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# New Interior Transmission Problem Applied to a Single Floquet-Bloch Mode Imaging of Local Perturbations in Periodic Media

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## Abstract.

This paper considers the imaging of local perturbations of an infinite penetrable periodic layer. A cell of this periodic layer consists of several bounded inhomogeneities situated in a known homogeneous media. We use a differential linear sampling method to reconstruct the support of perturbations without using the Green's function of the periodic layer nor reconstruct the periodic background inhomogeneities. The justification of this imaging method relies on the well-posedness of a nonstandard interior transmission problem, which until now was an open problem except for the special case when the local perturbation **did not** intersect the background inhomogeneities. The analysis of this new interior transmission problem is the main focus of this paper. We then complete the justification of our inversion method and present some numerical examples that confirm the theoretical behavior of the differential indicator function determining the reconstructable regions in the periodic layer.

## 1. Introduction

Nondestructive testing of period media is an important problem with grown interest since periodic material are part of many fascinating engineering structures with many technological use such as nanoglass. In many situation the periodicity of the healthy periodic material is complicated or difficult to model mathematically, hence computing its Green's function is computationally expensive or even impossible. On the other hand, when looking for flows in such complex media, the option of reconstructing everything, i.e. both periodic structure and defects, may not be viable. The approach used in this paper provides a criteria to reconstruct the support of anomalies without explicitly know or reconstruct the background. The imaging method is based on the generalized

linear sampling method which was first introduced in [3], [5]. This method falls in the class of qualitative approaches to inverse scattering. We refer the reader to [13] and [7] for a description of various aspects of such approaches. Qualitative methods have been applied to the imaging of many periodic structure, see [1], [2], [6], [9], [15], [11], [16], [17] for a sample of work. In the case of our problem, we use an adapted version of so-called differential linear sampling method which process the measured data against the data coming from the healthy background. The idea of using differential measurements for sampling methods was first introduced in [4] where the response of the background was measured, and was adapted to the case of locally perturbed periodic layers in [11], [18]. For the latter, the response of the periodic background does not need **to be measured**. It is replaced by the extraction of measurements associated with a single Floquet-Bloch mode to encode some differential behavior for the indicator functions. This extraction requires information only on the period size of the background. The justification of this method makes essential use of a non-standard interior transmission problem whose well-posedness was open, and this limited its use to the case when the defect does not intersect the inhomogeneous components of the background. In this paper we provide sufficient conditions for solvability of this non-standard interior transmission problem which allow us to design a differential imaging function for more general location of defects. Let us introduce the problem we consider here.

More specifically, we are concerned with nondestructive testing of a penetrable infinite layer in  $\mathbb{R}^d$ ,  $d = 2, 3$  which is periodic with respect to  $d - 1$  first variables. Let  $L_1, \dots, L_{d-1}, L_j > 0$ ,  $j = 1, \dots, d-1$  denote the periods of each of these  $d-1$  variables, respectively. The  $d - 1$  periodic refractive index of this periodic layer, denoted here by  $n_p$ , from physical consideration is a bounded function, has positive real part  $\text{Re}(n_p)$  and nonnegative imaginary part  $\text{Im}(n_p) \geq 0$ . Furthermore for simplicity we assume that this periodic layer is embedded in a homogeneous background with refractive index normalized to one, i.e.  $n_p = 1$  for  $|x_d| > h$  for some fixed  $h > 0$ . This is what we refer to as the healthy material. We assume that one or finitely many cells of the layer are locally damaged. This means that in a compactly supported region  $\omega$  (which can have multiple connected components) the refractive index differs from  $n_p$ . Let us call  $n$  the refractive index of the damaged layer (which is not any longer periodic), i.e.  $n \neq n_p$  only in  $\omega$ . The goal is to determine the support of the damaged region  $\omega$  by using the measured scattered field outside the layer due to appropriate incident fields (to become precise later). The challenging task however is to resolve  $\omega$  without an explicit knowledge of  $n_p$  (**which in practice can have a complicated form**) nor reconstructing it, but just using the fact that  $n_p$  is  $d - 1$  periodic with known periods  $L_1, \dots, L_{d-1}$  under some technical restriction which will be explained in the paper. **More specifically, our analysis can be carried through for a  $M$ -period truncation of the infinite layer (containing the local defect), for  $M$  large enough, which is then extended as  $ML$ -periodic layer. As it is shown in [10], this truncation is equivalent to approximating the problem in the Floquet-Bloch domain using uniform discretization of the Floquet-Bloch variable and a trapezoidal rule to approximate the discretized solution. However, let us remark that from the numerical**

point of view, our inversion algorithm does not require this  $ML$ -periodicity assumption. This is why it is indeed desirable to remove this technical assumption in our analysis. We also note that this assumption is not needed in [6] and [16] for the analysis of the inverse problem. The reason is that in these papers the local defect is reconstructed assuming that the periodic background is known, which is not the case in our paper, as mentioned above.

The paper is configured as follows. In the next section we formulate the direct and inverse problem, define the measurements operator and recall some of its properties which are essential to our imaging method. Section 3 is devoted to introducing the near field operator corresponding to a single Floquet-Bloch mode that enables us to use a differential imaging approach. Most importantly here we study the properties of this operator which bring up the new interior transmission problem. Section 4 deals with the analysis of this new interior transmission problem. In the last section we build the differential imaging function and study its behavior for various positions of defective regions. In addition, here we provide some numerical example showing the viability of our inversion method.

## 2. Formulation of the Problem

In this section we give a rigorous formulation of the direct and inverse scattering problem we consider here. In order to motivate the new interior transmission problem which is our main concern, we recall the differential linear sampling method that was first introduced in [18] (see also [11]). This method recovers the support of local perturbations of a periodic layer without needing to compute the Green's function of the periodic layer. However to do so we must make some technical mathematical restrictions aimed to preserve some kind of periodicity for the damaged layer. In particular, we truncate the damaged infinite periodic layer by considering  $M$  periods (with  $M$  large enough to contain the defect) and extend it periodically, yielding to a  $ML := (ML_1, \dots, ML_{d-1})$ -periodic layer. We call again  $n$  the refractive index of the  $ML$ -periodic extension of the truncated part. In this section we formulate rigorously this construction where we base our inversion algorithm. We remark that it is highly desirable to remove this mathematical artifact.

### 2.1. The Direct Scattering Problem

Here we adopt the notations from [11]. Recall that the parameter  $L := (L_1, \dots, L_{d-1}) \in \mathbb{R}^{d-1}$ ,  $L_j > 0$ ,  $j = 1, \dots, d-1$  refers to the periodicity of the media with respect to the first  $d-1$  variables and  $M := (M_1, \dots, M_{d-1}) \in \mathbb{N}^{d-1}$  refers to the number of periods in the truncated domain. A function defined in  $\mathbb{R}^d$  is called  $L$  periodic if it is periodic with period  $L$  with respect to the  $d-1$  first variables. We consider in the following  $ML$ -periodic Helmholtz equation (vector multiplications is to be understood component wise, i.e.  $ML = (M_1L_1, \dots, M_{d-1}L_{d-1})$ ). In this problem, the total field  $u$

satisfies

$$\begin{cases} \Delta u + k^2 n u = 0 & \text{in } \mathbb{R}^d, \quad d = 2, 3 \\ u \text{ is } ML\text{-periodic} \end{cases} \quad (1)$$

where  $k > 0$  is the *wave number*. We assume that the index of refraction  $n \in L^\infty(\mathbb{R}^d)$

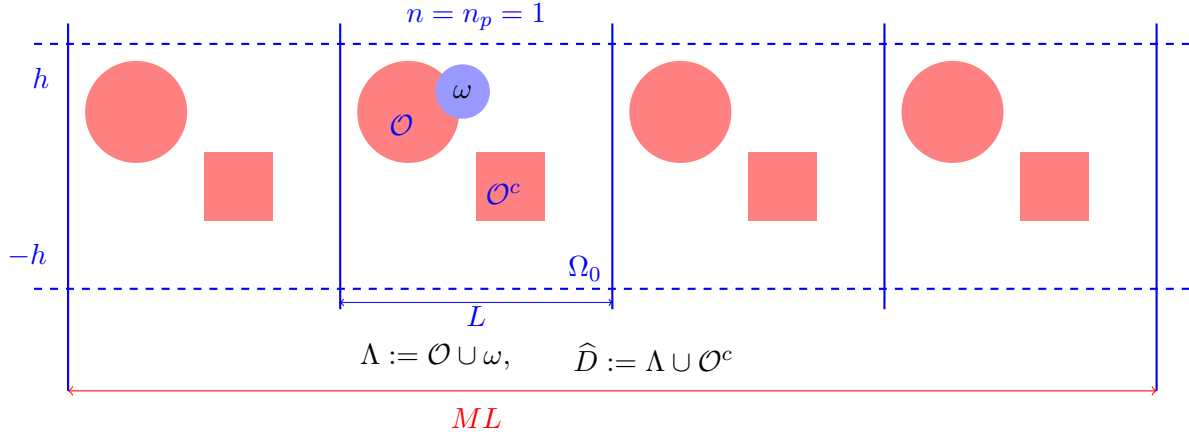


Figure 1: Sketch of the geometry for the  $ML$ -periodic problem

satisfies  $\text{Re}(n) \geq n_0 > 0$ ,  $\text{Im}(n) \geq 0$  and is  $ML$ -periodic. Furthermore  $n = n_p$  outside a compact domain  $\omega$  where  $n_p \in L^\infty(\mathbb{R}^d)$  is  $L$ -periodic, and in addition there exists  $h > 0$  such that  $n = 1$  for  $|x_d| > h$  (see Fig. 1). Thanks to the  $ML$ -periodicity, solving equation (1) in  $\mathbb{R}^d$  is equivalent to solving it in the period

$$\Theta := \bigcup_{m \in \mathbb{Z}_M^{d-1}} \Omega_m = \llbracket M_L^-, M_L^+ \rrbracket \times \mathbb{R}$$

with  $\Omega_m := \llbracket -\frac{L}{2} + mL, \frac{L}{2} + mL \rrbracket \times \mathbb{R}$ ,  $M_L^- := (\lfloor -\frac{M}{2} \rfloor + \frac{1}{2})L$ ,  $M_L^+ := (\lfloor \frac{M}{2} \rfloor + \frac{1}{2})L$ , and  $\mathbb{Z}_M^{d-1} := \{m \in \mathbb{Z}^{d-1}, \lfloor -\frac{M_\ell}{2} \rfloor + 1 \leq m_\ell \leq \lfloor \frac{M_\ell}{2} \rfloor, \ell = 1, \dots, d-1\}$ , where we use the notation  $\llbracket a, b \rrbracket := [a_1, b_1] \times \dots \times [a_{d-1}, b_{d-1}]$  and  $\lfloor \cdot \rfloor$  denotes the floor function. We also shall use the notation  $\llbracket a \rrbracket := |a_1 \cdot a_2 \cdot \dots \cdot a_{d-1}|$ . Without loss of generality we assume that there is a local perturbation  $\omega$  located in only one period, say  $\Omega_0$  (note the case when more periods are defective, the assumption holds true by grouping these cells as one cell with different period). This problem is treated in [18] under a strict assumption that the local perturbation does not intersect with the periodic background. In this work, we remove this assumption, and allow for the local perturbation to be located everywhere in  $\Omega_0$ . We call  $D_p$  the support of  $n_p - 1$  and  $D = D_p \cup \omega$ , note that  $n = 1$  outside  $D$ . For the justification of our inversion method (that relies on a unique continuation argument) we make the assumption that  $\mathbb{R}^d \setminus D$  is connected.

We consider down-to-up or up-to-down incident plane waves of the form

$$u^{i,\pm}(x, j) = \frac{-i}{2\beta_\#(j)} e^{i\alpha_\#(j) \cdot \bar{x} \pm i\bar{\beta}_\#(j) x_d} \quad (2)$$

where

$$\alpha_{\#}(j) := \frac{2\pi}{ML}j \quad \text{and} \quad \beta_{\#}(j) := \sqrt{k^2 - |\alpha_{\#}(j)|^2}, \quad \text{Im}(\beta_{\#}(j)) \geq 0, \quad j \in \mathbb{Z}^{d-1}$$

and  $x = (\bar{x}, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ . (Note that  $\alpha_{\#}(j)$  is a vector defined component-wise). Then the scattered field  $u^s = u - u^i$  satisfies

$$\begin{cases} \Delta u^s + k^2 n u^s = -k^2(n-1)u^i & \text{in } \mathbb{R}^d, \\ u^s \text{ is } ML\text{-periodic} \end{cases} \quad (3)$$

and we impose as a radiation condition the Rayleigh expansions:

$$\begin{cases} u^s(\bar{x}, x_d) = \sum_{\ell \in \mathbb{Z}^{d-1}} \widehat{u}^{s+}(\ell) e^{i(\alpha_{\#}(\ell) \cdot \bar{x} + \beta_{\#}(\ell)(x_d - h))}, & \forall x_d > h, \\ u^s(\bar{x}, x_d) = \sum_{\ell \in \mathbb{Z}^{d-1}} \widehat{u}^{s-}(\ell) e^{i(\alpha_{\#}(\ell) \cdot \bar{x} - \beta_{\#}(\ell)(x_d + h))}, & \forall x_d < -h, \end{cases} \quad (4)$$

where the Rayleigh coefficients  $\widehat{u}^{s\pm}(\ell)$  are given by

$$\begin{aligned} \widehat{u}^{s+}(\ell) &:= \frac{1}{\| [M_L^-, M_L^+] \|} \int_{[M_L^-, M_L^+]} u^s(\bar{x}, h) e^{-i\alpha_{\#}(\ell) \cdot \bar{x}} d\bar{x}, \\ \widehat{u}^{s-}(\ell) &:= \frac{1}{\| [M_L^-, M_L^+] \|} \int_{[M_L^-, M_L^+]} u^s(\bar{x}, -h) e^{-i\alpha_{\#}(\ell) \cdot \bar{x}} d\bar{x}. \end{aligned} \quad (5)$$

We shall use the notation

$$\Theta^h := [ [M_L^-, M_L^+] \times ] - h, h[$$

$$\Gamma_M^h := [ [M_L^-, M_L^+] \times \{h\}, \quad \Gamma_M^{-h} := [ [M_L^-, M_L^+] \times \{-h\}.$$

For integer  $m$ , we denote by  $H_{\#}^m(\Theta^h)$  the restrictions to  $\Theta^h$  of functions that are in  $H_{\text{loc}}^m(|x_d| \leq h)$  and are  $ML$ -periodic. The space  $H_{\#}^{1/2}(\Gamma_M^h)$  is then defined as the space of traces on  $\Gamma_M^h$  of functions in  $H_{\#}^1(\Theta^h)$  and the space  $H_{\#}^{-1/2}(\Gamma_M^h)$  is defined as the dual of  $H_{\#}^{1/2}(\Gamma_M^h)$ . Similar definitions are used for  $H_{\#}^{\pm 1/2}(\Gamma_M^{-h})$ .

More generally for a given  $f \in L^2(\Omega_M^h)$ , we consider the following problem: Find  $w \in H_{\#}^1(\Theta^h)$  satisfying

$$\Delta w + k^2 n w = k^2(1-n)f \quad (6)$$

together with the Rayleigh radiation condition (4). Then we make the following assumption:

**Assumption 1.** *The refractive index  $n$  and  $k > 0$  are such that (6) with  $n$  and with  $n$  replaced by  $n_p$  are both well-posed for all  $f \in L^2(\Omega_M^h)$ .*

We remark that the solution  $w \in H_{\#}^1(\Theta^h)$  of (6) can be extended to a function in  $\Theta$  satisfying  $\Delta w + k^2 n w = k^2(1-n)f$ , using the Rayleigh expansion (4). We denote by  $\Phi(n_p; \cdot)$  the fundamental solution to

$$\begin{cases} \Delta \Phi(n_p; \cdot) + k^2 n_p \Phi(n_p; \cdot) = -\delta_0, \\ \Phi(n_p; \cdot) \text{ is } ML\text{-periodic}, \\ \text{and the Rayleigh radiation condition (4)}. \end{cases} \quad (7)$$

Then  $w$  has the representation as

$$w(x) = - \int_D \left( k^2(n_p - n)w + k^2(1 - n)f \right)(y) \Phi(n_p; x - y) dy. \quad (8)$$

For sufficient conditions that **guarantee** Assumption 1 we refer the reader to [10], [14], [18] and the references therein.

