemma 104 will be computed over the level sets of f . For this sake, we need the lemma below.

Lemma 105. *The function f from Lemma 104 is Lipschitz with $|Df| = 1$ almost everywhere outside $\overline{Q_1}$. Moreover there are $c_1, \dots, c_d > 0$ such that the Lebesgue and Hausdorff measures of the level sets satisfy: For any $r \geq 0$,*

$$\text{meas}\{f \leq r\} = \text{meas}(\overline{Q_1}) + \sum_{i=1}^d c_i r^i \quad \text{and} \quad \mathcal{H}^{d-1}\{f = r\} = \sum_{i=1}^d i c_i r^{i-1}.$$

Proof. The first assertion is classical, cf. e.g. the proof of Theorem 3.14 of [29]. Now for any $\emptyset \neq I \subseteq \{1, \dots, d\}$ define

$$A_I := \{x : \min_{i \in I} |x_i| \geq 1, \max_{i \notin I} |x_i| < 1, \text{ and } 0 < \sqrt{\sum_{i \in I} (|x_i| - 1)^2} \leq r\}$$

and note that $\{0 < f \leq r\} = \cup_I A_I$ with I such as above.⁷ Since the union is disjoint, $\text{meas}\{0 < f \leq r\} = \sum_I \text{meas}(A_I)$. Moreover, cutting A_I and appropriately changing the variable on each part, we see that

$$\text{meas}(A_I) = \text{meas}\{x : \max_{i \notin I} |x_i| < 1 \text{ and } \sqrt{\sum_{i \in I} |x_i|^2} \leq r\},$$

roughly speaking because the connected components of A_I can be translated to reconstitute a partition of the new set above. The Fubini theorem then implies that $\text{meas}(A_I) = c_I r^{\text{card}(I)}$ for some $c_I > 0$. We deduce that $\text{meas}\{0 < f \leq r\} = \sum_{i=1}^d c_i r^i$ for some $c_1, \dots, c_d > 0$. The coarea formula e.g. Theorem 3.14 of [29] then implies that

$$\text{meas}\{f \leq r\} = \text{meas}\{f = 0\} + \text{meas}\{0 < f \leq r\} = \text{meas}(\overline{Q_1}) + \int_0^r \mathcal{H}^{d-1}\{f = \tau\} d\tau$$

and this gives the desired results. \square

Proof of the second liminf in Proposition 98. By Lemma 102 with $\|\cdot\| = \|\cdot\|$,

$$(92) \quad \text{Lip}(G_t, \|\cdot\|) \geq M_t \geq \|\overline{G}_t \mathbf{1}_{\{0\}}\| / \|\mathbf{1}_{\{0\}}\|.$$

Let us derive lower bounds of this ratio. By Lemma 104, coarea formula Theorem 3.13 of [29], and Lemma 105,

$$(93) \quad \begin{aligned} & \|\overline{G}_s^{\text{mod}} \overline{G}_t \mathbf{1}_{\{0\}}\|_{\text{int}} \\ &= \int_{\mathbb{R}^d} U((f(x) - \eta t - s)^+ / \sqrt{\nu t + s}) dx \\ &= \int_{f \leq \eta t + s} U(0 / \sqrt{\nu t + s}) dx + \int_{\eta t + s}^{\infty} \int_{f=r} U((r - \eta t - s) / \sqrt{\nu t + s}) d\mathcal{H}^{d-1} dr \\ &= \text{meas}\{f \leq \eta t + s\} + \int_{\eta t + s}^{\infty} U((r - \eta t - s) / \sqrt{\nu t + s}) \mathcal{H}^{d-1}\{f = r\} dr \\ &= \text{meas}(\overline{Q_1}) + \sum_{i=1}^d c_i (\eta t + s)^i + \underbrace{\sum_{i=1}^d i c_i \int_{\eta t + s}^{\infty} r^{i-1} U((r - \eta t - s) / \sqrt{\nu t + s}) dr}_{=: I_i} \end{aligned}$$

⁷To see this, take the orthogonal projection $P : \mathbb{R}^d \rightarrow \mathbb{R}^d$ over $\overline{Q_1}$, note that $\partial \overline{Q_1} = \cup_I B_I$ with $B_I = \{x : |x_i| = 1 \text{ for } i \in I \text{ and } \max_{i \notin I} |x_i| < 1\}$ which implies that the collection of sets $P^{-1}(B_I) = \{\min_{i \in I} |x_i| \geq 1 \text{ and } \max_{i \notin I} |x_i| < 1\}$ is a partition of Q_1^c , and conclude that the collection of $A_I = P^{-1}(B_I) \cap \{0 < f \leq r\}$ is a partition of $\{0 < f \leq r\}$.

