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Quantum Kählerian Lie groups from multiplicative unitaries

P. Bieliavsky*, Ph. Bonneau†, F. D’Andrea‡ and V. Gayral§

Abstract

We show that the deformation theory of Fréchet algebras for actions of Kählerian Lie groups developed by two of us in [6], leads in a natural way to examples of non-compact locally compact quantum groups. This is achieved by constructing a manageable multiplicative unitary out of the Fréchet deformation of $C_0(G)$ for the action $\lambda \otimes \rho$ of $G \times G$ and the undeformed coproduct. We also prove that these quantum groups are isomorphic to those constructed out of the unitary dual 2-cocycle discovered by Neshveyev and Tuset in [24] and associated with Bieliavsky’s covariant \star -product [3], via the De Commer’s results [10].

Keywords: Locally compact quantum group, Manageable multiplicative unitary, Covariant quantization, Deformation of Fréchet algebras, Unitary dual 2-cocycle

Contents

1	Introduction	2
2	Deformation quantization for actions of Kählerian Lie groups	3
2.1	Notations	4
2.2	Negatively curved Kählerian Lie groups	5
2.3	Functions spaces	6
2.4	Oscillatory integrals and deformation of Fréchet algebras	9
3	Quantum Kählerian Lie groups	13
3.1	Deformations of the Schwartz algebra	13
3.2	The deformed Kac-Takesaki operator	15
3.3	The coproduct	17
3.4	The invariant weight	20
3.5	Unitarity and multiplicativity	24
4	Properties of the multiplicative unitary	25
4.1	The legs of V_θ	25
4.2	Manageability and the antipode	27
4.3	Equivalence with De Commer’s approach	29
A	Proof of Theorem 2.5	33

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1 Introduction

Locally compact quantum groups in the setting of von Neumann algebras [17, 18] (LCQG in short) is certainly the most comprehensive and well established theory of quantum groups in the framework of operator algebras. However, the theory still suffers from lack of examples. Until recently, there were only five individual examples and two general procedures to construct new locally compact quantum group out of a given one (see for instance [31, Section 8.4]). The situation changed recently after the seminal work of De Commer [10] which, in fact, reveals a strong link between equivariant quantization and LCQG. (The relationships between equivariant quantization on groups and operator algebras has a long history and was first observed by Landstad and Raeburn [19, 20, 21, 22] and by Rieffel [26, 27, 28].) In turn, this paper fits in a research program [5, 6, 7, 8, 14, 15] aiming to construct operator algebraic objects from equivariant quantization on groups.

As a special instance of his general machinery, De Commer promoted cocycle deformation to the von Neumann algebraic setting: one can produce a new LCQG out of a given one and of a unitary 2-cocycle. We are mainly concerned here with unitary *dual* 2-cocycle on a LCQG (i.e. unitary 2-cocycle on the dual LCQG) which, by combination with quantum group duality [17, 18], provides a deformation process for the direct LCQG too. Already for a genuine locally compact group, an explicit construction of a unitary dual 2-cocycle is a very difficult task (while a unitary 2-cocycle is nothing but an ordinary \mathbb{T} -valued 2-cocycle on the group). To our knowledge, only one example of a non-classical dual 2-cocycle exists yet. By non-classical, we mean an example which is not associated to an Abelian group (or subgroup) nor to the (quantum group) dual of a non-Abelian group. In both cases, dual 2-cocycles correspond to ordinary group 2-cocycles (see for example [12, 13, 16, 19, 28, 32] for applications of classical dual 2-cocycle deformations). (We shall also mention [11], where a unitary 2-cocycle is constructed on the compact quantum group $SU_q(2)$.) Now, the point is that a (nontrivial) dual 2-cocycle on a group is exactly the same thing than a (non commutative) associative and equivariant product on functions on that group, an object which is naturally produced by equivariant quantization on a group. This non-classical example of unitary dual 2-cocycle we mentioned above, comes from the work of two of us [6], where a deformation theory for C^* -algebras (generalizing Rieffel's construction [26]) for actions of Kählerian Lie groups (with negative sectional curvature) is obtained. (In fact, the observation that the main object we were working with in [6] – the deformed product – was in fact associated to a *unitary* dual 2-cocycle is due to Neshveyev and Tuset [24, Section 5].)

Our construction elaborates on Rieffel's one and we shall describe the main lines of it now. Assume that we have a locally compact group G and (\mathcal{H}, π) an irreducible unitary projective representation of G . (Projectivity is fundamental to get nontrivial constructions even when G is Abelian.) Then, by a G -covariant quantization map on the group G , we mean a continuous linear map:

$$\Omega : \mathcal{D}(G) \rightarrow \mathcal{B}(\mathcal{H}) \quad \text{such that} \quad \pi(g)\Omega(f)\pi(g)^* = \Omega(\lambda_g f), \quad \forall g \in G,$$

where λ is the left regular representation and $\mathcal{D}(G)$ is the Bruhat space of test function (which is $C_c^\infty(G)$ if G is a Lie group). When Ω extends to a unitary operator from $L_\lambda^2(G)$ to $\mathcal{L}^2(\mathcal{H})$ (the Hilbert space of Hilbert-Schmidt operators on \mathcal{H}), we can then give to $L_\lambda^2(G)$ the structure of a (unimodular) Hilbert algebra, for the transported product:

$$f_1 \star f_2 := \Omega^*(\Omega(f_1)\Omega(f_2)).$$

(All the examples of quantization on groups we have encountered meet the unitarity property but, in fact, it is its invertibility that really matters. For instance, one can easily imagine to work the GNS space of an NSF weight instead of the Hilbert-Schmidt operators.) Since the product \star is G -equivariant on the left, it is

easy to see that there exists a distribution K on $G \times G$ (in the sense of Bruhat) such that

$$f_1 \star f_2(g) = \langle K | \lambda_{g^{-1}} f_1 \otimes \lambda_{g^{-1}} f_2 \rangle, \quad \forall f_1, f_2 \in \mathcal{D}(G).$$

This is precisely the distribution K which is the object of main interest for us, for at least two reasons.

First, K allows to construct a natural candidate for a unitary dual 2-cocycle on the group G . Indeed, define (with a little abuse of notations):

$$F_\lambda := \int_{G \times G} K(g_1, g_2) \lambda_{g_1} \otimes \lambda_{g_2} d^\lambda(g_1) d^\lambda(g_2),$$

as an operator (formally) affiliated with the group von Neumann algebra $W^*(G \times G)$. Then, the associativity of the deformed product \star is equivalent to the 2-cocycle relation for F_λ^* , the formal adjoint of F_λ on $L_\lambda^2(G \times G)$. That is to say, with $\widehat{\Delta}$ the coproduct of the group von Neumann algebra $W^*(G)$, we have

$$(F_\lambda^* \otimes 1) (\widehat{\Delta} \otimes \text{Id}) (F_\lambda^*) = (1 \otimes F_\lambda^*) (\text{Id} \otimes \widehat{\Delta}) (F_\lambda^*).$$

Hence, for a G -equivariant unitary quantization on a group G , to produce a unitary dual 2-cocycle on G it only remains to check the unitarity of F_λ . Examples (beyond negatively curved Kählerian Lie groups) where the unitarity property for the dual 2-cocycle holds true, will be presented in [7].

Second, the distribution K allows to construct a natural candidate for a deformation theory of C^* -dynamical systems for G (that is a generalization of Rieffel's construction [26]). For that, we need to assume that K is regular in the sense of Bruhat (that is, smooth for Lie groups). Let then (A, α) be a C^* -algebra endowed with a continuous action of G . One can then try to define a new associative product on A by the formula:

$$a_1 \star_\alpha a_2 = \int_{G \times G} K(g_1, g_2) \alpha_{g_1}(a_1) \alpha_{g_2}(a_2) d^\lambda(g_1) d^\lambda(g_2).$$

Of course, there is little chance to give a direct meaning to \star_α since the two-point function K is generically unbounded. In practice, we first work with oscillatory integrals on a dense Fréchet subspace of A (typically the space of smooth vectors when G is a Lie group) and then define a deformed C^* -norm. This program has been successfully carried out in [6], fully generalizing Rieffel's construction [26], for all Kählerian Lie groups with negative sectional curvature. (See also [5, 14] for super-symmetric and p -adic Abelian groups.)

The aim of this paper is to construct a quantum version of any negatively curved Kählerian Lie groups from the deformation of $C_0(G)$ through the action $\lambda \otimes \rho$ of $G \times G$ and from the undeformed coproduct and Haar weight. Our construction is conceptually similar to Rieffel's one [28]. But on the technical side, we had to choose a different strategy because Rieffel construction used in a crucial way the commutativity of the group to define the coproduct at the C^* -level. Instead, and this is our first main result, we use the deformation theory (of [6]) at the level of Fréchet algebras only to construct directly a multiplicative unitary that we prove to be manageable in the sense of Woronowicz [33]. An important feature of this construction is to setup a more general strategy allowing to construct quantum groups from covariant quantizations on locally compact groups, when the underlying dual 2-cocycle is no longer unitary. We also prove, and this is our second main result, that the resulting LCQG is (unitarily) equivalent to the one associated with the underlying unitary dual 2-cocycle via De Commer's construction.