## 2.2. The Inverse Problem

The inversion method is based on the so-called the Generalized Linear Sampling Method, which was first introduced in [3], [5] (see also [7, Chapter 2]), augmented with the idea of differential imaging introduced in [4] which was adapted to this problem in [18].

As described above we have two choices of interrogating waves. If we use down-to-up (scaled) incident plane waves  $u^{i,+}(x; j)$  defined by (2), then our measurements (data for the inverse problem) are given by the Rayleigh sequences

$$\widehat{u}^{s+}(\ell; j), \quad (j, \ell) \in \mathbb{Z}^{d-1} \times \mathbb{Z}^{d-1},$$

whereas if we use up-to-down (scaled) incident plane waves  $u^{i,-}(x; j)$  defined by (2) then our measurements are given the Rayleigh sequences

$$\widehat{u}^{s-}(\ell; j), \quad (j, \ell) \in \mathbb{Z}^{d-1} \times \mathbb{Z}^{d-1}.$$

These measurements define the so-called near field (or data) operator which is used to derive the indicator function of the defect. More specifically, let us consider the (Herglotz) operators  $\mathcal{H}^+ : \ell^2(\mathbb{Z}^{d-1}) \rightarrow L^2(D)$  and  $\mathcal{H}^- : \ell^2(\mathbb{Z}^{d-1}) \rightarrow L^2(D)$  defined by

$$\mathcal{H}^\pm a := \sum_{j \in \mathbb{Z}^{d-1}} a(j) u^{i,\pm}(\cdot; j)|_D, \quad \forall a = \{a(j)\}_{j \in \mathbb{Z}^{d-1}} \in \ell^2(\mathbb{Z}^{d-1}). \quad (9)$$

Then  $\mathcal{H}^\pm$  is compact and its adjoint  $(\mathcal{H}^\pm)^* : L^2(D) \rightarrow \ell^2(\mathbb{Z}^{d-1})$  is given by [11]

$$(\mathcal{H}^\pm)^* \varphi := \{\widehat{\varphi}^\pm(j)\}_{j \in \mathbb{Z}^{d-1}}, \quad \text{where} \quad \widehat{\varphi}_j^\pm := \int_D \varphi(x) \overline{u^{i,\pm}(\cdot; j)}(x) dx. \quad (10)$$

Let us denote by  $H_{\text{inc}}^\pm(D)$  the closure of the range of  $\mathcal{H}^\pm$  in  $L^2(D)$ . We then consider the (compact) operator  $G^\pm : H_{\text{inc}}^\pm(D) \rightarrow \ell^2(\mathbb{Z}^{d-1})$  defined by

$$G^\pm(f) := \{\widehat{w}^\pm(\ell)\}_{\ell \in \mathbb{Z}^{d-1}}, \quad (11)$$

where  $\{\widehat{w}^\pm(\ell)\}_{\ell \in \mathbb{Z}^{d-1}}$  is the Rayleigh sequence of  $w \in H_{\#}^1(\Theta^h)$  the solution of (6). We now define the sampling operators  $N^\pm : \ell^2(\mathbb{Z}^{d-1}) \rightarrow \ell^2(\mathbb{Z}^{d-1})$  by

$$N^\pm(a) = G^\pm \mathcal{H}^\pm(a). \quad (12)$$

By linearity of the operators  $G^\pm$  and  $\mathcal{H}^\pm$  we also get an equivalent definition of  $N^\pm$  directly in terms of measurements as

$$[N^\pm(a)]_\ell = \sum_{j \in \mathbb{Z}^{d-1}} a(j) \widehat{u}^{s,\pm}(\ell; j), \quad \ell \in \mathbb{Z}^{d-1}. \quad (13)$$

The following properties of  $G^\pm$  and  $\mathcal{H}^\pm$  are crucial to our inversion method. To state them, we must recall the standard *interior transmission problem*:  $(u, v) \in L^2(D) \times L^2(D)$  such that  $u - v \in H^2(D)$  and

$$\begin{cases} \Delta u + k^2 n u = 0 & \text{in } D, \\ \Delta v + k^2 v = 0 & \text{in } D, \\ u - v = \varphi & \text{on } \partial D, \\ \partial(u - v)/\partial\nu = \psi & \text{on } \partial D, \end{cases} \quad (14)$$

for given  $(\varphi, \psi) \in H^{3/2}(\partial D) \times H^{1/2}(\partial D)$  where  $\nu$  denotes the outward normal on  $\partial D$ .  $k$  is called a *transmission eigenvalue* if the homogeneous problem (14), i.e. with  $\varphi = 0$  and  $\psi = 0$ , has non-trivial solutions. Up-to-date results on this problem can be found in [7, Chapter 3] where in particular one finds sufficient solvability conditions. In the sequel we make the following assumption. If the boundary of  $D$  intersects the boundary of  $\Omega_0$ , then the previous interior transmission problem should be augmented with periodicity conditions on  $\partial D \cap \partial\Theta$ . Since this condition does not affect the assumptions on the solvability of the interior transmission problem (in  $H^2(D)$  with periodic conditions on  $\partial D \cap \partial\Theta$ ) nor requires any substantial modification of the arguments below (other than changing the solution space), we make the choice of simplifying this technicality and assume that  $\partial D \cap \partial\Omega_0 = \emptyset$ .

**Assumption 2.**  $\partial D \cap \partial\Omega_0 = \emptyset$  and the refractive index  $n$  and the wave number  $k > 0$  are such that (14) has a unique solution.

In particular, if  $\operatorname{Re}(n - 1) > 0$  or  $-1 < \operatorname{Re}(n - 1) < 0$  uniformly in a neighborhood of  $\partial D$  inside  $D$  the interior transmission problem (14) satisfies the Fredholm alternative, and the set of real standard transmission eigenvalues is discrete (possibly empty). Thus Assumption 2 holds as long as  $k > 0$  is not a transmission eigenvalue.

From now on, for  $z \in \Theta^h$ , we denote by  $\widehat{\Phi}^\pm(\cdot; z) := \{\widehat{\Phi}^\pm(\ell; z)\}_{\ell \in \mathbb{Z}^{d-1}}$  the Rayleigh sequences of  $\Phi(n_p, z)$  with  $n_p = 1$  define in (7) given by

$$\widehat{\Phi}^\pm(\ell; z) := \frac{i}{2\llbracket ML \rrbracket \beta_\#(\ell)} e^{-i(\alpha_\#(\ell)\bar{z} - \beta_\#(\ell)|z_d \mp h|)}. \quad (15)$$

**Lemma 2.1** (Lemma 3.3 in [11]). *The operator  $\mathcal{H}^\pm$  is compact and injective. Let  $H_{\text{inc}}^\pm(D)$  be the closure of the range of  $\mathcal{H}^\pm$  in  $L^2(D)$ . Then*

$$H_{\text{inc}}^\pm(D) = H_{\text{inc}}(D) := \{v \in L^2(D) : \Delta v + k^2 v = 0 \text{ in } D\}. \quad (16)$$

**Theorem 2.2** (Theorem 3.5 in [11]). *Assume that Assumptions 1 and 2 hold. Then the operator  $G^\pm : H_{\text{inc}}(D) \rightarrow \ell^2(\mathbb{Z}^{d-1})$  defined by (11) is injective with dense range. Moreover  $\widehat{\Phi}^\pm(\cdot; z)$  belongs to  $\mathcal{R}(G^\pm)$  if and only if  $z \in D$ .*

Another main ingredient is a symmetric factorization of an appropriate operator given in terms of  $N^\pm$ . To this end, for a generic operator  $F : H \rightarrow H$ , where  $H$  is a Hilbert space, with adjoint  $F^*$  we define

$$F_\# := |\operatorname{Re}(F)| + |\operatorname{Im}(F)| \quad (17)$$



where  $\operatorname{Re}(F) := \frac{1}{2}(F + F^*)$ ,  $\operatorname{Im}(F) := \frac{1}{2i}(F - F^*)$ .  
 Now if  $T : L^2(D) \rightarrow L^2(D)$  is defined by

$$Tf := k^2(n-1)(f + w|_D) \quad (18)$$

with  $w$  being the solution of (6), we have the following result under **Assumptions 1 and 2**.

**Theorem 2.3** (Theorem 4.2 in [11]). *The following factorization holds*

$$N_{\#}^{\pm} = (\mathcal{H}^{\pm})^* T_{\#} \mathcal{H}^{\pm}, \quad (19)$$

where  $T_{\#} : L^2(D) \rightarrow L^2(D)$  is self-adjoint and coercive on  $H_{\text{inc}}(D)$ . Moreover,  $z \in D$  if and only if  $\widehat{\Phi}^{\pm}(\cdot; z) \in \mathcal{R}((N_{\#}^{\pm})^{1/2})$ .

The above theorem provides a rigorous method to recover the support of  $D$ . However this is not satisfactory since the aim is find only the support of  $\omega$  and trying to reconstruct everything may not be feasible due to possible complicated structure of the periodic media and even useless if  $\omega \subset D_p$ . Our goal is to derive an imaging method that resolves only  $\omega$  without knowing or recovering  $D_p$ . This leads us to introducing next the sampling operator for a single Floquet-Bloch mode whose analysis will bring up a new interior transmission problem.

We end this section by introducing some more notations to be used in the sequel.

**Definition 2.4.** *A function  $u$  is called quasi-periodic with parameter  $\xi = (\xi_1, \dots, \xi_{d-1})$  and period  $L = (L_1, \dots, L_{d-1})$ , with respect to the first  $d-1$  variables (briefly denoted as  $\xi$ -quasi-periodic with period  $L$ ) if:*

$$u(\bar{x} + jL, x_d) = e^{i\xi \cdot (jL)} u(\bar{x}, x_d), \quad \forall j \in \mathbb{Z}^{d-1}.$$

Let  $q$  be a fixed parameter in  $\mathbb{Z}_M^{d-1}$ , we denote by  $\Phi_q(x)$  the solution to

$$\Delta \Phi_q(x) + k^2 \Phi_q(x) = -\delta_0 \quad \text{in } \Omega_0 \quad (20)$$

and is  $\alpha_q$  quasi-periodic with period  $L$  with  $\alpha_q := 2\pi q / (ML)$ . Then  $\Phi_q(\cdot - z)$  remains the solution to

$$\Delta \Phi_q(\cdot - z) + k^2 \Phi_q(\cdot - z) = -\delta_z \quad \text{in } \Omega_0$$

for all  $z \in \mathbb{R}^d$ . The Rayleigh coefficients  $\widehat{\Phi}_q^{\pm}(\cdot; z)$  of  $\Phi_q(\cdot - z)$  are given by

$$\widehat{\Phi}_q^{\pm}(j; z) = \begin{cases} \frac{i}{2\llbracket L \rrbracket \beta_{\#}(q+M\ell)} e^{-i(\alpha_{\#}(q+M\ell)\bar{z} - \beta_{\#}(q+M\ell)|z_d \mp h|)} & \text{if } j = q + M\ell, \ell \in \mathbb{Z}^{d-1}, \\ 0 & \text{if } j \neq q + M\ell, \ell \in \mathbb{Z}^{d-1}. \end{cases} \quad (21)$$

Furthermore we assume that each period of  $D_p$  is composed by  $J \in \mathbb{N}$  disconnected components and the defect  $\omega$  as well as the components that contains or have non-empty intersection with  $\omega$  are in one cell, which we denote by  $\Omega_0$  (otherwise we could rearrange the period). For convenience, we now introduce some additional notations. We denote

by  $\mathcal{O}$  the union of the components of  $D_p \cap \Omega_0$  that have nonempty intersection with  $\omega$ , and by  $\mathcal{O}^c$  its complement in  $D_p \cap \Omega_0$ , i.e the union of all the components of  $D_p \cap \Omega_0$  that do not intersect  $\omega$ . Furthermore, we denote by  $\Lambda := \mathcal{O} \cup \omega$  and by  $\widehat{D} := \Lambda \cup \mathcal{O}^c$ . Obviously,  $\widehat{D} = D \cap \Omega_0$ . (see also Fig. 1 and note that if  $\omega$  does not intersect with  $D_p$  then  $\mathcal{O} \equiv \emptyset$ ,  $\mathcal{O}^c \equiv D_p \cap \Omega_0$  and  $\Lambda = \omega$ ). Let us denote by  $\nu_m := (mL, 0) \in \mathbb{R}^d$ , the translate vector from  $\Omega_0$  to  $\Omega_m$ ,  $m \in \mathbb{Z}^{d-1}$ . We consider the following  $L$ -periodic copies of the aforementioned regions

$$\mathcal{O}_p^c = \bigcup_{m \in \mathbb{Z}_M^{d-1}} \mathcal{O}^c + \nu_m, \quad \Lambda_p := \bigcup_{m \in \mathbb{Z}_M^{d-1}} \Lambda + \nu_m \quad \text{and} \quad \widehat{D}_p := \bigcup_{m \in \mathbb{Z}_M^{d-1}} \widehat{D} + \nu_m \quad (22)$$

Remark that  $\widehat{D}_p \equiv D_p \cup (\cup_{m \in \mathbb{Z}_M^{d-1}} \omega + \nu_m)$  contains  $D$  and the  $L$ -periodic copies of  $\omega \setminus D_p$ . We remark that  $n = n_p = 1$  in  $\widehat{D}_p \setminus D$ .

### 3. The Near Field Operator for a Single Floquet-Bloch Mode

Let us define the operator  $I_q : \ell^2(\mathbb{Z}^{d-1}) \rightarrow \ell^2(\mathbb{Z}^{d-1})$  given by

$$(I_q a)(\ell) = \begin{cases} a(j), & \text{if } \ell = q + jM \\ 0, & \text{else.} \end{cases} \quad (23)$$

The adjoint  $I_q^* : \ell^2(\mathbb{Z}^{d-1}) \rightarrow \ell^2(\mathbb{Z}^{d-1})$  is then  $(I_q^* b)(j) = b(q + jM)$ . The single Floquet-Bloch mode Herglotz operator  $\mathcal{H}_q^\pm : \ell^2(\mathbb{Z}^{d-1}) \rightarrow L^2(D)$  is defined by

$$\mathcal{H}_q^\pm a := \mathcal{H}^\pm I_q a = \sum_j a(j) u^{i,\pm}(\cdot; q + jM)|_D \quad (24)$$

and the single Floquet-Bloch mode near field (or data) operator  $N_q^\pm : \ell^2(\mathbb{Z}^{d-1}) \rightarrow \ell^2(\mathbb{Z}^{d-1})$  is defined by

$$N_q^\pm a = I_q^* N^\pm I_q a. \quad (25)$$

We remark that  $\mathcal{H}_q^\pm a$  is an  $\alpha_q$ -quasi-periodic function with period  $L$ . The sequence  $N_q^\pm a$  corresponds to the Fourier coefficients of the  $\alpha_q$ -quasi-periodic component of the scattered field in the decomposition (30). This operator is then somehow associated with  $\alpha_q$ -quasi-periodicity. One immediately sees from the factorization  $N^\pm = (\mathcal{H}^\pm)^* T \mathcal{H}^\pm$  that the following factorization holds.

$$N_q^\pm = (\mathcal{H}_q^\pm)^* T \mathcal{H}_q^\pm. \quad (26)$$

For later use we also define the operator  $G_q^\pm : \overline{\mathcal{R}(\mathcal{H}_q^\pm)} \rightarrow \ell^2(\mathbb{Z}^{d-1})$  by

$$G_q^\pm = (\mathcal{H}_q^\pm)^* T|_{\overline{\mathcal{R}(\mathcal{H}_q^\pm)}} \quad (27)$$

where the operator  $T$  is defined by (18).