where $I_i = \sqrt{\nu t + s} \int_0^\infty (r\sqrt{\nu t + s} + \eta t + s)^{i-1} U(r) dr$. If $\eta = \nu = 0$, the semigroup G_t of (87) is the identity and (93) reads as follows:

$$(94) \quad \|\overline{G}_s^{\text{mod}} \mathbf{1}_{\{0\}}\|_{\text{int}} = \text{meas}(\overline{Q}_1) + \sum_{i=1}^d c_i s^i + \sum_{i=1}^d i c_i \sqrt{s} \int_0^\infty (r\sqrt{s} + s)^{i-1} U(r) dr.$$

Each term of (93) is greater or equal to the corresponding term in (94), and using this fact for $i \neq 1$ we find that

$$(95) \quad \|\overline{G}_s^{\text{mod}} \overline{G}_t \mathbf{1}_{\{0\}}\|_{\text{int}} \geq \|\overline{G}_s^{\text{mod}} \mathbf{1}_{\{0\}}\|_{\text{int}} + c_1 \eta t + (\sqrt{\nu t + s} - \sqrt{s}) c_1 \int_0^\infty U.$$

Consider now \bar{s} such that

$$\|\mathbf{1}_{\{0\}}\| = \sup_{s \geq 0} e^{-s} \|\overline{G}_s^{\text{mod}} \mathbf{1}_{\{0\}}\|_{\text{int}} = e^{-\bar{s}} \|\overline{G}_{\bar{s}}^{\text{mod}} \mathbf{1}_{\{0\}}\|_{\text{int}}.$$

This \bar{s} exists and is positive since by (94),

$$\|\overline{G}_s^{\text{mod}} \mathbf{1}_{\{0\}}\|_{\text{int}} = \text{meas}(\overline{Q}_1) + \sqrt{s} c_1 \int_0^\infty U + o(\sqrt{s}) \quad \text{as } s \rightarrow 0^+$$

with $\text{meas}(\overline{Q}_1) = \|\overline{G}_{s=0}^{\text{mod}} \mathbf{1}_{\{0\}}\|_{\text{int}}$. Using then that $\sqrt{\nu t + \bar{s}} - \sqrt{\bar{s}} \geq \nu t / (2\sqrt{\nu t + \bar{s}})$ in (95) with $s = \bar{s}$,

$$\|\overline{G}_t \mathbf{1}_{\{0\}}\| \geq e^{-\bar{s}} \|\overline{G}_{\bar{s}}^{\text{mod}} \overline{G}_t \mathbf{1}_{\{0\}}\|_{\text{int}} \geq \|\mathbf{1}_{\{0\}}\| + e^{-\bar{s}} (c_1 \eta t + \tilde{c}_1 \nu t / \sqrt{\nu t + \bar{s}})$$

with $\tilde{c}_1 = c_1 \int_0^\infty U > 0$. Hence

$$\|\overline{G}_t \mathbf{1}_{\{0\}}\| / \|\mathbf{1}_{\{0\}}\| \geq 1 + e^{-\bar{s}} (c_1 \eta t + \tilde{c}_1 \nu t / \sqrt{\nu t + \bar{s}}) / \|\mathbf{1}_{\{0\}}\|$$

and (92) implies the second liminf in Proposition 98. \square

It only remains to complete the proof of the second result.

Proof of the first part of Proposition 99. We will use (93) with appropriately chosen parameters to bound $\omega_d(\cdot)$ from below. Let us take $\eta = 0$, $\nu = 1$, and $s = 0$. Then the maximal solution operator \overline{G}_t associated to (87) is the operator $\overline{G}_t^{\text{diff}}$ associated to (49), and (93) with $s = 0$ and $\overline{G}_{s=0}^{\text{mod}} = \text{Id}$ then implies

$$\|\overline{G}_t^{\text{diff}} \mathbf{1}_{\{0\}}\|_{\text{int}} = \text{meas}(\overline{Q}_1) + \sum_{i=1}^d i c_i \sqrt{t} \int_0^\infty (r\sqrt{t})^{i-1} U(r) dr$$

where $\text{meas}(\overline{Q}_1) = \|\mathbf{1}_{\{0\}}\|_{\text{int}}$. By (91) with $\varphi_0 = \mathbf{1}_{\{0\}}$,

