



HAL
open science

Optimal and dual stability results for L^1 viscosity and L^∞ entropy solutions

Nathaël Alibaud, Jørgen Endal, Espen Robstad Jakobsen

► **To cite this version:**

Nathaël Alibaud, Jørgen Endal, Espen Robstad Jakobsen. Optimal and dual stability results for L^1 viscosity and L^∞ entropy solutions. 2018. hal-01945687v1

HAL Id: hal-01945687

<https://hal.science/hal-01945687v1>

Preprint submitted on 5 Dec 2018 (v1), last revised 12 Apr 2024 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

OPTIMAL AND DUAL STABILITY RESULTS FOR L^1 VISCOSITY AND L^∞ ENTROPY SOLUTIONS

NATHAËL ALIBAUD, JØRGEN ENDAL, AND ESPEN R. JAKOBSEN

ABSTRACT. We revisit stability results for two central notions of weak solutions for nonlinear PDEs: entropy and viscosity solutions originally introduced for scalar conservation laws and Hamilton-Jacobi equations. Here, we consider two second order model equations, the Hamilton-Jacobi-Bellman (HJB) equation

$$\partial_t \varphi = \sup_{\xi} \{b(\xi) \cdot D\varphi + \text{tr}(a(\xi)D^2\varphi)\},$$

and the anisotropic degenerate parabolic equation

$$\partial_t u + \text{div}F(u) = \text{div}(A(u)Du).$$

The viscosity solutions of the first equation and the entropy solutions of the second satisfy contraction principles in L^∞ and L^1 respectively. Our aim is to get similar results for viscosity solutions in L^1 and entropy solutions in L^∞ . For the first equation, we identify the smallest Banach topology which is stronger than L^1 for which we have stability. We then construct a norm such that a quasicontraction principle holds. For the second equation, we propose a new weighted L^1 contraction principle allowing for pure L^∞ solutions. Our main contribution is to show that the solutions of the HJB equation can be used as weights and that this choice is optimal. Interestingly, this reveals a new type of duality between entropy and viscosity solutions.

CONTENTS

1. Introduction	2
2. Preliminaries	4
2.1. Viscosity solutions of (1)	4
2.2. Entropy solutions of (2)	6
2.3. The function space L_{int}^∞	7
2.4. Semi-norms $ \cdot _{\text{conv}}$ and $ \cdot _{\text{diff}}$	7
3. Main results	8
3.1. Some L^1 instabilities for (1)	8
3.2. Optimal L^1 framework for (1)	9
3.3. Weighted L^1 contraction for (2)	12
3.4. Optimal weight and duality between (1) and (2)	13
4. Proofs	14
4.1. More on viscosity solutions of (1)	14
4.2. L^1 instability: Proofs of Propositions 24, 25 and 26	16
4.3. L_{int}^∞ stability: Proof of Theorem 31	18
4.4. L_{int}^∞ stability: Proof of Theorem 30	26
4.5. Consequences: Proofs of Theorem 28 and Corollary 33	27
4.6. L_{int}^∞ quasicontraction: Proofs of Proposition 34 and Theorem 35	29
4.7. Weighted L^1 contraction: Proof of Theorem 38	36

2010 *Mathematics Subject Classification.* 35K65, 35L65, 35B35, 35B30, 35D30, 35D40.

Key words and phrases. Hamilton-Jacobi-Bellman equations, degenerate parabolic equations, conservation laws, L^1 viscosity solutions, L^∞ entropy solutions, quantitative stability, contraction principle, quasicontraction principle, weighted L^1 contraction principle, duality.

4.8. Duality: Proofs of Theorem 44 and Corollaries 46 and 47	37
Acknowledgements	45
Appendix A. Technical proofs	45
A.1. Min and max viscosity solutions	45
A.2. Representation formulas	47
A.3. L_{int}^∞ strong continuity in time	48
A.4. Main properties of $ \cdot _{\text{conv}}$ and $ \cdot _{\text{diff}}$	48
A.5. An optimal L_{int}^∞ inequality	50
Appendix B. Complementary proofs for entropy solutions	52
References	56

1. INTRODUCTION

This paper is concerned with nonstandard stability results for two central notions of weak solutions for nonlinear PDEs of hyperbolic or degenerate parabolic type: entropy and viscosity solutions. Originally, these solution concepts were introduced for scalar conservation laws [38] and Hamilton-Jacobi (HJ) equations [22]; see [37, 36, 18, 20] for extensions to second order convection-diffusion and fully nonlinear equations. Conservation laws are divergence form equations arising in continuum physics [24], while HJ equations are nondivergence form equations e.g. from differential geometry and optimal control theory [29, 5, 4]. The well-posedness of such PDEs is a challenging issue that requires entropy and viscosity solution theories in general. By now, the literature is enormous and includes lots of applications. For reference books and states-of-the-art, see [29, 25, 5, 4, 46, 24, 21].

Here, we consider two model problems, initial-value problems for the Hamilton-Jacobi-Bellman (HJB) equation

$$(1a) \quad \partial_t \varphi = \sup_{\xi \in \mathcal{E}} \{b(\xi) \cdot D\varphi + \text{tr}(a(\xi)D^2\varphi)\} \quad x \in \mathbb{R}^d, t > 0,$$

$$(1b) \quad \varphi(x, 0) = \varphi_0(x) \quad x \in \mathbb{R}^d,$$

and the anisotropic degenerate parabolic convection-diffusion equation

$$(2) \quad \partial_t u + \text{div}F(u) = \text{div}(A(u)Du) \quad x \in \mathbb{R}^d, t > 0,$$

$$u(x, 0) = u_0(x) \quad x \in \mathbb{R}^d,$$

where “ D ,” “ D^2 ” and “ div ” respectively denote the gradient, the Hessian and the divergence in x , and “ tr ” is the trace. We assume that

$$(H1) \quad \begin{cases} \mathcal{E} \text{ is a nonempty set,} \\ b : \mathcal{E} \rightarrow \mathbb{R}^d \text{ a bounded function,} \\ a = \sigma^a (\sigma^a)^\top \text{ for some bounded } \sigma^a : \mathcal{E} \rightarrow \mathbb{R}^{d \times K}, \end{cases}$$

$$(H2) \quad F \in W_{\text{loc}}^{1,\infty}(\mathbb{R}, \mathbb{R}^d) \quad \text{and} \quad A = \sigma^A (\sigma^A)^\top \quad \text{for} \quad \sigma^A \in L_{\text{loc}}^\infty(\mathbb{R}, \mathbb{R}^{d \times K}),$$

where K is the maximal rank of $a(\xi)$ and $A(u)$. Problems (1) and (2) are well-posed for L^∞ viscosity¹ and L^1 entropy solutions (cf. [37, 36, 20]) and satisfy the contraction principles in L^∞ and L^1 respectively:

$$(3) \quad \|(\varphi - \psi)(t)\|_\infty \leq \|\varphi_0 - \psi_0\|_\infty \quad \text{and} \quad \|(u - v)(t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1}.$$

Our aim is to get the most natural and optimal versions of these estimates for viscosity solutions in L^1 and entropy solutions in L^∞ . Our results will reveal new duality relations between entropy and viscosity solutions.

¹Strictly speaking, we mean the space of continuous and bounded functions.

Let us first discuss L^1 viscosity solutions of (1). We refer to [17] for L^p viscosity solutions of uniformly elliptic PDEs, with p large enough. See also [41, 6, 16] for first order HJ equations, and [1, 26] for second order degenerate PDEs. The two last papers are not focused on L^1 viscosity solutions but contain related estimates. Important differences with us are the linearity of the diffusion in [1] and the restriction to compactly supported initial data in [26]. Here, we consider the smallest topology which is stronger than L^1 for which we have stability for all equations of the form (1a). We show that it is generated by the norm

$$\|\varphi_0\|_{\text{int}} := \int \sup_{x+[-1,1]^d} |\varphi_0| \, dx$$

(cf. Theorem 28) which is the norm of L_{int}^∞ as defined in [2, 1]. Next, we look for contraction properties. They depend on the nonlinearities in (1a) since we have an L^1 contraction principle if (1a) is linear, but not in the general case. To measure these nonlinearities, we introduce the norms $|H|_{\text{conv}}$ and $|H|_{\text{diff}}$ on $D_p^2 H$ and $D_X^2 H$ (see Section 2.4) where

$$(4) \quad H : (p, X) \in \mathbb{R}^d \times \mathbb{S}_d \mapsto \sup_{\mathcal{E}} \{b \cdot p + \text{tr}(aX)\} \in \mathbb{R},$$

is the Hamiltonian of (1a), i.e.,

$$(1a) \quad \iff \quad \partial_t \varphi = H(D\varphi, D^2\varphi).$$

Now we have two main estimates for (1) (cf. Theorems 31 and 35). The first is

$$(5) \quad \|(\varphi - \psi)(t)\|_{\text{int}} \leq (1 + \omega_d(t|H|_{\text{diff}})) (1 + t|H|_{\text{conv}})^d \|\varphi_0 - \psi_0\|_{\text{int}} \quad \forall t \geq 0,$$

for some modulus of continuity ω_d (depending only on the dimension). The second states that there is an equivalent norm $\|\cdot\|$ of L_{int}^∞ such that

$$(6) \quad \|(\varphi - \psi)(t)\| \leq e^{(|H|_{\text{conv}} \vee |H|_{\text{diff}})t} \|\varphi_0 - \psi_0\| \quad \forall t \geq 0$$

(where $a \vee b := \max\{a, b\}$). Contrarily to (5), (6) is a true quasicontraction estimate as defined in semigroup theory [32, 27, 13] with the constant 1 in front of the exponential. Estimate (5) can not be written in this form since ω_d is typically an n -th root. Contraction and quasicontraction semigroups are important for their relations to accretive operators, product formulas, splitting methods, etc. Estimate (5) is interesting since it involves the simple norm $\|\cdot\|_{\text{int}}$ of L_{int}^∞ , while (6) is a quasicontraction result for the semigroup associated to (1) in this optimal L^1 type Banach space setting.

Let us now focus on L^∞ entropy solutions of (2). The well-posedness in the pioneering work [38] is written for pure L^∞ solutions. Kinetic and renormalized solution theories to handle pure L^1 were developed later in [42, 43, 12]. For second order PDEs, isotropic diffusions were treated in [18], and anisotropic diffusion in [20] via the L^1 kinetic approach. Well-posedness of (2) in L^∞ is not standard, but results exist in [30]: see also [19, 11, 3, 26]. Our contribution concerns the quantitative stability in L^∞ . There can be no L^∞ contraction since the solutions can develop discontinuities in finite time. In the literature, the stability is therefore quantified by weighted L^1 contraction principles like the well-known finite speed of propagation property for first order PDEs:

$$(7) \quad \int_{|x-x_0|<R} |u(x,t) - v(x,t)| \, dx \leq \int_{|x-x_0|<R+Ct} |u_0(x) - v_0(x)| \, dx,$$

see [38, 44]. An example for second order PDEs (cf. [14, 19, 48, 30]) is the following estimate:

$$(8) \quad \int |u(x,t) - v(x,t)| e^{-\sqrt{1+|x|^2}} \, dx \leq e^{Ct} \int |u_0(x) - v_0(x)| e^{-\sqrt{1+|x|^2}} \, dx.$$

Our main result is a very accurate weighted L^1 contraction principle for (2) (cf. Theorem 38). It retains as much information as possible when the weights are required not to depend on the solutions u and v . Our new weight is the solution of a “dual” equation of the form (1). Using a representation formula for this solution from stochastic control theory [29], we can write the result in the following form (cf. Corollary 42):

$$(9) \quad \int_{|x-x_0|<R} |u(x,t) - v(x,t)| dx \leq \int |u_0(x) - v_0(x)| \sup_{\xi} \mathbb{P}(|\mathbf{X}_t^x - x_0| < R) dx,$$

where \mathbb{P} is the probability, ξ_s is an adapted control, and for a Brownian motion \mathbf{B}_s , \mathbf{X}_s^x is an Ito process satisfying

$$d\mathbf{X}_s^x = F'(\xi_s) ds + \sqrt{2} \sigma^A(\xi_s) d\mathbf{B}_s, \quad \mathbf{X}_{s=0}^x = x.$$

This is the natural second order extension of (7). The idea to construct weights via viscosity solutions was introduced in [26], and another extension of (7) is given there for isotropic diffusion. But this result is not so accurate, and valuable information like the global L^1 contraction in (3) cannot be recovered (cf. Remarks 2.7 (b) and 2.11 (e) in [26]). Here we not only generalize (3), (7), (8), and [26] but we also prove that our weight is optimal in a certain class satisfying a semigroup property (cf. Corollary 46). This is our most original contribution. Interestingly, this optimality result also reveals rigorous duality relations between the semigroups associated to (1) and (2) (cf. Theorem 44 and Corollary 47).

The rest of the paper is organized as follows: Preliminary results are given in Section 2, in Sections 3 and 4 we state and prove our main results, and at the end there are appendices containing complementary results and technical proofs.

General notation. \mathbb{R}^+ (resp. \mathbb{R}^-) denotes nonnegative (resp. nonpositive) real numbers and \mathbb{S}_d^+ nonnegative symmetric $d \times d$ matrices. A modulus of continuity is a function $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\omega(t) \rightarrow 0$ as $t \rightarrow 0^+$. Finally, the symbols \vee and \wedge denote max and min respectively.

2. PRELIMINARIES

In this section, we recall some basic facts on viscosity and entropy solutions; for proofs, see for instance [21, 29, 5, 4] and [20, 11, 24] respectively. We also define the space L_{int}^∞ and the semi-norms $|\cdot|_{\text{conv}}$ and $|\cdot|_{\text{diff}}$.

2.1. Viscosity solutions of (1). Let φ^* (resp. φ_*) denote the upper (resp. lower) semicontinuous envelope of φ .

Definition 1 (Viscosity solutions). *Assume (H1) and $\varphi_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ is bounded.*

(a) *A locally bounded function $\varphi : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is a viscosity subsolution (resp. supersolution) of (1) if*

i) *for every $\phi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^+)$ and local maximum $(x,t) \in \mathbb{R}^d \times (0,\infty)$ of $\varphi^* - \phi$ (resp. minimum),*

$$\partial_t \phi(x,t) \leq H(D\phi(x,t), D^2\phi(x,t)) \quad (\text{resp. } \geq),$$

ii) *and for every $x \in \mathbb{R}^d$,*

$$\varphi^*(x,0) \leq (\varphi_0)^*(x) \quad (\text{resp. } \varphi_*(x,0) \geq (\varphi_0)_*(x)).$$

(b) *A function φ is a viscosity solution if it is both a sub and supersolution.*

Remark 2. We say that φ is a viscosity subsolution (resp. supersolution) of (1a) if (ai) holds.

We recall the well-known comparison and the well-posedness for (1).

Theorem 3 (Comparison principle). *Assume (H1). If φ and ψ are bounded sub and supersolutions of (1a), and*

$$\varphi^*(x, 0) \leq \psi_*(x, 0) \quad \forall x \in \mathbb{R}^d,$$

then $\varphi^ \leq \psi_*$ on $\mathbb{R}^d \times \mathbb{R}^+$.*

Theorem 4 (Existence and uniqueness). *Assume (H1) and $\varphi_0 \in C_b(\mathbb{R}^d)$. Then there exists a unique viscosity solution $\varphi \in C_b(\mathbb{R}^d \times \mathbb{R}^+)$ of (1).*

Remark 5. We also have the maximum principle $\inf \varphi_0 \leq \varphi \leq \sup \varphi_0$, and for solutions φ and ψ with initial data φ_0 and ψ_0 ,

$$\|\varphi(\cdot, t) - \psi(\cdot, t)\|_\infty \leq \|\varphi_0 - \psi_0\|_\infty \quad \forall t \geq 0.$$

We may take φ_0 to be discontinuous as in (9). In that case, we loose uniqueness and we have to work with minimal and maximal solutions [23, 10, 31] (see also [4] for bilateral solutions). From now on, we denote by *BLSC* (resp. *BUSC*) bounded and lower (resp. upper) semicontinuous functions.

Theorem 6 (Minimal and maximal solutions). *Assume (H1) and $\varphi_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ is bounded. Then there exists a pair of viscosity solutions of (1),*

$$(\underline{\varphi}, \overline{\varphi}) \in \text{BLSC}(\mathbb{R}^d \times \mathbb{R}^+) \times \text{BUSC}(\mathbb{R}^d \times \mathbb{R}^+),$$

where $\underline{\varphi}$ is minimal and $\overline{\varphi}$ is maximal in the sense that

$$\underline{\varphi} \leq \varphi \leq \overline{\varphi} \quad \text{for any bounded viscosity solution } \varphi \text{ of (1).}$$

Moreover, at $t = 0$,

$$\underline{\varphi}(x, 0) = (\varphi_0)_*(x) \quad \text{and} \quad \overline{\varphi}(x, 0) = (\varphi_0)^*(x) \quad \forall x \in \mathbb{R}^d.$$

Note that $\underline{\varphi}$ and $\overline{\varphi}$ are unique by definition.

Proposition 7 (Comparison). *Assume (H1) and $\varphi_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ is bounded.*

(i) *For any bounded supersolution φ of (1) (resp. subsolution).*

$$\underline{\varphi} \leq \varphi_* \quad (\text{resp. } \overline{\varphi} \geq \varphi^*).$$

(ii) *In particular, $\underline{\varphi} \leq \underline{\psi}$ and $\overline{\varphi} \leq \overline{\psi}$ for any bounded initial data $\varphi_0 \leq \psi_0$.*

For completeness, the proofs of Theorems 6 and Propotision 7 are given in Appendix A.1 because [23, 10, 31, 4] consider slightly different problems. Let us continue with representation formulas for the solution $\underline{\varphi}$ from control theory [29, 4, 33, 34]. Throughout, “co” denote the convex hull and “Im” the image.

Proposition 8 (First order). *Assume (H1), $a \equiv 0$, and $\varphi_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ bounded. Then the minimal viscosity solution of (1) is given by*

$$\underline{\varphi}(x, t) = \sup_{x+t\mathcal{C}} (\varphi_0)_* \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+,$$

where $\mathcal{C} = \overline{\text{co}\{\text{Im}(b)\}}$.

In the second order case, we need a probabilistic framework. For simplicity, we fix for the rest of this paper

$$(10) \quad \begin{cases} \text{a complete filtered probability space } (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}), \text{ and} \\ \text{a standard } d\text{-dimensional Brownian } \mathbf{B}_t \text{ on this filtration.} \end{cases}$$

We will assume without mention that all stochastic processes in this paper are defined on this filtered probability space, and that whenever we need a Brownian motion, then we take the above Brownian motion. Let us denote the expectation by \mathbb{E} . Then:

Proposition 9 (Second order). *Assume (H1), $\varphi_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ is bounded, and*

(11) *the set \mathcal{E} is compact and the functions $b(\cdot)$ and $\sigma^a(\cdot)$ are continuous.*

Then the minimal viscosity solution of (1) is given by

$$\underline{\varphi}(x, t) = \sup_{\xi} \mathbb{E} \{ (\varphi_0)_*(\mathbf{X}_t^x) \},$$

where ξ_s is a progressively measurable \mathcal{E} -valued control and \mathbf{X}_s^x an Ito process satisfying the SDE

$$\begin{cases} d\mathbf{X}_s^x = b(\xi_s) ds + \sqrt{2} \sigma^a(\xi_s) d\mathbf{B}_s, & s > 0, \\ \mathbf{X}_{s=0}^x = x. \end{cases}$$

These results are standard for continuous viscosity solutions [29, 4], see also [4, 33, 34] for maximal solutions. For minimal solutions, we did not find any reference so we provide the proofs in Appendix A.2.

2.2. Entropy solutions of (2). In the nonstandard pure L^∞ setting, the well-posedness of (2) is essentially considered in [30] for smooth fluxes, see also [20, 11] for L^1 . Let us now recall these results in the form we need and provide complementary proofs in Appendix B for completeness.

Definition 10 (Entropy-entropy flux triple). *We say that (η, q, r) is an entropy-entropy flux triple if $\eta \in C^2(\mathbb{R})$ is convex, $q' = \eta' F'$ and $r' = \eta' A$.*

Given $\beta \in C(\mathbb{R})$, we also need the notation

$$\zeta_{ik}(u) := \int_0^u \sigma_{ik}^A(\xi) d\xi \quad \text{and} \quad \zeta_{ik}^\beta(u) := \int_0^u \sigma_{ik}^A(\xi) \beta(\xi) d\xi.$$

Definition 11 (Entropy solutions). *Assume (H2) and $u_0 \in L^\infty(\mathbb{R}^d)$. A function $u \in L^\infty(\mathbb{R}^d \times \mathbb{R}^+) \cap C(\mathbb{R}^+; L_{\text{loc}}^1(\mathbb{R}^d))$ is an entropy solution of (2) if*

- (a) $\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) \in L_{\text{loc}}^2(\mathbb{R}^d \times \mathbb{R}^+)$ for any $k = 1, \dots, K$,
- (b) for any $k = 1, \dots, K$ and any $\beta \in C(\mathbb{R})$

$$\sum_{i=1}^d \partial_{x_i} \zeta_{ik}^\beta(u) = \beta(u) \sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) \in L_{\text{loc}}^2(\mathbb{R}^d \times \mathbb{R}^+),$$

- (c) and for all entropy-entropy flux triples (η, q, r) and $0 \leq \phi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+)$,

$$\begin{aligned} & \iint_{\mathbb{R}^d \times \mathbb{R}^+} \left(\eta(u) \partial_t \phi + \sum_{i=1}^d q_i(u) \partial_{x_i} \phi + \sum_{i,j=1}^d r_{ij}(u) \partial_{x_i x_j}^2 \phi \right) dx dt \\ & + \int_{\mathbb{R}^d} \eta(u_0(x)) \phi(x, 0) dx \geq \iint_{\mathbb{R}^d \times \mathbb{R}^+} \eta''(u) \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) \right)^2 \phi dx dt. \end{aligned}$$

Theorem 12 (Existence and uniqueness). *Assume (H2) and $u_0 \in L^\infty(\mathbb{R}^d)$. Then there exists a unique entropy solution $u \in L^\infty(\mathbb{R}^d \times \mathbb{R}^+) \cap C(\mathbb{R}^+; L_{\text{loc}}^1(\mathbb{R}^d))$ of (2).*

See [30, Theorem 1.1] or Appendix B for the proof.

Remark 13. (a) In the L^1 settings of [20, 11], the following contraction principle holds: For solutions u and v of (2) with initial data u_0 and v_0 ,

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1} \quad \forall t \geq 0.$$

- (b) In the L^∞ setting of [30], uniqueness is based on the weighted L^1 contraction principle (8), see also Lemma 93 in Appendix B.

(c) In all cases, we have comparison and maximum principles as stated in Lemma 95 in Appendix B.

In L^∞ , uniqueness is based on a doubling of variables arguments developed in [38, 18, 11]. This argument leads to inequality (12) below, and this inequality will be the starting point of our analysis of (2).

Lemma 14 (Kato inequality). *Assume (H2) and u, v are entropy solutions of (2) with initial data $u_0, v_0 \in L^\infty(\mathbb{R}^d)$. Then for all $T \geq 0$ and nonnegative test functions $\phi \in C_c^\infty(\mathbb{R}^d \times [0, T])$,*

$$(12) \quad \int_{\mathbb{R}^d} |u - v|(x, T) \phi(x, T) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \phi(x, 0) \, dx \\ + \iint_{\mathbb{R}^d \times (0, T)} \left(|u - v| \partial_t \phi + \sum_{i=1}^d q_i(u, v) \partial_{x_i} \phi + \sum_{i,j=1}^d r_{ij}(u, v) \partial_{x_i x_j}^2 \phi \right) \, dx \, dt,$$

where

$$q_i(u, v) = \text{sign}(u - v) \int_v^u F_i'(\xi) \, d\xi, \quad r_{ij}(u, v) = \text{sign}(u - v) \int_v^u A_{ij}(\xi) \, d\xi.$$

See Appendix B for a sketch of the proof of this lemma with references to computations in [11].

2.3. The function space L_{int}^∞ . Consider the following normed space

$$L_{\text{int}}^\infty(\mathbb{R}^d) := \{ \varphi_0 \in L^1 \cap L^\infty(\mathbb{R}^d) \text{ s.t. } \|\varphi_0\|_{L_{\text{int}}^\infty} < \infty \}$$

where $\|\varphi_0\|_{L_{\text{int}}^\infty} := \int \text{ess sup}_{\overline{Q}_1(x)} |\varphi_0| \, dx$ and $\overline{Q}_r(x) := x + [-r, r]^d$ for $r > 0$.

Theorem 15. *This is a Banach space continuously embedded into $L^1 \cap L^\infty(\mathbb{R}^d)$.*

For the proof, see [2, 1] from which we have taken the above notation. From now on, we would rather consider the pointwise sup, and to avoid confusion we then use the notation

$$\|\varphi_0\|_{\text{int}} := \int \sup_{\overline{Q}_1(x)} |\varphi_0| \, dx.$$

Note that $\|\varphi_0\|_{\text{int}} = \|\varphi_0\|_{L_{\text{int}}^\infty}$ if φ_0 is continuous. Here is another result of [2, 1], see for instance [1, Lemma 2.5.1].

Lemma 16. *For any $r > 0$ and $\varepsilon \geq 0$, there is a constant $C_{r,\varepsilon} \geq 0$ such that*

$$\int \sup_{\overline{Q}_{r+\varepsilon}(x)} |\varphi_0| \, dx \leq C_{r,\varepsilon} \int \sup_{\overline{Q}_r(x)} |\varphi_0| \, dx \quad \forall \varphi_0 : \mathbb{R}^d \rightarrow \mathbb{R}.$$

Remark 17. (a) This result will be used with the pointwise sup for discontinuous φ_0 , typically lower or uppersemicontinuous.

(b) A more precise and possibly new result is given in Lemma 90 in Appendix A.5.

2.4. Semi-norms $|\cdot|_{\text{conv}}$ and $|\cdot|_{\text{diff}}$. The set of Hamiltonians associated to equations of the form (1) is a convex cone which we denote by

$$\Gamma := \{ H : \mathbb{R}^d \times \mathbb{S}_d \rightarrow \mathbb{R} \text{ s.t. } H \text{ satisfy (4) for some } (\mathcal{E}, b, a) \text{ satisfying (H1)} \}.$$

Note that the same $H \in \Gamma$ can be represented by different (\mathcal{E}, b, a) satisfying (H1) as long as $\text{Im}(b, a)$ is the same (see Lemma 87 or Remark 89(a)). Let us endow Γ with semi-norms.

Definition 18. For each $H \in \Gamma$, set

$$|H|_{\text{conv}} := \inf_{b_0 \in \mathbb{R}^d} \sup_{\xi \in \mathcal{E}} |b(\xi) - b_0| \quad \text{and} \quad |H|_{\text{diff}} := \inf_{\mathbb{S}_d^+ \ni a_0 \leq \text{Im}(a)} \sup_{\xi \in \mathcal{E}} |\text{tr}(a(\xi) - a_0)|,$$

where (\mathcal{E}, b, a) is a representative of H , i.e. a triplet satisfying (H1) such that (4) holds.

Remark 19. (a) In the second inf, “ $a_0 \leq \text{Im}(a)$ ” means that $a_0 \leq a(\xi)$, $\forall \xi \in \mathcal{E}$.

(b) You may think that these quantities depend of the choice of the representative (\mathcal{E}, b, a) , but this is not the case: see Lemma 88 in Appendix A.4.

These semi-norms roughly speaking measure the nonlinearities of the Hamiltonians.

Proposition 20. The above $|\cdot|_{\text{conv}}$ and $|\cdot|_{\text{diff}}$ are semi-norms. Moreover, for each Hamiltonian $H = H(p, X) \in \Gamma$, we have:

$$(13) \quad \begin{cases} |H|_{\text{conv}} = 0 & \iff H \text{ is affine in the gradient } p, \text{ and} \\ |H|_{\text{diff}} = 0 & \iff H \text{ is affine in the Hessian } X. \end{cases}$$

See Appendix A.4 for the proof.

Remark 21. Let us give an example. Let $d = 1$ and

$$\mathcal{E} := \{c, d\} \times \{e, f\}$$

for some $c, d, e, f \in \mathbb{R}$ such that $e, f \geq 0$. Let $b(\xi) = \xi_1$ and $a(\xi) = \xi_2$ where $\xi = (\xi_1, \xi_2)$, and

$$H(p, X) := \sup_{\xi \in \mathcal{E}} \{b(\xi) \cdot p + \text{tr}(a(\xi)X)\}.$$

Then

$$|H|_{\text{conv}} = \frac{|c - d|}{2} \quad \text{and} \quad |H|_{\text{diff}} = e \vee f - e \wedge f.$$

Note also that

$$H(p, X) = \max\{cp, dp\} + \max\{eX, fX\},$$

so that for this particular example,

$$|H|_{\text{conv}} = \frac{1}{2} \int_{\mathbb{R}} |\partial_{pp}^2 H| \quad \text{and} \quad |H|_{\text{diff}} = \int_{\mathbb{R}} |\partial_{XX}^2 H|$$

in the sense of the total variations. For general dimensions and Hamiltonians, these total variations may be infinite, but $|H|_{\text{conv}}$ and $|H|_{\text{diff}}$ are always finite under our assumptions.