Observing that

$$\varphi(j; \bar{x}) := e^{i\alpha_\#(j) \cdot \bar{x}} = e^{i\frac{2\pi}{ML} j \cdot \bar{x}}, \quad j \in \mathbb{Z}^{d-1}$$

is a Fourier **basis** of  $ML$  periodic function in  $L^2(\Theta)$ , we have that any  $w \in L^2(\Theta)$  which is  $ML$  periodic, has the expansion

$$w(x) = \sum_{j \in \mathbb{Z}^{d-1}} \widehat{w}(j, x_d) \varphi(j; \bar{x}), \quad \text{where} \quad \widehat{w}(j, x_d) := \frac{1}{\llbracket ML \rrbracket} \int_{\Theta} w(x) \overline{\varphi(j; \bar{x})} \, d\bar{x}. \quad (28)$$

Splitting  $j$  by module  $M$  we can arrange the expansion of  $w$  as

$$w(x) = \sum_{q \in \mathbb{Z}_M^{d-1}} \left( \sum_{\ell \in \mathbb{Z}^{d-1}} \widehat{w}(q + M\ell, x_d) \varphi(q + M\ell; \bar{x}) \right), \quad (29)$$

where  $\varphi(q + M\ell; \bar{x})$  is  $\alpha_q$ -quasi-periodic with period  $L$ , here  $\alpha_q := \frac{2\pi}{ML}q$ . Letting

$$w_q := \sum_{\ell \in \mathbb{Z}} \widehat{w}(q + M\ell, x_d) \varphi(q + M\ell; \bar{x})$$

we have that  $w_q$  is  $\alpha_q$ -quasi-periodic with period  $L$ . Thus any  $ML$ -periodic function  $w \in L^2(\Theta)$  can be decomposed

$$w = \sum_{q \in \mathbb{Z}_M} w_q \quad (30)$$

where  $w_q$  is  $\alpha_q$ -quasi-periodic with period  $L$ . Moreover, by the orthogonality of the Fourier **basis**  $\{\varphi(j; \cdot)\}_{j \in \mathbb{Z}}$ , we have that

$$\widehat{w}_q^\pm(j) = 0 \quad \text{if } j \neq q + M\ell, \ell \in \mathbb{Z} \quad \text{and} \quad \widehat{w}^\pm(q + M\ell) = \widehat{w}_q^\pm(q + M\ell) \quad (31)$$

where  $\widehat{w}_q^\pm(j)$  the Rayleigh sequence of  $w_q$  defined in (5). By definition of  $G_q^\pm$ , we see that  $G_q^\pm(f)$  is a Rayleigh sequence of  $\widehat{w}^\pm(j)$  at all indices  $j = q + M\ell, \ell \in \mathbb{Z}$ , where  $w$  is solution of (6). Seeing also the line above that these coefficients come from the Rayleigh sequence of  $w_q$  where  $w_q$  is one of the component of  $w$  using the decomposition (30), which is  $\alpha_q$ -quasi-periodic. We now assume that  $f|_{D_p}$  is  $\alpha_q$ -quasi-periodic. Then, using the decomposition (30) for  $w$ , and that fact that  $n_p$  is periodic,  $f$  is  $\alpha_q$ -quasi-periodic and  $n - n_p$  is compactly supported in one period  $\Omega_0$ , (6) becomes

$$\Delta w_q + k^2 n_p w_q = k^2 (n_p - n) w + k^2 (1 - n) f \quad \text{in } \Omega_0. \quad (32)$$

Denoting by  $\widetilde{w} := w - w_q$ , the previous equation is equivalent to

$$\Delta w_q + k^2 n w_q = k^2 (n_p - n) \widetilde{w} + k^2 (1 - n) f \quad \text{in } \Omega_0. \quad (33)$$

Therefore, operator  $G_q^\pm : \overline{\mathcal{R}(\mathcal{H}_q^\pm)} \rightarrow \ell^2(\mathbb{Z}^{d-1})$  can be equivalently defined as

$$G_q^\pm(f) := \mathbf{I}_q^* \{ \widehat{w}_q^\pm(\ell) \}_{\ell \in \mathbb{Z}^{d-1}}, \quad (34)$$

where  $w_q$  solution of (33) and  $w_q + \widetilde{w}$  is solution of (6). This definition is helpful for proving the following properties for  $\mathcal{H}_q^\pm$  and  $G_q^\pm$  that are the counterpart results to Lemma 2.1 and Lemma 2.2, now needed for the operator  $N_q^\pm$ .

**Lemma 3.1.** *The operator  $\mathcal{H}_q^\pm$  is injective and*

$$\overline{\mathcal{R}(\mathcal{H}_q^\pm)} = H_{\text{inc}}^q(D) := \{v \in L^2(D), \quad \Delta v + k^2 v = 0 \text{ in } D \text{ and } v|_{D_p} \text{ is } \alpha_q\text{-quasi-periodic}\}.$$

*Proof.* The proof of this lemma follows the lines of the proof of Lemma 5.1 in [11] slightly modified to account for more general location of the defect.  $\mathcal{H}_q^\pm$  is injective since  $\mathcal{H}^\pm$  is injective and  $I_q$  is injective. We now prove that  $(\mathcal{H}_q^\pm)^*$  is injective on  $H_{\text{inc}}^q(D)$ . Let  $\varphi \in H_{\text{inc}}^q(D)$  and assume  $(\mathcal{H}_q^\pm)^*(\varphi) = 0$ . We define

$$u(x) := \frac{1}{\llbracket M \rrbracket} \int_D \Phi_q(x-y)\varphi(y) \, dy.$$

where  $\Phi_q(x)$  has expansion

$$\Phi_q(x) = \frac{i}{2\llbracket L \rrbracket} \sum_{\ell \in \mathbb{Z}^{d-1}} \frac{1}{\beta_\#(q+M\ell)} e^{i\alpha_\#(q+M\ell)\cdot\bar{x} + i\beta_\#(q+M\ell)|x_d|}. \quad (35)$$

with  $\alpha_\#(q+M\ell) = \frac{2\pi}{ML}(q+M\ell)$  and  $\beta_\#(q+M\ell) = \sqrt{k^2 - |\alpha_\#(q+M\ell)|^2}$ . By definition of  $u$  and the expansion of  $\Phi_q$  we have that

$$\begin{aligned} \widehat{u}^+(j) &= \frac{1}{\llbracket M_L^-, M_L^+ \rrbracket} \int_{\llbracket M_L^-, M_L^+ \rrbracket} \left[ \frac{1}{\llbracket M \rrbracket} \int_D \Phi_q((\bar{x}, h) - y)\varphi(y) \, dy \right] e^{-i\alpha_\#(j)\cdot\bar{x}} \, d\bar{x} \\ &= \begin{cases} \int_D \varphi(y) \frac{i}{2\beta_\#(j)} e^{i\alpha_\#(j) - i\beta_\#(j)(x_d - h)} = ((\mathcal{H}^+)^*(\varphi))(j), & \text{if } j = q + M\ell \\ 0 & \text{if } j \neq q + M\ell \end{cases} \end{aligned} \quad (36)$$

which implies that  $\widehat{u}^\pm(j) = 0$  for all  $j \neq q + M\ell$  and  $\widehat{u}^\pm(q + M\ell) = ((\mathcal{H}^\pm)^*(\varphi))(q + M\ell) = ((\mathcal{H}_q^\pm)^*(\varphi))(\ell) = 0$ . Therefore  $u$  has all Rayleigh coefficients equal 0, which implies that

$$u = 0, \quad \text{for } \pm x_d > h.$$

We now observe that for all  $y \in D$ ,  $\Delta\Phi_q(\cdot - y) + k^2\Phi_q(\cdot - y) = 0$  in the complement of  $\widehat{D}_p$ . This implies that

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^d \setminus \widehat{D}_p.$$

Using a unique continuation argument we infer that  $u = 0$  in  $\Theta \setminus \widehat{D}_p$ . Therefore,  $u \in H_0^2(\widehat{D}_p)$  by the regularity of volume potentials. We now consider two cases:

If  $\omega \subset D_p$ , then  $\widehat{D}_p \equiv D_p$ , i.e  $u \in H_0^2(D_p)$ . Moreover, by definition,  $u$  verifies  $\Delta u + k^2 u = -\varphi$  in  $D_p$  and using the fact that  $\Delta\varphi + k^2\varphi = 0$  in  $D_p$  we finally have

$$-\|\varphi\|_{L^2(D_p)}^2 = \int_{D_p} (\Delta u + k^2 u)\bar{\varphi} \, dx = \int_{D_p} u(\Delta\bar{\varphi} + k^2\bar{\varphi}) \, dx = 0. \quad (37)$$

This proves that  $\varphi = 0$  and yields the injectivity of  $(\mathcal{H}_q^\pm)^*$  on  $H_{\text{inc}}^q(D)$ .

If  $\omega \not\subset D_p$ , let denote by  $\omega^c := \omega \setminus D_p$  then  $\omega^c \neq \emptyset$ . We then rewrite  $u(x)$  as

$$u(x) := \frac{1}{\llbracket M \rrbracket} \int_{\omega^c} \Phi_q(x-y)\varphi(y) \, dy + \frac{1}{\llbracket M \rrbracket} \int_{D_p} \Phi_q(x-y)\varphi(y) \, dy.$$

Since  $\varphi|_{D_p}$  and  $\Phi_q(\cdot)$  are  $\alpha_q$ -quasi-periodic functions with period  $L$ , then for  $x \in D_p$  we have

$$\int_{D_p \cap \Omega_m} \Phi_q(x-y)\varphi(y) \, dy = \int_{D_p \cap \Omega_0} \Phi_q(x-y)\varphi(y) \, dy, \quad \forall m \in \mathbb{Z}_M^{d-1}.$$

Therefore, for  $x \in D_p \cap \Omega_m$ .

$$u(x) := \frac{1}{\llbracket M \rrbracket} \int_{\omega^c} \Phi_q(x-y)\varphi(y) \, dy + \int_{D_p \cap \Omega_m} \Phi_q(x-y)\varphi(y) \, dy \quad (38)$$

We recall that  $\Delta\Phi_q(\cdot-x) + k^2\Phi_q(\cdot-x) = -\delta_x$  in  $\Omega_m$  and  $\Delta\Phi_q(\cdot-x) + k^2\Phi_q(\cdot-x) = 0$  in  $\omega^c$  (by quasi-periodicity of  $\Phi_q(\cdot-x)$ ). Hence, from (38) we obtain that for  $m \in \mathbb{Z}_M^{d-1}$ ,

$$\Delta u(x) + k^2 u(x) = -\varphi(x) \text{ in } D_p \cap \Omega_m. \quad (39)$$

Let us set for  $m \in \mathbb{Z}_M^{d-1}$

$$\varphi_m(x) := e^{i\alpha_q \cdot mL} \varphi(x - mL) \text{ for } x \in \omega^c + mL.$$

Then we have, using the  $\alpha_q$ -quasi-periodicity of  $\Phi_q(\cdot)$  that for  $x \in \omega^c + mL$

$$u(x) := \frac{1}{\llbracket M \rrbracket} \int_{\omega^c + mL} \Phi_q(x-y)\varphi_m(y) \, dy + \frac{1}{\llbracket M \rrbracket} \int_{D_p} \Phi_q(x-y)\varphi(y) \, dy$$

where  $\Phi_q(\cdot-x) + k^2\Phi_q(\cdot-x) = -0$  in  $D_p$  and  $\Delta\Phi_q(\cdot-x) + k^2\Phi_q(\cdot-x) = -\delta_x$  in  $\omega^c + mL$ . We then get

$$\Delta u(x) + k^2 u(x) = -\varphi_m \text{ in } \omega^c + mL. \quad (40)$$

Now define the function  $\tilde{\varphi}$  by

$$\tilde{\varphi} = \varphi \text{ in } D_p \text{ and } \tilde{\varphi} = \varphi_m \text{ in } \omega^c + mL \quad \text{with} \quad m \in \mathbb{Z}_M^{d-1}.$$

Clearly, we have

$$\Delta \tilde{\varphi} + k^2 \tilde{\varphi} = 0 \text{ in } \widehat{D}_p.$$

Since  $u \in H_0^2(\widehat{D}_p)$  we then have

$$\int_{D_p} (\Delta u + k^2 u) \tilde{\varphi} = 0.$$

This implies according to (39) and (40) that

$$\int_{D_p} |\varphi|^2 \, dx + \llbracket M \rrbracket \int_{\omega^c} |\varphi|^2 = 0,$$

which implies  $\varphi = 0$  in  $D$ . This proves the injectivity of  $(\mathcal{H}^\pm)^*$  on  $H_{\text{inc}}^q(D)$  and hence proves the Lemma.  $\square$

The following theorem concerning the properties  $G_q^\pm$  requires the solvability of a new interior transmission problem (to be formulated later), which up to this study was an open problem except for the case when  $\omega \cap D_p = \emptyset$  investigated in [18].

**Assumption 3.** *The refractive index  $n$  and  $k > 0$  are such that the new interior transmission problem defined in Definition 3.3 has a unique solution.*

Section 4 is dedicated to derive sufficient conditions for which Assumption 3 holds true.

**Theorem 3.2.** *Suppose that Assumptions 1, 2 and 3 hold. Then the operator  $G_q^\pm : H_{\text{inc}}^q(D) \rightarrow \ell^2(\mathbb{Z}^{d-1})$  is injective with dense range.*

*Proof.* To prove the injectivity of  $G_q$  we assume that  $f \in H_{\text{inc}}^q(D)$  such that  $G_q(f) = 0$ . Let  $w$  be solution of (6) with data  $f$ . Since  $f|_{D_p}$  is  $\alpha_q$ -quasi-periodic and  $n_p - n$  is compactly supported of period  $\Omega_0$  we then have from (34) that

$$G_q^\pm(f) := \mathbb{I}_q^* \{ \widehat{w}_q^\pm(\ell) \}_{\ell \in \mathbb{Z}^{d-1}},$$

where  $\{ \widehat{w}_q^\pm(\ell) \}_{\ell \in \mathbb{Z}^{d-1}}$  is the Rayleigh sequence of  $w_q$  and  $w_q$  is solution to

$$\Delta w_q + k^2 n w_q = k^2 (n_p - n)(w - w_q) + k^2 (1 - n)f \quad \text{in } \Omega_0 \quad (41)$$

In particular we have that  $\Delta w_q + k^2 w_q = 0$  in  $\Theta \setminus \widehat{D}_p$  (we recall notations in (22)). Using a similar unique continuation argument as at the beginning of the proof of Lemma 3.1 we deduce that

$$w_q = 0 \quad \text{in } \Theta \setminus \widehat{D}_p.$$

Actually, if  $\omega \subset D_p$  then  $\widehat{D}_p \equiv D_p$  thus  $f|_{\widehat{D}_p}$  is  $\alpha_q$ -quasi-periodic. However, if  $\omega \not\subset D_p$ ,  $f|_{\widehat{D}_p}$  is not  $\alpha_q$ -quasi-periodic. To restore  $\alpha_q$ -quasi-periodicity, we introduce **another** function  $\widetilde{f}$  by

$$\widetilde{f} := \begin{cases} f & \text{in } \Theta \setminus \Lambda_p \\ e^{i\alpha_q m L} f|_\Lambda & \text{in } \mathcal{O} + mL, \quad \forall m \in \mathbb{Z}_M. \end{cases} \quad (42)$$

i.e we keep  $f$  the same outside  $\Lambda_p$  and extend the values of  $f$  in  $\Lambda$  by  $\alpha_q$ -quasi-periodicity to  $\Lambda_p$ . Since  $f|_{D_p}$  is  $\alpha_q$ -quasi-periodic then the definition of  $\widetilde{f}$  implies that  $\widetilde{f} = f$  also in  $\Lambda_p \cap D_p$ . Therefore  $\widetilde{f} = f$  in  $D$  and  $\widetilde{f} = f$  in  $\Theta \setminus \widehat{D}_p$  (in other words  $\widetilde{f} \neq f$  in  $\widehat{D}_p \setminus \overline{D}$ ) and  $\widetilde{f}|_{\widehat{D}_p}$  is  $\alpha_q$ -quasi-periodic with period  $L$ . We remark that in the special case when  $\omega \subset D_p$ ,  $\widetilde{f} \equiv f$  and  $\widehat{D}_p \equiv D_p$ . We now write (41) in terms of  $\widetilde{f}$  (since  $\widetilde{f} \equiv f$  in  $D$ )

$$\Delta w_q + k^2 n w_q = k^2 (n_p - n)(w - w_q) + k^2 (1 - n)\widetilde{f} \quad \text{in } \Omega_0, \quad (43)$$

hence it is enough to prove that  $\widetilde{f} = 0$  in  $\widehat{D}_p$ . To this end, using the fact that  $\widetilde{f} = f$  in  $\widehat{D}$  (recall that  $\widehat{D} = \Lambda \cup \mathcal{O}^c = \widehat{D}_p \cap \Omega_0 = D \cap \Omega_0$ ) then  $\widetilde{f}$  verifies

$$\Delta \widetilde{f} + k^2 \widetilde{f} = 0 \quad \text{in } \widehat{D}_p$$

and by the  $\alpha_q$ -quasi-periodicity of  $w_q$  and  $\tilde{f}$ , it is sufficient to prove that  $\tilde{f} = 0$  in the cell  $\Omega_0$ , i.e, proving that the following problem

$$\begin{cases} \Delta w_q + k^2 n w_q = k^2 (n_p - n)(w - w_q) + k^2 (1 - n) \tilde{f} & \text{in } \widehat{D}, \\ \Delta \tilde{f} + k^2 \tilde{f} = 0 & \text{in } \widehat{D}, \end{cases} \quad (44)$$

has trivial solution  $(w_q, \tilde{f}) \in H_0^2(\widehat{D}) \times L^2(\widehat{D})$ . Since in  $\mathcal{O}^c$ ,  $n = n_p$  and  $\mathcal{O}^c \cap \Lambda = \emptyset$ , then  $(w_q, f) \in w_q \in H_0^2(\mathcal{O}^c) \times L^2(\mathcal{O}^c)$  verifies

$$\begin{cases} \Delta w_q + k^2 n w_q = k^2 (1 - n) f & \text{in } \mathcal{O}^c, \\ \Delta f + k^2 f = 0 & \text{in } \mathcal{O}^c. \end{cases} \quad (45)$$