2 Deformation quantization for actions of Kählerian Lie groups

In this section, we review (and extend) the deformation theory for actions of Kählerian Lie groups built in [6], but we only need the deformation theory of Fréchet algebras and not of C^* -algebras. The only exception

is Corollary 4.15. At the level of Fréchet algebras, the results of [6] are essentially based on a construction of an oscillatory integral (for each Kählerian Lie group) together with previous works of one of us [3, 4, 2]. Both aspects rely on the geometric structures that Kählerian Lie groups are endowed with. Before explaining all this, we start by fixing general conventions.

2.1 Notations

Let G be a Lie group with Lie algebra \mathfrak{g} . We fix a left-invariant Haar measure on G , which we denote by $d^\lambda(g)$ and we associated to it a right-invariant Haar measure by $d^\rho(g) := d^\lambda(g^{-1})$. We let $L_\lambda^p(G)$ and $L_\rho^p(G)$ the L^p -spaces for the left and right Haar measures. We define the modular function¹, χ_G , to be such that the following relation holds true:

$$\chi_G(g) d^\lambda(g) := d^\rho(g). \quad (2.1)$$

By λ and ρ , we mean the left and right regular actions of G , defined for a complex valued function f by

$$\lambda_g f(g') := f(g^{-1}g'), \quad \rho_g f(g') := f(g'g).$$

Of course, λ is unitary on $L_\lambda^2(G)$ and ρ is unitary on $L_\rho^2(G)$. When f is in $L_\lambda^1(G)$ or when f is a distribution on G (whenever it makes sense), we let $\lambda(f)$ and $\rho(f)$ be the integrated representations, always with respect to the left Haar measure. By \tilde{X} and \underline{X} , we mean the left-invariant and right-invariant vector fields on G associated to the elements X and $-X$ of the Lie algebra \mathfrak{g} of G :

$$\tilde{X} := \left. \frac{d}{dt} \right|_{t=0} \rho(e^{tX}), \quad \underline{X} := \left. \frac{d}{dt} \right|_{t=0} \lambda(e^{tX}).$$

Given an element X of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, we adopt the same notations \tilde{X} and \underline{X} for the associated left- and right-invariant differential operators on G .

When looking at a group as a locally compact quantum group in the von Neumann algebraic setting [17, 18], we use standard notations (see e.g. [24, Section 1.1] for a quick summary). In particular, we let $\Delta : L^\infty(G) \rightarrow L^\infty(G \times G)$ and $S : L^\infty(G) \rightarrow L^\infty(G)$ be the classical coproduct and antipode, defined by

$$\Delta f(g, g') := f(gg'), \quad S f(g) := f(g^{-1}).$$

The modular conjugations of the group G and of its dual (*quantum group*) \widehat{G} are given by:

$$J f(g) := \overline{f(g)}, \quad \widehat{J} f(g) := \chi_G^{1/2}(g) \overline{f(g^{-1})}, \quad \forall f \in L_\lambda^2(G), \quad (2.2)$$

V and W are the multiplicative unitaries of G , acting respectively on $L_\rho^2(G \times G)$ and $L_\lambda^2(G \times G)$, given by

$$V f(g_1, g_2) = f(g_1 g_2, g_2) \quad \text{and} \quad W f(g_1, g_2) = f(g_1, g_1^{-1} g_2). \quad (2.3)$$

Algebraic tensor products will be denoted by \otimes , while $\bar{\otimes}$ will be used for completed tensor products (that will be specified in each context: von Neumann, Hilbert, C^* , Fréchet ...). Our convention for scalar products $\langle \cdot, \cdot \rangle$ of Hilbert spaces is to be conjugate linear on the left. By a multiplier of a Fréchet algebra \mathcal{A} , we mean a pair (L, R) of continuous linear operators on \mathcal{A} satisfying $L(ab) = L(a)b$, $R(ab) = aR(b)$ and $aL(b) = R(a)b$, for all $a, b \in \mathcal{A}$.

¹We insist on the terminology “modular function” because “modular weight” will be used later to define another function.

2.2 Negatively curved Kählerian Lie groups

Let G be a Kählerian group. By this we mean that G is a Lie group which, as a manifold, is endowed with a left-invariant Kählerian structure. From the work of Pyatetskii-Shapiro [25], one knows that every Kählerian Lie group whose sectional curvature is negative (negatively curved Kählerian group in short) is an iterated split extension

$$G = (\mathbb{S}_N \times \dots) \times \mathbb{S}_1, \quad (2.4)$$

where each elementary block \mathbb{S} is isomorphic to the Iwasawa factor AN of the simple Lie group $SU(1, n)$. As a manifold, \mathbb{S} is isomorphic to $\mathbb{R} \times V \times \mathbb{R}$, where (V, ω) is a symplectic vector space, with group law

$$(a, v, t) \cdot (a', v', t') = \left(a + a', e^{-a'}v + v', e^{-2a'}t + t' + \frac{1}{2}e^{-a'}\omega(v, v') \right). \quad (2.5)$$

The Lebesgue measure on $\mathbb{R} \times V \times \mathbb{R}$ therefore defines a left invariant Haar measure on \mathbb{S} and the modular function reads $\chi_{\mathbb{S}}(a, v, t) = e^{\dim(\mathbb{S})a}$.

In particular, every negatively curved Kählerian group is connected and simply connected, solvable, non-unimodular and exponential (by which we mean that $\exp : \mathfrak{g} \rightarrow G$ is a global diffeomorphism). One of the most important feature (here) about Pyatetskii-Shapiro's theory, is that the extension homomorphisms at each steps in the decomposition (2.4) of a negatively curved Kählerian group in elementary blocks:

$$\mathbf{R}^j \in \text{Hom}((\mathbb{S}_N \times \dots) \times \mathbb{S}_{j+1}, \text{Aut}(\mathbb{S}_j)), \quad j = 1, \dots, N-1, \quad (2.6)$$

take values in the linear symplectic group $\text{Sp}(V_j, \omega_j)$. Here (V_j, ω_j) denotes the symplectic vector space attached to \mathbb{S}_j . In particular, the associated automorphisms of \mathbb{S}_j , $\mathbf{R}_{g'}^j$, $g' \in (\mathbb{S}_N \times \dots) \times \mathbb{S}_{j+1}$, preserve both left and right Haar measures on \mathbb{S}_j . This implies that the product of Lebesgue measures on the \mathbb{S}_j 's is a left Haar measure on G in both parametrizations $g = g_1 \dots g_N$ and $g = g_N \dots g_1$ where $g \in G$ and $g_j \in \mathbb{S}_j$. This also implies that the modular function of G is $\chi_G(g) = e^{\sum_{j=1}^N \dim(\mathbb{S}_j)a_j}$.

Each elementary factor \mathbb{S} possesses another important geometric structure, namely the formula

$$s(a, v, t; a', v', t') := (2a - a', 2v \cosh(a - a') - v', 2t \cosh(2a - 2a') - t' + \omega(v, v') \sinh(a - a')), \quad (2.7)$$

endows the manifold \mathbb{S} with the structure of a \mathbb{S} -equivariant symplectic symmetric space, for the left-invariant symplectic form given in coordinates (2.5) by $\Omega := 2da \wedge dt + \omega$. This means that $s : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{S}$ is a smooth map such that the associated symmetries

$$s_g : \mathbb{S} \rightarrow \mathbb{S}, \quad g' \mapsto s(g, g'), \quad \forall g \in \mathbb{S},$$

are involutive diffeomorphisms of \mathbb{S} , admitting g as an isolated fixed point, satisfying the relations

$$s_g \circ s_{g'} \circ s_g = s_{s_g(g')} \quad \text{and} \quad s(gg', gg'') = gs(g, g'),$$

and leaving the symplectic form Ω invariant.

The automorphism group $\text{Aut}(\mathbb{S}, s, \Omega)$ of the symplectic symmetric space (\mathbb{S}, s, Ω) is defined as the subgroup of symplectomorphisms $\varphi \in \text{Symp}(\mathbb{S}, \Omega)$ which are covariant under the symmetries:

$$\varphi \circ s_g = s_{\varphi(g)} \circ \varphi, \quad \forall g \in \mathbb{S}.$$

It is a Lie subgroup of $\text{Symp}(\mathbb{S}, \Omega)$ that acts transitively on \mathbb{S} . It contains \mathbb{S} via left multiplication and the linear symplectic group $\text{Sp}(V, \omega)$. In fact, $\text{Sp}(V, \omega) \simeq \text{Aut}(\mathbb{S}) \cap \text{Aut}(\mathbb{S}, s, \Omega)$.