$$\liminf_{t \rightarrow 0^+} \frac{\omega_d(t)}{\sqrt{t}} \geq \lim_{t \rightarrow 0^+} \frac{\|\overline{G}_t^{\text{diff}} \mathbf{1}_{\{0\}}\|_{\text{int}} / \|\mathbf{1}_{\{0\}}\|_{\text{int}} - 1}{\sqrt{t}} = \frac{c_1 \int_0^\infty U}{\|\mathbf{1}_{\{0\}}\|_{\text{int}}} > 0$$

and the proof is complete. \square

REFERENCES

- [1] N. Alibaud and C. Imbert. Fractional semi-linear parabolic equations with unbounded data. *Trans. Amer. Math. Soc.*, 361(5):2527–2566, 2009.
- [2] O. Alvarez, P. Hoch, Y. Le Bouar and R. Monneau. Dislocation Dynamics: Short-time Existence and Uniqueness of the Solution. *Arch. Ration. Mech. Anal.*, 181(3):449–504, 2006.
- [3] B. Andreianov and M. Maliki. A note on uniqueness of entropy solutions to degenerate parabolic equations in \mathbb{R}^d . *NoDEA Nonlinear Differential Equations Appl.*, 17:109–118, 2010.
- [4] M. Bardi and I. Capuzzo-Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellmann equations*. Systems & Control: Foundations & Applications, Birkhäuser, Boston, Basel, Berlin, 1997.
- [5] G. Barles. *Solutions de viscosité des équations de Hamilton-Jacobi*. Springer-Verlag, Paris, 1994.
- [6] G. Barles, P. Cardaliaguet, O. Ley and A. Monteillet. Uniqueness results for nonlocal Hamilton-Jacobi equations. *J. Funct. Anal.*, 257(5):1261–1287, 2009.

- [7] G. Barles and E. R. Jakobsen. On the convergence rate of approximation schemes for Hamilton-Jacobi-Bellman equations. *M2AN Math. Model. Numer. Anal.*, 36(1):33–54, 2002.
- [8] G. Barles and E. R. Jakobsen. Error bounds for monotone approximation schemes for parabolic Hamilton-Jacobi-Bellman equations. *Math. Comp.*, 76(240): 1861–1893, 2007.
- [9] G. Barles and B. Perthame. Exit time problems in optimal control and vanishing viscosity method. *SIAM J. Control Optim.*, 26(5):1133–1148, 1988.
- [10] G. Barles, H. M. Soner and P. E. Souganidis. Front propagation and phase field theory. *SIAM J. Control Optim.*, 31(2):439–469, 1993.
- [11] M. Bendahmane and K. H. Karlsen. Renormalized entropy solutions for quasi-linear anisotropic degenerate parabolic equations. *SIAM J. Math. Anal.*, 36(2): 405–422, 2004.
- [12] P. Bénilan, J. Carrillo and P. Wittbold. Renormalized entropy solutions of scalar conservation laws. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 29(2):313–327, 2000.
- [13] Ph. Bénilan, M. G. Crandall and A. Pazy. *Nonlinear Evolution Equations Governed by Accretive Operators*. Book Manuscript, Besançon, 2001.
- [14] Ph. Bénilan, M. G. Crandall and M. Pierre. Solutions of the porous medium equation in R^N under optimal conditions on initial values. *Indiana Univ. Math. J.*, 33(1):51–87, 1984.
- [15] C. Berg and G. Forst. *Potential theory on locally compact abelian groups*. Additional book information: Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 87, Springer-Verlag, Berlin, Heidelberg, New York, 1975.
- [16] O. Bokanowski, N. Forcadel and H. Zidani. L^1 -error estimates for numerical approximations of Hamilton-Jacobi-Bellman equations in dimension 1. *Math. Comp.*, 79(271):1395–1426, 2010.
- [17] L. Caffarelli, M. G. Crandall, M. Kocan and A. Święch. On viscosity solutions of fully nonlinear equations with measurable ingredients. *Comm. Pure Appl. Math.*, 49(4):365–397, 1996.
- [18] E. Carlini, N. Forcadel and R. Monneau. A generalized fast marching method for dislocation dynamics. *SIAM J. Numer. Anal.*, 49(6):2470–2500, 2011.
- [19] J. Carrillo. Entropy Solutions for nonlinear Degenerate Problems. *Arch. Ration. Mech. Anal.*, 147(4):269–361, 1999.
- [20] G.-Q. Chen and E. DiBenedetto. Stability of entropy solutions to the Cauchy problem for a class of nonlinear hyperbolic-parabolic equations. *SIAM J. Math. Anal.*, 33(4):751–762, 2001.
- [21] G.-Q. Chen and B. Perthame. Well-posedness for non-isotropic degenerate parabolic-hyperbolic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 20(4):645–668, 2003.
- [22] M. G. Crandall, H. Ishii and P.-L. Lions. User’s guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.*, 27(1):1–67, 1992.
- [23] M. G. Crandall and P.-L. Lions. Viscosity solutions of Hamilton-Jacobi Equations. *Trans. Amer. Math. Soc.*, 277:1–42, 1983.
- [24] M. G. Crandall, P.-L. Lions and P. E. Souganidis. Maximal solutions and universal bounds for some partial differential equations of evolution. *Arch. Rational Mech. Anal.*, 105(2):163–190, 1989.
- [25] C. Dafermos. *Hyperbolic Conservation Laws in Continuum Physics*. Springer, 3rd ed., 2010.
- [26] E. DiBenedetto. *Degenerate parabolic equations*. Springer-Verlag, New-York, 1993.
- [27] J. Endal and E. R. Jakobsen. L^1 Contraction for Bounded (Non-integrable) Solutions of Degenerate Parabolic Equations. *SIAM J. Math. Anal.*, 46(6):3957–3982, 2014.
- [28] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*. Graduate Texts in Mathematics, 194. Springer-Verlag, New York, 2000.
- [29] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions. Revised edition*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, 2015.
- [30] W. Fleming and H.M. Soner. *Controlled Markov Processes and Viscosity Solutions*. Springer-Verlag, New York, 1993.
- [31] H. Frid. Decay of Almost Periodic Solutions of Anisotropic Degenerate Parabolic-Hyperbolic Equations. In *Non-linear partial differential equations, mathematical physics, and stochastic analysis*, EMS Ser. Congr. Rep., 183–205, Eur. Math. Soc., Zürich, 2018.
- [32] Y. Giga and M.-H. Sato. A level set approach to semicontinuous viscosity solutions for Cauchy problems. *Comm. Partial Differential Equations*, 26(5-6):813–839, 2001.
- [33] J. Goldstein. *Semigroups of Linear Operators and Applications*. Oxford U. Press, New York, 1985.
- [34] D. Goreac and O.-S. Serea. Discontinuous control problems for non-convex dynamics and near viability for singularly perturbed control systems. *Nonlinear Anal.*, 73(8):2699–2713, 2010.
- [35] D. Goreac and O.-S. Serea. Mayer and optimal stopping stochastic control problems with discontinuous cost. *J. Math. Anal. Appl.*, 380:327–342, 2011.
- [36] J.-B. Hiriart-Urruty and C. Lemaréchal. *Fundamentals of Convex Analysis*. Springer, 2001.
- [37] H. Ishii. On uniqueness and existence of viscosity solutions of fully nonlinear second order elliptic PDEs. *Comm. Pure Appl. Math.*, 42:14–45, 1989.