See Remark 89 in Appendix A.4, for connections to support functions and convex analysis.

3. MAIN RESULTS

We are ready to state our main results. The long proofs are given in Section 4.

3.1. Some L^1 instabilities for (1). Let us begin with (1) and explain why we can not have pure L^1 stability results. Let us consider the unique viscosity solution of the eikonal equation

$$(14) \quad \partial_t \varphi = \sum_{i=1}^d |\partial_{x_i} \varphi|,$$

with a given initial data $\varphi_0 \in C_b(\mathbb{R}^d)$. Under which condition is it integrable?

Proposition 22 (Integrability condition). *We have*

$$[\varphi(\cdot, t) \in L^1(\mathbb{R}^d) \quad \forall t \geq 0] \iff [\varphi_0^- \in L^1(\mathbb{R}^d) \text{ and } \varphi_0^+ \in L_{\text{int}}^\infty(\mathbb{R}^d)].$$

Proof. Use that $\varphi(x, t) = \sup_{\overline{Q}_t(x)} \varphi_0$ by Proposition 8, and then Lemma 16. \square

We can therefore not expect general L^1 stability because of the positive parts.

Proposition 23 (L^1 instability for nonnegative solutions). *Let $\varphi_0^n(x) := (1-n|x|)^+$ for all $n \in \mathbb{N}$, and φ_n be the solution of (14) with φ_0^n as initial data. Then $\varphi_0^n \in C_b \cap L^1(\mathbb{R}^d)$ and*

$$\lim_{n \rightarrow \infty} \varphi_0^n = 0 \quad \text{in } L^1(\mathbb{R}^d),$$

but

$$\lim_{n \rightarrow \infty} \varphi_n(\cdot, t) = \mathbf{1}_{\overline{Q}_t}(\cdot) \neq 0 \quad \text{in } L^1(\mathbb{R}^d), \quad \forall t > 0.$$

Proof. Use again that $\varphi_n(x, t) = \sup_{\overline{Q}_t(x)} \varphi_0^n$. \square

You might expect L^1 stability for nonpositive solutions, because the negative parts behave better in Proposition 22. But this is not the case either:

Proposition 24 (L^1 instability for nonpositive solutions). *There exist nonpositive $\varphi_0, \varphi_0^n \in C_b \cap L^1(\mathbb{R}^d)$ such that*

$$\lim_{n \rightarrow \infty} \varphi_0^n = \varphi_0 \quad \text{in } L^1(\mathbb{R}^d),$$

but the solutions φ and φ_n of (14) with initial data φ_0 and φ_0^n satisfy

$$\liminf_{n \rightarrow \infty} \|\varphi_n(\cdot, t) - \varphi(\cdot, t)\|_{L^1} > 0 \quad \forall t > 0 \quad (\text{small enough}).$$

See Section 4.2 for the proof. To see that L^1 is not good for pure diffusion equations of the form (1a) either, we consider an equation in one space dimension

$$(15) \quad \partial_t \varphi = (\partial_{xx}^2 \varphi)^+.$$

Here again, to have L^1 solutions, we need $\varphi_0^+ \in L_{\text{int}}^\infty$.

Proposition 25 (L_{int}^∞ and nonlinear diffusions). *Let $\varphi_0 \in C_b(\mathbb{R})$ be nonnegative and φ be the solution of (15) with φ_0 as initial data. Then,*

$$[\varphi(\cdot, t) \in L^1(\mathbb{R}) \quad \forall t \geq 0] \iff \varphi_0 \in L_{\text{int}}^\infty(\mathbb{R}).$$

See Section 4.2 for the proof. To simplify, we omit the analysis of nonpositive solutions of (15) and discuss instead the lack of a fundamental solution. Consider solutions φ_n of (15) with an approximate delta-functions as initial data:

$$(16) \quad \varphi_n(x, t = 0) = n\rho(nx),$$

where $0 \leq \rho \in C_c(\mathbb{R})$ is nontrivial. Then:

Proposition 26 (Blow-up everywhere). $\lim_{n \rightarrow \infty} \varphi_n(x, t) = \infty, \forall x \in \mathbb{R}, \forall t > 0$.

See Section 4.2 for the proof. For linear diffusion equations, φ_n would converge to the fundamental solution, but here it explodes pointwise and in all L_{loc}^p .

3.2. Optimal L^1 framework for (1). We now look for the smallest topology which is stronger than L^1 for which we have stability. To get a general theory, we restrict to normed spaces and results that hold for all equations of the form (1a). We will obtain a quasicontraction result for the corresponding semigroups. Let us recall some definitions from [32, 27, 13].

Definition 27. *Let E be a normed space. We say that G_t is a semigroup on E if it is a family of maps $G_t : E \rightarrow E$, parametrized by $t \geq 0$, satisfying*

$$\begin{cases} G_{t=0} = \text{id} \text{ (the identity), and} \\ G_{t+s} = G_t G_s \text{ (meaning the composition) for any } t, s \geq 0. \end{cases}$$

It is a contraction semigroup if

$$\|G_t\varphi_0 - G_t\psi_0\|_E \leq \|\varphi_0 - \psi_0\|_E,$$

and a quasicontraction semigroup if there is some $\gamma \in \mathbb{R}$ such that

$$\|G_t\varphi_0 - G_t\psi_0\|_E \leq e^{\gamma t} \|\varphi_0 - \psi_0\|_E,$$

for any $\varphi_0, \psi_0 \in E$ and $t \geq 0$.

For example, let φ be the unique viscosity solution of (1) and define

$$(17) \quad G_t : \varphi_0 \in C_b(\mathbb{R}^d) \mapsto \varphi(\cdot, t) \in C_b(\mathbb{R}^d).$$

Then G_t is a contraction semigroup. Consider now spaces E such that

$$(18) \quad \begin{cases} E \text{ is a vector subspace of } C_b \cap L^1(\mathbb{R}^d), \\ E \text{ is a normed space,} \\ E \text{ is continuously embedded}^2 \text{ into } L^1(\mathbb{R}^d), \end{cases}$$

and for any triplet of data (\mathcal{E}, b, a) satisfying (H1), the associated semigroup (17) is such that for any $t \geq 0$,

$$(19) \quad G_t \text{ maps } E \text{ into itself and } G_t : E \rightarrow E \text{ is continuous.}$$

The best possible E is given below.

Theorem 28 (Optimal L^1 setting for HJB equations). *The space $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ is a Banach space satisfying the properties (18)–(19). Moreover, any other space E satisfying (18)–(19) is continuously embedded² into $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$.*

Remark 29. We have considered spaces E of continuous functions in order to avoid uniqueness issues. But this is not restrictive since the best E above is a complete space.

Let us now focus on explicit estimates. We first give a rough L_{int}^∞ a priori estimate.

Theorem 30 (General L_{int}^∞ stability). *Assume (H1) and $T \geq 0$. For any bounded subsolution φ and supersolution ψ of (1a),*

$$(20) \quad \int_{\overline{Q_1(x)} \times [0, T]} \sup (\varphi^* - \psi_*)^+ dx \leq C \int_{\overline{Q_1(x)}} \sup (\varphi^* - \psi_*)^+(\cdot, 0) dx,$$

for some constant $C = C(d, \|a\|_\infty, \|b\|_\infty, T) \geq 0$.

Note that the supremum in time is inside the integral. In the second estimate, we precisely quantify the influence of the convective and diffusive nonlinearities.

Theorem 31 (Quantitative L_{int}^∞ stability). *Assume (H1) and let $\mathcal{C} = \overline{\text{co}\{\text{Im}(b)\}}$. There exists a modulus of continuity $\omega_d(\cdot)$ only depending on the dimension d such that, for any bounded subsolution φ and supersolution ψ of (1a),*

$$(21) \quad \int_{\overline{Q_1(x)}} \sup (\varphi^* - \psi_*)^+(\cdot, t) dx \leq (1 + \omega_d(t|H|_{\text{diff}})) \int_{\overline{Q_1(x) + t\mathcal{C}}} \sup (\varphi^* - \psi_*)^+(\cdot, 0) dx \quad \forall t \geq 0.$$

The effect of convection is seen in the set $t\mathcal{C}$ and the diffusion in $t|H|_{\text{diff}}$.

²Let X, Y be normed spaces such that $X \subseteq Y$. X is continuously embedded into Y if there is $C \geq 0$ such that $\|x\|_Y \leq C\|x\|_X$ for all $x \in X$.

Remark 32. (a) For each $r \geq 0$, $\omega_d(r)$ will be defined during the proof as the smallest real number satisfying

$$\int \sup_{\overline{Q}_1(x)} \varphi^*(\cdot, r) \, dx \leq (1 + \omega_d(r)) \int \sup_{\overline{Q}_1(x)} \varphi^*(\cdot, 0) \, dx,$$

for any bounded and nonnegative subsolution φ of the equation

$$\partial_t \varphi = \sup_{\lambda \in \text{Sp}(D^2 \varphi)} \lambda^+,$$

where $\text{Sp}(X)$ is the spectrum of X . In particular, the modulus $\omega_d(\cdot)$ is optimal for this prototypical diffusive equation with $|H|_{\text{diff}} = 1$ and $\mathcal{C} = \{0\}$.

(b) Formal computations suggest that $\omega_d(r) = C\sqrt{r}$, but we are not able to prove it.

Let us now consider the semigroup (17).

Corollary 33 (Contraction like estimate in $\|\cdot\|_{\text{int}}$). *Assume (H1) and $\omega_d(\cdot)$ is defined in Theorem 31. Then, for any $\varphi_0, \psi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ and $t \geq 0$,*

$$\|G_t \varphi_0 - G_t \psi_0\|_{\text{int}} \leq (1 + \omega_d(t|H|_{\text{diff}})) (1 + t|H|_{\text{conv}})^d \|\varphi_0 - \psi_0\|_{\text{int}}.$$

By this result and Proposition 20, the semigroup is quasicontractive or contractive (for the norm $\|\cdot\|_{\text{int}}$) if the diffusion or the whole equation is linear, respectively. Without this linearity, the contraction properties may fail. Let us give counterexamples for the case $d = 1$ to simplify. To emphasize that only the linearity counts, consider uniformly parabolic PDEs

$$(22) \quad \partial_t \varphi = |\partial_x \varphi| + \partial_{xx}^2 \varphi \quad \text{and}$$

$$(23) \quad \partial_t \varphi = (\partial_{xx}^2 \varphi)^+ + \partial_{xx}^2 \varphi.$$

Proposition 34 (Lack of contraction property). *The semigroup associated to (22) (resp. (23)) is not contractive (resp. quasicontractive) on $C_b \cap L_{\text{int}}^\infty(\mathbb{R})$ endowed with the norm $\|\cdot\|_{\text{int}}$.*

To get a quasicontraction principle for fully nonlinear PDEs, we need to change the norm.

Theorem 35 (General quasicontraction). *For any $\varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$, define*

$$\|\varphi_0\| := \sup_{t \geq 0} e^{-t} \|G_t^{\text{mod}} \varphi_0\|_{\text{int}},$$

where G_t^{mod} is the semigroup on $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ associated to the model equation

$$\partial_t \varphi = |D\varphi| + \sup_{\lambda \in \text{Sp}(D^2 \varphi)} \lambda^+.$$

Then $\|\cdot\|$ is a norm on $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ which is equivalent to $\|\cdot\|_{\text{int}}$. Moreover, under (H1), the semigroup (17) is $\|\cdot\|$ -quasicontractive with

$$(24) \quad \|G_t \varphi_0 - G_t \psi_0\| \leq e^{(|H|_{\text{conv}} \vee |H|_{\text{diff}})t} \|\varphi_0 - \psi_0\|,$$

for any $\varphi_0, \psi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ and $t \geq 0$.

Remark 36. The choice of $\|\cdot\|$ is inspired by linear semigroup theory, see Theorem 2.13 in [32]. Here, our semigroup is nonlinear, and remarkably $\|\cdot\|$ does not depend on the particular semigroup G_t that we consider as in [32]. More precisely, G_t^{mod} is a fixed model semigroup, and $\|\cdot\|$ is a fixed norm in which all semigroups of equations of the form (1a) are quasicontractive.

Remark 37. This is the L^1 counterpart of the L^∞ contraction for (1),

$$\|(\varphi - \psi)(t)\|_\infty \leq \|\varphi_0 - \psi_0\|_\infty.$$

In the L^1 setting we have

$$\|(\varphi - \psi)(t)\| \leq e^{(|H|_{\text{conv}} \vee |H|_{\text{diff}})t} \|\varphi_0 - \psi_0\|,$$

where, for some constant $M \geq 1$ and any $\phi = \phi(x)$,

$$\|\phi\|_{L^1} \leq \int \sup_{\overline{Q}_1(x)} |\phi| \, dx \leq \|\phi\| \leq M \int \sup_{\overline{Q}_1(x)} |\phi| \, dx.$$

The proofs of all results in this section can be found in Sections 4.3–4.6.

3.3. Weighted L^1 contraction for (2). Let us continue with Problem (2) and state a new weighted L^1 contraction principle for L^∞ entropy solutions. The weight will be the viscosity solution of the following problem:

$$(25a) \quad \partial_t \varphi = \operatorname{ess\,sup}_{m \leq \xi \leq M} \{F'(\xi) \cdot D\varphi + \operatorname{tr}(A(\xi)D^2\varphi)\} \quad x \in \mathbb{R}^d, t > 0,$$

$$(25b) \quad \varphi(x, 0) = \varphi_0(x) \quad x \in \mathbb{R}^d,$$

for given $m < M$ and φ_0 . Here is the precise statement.

Theorem 38 (Weighted L^1 contraction). *Assume (H2), $m < M$, and take measurable $u_0 = u_0(x)$ and $v_0 = v_0(x)$ with values in $[m, M]$. Take also a nonnegative initial weight $\varphi_0 \in BLSC(\mathbb{R}^d)$. Then, the associated entropy solutions u and v of (2) and viscosity minimal solution $\underline{\varphi}$ of (25) satisfy*

$$(26) \quad \int_{\mathbb{R}^d} |u - v|(x, t) \varphi_0(x) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \underline{\varphi}(x, t) \, dx \quad \forall t \geq 0.$$

Remark 39. (a) Problem (25) is of the form (1) with a pointwise sup taken over the Lebesgue points of F' and A .

(b) The right-hand side of (26) can be infinite. To get finite integrals, it suffices to take $\varphi_0 \in L_{\text{int}}^\infty$ or $u_0 - v_0 \in L^1$.

(c) The same result holds when $\underline{\varphi}$ is replaced by any measurable supersolution of (25), since it is greater than $\underline{\varphi}$.

Let us be more explicit. By the representation formula for first order PDEs, we obtain the following estimate on the domain of dependence:

Corollary 40 (First order equations). *Assume (H1) with $A \equiv 0$, $m < M$, u_0 and v_0 are measurable functions with values in $[m, M]$, and u and v are entropy solutions of (2) with initial data u_0 and v_0 . Then*

$$\int_B |u - v|(x, t) \, dx \leq \int_{B-t\mathcal{C}} |u_0 - v_0|(x) \, dx$$

for any Borel set $B \subseteq \mathbb{R}^d$ and $t \geq 0$, where

$$\mathcal{C} = \overline{\operatorname{co} \left\{ \operatorname{ess\,Im} \left((F')|_{[m, M]} \right) \right\}}$$

and $\operatorname{ess\,Im}$ is the essential image.

Proof. Let $U \supseteq B$ be an open set and take $\varphi_0 = \mathbf{1}_U$, the indicator function of U . By Proposition 8, the minimal solution of (25) is $\underline{\varphi}(x, t) = \mathbf{1}_{U-t\mathcal{C}}(x)$. Apply then Theorem 38 and take the infimum over all open $U \supseteq B$. \square

Remark 41. The above result is also a consequence of [44]. In that recent paper, the author gives similar estimates for (x, t) -dependent first order PDEs using different techniques based on differential inclusions.

For second order equations, we have the following result.

Corollary 42 (Second order equations). *Assume (H1), $F'(\cdot)$ and $\sigma^A(\cdot)$ continuous, $m < M$, u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$, and u and v solutions of (2) with u_0 and v_0 as initial data. Then for any open $U \subseteq \mathbb{R}^d$ and $t \geq 0$,*

$$\int_U |u - v|(x, t) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \sup_{\xi} \mathbb{P}(\mathbf{X}_t^x \in U) \, dx,$$

where ξ_s is a progressively measurable $[m, M]$ -valued process and \mathbf{X}_s^x is an Ito process satisfying the SDE

$$(27) \quad \begin{cases} d\mathbf{X}_s^x = F'(\xi_s) \, ds + \sqrt{2} \sigma^A(\xi_s) \, d\mathbf{B}_s, & s > 0, \\ \mathbf{X}_{s=0}^x = x. \end{cases}$$

Remark 43. \mathbf{X}_s^x is a stochastic process starting from x at time $s = 0$. The dynamics of \mathbf{X}_s^x is given by the controlled SDE (27) where the control is determined to maximize the probability for the process to reach U at time $s = t$. Equation (25) is the dynamic programming equation for this control problem.

Proof. Take $\varphi_0 = \mathbf{1}_U$ and apply Proposition 9 to compute $\underline{\varphi}$ in Theorem 38. \square

The proof of Theorem 38 is given in Section 4.7.

3.4. Optimal weight and duality between (1) and (2). Let us discuss the optimality of Theorem 38. First we give a reformulation of the definition of viscosity supersolutions in terms of weights in L^1 contraction estimates for (2).

Theorem 44 (Weights and supersolutions). *Assume (H2), $m < M$, and $0 \leq \varphi \in BLSC(\mathbb{R}^d \times \mathbb{R}^+)$. Then the assertions below are equivalent.*

(I) *For any measurable functions u_0 and v_0 with values in $[m, M]$ and entropy solutions u and v of (2) with initial data u_0 and v_0 ,*

$$\int_{\mathbb{R}^d} |u - v|(x, t) \varphi(x, s) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x, t + s) \, dx \quad \forall t, s \geq 0.$$

(II) *The function*

$$\varphi_{\#}(x, t) := \liminf_{\substack{r \rightarrow 0^+ \\ y \rightarrow x}} \frac{1}{\text{meas}(B_r(y))} \int_{B_r(y)} \varphi(z, t) \, dz$$

is a viscosity supersolution of (25a).

We have used the notation $B_r(y) := \{z : |z - y| < r\}$.

Remark 45. The function $\varphi_{\#}$ satisfies (I) if and only if φ does since it is an a.e. representative in space of φ .

Our weight is therefore optimal in the class of weights

$$\mathscr{W}_{m, M, \varphi_0} := \{0 \leq \varphi \in BLSC(\mathbb{R}^d \times \mathbb{R}^+) \text{ satisfying (I) and } \varphi(t=0) \geq \varphi_0\}.$$

Corollary 46 (Optimality of the weight). *Assume (H2), $m < M$, and $0 \leq \varphi_0 \in BLSC(\mathbb{R}^d)$. Then the weight $\underline{\varphi}$ from Theorem 38 belongs to the class $\mathscr{W}_{m, M, \varphi_0}$ and satisfies*

$$(\underline{\varphi})_{\#}(x, t) = \inf \{\varphi_{\#}(x, t) : \varphi \in \mathscr{W}_{m, M, \varphi_0}\} \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+.$$

Property (I) is stronger than (26) since it holds for any $s \geq 0$. This may be interpreted as a semigroup property. In that context the above result reflects some form of duality. For each $t \geq 0$, let

$$S_t : u_0 \in L^\infty(\mathbb{R}^d) \mapsto u(\cdot, t) \in L^\infty(\mathbb{R}^d)$$

where u is the entropy solution of (2), and let

$$G_t : \varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d) \mapsto \varphi(\cdot, t) \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$$

where φ is the viscosity solution of (25). We have the following result.

Corollary 47 (A form of duality). *Assume (H2), $m < M$, and S_t, G_t defined as above. Then G_t is the smallest strongly continuous semigroup on $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ satisfying*

$$\int_{\mathbb{R}^d} |S_t u_0 - S_t v_0| \varphi_0 \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0| G_t \varphi_0 \, dx,$$

for every u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$, $0 \leq \varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$, and $t \geq 0$.

Remark 48. (a) Here “smallest” means that any other semigroup H_t satisfying the same properties is such that

$$G_t \varphi_0 \leq H_t \varphi_0 \quad \forall t \geq 0, \forall \varphi_0 \geq 0.$$

(b) A “strongly continuous semigroup” means that $t \geq 0 \mapsto G_t \varphi_0$ is continuous for the strong or norm topology. This result is proven in Lemma 86.

The proofs of Theorem 44 and Corollaries 46 and 47 are given in Section 4.8.

Open questions. Denote by $G_t^{m, M}$ the previous semigroup G_t to emphasize its dependence on these parameters. Then S_t is a weakly- \star continuous semigroup on $L^\infty(\mathbb{R}^d)$ satisfying:

$$\int_{\mathbb{R}^d} |S_t u_0 - S_t v_0| \varphi_0 \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0| G_t^{m, M} \varphi_0 \, dx,$$

for every $m < M$, u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$, $0 \leq \varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$, and $t \geq 0$.

Here are natural open questions on the reciprocal duality:

- (1) Is S_t the only semigroup satisfying the above properties?
- (2) If no, can we characterize all the other semigroups?

4. PROOFS

Let us now prove the previous results. We begin with Problem (1).

4.1. More on viscosity solutions of (1). We need further classical results that can be found in [21, 29, 5, 4]. Let us begin with stability.

Proposition 49 (Stability by sup and inf). *Assume (H1) and $\mathcal{F} \neq \emptyset$ is a uniformly locally bounded family of viscosity subsolutions (resp. supersolutions) of (1a). Then, the function*

$$(x, t) \mapsto \sup\{\varphi(x, t) : \varphi \in \mathcal{F}\}$$

is a viscosity subsolution of (1a) (resp. supersolution but with the “inf”).

The second result concerns relaxed limits defined as follows:

$$\limsup^* \varphi_\varepsilon(x, t) := \limsup_{\substack{(y, s) \rightarrow (x, t) \\ \varepsilon \rightarrow 0^+}} \varphi_\varepsilon(y, s) \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+,$$

and $\liminf_* \varphi_\varepsilon := -\limsup^*(-\varphi_\varepsilon)$.

Proposition 50 (Stability with respect to relaxed limits). *Assume (H1) and $(\varphi_\varepsilon)_{\varepsilon > 0}$ is a family of uniformly locally bounded viscosity subsolutions (resp. supersolutions) of (1a). Then $\limsup^* \varphi_\varepsilon$ (resp. $\liminf_* \varphi_\varepsilon$) is a subsolution of (1a) (resp. supersolution).*

Remark 51. The notion of solution is thus stable under local uniform convergence (equivalent to $\limsup^* \varphi_\varepsilon = \liminf_* \varphi_\varepsilon$).

Here is the case of extremal solutions, see Appendix A.1 for the proof.

Proposition 52 (Stability of extremal solutions). *Assume (H1) and $(\varphi_0^n)_n$ is a nondecreasing (resp. nonincreasing) uniformly locally bounded sequence. If $\underline{\varphi}_n$ (resp. $\overline{\varphi}_n$) is the min (resp. max) solution of (1) with φ_0^n as initial data, then*

$$\sup_n \underline{\varphi}_n \quad (\text{resp. } \inf_n \overline{\varphi}_n)$$

is the minimal (resp. maximal) solution of (1) with initial data

$$\varphi_0 := \sup_n \varphi_0^n \quad (\text{resp. } \inf_n \varphi_0^n).$$

Let us continue with regularization procedures. Given $\varphi = \varphi(x, t)$ and $\varepsilon > 0$, we define the space supconvolution of φ as follows:

$$(28) \quad \varphi^\varepsilon(x, t) := \sup_{y \in \mathbb{R}^d} \left\{ \varphi^*(y, t) - \frac{|x - y|^2}{2\varepsilon^2} \right\}.$$

For standard properties of supconvolution, see e.g. [21, 29, 5, 4].

Lemma 53. *Assume (H1) and φ is a bounded subsolution of (1a). Then φ^ε is a subsolution of the same equation such that*

$$\varphi^\varepsilon \in \text{BUSC}(\mathbb{R}^d \times \mathbb{R}^+) \cap L^\infty(\mathbb{R}^+; W^{1,\infty}(\mathbb{R}^d))$$

and $\varphi^*(x, t) = \lim_{\varepsilon \downarrow 0} \downarrow \varphi^\varepsilon(x, t)$ for any $x \in \mathbb{R}^d$ and $t \geq 0$.

For supersolutions, we can regularize by convolution because of the convexity of the Hamiltonian, see [7, 8] (the ideas were introduced in [39]).

Lemma 54. *Assume (H1), $\varphi \in \text{BLSC}(\mathbb{R}^d \times \mathbb{R}^+)$ is a supersolution of (1a), and $0 \leq f \in L^1(\mathbb{R}^d \times \mathbb{R}^-)$. Then $\varphi *_{x,t} f$ is a supersolution of (1a).*

Here and throughout we extend the functions by zero to all $t \in \mathbb{R}$ to give a meaning to the convolutions in time. Below is another version that will be needed. We use the notation $M^1 = (C_b)'$ for spaces of bounded Borel measures.

Lemma 55. *Assume (H1), $\varphi \in C_b(\mathbb{R}^d \times \mathbb{R}^+)$ is a supersolution of (1a), and $0 \leq \mu \in M^1(\mathbb{R}^d)$. Then $\varphi * \mu \in C_b(\mathbb{R}^d \times \mathbb{R}^+)$ is also a supersolution (with $*$ = $*_x$ for short).*

The latter lemma is not proven in [7, 8], but can be obtained via a standard approximation procedure. Let us give it for completeness. Throughout this paper ρ_ν is a space approximate unit as $\nu \rightarrow 0^+$ of the form

$$(29) \quad \rho_\nu(x) := \frac{1}{\nu^d} \rho\left(\frac{x}{\nu}\right),$$

where $0 \leq \rho \in C_c^\infty(\mathbb{R}^d)$ and $\int \rho = 1$. Moreover θ_δ is a time approximate unit of the form

$$(30) \quad \theta_\delta(t) := \frac{1}{\delta} \theta\left(\frac{t}{\delta}\right),$$

where $0 \leq \theta \in C_c^\infty(\mathbb{R}^-)$ and $\int \theta = 1$. Below $\delta = \nu$ but these parameters may be taken differently later.

Proof of Lemma 55. By Lemma 54, $\varphi_\nu := \varphi *_{x,t} \mu *_{x,t} \rho_\nu *_{x,t} \theta_\nu$ is a supersolution of (1a). It remains to pass to the limit as $\nu \rightarrow 0^+$. We will show that the convergence is local uniform towards $\varphi *_{x,t} \mu$, which will be sufficient by stability of the equation. With the assumed regularity on φ ,

$$\lim_{\nu \rightarrow 0^+} \varphi *_{x,t} \rho_\nu *_{x,t} \theta_\nu = \varphi \quad \text{locally uniformly,}$$

and $\|\varphi *_x \rho_\nu *_t \theta_\nu\|_\infty \leq \|\varphi\|_\infty$. Moreover, for any $x \in \mathbb{R}^d$, $t \geq 0$ and $R \geq 0$,

$$\begin{aligned} |\varphi_\nu - \varphi *_x \mu|(x, t) &\leq |\varphi *_x \rho_\nu *_t \theta_\nu - \varphi *_x \mu|(x, t) \\ &\leq \left(\sup_{|y| \leq R} |\varphi *_x \rho_\nu *_t \theta_\nu - \varphi|(x - y, t) \right) \int_{|y| \leq R} d\mu(y) \\ &\quad + 2\|\varphi\|_\infty \int_{|y| > R} d\mu(y). \end{aligned}$$

This is enough to conclude since $\lim_{R \rightarrow \infty} \int_{|y| > R} d\mu(y) = 0$. \square

4.2. L^1 instability: Proofs of Propositions 24, 25 and 26.

Proof of Proposition 24. Let us first consider the case $d = 1$ (i.e. $x \in \mathbb{R}$). The eikonal equation (14) reduces to

$$(31) \quad \partial_t \varphi = |\partial_x \varphi|.$$

Let φ_0 be a continuous approximation of $-\mathbf{1}_{(-1,1)}$, e.g. $\varphi_0(x) := -g(|x|)$ where

$$g(r) := \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 2 - r & \text{if } 1 < r \leq 2, \\ 0 & \text{if } r > 2, \end{cases}$$

and take $\varphi_0^n := \varphi_0 + \mathbf{1}_{[-1/n, 1/n]} \leq 0$ which is discontinuous. With this choice, $\varphi_0^n \rightarrow \varphi_0$ in $L^1(\mathbb{R})$ as $n \rightarrow \infty$. Let now φ and $\underline{\varphi}_n$ be the exact and minimal viscosity solutions of (31) with φ_0 and φ_0^n as initial data. By Proposition 8,

$$\begin{aligned} \varphi(x, t) &= \sup_{x+[-t, t]} \varphi_0 = -g(|x| + t) \quad \text{and} \\ \underline{\varphi}_n(x, t) &= \sup_{x+[-t, t]} (\varphi_0^n)_* = -g(|x| + t) + \mathbf{1}_{[-t-1/n, t+1/n]}(x) \end{aligned}$$

for $t \leq (1 - 1/n)/2$. Thus for $0 < t < 1/4$ and $n \geq 2$, $\underline{\varphi}_n \geq \varphi$ by Proposition 7(ii) and

$$\int (\underline{\varphi}_n - \varphi)(x, t) dx = \int_{-t-1/n}^{t+1/n} dx \geq 2t.$$

Next we take continuous $(\tilde{\varphi}_0^n)_n$ such that

$$\tilde{\varphi}_0^n \geq \varphi_0^n \quad \text{and} \quad \lim_{n \rightarrow \infty} \|\tilde{\varphi}_0^n - \varphi_0^n\|_{L^1} = 0.$$

Clearly, we have $\tilde{\varphi}_0^n \rightarrow \varphi_0$ in $L^1(\mathbb{R})$ as $n \rightarrow \infty$, and the corresponding solutions of (31) satisfy $\tilde{\varphi}_n \geq \underline{\varphi}_n$ by Proposition 7(ii). Hence, for any $0 < t < 1/4$ and $n \geq 2$,

$$\int (\tilde{\varphi}_n - \varphi)(x, t) dx \geq \int (\underline{\varphi}_n - \varphi)(x, t) dx \geq 2t.$$

Hence, $\liminf_{n \rightarrow \infty} \|\underline{\varphi}_n(\cdot, t) - \varphi(\cdot, t)\|_{L^1} > 0$ and the proof for $d = 1$ is complete. In d dimensions, we replace $\mathbf{1}_{(-1,1)}$ by $\mathbf{1}_{(-1,1)^d}$. The argument is the same and we leave the details to the reader. \square

To prove Propositions 25 and 26, we need the lemma below.