Moreover, the partial maps $s^{g'} : \mathbb{S} \rightarrow \mathbb{S}$, $g \mapsto s_g(g')$ are global diffeomorphisms and this implies that the symplectic symmetric space (\mathbb{S}, Ω, s) possesses a (unique) midpoint map

$$\text{mid} : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{S},$$

that is a smooth map such that $s_{\text{mid}(g,g')}(g) = g'$ for all $g, g' \in \mathbb{S}$. It is given by $\text{mid}(g, g') := (s^g)^{-1}(g')$. Since every $\varphi \in \text{Aut}(\mathbb{S}, s, \Omega)$ intertwines the midpoints:

$$\varphi(\text{mid}(g, g')) = \text{mid}(\varphi(g), \varphi(g')) ,$$

we deduce that the ‘‘medial triangle’’ three-point function

$$\Phi : \mathbb{S}^3 \rightarrow \mathbb{S}^3 , \quad (g_1, g_2, g_3) \mapsto (\text{mid}(g_1, g_2), \text{mid}(g_2, g_3), \text{mid}(g_3, g_1)) ,$$

is invariant under the diagonal left action of \mathbb{S} . Being moreover a global diffeomorphism of \mathbb{S}^3 , we can therefore define

$$S_{\mathbb{S}}(g_1, g_2) := \text{Area}(\Phi^{-1}(e, g_1, g_2)) \quad \text{and} \quad A_{\mathbb{S}}(g_1, g_2) := \text{Jac}_{\Phi^{-1}}^{1/2}(e, g_1, g_2) \quad \text{in} \quad C^\infty(\mathbb{S} \times \mathbb{S}, \mathbb{R}) ,$$

where $\text{Area}(g_1, g_2, g_3)$ denotes the symplectic area of the geodesic triangle in \mathbb{S} with edges g_1, g_3, g_3 . In coordinates (2.5), we have with ω the symplectic form on V :

$$S_{\mathbb{S}}(a_1, v_1, t_1; a_2, v_2, t_2) = t_2 \sinh 2a_1 - t_1 \sinh 2a_2 + \omega(v_1, v_2) \cosh a_1 \cosh a_2 ,$$

$$A_{\mathbb{S}}(a_1, v_1, t_1; a_2, v_2, t_2) = (\cosh a_1 \cosh a_2 \cosh(a_1 - a_2))^{\dim(V)/2} (\cosh 2a_1 \cosh 2a_2 \cosh(2a_1 - 2a_2))^{1/2} .$$

In the case of an arbitrary negatively curved Kählerian group G , with decomposition (2.4), and parametrizing elements of G as $g = g_1 \dots g_N$, $g_j \in \mathbb{S}_j$ (that is with the reversed order), we set

$$S_G(g, g') = \sum_{j=1}^N S_{\mathbb{S}_j}(g_j, g'_j) \quad \text{and} \quad A_G(g, g') = \prod_{j=1}^N A_{\mathbb{S}_j}(g_j, g'_j) .$$

For $\theta \in \mathbb{R}^*$, consider the two-point function on G

$$K_\theta(g_1, g_2) = \frac{4^N}{(\pi\theta)^{\dim(G)}} A_G(g_1, g_2) \exp \left\{ \frac{2i}{\theta} S_G(g_1, g_2) \right\} . \quad (2.8)$$

It has been shown by one of us [2], that the following formula

$$f_1 \tilde{\star}_\theta f_2 := \int_{G \times G} K_\theta(g_1, g_2) (\rho_{g_1} f_1) (\rho_{g_2} f_2) d^\lambda(g_1) d^\lambda(g_2) ,$$

initially defined on $C_c^\infty(G)$, extends uniquely to an associative, continuous and left- G -equivariant product on $L_\lambda^2(G)$, for which the complex conjugation is an involution.

2.3 Functions spaces

In [6], we constructed two important functions spaces on a negatively curved Kählerian group: $\mathcal{S}(G)$ and $\mathcal{B}^\mu(G)$. We review it now.

The first one, $\mathcal{S}(G)$, is an analogue of the Euclidean Schwartz space where regularity is defined in term of left invariant differential operators and decay is measured by the modular weight² \mathfrak{d}_G defined by:

$$\mathfrak{d}_G(g) := (1 + |\text{Ad}_g|^2 + |\text{Ad}_{g^{-1}}|^2)^{1/2} . \quad (2.9)$$

Here, $|\text{Ad}_g|$ is the operator norm of the adjoint action of G on \mathfrak{g} , for a chosen Euclidean structure on \mathfrak{g} . In the case of an elementary negatively curved Kählerian group \mathbb{S} and within the coordinates (2.5), $\mathfrak{d}_{\mathbb{S}}$ behaves like the function (see [6, Lemma 3.27]):

$$(a, v, t) \mapsto \cosh a + \cosh 2a + |v|(1 + e^{2a} + \cosh a) + |t|(1 + e^{2a}) .$$

² \mathfrak{d}_G which should not be confused with the modular function χ_G .

In the general case of a negatively curved Kählerian group G , with Pyatetskii-Shapiro decomposition (2.4) and under the parametrization $g = g_1 \dots g_N \in G$, $g_j \in \mathbb{S}_j$, we have the lower-bound (see [6, Lemma 3.31]):

$$\mathfrak{d}_G(g) \geq C \sum_{j=1}^N \mathfrak{d}_{\mathbb{S}_j}(g_j).$$

In particular, $\mathfrak{d}_G^{-p} \in L_\lambda^1(G) \cap L_\rho^1(G)$ for all $p > \dim(G)$. Our Schwartz space is then defined by:

$$\mathcal{S}(G) := \{f \in C^\infty(G) : \forall X \in \mathcal{U}(\mathfrak{g}), \forall n \in \mathbb{N}, \|\mathfrak{d}_G^n \tilde{X}f\|_\infty < \infty\}.$$

To a given ordered basis $\{X_1, \dots, X_{\dim(G)}\}$ of \mathfrak{g} , we let $\{X^\beta := X_1^{\beta_1} X_2^{\beta_2} \dots X_{\dim(G)}^{\beta_{\dim(G)}}, \beta \in \mathbb{N}^{\dim(G)}\}$ be the associated PBW basis of $\mathcal{U}(\mathfrak{g})$. This induces a filtration $\mathcal{U}(\mathfrak{g}) = \bigcup_{k \in \mathbb{N}} \mathcal{U}_k(\mathfrak{g})$ in terms of the subspaces $\mathcal{U}_k(\mathfrak{g}) := \{\sum_{|\beta| \leq k} C_\beta X^\beta, C_\beta \in \mathbb{R}\}$, where $|\beta| := \beta_1 + \dots + \beta_{\dim(G)}$. We then endow the finite dimensional vector space $\mathcal{U}_k(\mathfrak{g})$, with the ℓ^1 -norm $|\cdot|_k$ within the basis $\{X^\beta, |\beta| \leq k\}$:

$$|X|_k := \sum_{|\beta| \leq k} |C_\beta| \quad \text{if} \quad X = \sum_{|\beta| \leq k} C_\beta X^\beta \in \mathcal{U}_k(\mathfrak{g}), \quad (2.10)$$

and we let $S_k(\mathfrak{g})$ be the unit sphere of $\mathcal{U}_k(\mathfrak{g})$ for this norm. Then, one can define a topology on $\mathcal{S}(G)$ from the following countable set of seminorms:

$$\|f\|_{k,n}^\lambda := \sup_{X \in S_k(\mathfrak{g})} \|\mathfrak{d}_G^n \tilde{X}f\|_\infty, \quad k, n \in \mathbb{N}. \quad (2.11)$$

Of course, this topology is independent of the basis chosen. It is proven in [6, Lemma 2.41] that the Schwartz space $\mathcal{S}(G)$ is then Fréchet and nuclear. It is not hard to see that S , the antipode of $L^\infty(G)$, is an homeomorphism of $\mathcal{S}(G)$. Hence, the topology of $\mathcal{S}(G)$ can be equally described using a variant of the seminorms (2.11) constructed with right-invariant differential operators instead of left-invariant ones. We will freely use this fact and we denote this new seminorms by $\|\cdot\|_{k,n}^\rho$, $k, n \in \mathbb{N}$. Finally, note that both parametrizations of $g \in G = (\mathbb{S}_N \times \dots) \times \mathbb{S}_1$, given by $g = g_1 \dots g_N$ or by $g = g_N \dots g_1$, $g_j \in \mathbb{S}_j$, yield topological isomorphisms $\mathcal{S}(G) \simeq \mathcal{S}(\mathbb{S}_N \times \dots \times \mathbb{S}_1)$.