Assumption 2 implies that equation (45) has a trivial solution, and therefore

$$w_q = f = 0 \quad \text{in } \mathcal{O}^c.$$

It remains to prove that  $f = 0$  in  $\Lambda$ . We can now rewrite (44) as a problem only in  $\Lambda$

$$\begin{cases} w_q \in H_0^2(\Lambda) \text{ and } \tilde{f} \in L^2(\Lambda) \\ \begin{cases} \Delta w_q + k^2 n w_q = k^2 (n_p - n)(w - w_q) + k^2 (1 - n) \tilde{f} & \text{in } \Lambda, \\ \Delta \tilde{f} + k^2 \tilde{f} = 0 & \text{in } \Lambda. \end{cases} \end{cases} \quad (46)$$

To deal with this problem, we first express the quantity  $w - w_q$  in terms of  $\tilde{f}$  using the property that  $\tilde{f} = 0$  outside  $\Lambda$ . To this end, recalling that  $\tilde{f} = f$  in  $D$ , we can write (6) in terms of  $\tilde{f}$  as

$$\Delta w + k^2 n_p w = k^2 (n_p - n) w + k^2 (1 - n) \tilde{f} \quad (47)$$

and then have

$$w(x) = - \int_D \left( k^2 (n_p - n) w + k^2 (1 - n) \tilde{f} \right) (y) \Phi(n_p; x - y) dy. \quad (48)$$

Using the facts that  $\tilde{f} = 0$  and  $n = n_p$  in  $\mathcal{O}_p^c$ , i.e.  $n_p = n = 1$  in  $\Lambda_p \setminus D$  we have

$$\begin{aligned} w(x) &= - \int_{D \setminus \mathcal{O}_p^c} \left( k^2 (n_p - n) w + k^2 (1 - n) \tilde{f} \right) (y) \Phi(n_p; x - y) dy. \\ &= - \int_{\Lambda_p} \left( k^2 (n_p - n) w + k^2 (1 - n) \tilde{f} \right) (y) \Phi(n_p; x - y) dy. \\ &= - k^2 \int_{\Lambda} \left( (n_p - n) w + (1 - n) \tilde{f} \right) (y) \Phi(n_p; x - y) dy \\ &\quad - k^2 \int_{\Lambda_p \setminus \Lambda} (1 - n_p) \tilde{f} (y) \Phi(n_p; x - y) dy. \end{aligned} \quad (49)$$

Moreover, since  $w_q \in H_0^2(\Lambda)$ , then for all  $\theta \in H^2(\Lambda)$  satisfying  $\Delta \theta + k^2 n_p \theta = 0$  we have

$$\int_{\Lambda} \left( \Delta w_q + k^2 n_p w_q \right) \theta dx = 0, \quad (50)$$

implying from equation (46), that

$$\int_{\Lambda} \left( (n_p - n)w + (1 - n)\tilde{f} \right) \theta \, dx = 0. \quad (51)$$

Remark that for  $x \notin \Lambda$ ,  $\Delta\Phi(n_p; x - y) + k^2 n_p \Phi(n_p; x - y) = 0$  for all  $y \in \Lambda$ . Combined with  $\tilde{f} = 0$  outside  $\Lambda_p$ , we then conclude from (49) that

$$w(x) = - \int_{\Lambda_p \setminus \Lambda} k^2 (1 - n_p) \tilde{f}(y) \Phi(n_p; x - y) \, dy \quad \text{for } x \notin \Lambda. \quad (52)$$

Next we define

$$\tilde{w}(x) = - \int_{\Lambda_p \setminus \Lambda} k^2 (1 - n_p) \tilde{f}(y) \Phi(n_p; x - y) \, dy, \quad x \in \Theta. \quad (53)$$

then  $w - \tilde{w} \in H_0^2(\Lambda)$  and  $\Delta\tilde{w} + k^2 n_p \tilde{w} = 0$  in  $\Lambda$ . We now keep  $w$  and  $w_q$  as above and let  $\hat{w} := w_q + \tilde{w}$  in  $\Lambda$  which obviously verifies

$$\Delta\hat{w} + k^2 n \hat{w} = k^2 (1 - n) \tilde{f} \quad \text{in } \Lambda.$$

By Assumption 1 we have  $w = \hat{w}$  in  $\Lambda$ . This proves that  $\tilde{w} = w - w_q$  in  $\Lambda$ . Moreover, using the  $\alpha_q$  quasi-periodicity of  $\tilde{f}$  in  $\Lambda_p$  and the periodicity of  $n_p$  we have that

$$\begin{aligned} \int_{\Lambda + mL} k^2 (1 - n_p) \tilde{f}(y) \Phi(n_p; x - y) \, dy &= \int_{\Lambda} k^2 (1 - n_p) \tilde{f}(y + mL) \Phi(n_p; x - (y + mL)) \, dy \\ &= e^{i\alpha_q mL} \int_{\Lambda} k^2 (1 - n_p) \tilde{f}(y) \Phi(n_p; x - mL - y) \, dy. \end{aligned}$$

Letting for  $y \in \Lambda$

$$\tilde{\Phi}(x, y) := \sum_{0 \neq m \in \mathbb{Z}_M} e^{i\alpha_q mL} \Phi(n_p; x - mL - y), \quad (54)$$

we see that  $\tilde{w}$  defined by (53) is equivalent to

$$\tilde{w}(x) = - \int_{\Lambda} k^2 (1 - n_p) \tilde{f}(y) \tilde{\Phi}(x, y) \, dy. \quad (55)$$

This leads us to define the operator  $\tilde{\mathcal{S}}_k : L^2(\Lambda) \rightarrow L^2(\Lambda)$

$$\tilde{\mathcal{S}}_k : f \mapsto - \int_{\Lambda} k^2 (1 - n_p) f(y) \tilde{\Phi}(x, y) \, dy. \quad (56)$$

From the smoothing property of the volume potential we have that the operator  $\tilde{\mathcal{S}}_k$  is compact and  $\tilde{\mathcal{S}}_k(f)$  satisfies

$$\Delta\tilde{\mathcal{S}}_k(f) + k^2 n_p \tilde{\mathcal{S}}_k(f) = 0 \quad \text{in } \mathbb{R}^d \setminus \Lambda.$$



From the above, we write  $w - w_q$  in terms  $\tilde{\mathcal{S}}_k(f)$  then reformulate (46) for  $w_0 := w_q$  and  $f$ , we finally obtain

$$\left\{ \begin{array}{l} w_0 \in H_0^2(\Lambda), f \in L^2(\Lambda), \\ \left\{ \begin{array}{ll} \Delta w_0 + k^2 n w_0 = k^2(1-n)f + k^2(n_p - n)\tilde{\mathcal{S}}_k(f) & \text{in } \Lambda, \\ \Delta f + k^2 f = 0 & \text{in } \Lambda. \end{array} \right. \end{array} \right. \quad (57)$$

This problem is the homogeneous version of the new interior transmission problem defined in Definition 3.3 below, where  $u := w_0 + f$ . Assumption 3 now implies that  $w_0 = f = 0$  in  $\Lambda$ , which proves the injectivity of  $G_q$ .  $\square$

**Definition 3.3** (The new interior transmission problem). Find  $(u, f) \in L^2(\Lambda) \times L^2(\Lambda)$  such that  $u - f \in H^2(\Lambda)$  and

$$\left\{ \begin{array}{ll} \Delta u + k^2 n u = k^2(n_p - n)\tilde{\mathcal{S}}_k(f) & \text{in } \Lambda, \\ \Delta f + k^2 f = 0 & \text{in } \Lambda, \\ u - f = \varphi & \text{on } \partial\Lambda, \\ \partial(u - f)/\partial\nu = \psi & \text{on } \partial\Lambda, \end{array} \right. \quad (58)$$

for given  $(\varphi, \psi) \in H^{3/2}(\partial\Lambda) \times H^{1/2}(\partial\Lambda)$  where

$$\begin{aligned} \tilde{\mathcal{S}}_k : L^2(\Lambda) &\rightarrow L^2(\Lambda) : \\ f &\mapsto - \int_{\Lambda} k^2(1 - n_p)f(y) \left( \sum_{0 \neq m \in \mathbb{Z}_M} e^{i\alpha_q m L} \Phi(n_p; x - mL - y) \right) dy, \end{aligned} \quad (59)$$

$\Phi(n_p; \cdot)$  is the  $ML$ -periodic fundamental solution given by (7), and  $\nu$  denotes the unit normal on  $\partial\Lambda$  outward to  $\Lambda$ .

**Definition 3.4.** Values of  $k \in \mathbb{C}$  for which the **homogeneous** problem with  $\varphi = \psi = 0$ , are called *new transmission eigenvalues*.

For sake of completeness, we end this section with proving a range statement for the operator  $G_q^\pm$  which is used in the imaging algorithm. To this end, we recall  $\Phi_q(\cdot - z)$  defined by (20),  $I_q$  given by (23) with adjoint  $I_q^*$ .

**Theorem 3.5.** *Suppose that Assumptions 1, 2 and 3 hold. Then,  $I_q^* \widehat{\Phi}_q^\pm(\cdot; z) \in \mathcal{R}(G_q^\pm)$  if and only if  $z \in \widehat{D}_p$ .*

*Proof.* We consider two cases:

*Case 1:*  $z \in \widehat{D}_p = \Lambda_p \cup \mathcal{O}_p^c$ .

(i) If  $z \in \mathcal{O}_p^c$ : Let  $(u, v) \in L^2(D) \times L^2(D)$  be the unique solution of (14) with  $\varphi := \Phi_q(\cdot - z)|_{\partial D}$  and  $\psi := \partial\Phi_q(\cdot - z)/\partial\nu|_{\partial D}$  and define

$$w = \begin{cases} u - v & \text{in } \mathcal{O}_p^c \\ \Phi_q & \text{in } \Theta \setminus \mathcal{O}_p^c. \end{cases}$$

Then  $w \in H_{\text{loc}}^2(\Theta)$  and verifies equation (6) with  $f = v$  in  $\mathcal{O}_p^c$  and  $f = -\Phi_q$  in  $\Theta \setminus \mathcal{O}_p^c$ . Therefore  $G^\pm(f) = \widehat{\Phi}_q^\pm(\cdot; z)$ . Furthermore  $u|_{\mathcal{O}_p^c}$  and  $v|_{\mathcal{O}_p^c}$  are  $\alpha_q$ -quasi-periodic (due to the periodicity of domain  $\mathcal{O}_p^c$  and  $\alpha_q$ -quasi-periodicity of the data), hence  $f$  is also  $\alpha_q$  quasi-periodic. This implies  $f \in H_{\text{inc}}^q(D)$  and  $G_q^\pm(f) = \mathbf{I}_q^* G_q^\pm(f) = \mathbf{I}_q^* \widehat{\Phi}_q^\pm(\cdot; z)$ .

(ii) If  $z \in \Lambda_p$ : We first consider that  $z \in \Lambda = \Lambda_p \cap \Omega_0$ . Let  $(u, v) \in L^2(\Lambda_p) \times L^2(\Lambda_p)$  be the  $\alpha_q$ -quasi-periodic extension of  $(u_\Lambda, v_\Lambda)$ , where  $(u_\Lambda := u, v_\Lambda := f)$  is the solution of the new interior transmission problem in Definition 3.3 with  $\varphi := \Phi_q(\cdot - z)|_{\partial\Lambda}$  and  $\psi := \partial\Phi_q(\cdot - z)/\partial\nu|_{\partial\Lambda}$ . We then define

$$w_q = \begin{cases} u - v & \text{in } \Lambda_p \\ \Phi_q & \text{in } \Theta \setminus \Lambda_p. \end{cases}$$

Let  $f := v$  in  $\Lambda_p$  and  $f := -\Phi_q$  in  $\Theta \setminus \Lambda_p$  then  $f \in H_{\text{inc}}^q(D)$  and  $w_q \in H_{\text{loc}}^2(\Theta)$  satisfies the scattering problem (33) with data  $f$ . Furthermore,  $w$  defined such as  $w := w_q + \widetilde{\mathcal{S}}_k(f)$  in  $\Lambda$  and  $w := w_q$  in  $D \setminus \Lambda$  is solution to (6) with data  $f$ . Therefore  $G_q^\pm(f) = \mathbf{I}_q^* \widehat{\Phi}_q^\pm(\cdot; z)$ . We next consider  $z \in \Lambda + mL$  with  $0 \neq m \in \mathbb{Z}_M^{d-1}$ , and recall that  $\widehat{\Phi}_q^\pm(\cdot; z) = e^{imL \cdot \alpha_q} \widehat{\Phi}_q^\pm(\cdot; z - mL)$ . If we take  $f \in H_{\text{inc}}^q(D)$  such that  $G_q^\pm(f) = \mathbf{I}_q^* \widehat{\Phi}_q^\pm(\cdot; z - mL)$ , which is possible by the previous step since  $z - mL \in \Lambda$ , then

$$G_q^\pm(e^{imL \cdot \alpha_q} f) = \mathbf{I}_q^* (\widehat{\Phi}_q^\pm(\cdot; z)).$$

*Case 2:*  $z \notin \widehat{D}_p$ . If  $G_q^\pm(v) = \mathbf{I}_q^* \widehat{\Phi}_q^\pm(\cdot; z)$ , then using the same unique continuation argument as in the proof of Lemma 3.2 we obtain  $w_q = \Phi_q$  in  $\Theta \setminus \widehat{D}_p$  where  $w_q$  is defined by (30) with  $w$  being the solution of (6) with  $f = v$ . This gives a contradiction since  $w_q$  is locally  $H^2$  in  $\Theta \setminus \widehat{D}_p$  while  $\Phi_q(\cdot - z)$  is not.  $\square$

#### 4. The Analysis of the New Interior Transmission Problem

This section is devoted to the study of the solvability of the new interior transmission problem in Definition 3.3. It provides sufficient conditions on  $n$  and  $k$  for which this problem is well-posed, i.e. such that Assumption 3 holds. As described in the previous section the solvability of the new interior transmission problem is fundamental to ensuring the properties needed for the imaging of the defect  $\omega$  with a single Floquet-Bloch mode. Up to now the only case that could be handled was when  $\omega \cap D = \emptyset$  [18] (see also [11]). Here we provide a general analysis that cover all possible cases. Our approach generalizes [12] and [19].

We start with proving the following technical lemma:

**Lemma 4.1.** *Assume that  $n_p > \alpha > 0$  on  $\mathbb{R}^d$ . Then there exists  $\theta > 0$  and  $C > 0$  such that*

$$\|\widetilde{\mathcal{S}}_{i\kappa}(f)\|_{L^2(\Lambda)} \leq C e^{-\theta\kappa} \|f\|_{L^2(\Lambda)}, \quad \kappa > 0$$

for all  $f \in L^2(\Lambda)$ .