Lemma 56. *For any $(x, t) \in \mathbb{R} \times \mathbb{R}^+$, let*

$$\psi(x, t) := \begin{cases} U(|x|/\sqrt{t}) & \text{if } t > 0, \\ \mathbf{1}_{\{0\}}(x) & \text{if } t = 0, \end{cases}$$

where

$$U(r) := c_0 \int_r^\infty e^{-\frac{s^2}{4}} ds \quad \text{with} \quad c_0 := \left(\int_0^\infty e^{-\frac{s^2}{4}} ds \right)^{-1}.$$

Then ψ is a *BUSC* subsolution of (15) in $\mathbb{R} \times \mathbb{R}^+$.

Proof of Proposition 25. In addition to the above lemma, we also need Theorem 28. Theorem 28 says that $E = C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ satisfies (18)–(19). Hence, the semigroup G_t associated to (15) is such that

$$G_t : C_b \cap L_{\text{int}}^\infty(\mathbb{R}) \rightarrow C_b \cap L_{\text{int}}^\infty(\mathbb{R}) \subset L^1(\mathbb{R}) \quad \forall t \geq 0.$$

This immediately implies the if-part of Proposition 25. Let us continue with the only-if-part. It is based on the following pointwise lower bound:

$$(32) \quad \varphi(x, t) \geq U \left(1/\sqrt{t} \right) \sup_{x+[-1,1]} \varphi_0 \quad \forall x \in \mathbb{R}, \forall t > 0,$$

where U is the profile from the previous lemma, $0 \leq \varphi_0 \in C_b(\mathbb{R}^d)$ and φ is the solution of (15) with φ_0 as initial data. Let us prove (32). Fix x and t . The sup on the right-hand side is attained at some $x_0 \in x + [-1, 1]$. By the previous lemma,

$$(y, s) \mapsto \varphi_0(x_0)U \left(|y - x_0|/\sqrt{s} \right)$$

is a *BUSC* subsolution of (15). At $s = 0$, it equals the function

$$y \mapsto \varphi_0(x_0)\mathbf{1}_{\{x_0\}}(y)$$

which is less or equal to $\varphi_0 = \varphi_0(y)$. By the comparison principle (Theorem 3),

$$\varphi(y, s) \geq \varphi_0(x_0)U \left(|y - x_0|/\sqrt{s} \right) \quad \forall y \in \mathbb{R}, \forall s > 0.$$

Taking $(y, s) = (x, t)$, we then get that

$$\varphi(x, t) \geq \underbrace{\varphi_0(x_0)}_{= \sup_{x+[-1,1]} \varphi_0} \underbrace{U \left(|x - x_0|/\sqrt{t} \right)}_{\geq U(1/\sqrt{t})}.$$

This completes the proof of (32). From that bound the only-if-part of Proposition 25 is obvious since $U(1/\sqrt{t})$ is positive for $t > 0$. \square

Proof of Proposition 26. Let $x_0 \in \mathbb{R}$ and $c > 0$ be such that

$$\rho \geq c\mathbf{1}_{\{x_0\}},$$

where ρ is defined in (16), and define

$$\psi_n(x, t) := nc\psi \left(nx - x_0, n^2t \right),$$

where ψ is given by Lemma 56. It is easy to see that ψ_n remains a subsolution of (15). Moreover, it is *BUSC* with

$$\varphi_0^n(x) \geq \psi_n(x, 0) \quad \forall x \in \mathbb{R}$$

where φ_0^n is defined in (16). The solution φ_n of (15) associated to the initial data φ_0^n is thus such that $\varphi_n \geq \psi_n$, by the comparison principle. Hence, it suffices to show that

$$\lim_{n \rightarrow \infty} \psi_n(x, t) = \infty \quad \forall x \in \mathbb{R}, \forall t > 0.$$

But this is quite easy because

$$\psi_n(x, t) = ncU \left(\left| x - \frac{x_0}{n} \right| / \sqrt{t} \right),$$

for any $x \in \mathbb{R}$ and $t > 0$, and both the constant c and the profile $U(\cdot)$ are positive. The proof of Proposition 26 is complete. \square

Let us now prove Lemma 56.

Proof of Lemma 56. Let us first prove that ψ , defined as in the lemma, is a subsolution of (15). In the domain $\{x \neq 0, t > 0\}$, we have

$$\partial_t \psi = \partial_{xx}^2 \psi = (\partial_{xx}^2 \psi)^+$$

in the classical sense, by direct computations based on the explicit formula for $U(\cdot)$. If now $x = 0$, we have

$$\partial_t \psi(0, \cdot) = 0 \leq (\partial_{xx}^2 \psi(0, \cdot))^+$$

since $\psi(0, \cdot)$ is constant in time.

Let us now show that ψ is *BUSC*. It is clearly continuous for positive t and it only remains to prove that

$$\mathbf{1}_{x=0} \geq \limsup_{\mathbb{R} \times \mathbb{R}^+(y,t) \rightarrow (x,0)} U(|y|/\sqrt{t}),$$

for any $x \in \mathbb{R}$. If $x = 0$, the result follows since $U(r) \leq U(0) = 1$, for any $r \geq 0$, by the choice of c_0 in the statement of the lemma. If $x \neq 0$, then we use that

$$|y|/\sqrt{t} \rightarrow \infty \quad \text{as } (y,t) \rightarrow (x, 0^+)$$

together with the fact that $\lim_{r \rightarrow \infty} U(r) = 0$. The proof of Lemma 56 is now complete. \square

4.3. L_{int}^∞ stability: Proof of Theorem 31.

4.3.1. *Preliminary technical results.* Let us give several lemmas that will be needed. They consist in constructing special supersolutions of (1a) or variants. The first one involves the successive resolution of the pure convective and diffusive PDEs.

Lemma 57. *Assume (H1) and φ is a bounded subsolution of (1a). Then, for any $x \in \mathbb{R}^d$ and $t_0 \geq 0$,*

$$\varphi^*(x, t_0) \leq \bar{\varphi}(x, t_0),$$

where $\bar{\varphi}$ is the maximal solution of

$$(33) \quad \partial_t \bar{\varphi} = \sup_{\xi \in \mathcal{E}} \text{tr} (a(\xi) D^2 \bar{\varphi}), \quad \bar{\varphi}_0(x) := \sup_{x+t_0 \mathcal{C}} \varphi^*(\cdot, 0)$$

(with $\mathcal{C} = \overline{\text{co}\{\text{Im}(b)\}}$).

The proof uses another version of Proposition 9 which holds without assumption (11). It involves the new control set

$$(34) \quad \mathcal{U} := \{\text{progressively measurable } \mathcal{U}\text{-valued processes } (q_t, \sigma_t)\},$$

where $\mathcal{U} := \overline{\{(q, \sigma) \in \mathbb{R}^d \times \mathbb{S}_d : (q, \sigma^2) \in \text{co}\{\text{Im}(b, a)\}\}}$. Here is this version:

Proposition 58. *Assume (H1) and $\varphi_0 \in C_b(\mathbb{R}^d)$. Then the viscosity solution of (1) satisfies*

$$\varphi(x, t) = \sup_{(q, \sigma) \in \mathcal{U}} \mathbb{E} \{(\varphi_0)_*(\mathbf{X}_t^x)\} \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+,$$

where the Ito process \mathbf{X}_s^x satisfies the Ito SDE

$$\begin{cases} d\mathbf{X}_s^x = \mathbf{q}_s ds + \sqrt{2} \sigma_s d\mathbf{B}_s, & s > 0, \\ \mathbf{X}_{s=0}^x = x. \end{cases}$$

Proof. By Lemma 87 in Appendix A.4, we can rewrite (1a) as follows:

$$\partial_t \varphi = \sup_{(q, \sigma) \in \mathcal{U}} \{q \cdot D\varphi + \text{tr}(\sigma^2 D^2 \varphi)\},$$

where \mathcal{U} is defined below (34). Since \mathcal{U} is compact and $(q, \sigma) \in \mathcal{U} \mapsto q \in \mathbb{R}^d$ and $(q, \sigma) \in \mathcal{U} \mapsto \sigma \in \mathbb{S}_d$ are continuous, we can apply Proposition 9 (or directly standard results from stochastic control theory [29]). \square

Proof of Lemma 57. Take the supconvolution in space, φ^ε for $\varepsilon > 0$, see (28). By Lemma 53, it is a subsolution of (1a), Lipschitz continuous in x , and *BUSC* in (x, t) . Let us first derive an inequality for φ^ε . By the representation formula Proposition 58 and the comparison principle, we have

$$\varphi^\varepsilon(x, t_0) \leq \sup_{(\mathbf{q}, \boldsymbol{\sigma}) \in \mathcal{U}} \mathbb{E} \{ \varphi^\varepsilon(\mathbf{X}_{t_0}^x, 0) \}$$

where $\mathbf{X}_{t_0}^x = x + \int_0^{t_0} \mathbf{q}_t dt + \sqrt{2} \int_0^{t_0} \boldsymbol{\sigma}_t d\mathbf{B}_t$; indeed, the right-hand side is the unique solution with initial data $\varphi^\varepsilon(\cdot, 0)$. By the definition of \mathcal{U} , see (34), we infer that

$$\int_0^{t_0} \mathbf{q}_t(\omega) dt \in t_0 \mathcal{C} \quad \forall (\mathbf{q}_t, \boldsymbol{\sigma}_t) \in \mathcal{U}, \forall \omega \in \Omega,$$

where Ω is the sample space of (10). Hence, for such given controls,

$$\varphi^\varepsilon(\mathbf{X}_{t_0}^x, 0) \leq \sup_{\mathbf{Y}_{t_0}^x + t_0 \mathcal{C}} \varphi^\varepsilon(\cdot, 0)$$

where $\mathbf{Y}_{t_0}^x := x + \sqrt{2} \int_0^{t_0} \boldsymbol{\sigma}_t d\mathbf{B}_t$. Let ϕ_0^ε be the supconvolution of ϕ_0 , then

$$\begin{aligned} \varphi^\varepsilon(\mathbf{X}_{t_0}^x, 0) &\leq \sup_{y \in \mathbf{Y}_{t_0}^x + t_0 \mathcal{C}} \sup_{z \in \mathbb{R}^d} \left\{ \varphi^*(y + z, 0) - \frac{|z|^2}{2\varepsilon^2} \right\} \\ &= \sup_{z \in \mathbb{R}^d} \left[\left\{ \sup_{y \in \mathbf{Y}_{t_0}^x + t_0 \mathcal{C}} \varphi^*(y + z, 0) \right\} - \frac{|z|^2}{2\varepsilon^2} \right] \\ &= \sup_{z \in \mathbb{R}^d} \left\{ \phi_0(\mathbf{Y}_{t_0}^x + z) - \frac{|z|^2}{2\varepsilon^2} \right\} \\ &= \phi_0^\varepsilon(\mathbf{Y}_{t_0}^x). \end{aligned}$$

Taking expectation and sup over the controls, we get

$$(35) \quad \varphi^\varepsilon(x, t_0) \leq \sup_{\boldsymbol{\sigma} \in \sqrt{\mathcal{D}}} \mathbb{E} \{ \phi_0^\varepsilon(\mathbf{Y}_{t_0}^x) \}$$

for $\mathbf{Y}_{t_0}^x = x + \sqrt{2} \int_0^{t_0} \boldsymbol{\sigma}_t d\mathbf{B}_t$,

$$\sqrt{\mathcal{D}} := \left\{ \text{progressively measurable } \boldsymbol{\sigma}_t \text{ with values in } \sqrt{\mathcal{D}} \right\}$$

and

$$\sqrt{\mathcal{D}} := \overline{\{ \boldsymbol{\sigma} \in \mathbb{S}_d : \boldsymbol{\sigma}^2 \in \text{co} \{ \text{Im}(a) \} \}}.$$

Here we have used that if $(\mathbf{q}_t, \boldsymbol{\sigma}_t) \in \mathcal{U}$ then necessarily $\boldsymbol{\sigma}_t \in \sqrt{\mathcal{D}}$. But using again the representation of Proposition 58, the right-hand side of (35) is the solution ϕ_ε of (33) with initial data ϕ_0^ε . Hence,

$$\varphi^\varepsilon(x, t_0) \leq \phi_\varepsilon(x, t_0) \quad \forall x \in \mathbb{R}^d,$$

and we deduce the result as $\varepsilon \downarrow 0$. Indeed, since $\phi_0^\varepsilon \downarrow \phi_0$ (pointwise) by standard properties, $\phi_\varepsilon \downarrow \bar{\phi}$ by the stability result for extremal solutions in Proposition 52. The proof is complete (since $\varphi^\varepsilon \downarrow \varphi$ too). \square

The next lemma involves the convolution semigroup of the equation

$$(36) \quad \partial_t \varphi = \text{tr} (a_0 D^2 \varphi),$$

with a fixed a_0 . Let us recall its definition, see [15] and the references therein.

Proposition 59 (Convolution semigroups). *For any $0 \leq a_0 \in \mathbb{S}_d$, there exists a family of measures $\mu_t \in M^1(\mathbb{R}^d)$, parametrized by $t \geq 0$, such that*

- (i) $\mu_{t=0}$ is the Dirac delta at $x = 0$,
- (ii) $\mu_{t+s} = \mu_t * \mu_s$ for any $t, s \geq 0$,

(iii) $t \geq 0 \mapsto \mu_t$ is weakly- \star continuous, i.e.

$$\lim_{t \rightarrow t_0} \int \varphi_0 d\mu_t \rightarrow \int \varphi_0 d\mu_{t_0} \quad \forall t_0 \in \mathbb{R}^+, \forall \varphi \in C_b(\mathbb{R}^d),$$

(iv) $\mu_t \geq 0$ and $\mu_t(\mathbb{R}^d) \leq 1$ for any $t \geq 0$, and

(v) for any $\varphi_0 \in C_b^\infty(\mathbb{R}^d)$,

$$\varphi(x, t) := \mu_t * \varphi_0(x) \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^+)$$

solves (36) and satisfies $\varphi(\cdot, t = 0) = \varphi_0(\cdot)$.

Lemma 60. Assume $\tilde{b} = \tilde{b}(\xi)$ and $\tilde{a} = \tilde{a}(\xi)$ satisfy (H1), and that $a_0 \in \mathbb{S}_d$ is a nonnegative matrix with convolution semigroup μ_t . Then for any bounded and nonnegative subsolution ϕ of the equation

$$(37) \quad \partial_t \phi = \sup_{\xi \in \mathcal{E}} \left\{ \tilde{b}(\xi) \cdot D\phi + \text{tr}(\tilde{a}(\xi) D^2 \phi) \right\} + \text{tr}(a_0 D^2 \phi)$$

and any time $t \geq 0$,

$$(38) \quad \phi^*(\cdot, t) \leq \mu_t * \bar{\varphi}(\cdot, t),$$

where $\bar{\varphi}$ is the maximal solution of

$$(39) \quad \partial_t \varphi = \sup_{\xi \in \mathcal{E}} \left\{ \tilde{b}(\xi) \cdot D\varphi + \text{tr}(\tilde{a}(\xi) D^2 \varphi) \right\}, \quad \varphi(\cdot, 0) = \phi^*(\cdot, 0).$$

For the proof, we need a classical result for viscosity solutions (see e.g. [21]). Let BUC be the spaces of bounded uniformly continuous functions.

Lemma 61. Assume (H1) and $\varphi_0 \in BUC(\mathbb{R}^d)$. Then the viscosity solution φ of (1) satisfies $\varphi \in BUC(\mathbb{R}^d \times \mathbb{R}^+)$.

Let us prove the previous lemma.

Proof of Lemma 60. Take the space approximate unit (29) and consider

$$(x, t) \mapsto \mu_t * \rho_\nu(x).$$

By Proposition 59, it is nonnegative, C_b^∞ in space-time, and solves:

$$\partial_t(\mu_t * \rho_\nu) = \text{tr}(a_0 D^2(\mu_t * \rho_\nu)), \quad \mu_{t=0} * \rho_\nu = \rho_\nu.$$

Then take the supconvolution ϕ^ε in space of ϕ , as in (28). By Lemma 53, it is a $BUSC$ subsolution of the same equation as ϕ , that is (37). It is also Lipschitz continuous in space, and hence by Lemma 61 there exists a unique solution $\varphi_\varepsilon \in BUC(\mathbb{R}^d \times \mathbb{R}^+)$ of the PDE part of (39) with initial data

$$\varphi_\varepsilon(\cdot, 0) = \phi^\varepsilon(\cdot, 0).$$

By the comparison principle, and since $\phi \geq 0$, we know that φ_ε is nonnegative. Now let θ_ν be the time approximate unit (30) with $\delta = \nu$, and define

$$\varphi_{\varepsilon, \nu} := \varphi_\varepsilon *_{x,t} (\rho_\nu \theta_\nu).$$

This function is nonnegative, C_b^∞ in space-time, and by Lemma 54 a supersolution of the PDE part of (39). Now, we claim that:

$$(40) \quad \text{The function } \phi_{\varepsilon, \nu}(x, t) := \mu_t * \rho_\nu * \varphi_{\varepsilon, \nu}(\cdot, t)(x) \text{ is a supersolution of (37).}$$

This follows from the computations

$$\begin{aligned} \partial_t \phi_{\varepsilon, \nu} &= \mu_t * \rho_\nu * \partial_t \varphi_{\varepsilon, \nu} + \varphi_{\varepsilon, \nu} * \partial_t(\mu_t * \rho_\nu) \\ &\geq \mu_t * \rho_\nu * \sup_{\mathcal{E}} \left\{ \tilde{b} \cdot D\varphi_{\varepsilon, \nu} + \text{tr}(\tilde{a} D^2 \varphi_{\varepsilon, \nu}) \right\} + \varphi_{\varepsilon, \nu} * \text{tr}(a_0 D^2(\mu_t * \rho_\nu)), \end{aligned}$$

by the equations satisfied by $\varphi_{\varepsilon,\nu}$ and $\mu_t * \rho_\nu$. It follows that

$$\begin{aligned} \partial_t \phi_{\varepsilon,\nu} &\geq \sup_{\mathcal{E}} \left\{ \tilde{b} \cdot (\mu_t * \rho_\nu * D\varphi_{\varepsilon,\nu}) + \operatorname{tr} (\tilde{a} (\mu_t * \rho_\nu * D^2\varphi_{\varepsilon,\nu})) \right\} \\ &\quad + \operatorname{tr} (a_0 (\varphi_{\varepsilon,\nu} * D^2(\mu_t * \rho_\nu))) \\ &= \sup_{\mathcal{E}} \left\{ \tilde{b} \cdot D\phi_{\varepsilon,\nu} + \operatorname{tr} (\tilde{a} D^2\phi_{\varepsilon,\nu}) \right\} + \operatorname{tr} (a_0 D^2\phi_{\varepsilon,\nu}), \end{aligned}$$

which completes the proof of (40).

It remains to pass to the limits to obtain (38). Since $\varphi_\varepsilon \in BUC(\mathbb{R}^d \times \mathbb{R}^+)$, it is standard that $\varphi_{\varepsilon,\nu} = \varphi_\varepsilon *_{x,t} (\rho_\nu \theta_\nu)$ converges uniformly towards φ_ε as $\nu \rightarrow 0^+$. Then $\phi_{\varepsilon,\nu}(x, t)$ also converges uniformly towards $\mu_t * \varphi_\varepsilon(\cdot, t)(x)$ as $\nu \rightarrow 0^+$. Indeed,

$$\begin{aligned} \|\phi_{\varepsilon,\nu} - \mu_t * \varphi_\varepsilon\|_\infty &= \|\mu_t * \rho_\nu * \varphi_{\varepsilon,\nu} - \mu_t * \varphi_\varepsilon\|_\infty \\ &= \|\mu_t * (\rho_\nu * \varphi_{\varepsilon,\nu} - \varphi_\varepsilon)\|_\infty \\ &\leq \|\rho_\nu * \varphi_{\varepsilon,\nu} - \varphi_\varepsilon\|_\infty, \end{aligned}$$

where the infinity norm is in space-time and we used that $\mu_t(\mathbb{R}^d) \leq 1$, see Proposition 59. By stability, see e.g. Proposition 50, we infer that $\mu_t * \varphi_\varepsilon$ is a *BUC* viscosity supersolution of (37) since $\phi_{\varepsilon,\nu}$ was by (40). Moreover, at the initial time,

$$\mu_{t=0} * \varphi_\varepsilon(\cdot, 0) = \varphi_\varepsilon(\cdot, 0) = \phi^\varepsilon(\cdot, 0) \geq \phi^*(\cdot, 0).$$

Comparing the sub and supersolutions ϕ and $\mu_t * \varphi_\varepsilon$ of (37) implies that

$$(41) \quad \phi^*(\cdot, t) \leq \mu_t * \varphi_\varepsilon(\cdot, t) \quad \forall t \geq 0.$$

We now conclude by the stability of extremal solutions, see Proposition 52. Indeed, recall that $\varphi_\varepsilon(\cdot, 0) = \phi^\varepsilon(\cdot, 0) \downarrow \phi^*(\cdot, 0)$ as $\varepsilon \downarrow 0$, so that the corresponding solutions of (39) satisfy $\varphi_\varepsilon \downarrow \bar{\varphi}$. We then get (38) by passing to the limit in (41) using the dominated convergence theorem. Let us justify this. Since the measure μ_t is bounded, it suffices to derive an upper bound on φ_ε uniform in ε . To do so, use the maximum principle $\varphi_\varepsilon \leq \sup \varphi_\varepsilon(\cdot, 0)$ and recall that $\varphi_\varepsilon(\cdot, 0) = \phi^\varepsilon(\cdot, 0)$ to find that $\varphi_\varepsilon \leq \sup \phi$. The proof is complete. \square

4.3.2. Proof of Theorem 31. We are now ready to focus on the proof of Theorem 31, that is to say of Estimate (21). First we will roughly speaking reduce the proof to considering the pure diffusive equation

$$(42) \quad \partial_t \phi = \sup_{\xi \in \mathcal{E}} \operatorname{tr} (a(\xi) D^2 \phi).$$

Define $c_a(t)$ to be the smallest $c \in [1, \infty]$ such that for any bounded and nonnegative subsolution ϕ of (42),

$$(43) \quad \int \sup_{\bar{Q}_1(x)} \phi^*(\cdot, t) \, dx \leq c \int \sup_{\bar{Q}_1(x)} \phi^*(\cdot, 0) \, dx.$$

Henceforth it is agreed that (43) holds whenever the left-hand side is zero or the right-hand side is not finite. Note then that $c = \infty$ satisfies (43) because $\phi^* \equiv 0$ whenever $\phi^*(\cdot, 0) \equiv 0$, by the comparison principle. Actually, we shall see that $c_a(t)$ is always finite; but, let us first explain how to reduce the proof of (21) to the qualitative analysis of this constant.

Lemma 62. *Assume (H1), and φ and ψ are bounded sub and supersolutions of (1a). Then for all $t \geq 0$,*

$$(44) \quad \int \sup_{\bar{Q}_1(x)} (\varphi^* - \psi_*)^+(\cdot, t) \, dx \leq c_a(t) \int \sup_{\bar{Q}_1(x)+tC} (\varphi^* - \psi_*)^+(\cdot, 0) \, dx.$$

The proof uses the elementary result:

Lemma 63. *Assume (H1), and φ and ψ are bounded sub and supersolutions of (1a). Then $\varphi^* - \psi_*$ is a subsolution.*

Sketch of the proof. Let

$$H_\xi(\varphi) := b(\xi) \cdot D\varphi + \text{tr}(a(\xi)D^2\varphi)$$

and note that then

$$\partial_t(\varphi - \psi) = \sup_{\xi \in \mathcal{E}} H_\xi(\varphi) - \sup_{\xi \in \mathcal{E}} H_\xi(\psi) \leq \sup_{\xi \in \mathcal{E}} (H_\xi(\varphi) - H_\xi(\psi)),$$

which gives us the result. \square

Rigourously, we need to argue with a test function. This can be done with the help of the Ishii lemma and semijets, see [21, Theorem 8.3]. Such a justification is standard and left to the reader.

Proof of Lemma 62. Lemma 63 implies that $\varphi^* - \psi_*$ is a subsolution of (1a); hence, Lemma 57 implies that

$$(\varphi^* - \psi_*)(x, t_0) \leq \bar{\phi}(x, t_0),$$

for any $x \in \mathbb{R}^d$ and $t_0 \geq 0$, where $\bar{\phi}$ is the maximal solution of (42) with initial data $\phi_0(x) = \sup_{x+t_0\mathcal{C}}(\varphi^* - \psi_*)(\cdot, 0)$. Taking the sup in x over cubes, we get

$$\sup_{\bar{Q}_1(x)} (\varphi^* - \psi_*)^+(\cdot, t_0) \leq \sup_{\bar{Q}_1(x)} (\bar{\phi})^+(\cdot, t_0).$$

Note that $(\bar{\phi})^+$ is still a subsolution of (42), by the stability by supremum of viscosity subsolutions, see Proposition 49. In particular,

$$\int \sup_{\bar{Q}_1(x)} (\bar{\phi})^+(\cdot, t_0) dx \leq c_a(t_0) \int \sup_{\bar{Q}_1(x)} (\phi_0)^+ dx$$

by the definition of $c_a(t_0)$. We then readily get the desired result from the previous inequalities and using the definition of ϕ_0 . \square

We now estimate $c_a(t)$ in terms of $|H|_{\text{diff}}$ from Definition 18. We will need the lemmas below.

Lemma 64. *Let $a = a(\xi)$ satisfy (H1), $a_0 \in \mathbb{S}_d$, $0 \leq a_0 \leq \text{Im}(a)$, and ϕ be a bounded and nonnegative subsolution of (42). Then for any $t \geq 0$,*

$$(45) \quad \int \sup_{\bar{Q}_1(x)} \phi^*(\cdot, t) dx \leq \int \sup_{\bar{Q}_1(x)} \bar{\varphi}(\cdot, t) dx,$$

where $\bar{\varphi}$ is the maximal solution of

$$\partial_t \varphi = \sup_{\xi \in \mathcal{E}} \text{tr}((a(\xi) - a_0)D^2\varphi), \quad \varphi(\cdot, 0) = \phi^*(\cdot, 0).$$

Proof. We can rewrite (42) as follows:

$$\partial_t \phi = \sup_{\mathcal{E}} \{ \text{tr}((a - a_0)D^2\phi) \} + \text{tr}(a_0 D^2\phi).$$

We recognize Equation (37) with $\tilde{b} \equiv 0$ and $\tilde{a}(\xi) = a(\xi) - a_0$. Applying Lemma 60, we obtain that

$$\phi^*(\cdot, t) \leq \mu_t * \bar{\varphi}(\cdot, t),$$

for any $t \geq 0$ and some nonnegative measure μ_t with mass $\mu_t(\mathbb{R}^d) \leq 1$. The desired inequality (45) then readily follows by a simple Young type inequality for convolutions. \square

Corollary 65. *Assume $a = a(\xi)$ satisfies (H1) and $c_a(t)$ is defined in (43). Then for any nonnegative $a_0 \in \mathbb{S}_d$ such that $a_0 \leq \text{Im}(a)$ and any $t \geq 0$,*

$$c_a(t) \leq c_{a-a_0}(t).$$

Proof. This is immediate from Lemma 64. \square

Let us now define the modulus below.

Lemma 66. *For any $r \geq 0$, let $\omega_d(r)$ be the smallest $\omega \in [0, \infty]$ satisfying*

$$(46) \quad \int \sup_{\overline{Q}_1(x)} \varphi^*(\cdot, r) \, dx \leq (1 + \omega) \int \sup_{\overline{Q}_1(x)} \varphi^*(\cdot, 0) \, dx,$$

for any nonnegative and bounded subsolution of

$$(47) \quad \partial_t \varphi = \sup_{\lambda \in \text{Sp}(D^2 \varphi)} \lambda^+.$$

Then, for any $a = a(\xi)$ satisfying (H1) and $t \geq 0$,

$$(48) \quad c_a(t) \leq 1 + \omega_d(t|H|_{\text{diff}}).$$

The proof is based on the Ky Fan type inequality (see e.g. [45]),

$$(49) \quad \text{tr}(XY) \leq \sum_{i=1}^d \lambda_i(X) \lambda_i(Y) \quad \forall X, Y \in \mathbb{S}_d,$$

where $\lambda_1 \leq \dots \leq \lambda_d$ denote the ordered eigenvalues.