The second important function space is a non-Abelian and weighted analogue of Laurent Schwartz's space $\mathcal{B}(\mathbb{R}^n)$. To define it, we need first to recall the notion of weights in the sense of [6]. A function $\mu > 0$ on G is called a weight if for all $X \in \mathcal{U}(\mathfrak{g})$ there exists $C_1 > 0$ such that $|\tilde{X}\mu| + |\underline{X}\mu| \leq C_1\mu$ and if there exists $C_2, L, R > 0$ such that for all $g, g' \in G$ we have $\mu(gg') \leq C_2\mu^L(g)\mu^R(g')$. The basic example of a weight is precisely given by the modular weight \mathfrak{d}_G (see [6, Lemma 2.4]) where $C_2 = L = R = 1$. Given a tempered (which means bounded by a power of \mathfrak{d}_G) weight μ , one can consider the space

$$\mathcal{B}_\lambda^\mu(G) := \{f \in C^\infty(G) : \forall X \in \mathcal{U}(\mathfrak{g}), \|\mu^{-1} \tilde{X}f\|_\infty < \infty\}. \quad (2.12)$$

The natural topology that $\mathcal{B}_\lambda^\mu(G)$ may be endowed with, underlies the sequence of seminorms

$$\|f\|_{k;\mu}^\lambda := \sup_{X \in S_k(\mathfrak{g})} \|\mu^{-1} \tilde{X}f\|_\infty, \quad k \in \mathbb{N}. \quad (2.13)$$

It is shown in [6, Lemma 2.8] that $\mathcal{B}_\lambda^\mu(G)$ is Fréchet. For instance, it coincides for $\mu = 1$ with the space of smooth vectors for the right regular action within the C^* -algebra of right-uniformly continuous and bounded functions on G . (Our convention for the right uniform structure on a group is the one that yields strong continuity for the right regular action.) However, contrary to the Schwartz space, $\mathcal{B}_\lambda^\mu(G)$ is not stable under the group inversion and one cannot use right-invariant vector fields to define its topology. So we also need the right-invariant version of that space, namely

$$\mathcal{B}_\rho^\mu(G) := \{f \in C^\infty(G) : \forall X \in \mathcal{U}(\mathfrak{g}), \|\mu^{-1} \underline{X}f\|_\infty < \infty\}, \quad (2.14)$$

endowed with the sequence of seminorms

$$\|f\|_{k;\mu}^\rho := \sup_{X \in S_k(\mathfrak{g})} \|\mu^{-1} \underline{X}f\|_\infty, \quad k \in \mathbb{N}. \quad (2.15)$$

Since the antipode S intertwine left- and right-invariant vector fields, we deduce that $\mathcal{B}_\rho^\mu(G)$ is Fréchet and for $\mu = 1$, that it coincides with the set of smooth vectors for the left regular action within the C^* -algebra of left-uniformly continuous and bounded functions on G .

We then need the vector valued versions of these functions spaces. So, let \mathcal{E} be any Fréchet space. Since $\mathcal{S}(G)$ is nuclear, $\mathcal{S}(G, \mathcal{E})$ can be unambiguously defined as the completed tensor product $\mathcal{S}(G) \otimes \mathcal{E}$. It is convenient to consider on $\mathcal{S}(G, \mathcal{E})$ the cross-seminorms:

$$\|f\|_{k,n;j}^\lambda := \sup_{X \in S_k(\mathfrak{g})} \sup_{g \in G} \mathfrak{d}_G^n(g) \|\tilde{X}f(g)\|_j, \quad k, n \in \mathbb{N}, j \in J,$$

if $\{\|\cdot\|_j\}_{j \in J}$ is a countable set of seminorms defining the topology of \mathcal{E} . Of course, we may also consider the equivalent family of cross-seminorms:

$$\|f\|_{k,n;j}^\rho := \sup_{X \in S_k(\mathfrak{g})} \sup_{g \in G} \mathfrak{d}_G^n(g) \|\underline{X}f(g)\|_j, \quad k, n \in \mathbb{N}, j \in J.$$

To define vector valued versions of $\mathcal{B}_\lambda^\mu(G)$ and $\mathcal{B}_\rho^\mu(G)$, there is much more degree of freedom and we proceed as follows. We fix a family of tempered weights $\underline{\mu} := \{\mu_j\}_{j \in J}$, labelled by the same countable set that the one labelling the family of seminorms of \mathcal{E} . We then set:

$$\mathcal{B}_\lambda^\mu(G, \mathcal{E}) := \{f \in C^\infty(G, \mathcal{E}) : \forall X \in \mathcal{U}(\mathfrak{g}), \forall j \in J, \sup_{g \in G} \mu_j^{-1}(g) \|\tilde{X}f(g)\|_j < \infty\}. \quad (2.16)$$

Endowed with the seminorms:

$$\|f\|_{k,j;\underline{\mu}}^\lambda := \sup_{X \in S_k(\mathfrak{g})} \sup_{g \in G} \mu_j^{-1}(g) \|\tilde{X}f(g)\|_j, \quad k \in \mathbb{N}, j \in J, \quad (2.17)$$

$\mathcal{B}_\lambda^\mu(G, \mathcal{E})$ turns to be Fréchet too (see [6, Lemma 2.12]). The space $\mathcal{B}_\rho^\mu(G, \mathcal{E})$ is defined in a similar way.

Since the antipode S preserves tempered weights, we observe the following obvious but nevertheless important fact:

Lemma 2.1. *Let \mathcal{E} a complex Fréchet space, $\underline{\mu} = \{\mu_j\}_{j \in J}$ a family of tempered weights and $f \in C^\infty(G, \mathcal{E})$. Then f belongs to $\mathcal{B}_\lambda^\mu(G, \mathcal{E})$ if and only if Sf belongs to $\mathcal{B}_\rho^{S\underline{\mu}}(G, \mathcal{E})$, where $S\underline{\mu} := \{S\mu_j\}_{j \in J}$.*

Finally, we will need the two-sided version of the spaces $\mathcal{B}_\lambda^\mu(G, \mathcal{E})$ and $\mathcal{B}_\rho^\mu(G, \mathcal{E})$:

$$\mathcal{B}_\lambda^\mu \cap \mathcal{B}_\rho^\nu(G, \mathcal{E}) := \mathcal{B}_\lambda^\mu(G, \mathcal{E}) \cap \mathcal{B}_\rho^\nu(G, \mathcal{E}),$$

endowed with the topology associated to the sum seminorms:

$$\|\cdot\|_{k,j;\underline{\mu},\underline{\nu}}^{\lambda,\rho} := \|\cdot\|_{k,j;\underline{\mu}}^\lambda + \|\cdot\|_{k,j;\underline{\nu}}^\rho, \quad k \in \mathbb{N}, j \in J.$$

The space $\mathcal{B}_\lambda^\mu \cap \mathcal{B}_\rho^\nu(G, \mathcal{E})$ is also Fréchet and for $\mathcal{E} = \mathbb{C}$ we denote it by $\mathcal{B}_\lambda^\mu \cap \mathcal{B}_\rho^\nu(G)$. Note that when we have $\mu = \nu = 1$, it coincides with the space of smooth vectors for the action $\lambda \otimes \rho$ of $G \times G$, within the C^* -algebra of both left and right uniformly continuous and bounded functions on G . Note also that, in general, the space $\mathcal{B}_\lambda^\mu \cap \mathcal{B}_\rho^\nu(G)$ does not contain many interesting functions, besides elements of $\mathcal{S}(G)$. However, it follows from [6, Definition 2.1] that $\mu \in \mathcal{B}_\lambda^\mu \cap \mathcal{B}_\rho^\nu(G)$ whenever μ is a sub-multiplicative weight on G . This is in particular the case for the modular weight \mathfrak{d}_G and for the modular function χ_G (see Lemma 2.4 and the discussion preceding Definition 2.6 in [6]). The two legs versions of these spaces, namely $\mathcal{S}(G \times G, \mathcal{E})$, $\mathcal{B}_\lambda^\mu(G \times G, \mathcal{E})$, $\mathcal{B}_\rho^\nu(G \times G, \mathcal{E})$ and $\mathcal{B}_\lambda^\mu \cap \mathcal{B}_\rho^\nu(G \times G, \mathcal{E})$ are defined in a similar way.

2.4 Oscillatory integrals and deformation of Fréchet algebras

The main results of [6], concerning the deformation of Fréchet algebras, are summarised in the next two theorems.