*Proof.* Denoting  $\tilde{w} := \tilde{\mathcal{S}}_{i\kappa}(f)$  and  $\tilde{f}$  the extension of  $f$  as  $\alpha_q$ -quasi-periodic in  $\Lambda_p$ , we have that

$$\tilde{w}(x) = \kappa^2 \int_{\Lambda_p \setminus \Lambda} (1 - n_p) \tilde{f}(y) \Phi(n_p; x - y) dy.$$

where  $\Phi(n_p; \cdot)$  denotes here the  $ML$ -periodic fundamental solution associated with  $k = i\kappa$ . An application of the Cauchy-Schwarz inequality and the Fubini theorem implies

$$\begin{aligned} \|\tilde{w}\|_{L^2(\Lambda)}^2 &\leq \kappa^4 |\Lambda_p \setminus \Lambda| \int_{\Lambda} \int_{\Lambda_p \setminus \Lambda} |(n_p - 1) \tilde{f}(y) \Phi(n_p; x - y)|^2 dy dx \\ &= \kappa^4 |\Lambda_p \setminus \Lambda| \int_{\Lambda_p \setminus \Lambda} |(n_p - 1) \tilde{f}(y)|^2 \int_{\Lambda} |\Phi(n_p; x - y)|^2 dx dy. \end{aligned}$$

Next we let

$$\Sigma = \{z := x - y, \quad x \in \Lambda, \quad y \in \Lambda_p \setminus \Lambda\},$$

$$d_{max} \in \mathbb{R} : \quad d_{max} > \sup\{|z|, z \in \Sigma\} \quad \text{and} \quad d := d(\Lambda, \Lambda_p \setminus \Lambda),$$

and remark that  $\forall x \in \Lambda, \forall y \in \Lambda_p \setminus \Lambda, |x - y| > d$ , hence

$$\Sigma \subset \mathcal{B} := B(0, d_{max}) \setminus B(0, d) \tag{60}$$

where  $B(0, d)$  is a ball of radii  $d$  and centered at the origin. Remark that  $d > 0$  by Assumption 2. We then have

$$\|\tilde{w}\|_{L^2(\Lambda)}^2 \leq \kappa^4 |\Lambda_p \setminus \Lambda| \int_{\Lambda_p \setminus \Lambda} |(n_p - 1) \tilde{f}(y)|^2 dy \int_{\mathcal{B}} |\Phi(n_p; z)|^2 dz.$$

Since  $\tilde{f} = f$  in  $\Lambda$ ,  $\tilde{f}$  is quasi-periodic and  $n_p$  is periodic in  $\Lambda_p$ , then

$$\int_{\Lambda_p \setminus \Lambda} |(n_p - 1) \tilde{f}(y)|^2 dy = (|M| - 1) \int_{\Lambda} |(n_p - 1) \tilde{f}(y)|^2 dy \leq (|M| - 1) \sup_{\Lambda} |1 - n_p| \|f\|_{L^2(\Lambda)}^2.$$

We now prove that, there exists  $\theta > 0$  and  $C_0 > 0$  such that

$$\int_{\mathcal{B}} |\Phi(n_p; z)|^2 dz \leq C_0 e^{-\theta\kappa}. \tag{61}$$

We recall that  $\Phi(n_p; z)$  is  $ML$ -periodic and satisfies

$$(\Delta - \kappa^2 n_p) \Phi(n_p; \cdot) = -\delta_0 \quad \text{in } \Theta \tag{62}$$

and  $\Phi(n_p; \cdot) \in L^2(\Theta)$  (or equivalently the Rayleigh radiation condition (4) with  $k = i\kappa$ ). Consider the fundamental solution  $\Psi \in L^2(\mathbb{R}^d)$  satisfying

$$(\Delta - \kappa^2 n_p) \Psi = -\delta_0 \quad \text{in } \mathbb{R}^d.$$

Since  $n_p$  is positive definite, then one can prove that  $e^{\gamma\kappa|x|} \Psi(x) \in L^2(\mathbb{R}^d)$  for  $\gamma > 0$  sufficiently small (following the lines of the proof of Theorem 4.4 in [10]). The function  $u^s := \Phi(n_p; \cdot) - \Psi$  satisfies

$$\Delta u^s - \kappa^2 n_p u^s = 0 \quad \text{in } \Theta$$

and application of the Green formula and using the periodicity conditions of  $\Phi(n_p; \cdot)$  imply

$$\int_{\Theta} \left( |\nabla u^s|^2 + \kappa^2 n_p |u^s|^2 \right) dx \leq C \left( \|\Psi\|_{H^{\frac{1}{2}}(\partial\Theta)} \left\| \frac{\partial u^s}{\partial \nu} \right\|_{H^{-\frac{1}{2}}(\partial\Theta)} + \|u^s\|_{H^{\frac{1}{2}}(\partial\Theta)} \left\| \frac{\partial \Psi}{\partial \nu} \right\|_{H^{-\frac{1}{2}}(\partial\Theta)} \right)$$

for some constant  $C$  independent from  $\kappa$ . The decay property of  $\Psi$  implies that

$$\left( \|\Psi\|_{H^{\frac{1}{2}}(\partial\Theta)} + \left\| \frac{\partial \Psi}{\partial \nu} \right\|_{H^{-\frac{1}{2}}(\partial\Theta)} \right) \leq e^{-\gamma_1 \kappa}$$

for  $\kappa$  sufficiently large and  $\gamma_1 > 0$  a constant independent from  $\kappa$ . Therefore, by classical continuity properties for traces and normal traces and the fact that  $\Delta u^s = \kappa^2 n_p u^s$ , one conclude that for  $\kappa$  sufficiently large

$$\int_{\Theta} (|\nabla u^s|^2 + |u^s|^2) dx \leq C e^{-\gamma_1 \kappa}$$

for some constant  $C$  independent from  $\kappa$ . **Since  $\mathcal{B}$  is contained in the union of  $\Theta$  and two  $ML$  periods, one to the left of  $\Theta$  and the other to the right, using the  $ML$ -periodicity of  $u^s$  we have that  $\int_{\mathcal{B}} (|\nabla u^s|^2 + |u^s|^2) dx \leq 3C e^{-\gamma_1 \kappa}$ .** One then obtain the desired estimate (61) by writing  $\Phi(n_p; \cdot) = \Psi + u^s$  and combining the previous estimate with the exponential decay of  $\Psi$ .  $\square$

We now turn our attention to the analysis of the new interior transmission problem in Definition 3.3. For the given  $(\varphi, \psi) \in H^{3/2}(\Lambda) \times H^{1/2}(\Lambda)$  in (58) we construct a lifting function  $u_0 \in H^2(\Lambda)$  such that  $u_0|_{\partial\Lambda} = \varphi$  and  $\partial u_0 / \partial \nu|_{\partial\Lambda} = \psi$ . Hence  $w := u - u_0 - f \in H_0^2(\Lambda)$  and we let  $F := (\Delta u_0 + k^2 n u_0) / k^2 \in L^2(\Lambda)$ . To further simplify notation, we set  $\lambda := -k^2 \in \mathbb{C}$ ,  $q := n - 1$ ,  $p := n - n_p$  and  $v := -w / k^2 \in H_0^2(\Lambda)$ . In these notations, the problem we need to solve reads:

$$\left\{ \begin{array}{l} v \in H_0^2(\Lambda) \text{ and } f \in L^2(\Lambda), \\ \left\{ \begin{array}{l} \Delta v - \lambda(q+1)v = qf + p\tilde{\mathcal{S}}_{\sqrt{-\lambda}}(f) + F \\ \Delta f - \lambda f = 0 \end{array} \right. \quad \begin{array}{l} \text{in } \Lambda, \\ \text{in } \Lambda, \end{array} \end{array} \right. \quad (63)$$

for a given  $F \in L^2(\Lambda)$ , where  $\tilde{\mathcal{S}}_{\sqrt{-\lambda}}(f)$  is given by (59). We remark that (63) is a modification of the problem considered in [12] (see also [7, Section 3.1.3]). We write this problem in an equivalent variational form. To this end, let us denote  $\mathbf{X} := H_0^2(\Lambda) \times L^2(\Lambda)$  with the norm  $\|(v, f)\|_{\mathbf{X}}^2 := \|v\|_{H_0^2(\Lambda)}^2 + \|f\|_{L^2(\Lambda)}^2$  and the corresponding inner product  $\langle \cdot, \cdot \rangle_{\mathbf{X}}$ . Then we define the sesquilinear form  $a_k : \mathbf{X} \times \mathbf{X} \rightarrow \mathbb{C}$  by

$$a_{\lambda}(v, f; \phi, \psi) = \int_{\Lambda} (\Delta \bar{\phi} - \lambda \bar{\phi}) f \, dx + \int_{\Lambda} (\Delta v - \lambda(q+1)v) \bar{\psi} - (qf + p\tilde{\mathcal{S}}_{\sqrt{-\lambda}} f) \bar{\psi} \, dx, \quad (64)$$

for  $(v, f) \in \mathbf{X}$  and  $(\phi, \psi) \in \mathbf{X}$  and the bounded linear operator  $A_{\lambda} : \mathbf{X} \rightarrow \mathbf{X}$  defined by means of the Riesz's representation theorem

$$a_{\lambda}(v, f; \phi, \psi) = \langle A_{\lambda}(v, f), (\phi, \psi) \rangle_{\mathbf{X}}. \quad (65)$$

Letting  $\ell \in \mathbf{X}$  be the Riesz's representative of the conjugate linear functional

$$(\ell, (\phi, \psi))_{\mathbf{X}} = \int_{\Lambda} F \bar{\psi} \, dx,$$

solving (63) is equivalent to solving

$$A_{\lambda}(v, f) = \ell \quad \text{or} \quad a_{\lambda}(v, f; \phi, \psi) = \int_{\Lambda} F \bar{\psi} \, dx, \quad \forall (\phi, \psi) \in \mathbf{X}. \quad (66)$$

In a similar fashion, we define the sesquilinear form  $b_{\lambda} : \mathbf{X} \times \mathbf{X} \rightarrow \mathbb{C}$  by

$$b_{\lambda}(v, f; \phi, \psi) = \int_{\Lambda} (\Delta \bar{\phi} - \lambda \bar{\phi}) f \, dx + \int_{\Lambda} (\Delta v - \lambda v - qf) \bar{\psi} \, dx$$

with the associated bounded linear operator  $B_{\lambda} : \mathbf{X} \rightarrow \mathbf{X}$  such that

$$b_{\lambda}(v, f; \phi, \psi) = \langle B_{\lambda}(v, f), (\phi, \psi) \rangle_{\mathbf{X}},$$

**Lemma 4.2.** *For any  $\lambda, \mu \in \mathbb{C}$ , the differences  $A_{\lambda} - B_{\mu}$  and  $A_{\lambda} - A_{\mu}$  are compact.*

*Proof.* Let  $(v_j, f_j) \in \mathbf{X}$  be an arbitrary sequence converging weakly to  $(0, 0)$  in  $\mathbf{X}$ . We must show that  $(A_{\lambda} - B_{\mu})(v_j, f_j)$  converges  $(0, 0)$  strongly in  $\mathbf{X}$ . Recall that

$$\langle (A_{\lambda} - B_{\mu})(v, f), (\phi, \psi) \rangle_{\mathbf{X}} = (a_{\lambda} - b_{\mu})(v, f; \phi, \psi)$$

where

$$(a_{\lambda} - b_{\mu})(v_j, f_j; \phi, \psi) = (\mu - \lambda) \int_{\Lambda} f_j \bar{\phi} + \int_{\Lambda} (\mu - \lambda(q+1)) v_j \bar{\psi} - \int_{\Lambda} p(\tilde{\mathcal{S}}_{\sqrt{-\lambda}} f_j - \tilde{\mathcal{S}}_{\sqrt{-\mu}} f_j) \bar{\psi}.$$

Let us take  $(\phi, \psi) \in \mathbf{X}$  such that  $\|(\phi, \psi)\|_{\mathbf{X}} = 1$  and define  $g_j \in H^1(\Lambda)$  such that  $\Delta g_j = f_j$  and  $g_j = 0$  on  $\partial\Lambda$ . Hence  $g_j \rightarrow 0$  in  $H^1(\Lambda)$  implying that  $g_j \rightarrow 0$  in  $L^2(\Lambda)$  by compact embedding of  $H^1(\Lambda)$  in  $L^2(\Lambda)$ . Obviously,

$$\left| \int_{\Lambda} f_j \bar{\phi} \right| = \left| \int_{\Lambda} \Delta g_j \bar{\phi} \right| = \left| \int_{\Lambda} g_j \Delta \bar{\phi} \right| \leq \|\Delta \bar{\phi}\|_{L^2(\Lambda)} \|g_j\|_{L^2(\Lambda)} \leq \|g_j\|_{L^2(\Lambda)}.$$

Similarly, since  $v_j \rightarrow 0$  in  $H^2(\Lambda)$ , then  $v_j \rightarrow 0$  in  $L^2(\Lambda)$ , and

$$\left| \int_{\Lambda} (\mu - \lambda(q+1)) v_j \bar{\psi} \right| \leq \sup_{\Lambda} |\mu - \lambda(q+1)| \|v_j\|_{L^2(\Lambda)} \|\psi_j\|_{L^2(\Lambda)} \leq \sup_{\Lambda} |\mu - \lambda(q+1)| \|v_j\|_{L^2(\Lambda)}$$

and

$$\begin{aligned} \left| \int_{\Lambda} p(\tilde{\mathcal{S}}_{\sqrt{-\lambda}} f_j - \tilde{\mathcal{S}}_{\sqrt{-\mu}} f_j) \bar{\psi} \right| &\leq \sup_{\Lambda} |p| \|\tilde{\mathcal{S}}_{\sqrt{-\lambda}} f_j - \tilde{\mathcal{S}}_{\sqrt{-\mu}} f_j\|_{L^2(\Lambda)} \|\psi_j\|_{L^2(\Lambda)} \\ &\leq \sup_{\Lambda} |p| \|(\tilde{\mathcal{S}}_{\sqrt{-\lambda}} - \tilde{\mathcal{S}}_{\sqrt{-\mu}}) f_j\|_{L^2(\Lambda)}. \end{aligned}$$

Since  $(\tilde{\mathcal{S}}_{\sqrt{-\lambda}} - \tilde{\mathcal{S}}_{\sqrt{-\mu}}) : L^2(\Lambda) \rightarrow L^2(\Lambda)$  is a compact linear operator, then  $\|(\tilde{\mathcal{S}}_{\sqrt{-\lambda}} - \tilde{\mathcal{S}}_{\sqrt{-\mu}}) f_j\|_{L^2(\Lambda)}$  converge to 0 strongly in  $L^2(\Lambda)$ . Therefore

$$\begin{aligned} \|(A_{\lambda} - B_{\mu})(v_j, f_j)\|_{\mathbf{X} \rightarrow \mathbf{X}} &= \sup_{\|(\phi, \psi)\|_{\mathbf{X}}=1} |(a_{\lambda} - b_{\mu})(v_j, f_j; \phi, \psi)| \\ &\leq c \left( \|v_j\|_{L^2(\Lambda)} + \|g_j\|_{L^2(\Lambda)} + \|\tilde{\mathcal{S}}_{\lambda} f_j - \tilde{\mathcal{S}}_{\mu} f_j\|_{L^2(\Lambda)} \right) \rightarrow 0. \end{aligned}$$

The proof for  $A_{\lambda} - A_{\mu}$  follows exactly the same lines.  $\square$

At this point we need to assume a sign condition on  $q := n - 1$ . To this end, let  $R$  be a neighborhood of  $\partial\Lambda$  in  $\Lambda$  and denote by

$$q_{min} := \inf_{\Lambda} \operatorname{Re}(q) > -1, \quad q_{\star} := \inf_R \operatorname{Re}(q), \quad q^{\star} := \sup_R \operatorname{Re}(q). \quad (67)$$

The following lemmas, which are proven first in [12] for real refractive index and adapted to the case of complex refractive index in [7, Section 3.1.3]), play an important role in our analysis.

**Lemma 4.3.** *Assume that  $q \in L^{\infty}(\Lambda)$  is such that  $q_{min} + 1 > 0$  and either  $q_{\star} > 0$  or  $q^{\star} < 0$ . Then there exists  $c > 0$  and  $d > 0$  such that for all  $\lambda > 0$  the following estimates holds*

$$\int_{\Lambda \setminus R} |f|^2 dx \leq ce^{-2d\sqrt{\lambda}} \int_R |\operatorname{Re}(q)||f|^2 dx \leq ce^{-2d\sqrt{\lambda}} \int_R |q||f|^2 dx \quad (68)$$

for all  $f \in L^2(\Lambda)$  solution of  $\Delta f - \lambda f = 0$ .