Proof of Lemma 66. Let ϕ be a nonnegative bounded subsolution of (42). By (49),

$$\partial_t \phi \leq \|\text{tr}(a)\|_\infty \sup_{\lambda \in \text{Sp}(D^2 \phi)} \lambda^+$$

in the viscosity sense. We recognize Equation (47) up to a rescaling. Indeed, set

$$\varphi(x, t) := \phi \left(x, \frac{t}{\|\text{tr}(a)\|_\infty} \right)$$

and observe that φ is a subsolution of (47). By the definition of $\omega_d(\cdot)$ in (46), we deduce that

$$\begin{aligned} \int \sup_{\overline{Q}_1(x)} \phi^*(\cdot, t) \, dx &= \int \sup_{\overline{Q}_1(x)} \varphi^*(\cdot, t \|\text{tr}(a)\|_\infty) \, dx \\ &\leq (1 + \omega_d(t \|\text{tr}(a)\|_\infty)) \int \sup_{\overline{Q}_1(x)} \varphi^*(\cdot, 0) \, dx \\ &= (1 + \omega_d(t \|\text{tr}(a)\|_\infty)) \int \sup_{\overline{Q}_1(x)} \phi^*(\cdot, 0) \, dx. \end{aligned}$$

Recalling that ϕ is an arbitrary nonnegative and bounded subsolution of (42), it follows that

$$c_a(t) \leq 1 + \omega_d(t \|\text{tr}(a)\|_\infty)$$

(this constant being the smallest one satisfying (43)). Taking now $a_0 \in \mathbb{S}_d^+$ such that $a_0 \leq \text{Im}(a)$ and repeating the above arguments, we also have

$$c_{a-a_0}(t) \leq 1 + \omega_d(t \|\text{tr}(a - a_0)\|_\infty).$$

Using in addition Corollary 65 and taking the infimum in a_0 , we infer that

$$c_a(t) \leq 1 + \omega_d \left(t \inf_{a_0 \text{ such as above}} \|\text{tr}(a - a_0)\|_\infty \right),$$

by the monotonicity of $\omega_d(\cdot)$. This function is nondecreasing because the right-hand side of (47) is nonnegative such that the solutions of (47) are nondecreasing in time. We recognize the above inf as the quantity $|H|_{\text{diff}}$ from Definition 18 and this gives us the desired inequality (48).

Note finally that we have assumed $\|\text{tr}(a - a_0)\|_\infty$ to be positive when dividing by it. If this is not the case then, $a \equiv a_0$ for some a_0 and Equation (42) becomes

linear. Its solutions are then given by Proposition 59, and it is readily seen that $c_a(t) = 1$. This completes the proof. \square

In order to establish Theorem 31, it remains to show that $\omega_d(\cdot)$ is a modulus of continuity. Here is a technical lemma. It involves again the profile

$$U(r) = c_0 \int_r^\infty e^{-\frac{s^2}{4}} ds \quad \text{with} \quad c_0 = \left(\int_0^\infty e^{-\frac{s^2}{4}} ds \right)^{-1}.$$

Lemma 67. *For any $(x, t) \in \mathbb{R}^d \times \mathbb{R}^+$, define*

$$\Phi(x, t) := \begin{cases} U\left((|x| - 1)^+ / \sqrt{t}\right) & \text{if } t > 0, \\ \mathbf{1}_{|x| < 1} & \text{if } t = 0. \end{cases}$$

Then Φ is a supersolution of (47) which is BLSC on $\mathbb{R}^d \times \mathbb{R}^+$.

Proof. The lower semicontinuity up to $t = 0$ is immediate. Arguing as in Lemma 56, we observe that (47) is satisfied for $|x| < 1$, since Φ is constant in that region. It is satisfied if $|x| = 1$, because the subsets are empty. Now if $|x| > 1$,

$$\Phi(x, t) = U\left((|x| - 1) / \sqrt{t}\right),$$

so that

$$\partial_t \Phi = -\frac{|x| - 1}{2t^{\frac{3}{2}}} U' \quad \text{and} \quad \partial_{x_i x_j}^2 \Phi = \left(\frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3} \right) \frac{U'}{\sqrt{t}} + \frac{x_i x_j}{|x|^2} \frac{U''}{t}.$$

Since $U' \leq 0$ and $U'' \geq 0$, we have

$$\sum_{i,j=1}^d \partial_{x_i x_j}^2 \Phi h_i h_j \leq \frac{U''}{t}$$

for any $h = (h_i)$ with $|h| = 1$ (we have equality if $h = x/|x|$). It follows that

$$\partial_t \Phi - \sup_{\lambda \in \text{Sp}(D^2 \Phi)} \lambda^+ \geq -\frac{1}{t} \left(\frac{r U'(r)}{2} + U''(r) \right)$$

with $r := (|x| - 1) / \sqrt{t}$. But the right-hand side is zero because of the definition of U . \square

Here are some properties on Φ .

Lemma 68. *Let Φ be defined in Lemma 67. Then Φ is nonnegative, nondecreasing with respect to t , and integrable with respect to x .*

Proof. The nonnegativity and monotonicity are clear from the definition of Φ . Let us compute its integral in x to check that it is finite—we provide the details because the explicit result will be useful later. We have

$$\begin{aligned} \int_{\mathbb{R}^d} \Phi(x, t) dx &= \int_{\mathbb{R}^d} U\left((|x| - 1)^+ / \sqrt{t}\right) dx \\ &\leq \left(U(0) \int_{|x| < 1} dx + \int_{|x| \geq 1} U\left((|x| - 1) / \sqrt{t}\right) dx \right). \end{aligned}$$

Using that $U(0) = 1$ and polar coordinates to compute the second integral,

$$(50) \quad \int_{\mathbb{R}^d} \Phi(x, t) dx \leq c_d \left(1 + \underbrace{\int_1^\infty r^{d-1} U\left(r / \sqrt{t}\right) dr}_{= t^{\frac{d}{2}} \int_{1/\sqrt{t}}^\infty s^{d-1} U(s) ds} \right),$$

for a constant c_d which only depends on the dimension d . Recalling now the definition of U above Lemma 67, the last integral is finite. This completes the proof. \square

We are now in position to prove the result below.

Lemma 69. *Let $\omega_d(\cdot)$ be defined as in Lemma 66. Then $\omega_d(\cdot)$ is a modulus of continuity in the sense that it is everywhere finite and has limit zero at zero.*

Proof. We would like to derive an upper bound on $\omega_d(r)$ for any $r \geq 0$. Take an arbitrary bounded and nonnegative subsolution φ of (47) and let us try to get an estimate of the form (46) with a finite ω independent of φ . We have to be precise enough to conclude in the end that $\omega_d(r) \rightarrow 0$ as $r \rightarrow 0^+$. If $\int \sup_{\overline{Q_1(x)}} \varphi^*(\cdot, 0) dx = \infty$, then (46) holds for any $\omega \geq 0$. The interesting case is therefore when

$$(51) \quad \int \sup_{\overline{Q_1(x)}} \varphi^*(\cdot, 0) dx < \infty,$$

which we assume from now on. The idea to get (46) is to construct an integrable supersolution of (47), see also [26]. To do so, we need mollifiers ρ_ν in space, as in (29), and it will be important to choose them with supports

$$(52) \quad \text{supp}(\rho_\nu) \subset B_\nu(0).$$

For any $\nu > 0$, let Ψ_ν be the solution of (47) with initial data

$$\Psi_\nu(\cdot, 0) = \rho_\nu(\cdot).$$

Next, for any $\delta > 0$, we introduce

$$\psi_{\nu, \delta} := \Psi_\nu *_{x,t} \left(\theta_\delta \sup_{\overline{Q_\nu(\cdot)}} \varphi^*(\cdot, 0) \right)$$

with the time approximate unit θ_δ from (30). This notation means that

$$\psi_{\nu, \delta}(x, t) := \iint_{\mathbb{R}^d \times \mathbb{R}^-} \Psi_\nu(x - y, t - s) \theta_\delta(s) \sup_{\overline{Q_\nu(y)}} \varphi^*(\cdot, 0) dy ds.$$

Note that $\psi_{\nu, \delta} \in BUC(\mathbb{R}^d \times \mathbb{R}^+)$ since $\Psi_\nu \in BUC(\mathbb{R}^d \times \mathbb{R}^+)$ and

$$(x, t) \mapsto \theta_\delta(t) \sup_{\overline{Q_\nu(x)}} \varphi^*(\cdot, 0) \in L^1(\mathbb{R}^d \times \mathbb{R}^-),$$

by Lemma 61 and (51). It is also standard that

$$\lim_{\delta \rightarrow 0^+} \psi_{\nu, \delta} = \psi_\nu := \Psi_\nu * \sup_{\overline{Q_\nu(\cdot)}} \varphi^*(\cdot, 0) \quad \text{uniformly on } \mathbb{R}^d \times \mathbb{R}^+$$

(with $* = *_x$). The latter limit will be the supersolution of (47) that we will use to derive an L_{int}^∞ a priori estimate of the form (46).

Let us first compare φ with ψ_ν . Note that ψ_ν is indeed a supersolution of (47) because so were all the $\psi_{\nu, \delta}$ by Lemma 54. It is also BUC up to $t = 0$ as a uniform limit of BUC functions. Since, by (52),

$$\psi_\nu(x, 0) = \int \rho_\nu(y) \sup_{\overline{Q_\nu(x-y)}} \varphi^*(\cdot, 0) dy \geq \varphi^*(x, 0) \quad \forall x \in \mathbb{R}^d,$$

the comparison principle implies that

$$\varphi^* \leq \psi_\nu.$$

It follows that for any $r \geq 0$ and $x \in \mathbb{R}^d$,

$$\begin{aligned} \sup_{\overline{Q_1(x)}} \varphi^*(\cdot, r) &\leq \sup_{z \in [-1, 1]^d} \int \Psi_\nu(y, r) \sup_{\overline{Q_\nu(x+z-y)}} \varphi^*(\cdot, 0) dy \\ &\leq \int \Psi_\nu(y, r) \sup_{\overline{Q_{1+\nu}(x-y)}} \varphi^*(\cdot, 0) dy, \end{aligned}$$

so that

$$\int \sup_{\overline{Q}_1(x)} \varphi^*(\cdot, r) \, dx \leq \int_{\mathbb{R}^d} \Psi_\nu(y, r) \, dy \int \sup_{\overline{Q}_{1+\nu}(x)} \varphi^*(\cdot, 0) \, dx.$$

Applying Lemma 90 of the appendix, we get that

$$\int \sup_{\overline{Q}_1(x)} \varphi^*(\cdot, r) \, dx \leq (1 + \omega) \int \sup_{\overline{Q}_1(x)} \varphi^*(\cdot, 0) \, dx,$$

where

$$\omega := (1 + \nu)^d \int_{\mathbb{R}^d} \Psi_\nu(y, r) \, dy - 1.$$

This ω is nonnegative since Ψ_ν is nondecreasing in time as a solution of (47). It is also independent of φ so we can conclude that

$$(53) \quad \omega_d(r) \leq \inf_{\nu > 0} (1 + \nu)^d \int_{\mathbb{R}^d} \Psi_\nu(y, r) \, dy - 1 \quad \forall r \geq 0,$$

by the definition of $\omega_d(r)$ given in Lemma 66. From this inequality, it is sufficient to prove that $\Psi_\nu \in C(\mathbb{R}^+; L^1(\mathbb{R}^d))$ for any $\nu > 0$, to conclude that $\omega_d(\cdot)$ is a modulus of continuity. Indeed, $\omega_d(\cdot)$ will then be everywhere finite and have limit zero at zero since $\int \Psi_\nu(y, 0) \, dy = \int \rho_\nu = 1$ and ν can be taken arbitrary small.

Let us check that $\Psi_\nu \in C(\mathbb{R}^+; L^1(\mathbb{R}^d))$ to finish the proof. We need the supersolution Φ from Lemma 67. Let

$$(54) \quad \Phi_\nu(x, t) := \frac{c}{\nu^d} \Phi\left(\frac{x}{\nu}, \frac{t}{\nu^2}\right),$$

for some constant $c > 0$ and note that this is a supersolution of (47) since so was Φ . Take c so large that $c \mathbf{1}_{|\cdot| < 1} \geq \rho(\cdot)$ which implies that

$$\Phi_\nu(x, 0) = \frac{c}{\nu^d} \mathbf{1}_{|x| < \nu} \geq \frac{1}{\nu^d} \rho\left(\frac{x}{\nu}\right) = \Psi_\nu(x, 0),$$

for any $x \in \mathbb{R}^d$. It follows that $\Psi_\nu \leq \Phi_\nu$ by the comparison principle. The continuity of Ψ_ν in time with values in $L^1(\mathbb{R}^d)$ is then an immediate consequence of the properties of Φ stated in Lemma 68. Indeed, fixing for instance $T > 0$, we have

$$(55) \quad \Psi_\nu(x, t) \leq \Phi_\nu(x, T) \quad \forall x \in \mathbb{R}^d, \forall t \in [0, T],$$

since Φ is nondecreasing in time. The fixed integrable function $\Phi_\nu(\cdot, T)$ can then serve as a dominating function and continuity follows by the dominated convergence theorem. The proof is complete. \square

End of the proof of Theorem 31. Estimates (44) and (48) imply the desired inequality (21), with an $\omega_d(\cdot)$ being indeed a modulus of continuity by Lemma 69. \square

4.4. L_{int}^∞ stability: Proof of Theorem 30.

Proof of Theorem 30. Let $L_b := \|b\|_\infty$ and $L_a := \|\text{tr}(a)\|_\infty$ and assume $L_a > 0$. By the Ky Fan inequality (49), the Hamiltonian (4) of Equation (1a) satisfies

$$H(p, X) \leq L_b |p| + L_a \sup_{\lambda \in \text{Sp}(X)} \lambda^+,$$

and hence, any subsolution of (1a) is a subsolution of the equation

$$(56) \quad \partial_t \varphi = L_b |D\varphi| + L_a \sup_{\lambda \in \text{Sp}(D^2\varphi)} \lambda^+.$$

By Lemma 63, if φ and ψ are bounded sub and supersolutions of (1a), then $\varphi^* - \psi_*$ is a subsolution of the same equation. In particular, $\varphi^* - \psi_*$ is a subsolution of (56). To prove Estimate (20), we will construct an integrable supersolution of (56).

Consider again

$$U : r \geq 0 \mapsto c_0 \int_r^\infty e^{-\frac{s^2}{4}} ds$$

where $c_0 > 0$ is such that $U(0) = 1$, and

$$(57) \quad \Psi(x, t) := U\left(\frac{(|x| - 1 - L_b t)^+}{\sqrt{L_a t}}\right) \quad \text{with} \quad \Psi(x, t = 0) := \mathbf{1}_{|x| < 1}.$$

Arguing as in the proof of Lemma 67, we see that Ψ is a supersolution of (56). Define now

$$\psi := \Psi *_x \sup_{\overline{Q}_1(\cdot)} \phi_0,$$

where $\phi_0(x) := (\varphi^* - \psi_*)^+(x, t = 0)$. We will use Lemma 54 to show that ψ is a supersolution of (56). To do this, Ψ should be bounded and continuous, and $\sup_{\overline{Q}_1(\cdot)} \phi_0$ integrable. The latter condition can be assumed since (20) trivially holds if not. Since Ψ is bounded continuous only for $t > 0$, we apply Lemma 54 for $t > \varepsilon > 0$. Since ε is arbitrary, we conclude that ψ is a supersolution of (56). Since $\Psi \in C(\mathbb{R}^+, L^1)$ and $\sup_{\overline{Q}_1(\cdot)} \phi_0 \in L^\infty(\mathbb{R}^d)$, this supersolution is continuous up to $t = 0$ and satisfies

$$\psi(x, 0) = \int \mathbf{1}_{|y| < 1} \sup_{\overline{Q}_1(x-y)} \underbrace{\phi_0}_{=(\varphi^* - \psi_*)(t=0)} dy \geq (\varphi^* - \psi_*)(x, 0).$$

Since $\varphi^* - \psi_*$ is a subsolution of (56), $(\varphi^* - \psi_*)^+ \leq \psi$ everywhere (where we can take the positive part because $\psi \geq 0$). Hence

$$\begin{aligned} \int \sup_{\overline{Q}_1(x) \times [0, T]} (\varphi^* - \psi_*)^+ dx &\leq \int \sup_{\overline{Q}_1(x) \times [0, T]} \psi dx \\ &\leq \int \sup_{t \in [0, T]} \Psi(y, t) dy \int \sup_{\overline{Q}_2(x)} \phi_0 dx, \end{aligned}$$

by the Fubini theorem, etc. The first integral satisfies

$$\int \sup_{t \in [0, T]} \Psi(y, t) dy \leq \int U\left(\frac{(|y| - 1 - L_b T)^+}{\sqrt{L_a T}}\right) dy < \infty,$$

by (57) and since U is nondecreasing and integrable. For the second integral, Lemma 16 implies that

$$\int \sup_{\overline{Q}_2(x)} \phi_0 dx \leq C \int \sup_{\overline{Q}_1(x)} (\varphi^* - \psi_*)^+(\cdot, 0) dx,$$

for a constant C which only depends on d . Combining the three inequalities above completes the proof of (20) when $L_a = \|\text{tr}(a)\|_\infty > 0$. If $L_a = 0$, there is no diffusive part in (1a), and (20) follows from Proposition 8 and Lemma 16. \square

4.5. Consequences: Proofs of Theorem 28 and Corollary 33.

Lemma 70. *Assume (H1) and φ and ψ are continuous viscosity solutions of (1a). Then $|\varphi - \psi|$ is a subsolution of the same PDE.*

Proof. By Lemma 63, both $\varphi - \psi$ and $\psi - \varphi$ are subsolutions. Thus, by the stability by sup, $(\varphi - \psi)^+ = \max\{0, \varphi - \psi\}$ and $(\psi - \varphi)^+$ are subsolutions and then also $|\varphi - \psi| = \max\{(\varphi - \psi)^+, (\psi - \varphi)^+\}$. \square

In the sequel, G_t is the semigroup on $C_b(\mathbb{R}^d)$ of (1) defined in (17).

Proof of Theorem 28. Let us first prove that $E = C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ satisfies (18)–(19). Property (18) follows from Theorem 15. The fact that G_t maps $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ into itself follows from Theorem 31. Indeed, if $\varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$, then the function $(x, t) \mapsto |G_t \varphi_0(x)|$ is a bounded subsolution of (1a), by Lemma 70 with $\psi \equiv 0$. Estimate (21) then implies that for any $t \geq 0$,

$$\begin{aligned} \|G_t \varphi_0\|_{\text{int}} &= \int \sup_{\overline{Q_1(x)}} |G_t \varphi_0| \, dx \\ &\leq (1 + \omega_d(t|H|_{\text{diff}})) \int \sup_{\overline{Q_1(x)+t\mathcal{C}}} |\varphi_0| \, dx. \end{aligned}$$

By the boundedness of $\mathcal{C} = \overline{\text{co}\{\text{Im}(b)\}}$ and Lemma 16, these integrals are finite whenever $\varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$. It remains to prove that

$$G_t : C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d) \rightarrow C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$$

is continuous for any $t \geq 0$. Let us apply again (21) to

$$(x, t) \mapsto |G_t \varphi_0(x) - G_t \psi_0(x)|,$$

which is a subsolution of (1a) by Lemma 70. As above we get that

$$(58) \quad \|G_t \varphi_0 - G_t \psi_0\|_{\text{int}} \leq (1 + \omega_d(t|H|_{\text{diff}})) \int \sup_{\overline{Q_1(x)+t\mathcal{C}}} |\varphi_0 - \psi_0| \, dx.$$

Arguing as above we deduce the desired continuity. This completes the proof of the fact that $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ satisfies (18)–(19).

Let now E be another normed space satisfying such properties and let us prove that it is continuously embedded into $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$. We will need the following lemma (whose proof is elementary).

Lemma 71. *For any $\varphi_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\sup |\varphi_0| \leq |\sup \varphi_0| + |\inf \varphi_0|$.*

Recall then that (19) is required to hold for any data $b = b(\xi)$ and $a = a(\xi)$ satisfying (H1). Choose for instance the eikonal equation

$$\partial_t \varphi = \sum_{i=1}^d |\partial_{x_i} \varphi|$$

and denote by G_t^e its semigroup. By the representation Proposition 8,

$$G_t^e \varphi_0(x) = \sup_{x+t[-1,1]^d} \varphi_0.$$

Since $G_{t=1}^e$ maps $E \subset L^1(\mathbb{R}^d)$ into itself by assumption, the function

$$x \mapsto \sup_{x+[-1,1]^d} \varphi_0$$

belongs to $L^1(\mathbb{R}^d)$ for any $\varphi_0 \in E$. Using that E is a vector space, $-\varphi_0 \in E$, and the function

$$x \mapsto \inf_{x+[-1,1]^d} \varphi_0$$

also belongs to $L^1(\mathbb{R}^d)$. By Lemma 71, we conclude that $E \subseteq C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$. Finally we use that

$$G_{t=1}^e : E \rightarrow E$$

is continuous at $\varphi_0 \equiv 0$ to obtain that for any $\|\varphi_0^n\|_E \rightarrow 0$,

$$\|G_{t=1}^e \varphi_0^n\|_E \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Combining this with the continuity of the embedding $E \subset L^1(\mathbb{R}^d)$, we obtain that

$$\left\| \sup_{x+[-1,1]^d} \varphi_0^n \right\|_{L_x^1} \rightarrow 0.$$

Using once again that E is a normed space, the same holds with $-\varphi_0$, that is

$$\left\| \inf_{x+[-1,1]^d} \varphi_0^n \right\|_{L_x^1} \rightarrow 0.$$

By Lemma 71, we conclude that $\|\varphi_0^n\|_{\text{int}} \rightarrow 0$ which completes the proof. \square

Before proving Corollary 33, we need a technical result.

Lemma 72. *Assume (H1) and let $|H|_{\text{conv}}$ be defined in Definition 18. Then there exists $b_0 \in \mathbb{R}^d$ such that $\mathcal{C} = \overline{\text{co}\{\text{Im}(b)\}}$ is included in the cube $\overline{Q}_{|H|_{\text{conv}}}(b_0)$.*

Proof. Let $(b_0^n)_n$ be a minimizing sequence in \mathbb{R}^d for the infimum

$$|H|_{\text{conv}} = \inf_{b_0 \in \mathbb{R}^d} \underbrace{\|b - b_0\|_{\infty}}_{=\sup_{\xi \in \mathcal{E}} |b(\xi) - b_0|}.$$

The sequence is bounded since b is bounded, and (a subsequence of) it converges to some b_0 . But then $|H|_{\text{conv}} = \lim_{n \rightarrow \infty} \|b - b_0^n\|_{\infty} = \|b - b_0\|_{\infty}$. Hence,

$$\text{Im}(b) \subseteq \overline{Q}_{\|b - b_0\|_{\infty}}(b_0) = \overline{Q}_{|H|_{\text{conv}}}(b_0)$$

because for any $\xi \in \mathcal{E}$ and $i = 1, \dots, d$,

$$|(b(\xi))_i - (b_0)_i| \leq \sup_{\mathcal{E}} \sqrt{\sum_{j=1}^d (b_j - (b_0)_j)^2} = \|b - b_0\|_{\infty}.$$

The convex envelope \mathcal{C} is therefore also included in this cube. \square

Proof of Corollary 33. Let us continue the computations from (58). We have, for any φ_0 and ψ_0 in $C_b \cap L_{\text{int}}^{\infty}(\mathbb{R}^d)$, and any $t \geq 0$,

$$\begin{aligned} \|G_t \varphi_0 - G_t \psi_0\|_{\text{int}} &\leq (1 + \omega_d(t|H|_{\text{diff}})) \int \sup_{\overline{Q}_1(x) + t\mathcal{C}} |\varphi_0 - \psi_0| dx \\ &\leq (1 + \omega_d(t|H|_{\text{diff}})) \int \sup_{\overline{Q}_1(x) + t\overline{Q}_{|H|_{\text{conv}}}(b_0)} |\varphi_0 - \psi_0| dx, \end{aligned}$$

for some center $b_0 \in \mathbb{R}^d$ given by Lemma 72. We change the variables $x + tb_0 \mapsto x$ to find that

$$\|G_t \varphi_0 - G_t \psi_0\|_{\text{int}} \leq (1 + \omega_d(t|H|_{\text{diff}})) \int \sup_{\overline{Q}_{1+t|H|_{\text{conv}}}(x)} |\varphi_0 - \psi_0| dx.$$

Applying the inequality given by Lemma 90 in the appendix, then leads to

$$\|G_t \varphi_0 - G_t \psi_0\|_{\text{int}} \leq (1 + \omega_d(t|H|_{\text{diff}})) (1 + t|H|_{\text{conv}})^d \|\varphi_0 - \psi_0\|_{\text{int}}. \quad \square$$

4.6. L_{int}^{∞} quasicontraction: Proofs of Proposition 34 and Theorem 35.

Proof of Proposition 34. Let us prove that the semigroup associated to (22) is not contractive. We proceed by contradiction assuming (22) is contractive. From now, let us fix an initial data $0 \leq \varphi_0 \in C_b^{\infty} \cap L_{\text{int}}^{\infty}(\mathbb{R})$ satisfying:

$$(59) \quad \varphi_0 \text{ is even and nonincreasing on } \mathbb{R}^+.$$

We shall see later why this is useful. Let φ be the solution of (22) with φ_0 as initial data. Contractiveness then implies that

$$(60) \quad \int \sup_{x+[-1,1]} \varphi(\cdot, t) dx \leq \int \sup_{x+[-1,1]} \varphi_0 dx \quad \forall t \geq 0.$$

Our plan is to integrate Equation (22) to contradict this inequality. To easily justify the computations, let us regularize the drift. Take for instance

$$(61) \quad \partial_t \varphi_\varepsilon = \sqrt{\varepsilon^2 + (\partial_x \varphi_\varepsilon)^2} - \varepsilon + \partial_{xx}^2 \varphi_\varepsilon, \quad \varphi_\varepsilon(t=0) = \varphi_0.$$

For any $\varepsilon > 0$, the new solution is C_b^∞ up to $t = 0$, see [40]. Since

$$H_\varepsilon(p) := \sqrt{\varepsilon^2 + p^2} - \varepsilon \leq |p| \quad \forall p \in \mathbb{R},$$

this φ_ε is also a subsolution of (22). Hence, by the comparison principle and (60),

$$\int \sup_{x+[-1,1]} \varphi_\varepsilon(\cdot, t) dx \leq \int \sup_{x+[-1,1]} \varphi_0 dx \quad \forall t \geq 0.$$

But from our choice of φ_0 , see (59), we can compute the right-hand side and find that

$$(62) \quad \int \sup_{x+[-1,1]} \varphi_\varepsilon(\cdot, t) dx \leq 2\varphi_0(0) + \int \varphi_0 \quad \forall t \geq 0.$$

Moreover, we claim that φ_ε satisfies the properties of φ_0 in (59) for any fixed time. It is indeed clear that it is even in x because $(x, t) \mapsto \varphi_\varepsilon(-x, t)$ is a solution of (61) which is equal to φ_ε by uniqueness. Checking that φ_ε is nonincreasing in x on $\mathbb{R}^+ \times \mathbb{R}^+$ is equivalent to showing that $\partial_x \varphi_\varepsilon$ is nonpositive there. By the assumptions on φ_0 , this is true for $t = 0$. Moreover, $\partial_x \varphi_\varepsilon$ is odd in x and therefore is zero at $x = 0$. Take then the parabolic PDE satisfied by $\partial_x \varphi_\varepsilon$ and apply the maximum principle in the half-space $\mathbb{R}^+ \times \mathbb{R}^+$. Since $\partial_x \varphi_\varepsilon$ is nonpositive on $\{t = 0, x > 0\} \cup \{x = 0\}$, it is nonpositive everywhere in the half-space. This completes the proof of the claim. Hence,

$$\int \sup_{x+[-1,1]} \varphi_\varepsilon(\cdot, t) dx = 2\varphi_\varepsilon(0, t) + \int \varphi_\varepsilon(x, t) dx$$

and then by (62),

$$(63) \quad \int \varphi_\varepsilon(x, t) dx \leq 2(\varphi_0(0) - \varphi_\varepsilon(0, t)) + \int \varphi_0.$$

Let us now integrate the equation for φ_ε . We obtain that

$$\begin{aligned} \int_{-R}^R \varphi_\varepsilon(x, t) dx &= \int_{-R}^R \varphi_0(x) dx \\ &+ \int_0^t \int_{-R}^R H_\varepsilon(\partial_x \varphi_\varepsilon(x, s)) dx ds + \int_0^t \int_{-R}^R \partial_{xx}^2 \varphi_\varepsilon(x, s) dx ds, \end{aligned}$$

for any $R \geq 0$ and $t \geq 0$. The last term equals

$$2 \int_0^t \partial_x \varphi_\varepsilon(R, s) ds.$$

We claim that it vanishes as $R \rightarrow \infty$. This can be justified by the dominated convergence theorem. To find a dominating function, we use that φ_ε is C_b^∞ . To see that $\partial_x \varphi_\varepsilon(R, s) \rightarrow 0$ as $R \rightarrow \infty$, for any fixed $s \geq 0$, we use that $\partial_x \varphi_\varepsilon$ is

uniformly continuous and integrable in x . It is indeed integrable since φ_ε satisfies the properties of φ_0 in (59) and thus

$$\int |\partial_x \varphi_\varepsilon| dx = -2 \int_0^\infty \partial_x \varphi_\varepsilon dx,$$

which is finite because φ_ε is bounded by the maximum principle for Problem (61).