Fix $\theta \in \mathbb{R}^*$, \mathcal{E} a Fréchet space and $\underline{\mu}$ an associated family of tempered weights on a negatively curved Kählerian Lie groups G . (That is to say, $\underline{\mu} = \{\mu_j\}_{j \in J}$ is indexed by the same countable set than the one indexing the seminorms $\{\|\cdot\|_j\}_{j \in J}$ defining the topology of \mathcal{E} .) Based on very specific properties of the two-point kernels K_θ given in (2.8), it is proven in [6] (combine Definition 2.31 and Theorem 3.35) that:

Theorem 2.2. *The oscillatory integral*

$$\mathcal{S}(G \times G, \mathcal{E}) \rightarrow \mathcal{E}, \quad f \mapsto \int K_\theta(g_1, g_2) f(g_1, g_2) d^\lambda(g_1) d^\lambda(g_2),$$

extends to a continuous linear map

$$\mathcal{B}_\lambda^\mu(G \times G, \mathcal{E}) \rightarrow \mathcal{E}, \quad f \mapsto \widetilde{\int K_\theta(g_1, g_2) f(g_1, g_2) d^\lambda(g_1) d^\lambda(g_2)}.$$

The main ideas behind Theorem 2.2 can be carried out for more general Lie groups than those considered here and go as follow. We decompose the kernel K_θ in phase and amplitude $A e^{iS}$ and we realise the Fréchet space \mathcal{E} as a countable projective limit of Banach spaces $\prod_{j \in J} \mathcal{E}_j$. Then, for each family of tempered weights $\underline{\mu} = \{\mu_j\}_{j \in J}$, one looks for a family $\underline{\mathbf{D}} := \{\mathbf{D}_j\}_{j \in J}$ of differential operators on $G \times G$ satisfying the following two properties. First, it should be such that the map

$$f \mapsto \mathbf{D}_j(A f),$$

sends $\mathcal{B}^{\mu_j}(G \times G, \mathcal{E}_j)$ to $L^1(G \times G, \mathcal{E}_j)$ continuously. With \mathbf{D}_j^* the formal adjoint of \mathbf{D}_j on $L_\lambda^2(G \times G)$, the second required property is that

$$\mathbf{D}_j^* e^{iS} = e^{iS}, \quad \forall j \in J.$$

The oscillatory integral is then defined as the sequence of \mathcal{E}_j -valued Böchner integrals:

$$\widetilde{\int K_\theta(g_1, g_2) f(g_1, g_2) d^\lambda(g_1) d^\lambda(g_2)} := \left\{ \int_{G \times G} e^{iS(g_1, g_2)} \mathbf{D}_j(A f)(g_1, g_2) d^\lambda(g_1) d^\lambda(g_2) \right\}_{j \in J}.$$

At a more concrete level, the operators \mathbf{D}_j constructed in [6] are finite products of differential operators of the form

$$f \mapsto \tilde{X} \left(\frac{f}{\alpha_X} \right), \tag{2.18}$$

where the \tilde{X} 's are (specific!) left-invariant differential operators and the α_X 's are strictly positive smooth functions defined by

$$\alpha_X := e^{-iS} \tilde{X} e^{iS}.$$

Remark 2.3. In [6], we used $C_c^\infty(G)$ instead of $\mathcal{S}(G)$ for the initial domain of the oscillatory integral but this makes no difference at all. Also, we have shown in [6, Proposition 1.32] that this extension of the oscillatory integral depends (essentially) only on $\underline{\mu}$. We however do not claim that this is the unique continuous extension.

Let now (\mathcal{A}, α) be a Fréchet algebra endowed with a strongly continuous action of a negatively curved Kählerian group G . Fix $\{\|\cdot\|_j\}_{j \in J}$ a countable family of seminorms defining the topology of \mathcal{A} . We say that

the action α is tempered if there exists a family $\underline{\mu} = \{\mu_j\}_{j \in J}$ of tempered weights on G such that for all $(g, a) \in G \times \mathcal{A}$ we have

$$\|\alpha_g(a)\|_j \leq \mu_j(g) \|a\|_j .$$

Important examples of tempered actions are λ and ρ on $\mathcal{S}(G)$ (see the proof of Lemma 2.8 below).

Let \mathcal{A}^∞ be the set of smooth vectors for the action α . By strong continuity, \mathcal{A}^∞ is dense in \mathcal{A} . On \mathcal{A}^∞ , we consider the infinitesimal form of the action α , given for $X \in \mathfrak{g}$ by:

$$X_\alpha(a) := \left. \frac{d}{dt} \right|_{t=0} \alpha_{e^{tX}}(a), \quad a \in \mathcal{A}^\infty ,$$

and we extend it to the whole universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, by declaring the map $\mathcal{U}(\mathfrak{g}) \rightarrow \text{End}(\mathcal{A}^\infty)$, $X \mapsto X_\alpha$ to be an algebra homomorphism. The subspace \mathcal{A}^∞ carries a finer topology which is associated with the seminorms:

$$\|a\|_{j,k} := \sup_{X \in \mathcal{S}_k(\mathfrak{g})} \|X_\alpha(a)\|_j, \quad j \in J, k \in \mathbb{N} . \quad (2.19)$$

The point is that the action α on \mathcal{A}^∞ is still tempered [6, Lemma 5.3] and that the function $[(g_1, g_2) \mapsto \alpha_{g_1}(a_1) \alpha_{g_2}(a_2)]$ belongs to $\mathcal{B}_\lambda^\mu(G \times G, \mathcal{A}^\infty)$. The main result of [6, Section 5] can be summarized as follows:

Theorem 2.4. *Let (\mathcal{A}, α, G) be a Fréchet algebra endowed with a strongly continuous and tempered action of a negatively curved Kählerian group. Then, there exists a family $\underline{\mu}$ of tempered weights on $G \times G$, such that we have a continuous bilinear mapping:*

$$\mathcal{A}^\infty \times \mathcal{A}^\infty \rightarrow \mathcal{B}_\lambda^\mu(G \times G, \mathcal{A}^\infty), \quad (a_1, a_2) \mapsto [(g_1, g_2) \mapsto \alpha_{g_1}(a_1) \alpha_{g_2}(a_2)] .$$

Moreover, for any value of the parameter $\theta \in \mathbb{R}^*$, the bilinear mapping

$$\star_\theta^\alpha : \mathcal{A}^\infty \times \mathcal{A}^\infty \rightarrow \mathcal{A}^\infty, \quad (a_1, a_2) \mapsto \widetilde{\int} K_\theta(g_1, g_2) \alpha_{g_1}(a_1) \alpha_{g_2}(a_2) d^\lambda(g_1) d^\lambda(g_2) ,$$

is continuous and associative. We call $(\mathcal{A}^\infty, \star_\theta^\alpha)$ the Fréchet deformation of the Fréchet algebra \mathcal{A} .

We now give a few results not proven in [6] and that we will need here. The first one concerns the right-counterpart of Theorem 2.2, that is an extension of the oscillatory integral from $\mathcal{S}(G \times G, \mathcal{E})$ to $\mathcal{B}_\rho^\mu(G \times G, \mathcal{E})$. To do so, we employ the same strategy as the one sketched right after Theorem 2.2 but with right-invariant differential operators instead of left-invariant one in (2.18), to construct the \mathbf{D}_j 's. The proof being purely technical, it is relegated to the Appendix A.

Theorem 2.5. *Let \mathcal{E} be a complex Fréchet space and $\underline{\mu}$ be an associated family of tempered weights on G . Then, the oscillatory integral*

$$\mathcal{S}(G \times G, \mathcal{E}) \rightarrow \mathcal{E}, \quad f \mapsto \int K_\theta(g_1, g_2) f(g_1, g_2) d^\lambda(g_1) d^\lambda(g_2) ,$$

extends to a continuous map

$$\mathcal{B}_\rho^\mu(G \times G, \mathcal{E}) \rightarrow \mathcal{E}, \quad f \mapsto \widetilde{\int} K_\theta(g_1, g_2) f(g_1, g_2) d^\lambda(g_1) d^\lambda(g_2) .$$

Remark 2.6. We keep the same notation for this second extension of the oscillatory integral, because they both coincide on $\mathcal{B}_\lambda^\mu \cap \mathcal{B}_\rho^\mu(G \times G, \mathcal{E})$.

We need now to consider the non formal Drinfel'd twists on $\mathcal{S}(G \times G)$ given by the oscillatory integral mappings. To this hand, we fix a Fréchet space \mathcal{E} together with an associated family of sub-multiplicative (to simplify the picture) tempered weights on $G \times G$, and we make the following observations. First, we see by definition, that $\mathcal{B}_\lambda^\mu(G \times G, \mathcal{E})$ is its own space of smooth vectors for the right regular action $\rho \otimes \rho$ of $G \times G$. Similarly, $\mathcal{B}_\rho^\mu(G \times G, \mathcal{E})$ is its own space of smooth vectors for the left regular action $\lambda \otimes \lambda$ of $G \times G$. It is then not difficult to see that these actions are tempered. Hence, one may apply [6, Lemma 5.5] (or [6, Lemma 2.15] applied to $G \times G$ instead of G) to deduce that there exists a family of tempered weights $\underline{\nu}$ such that the map

$$f \mapsto [(g_1, g_2) \mapsto (\rho_{g_2} \otimes \rho_{g_2})f],$$

is continuous from $\mathcal{B}_\lambda^\mu(G \times G, \mathcal{E})$ to $\mathcal{B}_\lambda^{\underline{\nu}}(G \times G, \mathcal{B}_\lambda^\mu(G \times G, \mathcal{E}))$. Similarly, one deduces that the map

$$f \mapsto [(g_1, g_2) \mapsto (\lambda_{g_1} \otimes \lambda_{g_2})f],$$

is continuous from $\mathcal{B}_\rho^\mu(G \times G, \mathcal{E})$ to $\mathcal{B}_\lambda^{\underline{\nu}}(G \times G, \mathcal{B}_\rho^\mu(G \times G, \mathcal{E}))$ too (with the same $\underline{\nu}$ as above). Combining this with Theorem 2.2, one gets the following well-defined notion of non-formal Drinfel'd twists:

Definition 2.7. For $\underline{\mu}$ any family of sub-multiplicative and tempered weights on G , we set

$$F_\theta^\rho : \mathcal{B}_\lambda^\mu(G \times G, \mathcal{E}) \rightarrow \mathcal{B}_\lambda^\mu(G \times G, \mathcal{E}), \quad f \mapsto \widetilde{\int} K_\theta(g_1, g_2) (\rho_{g_1} \otimes \rho_{g_2})f \, d^\lambda(g_1) \, d^\lambda(g_2),$$

and

$$F_\theta^\lambda : \mathcal{B}_\rho^\mu(G \times G, \mathcal{E}) \rightarrow \mathcal{B}_\rho^\mu(G \times G, \mathcal{E}), \quad f \mapsto \widetilde{\int} K_\theta(g_1, g_2) (\lambda_{g_1} \otimes \lambda_{g_2})f \, d^\lambda(g_1) \, d^\lambda(g_2).$$

The next statement (mainly based on the results of [6]) shows that these twists (as well as variants of them) preserves the Schwartz space $\mathcal{S}(G \times G)$:

Lemma 2.8. Both twists F_θ^λ and F_θ^ρ defines continuous linear operators on $\mathcal{S}(G \times G)$. The same is true if in F_θ^λ one replaces $\lambda \otimes \lambda$ by $\lambda \otimes \rho$ or by $\rho \otimes \lambda$, or even if one replaces simultaneously the actions λ or ρ by the anti-actions $[g \mapsto \lambda_{g^{-1}}]$ or $[g \mapsto \rho_{g^{-1}}]$.

Proof. The actions ρ and λ of G are clearly strongly continuous on $\mathcal{S}(G)$. Since right invariant vector fields are finite linear combination of left invariant vector fields with coefficients in the ring of tempered functions, and vice versa, (see [6, Remark 2.20]) $\mathcal{S}(G)$ is its own space of smooth vectors for both actions. Moreover, ρ and λ are tempered actions. This follows from [6, Lemma 5.3] together with the fact that $\mathcal{S}(G)$ coincides with the set of smooth vectors (for any of these actions) of the Fréchet completion of $C_c^\infty(G)$ for the topology underlying the seminorms $f \mapsto \|\mathfrak{d}_G^n f\|_\infty$, $n \in \mathbb{N}$ and on this Fréchet space, ρ and λ are tempered due to the sub-multiplicativity of the modular weight \mathfrak{d}_G (see [6, Lemma 2.4]). Hence, Lemma 5.5 of [6] entails that for $f \in \mathcal{S}(G)$, the maps $[g \mapsto \lambda_g f]$ and $[g \mapsto \rho_g f]$ belong to $\mathcal{B}_\lambda^\mu(G, \mathcal{S}(G))$ for a suitable family $\underline{\mu}$ of tempered weights. Obviously, we can repeat this reasoning in the two-legs case, showing that if $f \in \mathcal{S}(G \times G)$ then the maps $[(g_1, g_2) \mapsto (\lambda_{g_1} \otimes \lambda_{g_2})f]$, $[(g_1, g_2) \mapsto (\rho_{g_1} \otimes \rho_{g_2})f]$, $[(g_1, g_2) \mapsto (\lambda_{g_1} \otimes \rho_{g_2})f]$ and $[(g_1, g_2) \mapsto (\rho_{g_1} \otimes \lambda_{g_2})f]$ belong to $\mathcal{B}_\lambda^\mu(G \times G, \mathcal{S}(G \times G))$. Then, the first part of the proof follows from Theorem 2.2. The cases where one uses anti-actions follow from what precedes combined with (the two legs version of) Lemma 2.1 and Theorem 2.5. \square

It has been first observed in [24, Section 5] (see also [8, Appendix]) that F_θ^λ extends from $\mathcal{S}(G \times G)$ to $L_\lambda^2(G \times G)$ as a unitary operator. Since $F_\theta^\rho = (S \otimes S)F_\theta^\lambda(S \otimes S)$ on $\mathcal{S}(G \times G)$, it immediately implies that F_θ^ρ extends to a unitary operator on $L_\rho^2(G \times G)$. Using $\overline{K_\theta} = K_{-\theta}$, the unitarity of the twists combined with the last part of the Lemma 2.8 (that is, when one uses anti-actions), imply that the inverses of F_θ^λ and F_θ^ρ preserve $\mathcal{S}(G \times G)$ and are also expressible as oscillatory integrals:

Proposition 2.9. *The twists F_θ^λ and F_θ^ρ are homeomorphisms of $\mathcal{S}(G \times G)$ with inverses given by:*

$$\begin{aligned} (F_\theta^\lambda)^{-1} f &= \widetilde{\int} K_{-\theta}(g_1, g_2) \lambda_{g_1^{-1}} \otimes \lambda_{g_2^{-1}} f \, d^\lambda(g_1) \, d^\lambda(g_2), \\ (F_\theta^\rho)^{-1} f &= \widetilde{\int} K_{-\theta}(g_1, g_2) \rho_{g_1^{-1}} \otimes \rho_{g_2^{-1}} f \, d^\lambda(g_1) \, d^\lambda(g_2). \end{aligned}$$

Using Lemma 2.1, it follows from definition 2.7 that the inverses of the twists, $(F_\theta^\lambda)^{-1}$ and $(F_\theta^\rho)^{-1}$, sends $\mathcal{B}_\rho^\mu(G \times G, \mathcal{E})$ to itself continuously. Hence, we deduce in the scalar-valued case:

Proposition 2.10. *The twists F_θ^λ and F_θ^ρ are homeomorphism of $\mathcal{B}_\lambda^\mu \cap \mathcal{B}_\rho^\nu(G \times G)$, for any tempered weights μ, ν on $G \times G$.*

Remark 2.11. Unitarity of the twist F_θ^λ is a very important observation since associativity of \star_θ immediately implies that the adjoint of the twist defines a dual unitary 2-cocycle on G (see [10, 24] for more informations on dual cocycles for locally compact quantum groups). Now the point is that from De Commer's results [10], we can construct a quantum version (in the von Neumann algebraic setting) of any negatively curved Kählerian group from the dual unitary 2-cocycle $F_\theta^{\lambda^*}$. To our knowledge, this is the first and only example of a dual unitary 2-cocycle on a non-compact and non-Abelian group. One of our results here, is that the locally compact quantum group associated with De Commer's construction is unitarily equivalent to the one constructed here from a manageable multiplicative unitary (Theorem 4.12).

Let μ, ν be tempered weights on G . For $f_1, f_2 \in \mathcal{B}_\lambda^\mu \cap \mathcal{B}_\rho^\nu(G)$, the coproducts $\Delta f_1, \Delta f_2$ are a priori only defined as tempered functions on $G \times G$. But if we treat the first and the third variables or the second and the forth variables in $\Delta f_1 \otimes \Delta f_2$ as parameters, we obtain functions in $\mathcal{B}_\lambda^{\mu \otimes \mu} \cap \mathcal{B}_\rho^{\nu \otimes \nu}(G \times G)$. By Proposition 2.10 (and using usual legs numbering notations), the elements $(F_\theta^\rho)_{13}(\Delta f_1 \otimes \Delta f_2)$ and $(F_{-\theta}^\lambda)_{24}^{-1}(\Delta f_1 \otimes \Delta f_2)$ are therefore well defined for $f_1, f_2 \in \mathcal{B}_\lambda^\mu \cap \mathcal{B}_\rho^\nu(G)$ as a family of functions in $\mathcal{B}_\lambda^{\mu \otimes \mu} \cap \mathcal{B}_\rho^{\nu \otimes \nu}(G \times G)$ parametrized by $G \times G$. Now, since

$$(\rho_g \otimes \text{Id})\Delta f = (\text{Id} \otimes \lambda_{g^{-1}})\Delta f,$$

we deduce the following decisive fact:

Lemma 2.12. *For $f_1, f_2 \in \mathcal{B}_\lambda^\mu \cap \mathcal{B}_\rho^\nu(G)$, we have an equality of smooth functions on G^4 :*

$$(F_\theta^\rho)_{13}(\Delta f_1 \otimes \Delta f_2) = (F_{-\theta}^\lambda)_{24}^{-1}(\Delta f_1 \otimes \Delta f_2).$$

We finish this section by an important property of the twist:

Lemma 2.13. *Let $f \in \mathcal{S}(G \times G)$. Then, for $g_1, g_2 \in G$ fixed, the fonction*

$$\left[(h_1, h_2) \mapsto F_\theta^\lambda(\lambda_{h_1^{-1}} \otimes \lambda_{h_2^{-1}} f)(g_1, g_2) \right],$$

belongs to $\mathcal{S}(G \times G)$.