*Proof.* See Lemma 2.3 in [12] if  $q$  is real and Lemma 3.14 in [7] for  $q$  complex.  $\square$

**Lemma 4.4.** *Assume that  $q \in L^{\infty}(\Lambda)$  is such that  $q_{min} + 1 > 0$  and either  $q_{\star} > 0$  or  $q^{\star} < 0$ . Then for sufficiently large  $\lambda > 0$ , the operator  $B_{\lambda}$  is an isomorphism from  $\mathbf{X}$  onto itself.*

*Proof.* See Theorem 2.7 in [12] if  $q$  is real and Lemma 3.15 in [7] for  $q$  complex.  $\square$

We now proceed with proving that the operator  $A_{\lambda} : X \rightarrow X$  is an isomorphism for sufficiently large  $\lambda$ . We adopt two different approaches for the cases  $q^{\star} < 0$  (Theorem 4.5) and  $q_{\star} > 0$  (Theorem 4.6).

**Theorem 4.5.** *Assume that  $q \in L^{\infty}(\Lambda)$  is such that  $q_{min} + 1 > 0$  and  $q^{\star} < 0$ . Then, for sufficiently large  $\lambda > 0$ , the operator  $A_{\lambda} : X \rightarrow X$  is an isomorphism.*

*Proof.* It is sufficient to prove the injectivity of  $A_{\lambda}$  since  $B_{\lambda}$  is an isomorphism from Lemma 4.4 and  $A_{\lambda} - B_{\lambda}$  is compact from Lemma 4.2. To this end, assume that  $(v, f) \in \mathbf{X}$  is such that  $A_{\lambda}(v, f) = 0$ , i.e.,  $a_{\lambda_j}(v, f; \phi, \psi) = 0$  for all  $(\phi, \psi) \in \mathbf{X}$ . As such,  $(v, f)$  satisfies

$$\begin{cases} \Delta v - \lambda(q+1)v &= qf + p\tilde{\mathcal{S}}_{i\sqrt{\lambda}}(f) \\ \Delta f - \lambda f &= 0 \end{cases} \quad \text{in } \Lambda. \quad (69)$$

Multiplying the first equation of (69) with  $\bar{f}$  we obtain

$$\int_{\Lambda} q|f|^2 + \int_{\Lambda} p\tilde{\mathcal{S}}_{i\sqrt{\lambda}}(f)\bar{f} = \int_{\Lambda} (\Delta v - \lambda(q+1)v)\bar{f} = \int_{\Lambda} (\Delta v - \lambda v)\bar{f} - \int_{\Lambda} qv\bar{f}.$$

First observe that  $\int_{\Lambda} (\Delta v - \lambda v)\bar{f} = 0$  since  $v \in H_0^2(\Lambda)$  and  $\Delta f - \lambda f = 0$  in  $\Lambda$ . Therefore

$$\int_{\Lambda} q|f|^2 = - \int_{\Lambda} p\tilde{\mathcal{S}}_{i\sqrt{\lambda}}(f)\bar{f} - \lambda \int_{\Lambda} qv\bar{f}. \quad (70)$$

Multiplying now the first equation of (69) with  $\bar{v}$  and integrating by parts we obtain

$$\int_{\Lambda} |\nabla v|^2 + \lambda(q+1)|v|^2 dx = - \int_{\Lambda} qf\bar{v} dx - \int_{\Lambda} p\tilde{\mathcal{S}}_{i\sqrt{\lambda}}(f)\bar{v} dx. \quad (71)$$

Now assume that  $q^* < 0$  and taking the real part of the above we write

$$\int_{\Lambda} |\nabla v|^2 + \lambda(\operatorname{Re}(q) + 1)|v|^2 = \operatorname{Re} \left( - \int_{\Lambda} qf\bar{v} - \int_{\Lambda} p\tilde{\mathcal{S}}_{i\sqrt{\lambda}}(f)\bar{v} \right). \quad (72)$$

From (70) and (72) noting that  $\lambda$  is real, we deduce that

$$\begin{aligned} \int_{\Lambda} |\nabla v|^2 + \lambda(\operatorname{Re}(q) + 1)|v|^2 + \frac{1}{\lambda} \int_{\Lambda} -\operatorname{Re}(q)|f|^2 dx \\ = \operatorname{Re} \left( \int_{\Lambda} p\tilde{\mathcal{S}}_{i\sqrt{\lambda}}(f) \left( \frac{1}{\lambda} \bar{f} - \bar{v} \right) dx \right). \end{aligned} \quad (73)$$

For large enough  $\lambda > 0$ , we set  $\rho := \max \left( \|q\|_{\infty} c e^{-2d\sqrt{\lambda}}, c e^{-2d\sqrt{\lambda}} \right) < 1$ . Then Lemma 4.3 implies

$$(1 - \rho) \int_R |\operatorname{Re}(q)||f|^2 dx \leq \int_{\Lambda} -\operatorname{Re}(q)|f|^2 dx \leq (1 + \rho) \int_R |\operatorname{Re}(q)||f|^2 dx. \quad (74)$$

The latter inequalities implies

$$\begin{aligned} \int_{\Lambda} \lambda(\operatorname{Re}(q) + 1)|v|^2 + \frac{1}{\lambda} \int_{\Lambda} -\operatorname{Re}(q)|f|^2 dx \\ \geq \lambda \int_{\Lambda} (\operatorname{Re}(q) + 1)|v|^2 + \frac{1 - \rho}{\lambda} \int_R |\operatorname{Re}(q)||f|^2 dx. \end{aligned} \quad (75)$$

From Lemma 4.1 we infer the existence of  $c > 0$  such that  $\|p\tilde{\mathcal{S}}_{i\sqrt{\lambda}}(f)\|_{L^2(\Lambda)} \leq c e^{-\theta\sqrt{\lambda}} \|f\|_{L^2(\Lambda)}$ . We then have

$$\left| \int_{\Lambda} \operatorname{Re} \left[ p\tilde{\mathcal{S}}_{i\sqrt{\lambda}}(f) \left( \frac{1}{\lambda} \bar{f} - \bar{v} \right) \right] dx \right| \leq c e^{-\theta\sqrt{\lambda}} \|f\|_{L^2(\Lambda)} \left( \|v\|_{L^2(\Lambda)} + \frac{1}{\lambda} \|f\|_{L^2(\Lambda)} \right)$$

Using Young's inequality then splitting the integral of  $|f|^2$  into the domains  $R$  and  $\Lambda \setminus R$  we finally have

$$\begin{aligned} \left| \int_{\Lambda} \operatorname{Re} \left( p\tilde{\mathcal{S}}_{i\sqrt{\lambda}}(f) \left( \frac{1}{\lambda} \bar{f} - \bar{v} \right) \right) dx \right| &\leq \frac{2c}{\lambda} e^{-\theta\sqrt{\lambda}} \|f\|_{L^2(\Lambda)}^2 + \frac{c\lambda}{4} e^{-\theta\sqrt{\lambda}} \|v\|_{L^2(\Lambda)}^2 \\ &\leq \frac{2c}{\lambda} e^{-\theta\sqrt{\lambda}} (1 + \rho) \|f\|_{L^2(R)}^2 + \frac{c\lambda}{4} e^{-\theta\sqrt{\lambda}} \|v\|_{L^2(\Lambda)}^2 \\ &\leq \frac{2c(1 + \rho)}{\lambda q_{\star}} e^{-\theta\sqrt{\lambda}} \|\sqrt{|\operatorname{Re}(q)|} f\|_{L^2(R)}^2 + \frac{c\lambda}{4(q_{\min} + 1)} e^{-\theta\sqrt{\lambda}} \|\sqrt{\operatorname{Re}(q) + 1} v\|_{L^2(\Lambda)}^2. \end{aligned} \quad (76)$$

For  $\lambda$  sufficiently large such that

$$\frac{2c(1 + \rho)}{\lambda q_{\star}} e^{-\theta\sqrt{\lambda}} \leq \frac{1 - \rho}{\lambda} - \varepsilon \quad \text{and} \quad \frac{c\lambda}{4(q_{\min} + 1)} e^{-\theta\sqrt{\lambda}} \leq \lambda - \varepsilon,$$

with  $\varepsilon > 0$  small enough we finally obtain from (73), (75) and (76)

$$\varepsilon \|\sqrt{\operatorname{Re}(q) + 1}v\|_{L^2(\Lambda)}^2 + \varepsilon \|\sqrt{|\operatorname{Re}(q)|}f\|_{L^2(R)}^2 \leq 0.$$

This prove that  $v = 0$  in  $\Lambda$  and  $f = 0$  in  $R$ . Inequality (74) then implies  $f = 0$  in  $\Lambda$  which ends the proof.  $\square$

**Theorem 4.6.** *Assume that  $q \in L^\infty(\Lambda)$  is such that  $q_{\min} + 1 > 0$  and  $q_* > \sup_R(\operatorname{Im}(q))^2 \geq 0$ . Then, for sufficiently large  $\lambda > 0$ , the operator  $A_\lambda : X \rightarrow X$  is an isomorphism.*

*Proof.* Again similarly to the proof of Theorem 4.5 it is sufficient to prove that  $A_\lambda$  is injection, which will do by contradiction. Assume to the contrary that there exists a sequence  $\lambda_j \rightarrow \infty$  and  $(v_j, f_j) \in \mathbf{X}$  with  $\|(v_j, f_j)\|_{L^2(\Lambda)} = 1$  and  $A_{\lambda_j}(v_j, f_j) = 0$ . Then  $v_j \in H_0^2(\Lambda)$  and  $f_j \in L^2(\Lambda)$  satisfy

$$\Delta v_j - \lambda_j(q + 1)v_j = qf_j + p\tilde{\mathcal{S}}_{i\sqrt{\lambda_j}}(f_j) \quad \text{and} \quad \Delta f_j - \lambda_j f_j = 0. \quad (77)$$

For given  $\varepsilon > 0$  small enough, from Lemma 4.1 and Lemma 4.3 and the fact that  $\|(v_j, f_j)\|_{L^2(\Lambda)} = 1$ , we have that for  $\lambda_j > 0$  large enough

$$\begin{aligned} \|p\tilde{\mathcal{S}}_{i\sqrt{\lambda_j}}(f_j)\|_{L^2(\Lambda)}\|f_j\|_{L^2(\Lambda)} + \|\sqrt{|q|}f_j\|_{L^2(\Lambda \setminus R)}^2 \\ + \lambda_j\|\sqrt{|q|}f_j\|_{L^2(\Lambda \setminus R)}\|\sqrt{|q|}v_j\|_{L^2(\Lambda \setminus R)} \leq \varepsilon^2. \end{aligned} \quad (78)$$

From (70), we observe that

$$\begin{aligned} \int_R \operatorname{Re}(q)|f_j|^2 dx &= \operatorname{Re} \left( -\lambda_j \int_R qv_j \bar{f}_j dx \right) - \int_{\Lambda \setminus R} \operatorname{Re}(q)|f_j|^2 dx \\ &\quad - \operatorname{Re} \left( \int_\Lambda p\tilde{\mathcal{S}}_{i\sqrt{\lambda}}(f_j)\bar{v}_j dx + \lambda_j \int_{\Lambda \setminus R} qv_j \bar{f}_j dx \right) \end{aligned}$$

Estimating the last three terms from (78) and using Cauchy-Schwarz inequality we now obtain

$$\int_R \operatorname{Re}(q)|f_j|^2 dx \leq \varepsilon^2 + \lambda_j \left( \int_R \operatorname{Re}(q)|f_j|^2 dx \right)^{1/2} \left( \int_R \frac{|q|^2}{\operatorname{Re}(q)}|v_j|^2 dx \right)^{1/2}.$$

This implies that

$$\left( \int_R \operatorname{Re}(q)|f_j|^2 dx \right)^{1/2} \leq \varepsilon + \lambda_j \left( \int_R \frac{|q|^2}{\operatorname{Re}(q)}|v_j|^2 dx \right)^{1/2}, \quad (79)$$

(where we used for  $A > 0$  and  $B > 0$ ,  $A^2 \leq \varepsilon^2 + AB$  implies  $(A - B/2)^2 \leq \varepsilon^2 + B^2/4 \leq (\varepsilon + B/2)^2$  and taking the square root yields  $A \leq \varepsilon + B$ ). From (71), we see that (using again  $\|(v_j, f_j)\|_{\mathbf{X}} = 1$ )

$$\begin{aligned} \lambda_j \int_R (1 + \operatorname{Re}(q))|v_j|^2 dx &\leq \lambda_j \int_\Lambda (1 + \operatorname{Re}(q))|v_j|^2 dx \\ &\leq \left( \int_R \operatorname{Re}(q)|f_j|^2 dx \right)^{1/2} \left( \int_R \frac{|q|^2}{\operatorname{Re}(q)}|v_j|^2 dx \right)^{1/2} + \varepsilon^2 \end{aligned} \quad (80)$$



since  $(1 + \operatorname{Re}(q)) > 0$  in  $\Lambda$ . Combining the last two inequalities yield

$$\lambda_j \int_R (1 + \operatorname{Re}(q)) |v_j|^2 dx \leq \lambda_j \int_R \frac{|q|^2}{\operatorname{Re}(q)} |v_j|^2 dx + \varepsilon \left( \int_R \frac{|q|^2}{\operatorname{Re}(q)} |v_j|^2 dx \right)^{1/2} + \varepsilon^2. \quad (81)$$

The assumption  $q_* > \sup_R (\operatorname{Im}(q))^2 \geq 0$  in particular implies  $\inf_R \frac{\operatorname{Re}(q) - (\operatorname{Im}(q))^2}{\operatorname{Re}(q)} = \delta_0 > 0$ , hence we have

$$\lambda_j \delta_0 \int_R |v_j|^2 dx \leq \varepsilon \left( \int_R \frac{|q|^2}{\operatorname{Re}(q)} |v_j|^2 dx \right)^{1/2} + \varepsilon^2.$$

Choosing  $\lambda_j$  such that  $4\lambda_j \delta_0 \geq \left\| \frac{|q|^2}{\operatorname{Re}(q)} \right\|_{L^\infty(R)}$  we obtain that

$$\lambda_j \delta_0 \int_R |v_j|^2 dx \leq 2\varepsilon \left( \lambda_j \delta_0 \int_R |v_j|^2 dx \right)^{1/2} + \varepsilon^2.$$

which implies

$$\left( \lambda_j \delta_0 \int_R |v_j|^2 dx \right)^{1/2} \leq 3\varepsilon$$

and proves that  $\lambda_j \delta_0 \int_R |v_j|^2 dx \rightarrow 0$  as  $j \rightarrow \infty$ . Coming back to (80) we deduce that

$$\int_\Lambda (1 + \operatorname{Re}(q)) |v_j|^2 dx \rightarrow 0 \text{ as } j \rightarrow \infty$$

and from (79), since  $\operatorname{Re}(q)$  is positive definite in  $R$ , we deduce that  $\int_R |f_j|^2 dx \rightarrow 0$  as  $j \rightarrow \infty$ . Lemma 4.3 then implies

$$\int_\Lambda |f_j|^2 dx \rightarrow 0 \text{ as } j \rightarrow \infty.$$

The last two zero limits contradict the fact that  $\|(v_j, f_j)\|_{L^2(\Lambda)} = 1$ , which proves injectivity.  $\square$

Recall that solving the new interior transmission problem formulated in Definition 3.3 is equivalent to solving

$$A_k(v, f) = \ell \quad \text{in } \mathbf{X} \quad (82)$$

(where we go back to the original eigen-parameter  $k \in \mathbb{C}$  which is such that  $\lambda = -k^2$ ). Let us fix  $\lambda_0 > 0$  such that Theorem 4.5 or Theorem 4.6 holds, i.e.  $A_{\lambda_0}$  is an isomorphism and let  $k_0 = \sqrt{\lambda_0}i$ . Then we can rewrite (82) as

$$(v, f) + A_{k_0}^{-1}(A_k - A_{k_0})(v, f) = A_{k_0}^{-1}\ell \quad \text{or} \quad (I - C_k)(v, f) = A_k^{-1}\ell. \quad (83)$$

where  $C_k := A_{k_0}^{-1}(A_k - A_{k_0})$  is compact from Lemma 4.2. Thus the Fredholm alternative applies to (82). In particular a solution of the new interior transmission problem exists provided  $k$  is not a new transmission eigenvalue defined in Definition 3.4. To show that the set of new transmission eigenvalues is discrete we appeal to Fredholm Analytic Theory (see e.g. [8]) since the mapping  $k \mapsto C_k$  is analytic in  $\mathbb{C}$ . Thus we have proven the following theorem:

**Theorem 4.7.** *Assume that the bounded function  $n$  satisfies  $\inf_{\Lambda} \operatorname{Re}(n) > 0$ ,  $\operatorname{Im}(n) \geq 0$ , and there exists a neighborhood  $R$  of  $\partial\Lambda$  inside  $\Lambda$  such that either  $\sup_R \operatorname{Re}(n-1) < 0$  or  $\inf_R \operatorname{Re}(n-1) > \sup_R (\operatorname{Im}(n))^2 \geq 0$ . Then the new interior transmission formulated in Definition 3.3 has a unique solution depending continuously on the data  $\varphi$  and  $\psi$  provided  $k \in \mathbb{C}$  is not a new transmission eigenvalue defined in Definition 3.4. In particular the set of new transmission eigenvalues in  $\mathbb{C}$  is discrete (possibly empty) with  $+\infty$  as the only possible accumulation point.*

Note that Theorem 4.7 provides sufficient conditions under which Assumption 3 hold. It is highly desirable to show if and when real new transmission eigenvalues exist, because for such real wave numbers our imaging algorithm introduced in the next section fails.