Passing to the limit, we then find that

$$\int \varphi_\varepsilon(x, t) dx = \int \varphi_0 + \int_0^t \int H_\varepsilon(\partial_x \varphi_\varepsilon(x, s)) dx ds,$$

by the monotone convergence theorem. Using also (63), we deduce that

$$2(\varphi_0(0) - \varphi_\varepsilon(0, t)) + \int \varphi_0 \geq \int \varphi_0 + \int_0^t \int_{\mathbb{R}} H_\varepsilon(\partial_x \varphi_\varepsilon(x, s)) dx ds.$$

Dividing next by t and letting $t \rightarrow 0^+$, we conclude that

$$-2\partial_t \varphi_\varepsilon(0, 0) \geq \int H_\varepsilon(\partial_x \varphi_0).$$

Here, the passage to the limit is justified by the regularity of φ_ε and Fatou's lemma. But, the regularity up to $t = 0$ also implies that the equation of φ_ε is satisfied at the initial time too. Hence,

$$-2H_\varepsilon(\partial_x \varphi_0(0)) - 2\partial_{xx}^2 \varphi_0(0) \geq \int H_\varepsilon(\partial_x \varphi_0).$$

Using that $\partial_x \varphi_0(0) = 0$ and $H_\varepsilon(0) = 0$, once again because of (59), we have

$$\int H_\varepsilon(\partial_x \varphi_0) \leq -2\partial_{xx}^2 \varphi_0(0).$$

By a final passage to the limit as $\varepsilon \rightarrow 0^+$, we conclude that

$$\int |\partial_x \varphi_0| \leq -2\partial_{xx}^2 \varphi_0(0),$$

for any nonnegative $\varphi_0 \in C_b^\infty \cap L_{\text{int}}^\infty(\mathbb{R})$ satisfying (59). Now choosing such a nontrivial function such that $\partial_{xx}^2 \varphi_0(0) = 0$, we get the desired contradiction. This completes the proof of the failure of the contraction principle for (22).

Let us now consider Equation (23) and show that its semigroup is not quasicontractive. We use similar computations. Fix then an arbitrary $C_b^\infty \cap L_{\text{int}}^\infty$ initial data satisfying (59), as before. Since the equation is uniformly parabolic and convex, the solution is smooth up to $t = 0$, at least C^2 by [28, 47] which will be enough. Arguing by contradiction we assume that

$$\int \sup_{x+[-1,1]} \varphi(\cdot, t) dx \leq e^{\gamma t} \int \sup_{x+[-1,1]} \varphi_0 dx \quad \forall t \geq 0,$$

for some $\gamma \in \mathbb{R}$ independent of φ_0 . Arguing as before, we can use again (59) to rewrite this inequality in the form:

$$(64) \quad \int \varphi(x, t) dx \leq \underbrace{2(\varphi_0(0) - \varphi(0, t)) + \int \varphi_0}_{\text{as before}} + \underbrace{2(e^{\gamma t} - 1)\varphi_0(0) + (e^{\gamma t} - 1) \int \varphi_0}_{\text{additional terms}}.$$

Moreover, integrating Equation (23) gives

$$\begin{aligned} \int_{-R}^R \varphi(x, t) dx &= \int_{-R}^R \varphi_0(x) dx \\ &\quad + \int_0^t \int_{-R}^R (\partial_{xx}^2 \varphi(x, s))^+ dx ds + \int_0^t \int_{-R}^R \partial_{xx}^2 \varphi(x, s) dx ds, \end{aligned}$$

for any $R \geq 0$ and $t \geq 0$. The last term vanishes as $R \rightarrow \infty$, for the same reason as before, so letting $R \rightarrow \infty$ and injecting (64) leads to

$$\begin{aligned} 2(\varphi_0(0) - \varphi(0, t)) + 2(e^{\gamma t} - 1)\varphi_0(0) + (e^{\gamma t} - 1) \int \varphi_0 \\ \geq \int_0^t \int_{\mathbb{R}} (\partial_{xx}^2 \varphi(x, s))^+ dx ds. \end{aligned}$$

Dividing by t and letting $t \rightarrow 0^+$, we now get

$$-2\partial_t \varphi(0, 0) + 2\gamma \varphi_0(0) + \gamma \int \varphi_0 \geq \int (\partial_{xx}^2 \varphi_0)^+.$$

Using Equation (23) (which is valid up to $t = 0$) to compute the first term, we conclude that

$$(65) \quad \int (\partial_{xx}^2 \varphi_0)^+ \leq -2(\partial_{xx}^2 \varphi_0(0))^+ - 2\partial_{xx}^2 \varphi_0(0) + 2\gamma \varphi_0(0) + \gamma \int \varphi_0,$$

for any $0 \leq \varphi_0 \in C_b^\infty \cap L_{\text{int}}^\infty(\mathbb{R})$ satisfying (59) (and with γ independent of φ_0). To conclude, let us choose φ_0 as an approximation of $\mathbf{1}_{(-1,1)}$. Taking for instance the convolution $\varphi_0 := \rho_\nu * \mathbf{1}_{(-1,1)}$, with an even and radially nonincreasing mollifier satisfying (29), we obtain that

$$(\partial_{xx}^2 \varphi_0)^+(x) = -\partial_x \rho_\nu(|x| - 1)$$

for any $|x| \geq 1$ and ν small enough so that $\text{supp}(\rho_\nu) \subseteq [-1, 1]$. In particular,

$$\int (\partial_{xx}^2 \varphi_0)^+ \geq 2\rho_\nu(0) \rightarrow \infty \quad \text{as } \nu \rightarrow 0^+.$$

The left-hand side of (65) thus explodes where as the right-hand side remains bounded at this limit. This gives us the desired contradiction and completes the proof. \square

Now it only remains to prove Theorem 35. Recall that G_t^{mod} is the semigroup associated to the model HJB equation

$$(66) \quad \partial_t \varphi = |D\varphi| + \sup_{\lambda \in \text{Sp}(D^2 \varphi)} \lambda^+.$$

It is a semigroup of Lipschitz operators in $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ by the previous analysis. Recall also that

$$(67) \quad \|\varphi_0\| = \sup_{t \geq 0} e^{-t} \|G_t^{\text{mod}} \varphi_0\|_{\text{int}}.$$

To show that it is a norm, we need to know that the modulus from Theorem 31 does not increase faster than an exponential at infinity.

Lemma 73. *The modulus from Theorem 31 satisfies*

$$\omega_d(r) = O(r^{\frac{d}{2}}) \quad \text{as } r \rightarrow \infty.$$

Proof. We have established in (53) that

$$\omega_d(r) \leq \inf_{\nu > 0} (1 + \nu)^d \int_{\mathbb{R}^d} \Psi_\nu(y, r) \, dy - 1,$$

where, by (54) and (55), $\Psi_\nu(y, r) \leq c\nu^{-d}\Phi(y\nu^{-1}, r\nu^{-2})$ for a $c > 0$ independent of ν, y, r . Taking $\nu = 1$,

$$\omega_d(r) \leq 2^d c \int_{\mathbb{R}^d} \Phi(y, r) \, dy,$$

and the result easily follows from Estimate (50). \square

Let us now establish that:

Lemma 74. *The function $\|\cdot\|$ defined in (67) is a norm on $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ which is equivalent to the usual one $\|\cdot\|_{\text{int}}$.*

Proof. Assume first that it is a norm and let us prove the equivalence. Simple computations show that $|H|_{\text{conv}} = |H|_{\text{diff}} = 1$ for the Hamiltonian of (66).³ Applying Corollary 33 to that equation, we deduce that

$$\|G_t^{\text{mod}}|\varphi_0|\|_{\text{int}} \leq (1 + \omega_d(t)) (1 + t)^d \|\varphi_0\|_{\text{int}},$$

for any $\varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ and $t \geq 0$. Hence, by Lemma 73,

$$\|G_t^{\text{mod}}|\varphi_0|\|_{\text{int}} \leq Me^t \|\varphi_0\|_{\text{int}}$$

for some constant $M = M(d) \geq 1$. From that, the equivalence between the norms is clear.

Let us now prove that $\|\cdot\|$ is a norm on $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$. Using again the previous inequality, we deduce that the sup in (67) is always finite. Moreover, given any initial data φ_0 and ψ_0 in $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$, the sum of the solutions

$$\varphi(x, t) := G_t^{\text{mod}}|\varphi_0|(x) \quad \text{and} \quad \psi(x, t) := G_t^{\text{mod}}|\psi_0|(x)$$

is a supersolution of the same equation; indeed,

$$\partial_t(\varphi + \psi) = \sup\{\dots\} + \sup\{\dots\} \geq \sup\{\dots + \dots\}.$$

Hence, the comparison principle implies that, for any $t \geq 0$,

$$G_t^{\text{mod}}|\varphi_0 + \psi_0| \leq G_t^{\text{mod}}(|\varphi_0| + |\psi_0|) \leq G_t^{\text{mod}}|\varphi_0| + G_t^{\text{mod}}|\psi_0|.$$

It is thus clear that $\|\varphi_0 + \psi_0\| \leq \|\varphi_0\| + \|\psi_0\|$. To obtain that $\|\alpha\varphi_0\| = |\alpha|\|\varphi_0\|$, for any φ_0 and scalar $\alpha \in \mathbb{R}$, we use that

$$G_t^{\text{mod}}|\alpha\varphi_0| = |\alpha|G_t^{\text{mod}}|\varphi_0|.$$

This follows from the invariance of Equation (66) with respect to multiplication of $|\alpha| \geq 0$. Finally, if $\|\varphi_0\| = 0$, then choosing $t = 0$ in its definition implies that $\|\varphi_0\| \geq \|\varphi_0\|_{\text{int}} = 0$ and so $\varphi_0 \equiv 0$. The proof is complete. \square

At this stage, it only remains to obtain (24). Here is a first inequality.

Lemma 75. *Assume (H1). Then for any $\varphi_0, \psi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ and $t, s \geq 0$,*

$$G_s^{\text{mod}}|G_t\varphi_0 - G_t\psi_0| \leq G_s^{\text{mod}}G_t|\varphi_0 - \psi_0|,$$

where G_t and G_s^{mod} are the semigroups associated to (1) and (66).

Proof. By Lemma 70, the difference $|G_t\varphi_0 - G_t\psi_0|$ is a subsolution of (1a). By comparison, we obtain that

$$|G_t\varphi_0 - G_t\psi_0| \leq G_t|\varphi_0 - \psi_0|,$$

for any $t \geq 0$. We conclude by next using the comparison for Equation (66). \square

³Use that the Hamiltonian of (66) is $H(p, X) = \sup_{\mathcal{E}} \{q \cdot p + \text{tr}((\bar{q} \otimes \bar{q})X)\}$, where the sup is taken over the set $\mathcal{E} = \{(q, \bar{q}) \in \mathbb{R}^d \times \mathbb{R}^d \text{ s.t. } |q|, |\bar{q}| \leq 1\}$.

In the next lemma we use the translation operator \mathcal{T}_h defined by $\mathcal{T}_h\varphi_0(\cdot) := \varphi_0(\cdot - h)$ for $h \in \mathbb{R}^d$.

Lemma 76. *Assume (H1), $b_0 \in \mathbb{R}^d$, $a_0 \in \mathbb{S}_d$, $0 \leq a_0 \leq \text{Im}(a)$, and μ_t is the convolution semigroup corresponding to a_0 given by Proposition 59. For any nonnegative $\phi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$,*

$$\mathcal{T}_{tb_0} G_s^{\text{mod}} G_t \phi_0 \leq \mu_t * \{G_{s+\gamma t}^{\text{mod}} \phi_0\} \quad \forall t, s \geq 0,$$

where $\gamma = \|b - b_0\|_\infty \vee \|\text{tr}(a - a_0)\|_\infty$.

Proof. We will show that $\phi(x, t) := G_t \phi_0(x)$ is a subsolution of an equation related to (66). Starting from its equation (1a), we have

$$\begin{aligned} \partial_t \phi &= \sup_{\mathcal{E}} \{(b - b_0) \cdot D\phi + \text{tr}((a - a_0)D^2\phi)\} + b_0 \cdot D\phi + \text{tr}(a_0 D^2\phi) \\ &\leq \sup_{\mathcal{E}} \{(b - b_0) \cdot D\phi\} + \sup_{\mathcal{E}} \{\text{tr}((a - a_0)D^2\phi)\} + b_0 \cdot D\phi + \text{tr}(a_0 D^2\phi). \end{aligned}$$

For the first term, we have

$$\sup_{\mathcal{E}} \{(b - b_0) \cdot D\phi\} \leq \|b - b_0\|_\infty |D\phi|,$$

and for the second one,

$$\sup_{\mathcal{E}} \{\text{tr}((a - a_0)D^2\phi)\} \leq \|\text{tr}(a - a_0)\|_\infty \sup_{\lambda \in \text{Sp}(D^2\phi)} \lambda^+,$$

by the Ky Fan inequality (49). Hence

$$(68) \quad \partial_t \phi \leq \|b - b_0\|_\infty |D\phi| + \|\text{tr}(a - a_0)\|_\infty \sup_{\lambda \in \text{Sp}(D^2\phi)} \lambda^+ + b_0 \cdot D\phi + \text{tr}(a_0 D^2\phi).$$

We almost recognize (66). To get rid of the linear convection term, we use the change of variables:

$$\tilde{\phi}(x, t) := \phi(x - tb_0, t) = \mathcal{T}_{tb_0} \phi(x, t),$$

and we rescale to normalize the coefficients:

$$\tilde{\tilde{\phi}}(x, t) := \tilde{\phi}(x, \gamma^{-1}t) \quad \text{with} \quad \gamma := \|b - b_0\|_\infty \vee \|\text{tr}(a - a_0)\|_\infty.$$

Then $\tilde{\tilde{\phi}}$ satisfies the PDE

$$\partial_t \tilde{\tilde{\phi}} \leq |D\tilde{\tilde{\phi}}| + \sup_{\lambda \in \text{Sp}(D^2\tilde{\tilde{\phi}})} \lambda^+ + \gamma^{-1} \text{tr}(a_0 D^2\tilde{\tilde{\phi}}),$$

in the viscosity sense, i.e. (66) plus a linear diffusion term. We have a PDE of the form (37). By Lemma 60, we deduce that for any $t \geq 0$,

$$\tilde{\tilde{\phi}}(\cdot, t) \leq \mu_{\gamma^{-1}t} * \{G_t^{\text{mod}} \tilde{\tilde{\phi}}(\cdot, 0)\} = \mu_{\gamma^{-1}t} * \{G_t^{\text{mod}} \phi(\cdot, 0)\}$$

for the convolution semigroup μ_t associated to the diffusion matrix a_0 . Note that by scaling, the semigroup associated to $\gamma^{-1}a_0$ is $\mu_{\gamma^{-1}t}$. Since

$$\tilde{\tilde{\phi}}(\cdot, t) = \tilde{\phi}(\cdot, \gamma^{-1}t) = \mathcal{T}_{\gamma^{-1}tb_0} \phi(\cdot, \gamma^{-1}t) = \mathcal{T}_{\gamma^{-1}tb_0} G_{\gamma^{-1}t} \phi_0,$$

we conclude that

$$\mathcal{T}_{\gamma^{-1}tb_0} G_{\gamma^{-1}t} \phi_0 \leq \mu_{\gamma^{-1}t} * \{G_t^{\text{mod}} \phi_0\},$$

or equivalently

$$\mathcal{T}_{tb_0} G_t \phi_0 \leq \mu_t * \{G_{\gamma t}^{\text{mod}} \phi_0\}.$$

At this stage, we fix t and let only s move. If we apply the semigroup G_s^{mod} to the previous inequality, we get that

$$G_s^{\text{mod}} \mathcal{T}_{tb_0} G_t \phi_0 \leq G_s^{\text{mod}} \{\mu_t * G_{\gamma t}^{\text{mod}} \phi_0\}$$

for any $s \geq 0$. This is a consequence of the comparison principle for (66). Using also the invariance of (66) with respect to the fixed translation $-tb_0$, we can commute the operators G_s^{mod} and \mathcal{T}_{tb_0} ,

$$(69) \quad \mathcal{T}_{tb_0} G_s^{\text{mod}} G_t \phi_0 \leq G_s^{\text{mod}} \{ \mu_t * G_{\gamma t}^{\text{mod}} \phi_0 \}.$$

Moreover, by Lemma 55,

$$(x, s) \mapsto \mu_t * \{ G_s^{\text{mod}} G_{\gamma t}^{\text{mod}} \phi_0 \}(x)$$

is a supersolution of (66). At $s = 0$ we have

$$G_{t=0}^{\text{mod}} \{ \mu_t * G_{\gamma t}^{\text{mod}} \phi_0 \} = \mu_t * G_{\gamma t}^{\text{mod}} \phi_0 = \mu_t * \{ G_{t=0}^{\text{mod}} G_{\gamma t}^{\text{mod}} \phi_0 \},$$

so we can use again the comparison principle for (66) to deduce that

$$(70) \quad G_s^{\text{mod}} \{ \mu_t * G_{\gamma t}^{\text{mod}} \phi_0 \} \leq \underbrace{\mu_t * \{ G_s^{\text{mod}} G_{\gamma t}^{\text{mod}} \phi_0 \}}_{= G_{s+\gamma t}^{\text{mod}} \phi_0},$$

for any $s \geq 0$. By (69) and (70) we can conclude the proof when $\gamma = \|b - b_0\|_\infty \vee \|\text{tr}(a - a_0)\|_\infty > 0$ (we divided by it when we rescaled). If $\gamma = 0$, Equation (68) is linear and the same reasoning applies without rescaling the time. \square

We are now ready to prove the theorem.

Proof of Theorem 35. The norms are equivalent by Lemma 74, so it remains to show Estimate (24):

$$(71) \quad e^{-s} \|G_s^{\text{mod}} |G_t \varphi_0 - G_t \psi_0|\|_{\text{int}} \leq e^{(|H|_{\text{conv}} \vee |H|_{\text{diff}})t} \sup_{\tau \geq 0} e^{-\tau} \|G_\tau^{\text{mod}} |\varphi_0 - \psi_0|\|_{\text{int}},$$

for arbitrary $\varphi_0, \psi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ and times t and s . By Lemma 75 with $\phi_0 := |\varphi_0 - \psi_0|$, it suffices to show that

$$(72) \quad e^{-s} \|G_s^{\text{mod}} G_t \phi_0\|_{\text{int}} \leq e^{(|H|_{\text{conv}} \vee |H|_{\text{diff}})t} \sup_{\tau \geq 0} e^{-\tau} \|G_\tau^{\text{mod}} \phi_0\|_{\text{int}},$$

for any t and s . Indeed that lemma shows that

$$G_s^{\text{mod}} |G_t \varphi_0 - G_t \psi_0| \leq G_s^{\text{mod}} G_t \phi_0,$$

where the left-hand side is nonnegative by the comparison principle for (66). Hence

$$\|G_s^{\text{mod}} |G_t \varphi_0 - G_t \psi_0|\|_{\text{int}} \leq \|G_s^{\text{mod}} G_t \phi_0\|_{\text{int}}$$

and (72) will imply (71). To show (72) we use Lemma 76. It implies that for any $b_0 \in \mathbb{R}^d$ and $a_0 \in \mathbb{S}_d^+$ such that $a_0 \leq \text{Im}(a)$,

$$\mathcal{T}_{tb_0} G_s^{\text{mod}} G_t \phi_0 \leq \mu_t * \{ G_{s+\gamma t}^{\text{mod}} \phi_0 \},$$

for some measure $\mu_t \geq 0$, $\mu_t(\mathbb{R}^d) \leq 1$, and

$$\gamma = \|b - b_0\|_\infty \vee \|\text{tr}(a - a_0)\|_\infty.$$

Since these functions are nonnegative by the comparison principle, Young's inequality implies that

$$\|\mathcal{T}_{tb_0} G_s^{\text{mod}} G_t \phi_0\|_{\text{int}} \leq \|G_{s+\gamma t}^{\text{mod}} \phi_0\|_{\text{int}},$$

for any t and s . Changing variables as before, we see that

$$\int \sup_{y \in [-1, 1]^d} G_s^{\text{mod}} G_t \phi_0(x + y - tb_0) dx = \int \sup_{y \in [-1, 1]^d} G_s^{\text{mod}} G_t \phi_0(x - y) dx,$$

which means that

$$\|G_s^{\text{mod}} G_t \phi_0\|_{\text{int}} = \|\mathcal{T}_{tb_0} G_s^{\text{mod}} G_t \phi_0\|_{\text{int}} \leq \|G_{s+\gamma t}^{\text{mod}} \phi_0\|_{\text{int}},$$

and multiplying by e^{-s} it easily follows that

$$e^{-s} \|G_s^{\text{mod}} G_t \phi_0\|_{\text{int}} \leq e^{(\|b - b_0\|_\infty \vee \|\text{tr}(a - a_0)\|_\infty)t} \sup_{\tau \geq 0} e^{-\tau} \|G_\tau^{\text{mod}} \phi_0\|_{\text{int}},$$

for any t, s, b_0 , and a_0 . Taking the infimum in b_0 and a_0 implies (72) and thus the desired estimate (24). \square

The proofs of the results of Section 3.2 are complete.

4.7. Weighted L^1 contraction: Proof of Theorem 38. We recall that we will use the “dual” problem (25), which can be written with a pointwise supremum taken over the Lebesgue points of F' and A . It is therefore of the form (1) and the viscosity solution theory applies.

Proof of Theorem 38. Recall that $m < M$, u and v are the solutions of (2) with initial data u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$, $0 \leq \varphi_0 \in BLSC(\mathbb{R}^d)$, and $\underline{\varphi}$ is the minimal solution of (25) with φ_0 as initial data.

We have to show that

$$(73) \quad \int_{\mathbb{R}^d} |u - v|(x, T) \varphi_0(x) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \underline{\varphi}(x, T) \, dx \quad \forall T \geq 0.$$

Let us use the Kato inequality (12). For almost every $x \in \mathbb{R}^d$ and $t \geq 0$,

$$\begin{aligned} & \left\{ \sum_{i=1}^d q_i(u, v) \partial_{x_i} \phi + \sum_{i,j=1}^d r_{ij}(u, v) \partial_{x_i x_j}^2 \phi \right\} (x, t) \\ &= \{q(u, v) \cdot D\phi + \text{tr}(r(u, v) D^2 \phi)\} (x, t) \\ &= \text{sign}(u(x, t) - v(x, t)) \int_{v(x, t)}^{u(x, t)} \{F'(\xi) \cdot D\phi(x, t) + \text{tr}(A(\xi) D^2 \phi(x, t))\} \, d\xi \\ &\leq |u(x, t) - v(x, t)| \text{ess sup}_{m \leq \xi \leq M} \{F'(\xi) \cdot D\phi(x, t) + \text{tr}(A(\xi) D^2 \phi(x, t))\}, \end{aligned}$$

where we have taken the sup over $[m, M]$ because of the maximum principle Lemma 95. Injecting into (12), we get that

$$(74) \quad \int_{\mathbb{R}^d} |u - v|(x, T) \phi(x, T) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \phi(x, 0) \, dx \\ + \iint_{\mathbb{R}^d \times (0, T)} |u - v| \left(\partial_t \phi + \text{ess sup}_{m \leq \xi \leq M} \{F'(\xi) \cdot D\phi + \text{tr}(A(\xi) D^2 \phi)\} \right) \, dx \, dt.$$

In the third integral, we recognize the backward in time version of (25a). The proof of (73) then consists in taking $\phi(x, t) := \underline{\varphi}(x, T - t)$.

Simplified case: $0 \leq \varphi_0 \in C_c(\mathbb{R}^d)$.

Now (25) has a unique viscosity solution φ which coincides with $\underline{\varphi}$. It belongs to $BUC(\mathbb{R}^d \times \mathbb{R}^+)$ by Lemma 61, and it is continuous in time with values in $L^1(\mathbb{R}^d)$ by Lemma 86. Let us regularize it by convolution

$$\varphi_\nu := \varphi *_{x,t} (\rho_\nu \theta_\nu),$$

with the mollifiers (29) and (30). It follows that

$$\varphi_\nu \in C^\infty(\mathbb{R}^d \times \mathbb{R}^+) \cap C(\mathbb{R}^+; L^1(\mathbb{R}^d))$$

along with all its derivatives. This is enough to take $\phi_\nu(x, t) := \varphi_\nu(x, T - t)$ as a test function in (74) by approximation. Note that φ_ν is a supersolution of the backward version of (25a) by Lemma 54, i.e.

$$\partial_t \phi_\nu + \text{ess sup}_{m \leq \xi \leq M} \{F'(\xi) \cdot D\phi_\nu + \text{tr}(A(\xi) D^2 \phi_\nu)\} \leq 0 \quad \text{for any } t < T.$$

Inequality (74) with the test function ϕ_ν then implies that

$$\int_{\mathbb{R}^d} |u - v|(x, T) \varphi_\nu(x, 0) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi_\nu(x, T) \, dx,$$

for any $T \geq 0$ and $\nu > 0$. By the $C(\mathbb{R}^+; L^1(\mathbb{R}^d))$ regularity of φ , the convolution $\varphi_\nu = \varphi *_{x,t} (\rho_\nu \theta_\nu)$ converges to φ in $C([0, T]; L^1(\mathbb{R}^d))$ as $\nu \rightarrow 0^+$. Passing to the limit as $\nu \rightarrow 0^+$ then yields (73).

General case: $0 \leq \varphi_0 \in BLSC(\mathbb{R}^d)$.

We would like to pointwise approximate φ_0 by a monotone sequence $\varphi_0^n \uparrow \varphi_0$ such that $0 \leq \varphi_0^n \in C_c(\mathbb{R}^d)$. Take

$$\varphi_0^n(x) := \inf_{y \in \mathbb{R}^d} \left\{ \varphi_0(y) \mathbf{1}_{|y| < n} + n^2 \frac{|x - y|^2}{2} \right\} \geq 0.$$

Then φ_0^n is continuous as an infconvolution. Also,

$$\varphi_0^n(x) \leq \varphi_0(x) \mathbf{1}_{|x| < n} \quad \forall x \in \mathbb{R}^d,$$

which implies that $\varphi_0^n \in C_c(\mathbb{R}^d)$. In the limit $n \rightarrow \infty$, we have $\varphi_0^n \uparrow (\varphi_0)_* = \varphi_0$. Let φ_n be the solution of (25) with initial data φ_0^n , then by the previous step,

$$\int_{\mathbb{R}^d} |u - v|(x, T) \varphi_0^n(x) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi_n(x, T) \, dx,$$

for any $T \geq 0$ and n . By the stability of extremal solutions (see Proposition 52), these solutions satisfy $\varphi_n \uparrow \underline{\varphi}$ pointwise. So we conclude the proof of (73) by passing to the limit as $n \rightarrow \infty$ using the monotone convergence theorem. \square

4.8. Duality: Proofs of Theorem 44 and Corollaries 46 and 47. We need some auxiliary lemmas. Here is a first classical result on entropy solutions.

Lemma 77. *Assume (H2) and $u_0 \in L^\infty(\mathbb{R}^d)$. Then, the entropy solution of (2) is a distributional solution of (2),*

$$\begin{aligned} \iint_{\mathbb{R}^d \times \mathbb{R}^+} \left(u \partial_t \phi + \sum_{i=1}^d F_i(u) \partial_{x_i} \phi + \sum_{i,j=1}^d \mathcal{A}_{ij}(u) \partial_{x_i x_j}^2 \phi \right) dx \, dt \\ + \int_{\mathbb{R}^d} u_0(x) \phi(x, 0) \, dx = 0 \quad \forall \phi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+), \end{aligned}$$

where $\mathcal{A}_{ij}(u) := \int_0^u A_{ij}(\xi) \, d\xi$.

Proof. Take $\eta(u) = \pm u$ successively in the entropy inequalities, Definition 11(c). \square

Here is another result on the continuity in time.

Lemma 78. *Assume (H2), $u_0, v_0 \in L^\infty(\mathbb{R}^d)$ with $u_0 - v_0 \in L^1(\mathbb{R}^d)$, u and v entropy solutions of (2) with initial data u_0 and v_0 . Then $u - v \in C(\mathbb{R}^+; L^1(\mathbb{R}^d))$.*

To prove it, we need the following result on viscosity solutions.