Proof. Since $f \in \mathcal{S}(G \times G)$, we have with absolutely convergent integrals:

$$F_\theta^\lambda(\lambda_{h_1^{-1}} \otimes \lambda_{h_2^{-1}})f(g_1, g_2) = \int_{G \times G} K_\theta(g_3, g_4) (\rho_{g_1} \otimes \rho_{g_2} f)(h_1 g_3^{-1}, h_2 g_4^{-1}) \, d^\lambda(g_3) \, d^\lambda(g_4).$$

Setting for fixed $g_1, g_2 \in G$, $\psi_{g_1, g_2} := \rho_{g_1} \otimes \rho_{g_2} f \in \mathcal{S}(G \times G)$ and returning to a non-evaluated expression (thus to oscillatory integrals), the function we need to control reads:

$$\widetilde{\int} K_\theta(g_3, g_4) \rho_{g_3^{-1}} \otimes \rho_{g_4^{-1}} \psi_{g_1, g_2} \, d^\lambda(g_3) \, d^\lambda(g_4).$$

The proof follows then by (the anti-action part of) Lemma 2.8. □

3 Quantum Kählerian Lie groups

This section is the core of the paper: we construct a multiplicative unitary out of the deformation theory of Fréchet algebras for actions of negatively curved Kählerian Lie group [6].

3.1 Deformations of the Schwartz algebra

We have seen in the proof of Lemma 2.8 that the actions λ and ρ of G on $\mathcal{S}(G)$ are strongly continuous and tempered. Obviously, the same is true for the product action $\lambda \otimes \rho$ of $G \times G$ on $\mathcal{S}(G)$. Hence, Theorem 2.4 yields three possible deformations of the Fréchet algebra $\mathcal{S}(G)$. The first two are parametrized by $\theta \in \mathbb{R}^*$ while the third is parametrized by $(\theta_1, \theta_2) \in \mathbb{R}^* \times \mathbb{R}^*$. We shall use the following notations:

- $\tilde{\star}_\theta$ for the product on $\mathcal{S}(G)$, deformed by the action ρ of G and parameter θ ,
- \star_θ for the product on $\mathcal{S}(G)$, deformed by the action λ of G and parameter $-\theta$,
- \star_θ for the product on $\mathcal{S}(G)$, deformed by the action $\lambda \otimes \rho$ of $G \times G$ and parameters $(-\theta, \theta)$.

By construction, $\tilde{\star}_\theta$ is equivariant under λ and we call this product the left invariant deformed product. Similarly, \star_θ is equivariant under ρ and we call it the right invariant deformed product. However, \star_θ has no (classical) equivariance property and we call it the doubly deformed product.

By [6, Proposition 4.16], the complex conjugation is an involution of the three Fréchet algebras $(\mathcal{S}(G), \tilde{\star}_\theta)$, $(\mathcal{S}(G), \star_\theta)$ and $(\mathcal{S}(G), \star_\theta)$.

Remark 3.1. We could have used generic parameters (θ_1, θ_2) to define a more general doubly deformed product. However, the only choice that makes the ordinary coproduct a morphism for such a (θ_1, θ_2) -doubly deformed product is $(-\theta, \theta)$. This comes from Lemma 2.12.

By compatibility of the oscillatory integral with continuous linear mapping (see [6, Lemma 2.37]), we get that these three different associative products on $\mathcal{S}(G)$ can be written directly in term of the non formal Drinfel'd twists of Definition 2.7:

Proposition 3.2. *With $\mu : \mathcal{S}(G \times G) \rightarrow \mathcal{S}(G)$, $f \mapsto [g \mapsto f(g, g)]$ the ordinary multiplication, we have for $f_1, f_2 \in \mathcal{S}(G)$:*

$$f_1 \tilde{\star}_\theta f_2 = \mu(F_\theta^\rho(f_1 \otimes f_2)), \quad f_1 \star_\theta f_2 = \mu(F_{-\theta}^\lambda(f_1 \otimes f_2)), \quad f_1 \star_\theta f_2 = \mu(F_{-\theta}^\lambda F_\theta^\rho(f_1 \otimes f_2)) = \mu(F_\theta^\rho F_{-\theta}^\lambda(f_1 \otimes f_2)).$$

By [6, Theorem 4.9], the group G still acts (strongly continuously and temperedly) on the left on $(\mathcal{S}(G), \tilde{\star}_\theta)$. Similarly, one can show that the group acts on the right on $(\mathcal{S}(G), \star_\theta)$. Hence, one can use Theorem 2.4 one more time to deform $(\mathcal{S}(G), \tilde{\star}_\theta)$ and $(\mathcal{S}(G), \star_\theta)$. As a special case of [6, Proposition 5.20], we know that these two deformations coincide with the deformation of $\mathcal{S}(G)$ for the action $\lambda \otimes \rho$ of $G \times G$:

Proposition 3.3. *The algebra $(\mathcal{S}(G), \star_\theta)$ coincides with the deformation of $(\mathcal{S}(G), \tilde{\star}_\theta)$ for the action λ and parameter $-\theta$ and also with the deformation of $(\mathcal{S}(G), \star_\theta)$ for the action ρ and parameter θ .*

We now give a more convenient expression for the doubly deformed product \star_θ :

Lemma 3.4. *For $f_1, f_2 \in \mathcal{S}(G)$ and for $g \in G$ fixed, we have the ordinary integral formulas:*

$$\begin{aligned} f_1 \star_\theta f_2(g) &= \int K_{-\theta}(g_1, g_2) (\lambda_{g_1} f_1) \tilde{\star}_\theta (\lambda_{g_2} f_2)(g) d^\lambda(g_1) d^\lambda(g_2) \\ &= \int K_\theta(g_1, g_2) (\rho_{g_1} f_1)(g) \star_\theta (\rho_{g_2} f_2)(g) d^\lambda(g_1) d^\lambda(g_2). \end{aligned}$$

Proof. We prove the first formula only, the arguments for the second being similar. By Proposition 3.3, one can view the algebra $(\mathcal{S}(G), \star_\theta)$ as the deformation of $(\mathcal{S}(G), \tilde{\star}_\theta)$ for the action λ of G and parameter $-\theta$. Hence by [6, Proposition 5.10], we have

$$f_1 \star_\theta f_2 = \int \widetilde{K_{-\theta}(g_1, g_2)} (\lambda_{g_1} f_1) \tilde{\star}_\theta (\lambda_{g_2} f_2) d^\lambda(g_1) d^\lambda(g_2). \quad (3.1)$$

Now, by Proposition 3.2, we have

$$(\lambda_{g_1} f_1) \tilde{\star}_\theta (\lambda_{g_2} f_2)(g) = F_\theta^\rho (\lambda_{g_1} f_1 \otimes \lambda_{g_2} f_2)(g, g),$$

which implies since F_θ^ρ commutes with left translations:

$$(\lambda_{g_1} f_1) \tilde{\star}_\theta (\lambda_{g_2} f_2)(g) = F_\theta^\rho (f_1 \otimes f_2)(g_1^{-1} g, g_2^{-1} g).$$

By Lemma 2.8, $F_\theta^\rho (f_1 \otimes f_2)$ belongs to $\mathcal{S}(G \times G)$. Since $\mathcal{S}(G \times G)$ is stable under the group inversion and right translations, we therefore deduce that for g fixed, the function $[(g_1, g_2) \mapsto (\lambda_{g_1} f_1) \tilde{\star}_\theta (\lambda_{g_2} f_2)(g)]$ belongs to $\mathcal{S}(G \times G)$. Hence, we can replace the oscillatory integrals in (3.1) by ordinary one and we are done. \square

The following property is a key step to prove the morphism property of the undeformed coproduct for the doubly deformed product \star_θ .

Lemma 3.5. *Let $f_1, f_2 \in \mathcal{S}(G)$. Then, for a fixed $g \in G$, the function*

$$\left[(g_1, g_2) \mapsto (\lambda_{g_1^{-1}} f_1) \star_\theta (\lambda_{g_2^{-1}} f_2)(g) \right],$$

belongs to $\mathcal{S}(G \times G)$.

Proof. We have by Proposition 3.2 and the fact that F_θ^ρ commutes with left translations:

$$(\lambda_{g_1^{-1}} f_1) \star_\theta (\lambda_{g_2^{-1}} f_2)(g) = (F_{-\theta}^\lambda F_\theta^\rho (\lambda_{g_1^{-1}} \otimes \lambda_{g_2^{-1}})(f_1 \otimes f_2))(g, g) = (F_{-\theta}^\lambda (\lambda_{g_1^{-1}} \otimes \lambda_{g_2^{-1}}) F_\theta^\rho (f_1 \otimes f_2))(g, g).$$

By Lemma 2.8, $F_\theta^\rho (f_1 \otimes f_2)$ belongs to $\mathcal{S}(G \times G)$, hence we may apply Lemma 2.13 to get the result. \square

The following useful property survives:

Proposition 3.6. *The undeformed antipode is anti-multiplicative on the Fréchet algebra $(\mathcal{S}(G), \star_\theta)$:*

$$S(f_1 \star_\theta f_2) = (Sf_2) \star_\theta (Sf_1), \quad \forall f_1, f_2 \in \mathcal{S}(G).$$

At first glance this preserved property may be surprising but it is not: S will appear to be the unitary antipode (but not the antipode).