- [38] R. Jensen. The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations. *Arch. Rational Mech. Anal.*, 101:1–27, 1988.
- [39] S. N. Kružkov. First order quasilinear equations with several independent variables. *Math. Sb. (N.S.)* (in Russian), 81(123):228–255, 1970.
- [40] N.V. Krylov. On the rate of convergence of finite-difference approximations for Bellman’s equations. *St. Petersburg Math. J.*, 9(3):639–650, 1997.
- [41] C.-T. Lin and E. Tadmor. L^1 -stability and error estimates for approximate Hamilton-Jacobi solutions. *Numer. Math.*, 87(4):701–735, 2001.
- [42] P.-L. Lions, B. Perthame and E. Tadmor.: Formulation cinétique des lois de conservation scalaires multidimensionnelles. *C. R. Acad. Sci. Paris, Série I Math.* 312:97–102, 1991.
- [43] P.-L. Lions, B. Perthame and E. Tadmor. A kinetic formulation of multidimensional scalar conservation laws and related local equations. *J. Amer. Math. Soc.* 7:169–191, 1994.
- [44] E. Y. Panov, To the theory of entropy sub-solutions of degenerate non-linear parabolic equations. Preprint, arXiv:1910.08739, 2019.
- [45] N. Pogodaev. Estimates of the domain of dependence for scalar conservation laws. *J. Differ. Equ.*, 265(4):1654–1677, 2018.
- [46] C. M. Theobald. An inequality for the trace of the product of two symmetric matrices. *Math. Proc. Cambridge Philos. Soc.*, 77:265–267, 1975.
- [47] J. L. Vázquez. *The porous medium equation. Mathematical theory.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007.
- [48] Z. Wu, J. Zhao, J. Yin and H. Li. *Nonlinear diffusion equations.* World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

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