Proposition 79 (Limiting initial data). *Assume (H1) and $(\varphi_\varepsilon)_{\varepsilon > 0}$ is a uniformly locally bounded family of viscosity subsolutions (resp. supersolutions) of (1a). Then $\limsup^* \varphi_\varepsilon$ (resp. $\liminf_* \varphi_\varepsilon$) satisfies*

$$\{\limsup^* \varphi_\varepsilon\}(x, 0) = \{\limsup^*(\varphi_\varepsilon)^*(\cdot, 0)\}(x) \quad \forall x \in \mathbb{R}^d$$

(resp. $\{\liminf_* \varphi_\varepsilon\}(x, 0) = \{\liminf_*(\varphi_\varepsilon)_*(\cdot, 0)\}(x)$).

Remark 80. For subsolutions this means that

$$\limsup_{\substack{\mathbb{R}^d \times \mathbb{R}^+ \ni (y,s) \rightarrow (x,0) \\ \varepsilon \rightarrow 0^+}} \varphi_\varepsilon(y,s) = \limsup_{\substack{\mathbb{R}^d \ni y \rightarrow x \\ \varepsilon \rightarrow 0^+}} (\varphi_\varepsilon)^*(y,0),$$

where $(\varphi_\varepsilon)^*$ is the upper semicontinuous envelope computed in (x,t) . The proof can be found in [9] and [5, Theorem 4.7].⁴

Proof of Lemma 78. By Theorem 38 with $\varphi_0 \equiv 1$, we have

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1} \quad \forall t \geq 0.$$

Since the left-hand side is finite, $u - v \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^d))$. By the continuity in time with values in $L^1_{\text{loc}}(\mathbb{R}^d)$ of these functions, it remains to prove that

$$(75) \quad \lim_{R \rightarrow \infty} \sup_{t \in [0, T]} \int_{|x| \geq R} |u(x, t) - v(x, t)| dx = 0 \quad \forall T \geq 0.$$

To do so, we will use again Theorem 38.

Fix $m < M$ such that u_0 and v_0 take their values in $[m, M]$, and consider

$$\varphi_0^R(x) := \varphi_0\left(\frac{x}{R}\right), \quad R > 0,$$

where $\varphi_0 = \varphi_0(x)$ is some kernel such that

$$\begin{cases} 0 \leq \varphi_0 \in C_b(\mathbb{R}^d), \\ \varphi_0(x) = 0 \text{ for } |x| \leq 1/2, \\ \text{and } \varphi_0(x) = 1 \text{ for } |x| \geq 1. \end{cases}$$

With that choice, $\varphi_0^R \rightarrow 0$ as $R \rightarrow \infty$ locally uniformly. We then claim that the solutions φ_R of (25) with initial data φ_0^R converge locally uniformly to zero too. This is a consequence of the method of relaxed semilimits [9]. Let us give details for completeness. By the maximum principle,

$$\|\varphi_R\|_\infty \leq \|\varphi_0^R\|_\infty = \|\varphi_0\|_\infty \quad \forall R > 0.$$

We can then apply Propositions 50 and 79 to $\limsup^* \varphi_R$ as $R \rightarrow \infty$ and get that it is a subsolution of (25a) satisfying

$$\limsup^* \varphi_R(x, 0) = \limsup^* \varphi_0^R(x) = 0 \quad \forall x \in \mathbb{R}^d.$$

Similarly, $\liminf_* \varphi_R$ is a supersolution of (1) with zero as initial data. The comparison principle then implies that

$$\limsup^* \varphi_R \leq \liminf_* \varphi_R.$$

Hence φ_R converges locally uniformly to the unique solution of (25) with zero initial data, that is zero itself.

Now we can show (75). By Theorem 38 with the previous m, M , and φ_0^R ,

$$\begin{aligned} \int_{|x| \geq R} |u(x, t) - v(x, t)| dx &\leq \int_{\mathbb{R}^d} |u(x, t) - v(x, t)| \varphi_0^R(x) dx \\ &\leq \int_{\mathbb{R}^d} |u_0(x) - v_0(x)| \varphi_R(x, t) dx \leq \int_{\mathbb{R}^d} |u_0(x) - v_0(x)| \sup_{s \in [0, T]} \varphi_R(x, s) dx, \end{aligned}$$

for any $T \geq t \geq 0$. The right-hand side vanishes as $R \rightarrow \infty$ by the discussion above. and the dominated convergence theorem. The proof of (75) is complete. \square

⁴Let us briefly recall the ideas for the reader's convenience. First consider $\varphi = \limsup^* \varphi_\varepsilon$, $\varphi_0(x) = \{\limsup^*(\varphi_\varepsilon)^*(\cdot, 0)\}(x)$, and show that $\min\{\partial_t \varphi - H(D\varphi, D^2\varphi), \varphi - \varphi_0\} \leq 0$ in the viscosity sense, fix then some x and use the viscosity inequalities at a max (\bar{y}, \bar{t}) of the function $\varphi(y, t) - |y - x|^2/\varepsilon - Ct$ with C large enough such that $\bar{t} = 0$. We get $\varphi(x, 0) \leq \varphi_0(\bar{y})$ and conclude as $C \rightarrow \infty$ and $\varepsilon \rightarrow 0^+$.

Let us continue by introducing a regularization procedure for the weights.

Lemma 81. *Assume (H2), $m < M$, ρ_ν and θ_ν are defined in (29) and (30), and $0 \leq \varphi \in BLSC(\mathbb{R}^d \times \mathbb{R}^+)$ satisfies (I) in Theorem 44. Then for any $\nu > 0$, the convolution*

$$\varphi_\nu := \varphi *_{x,t} (\rho_\nu \theta_\nu) \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^+)$$

also satisfies (I) in Theorem 44.

Proof. By assumption,

$$(76) \quad \int_{\mathbb{R}^d} |u - v|(x, t) \varphi(x, s) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x, t + s) \, dx,$$

for any $t, s \geq 0$, u_0 and v_0 with values in $[m, M]$, and entropy solutions u and v of (2) with u_0 and v_0 as initial data. Our aim is to get the same result for φ_ν . Let us use (76) not for u_0 and v_0 , but their translations $u_0(\cdot + y)$ and $v_0(\cdot + y)$ for some fixed $y \in \mathbb{R}^d$. Since the PDE part of (2) is invariant with respect to translation, the corresponding solutions are $u(x + y, t)$ and $v(x + y, t)$. Hence,

$$\int_{\mathbb{R}^d} |u - v|(x + y, t) \varphi(x, s) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x + y) \varphi(x, t + s) \, dx$$

for any $t, s \geq 0$. By changing the variable of integration, we obtain that

$$\int_{\mathbb{R}^d} |u - v|(x, t) \varphi(x - y, s) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x - y, t + s) \, dx.$$

Now we fix $\tau \leq 0$ and apply this formula, not for s but $s - \tau$. We deduce that

$$\int_{\mathbb{R}^d} |u - v|(x, t) \varphi(x - y, s - \tau) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x - y, t + s - \tau) \, dx.$$

Multiply then by $\rho_\nu(y) \theta_\nu(\tau)$ and integrate over $(y, \tau) \in \mathbb{R}^d \times \mathbb{R}^-$ to conclude. \square

We want to pass to the limit and compare the functions

$$\varphi_b := \liminf_* \varphi_\nu \quad \text{as } \nu \rightarrow 0^+$$

and

$$(77) \quad \varphi_\#(x, t) = \liminf_{\substack{r \rightarrow 0^+ \\ y \rightarrow x}} \frac{1}{\text{meas}(B_r(y))} \int_{B_r(y)} \varphi(z, t) \, dz.$$

To do so, we assume in addition that

$$(78) \quad \text{supp}(\rho_\nu) \subset B_\nu(0) \quad \text{and} \quad \text{supp}(\theta_\nu) \subset (-\nu, 0).$$

Let us first give fundamental properties on φ_b and $\varphi_\#$.

Lemma 82. *Under the assumptions of the previous lemma and (78), we have:*

- (i) *The limit φ_b is the pointwise largest function in $BLSC(\mathbb{R}^d \times \mathbb{R}^+)$ less than or equal φ a.e. Moreover $\varphi_b = \varphi$ a.e.*
- (ii) *For any $t \geq 0$, $\varphi_\#(\cdot, t)$ is the pointwise largest function in $BLSC(\mathbb{R}^d)$ less than or equal $\varphi(\cdot, t)$ a.e. Moreover $\varphi_\# = \varphi(\cdot, t)$ a.e.*

Remark 83. It is understood that ‘‘a.e.’’ holds in (x, t) in (i) and x in (ii).

Proof. Let us prove (i). Note first that φ_b is lower semicontinuous as a lower relaxed limit. To prove that it is not greater than φ a.e., it suffices to do it for the Lebesgue points of φ . Such points $(x, t) \in \mathbb{R}^d \times (0, \infty)$ satisfy

$$\lim_{\nu \rightarrow 0^+} \frac{1}{\nu^{d+1}} \iint_{B_\nu(x) \times (t-\nu, t+\nu)} |\varphi(y, s) - \varphi(x, t)| \, dy \, ds = 0,$$

so by the assumptions on the mollifiers, see (29), (30) and (78), we find that

$$|\varphi_\nu(x, t) - \varphi(x, t)| \leq \frac{1}{\nu^{d+1}} \iint_{B_\nu(x) \times (t, t+\nu)} |\varphi(y, s) - \varphi(x, t)| \cdot \rho\left(\frac{x-y}{\nu}\right) \theta\left(\frac{t-s}{\nu}\right) dy ds \rightarrow 0 \quad \text{as } \nu \rightarrow 0^+.$$

It follows that

$$\varphi_b(x, t) \leq \lim_{\nu \rightarrow 0^+} \varphi_\nu(x, t) = \varphi(x, t),$$

at any Lebesgue point. Moreover, for any fixed (x, t) , lower semicontinuity of φ implies that

$$\varphi_\nu(y, s) = \iint_{B_\nu(y) \times (s, s+\nu)} \underbrace{\varphi(z, \tau)}_{\geq \varphi(x, t) + o(1)} \rho\left(\frac{y-z}{\nu}\right) \theta\left(\frac{s-\tau}{\nu}\right) dz d\tau \geq \varphi(x, t) + o(1)$$

as $(y, s, \nu) \rightarrow (x, t, 0^+)$, and we conclude that

$$\varphi_b(x, t) = \liminf_* \varphi_\nu(x, t) \geq \varphi(x, t).$$

We conclude that $\varphi_b = \varphi$ a.e.

Now, to conclude the proof of (i), it remains to prove that $\varphi_b \geq \psi$ pointwise for any other $\psi \in BLSC(\mathbb{R}^d \times \mathbb{R}^+)$ such that $\psi \leq \varphi$ a.e. Given such a function, let

$$\psi_b := \liminf_* \psi *_{x,t} (\rho_\nu \theta_\nu).$$

As above, $\psi \leq \psi_b$ pointwise; but also $\psi_b \leq \varphi_b$ pointwise since

$$\psi *_{x,t} (\rho_\nu \theta_\nu) \leq \varphi *_{x,t} (\rho_\nu \theta_\nu).$$

This completes the proof of (i). The arguments for (ii) are similar. \square

Here is now a general inequality between φ_b and $\varphi_\#$.

Lemma 84. *Under the hypotheses of the previous lemma, $(\varphi_\#)_* \leq \varphi_b$ pointwise.*

Proof. Let us first prove that $\varphi_\#$ is measurable in (x, t) . We have

$$\varphi_\#(x, t) = \sup_{n \geq 1} \underbrace{\inf_{m \geq n} \inf_{\substack{\frac{1}{m} \leq r \leq \frac{1}{n} \\ |y| \leq \frac{1}{n}}} \frac{1}{\text{meas}(B_r(y))} \int_{B_r(y)} \varphi(x+z, t) dz}_{=: \varphi_{n,m}(x,t)} \quad \overbrace{\hspace{10em}}^{=: \varphi_n(x,t)},$$

where n and m are integers. For each $\frac{1}{m} \leq r \leq \frac{1}{n}$ and $|y| \leq \frac{1}{n}$, the function

$$(x, t) \mapsto \frac{1}{\text{meas}(B_r(y))} \int_{B_r(y)} \varphi(x+z, t) dz$$

is lower semicontinuous by Fatou's lemma and $\varphi \in BLSC$ (assumption in the previous lemmas). The infimum $\varphi_{n,m}$ remains lower semicontinuous, because r and y live in compact sets. Hence, $\varphi_n = \inf_{m \geq n} \varphi_{n,m}$ is measurable in (x, t) and so is $\varphi_\# = \sup_{n \geq 1} \varphi_n$.

We can now prove the lemma. For any $t \geq 0$, the measurable functions $\varphi, \varphi_\#$ satisfy $\varphi_\#(\cdot, t) = \varphi(\cdot, t)$ a.e., hence we may use the Fubini theorem to conclude that

$$\iint_{\mathbb{R}^d \times \mathbb{R}^+} \mathbf{1}_{\{\varphi_\# = \varphi\}} dx dt = \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^d} \mathbf{1}_{\{\varphi_\#(x,t) = \varphi(x,t)\}} dx \right) dt = 0.$$

This proves that $\varphi_\# = \varphi$ a.e. in (x, t) , so that $(\varphi_\#)_* \leq \varphi$ a.e. in (x, t) . Hence $(\varphi_\#)_* \leq \varphi_b$ pointwise by Lemma 82(i). \square

Here are further properties that we will need.

Lemma 85. *Let $\varphi, \psi \in BLSC(\mathbb{R}^d \times \mathbb{R}^+)$ and $\varphi_{\#}, \psi_{\#}$ as in (77). Then*

- (i) $\varphi \leq (\varphi_{\#})_*$ pointwise, and
- (ii) if $\varphi \leq \psi_{\#}$ pointwise, then $\varphi_{\#} \leq \psi_{\#}$ pointwise.

Proof. We can show that $\varphi \leq \varphi_{\#}$ from the definition of $\varphi_{\#}$ and the lower semi-continuity of φ , exactly as we showed that $\varphi \leq \varphi_b$ in the proof of Lemma 82. In particular, $\varphi \leq (\varphi_{\#})_*$ which is part (i). For part (ii), use Lemma 82(ii). It says that $\psi_{\#}(\cdot, t) = \psi(\cdot, t)$ a.e. in x , for each fixed $t \geq 0$. Hence, $\varphi(\cdot, t) \leq \psi(\cdot, t)$ a.e. and the desired inequality follows again from the definitions of $\varphi_{\#}$ and $\psi_{\#}$. \square

Proof of Theorem 44. Let us proceed in several steps.

1) (II) \implies (I).

By (II), $(\varphi_{\#})_*$ is a *BLSC* supersolution of (25a). In particular, for any fixed $s \geq 0$, the function

$$(x, t) \mapsto (\varphi_{\#})_*(x, t + s)$$

is also a supersolution of (25a). By Remark 39(c), we can apply Theorem 38 to this supersolution with the *BLSC* initial weight $(\varphi_{\#})_*(\cdot, s)$. The result is that

$$\int_{\mathbb{R}^d} |u - v|(x, t)(\varphi_{\#})_*(x, s) dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x)(\varphi_{\#})_*(x, t + s) dx,$$

for any $u_0 = u_0(x)$ and $v_0 = v_0(x)$ with values in $[m, M]$, u and v entropy solutions of (2) with u_0 and v_0 as initial data, and $t, s \geq 0$. This is exactly (I) but with $(\varphi_{\#})_*$ instead of φ . To replace $(\varphi_{\#})_*$ by φ , we use Lemma 85(i) for the left-hand side. For the right-hand side, we use that $(\varphi_{\#})_* \leq \varphi_{\#}$ pointwise and the fact that $\varphi_{\#}(x, t + s) = \varphi(x, t + s)$ for almost each x , see Lemma 82(ii). This implies (I) with φ , as desired.

2) (I) \implies (II) for smooth weights φ .

Let us prove the reverse implication when $0 \leq \varphi \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^+)$. We will appropriately choose u_0 and v_0 later. For the moment, we assume that

$$m \leq v_0 \leq u_0 \leq M \quad \text{and} \quad u_0 - v_0 \in L^1(\mathbb{R}^d).$$

By Lemmas 95 and 78, $0 \leq u - v \in C(\mathbb{R}^+; L^1(\mathbb{R}^d))$, and then we can use (I) to get

$$(79) \quad \int_{\mathbb{R}^d} (u - v)(x, T)\varphi(x, s) dx \leq \int_{\mathbb{R}^d} (u_0 - v_0)(x)\varphi(x, T + s) dx,$$

for any $T, s \geq 0$. Let us fix $s > 0$ and determine what PDE φ satisfies. This will be done by injecting the weak formulation of (2) into (79) and then pass to the limit as $T \rightarrow 0^+$. By Lemma 77,

$$\begin{aligned} \int_{\mathbb{R}^d} (u - v)(x, T)\phi(x, T) dx &= \iint_{\mathbb{R}^d \times (0, T)} \left((u - v)\partial_t \phi + \sum_{i=1}^d (F_i(u) - F_i(v))\partial_{x_i} \phi \right. \\ &\quad \left. + \sum_{i,j=1}^d (\mathcal{A}_{ij}(u) - \mathcal{A}_{ij}(v))\partial_{x_i x_j}^2 \phi \right) dx dt \\ &\quad + \int_{\mathbb{R}^d} (u_0 - v_0)(x)\phi(x, 0) dx, \end{aligned}$$

for any $\phi \in C_c^\infty(\mathbb{R}^d \times [0, T])$ and $\mathcal{A}'_{ij} = \mathcal{A}_{ij}$. Note that we have rewritten the equation given by Lemma 77 with integrals in $t < T$ and an additional final term at $t = T$. This follows from standard arguments using the L^1_{loc} continuity in time of u and v . Since $\varphi \in C_b^\infty$, $u - v \in C_t(L^1_x)$ and $u, v \in L^\infty$, a standard approximation argument shows that we can take ϕ to be

$$\phi(x, t) = \varphi(x, t + s - T),$$

and get that

$$\begin{aligned}
(80) \quad \int_{\mathbb{R}^d} (u - v)(x, T) \varphi(x, s) \, dx &= \iint_{\mathbb{R}^d \times (0, T)} \left((u - v) \partial_t \varphi(t + s - T) \right. \\
&\quad + \sum_{i=1}^d (F_i(u) - F_i(v)) \partial_{x_i} \varphi(t + s - T) \\
&\quad + \sum_{i,j=1}^d (\mathcal{A}_{ij}(u) - \mathcal{A}_{ij}(v)) \partial_{x_i x_j}^2 \varphi(t + s - T) \left. \right) \, dx \, dt \\
&\quad + \int_{\mathbb{R}^d} (u_0 - v_0)(x) \varphi(x, s - T) \, dx.
\end{aligned}$$

Here we assume that $s > 0$ and T is so small that $s - T > 0$. Inserting (80) into (79), we get

$$\begin{aligned}
\int_{\mathbb{R}^d} (u_0 - v_0)(x) \varphi(x, s + T) \, dx - \int_{\mathbb{R}^d} (u_0 - v_0)(x) \varphi(x, s - T) \, dx \\
\geq \iint_{\mathbb{R}^d \times (0, T)} (\dots) \, dx \, dt.
\end{aligned}$$

We now would like to divide by $2T$ and pass to the limit as $T \rightarrow 0^+$. All the computations are justified, again because $\varphi \in C_b^\infty$, the solutions u and v are bounded, and $u - v \in C(\mathbb{R}^+; L^1(\mathbb{R}^d))$. We get that

$$\begin{aligned}
&\int_{\mathbb{R}^d} (u_0(x) - v_0(x)) \partial_s \varphi(x, s) \, dx \\
&\geq \frac{1}{2} \int_{\mathbb{R}^d} \left((u_0 - v_0) \partial_s \varphi(s) + \sum_{i=1}^d (F_i(u_0) - F_i(v_0)) \partial_{x_i} \varphi(s) \right. \\
&\quad \left. + \sum_{i,j=1}^d (\mathcal{A}_{ij}(u_0) - \mathcal{A}_{ij}(v_0)) \partial_{x_i x_j}^2 \varphi(s) \right) \, dx.
\end{aligned}$$

Subtracting the term $\int (u_0 - v_0) \partial_s \varphi(s) \, dx / 2$ of the right-hand side implies that

$$\begin{aligned}
(81) \quad &\int_{\mathbb{R}^d} (u_0(x) - v_0(x)) \partial_s \varphi(x, s) \, dx \\
&\geq \int_{\mathbb{R}^d} \left(\sum_{i=1}^d (F_i(u_0) - F_i(v_0)) \partial_{x_i} \varphi(s) + \sum_{i,j=1}^d (\mathcal{A}_{ij}(u_0) - \mathcal{A}_{ij}(v_0)) \partial_{x_i x_j}^2 \varphi(s) \right) \, dx \\
&= \int_{\mathbb{R}^d} \int_{v_0(x)}^{u_0(x)} \{ F'(\xi) \cdot D\varphi(x, s) + \text{tr}(\mathcal{A}_{ij}(\xi) D^2 \varphi(x, s)) \} \, d\xi \, dx,
\end{aligned}$$

for any $s > 0$ and $0 \leq u_0 - v_0 \in L^1(\mathbb{R}^d)$ such that both u_0 and v_0 take their values in the interval $[m, M]$. It remains to choose $u_0 - v_0$ as an approximate unit, up to some multiplicative constant.

Let us introduce new parameters: $x_0 \in \mathbb{R}^d$, $\varepsilon > 0$ and $m \leq a < b \leq M$. We would like to choose

$$u_0 - v_0 = (b - a) \mathbf{1}_{x_0 + (-\varepsilon, \varepsilon)^d},$$

with the constraint that both u_0 and v_0 only take the two values a and b . Writing $x = (x_i)$, take for instance

$$u_0(x) := \begin{cases} a & \text{if } x_1 > (x_0)_1 + \varepsilon, \\ b & \text{if not,} \end{cases}$$

and

$$v_0(x) := \begin{cases} a & \text{if } x_1 > (x_0)_1 + \varepsilon \text{ or } x \in x_0 + (-\varepsilon, \varepsilon)^d, \\ b & \text{if not.} \end{cases}$$

Then $m \leq v_0 \leq u_0 \leq M$ and $u_0 - v_0 \in L^1(\mathbb{R}^d)$ as required. Inserting our choice into (81) and dividing by $(b - a)\varepsilon^d$, we deduce that

$$\begin{aligned} & \frac{1}{\varepsilon^d} \int_{x_0 + (-\varepsilon, \varepsilon)^d} \partial_s \varphi(x, s) \, dx \\ & \geq \frac{1}{\varepsilon^d} \int_{x_0 + (-\varepsilon, \varepsilon)^d} \frac{1}{b - a} \int_a^b \{F'(\xi) \cdot D\varphi(x, s) + \text{tr}(A_{ij}(\xi) D^2 \varphi(x, s))\} \, d\xi \, dx. \end{aligned}$$

Let now $\xi \in (m, M)$ be any Lebesgue point of any arbitrarily chosen a.e. representative of (F', A) . Take first the limit as $a, b \rightarrow \xi$ such that ξ is the center of each $[a, b]$ in order to use the Lebesgue point property; take next the limit as $\varepsilon \rightarrow 0^+$. This gives us that

$$\partial_s \varphi(x_0, s) \geq F'(\xi) \cdot D\varphi(x_0, s) + \text{tr}(A_{ij}(\xi) D^2 \varphi(x_0, s)),$$

for any $x_0 \in \mathbb{R}^d$, $s > 0$, and Lebesgue point ξ . That is φ is a supersolution of (25). This completes the proof of the remaining implication in the case where φ is C_b^∞ (and then $\varphi_\# = \varphi$).

3) (I) \implies (II) for nonnegative BLSC weights φ .

In this case we use the regularization procedure of Lemma 81. By this lemma

$$\varphi_\nu = \varphi *_{x,t} (\rho_\nu \theta_\nu)$$

satisfies (I) since φ does by assumption. By the previous step we deduce that φ_ν is a supersolution of (25a). Hence

$$\varphi_b = \liminf_* \varphi_\nu$$

is also a supersolution by stability (cf. Proposition 50). But to prove (II), we need to show that $\varphi_\#$ is a supersolution. We will do this by showing that $\varphi_b = (\varphi_\#)_*$ pointwise (at least for positive times). We already have $(\varphi_\#)_* \leq \varphi_b$ by Lemma 84. To prove that $\varphi_b \leq (\varphi_\#)_*$, we need to use (I). By (I),

$$\int_{\mathbb{R}^d} |u - v|(x, t) \varphi(x, s) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x, t + s) \, dx,$$

for any u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$ and corresponding solutions u and v of (2) and $t, s \geq 0$. By Lemma 82(i), we also have that $\varphi_b = \varphi$ a.e. In particular, there is a null set $N \subset \mathbb{R}^+$ such that $\varphi(\cdot, s) = \varphi_b(\cdot, s)$ a.e., for any $s \notin N$.⁵ Fixing $T > 0$, there thus exists a sequence $s_n \rightarrow T^-$ such that $s_n \notin N$, for any n . Choosing moreover $t_n := T - s_n$, we deduce that

$$\int_{\mathbb{R}^d} |u - v|(x, t_n) \varphi_b(x, s_n) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x, T) \, dx.$$

Let us pass to the limit as $n \rightarrow \infty$ in the left-hand side. To do so, we use Fatou's lemma, which is possible because of the lower semicontinuity of φ_b and the continuity of entropy solutions with values in $L_{\text{loc}}^1(\mathbb{R}^d)$ which implies that

$$|u - v|(x, t_n) \rightarrow |u_0 - v_0|(x) \quad \text{for a.e. } x$$

⁵To find N use that $\iint_{\mathbb{R}^d \times \mathbb{R}^+} \mathbf{1}_{\{\varphi_b = \varphi\}} \, dx \, ds = 0 = \int_{\mathbb{R}^+} \text{meas}\{\varphi(\cdot, s) = \varphi_b(\cdot, s)\} \, ds$ by Fubini.

(along a subsequence). In the limit, it then follows that

$$\int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi_b(x, T) \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x, T) \, dx$$

for any u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$ and $T > 0$. To continue, we argue as in the previous step where we chose $0 \leq u_0 - v_0 \in L^1(\mathbb{R}^d)$ to be an approximate unit up to a multiplicative constant. The same arguments imply that for any $T > 0$,

$$\varphi_b(\cdot, T) \leq \varphi(\cdot, T) \quad \text{a.e.}$$

By Lemma 82(ii), we conclude that $\varphi_b \leq \varphi_\#$ pointwise (for positive times). Hence, $\varphi_b \leq (\varphi_\#)_*$ and then $\varphi_b = (\varphi_\#)_*$ (for positive times). This implies that $(\varphi_\#)_* = \varphi_b$ is a supersolution of (25a). The proof of Theorem 44 is complete. \square

Let us finally prove Corollaries 46 and 47.

Proof of Corollary 46. We already know that $\underline{\varphi} \in \mathscr{W}_{m, M, \varphi_0}$ by Theorem 38. Let us prove the formula with the inf. Take $\varphi \in \mathscr{W}_{m, M, \varphi_0}$, which means that $\varphi \in BLSC$ and satisfies Theorem 44(I) with $\varphi(t=0) \geq \varphi_0$. By this theorem, φ satisfies (II) as well, that is $\varphi_\#$ is a supersolution of (25a). Recall that $\varphi \leq (\varphi_\#)_*$ pointwise by Lemma 85(i). In particular

$$(\varphi_\#)_*(t=0) \geq \varphi(t=0) \geq \varphi_0.$$

Thus $\varphi_\#$ is a supersolution of the Cauchy problem (25), and $\underline{\varphi} \leq \varphi_\#$ by Proposition 7(i). Then Lemma 85(ii) implies that $(\underline{\varphi})_\# \leq \varphi_\#$, and we conclude that

$$(\underline{\varphi})_\#(x, t) = \inf \{ \varphi_\#(x, t) : \varphi \in \mathscr{W}_{m, M, \varphi_0} \} \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+$$

(with an equality because $\underline{\varphi} \in \mathscr{W}_{m, M, \varphi_0}$). The proof is complete. \square

Proof of Corollary 47. Let H_t be an arbitrary strongly continuous semigroup on $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ satisfying

$$\int_{\mathbb{R}^d} |S_t u_0 - S_t v_0| \varphi_0 \, dx \leq \int_{\mathbb{R}^d} |u_0 - v_0| H_t \varphi_0 \, dx,$$

for any u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$, $0 \leq \varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$, and $t \geq 0$. We have to prove that for any such φ_0 and t ,

$$G_t \varphi_0 \leq H_t \varphi_0,$$

where G_t is the semigroup corresponding to (25). For any such φ_0 , the minimal solution of (25) is the unique continuous solution, that is

$$\underline{\varphi}(x, t) = G_t \varphi_0(x) \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+.$$

Moreover, the above assumption on H_t implies that

$$H_t \varphi_0(x) \in \mathscr{W}_{m, M, \varphi_0}.$$

By Corollary 46 we deduce that for any $x \in \mathbb{R}^d$ and $t \geq 0$,

$$(G_t \varphi_0)_\#(x) \leq (H_t \varphi_0)_\#(x),$$

where we recall that

$$(G_t \varphi_0)_\#(x) = \liminf_{\substack{r \rightarrow 0^+ \\ y \rightarrow x}} \frac{1}{\text{meas}(B_r(y))} \int_{B_r(y)} G_t \varphi_0(z) \, dz$$

(and similarly for H). Since both $G_t \varphi_0(x)$ and $H_t \varphi_0(x)$ are continuous in x , we have $(G_t \varphi_0)_\# = G_t \varphi_0$ and $(H_t \varphi_0)_\# = H_t \varphi_0$ pointwise and the proof is complete. \square

ACKNOWLEDGEMENTS

E. R. Jakobsen and J. Endal were supported by the Toppforsk (research excellence) project Waves and Nonlinear Phenomena (WaNP), grant no. 250070 from the Research Council of Norway. N. Alibaud was supported by the French ANR project CoToCoLa (no. ANR-11-JS01-006-01). Part of this paper was completed the first half of 2016 during J. Endal's stay at Laboratory of Mathematics of Besançon (LMB), University of Bourgogne Franche-Comté (UBFC), and National Engineering Institute in Mechanics and Microtechnologies (ENSMM). We also thank Boris Andreianov for pointing out [44] to us and for other useful comments.