## 5. A Differential Imaging Algorithm

We now apply all the results of Section 2 and Section 4 above to design an algorithm that provides us with the support of the defect  $\omega$  without reconstructing  $D_p$  or computing the Green's function of the periodic media. We follow the idea proposed in [11] and build a differential imaging functional by comparing the application of the Generalized Linear Sampling algorithm to respectively the operators  $N^{\pm}$  and  $N_q^{\pm}$ . The new results obtained in Theorem 4.7 allow us to justify this algorithm for general location of  $\omega$  (possibly multi-component).

### 5.1. Description and analysis of the algorithm

Throughout this section we assume that Assumptions 1, 2 and 3 hold. For sake of simplicity of presentation we only state the results when the measurements operator  $N^+$  is available. Exactly the same holds for the operator  $N^-$  by changing everywhere the exponent  $+$  to  $-$ . For given  $\phi$  and  $a$  in  $\ell^2(\mathbb{Z}^{d-1})$  we define the functionals

$$\begin{aligned} J_{\alpha}^+(\phi, a) &:= \alpha(N_{\#}^+ a, a) + \|N^+ a - \phi\|^2, \\ J_{\alpha, q}^+(\phi, a) &:= \alpha(N_{q, \#}^+ a, a) + \|N_q^+ a - \phi\|^2 \end{aligned} \quad (84)$$

with  $N_{q, \#}^+ := I_q^* N_{\#}^+ I_q$ . Let  $a^{\alpha, z}$ ,  $a_q^{\alpha, z}$  and  $\tilde{a}_q^{\alpha, z}$  in  $\ell(\mathbb{Z}^{d-1})$  verify (i.e. are minimizing sequences)

$$\begin{aligned} J_{\alpha}^+(\widehat{\Phi}^+(\cdot; z), a^{\alpha, z}) &\leq \inf_{a \in \ell^2(\mathbb{Z}^{d-1})} J_{\alpha}^+(\widehat{\Phi}^+(\cdot; z), a) + c(\alpha) \\ J_{\alpha}^+(\widehat{\Phi}_q^+(\cdot; z), a_q^{\alpha, z}) &\leq \inf_{a \in \ell^2(\mathbb{Z}^{d-1})} J_{\alpha}^+(\widehat{\Phi}_q^+(\cdot; z), a) + c(\alpha) \\ J_{\alpha, q}^+(I_q^* \widehat{\Phi}_q^+(\cdot; z), \tilde{a}_q^{\alpha, z}) &\leq \inf_{a \in \ell^2(\mathbb{Z}^{d-1})} J_{\alpha, q}^+(I_q^* \widehat{\Phi}_q^+(\cdot; z), a) + c(\alpha) \end{aligned} \quad (85)$$

with  $\frac{c(\alpha)}{\alpha} \rightarrow 0$  as  $\alpha \rightarrow 0$ . Here  $\widehat{\Phi}^{\pm}(\cdot; z)$  are the Rayleigh coefficients of  $\Phi(1, z)$  (i.e.  $\Phi(n_p; z)$  defined by (7) with  $n_p = 1$ ) given by (15) and  $\widehat{\Phi}_q^{\pm}(\cdot; z)$  are the Rayleigh coefficients of  $\Phi_q(\cdot - z)$  given by (21).

The standard analysis of the generalized linear sampling method (see e.g. [7, Section 2.2]) making use of the factorization of  $N_{\#}$  in Theorem 2.3 along with all the properties of the involved operators developed in Section 2.2 and Section 3 imply the following results (see also [11] and [18] for detailed proofs)

**Lemma 5.1.** (i)  $z \in D$  if and only if  $\lim_{\alpha \rightarrow 0} (N_{\#}^+ a^{\alpha, z}, a^{\alpha, z}) < \infty$ . Moreover, if  $z \in D$  then  $\mathcal{H}^+ a^{\alpha, z} \rightarrow v_z$  in  $L^2(D)$  where  $(u_z, v_z)$  is the solution of problem (14) with  $\varphi = \Phi(1; z)$  and  $\psi = \partial\Phi(1; z)/\partial\nu$  on  $\partial D$ .

(ii)  $z \in D_p$  if and only if  $\lim_{\alpha \rightarrow 0} (N_{\#}^+ a_q^{\alpha, z}, a_q^{\alpha, z}) < \infty$ . Moreover, if  $z \in D_p$  then  $\mathcal{H}^+ a_q^{\alpha, z} \rightarrow v_z$  in  $L^2(D)$  where  $(u_z, v_z)$  is the solution of problem (14) with  $\varphi = \Phi_q(\cdot - z)$  and  $\psi = \partial\Phi_q(\cdot - z)/\partial\nu$  on  $\partial D$ .

(iii)  $z \in \widehat{D}_p$  if and only if  $\lim_{\alpha \rightarrow 0} (N_{q, \#}^+ \tilde{a}_q^{\alpha, z}, \tilde{a}_q^{\alpha, z}) < \infty$ . Moreover, if  $z \in \widehat{D}_p$  then  $\mathcal{H}_q^+ \tilde{a}_q^{\alpha, z} \rightarrow h_z$  in  $L^2(D)$  where  $h_z$  is defined by

$$h_z = \begin{cases} -\Phi_q(\cdot - z) & \text{in } \Lambda_p \\ v_z & \text{in } \mathcal{O}_p^c \end{cases} \quad \text{if } z \in \mathcal{O}_p^c \quad (86)$$

$$h_z = \begin{cases} \widehat{v}_z & \text{in } \Lambda_p \\ -\Phi_q(\cdot - z) & \text{in } \mathcal{O}_p^c \end{cases} \quad \text{if } z \in \Lambda_p$$

where  $(u_z, v_z)$  is the solution of problem (14) with  $\varphi = \Phi_q(\cdot - z)$  and  $\psi = \partial\Phi_q(\cdot - z)/\partial\nu$  on  $\partial D$  and  $(\widehat{u}_z, \widehat{v}_z)$  is  $\alpha_q$ -quasi-periodic extension of the solution  $(u, f)$  of the new interior transmission problem in Definition (3.3) with  $\varphi = \Phi_q(\cdot - z)$  and  $\psi = \partial\Phi_q(\cdot - z)/\partial\nu$  on  $\partial\Lambda$ .

*Proof.* The proof of the items (i) and (ii), we refer to [11]. The proof of items (iii) is a direct application of Theorem A.4 in [11] in combination with Theorem 3.5.  $\square$

We then consider the following imaging functional that characterizes  $\Lambda$ ,

$$\mathcal{I}_{\alpha}^+(z) = \left( (N_{\#}^+ a^{\alpha, z}, a^{\alpha, z}) \left( 1 + \frac{(N_{\#}^+ a^{\alpha, z}, a^{\alpha, z})}{D^+(a_q^{\alpha, z}, \tilde{a}_q^{\alpha, z})} \right) \right)^{-1} \quad (87)$$

where for  $a$  and  $b$  in  $\ell^2(\mathbb{Z}^{d-1})$ ,

$$D^+(a, b) := (N_{\#}^+(a - I_q b), (a - I_q b)).$$

Before giving the main theorem for the characterization of the defect we need the following assumption.

**Assumption 4.** The refractive indexes  $n$ ,  $n_p$  and wave-number  $k$  are such that

$$\begin{cases} \Delta u + k^2 n u = 0 & \text{in } \omega, \\ \Delta v + k^2 n_p v = 0 & \text{in } \omega, \\ u - v = 0 & \text{on } \partial\omega, \\ \partial(u - v)/\partial\nu = 0 & \text{on } \partial\omega \end{cases} \quad (88)$$

has only the trivial solution.

This assumption is satisfied if  $n - n_p$  does not change sign in a neighborhood of the boundary of domain  $\omega$ , or if either  $n$  or  $n_p$  have non-zero imaginary part (see e.g [7]).

**Theorem 5.2.** *Under Assumptions 1, 2,3 and 4, we have that*

$$z \in D \setminus \mathcal{O}_p^c \quad \text{if and only if} \quad \lim_{\alpha \rightarrow 0} \mathcal{I}_\alpha^+(z) > 0.$$

(Note that  $D \setminus \mathcal{O}_p^c = \omega \cup \mathcal{O}_p$  contains the physical defect  $\omega$  and  $\mathcal{O}_p := D_p \setminus \mathcal{O}_p^c$  the components of  $D_p$  which have nonempty intersection with the defect).

*Proof. Case 1:  $z \notin D \setminus \mathcal{O}_p^c$ .* If  $z \notin D$  then from Lemma 5.1(i) we have that  $(\mathbb{N}_\#^+ a^{\alpha,z}, a^{\alpha,z}) \rightarrow +\infty$  as  $\alpha \rightarrow 0$  and therefore  $\lim_{\alpha \rightarrow 0} \mathcal{I}_\alpha^+(z) = 0$ .

If  $z \in \mathcal{O}_p^c$ , let  $(u_z, v_z) \in L^2(D) \times L^2(D)$  be the unique solution of (14) with  $\varphi := \Phi_q(\cdot - z)|_{\partial D}$  and  $\psi = \partial \Phi_q(\cdot - z)/\partial \nu|_{\partial D}$  then from Lemma 5.1(ii),  $\mathcal{H}^+ a^{\alpha,z} \rightarrow v_z$ . Note that in this case  $u_z = 0$  and  $v_z := -\Phi_q(\cdot - z)$  outside  $\mathcal{O}_p^c$ .

Now let  $h_z \in L^2(\widehat{D}_p)$  defined by (86). Then from Lemma 5.1(iii),  $\mathcal{H}_q^+ \tilde{a}_q^{\alpha,z} \rightarrow h_z$  in  $L^2(D)$ . Furthermore, by definition of  $h_z$  and the fact that  $v_z = -\Phi_q(\cdot - z)$  in  $\Lambda_p$ , we have that  $v_z$  coincide with  $h_z$  in  $D$ . From the factorization of  $\mathbb{N}_\#^+$  and the definition of  $\mathcal{H}_q^+$  we have

$$\begin{aligned} D^+(a_q^{\alpha,z}, \tilde{a}_q^{\alpha,z}) &= (\mathbb{N}_\#^+(a_q^{\alpha,z} - I_q \tilde{a}_q^{\alpha,z}), a_q^{\alpha,z} - I_q \tilde{a}_q^{\alpha,z}) \\ &= (\mathbb{T}_\#(\mathcal{H}^+ a_q^{\alpha,z} - \mathcal{H}_q^+ \tilde{a}_q^{\alpha,z}), \mathcal{H}^+ a_q^{\alpha,z} - \mathcal{H}_q^+ \tilde{a}_q^{\alpha,z}) \\ &\leq \|\mathbb{T}_\#\| \|\mathcal{H}^+ a_q^{\alpha,z} - \mathcal{H}_q^+ \tilde{a}_q^{\alpha,z}\|_{L^2(D)}^2 \rightarrow 0, \quad \text{as } \alpha \rightarrow 0. \end{aligned}$$

Since from Lemma 5.1(i),  $(\mathbb{N}_\#^+ a^{\alpha,z}, a^{\alpha,z})$  remains bounded as  $\alpha \rightarrow 0$ , we can finally conclude from the above that

$$\lim_{\alpha \rightarrow 0} \mathcal{I}_\alpha^+(z) = 0 \quad \text{if } z \in \mathcal{O}_p^c.$$

*Case 2:  $z \in D \setminus \mathcal{O}_p^c \subset \Lambda_p$ .* Then again by Lemma 5.1(i),  $(\mathbb{N}_\#^+ a^{\alpha,z}, a^{\alpha,z})$  remains bounded and by Lemma 5.1(iii),  $(\mathbb{N}_{q,\#} \tilde{a}_q^{\alpha,z}, \tilde{a}_q^{\alpha,z})$  remain bounded. Using the factorization of  $\mathbb{N}_{q,\#}^+$  and the fact that  $\mathbb{N}_{q,\#}^+ = I_q^* \mathbb{N}_\#^+ I_q$  we can write

$$(\mathbb{N}_\#^+ I_q \tilde{a}_q^{\alpha,z}, I_q \tilde{a}_q^{\alpha,z}) \rightarrow (\mathbb{T}_\# h_z, h_z) < +\infty,$$

where  $h_z \in L^2(\widehat{D}_p)$  defined in (86). In this case, we need to consider the location of  $z$  in two sub-domain.

First, if  $z \in \omega \setminus D_p$  (the part of defect outside periodic domain), then by Lemma 5.1(ii)  $(\mathbb{N}_\#^+ a_q^{\alpha,z}, a_q^{\alpha,z}) \rightarrow +\infty$  as  $\alpha \rightarrow 0$ . This implies,

$$D^+(a_q^{\alpha,z}, \tilde{a}_q^{\alpha,z}) \geq |(\mathbb{N}_\#^+ a_q^{\alpha,z}, a_q^{\alpha,z}) - (\mathbb{N}_\#^+ I_q \tilde{a}_q^{\alpha,z}, I_q \tilde{a}_q^{\alpha,z})| \rightarrow +\infty.$$

We then conclude that

$$\lim_{\alpha \rightarrow 0} \mathcal{I}_\alpha^+(z) \neq 0 \quad \text{if } z \in \omega \setminus D_p.$$

Next, if  $z \in D_p \cap \Lambda_p = \mathcal{O}_p$ , again by Lemma 5.1(ii) ( $\mathbb{N}_\#^+ a_q^{\alpha, z}, a_q^{\alpha, z} \rightarrow (T_\# v_z, v_z)$  where  $v_z \in L^2(D)$  is defined in Lemma 5.1(ii). In this case  $v_z = -\Phi_q(\cdot - z)$  outside  $D \setminus \mathcal{O}_p^c$ , which implies that  $h_z = v_z$  in  $\mathcal{O}_p^c$ . On the other hand,  $h_z|_{\mathcal{O}_p}$  is  $\alpha_q$ -quasi-periodic with period  $L$  while  $v_z|_{\mathcal{O}_p}$  is not  $\alpha_q$ -quasi-periodic with period  $L$  (recall that  $(u_z, v_z)$  is a solution of (14) defined in  $\mathcal{O}_p^c$  with  $\varphi = \Phi_q(\cdot - z)$  and  $\psi = \partial\Phi_q(\cdot - z)/\partial\nu$  on  $\partial(D \setminus \mathcal{O}_p^c)$ ). Indeed, assume to the contrary that  $v_z|_{\mathcal{O}_p}$  is  $\alpha_q$ -quasi periodic with period  $L$ . For a fixed arbitrary  $0 \neq m \in \mathbb{Z}_M$ , let us define

$$\tilde{u}_z(x) := e^{-i\alpha_q m L} u_z(x + mL), \quad \text{for } x \in \mathcal{O},$$

hence  $(\tilde{u}_z, v_z)$  satisfy

$$\begin{cases} \Delta \tilde{u}_z + k^2 n_p \tilde{u}_z = 0 & \text{in } \mathcal{O}, \\ \Delta v_z + k^2 v_z = 0 & \text{in } \mathcal{O}, \\ \tilde{u}_z - v_z = \Phi_q(\cdot - z) & \text{on } \partial\mathcal{O}, \\ \partial(\tilde{u}_z - v_z)/\partial\nu = \partial/\partial\nu \Phi_q(\cdot - z) & \text{on } \partial\mathcal{O}. \end{cases} \quad (89)$$