APPENDIX A. TECHNICAL PROOFS

A.1. Min and max viscosity solutions. Here are the proofs of Theorem 6 and Propositions 7 and 52; the ideas are inspired by [23, 10, 31] and the details are given for completeness.

Proof of Theorem 6. Consider the inf and supconvolutions of φ_0 ,

$$(\varphi_0)_\varepsilon(x) := \inf_{y \in \mathbb{R}^d} \left\{ (\varphi_0)_*(y) + \frac{|x - y|^2}{2\varepsilon^2} \right\}$$

and

$$(\varphi_0)^\varepsilon(x) := \sup_{y \in \mathbb{R}^d} \left\{ (\varphi_0)^*(y) - \frac{|x - y|^2}{2\varepsilon^2} \right\}.$$

Recall that they are at least C_b with $\inf \varphi_0 \leq (\varphi_0)_\varepsilon \leq \varphi_0 \leq (\varphi_0)^\varepsilon \leq \sup \varphi_0$,

$$\lim_{\varepsilon \downarrow 0} \uparrow (\varphi_0)_\varepsilon = \sup_{\varepsilon > 0} (\varphi_0)_\varepsilon = (\varphi_0)_* \quad \text{and} \quad \lim_{\varepsilon \downarrow 0} \downarrow (\varphi_0)^\varepsilon = \inf_{\varepsilon > 0} (\varphi_0)^\varepsilon = (\varphi_0)^*,$$

see e.g. [21, 29, 5, 4]. Let φ_ε and φ^ε be the viscosity solutions of (1a) with initial data $(\varphi_0)_\varepsilon$ and $(\varphi_0)^\varepsilon$, whose well-posedness is ensured by Theorem 4. By the comparison and the maximum principles, see Theorem 3 and Remark 5, we have the bounds

$$\inf \varphi_0 \leq \varphi_\varepsilon \leq \varphi^\varepsilon \leq \sup \varphi_0.$$

We can then define real-valued and bounded functions by

$$\underline{\varphi} := \sup_{\varepsilon > 0} \varphi_\varepsilon \quad \text{and} \quad \overline{\varphi} := \inf_{\varepsilon > 0} \varphi^\varepsilon.$$

We will see that these are our desired extremal solutions. First note that $\underline{\varphi}$ and $\overline{\varphi}$ are respectively lower and upper semicontinuous, as sup and inf of continuous functions. Moreover, $\underline{\varphi} \leq \overline{\varphi}$, so that at the initial time we immediately have

$$\underline{\varphi}(t = 0) = (\varphi_0)_* \quad \text{and} \quad (\underline{\varphi})^*(t = 0) \leq \overline{\varphi}(t = 0) = (\varphi_0)^*.$$

This means that (the lower semicontinuous function) $\underline{\varphi}$ satisfies the desired discontinuous conditions of Definition 1(aii). Also, we have the exact pointwise initial data $\underline{\varphi}(t = 0) = (\varphi_0)_*$ as claimed in the theorem. We argue the same way to get the desired initial data for $\overline{\varphi}$. Let us now show that these functions solve (1a). By the stability of sup and inf, see Proposition 49, we already know that $\underline{\varphi}$ and $\overline{\varphi}$ are sub and supersolutions, respectively. To get the other viscosity inequalities, we need to use that, as $\varepsilon \rightarrow 0^+$,

$$(82) \quad \underline{\varphi} = \liminf_* \varphi_\varepsilon \quad \text{and} \quad \overline{\varphi} = \limsup^* \varphi^\varepsilon.$$

This follows by elementary arguments (see e.g. [5, 4]) since φ^ε (resp. φ_ε) increases (resp. decreases) as $\varepsilon \downarrow 0$, which again follows by comparison since $(\varphi_0)^\varepsilon$ (resp.

$(\varphi_0)_\varepsilon$) increases (resp. decreases) as $\varepsilon \downarrow 0$. For the reader's convenience, we do it for $\underline{\varphi}$. For any fixed (x, t) ,

$$\liminf_* \varphi_\varepsilon(x, t) \leq \lim_{\varepsilon \rightarrow 0^+} \varphi_\varepsilon(x, t) = \underline{\varphi}(x, t).$$

Moreover, for any sequence $(x_n, t_n, \varepsilon_n) \rightarrow (x, t, 0^+)$ such that $\varepsilon_n \leq \varepsilon_m$ for any $n \geq m$, we have $\varphi_{\varepsilon_n}(x_n, t_n) \geq \varphi_{\varepsilon_m}(x_n, t_n)$. Fixing m and taking the limit in n ,

$$\liminf_{n \rightarrow \infty} \varphi_{\varepsilon_n}(x_n, t_n) \geq \lim_{n \rightarrow \infty} \varphi_{\varepsilon_m}(x_n, t_n) = \varphi_{\varepsilon_m}(x, t)$$

by the continuity of φ_{ε_m} . Taking the limit in m ,

$$\liminf_{n \rightarrow \infty} \varphi_{\varepsilon_n}(x_n, t_n) \geq \lim_{m \rightarrow \infty} \varphi_{\varepsilon_m}(x, t) = \underline{\varphi}(x, t).$$

This proves the first part of (82) and the second part can be obtained similarly. By the stability with respect to lower and upper relaxed limits, see Proposition 50, we conclude that $\underline{\varphi}$ and $\overline{\varphi}$ are viscosity solutions of (1a), and thus of (1) since we already established (1b) in the sense of Definition 1(aii).

At this stage, it only remains to prove that these solutions are extremal. Let φ be another bounded discontinuous solution. Noting that

$$(\varphi_0)_\varepsilon \leq (\varphi_0)_* \leq \varphi_*(t=0) \leq \varphi^*(t=0) \leq (\varphi_0)^* \leq (\varphi_0)^\varepsilon,$$

we use once more the comparison principle to deduce that

$$\varphi_\varepsilon \leq \varphi \leq \varphi^\varepsilon,$$

for any $\varepsilon > 0$. In the limit as $\varepsilon \rightarrow 0^+$, we conclude that $\underline{\varphi} \leq \varphi \leq \overline{\varphi}$ and the proof is complete. \square

Proof of Proposition 7. To prove (i), we simply argue as in the end of the proof of Theorem 6. For instance, assume that φ is a bounded supersolution of (1). Then,

$$(\varphi_0)_\varepsilon \leq (\varphi_0)_* \leq \varphi_*(t=0)$$

and, by comparison, $\varphi_\varepsilon \leq \varphi$ for any $\varepsilon > 0$, etc. Part (ii) follows from part (i). \square

Proof of Proposition 52. We only give the proof for the minimal solutions, the other case is similar. Let $\underline{\varphi}$ denote the minimal solution of (1) with initial data $\varphi_0 := \sup_n \varphi_0^n$. We have to prove that $\underline{\varphi} = \sup_n \underline{\varphi}_n$, where $\underline{\varphi}_n$ is the minimal solution of (1) with initial data φ_0^n . By Proposition 7(ii), we have $\underline{\varphi}_n \leq \underline{\varphi}$ for any integer n . We thus already know that $\underline{\varphi} \geq \sup_n \underline{\varphi}_n$ and it only remains to prove the other inequality. To do so, it suffices to show that $\sup_n \underline{\varphi}_n$ is a supersolution of (1) (with initial data φ_0). Indeed, by Proposition 7(i), we then get $\underline{\varphi} \leq \sup_n \underline{\varphi}_n$. It is at this stage that we need to use monotonicity. Recall that $n \mapsto \varphi_0^n(x)$ is nondecreasing for any x . By the comparison principle, the same monotonicity holds for the solutions which means that $n \mapsto \varphi_n(x, t)$ is nondecreasing for any fixed x and t . Hence, as before, we can easily deduce that

$$\sup_n \underline{\varphi}_n = \liminf_* \underline{\varphi}_n \quad \text{as } n \rightarrow \infty.$$

By stability, see Propositions 50 and 79, we deduce that $\liminf_* \underline{\varphi}_n$ is a supersolution of (1a) with initial data

$$\liminf_* \underline{\varphi}_n(t=0) = \liminf_* \underline{\varphi}_0^n$$

(the first limit is taken in (x, t) and the second only in x). But this initial data is precisely

$$\liminf_* \underline{\varphi}_0^n = \sup_n \varphi_0^n = \varphi_0,$$

by the monotonicity of φ_0^n in n . This completes the proof. \square

A.2. Representation formulas. Let us prove Propositions 8 and 9. These results are classical in the control theory, but usually written for continuous or maximal solutions, see [29, 4, 33, 34]. Here we give the proofs for minimal solutions.

Proof of Proposition 8. By the assumption that $a \equiv 0$, (1a) is now

$$\partial_t \varphi = \sup_{\xi \in \mathcal{E}} \{b(\xi) \cdot D\varphi\} = \sup_{q \in \mathcal{C}} \{q \cdot D\varphi\},$$

where $\mathcal{C} = \overline{\{\text{Im}(b)\}}$ is compact. By control theory [5, 4] the viscosity solutions of (1) is given by

$$\varphi(x, t) = \sup_{x+t\mathcal{C}} \varphi_0$$

if $\varphi_0 \in BUC$. In the general case, consider the infconvolution

$$(\varphi_0)_\varepsilon(x) := \inf_{y \in \mathbb{R}^d} \left\{ \varphi_0(y) + \frac{|x-y|^2}{2\varepsilon^2} \right\}.$$

Recall that $(\varphi_0)_\varepsilon \in W^{1,\infty}(\mathbb{R}^d)$ and $(\varphi_0)_\varepsilon \uparrow (\varphi_0)_*$ pointwise as $\varepsilon \downarrow 0$. It follows that the solution of (1a) with $(\varphi_0)_\varepsilon$ as initial data is

$$\varphi_\varepsilon(x, t) = \sup_{x+t\mathcal{C}} (\varphi_0)_\varepsilon.$$

By Proposition 52, the minimal solution of (1) is thus

$$\underline{\varphi}(x, t) = \sup_{\varepsilon > 0} \varphi_\varepsilon(x, t) = \sup_{\varepsilon > 0} \sup_{x+t\mathcal{C}} (\varphi_0)_\varepsilon = \sup_{x+t\mathcal{C}} \sup_{\varepsilon > 0} (\varphi_0)_\varepsilon = \sup_{x+t\mathcal{C}} (\varphi_0)_*. \quad \square$$

Proof of Proposition 9. Equation (1a) is given by

$$\partial_t \varphi = \sup_{\xi \in \mathcal{E}} \left\{ b(\xi) \cdot D\varphi + \text{tr}(\sigma^a(\xi)(\sigma^a)^\top(\xi)D^2\varphi) \right\},$$

where \mathcal{E} is compact and the coefficients b and σ^a are continuous by (11). By stochastic control theory [29], the viscosity solution of (1) is given by

$$\varphi(x, t) = \sup_{\xi} \mathbb{E} \{ \varphi_0(\mathbf{X}_t^x) \},$$

if $\varphi_0 \in BUC$ and where ξ_s and \mathbf{X}_s^x are defined as in Proposition 9. Let us now repeat the argument of the proof of Proposition 8 considering the infconvolution $(\varphi_0)_\varepsilon$ and the corresponding solution of (1a)

$$\varphi_\varepsilon(x, t) = \sup_{\xi} \mathbb{E} \{ (\varphi_0)_\varepsilon(\mathbf{X}_t^x) \}.$$

We find that the minimal solution of (1) is

$$\underline{\varphi}(x, t) = \sup_{\varepsilon > 0} \varphi_\varepsilon(x, t) = \sup_{\xi} \sup_{\varepsilon > 0} \mathbb{E} \{ (\varphi_0)_\varepsilon(\mathbf{X}_t^x) \}.$$

Since $(\varphi_0)_\varepsilon \uparrow (\varphi_0)_*$ as $\varepsilon \downarrow 0$, we conclude the proof using the monotone convergence theorem:

$$\begin{aligned} & \sup_{\varepsilon > 0} \mathbb{E} \{ (\varphi_0)_\varepsilon(\mathbf{X}_t^x) \} \\ &= \lim_{\varepsilon \downarrow 0} \uparrow \mathbb{E} \{ (\varphi_0)_\varepsilon(\mathbf{X}_t^x) \} = \mathbb{E} \left\{ \lim_{\varepsilon \downarrow 0} \uparrow (\varphi_0)_\varepsilon(\mathbf{X}_t^x) \right\} = \mathbb{E} \{ (\varphi_0)_*(\mathbf{X}_t^x) \}. \quad \square \end{aligned}$$

A.3. L_{int}^∞ strong continuity in time. During the proofs, we have assumed that the solution of (1) is continuous in time with values in L_{int}^∞ . Let us prove it here.

Lemma 86. *Assume (H1), G_t is the semigroup associated to (1) defined in (17), and $\varphi_0 \in C_b \cap L^\infty(\mathbb{R}^d)$. Then the function $t \geq 0 \mapsto G_t \varphi_0 \in C_b \cap L^\infty(\mathbb{R}^d)$ is strongly continuous, i.e.*

$$\lim_{t \rightarrow t_0} \|G_t \varphi_0 - G_{t_0} \varphi_0\|_{\text{int}} = 0 \quad \forall t_0 \geq 0.$$

Proof. Fix $\varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ and let us prove that

$$t \geq 0 \mapsto G_t \varphi_0 \in C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$$

is continuous. That is to say, fix also $t_0 \in \mathbb{R}^+$ and let us show that

$$\int \sup_{\overline{Q}_1(x)} |G_t \varphi_0 - G_{t_0} \varphi_0| dx \rightarrow 0 \quad \text{as } t \rightarrow t_0.$$

The pointwise convergence follows from the continuity of $(x, t) \mapsto G_t \varphi_0(x)$ (this function being the continuous solution of (1)). Moreover, a dominating function is given by

$$x \mapsto \sup_{(y,s) \in \overline{Q}_1(x) \times [0, t_0+1]} |G_s \varphi_0(y)|.$$

It is indeed integrable by the uniform in time estimate of Theorem 30. \square

A.4. Main properties of $|\cdot|_{\text{conv}}$ and $|\cdot|_{\text{diff}}$. Let us prove what we have claimed in Section 2.4. We need support functions h_E of sets E (cf. e.g. [35] for their standard properties). For any $\emptyset \neq E \subseteq \mathbb{R}^d \times \mathbb{S}_d$, h_E is the function

$$h_E : (p, X) \in \mathbb{R}^d \times \mathbb{S}_d \mapsto \sup_{(q,Y) \in E} (q, p) \cdot (Y, X),$$

where $(q, Y) \cdot (p, X) := q \cdot p + \text{tr}(YX)$ is the inner product of $\mathbb{R}^d \times \mathbb{S}_d$.

Lemma 87. *Under (H1), the Hamiltonian of (1a) defined by (4) satisfies $H = h_{\mathcal{K}}$ where $\mathcal{K} = \overline{\text{co}\{\text{Im}(b, a)\}}$.*

Proof. Since $H = h_{\text{Im}(b,a)}$ by definition, it suffices to use that $h_E = h_{\overline{\text{co}(E)}}$ for any set E . \square

Let us now prove the assertion of Remark 19(b), that is the result below.

Lemma 88. *Let $H : \mathbb{R}^d \times \mathbb{S}_d \rightarrow \mathbb{R}$ be such that there are two different triplets (\mathcal{E}, b, a) and $(\tilde{\mathcal{E}}, \tilde{b}, \tilde{a})$ satisfying (H1) for which*

$$H(p, X) = \sup_{\mathcal{E}} \{b \cdot p + \text{tr}(aX)\} = \sup_{\tilde{\mathcal{E}}} \{\tilde{b} \cdot p + \text{tr}(\tilde{a}X)\}.$$

Then

$$|H|_{\text{conv}} = \inf_{b_0 \in \mathbb{R}^d} \sup_{\xi \in \mathcal{E}} |b(\xi) - b_0| = \inf_{b_0 \in \mathbb{R}^d} \sup_{\tilde{\xi} \in \tilde{\mathcal{E}}} |\tilde{b}(\tilde{\xi}) - b_0|$$

and

$$|H|_{\text{diff}} = \inf_{\mathbb{S}_d^+ \ni a_0 \leq \text{Im}(a)} \sup_{\xi \in \mathcal{E}} |\text{tr}(a(\xi) - a_0)| = \inf_{\mathbb{S}_d^+ \ni a_0 \leq \text{Im}(\tilde{a})} \sup_{\tilde{\xi} \in \tilde{\mathcal{E}}} |\text{tr}(\tilde{a}(\tilde{\xi}) - a_0)|.$$

Proof. By Lemma 87, $H = h_{\mathcal{K}} = h_{\tilde{\mathcal{K}}}$ where $\mathcal{K} = \overline{\text{co}\{\text{Im}(b, a)\}}$ and $\tilde{\mathcal{K}} = \overline{\text{co}\{\text{Im}(\tilde{b}, \tilde{a})\}}$. It follows that $\mathcal{K} = \tilde{\mathcal{K}}$ because any closed convex set is entirely determined by its support function. It thus suffices to prove that $|H|_{\text{conv}}$ and $|H|_{\text{diff}}$ only depend on \mathcal{K} . Let us do it for $|H|_{\text{diff}}$. Define the projection

$$\mathcal{D} := \text{proj}_{\mathbb{S}_d}(\mathcal{K}) = \overline{\text{co}\{\text{Im}(a)\}}.$$

Since $a_0 \leq \text{Im}(a)$ if and only if $a_0 \leq \mathcal{D}$,

$$|H|_{\text{diff}} = \inf_{\mathbb{S}_d^+ \ni a_0 \leq \mathcal{D}} \sup_{\mathcal{E}} \text{tr}(a - a_0),$$

where

$$\sup_{\mathcal{E}} \text{tr}(a - a_0) = \sup_{Y \in \text{Im}(a)} \text{tr}(Y - a_0) = \sup_{Y \in \overline{\text{co}\{\text{Im}(a)\}}} \text{tr}(Y - a_0).$$

It follows that

$$|H|_{\text{diff}} = \inf_{\mathbb{S}_d^+ \ni a_0 \leq \mathcal{D}} \sup_{Y \in \mathcal{D}} \text{tr}(Y - a_0) \quad \text{for } \mathcal{D} = \text{proj}_{\mathbb{S}_d}(\mathcal{K}).$$

We show in the same way that

$$|H|_{\text{conv}} = \inf_{b_0 \in \mathbb{R}^d} \sup_{q \in \mathcal{C}} |q - b_0| \quad \text{for } \mathcal{C} = \text{proj}_{\mathbb{R}^d}(\mathcal{K}).$$

This completes the proof since these formula only depend on \mathcal{K} . \square

Let us now prove the last result of Section 2.4.

Proof of Proposition 20. We only do the proof for $|\cdot|_{\text{diff}}$ since the argument is similar for $|\cdot|_{\text{conv}}$. Let us first prove that it is a semi-norm. It is clearly finitely valued because $a = a(\xi)$ is bounded. Given $H, \tilde{H} \in \Gamma$ with some respective triplets of data (\mathcal{E}, b, a) and $(\tilde{\mathcal{E}}, \tilde{b}, \tilde{a})$ satisfying (H1), we have

$$\begin{aligned} H(p, X) + \tilde{H}(p, X) &= \sup_{\mathcal{E}} \{b \cdot p + \text{tr}(aX)\} + \sup_{\tilde{\mathcal{E}}} \{\tilde{b} \cdot p + \text{tr}(\tilde{a}X)\} \\ &= \sup_{\mathcal{E} \times \tilde{\mathcal{E}}} \left\{ (b(\xi) + \tilde{b}(\tilde{\xi})) \cdot p + \text{tr} \left((a(\xi) + \tilde{a}(\tilde{\xi}))X \right) \right\}, \end{aligned}$$

for any $(p, X) \in \mathbb{R}^d \times \mathbb{S}_d$. This identifies some data which we can associate to the Hamiltonian $H + \tilde{H} \in \Gamma$. Let us denote the diffusion matrix by $a \oplus \tilde{a}$,

$$a \oplus \tilde{a} : (\xi, \tilde{\xi}) \mapsto a(\xi) + \tilde{a}(\tilde{\xi}),$$

and fix a_0 and \tilde{a}_0 in \mathbb{S}_d^+ such that $a_0 \leq \text{Im}(a)$ and $\tilde{a}_0 \leq \text{Im}(\tilde{a})$. It follows that $a_0 + \tilde{a}_0 \leq \text{Im}(a \oplus \tilde{a})$ and

$$\begin{aligned} |H + \tilde{H}|_{\text{diff}} &\leq \sup_{\mathcal{E} \times \tilde{\mathcal{E}}} \text{tr} \left((a(\xi) + \tilde{a}(\tilde{\xi})) - (a_0 + \tilde{a}_0) \right) \\ &= \sup_{\mathcal{E}} \text{tr}(a - a_0) + \sup_{\tilde{\mathcal{E}}} \text{tr}(\tilde{a} - \tilde{a}_0). \end{aligned}$$

Taking the infimum in a_0 and \tilde{a}_0 , we deduce that

$$|H + \tilde{H}|_{\text{diff}} \leq |H|_{\text{diff}} + |\tilde{H}|_{\text{diff}}.$$

We argue similarly to show that $|\alpha H|_{\text{diff}} = \alpha |H|_{\text{diff}}$, for any $\alpha \geq 0$.

Let us now prove (13). Take a minimizing sequence $(a_0^n)_n$ for the infimum

$$|H|_{\text{diff}} = \inf_{\mathbb{S}_d^+ \ni a_0 \leq \text{Im}(a)} \|\text{tr}(a - a_0)\|_{\infty}.$$

It is bounded since $a = a(\xi)$ is bounded and hence $a_0^n \rightarrow a_0$ for some matrix $a_0 \in \mathbb{S}_d^+$ up to a subsequence. In the limit, we get that

$$|H|_{\text{diff}} = \lim_{n \rightarrow \infty} \|\text{tr}(a - a_0^n)\|_{\infty} = \|\text{tr}(a - a_0)\|_{\infty}.$$

Hence if this semi-norm is zero, then $a \equiv a_0$ and the diffusion is linear. Conversely, assume that $H(p, X)$ is affine in X , for any p , and let us prove that $|H|_{\text{diff}} = 0$. Since

$$X \in \mathbb{S}_d \mapsto H(0, X) \in \mathbb{R}$$

is linear by assumption, Riesz representation theorem implies that there exists $a_0 \in \mathbb{S}_d$ such that $H(0, X) = \text{tr}(a_0 X)$ for any $X \in \mathbb{S}_d$. But we also have

$$H(0, X) = \sup_{\mathcal{E}} \text{tr}(aX) = \sup_{Y \in \overline{\text{co}\{\text{Im}(a)\}}} \text{tr}(YX)$$

and therefore $\overline{\text{co}\{\text{Im}(a)\}} = \{a_0\}$ (again because a closed convex set is entirely determined by its support function). In particular,

$$|H|_{\text{diff}} = \|\text{tr}(a - a_0)\|_{\infty} = 0$$

and this completes the proof. \square

During these proofs, we have seen some interesting connections with support functions. Let us summarize them for completeness.

Remark 89. (a) Recall the cone defined in Section 2.4:

$$\Gamma = \{H : \mathbb{R}^d \times \mathbb{S}_d \rightarrow \mathbb{R} \text{ s.t. } H \text{ satisfy (4) for some } (\mathcal{E}, b, a) \text{ satisfying (H1)}\}.$$

It satisfies

$$\Gamma = \{\text{support functions of compact convex sets } \mathcal{K} \subset \mathbb{R}^d \times \mathbb{S}_d^+\},$$

where the support function of \mathcal{K} is

$$h_{\mathcal{K}}(p, X) = \sup_{(q, Y) \in \mathcal{K}} (q, Y) \cdot (p, X)$$

for the inner product $(q, Y) \cdot (p, X) = q \cdot p + \text{tr}(YX)$.

(b) If $\mathcal{C} = \text{proj}_{\mathbb{R}^d}(\mathcal{K})$ and $\mathcal{D} = \text{proj}_{\mathbb{S}_d}(\mathcal{K})$ are the projected sets, then

i) $|h_{\mathcal{K}}|_{\text{conv}}$ is the minimal radius of balls containing \mathcal{C} , and

ii) $|h_{\mathcal{K}}|_{\text{diff}} = \inf_{\mathbb{S}_d^+ \ni Y_0 \leq \mathcal{D}} \sup_{Y \in \mathcal{D}} \text{tr}(Y - Y_0)$.

(c) For any triplet (\mathcal{E}, a, b) representing $H = h_{\mathcal{K}}$,

$$\mathcal{K} = \overline{\text{co}\{\text{Im}(b, a)\}}, \quad \mathcal{C} = \overline{\text{co}\{\text{Im}(b)\}} \quad \text{and} \quad \mathcal{D} = \overline{\text{co}\{\text{Im}(a)\}}.$$

A.5. An optimal L_{int}^{∞} inequality.

Lemma 90. *For any nonnegative $\varphi_0 \in USC(\mathbb{R}^d)$, $r > 0$ and $\varepsilon \geq 0$,*

$$\int \sup_{\overline{Q}_{r+\varepsilon}(x)} \varphi_0 \, dx \leq \left(\frac{r+\varepsilon}{r}\right)^d \int \sup_{\overline{Q}_r(x)} \varphi_0 \, dx.$$

Remark 91. This is Lemma 16 with an explicit estimate of the constant. The constant is optimal, for instance for $\varphi_0 = \mathbf{1}_{\{x_0\}}$ and any fixed $x_0 \in \mathbb{R}^d$.

Proof. Assume the result holds for $d = 1$, and let $x = (x_1, \hat{x}) \in \mathbb{R} \times \mathbb{R}^{d-1}$. Then by assumption

$$\begin{aligned} \int \sup_{\overline{Q}_{r+\varepsilon}(x_1, \hat{x})} \varphi_0 \, dx_1 &= \int \sup_{y_1 \in [-r-\varepsilon, r+\varepsilon]} \overbrace{\sup_{\hat{y} \in [-r-\varepsilon, r+\varepsilon]^{d-1}} \varphi_0(x_1 + y_1, \hat{x} + \hat{y})}^{\text{fixed } USC \text{ and nonnegative function of } x_1} \, dx_1 \\ &\leq \frac{r+\varepsilon}{r} \int \sup_{y_1 \in [-r, r]} \sup_{\hat{y} \in [-r-\varepsilon, r+\varepsilon]^{d-1}} \varphi_0(x_1 + y_1, \hat{x} + \hat{y}) \, dx_1, \end{aligned}$$

for any $\hat{x} \in \mathbb{R}^{d-1}$. Integrating in \hat{x} , we get

$$\int \sup_{\overline{Q}_{r+\varepsilon}(x)} \varphi_0 \, dx \leq \frac{r+\varepsilon}{r} \int \sup_{y \in [-r, r] \times [-r-\varepsilon, r+\varepsilon]^{d-1}} \varphi_0(x + y) \, dx.$$

Starting from the integral to the right, we interchange the roles of x_1 and x_2 and repeat the arguments. Continuing in this way leads to the result.