Next, we let

$$\hat{n} = \begin{cases} n_p & \text{in } \mathcal{O} \\ 1 & \text{in } \omega \setminus \mathcal{O} \end{cases} \quad \text{and} \quad \hat{u}_z = \begin{cases} \tilde{u}_z & \text{in } \mathcal{O} \\ \Phi_q(\cdot - z) + v_z & \text{in } \omega \setminus \mathcal{O} \end{cases}$$

and observe that  $(\hat{u}_z, v_z)$  verify

$$\begin{cases} \Delta \hat{u}_z + k^2 \hat{n} \hat{u}_z = 0 & \text{in } \Lambda, \\ \Delta v_z + k^2 v_z = 0 & \text{in } \Lambda, \\ \hat{u}_z - v_z = \Phi_q(\cdot - z) & \text{on } \partial\Lambda, \\ \partial(\hat{u}_z - v_z)/\partial\nu = \partial\Phi_q(\cdot - z)/\partial\nu & \text{on } \partial\Lambda, \end{cases} \quad (90)$$

where we use the fact that from (89) the Cauchy data of  $\tilde{u}_z$  and  $\Phi_q(\cdot - z) + v_z$  coincide on  $\partial\mathcal{O} \cap \omega$ . From the definition of  $u_z$  and (90) we have that  $(u_z, \hat{u}_z)$  is a solution to

$$\begin{cases} \Delta u_z + k^2 n u_z = 0 & \text{in } \Lambda, \\ \Delta \hat{u}_z + k^2 \hat{n} \hat{u}_z = 0 & \text{in } \Lambda, \\ u_z - \hat{u}_z = 0 & \text{on } \partial\Lambda, \\ \partial(u_z - \hat{u}_z)/\partial\nu = 0 & \text{on } \partial\Lambda. \end{cases} \quad (91)$$

Since  $n = n_p = \hat{n}$  in  $\Lambda \setminus \omega$ , then  $u_z$  and  $\hat{u}_z$  satisfy the same equation in  $\Lambda \setminus \omega$  and share the same Cauchy data on  $\partial\Lambda \setminus \bar{\omega}$ , hence by the unique continuation  $u_z = \hat{u}_z$  in  $\Lambda \setminus \omega$ . Therefore,  $(u_z, \hat{u}_z)$  is a solution of (88) and Assumption 4 implies that  $u_z = \hat{u}_z = 0$  in  $\omega$ , and consequently by definition  $v_z + \Phi_q(\cdot - z) = 0$  in  $\omega \setminus \mathcal{O}$  (recall  $\Lambda = \mathcal{O} \cup \omega$ ). On the other hand, we have that  $v := v_z + \Phi_q(\cdot - z)$  satisfies  $\Delta v + k^2 v = 0$  in  $\Lambda \setminus \{z\}$ , hence

unique continuation implies that  $v_z = -\Phi_q(\cdot - z)$  in  $\Lambda \setminus \{z\}$ . But this is a contradiction, since by the interior regularity of solutions of Helmholtz equation  $v_z$  is infinitely many times differentiable whereas  $\Phi$  has a singularity at  $z$ . This proves that  $v_z|_{\mathcal{O}_p}$  is  $\alpha_q$ -quasi periodic with period  $L$ , and hence  $v_z \neq h_z$  in  $D \setminus \mathcal{O}_p^c$ . We now have from the estimate

$$C \|\mathcal{H}^+ a_q^{\alpha,z} - \mathcal{H}_q^+ \tilde{a}_q^{\alpha,z}\|_{L^2(D)}^2 \leq D^+(a_q^{\alpha,z}, \tilde{a}_q^{\alpha,z}) \leq \|T_\# \| \|\mathcal{H}^+ a_q^{\alpha,z} - \mathcal{H}_q^+ \tilde{a}_q^{\alpha,z}\|_{L^2(D)}^2$$

where  $C$  is the coercivity constant associated with  $T_\#$ , that  $D^+(a_q^{\alpha,z}, \tilde{a}_q^{\alpha,z})$  is bounded and does not go to 0 as  $\alpha \rightarrow 0$ . Thus

$$\lim_{\alpha \rightarrow 0} \mathcal{I}_\alpha^+(z) \neq 0 \quad \text{if } z \in \mathcal{O}_p.$$

This ends the proof of the theorem.  $\square$

Theorem 5.2 shows that the functional  $\mathcal{I}_\alpha^+(z)$  provides an indicator function for  $D \setminus \mathcal{O}^c$ , i.e. the defect and the periodic components of the background that intersects  $\omega$ . However, the proof of Theorem 5.2 indicates that although the values of  $\mathcal{I}_\alpha^+(z)$  are positive only in  $D \setminus \mathcal{O}^c$ , they are smaller for  $z \in \mathcal{O}_p$  compared to the values of  $\mathcal{I}_\alpha^+(z)$  for  $z \in \omega \setminus D_p$ . Therefore, if the defect  $\omega$  has nonempty intersection with the periodic background then the reconstruction of  $\mathcal{O}_p$  (i.e. the components that have no nonempty intersection with the defect) are not displayed as clearly as the reconstruction of the part of defect outside  $D_p$ . This is illustrated in the following numerical experiments.

We recall that exactly the same can be shown for down-to-up incident field, by simply replacing the upper index  $+$  with  $-$ . It is also possible to handle the case with noisy data, and we refer the reader to [11] and [18] for more detailed discussion.

## 5.2. Numerical Experiments

We conclude by showing several numerical examples to test our differential imaging algorithm. We limit ourselves to examples in  $\mathbb{R}^2$ . The data is computed with both down-to-up and up-to-down plane-waves by solving the forward scattering problem based on the spectral discretization scheme of the volume integral formulation of the problem presented in [10].

Let us denote by

$$\mathbb{Z}_{inc}^{d-1} := \{j = q + M\ell, q \in \mathbb{Z}_M^{d-1}, \ell \in \mathbb{Z}^{d-1} \text{ and } \ell \in \llbracket -N_{min}, N_{max} \rrbracket\}$$

the set of indices for the incident waves (which is also the set of indices for measured Rayleigh coefficients). The values of all parameters used in our experiments will be indicated below. The discrete version of the operators  $N^\pm$  are given by the  $N_{inc} \times N_{inc}$  matrixes

$$N^\pm := \left( \widehat{u}^{s^\pm}(\ell; j) \right)_{\ell, j \in \mathbb{Z}_{inc}^{d-1}}. \quad (92)$$

Random noise is added to the data. More specifically, **in our computations we use**

$$N^{\pm, \delta}(j, \ell) := N^\pm(j, \ell)(1 + \delta A(j, \ell)), \quad \forall (j, \ell) \in \mathbb{Z}_{inc}^{d-1} \times \mathbb{Z}_{inc}^{d-1} \quad (93)$$

where  $A = (A(j, \ell))_{N_{inc} \times N_{inc}}$  is a matrix of uniform complex random variables with real and imaginary parts in  $[-1, 1]^2$  and  $\delta > 0$  is the noise level. In our examples we take  $\delta = 1\%$ .

For noisy data, one needs to redefine the functionals  $J_\alpha^+$  and  $J_{\alpha,q}^+$  as

$$\begin{aligned} J_\alpha^{+, \delta}(\phi, a) &:= \alpha \left( (N_\#^{+, \delta} a, a) + \delta \|N_\#^{+, \delta}\| \|a\|^2 \right) + \|N^{+, \delta} a - \phi\|^2, \\ J_{\alpha,q}^{+, \delta}(\phi, a) &:= \alpha \left( (N_\#^{+, \delta} I_q a, I_q a) + \delta \|N_\#^{+, \delta}\| \|a\|^2 \right) + \|N_q^{+, \delta} a - \phi\|^2 \end{aligned} \quad (94)$$

We then consider  $a_\delta^{\alpha, z}$ ,  $a_{q, \delta}^{\alpha, z}$  and  $\tilde{a}_{q, \delta}^{\alpha, z}$  in  $\ell(\mathbb{Z}^{d-1})$  as the minimizing sequence of, respectively,

$$J_\alpha^{+, \delta}(\widehat{\Phi}^+(\cdot; z), a), \quad J_{\alpha,q}^{+, \delta}(\widehat{\Phi}_q^+(\cdot; z), a) \quad \text{and} \quad J_{\alpha,q}^{+, \delta}(\widehat{\Phi}_q^+(\cdot; z), a).$$

The noisy indicator function takes the form

$$\mathcal{I}_\alpha^{+, \delta}(z) = \left( \mathcal{G}^{+, \delta}(a_\delta^{\alpha, z}) \left( 1 + \frac{\mathcal{G}^{+, \delta}(a_\delta^{\alpha, z})}{D^{+, \delta}(a_{q, \delta}^{\alpha, z}, \tilde{a}_{q, \delta}^{\alpha, z})} \right) \right)^{-1} \quad (95)$$

where for  $a$  and  $b$  in  $\ell^2(\mathbb{Z}^{d-1})$ ,

$$D^{+, \delta}(a, b) := \left( N_\#^{+, \delta}(a - I_q b), (a - I_q b) \right)$$

and

$$\mathcal{G}^{+, \delta}(a) := (N_\#^{+, \delta} a, a) + \delta \|N_\#^{+, \delta}\| \|a\|^2.$$

Defining in a similar way the indicator function  $\mathcal{I}^{-, \delta}(z)$  corresponding to up-to-down incident waves, we use the following indicator function in our numerical examples

$$\mathcal{I}^\delta(z) := \mathcal{I}^{+, \delta}(z) + \mathcal{I}^{-, \delta}(z).$$

In the following example, we consider the same periodic background  $D_p$ , in which each cell consists of two circular components, namely the discs with radii  $r_1, r_2$  specified below. The physical parameters are set as

$$k = 3.5\pi/3.14, \quad n_p = 2 \text{ inside the discs and } n_p = 1 \text{ otherwise.} \quad (96)$$

Letting  $\lambda := 2\pi/k$  be the wavelength, the geometrical parameters are set as

$$\text{the period } L = \pi\lambda, \text{ the width of the layer } h = 1.5\lambda, \quad r_1 = 0.3\lambda, \text{ and } r_2 = 0.4\lambda. \quad (97)$$

Finally we choose the truncated model

$$M = 3, \quad N_{min} = 5 \quad \text{and} \quad N_{max} = 5 \quad \text{and} \quad q = 1 \quad (98)$$

The reconstructions are displayed by plotting the indicator function  $\mathcal{I}^\delta(z)$ .

*Example 1:* In the first example, we consider the perturbation  $\omega$  be a disc of radius  $r_\omega = 0.25\lambda$  with the refractive index  $n = 4$  and located in the component of radii  $r_2$  (see Figure 2-left). In this case we reconstruct the whole component which contains the defect and its  $L$ -periodic copies (see Figure 2-right). We remark that this is the unfortunate case when it is not possible to determine in which period is the defect located. This is in accordance with what Theorem 5.2 predicts.

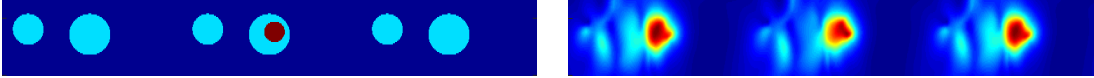


Figure 2: Left: The exact geometry for Example 1. Right: The reconstruction using  $z \mapsto \mathcal{I}^\delta(z)$ .

*Example 2 (a).* In the second example, we consider the perturbation  $\omega$  as in Example 1 but now located such that  $\omega$  has nonempty intersection with  $D_p$  but not included in  $D_p$  (see Figure 3-left). The reconstruction is represented in Figure 2-right. This example also illustrate that the value of  $\mathcal{I}^\delta(z)$  much bigger when  $z$  in  $\omega \setminus D_p$  than  $z$  in  $\mathcal{O}_p$ . Now we are able to find in which period the defect is located and recover clearly the part of the defect outside the component.

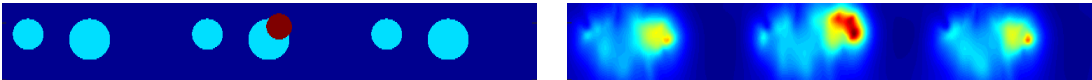


Figure 3: Left: The exact geometry for Example 2. Right: The reconstruction of the local perturbation using  $z \mapsto \mathcal{I}^\delta(z)$ .

*Example 2 (b).* In Figure 4 we consider same configuration as in Figure 3 and change only the refractive index of the defect which now is inhomogeneous. In particular, the refractive index of the defect is  $n = 4$  in  $\omega \cap D_p$  and  $n = 3$  in  $\omega \setminus D_p$ . The reconstruction is represented in Figure 4-right

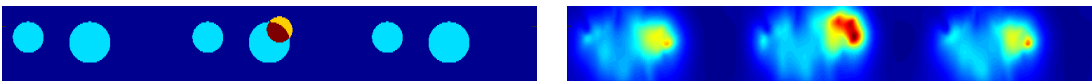


Figure 4: Left: The exact geometry for Example 2. Right: The reconstruction of the local perturbation using  $z \mapsto \mathcal{I}^\delta(z)$ .



*Example 3.* Example 3 just show that when the defect has no intersection with the periodic background, the indicator function  $\mathcal{I}^\delta(z)$  allows to reconstruct the true defect including its true location. Here the defect is a disc of  $r_\omega = 0.2\lambda$  with  $n = 3$ . More examples of this case can be found in [11].



Figure 5: Left: The exact geometry for Example 3. Right: The reconstruction using  $z \mapsto \mathcal{I}^\delta(z)$ .

The last two examples, present the case where the defect consists of two disconnected components.

*Example 4.* In this example the multicomponent defect has one component intersecting  $D_p$  and one component outside  $D_p$ . The true geometry is given in Figure 6-left and the reconstruction in Figure 6-right. Here  $\omega_1$  is the disc intersecting  $D_p$  and  $\omega_2$  the other. The parameters are  $r_{\omega_1} = r_{\omega_2} = 0.2\lambda$ .  $\omega_1$  is inhomogeneous with refractive index  $n = 4$  in the part inside  $D_p$  and  $n = 3$  in the part outside  $D_p$ , whereas the refractive index of  $\omega_2$  is  $n = 2.5$ .

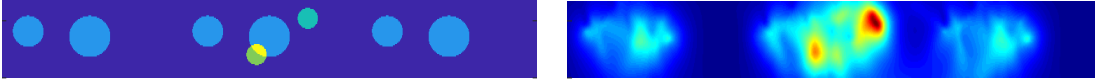


Figure 6: Left: The exact geometry for **Example 4**. Right: The reconstruction using  $z \mapsto \mathcal{I}^\delta(z)$ .

*Example 5.* In this last example, the defect has two disconnected components such that one is included in one component of  $D_p$  and the other lies outside  $D_p$ . An illustration of exact geometry is given in Figure 7-left and the reconstruction in Figure 7-right. Here keeping the same notations for  $\omega_1$  and  $\omega_2$  as in Example 5, we choose  $n = 4$  the refractive index of  $\omega_1$  and  $n = 2.5$  the refractive index of  $\omega_2$ .

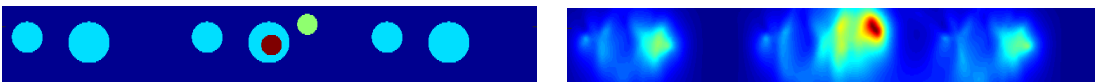


Figure 7: Left: The exact geometry for Example 5. Right: The reconstruction using  $z \mapsto \mathcal{I}^\delta(z)$ .

All our numerical examples validate the theoretical prediction provided by Theorem 5.2. As already mention the case when the defect is entirely included in a component of the periodic background is ambiguous in the sense that the actual defective period can not be determined.

We conclude by remarking that everything here can be adapted to the case when the background periodic layer is composed of inhomogeneities embedded in homogeneous media with constant refractive index  $n_0 \neq 1$ . The only difference is in the choice in the fundamental solution which in this case should correspond to a piecewise homogeneous media instead of  $n = 1$ .

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