Let us thus focus on the case $d = 1$. A standard approximation procedure allows to reduce the proof to the case where φ_0 is a step function,

$$\varphi_0(x) = \sum_{i=1}^n \alpha_i \mathbf{1}_{I_i}(x),$$

for some $\alpha_i \geq 0$ and bounded disjoint intervals I_i . To see this, it suffices to take the supconvolution followed by upper Riemann sum approximations of integrals. The details are rather standard and left to the reader. Let us focus on the step functions above. Let $0 < \tilde{\alpha}_1 < \dots < \tilde{\alpha}_m$ be all the possible different values taken by α_i , and

$$\varphi_0(x) = \sum_{i=1}^m \beta_i \mathbf{1}_{J_i}(x),$$

with nonnegative $\beta_1 := \tilde{\alpha}_1$, $\beta_2 := \tilde{\alpha}_2 - \tilde{\alpha}_1$, etc., and domains

$$J_i := \{\varphi_0 \geq \tilde{\alpha}_i\}$$

which are finite unions of intervals. For any $r > 0$, we have a similar representation for

$$\psi_0(x) := \sup_{x+[-r,r]} \varphi_0 = \sum_{i=1}^m \beta_i \mathbf{1}_{\{\psi_0 \geq \tilde{\alpha}_i\}}(x),$$

where by definition of ψ_0 ,

$$\{\psi_0 \geq \tilde{\alpha}_i\} = J_i + [-r, r].$$

In other words,

$$\sup_{x+[-r,r]} \varphi_0 = \sum_{i=1}^m \beta_i \mathbf{1}_{J_i+[-r,r]}(x).$$

By construction, it follows that

$$\int \sup_{x+[-r,r]} \varphi_0 \, dx = \sum_{i=1}^m \beta_i \text{meas}(J_i + [-r, r]).$$

The same formula holds for $r + \varepsilon$ and it thus suffices to prove that

$$\frac{\text{meas}(J_i + [-r - \varepsilon, r + \varepsilon])}{\text{meas}(J_i + [-r, r])} \leq \frac{r + \varepsilon}{r} \quad \forall i = 1, \dots, m.$$

To do so, rewrite

$$J_i + [-r, r] = \cup_{j=1}^k K_j$$

as a disjoint union of intervals K_j . In particular,

$$\text{meas}(J_i + [-r, r]) = \sum_{j=1}^k \text{meas}(K_j).$$

Observe also that

$$J_i + [-r - \varepsilon, r + \varepsilon] \leq \cup_{j=1}^k (K_j + [-\varepsilon, \varepsilon]),$$

where the union to the right no longer needs to be disjoint. Nevertheless, we have the estimate

$$\text{meas}(J_i + [-r - \varepsilon, r + \varepsilon]) \leq \sum_{j=1}^k \text{meas}(K_j + [-\varepsilon, \varepsilon])$$

and the proof further reduces to proving

$$(83) \quad \frac{\text{meas}(K_j + [-\varepsilon, \varepsilon])}{\text{meas}(K_j)} \leq \frac{r + \varepsilon}{r} \quad \forall j = 1, \dots, k,$$

since then

$$\frac{\text{meas}(J_i + [-r - \varepsilon, r + \varepsilon])}{\text{meas}(J_i + [-r, r])} \leq \frac{\sum_{j=1}^k \text{meas}(K_j + [-\varepsilon, \varepsilon])}{\sum_{j=1}^k \text{meas}(K_j)} \leq \frac{r + \varepsilon}{r}.$$

To prove (83), remember that $J_i + [-r, r] = \cup_{j=1}^k K_j$ is a disjoint union of intervals. Because of the addition of $[-r, r]$, each intervals K_j has a length greater than $2r$. To fix the ideas, let us say $K_j = [a, b]$ (the reasoning being the same with an open interval at a or b , etc.) Then $b - a \geq 2r$ and

$$\frac{\text{meas}(K_j + [-\varepsilon, \varepsilon])}{\text{meas}(K_j)} = \frac{b - a + 2\varepsilon}{b - a} \leq \frac{2r + 2\varepsilon}{2r} = \frac{r + \varepsilon}{r},$$

because the function $\tau > 0 \mapsto \frac{\tau + \varepsilon}{\tau}$ is nonincreasing. The proof is complete. \square

APPENDIX B. COMPLEMENTARY PROOFS FOR ENTROPY SOLUTIONS

For completeness, we recall the proof of Theorem 12 which is Theorem 1.1 in [30] under (H2). We will take the opportunity to give details, but we will not perform the doubling of variables to show Lemma 14 for which we will refer to [11].

Recall that [20, 11] proved the well-posedness of L^1 kinetic or renormalized solutions which are equivalent to entropy solutions in $L^1 \cap L^\infty$. The definition of entropy solutions in $L^1 \cap L^\infty$ uses the energy estimate (2.8) of [20],

$$\iint_{\mathbb{R}^d \times \mathbb{R}^+} \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) \right)^2 dx dt \leq \frac{1}{2} \|u_0\|_{L^2} < \infty \quad \text{if } u_0 \in L^1 \cap L^\infty,$$

where

$$\zeta_{ik}(u) = \int_0^u \sigma_{ik}^A(\xi) d\xi.$$

As a consequence “ L^2 ” was used e.g. in [11, Definition 2.2] instead of “ L_{loc}^2 ” in Definition 11. But we have the following result:

Lemma 92 (Local energy estimate). *Assume (H2), $u_0 \in L^\infty(\mathbb{R}^d)$, $0 \leq \phi \in C_c^\infty(\mathbb{R}^d)$, and $T \geq 0$. If u is an entropy solution of (2) in the sense of Definition 11 and*

$$\|u_0\|_{L^\infty} + \|u\|_{L^\infty} + \|\phi\|_{W^{2,1}} \leq M,$$

then there is a constant C only depending on T, M, F and A such that

$$\iint_{\mathbb{R}^d \times (0, T)} \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u(x, t)) \right)^2 \phi(x) dx dt \leq C.$$

Proof. We use Definition 11(c) with the entropy $\eta(u) = |u|^2$ and the corresponding fluxes

$$q(u) = 2 \int_0^u \xi F'(\xi) d\xi \quad \text{and} \quad r(u) = 2 \int_0^u \xi A(\xi) d\xi.$$

We also take a test function $\phi(x) \mathbf{1}_{[0, T]}(t)$ where $0 \leq \phi \in C_c^\infty(\mathbb{R}^d)$. It is not smooth in time but a standard approximation argument shows that it can be used in Definition 11(c) if we add also a final value term at $t = T$. Here we need the L_{loc}^1

continuity in time of entropy solutions. The result is

$$\begin{aligned} & \overbrace{\int_{\mathbb{R}^d} u^2(x, T) \phi(x) \, dx}^{\geq 0} + 2 \iint_{\mathbb{R}^d \times (0, T)} \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) \right)^2 \phi \, dx \, dt \\ & \leq \int_{\mathbb{R}^d} u_0^2(x) \phi(x) \, dx + \iint_{\mathbb{R}^d \times (0, T)} \left(\sum_{i=1}^d q_i(u) \partial_{x_i} \phi + \sum_{i,j=1}^d r_{ij}(u) \partial_{x_i x_j}^2 \phi \right) dx \, dt. \end{aligned}$$

By assumption $\|u_0\|_{L^\infty} + \|u\|_{L^\infty} + \|\phi\|_{W^{2,1}} \leq M$, so it follows that

$$\begin{cases} \|q(u)\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^+, \mathbb{R}^d)} \leq 2M^2 \operatorname{ess\,sup}_{-M \leq \xi \leq M} |F'(\xi)|, \text{ and} \\ \|r(u)\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^+, \mathbb{R}^{d \times d})} \leq 2M^2 \operatorname{ess\,sup}_{-M \leq \xi \leq M} |A(\xi)|. \end{cases}$$

With all these estimates, the conclusion readily follows. \square

Let us now show the Kato inequality.

Sketch of the proof of Lemma 14. Copy the proof of Theorem 3.1 of [11] with $l = \infty$ and zero renormalization measures $\mu_i^u \equiv \mu_i^v \equiv 0$. With the aid of the previous local energy estimate, check that every computation holds until (3.19)—even if u and v satisfy (a)–(b) of Definition 11 with L_{loc}^2 and not L^2 as in [11]. This gives (12) with $\phi \in C_c^\infty(\mathbb{R}^d \times (0, \infty))$. Use an approximation argument for $\phi(x) \mathbf{1}_{[0, T]}$ and the continuity in time with values in L_{loc}^1 to get initial and final terms. \square

To show the uniqueness of entropy solutions, it suffices to find a “good” ϕ in (12), for instance an exponential as in [19, 30]. This gives the result below.

Lemma 93. *Assume (H2) and u, v are L^∞ entropy solutions of (2) with initial data $u_0, v_0 \in L^\infty(\mathbb{R}^d)$. Then for any $t \geq 0$ and $m < M$ such that $m \leq u, v \leq M$,*

$$\int_{\mathbb{R}^d} |u - v|(x, t) e^{-|x|} \, dx \leq e^{(L_F + L_A)t} \int_{\mathbb{R}^d} |u_0 - v_0|(x) e^{-|x|} \, dx,$$

where $L_F := \operatorname{ess\,sup}_{[-m, M]} |F'|$ and $L_A := \operatorname{ess\,sup}_{[-m, M]} \operatorname{tr}(A)$.

Remark 94. We can take $m < M$ such that u_0 and v_0 take their values in $[m, M]$ by the maximum principle. At this stage of this appendix, this principle is only known in $L^1 \cap L^\infty$ (or L^1) by [20, 11] and it will follow later in L^∞ .

Sketch of the proof. The proof is inspired by [19, 30]. Consider

$$\phi_\varepsilon(x, t) := e^{(L_F + L_A)(T-t) - \sqrt{\varepsilon^2 + |x|^2}},$$

for some arbitrary $\varepsilon > 0$, and check that

$$\begin{aligned} |u - v| \partial_t \phi_\varepsilon + \sum_{i=1}^d q_i(u, v) \partial_{x_i} \phi_\varepsilon + \sum_{i,j=1}^d r_{ij}(u, v) \partial_{x_i x_j}^2 \phi_\varepsilon \\ \leq |u - v| \left\{ \partial_t \phi_\varepsilon + L_F |D\phi_\varepsilon| + L_A \sup_{\lambda \in \operatorname{Sp}(D^2 \phi_\varepsilon)} \lambda^+ \right\} \leq 0 \end{aligned}$$

by the Ky Fan inequality (49). Then by the Kato inequality (12) with ϕ_ε ,

$$\int_{\mathbb{R}^d} |u - v|(x, T) e^{-\sqrt{\varepsilon^2 + |x|^2}} \, dx \leq e^{(L_F + L_A)T} \int_{\mathbb{R}^d} |u_0 - v_0|(x) e^{-\sqrt{\varepsilon^2 + |x|^2}} \, dx$$

and the result follows in the limit $\varepsilon \rightarrow 0^+$. \square

Proof of Theorem 12. By Lemma 93, it remains to show the existence. The proof is inspired by [20, 11]. Given $u_0 \in L^\infty(\mathbb{R}^d)$, take a sequence of approximate data $(u_0^n)_n$ in $L^1 \cap L^\infty(\mathbb{R}^d)$ such that

$$(84) \quad \text{ess inf } u_0 \leq u_0^n \leq \text{ess sup } u_0 \quad \text{and} \quad u_0^n \rightarrow u_0 \text{ in } L^1_{\text{loc}}(\mathbb{R}^d).$$

Let u_n be the entropy solution of (2) with initial data u_0^n . By the maximum principle (in $L^1 \cap L^\infty$), we know that

$$(85) \quad \text{ess inf } u_0 \leq u_n \leq \text{ess sup } u_0.$$

Moreover, by Lemma 93, we have for any $R \geq 0$, $T \geq 0$, and integers n, m ,

$$\begin{aligned} & \|u_m - u_n\|_{C([0, T]; L^1(\{|x| < R\}))} \\ &= \sup_{t \in [0, T]} \int_{|x| < R} |u_m(x, t) - u_n(x, t)| \, dx \\ &\leq e^R \sup_{t \in [0, T]} \int_{\mathbb{R}^d} |u_m(x, t) - u_n(x, t)| e^{-|x|} \, dx \\ &\leq e^R e^{(L_F + L_A)T} \int_{\mathbb{R}^d} |u_0^m(x) - u_0^n(x)| e^{-|x|} \, dx, \end{aligned}$$

where the latter integral tends to zero as $n, m \rightarrow \infty$ by (84). Hence there exists some $u \in L^\infty(\mathbb{R}^d \times \mathbb{R}^+) \cap C(\mathbb{R}^+; L^1_{\text{loc}}(\mathbb{R}^d))$ such that

$$(86) \quad \lim_{n \rightarrow \infty} u_n = u \quad \text{in } C([0, T]; L^1_{\text{loc}}(\mathbb{R}^d)), \quad \forall T \geq 0.$$

It remains to show that u is an entropy solution with initial data u_0 .

We begin with the L^2_{loc} energy estimate of Definition 11(a). By Lemma 92 and the L^∞ bounds in (85), the sequence

$$\left\{ \sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u_n) \right\} \subset L^2(\mathbb{R}^d \times \mathbb{R}^+)$$

is uniformly bounded in $L^2(\mathcal{K})$, for any $k = 1, \dots, K$, and compact $\mathcal{K} \subset \mathbb{R}^d \times \mathbb{R}^+$. As a consequence, it weakly converges in $L^2(\mathcal{K})$ (up to some subsequence) to some function

$$f(x, t) \in L^2_{\text{loc}}(\mathbb{R}^d \times \mathbb{R}^+).$$

For any $\phi \in C^\infty_c(\mathbb{R}^d \times \mathbb{R}^+)$ and $k = 1, \dots, K$,

$$\iint_{\mathbb{R}^d \times \mathbb{R}^+} \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u_n) \right) \phi \, dx \, dt = - \iint_{\mathbb{R}^d \times \mathbb{R}^+} \left(\sum_{i=1}^d \zeta_{ik}(u_n) \partial_{x_i} \phi \right) \, dx \, dt.$$

Since

$$\zeta_{ik}(\cdot) = \int_0^\cdot \sigma_{ik}^A(\xi) \, d\xi \quad \text{for } \sigma_{ik}^A \in L^\infty_{\text{loc}}(\mathbb{R}, \mathbb{R}^{d \times K}),$$

the function $\zeta_{ik}(\cdot)$ is locally Lipschitz continuous. By the strong convergence stated in (86) and the L^∞ bounds in (85), we infer that $\zeta_{ik}(u_n)$ converges towards $\zeta_{ik}(u)$ in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^d))$ for all $T \geq 0$. Hence, at the limit, we deduce that

$$\iint_{\mathbb{R}^d \times \mathbb{R}^+} f \phi \, dx \, dt = - \iint_{\mathbb{R}^d \times \mathbb{R}^+} \left(\sum_{i=1}^d \zeta_{ik}(u) \partial_{x_i} \phi \right) \, dx \, dt.$$

By the definition of a weak derivatives, this means that

$$\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) = f \in L^2_{\text{loc}}(\mathbb{R}^d \times \mathbb{R}^+),$$

and the proof of part (a) in Definition 11 is complete. Moreover we have found that

$$\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u_n) \rightharpoonup \sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) \quad \text{in } L^2(\mathcal{K}),$$

for any $k = 1, \dots, K$ and compact $\mathcal{K} \subset \mathbb{R}^d \times \mathbb{R}^+$.

To show the chain rule in part (b) of Definition 11, we start from the chain rule for u_n ,

$$(87) \quad \sum_{i=1}^d \partial_{x_i} \zeta_{ik}^\beta(u_n) = \beta(u_n) \sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u_n) \in L^2(\mathbb{R}^d \times \mathbb{R}^+),$$

valid for any $\beta \in C(\mathbb{R})$, $k = 1, \dots, K$, and integer n . Recall also that

$$\zeta_{ik}^\beta(u_n) = \int_0^{u_n} \sigma_{ik}^A(\xi) \beta(\xi) d\xi.$$

By the previous convergence results and bounds, the right-hand side of (87) converges weakly to $\beta(u) \sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u)$ in $L^2(\mathcal{K})$. We can argue as before to show that the left-hand side converges weakly in $L^2(\mathcal{K})$ to $\sum_{i=1}^d \partial_{x_i} \zeta_{ik}^\beta(u)$. We thus get part (b) of Definition 11 in the limit. Moreover,

$$(88) \quad \sum_{i=1}^d \partial_{x_i} \zeta_{ik}^\beta(u_n) \rightharpoonup \sum_{i=1}^d \partial_{x_i} \zeta_{ik}^\beta(u) \quad \text{in } L^2(\mathcal{K}),$$

for any $\beta \in C(\mathbb{R})$, $k = 1, \dots, K$, and compact $\mathcal{K} \subset \mathbb{R}^d \times \mathbb{R}^+$.

Now, it remains to prove that the limiting function u satisfies the entropy inequalities in Definition 11(c). Let $\eta \in C^2(\mathbb{R})$ be a convex entropy with corresponding fluxes (q, r) , $\phi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+)$ a nonnegative test function, and consider the entropy inequality for u . From the previous results, it is standard to pass to the limit in the left-hand side of the inequality in Definition 11(c). The only difficulty concerns the right-hand side, because we have a quadratic term and mere weak convergence. Let us therefore only focus on this limit and leave the others to the reader. We take $\beta = \sqrt{\eta''}$ and apply the chain rule Definition 11(b),

$$\begin{aligned} & \iint_{\mathbb{R}^d \times \mathbb{R}^+} \eta''(u_n) \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u_n) \right)^2 \phi dx dt \\ &= \iint_{\mathbb{R}^d \times \mathbb{R}^+} \eta''(u_n) \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u_n) \right) \left(\sum_{j=1}^d \partial_{x_j} \zeta_{jk}(u_n) \right) \phi dx dt \\ &= \iint_{\mathbb{R}^d \times \mathbb{R}^+} \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}^{\sqrt{\eta''}}(u_n) \right) \left(\sum_{j=1}^d \partial_{x_j} \zeta_{jk}^{\sqrt{\eta''}}(u_n) \right) \phi dx dt \\ &= \iint_{\mathbb{R}^d \times \mathbb{R}^+} \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}^{\sqrt{\eta''}}(u_n) \sqrt{\phi} \right)^2 dx dt \\ &= \sum_{k=1}^K \left\| \sum_{i=1}^d \partial_{x_i} \zeta_{ik}^{\sqrt{\eta''}}(u_n) \sqrt{\phi} \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}^+)}^2. \end{aligned}$$

But, by (88), we have for any $k = 1, \dots, K$,

$$\sum_{i=1}^d \partial_{x_i} \zeta_{ik}^{\sqrt{\eta''}}(u_n) \sqrt{\phi} \rightharpoonup \sum_{i=1}^d \partial_{x_i} \zeta_{ik}^{\sqrt{\eta''}}(u) \sqrt{\phi} \quad \text{in } L^2(\mathbb{R}^d \times \mathbb{R}^+).$$

It follows that

$$\left\| \sum_{i=1}^d \partial_{x_i} \zeta_{ik} \sqrt{\eta''}(u) \sqrt{\phi} \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}^+)} \leq \liminf_{n \rightarrow \infty} \left\| \sum_{i=1}^d \partial_{x_i} \zeta_{ik} \sqrt{\eta''}(u_n) \sqrt{\phi} \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}^+)},$$

that is

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \iint_{\mathbb{R}^d \times \mathbb{R}^+} \eta''(u_n) \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u_n) \right)^2 \phi \, dx \, dt \\ & \geq \sum_{k=1}^K \left\| \sum_{i=1}^d \partial_{x_i} \zeta_{ik} \sqrt{\eta''}(u) \sqrt{\phi} \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}^+)}^2 \\ & = \iint_{\mathbb{R}^d \times \mathbb{R}^+} \eta''(u) \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) \right)^2 \phi \, dx \, dt, \end{aligned}$$

where similar chain rule computations have been used for u . This is enough to pass to the limit in the entropy inequalities of Definition 11(b) and the proof is complete. \square

As a byproduct of the previous proof, we get the lemma below.

Lemma 95. *Assume (H2), $u_0 \in L^\infty(\mathbb{R}^d)$, and u is the entropy solution of (2). Then $\text{ess inf } u_0 \leq u \leq \text{ess sup } u_0$. Moreover, if v is the entropy solution with initial data v_0 , then $u_0 \geq v_0$ implies $u \geq v$.*

Proof. Take the previous sequence of solutions u_n with initial data u_0^n . Recall that u_0^n converges to u_0 in $L^1_{\text{loc}}(\mathbb{R}^d)$ and has values between $\text{ess inf } u_0$ and $\text{ess sup } u_0$ (see (84)). We then showed that u_n converges to u in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^d))$, $T \geq 0$. We can assume this convergence to hold a.e. without loss of generality, up to taking a subsequence if necessary. The fact that $\text{ess inf } u_0 \leq u \leq \text{ess sup } u_0$ is then obtained at the limit, since $\text{ess inf } u_0^n \leq u_n \leq \text{ess sup } u_0^n$ by the maximum principle (in $L^1 \cap L^\infty$). To get the comparison principle, note that u_0^n can be chosen as

$$u_0^n(x) := \begin{cases} u_0(x) & \text{if } -n \leq u_0(x) \leq n, \\ \pm n & \text{if } \pm u_0(x) > n. \end{cases}$$

It follows that v_0^n , similarly constructed, is less than u_0^n if $u_0 \geq v_0$. Applying the comparison principle (in $L^1 \cap L^\infty$) and passing to the limits, as before, completes the proof. \square

REFERENCES

- [1] N. Alibaud and C. Imbert. Fractional semi-linear parabolic equations with unbounded data. *Trans. Amer. Math. Soc.*, 361(5):2527–2566, 2009.
- [2] O. Alvarez, P. Hoch, Y. Le Bouar and R. Monneau. Dislocation Dynamics: Short-time Existence and Uniqueness of the Solution. *Arch. Ration. Mech. Anal.*, 181(3):449–504, 2006.
- [3] B. Andreianov and M. Maliki. A note on uniqueness of entropy solutions to degenerate parabolic equations in \mathbb{R}^d . *NoDEA Nonlinear Differential Equations Appl.*, 17:109–118, 2010.
- [4] M. Bardi and I. Capuzzo-Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellmann equations*. Systems & Control: Foundations & Applications, Birkhäuser, Boston, Basel, Berlin, 1997.
- [5] G. Barles. *Solutions de viscosité des équations de Hamilton-Jacobi*. Springer-Verlag, Paris, 1994.
- [6] G. Barles, P. Cardaliaguet, O. Ley and A. Montillet. Uniqueness results for nonlocal Hamilton-Jacobi equations. *J. Funct. Anal.*, 257(5):1261–1287, 2009.
- [7] G. Barles and E. R. Jakobsen. On the convergence rate of approximation schemes for Hamilton-Jacobi-Bellman equations. *M2AN Math. Model. Numer. Anal.*, 36(1):33–54, 2002.
- [8] G. Barles and E. R. Jakobsen. Error bounds for monotone approximation schemes for parabolic Hamilton-Jacobi-Bellman equations. *Math. Comp.*, 76(240): 1861–1893, 2007.

- [9] G. Barles and B. Perthame. Exit time problems in optimal control and vanishing viscosity method. *SIAM J. Control Optim.*, 26(5):1133–1148, 1988.
- [10] G. Barles, H. M. Soner and P. E. Souganidis. Front propagation and phase field theory. *SIAM J. Control Optim.*, 31(2):439–469, 1993.
- [11] M. Bendahmane and K. H. Karlsen. Renormalized entropy solutions for quasi-linear anisotropic degenerate parabolic equations. *SIAM J. Math. Anal.*, 36(2): 405–422, 2004.
- [12] P. Bénilan, J. Carrillo and P. Wittbold. Renormalized entropy solutions of scalar conservation laws. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 29(2):313–327, 2000.
- [13] Ph. Bénilan, M. G. Crandall and A. Pazy. *Nonlinear Evolution Equations Governed by Accretive Operators*. Book Manuscript, Besançon, 2001.
- [14] Ph. Bénilan, M. G. Crandall and M. Pierre. Solutions of the porous medium equation in R^N under optimal conditions on initial values. *Indiana Univ. Math. J.*, 33(1):51–87, 1984.
- [15] C. Berg and G. Forst. *Potential theory on locally compact abelian groups*. Additional book information: Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 87, Springer-Verlag, Berlin, Heidelberg, New York, 1975.
- [16] O. Bokanowski, N. Forcadel and H. Zidani. L^1 -error estimates for numerical approximations of Hamilton-Jacobi-Bellman equations in dimension 1. *Math. Comp.*, 79(271):1395–1426, 2010.
- [17] L. Caffarelli, M. G. Crandall, M. Kocan and A. Święch. On viscosity solutions of fully nonlinear equations with measurable ingredients. *Comm. Pure Appl. Math.*, 49(4):365–397, 1996.
- [18] J. Carrillo. Entropy Solutions for nonlinear Degenerate Problems. *Arch. Ration. Mech. Anal.*, 147(4):269–361, 1999.
- [19] G.-Q. Chen and E. DiBenedetto. Stability of entropy solutions to the Cauchy problem for a class of nonlinear hyperbolic-parabolic equations. *SIAM J. Math. Anal.*, 33:751–762, 2001.
- [20] G.-Q. Chen and B. Perthame. Well-posedness for non-isotropic degenerate parabolic-hyperbolic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 20(4):645–668, 2003.
- [21] M. G. Crandall, H. Ishii and P.-L. Lions. User’s guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.*, 27(1):1–67, 1992.
- [22] M. G. Crandall and P.-L. Lions. Viscosity solutions of Hamilton-Jacobi Equations. *Trans. Amer. Math. Soc.*, 277:1–42, 1983.
- [23] M. G. Crandall, P.-L. Lions and P. E. Souganidis. Maximal solutions and universal bounds for some partial differential equations of evolution. *Arch. Rational Mech. Anal.*, 105(2):163–190, 1989.
- [24] C. Dafermos. *Hyperbolic Conservation Laws in Continuum Physics*. Springer, 3rd ed., 2010.
- [25] E. DiBenedetto. *Degenerate parabolic equations*. Springer-Verlag, New-York, 1993.
- [26] J. Endal and E. R. Jakobsen. L^1 Contraction for Bounded (Non-integrable) Solutions of Degenerate Parabolic Equations. *SIAM J. Math. Anal.*, 46(6):3957–3982, 2014.
- [27] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*. Graduate Texts in Mathematics, 194. Springer-Verlag, New York, 2000.
- [28] L. C. Evans and S. Lenhart. The parabolic Bellman equation. *Nonlinear Anal.*, 5(7):765–773, 1981.
- [29] W. Fleming and H.M. Soner. *Controlled Markov Processes and Viscosity Solutions*. Springer-Verlag, New York, 1993.
- [30] H. Frid. Decay of Almost Periodic Solutions of Anisotropic Degenerate Parabolic-Hyperbolic Equations. Preprint, arXiv:1610.07958v3 [math.AP], 2017.
- [31] Y. Giga and M.-H. Sato. A level set approach to semicontinuous viscosity solutions for Cauchy problems. *Comm. Partial Differential Equations*, 26(5-6):813–839, 2001.
- [32] J. Goldstein. *Semigroups of Linear Operators and Applications*. Oxford U. Press, New York, 1985.
- [33] D. Goreac and O.-S. Serea. Discontinuous control problems for non-convex dynamics and near viability for singularly perturbed control systems. *Nonlinear Anal.*, 73(8):2699–2713, 2010.
- [34] D. Goreac and O.-S. Serea. Mayer and optimal stopping stochastic control problems with discontinuous cost. *J. Math. Anal. Appl.*, 380:327–342, 2011.
- [35] J.-B. Hiriart-Urruty and C. Lemaréchal. *Fundamentals of Convex Analysis*. Springer, 2001.
- [36] H. Ishii. On uniqueness and existence of viscosity solutions of fully nonlinear second order elliptic pde’s. *Comm. pure Appl. Math.*, 42:14–45, 1989.
- [37] R. Jensen. The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations. *Arch. Rational Mech. Anal.*, 101:1–27, 1988.
- [38] S. N. Kruzkov. First order quasilinear equations with several independent variables. *Math. Sb. (N.S.)* (in Russian), 81(123):228–255, 1970.
- [39] N.V. Krylov. On the rate of convergence of finite-difference approximations for Bellman’s equations. *St. Petersburg Math. J.*, 9(3):639–650, 1997.

- [40] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva. *Linear and quasilinear equations of parabolic type*. Translations of Mathematical Monographs, Vol. 23 American Mathematical Society, Providence, R.I., 1968.
- [41] C.-T. Lin and E. Tadmor. L^1 -stability and error estimates for approximate Hamilton-Jacobi solutions. *Numer. Math.*, 87(4):701–735, 2001.
- [42] P.-L. Lions, B. Perthame and E. Tadmor.: Formulation cinétique des lois de conservation scalaires multidimensionnelles. *C. R. Acad. Sci. Paris, Série I Math.* 312:97–102, 1991.
- [43] P.-L. Lions, B. Perthame and E. Tadmor. A kinetic formulation of multidimensional scalar conservation laws and related local equations. *J. Amer. Math. Soc.* 7:169–191, 1994.
- [44] N. Pogodaev. Estimates of the domain of dependence for scalar conservation laws. *J. Differ. Equ.*, 265(4):1654–1677, 2018.
- [45] C. M. Theobald. An inequality for the trace of the product of two symmetric matrices. *Math. Proc. Cambridge Philos. Soc.*, 77:265–267, 1975.
- [46] J. L. Vázquez. *The porous medium equation. Mathematical theory*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007.
- [47] L. Wang. On the regularity theory of fully nonlinear parabolic equations. II. *Comm. Pure Appl. Math.*, 45(2):141–178, 1992.
- [48] Z. Wu, J. Zhao, J. Yin and H. Li. *Nonlinear diffusion equations*. World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

(N. Alibaud) ENSMM, 26 CHEMIN DE L'ÉPITAPHE, 25030 BESANÇON CEDEX, FRANCE, AND,
LMB, UMR CNRS 6623, UNIVERSITÉ DE BOURGOGNE FRANCHE-COMTÉ (UBFC), FRANCE
E-mail address: nathael.alibaud@ens2m.fr
URL: <https://lmb.univ-fcomte.fr/Alibaud-Nathael>

(J. Endal) DEPARTMENT OF MATHEMATICAL SCIENCES, NORWEGIAN UNIVERSITY OF SCIENCE
AND TECHNOLOGY (NTNU), N-7491 TRONDHEIM, NORWAY
E-mail address: jorgen.endal@ntnu.no
URL: <http://folk.ntnu.no/jorgeen>

(E. R. Jakobsen) DEPARTMENT OF MATHEMATICS, NORWEGIAN UNIVERSITY OF SCIENCE AND
TECHNOLOGY (NTNU), N-7491 TRONDHEIM, NORWAY
E-mail address: espen.jakobsen@ntnu.no
URL: <http://folk.ntnu.no/erj>