Proof. Note first that $K_\theta(g_1, g_2) = K_{-\theta}(g_2, g_1)$, which implies that we have for all $f_1, f_2 \in \mathcal{S}(G)$:

$$f_1 \tilde{\star}_\theta f_2 = f_2 \tilde{\star}_{-\theta} f_1, \quad f_1 \star_\theta f_2 = f_2 \star_{-\theta} f_1. \quad (3.2)$$

Since moreover the undeformed antipode satisfies $S\mu = \mu(S \otimes S)$ and intertwines F_θ^ρ with F_θ^λ , we get from Proposition 3.2:

$$S(f_1 \tilde{\star}_\theta f_2) = (Sf_1) \star_{-\theta} (Sf_2), \quad S(f_1 \star_\theta f_2) = (Sf_1) \tilde{\star}_{-\theta} (Sf_2), \quad S(f_1 \star_\theta f_2) = (Sf_1) \star_{-\theta} (Sf_2). \quad (3.3)$$

Combining the last equalities in (3.2) and in (3.3), we get the result. \square

Fix μ, ν two tempered weights on G . Note that the action $\lambda \otimes \rho$ of $G \times G$ on $\mathcal{B}_\lambda^\mu \cap \mathcal{B}_\rho^\nu(G)$ is tempered and strongly continuous and that $\mathcal{B}_\lambda^\mu \cap \mathcal{B}_\rho^\nu(G)$ is its own space of smooth vectors for this action. We can then proceed exactly like in [6, Proposition 4.0] to show $\mathcal{B}_\lambda^\mu \cap \mathcal{B}_\rho^\nu(G)$ acts continuously by \star_θ -multiplication (on the left and on the right) on $\mathcal{S}(G)$. Then:

Proposition 3.7. *Let $\alpha \in \mathbb{C}$. The pair of linear mappings*

$$f \mapsto \chi_G^\alpha \star_\theta f, \quad f \mapsto f \star_\theta \chi_G^\alpha,$$

defines a multiplier of the Fréchet algebra $(\mathcal{S}(G), \star_\theta)$. Moreover, the constant unit function is the unit of the multipliers algebra:

$$1 \star_\theta f = f \star_\theta 1 = f, \quad \forall f \in \mathcal{S}(G).$$

Proof. Continuity follows from the above discussion together from the fact that χ_G^α belongs to $\mathcal{B}_\lambda^{\chi_G^{\Re(\alpha)}} \cap \mathcal{B}_\rho^{\chi_G^{\Re(\alpha)}}(G)$. The fact the constant function is the unit of the multipliers algebra follows from [6, Proposition 4.11]. \square

Finally, we need the following density result:

Proposition 3.8. *The linear subspace $\mathcal{S}(G) \star_\theta \mathcal{S}(G)$ (finite sums of products) is dense in $\mathcal{S}(G)$.*

Proof. This follows by [6, Proposition 5.19] which shows that the Fréchet algebra $(\mathcal{S}(G), \star_\theta)$ possesses a bounded approximate unit. \square

3.2 The deformed Kac-Takesaki operator

Our starting point is the obvious observation that on elementary tensors, the classical Kac-Takesaki operator may be written as:

$$V(f_1 \otimes f_2)(g_1, g_2) = ((\lambda_{g_1^{-1}} f_1) f_2)(g_2).$$

Replacing the pointwise product by the doubly deformed product \star_θ leads to a natural deformation of V :

Proposition 3.9. *The deformed Kac-Takesaki operator*

$$V_\theta : f_1 \otimes f_2 \mapsto [(g_1, g_2) \mapsto (\lambda_{g_1^{-1}} f_1) \star_\theta f_2(g_2)],$$

sends $\mathcal{S}(G) \otimes \mathcal{S}(G)$ to $\mathcal{S}(G \times G)$ and extends uniquely to a continuous linear map from $\mathcal{S}(G \times G)$ to itself.

Proof. Note first that for $g_1 \in G$ fixed, $(\lambda_{g_1^{-1}} f_1) \star_\theta f_2$ is well defined as an element of $\mathcal{S}(G)$. Moreover, Proposition 3.2 and the fact that F_θ^ρ commutes with left translations, give:

$$(\lambda_{g_1^{-1}} f_1) \star_\theta f_2(g_2) = (F_{-\theta}^\lambda F_\theta^\rho (\lambda_{g_1^{-1}} \otimes \text{Id}) (f_1 \otimes f_2))(g_2, g_2) = (F_{-\theta}^\lambda (\lambda_{g_1^{-1}} \otimes \text{Id}) F_\theta^\rho (f_1 \otimes f_2))(g_2, g_2).$$

Hence, it suffices to show that the map

$$f \mapsto [(g_1, g_2) \mapsto (F_{-\theta}^\lambda (\lambda_{g_1^{-1}} \otimes \text{Id}) F_\theta^\rho f)(g_2, g_2)],$$

sends $\mathcal{S}(G \times G)$ to itself continuously (and the extension of V_θ to the whole $\mathcal{S}(G \times G)$ will be given by that formula). Note first that by Lemma 2.8, F_θ^ρ is continuous on $\mathcal{S}(G \times G)$ and thus it suffices to show that the map

$$f \mapsto [(g_1, g_2) \mapsto (F_{-\theta}^\lambda (\lambda_{g_1^{-1}} \otimes \text{Id}) f)(g_2, g_2)],$$

is continuous from $\mathcal{S}(G \times G)$ to itself. To see this fact, we first come back to the definition of the twist $F_{-\theta}^\lambda$ as oscillatory integrals, to get

$$(F_{-\theta}^\lambda (\lambda_{g_1^{-1}} \otimes \text{Id}) f)(g_2, g_2) = \int \widetilde{K_{-\theta}(g_3, g_4)} f(g_1 g_3^{-1} g_2, g_4^{-1} g_2) d^\lambda(g_3) d^\lambda(g_4) .$$

Note also that in term of V , the classical Kac-Takesaki operator (2.3), we then have

$$f(g_1 g_3^{-1} g_2, g_4^{-1} g_2) = ((\rho_{g_3^{-1} g_4} \otimes \lambda_{g_4}) V f)(g_1, g_2) .$$

Since V preserves $\mathcal{S}(G \times G)$, it suffices to prove that the map

$$f \mapsto \int \widetilde{K_{-\theta}(g_3, g_4)} (\rho_{g_3^{-1} g_4} \otimes \lambda_{g_4}) f d^\lambda(g_3) d^\lambda(g_4) ,$$

is continuous from $\mathcal{S}(G \times G)$ to itself. To see this, note first that for $f \in \mathcal{S}(G \times G)$ and for $g_1, g_2 \in G$ fixed, the map $[(g_3, g_4) \mapsto (\rho_{g_3^{-1} g_4} \otimes \lambda_{g_4}) f(g_1, g_2)]$ belongs to $\mathcal{S}(G \times G)$. Hence, provided one evaluates the previous expression at the point $(g_1, g_2) \in G \times G$, we may freely replace the oscillatory integral by the ordinary one. Performing then the translation $g_3 \mapsto g_4 g_3$, followed by the group inversion $g_4 \mapsto g_4^{-1}$, we are left with

$$\int_{G \times G} K_{-\theta}(g_4^{-1} g_3, g_4^{-1}) \chi_G(g_4) (\rho_{g_3^{-1}} \otimes \lambda_{g_4^{-1}}) f(g_1, g_2) d^\lambda(g_3) d^\lambda(g_4) .$$

As shown (for instance) in [8, Equation (2.2)], $K_{-\theta}(g_4^{-1} g_3, g_4^{-1}) = K_{-\theta}(g_4, g_3)$. Going back to a non-evaluated expression and hence back to oscillatory integrals, we see that the map we need to control coincides with

$$f \mapsto \int \widetilde{K_{-\theta}(g_4, g_3)} \chi_G(g_4) (\rho_{g_3^{-1}} \otimes \lambda_{g_4^{-1}}) f d^\lambda(g_3) d^\lambda(g_4) , \quad f \in \mathcal{S}(G \times G) .$$

The claim follows then from the last part of Lemma 2.8 and the fact that the modular function χ_G is tempered, hence it defines a continuous multiplier of the Fréchet algebra $\mathcal{S}(G)$ for the pointwise product. \square

From the proof of Proposition 3.9, we get the following:

Proposition 3.10. *The deformed Kac-Takesaki operator factorizes as:*

$$V_\theta = (1 \otimes \chi_G^{-1}) Y_\theta (1 \otimes \chi_G) V F_\theta^\rho , \tag{3.4}$$

where Y_θ is the continuous (by Lemma 2.8) linear map on $\mathcal{S}(G \times G)$, defined by:

$$Y_\theta f := \int \widetilde{K_{-\theta}(g_2, g_1)} (\rho_{g_1^{-1}} \otimes \lambda_{g_2^{-1}}) f d^\lambda(g_1) d^\lambda(g_2) ,$$

and where the modular function χ_G is identified with the associated pointwise multiplication operator.

From this factorization, we deduce:

Corollary 3.11. *The deformed Kac-Takesaki operator V_θ is an homeomorphism of $\mathcal{S}(G \times G)$.*

Proof. The claim follows because all the operators entering in the factorization (3.4) are homeomorphisms of $\mathcal{S}(G \times G)$. For $1 \otimes \chi_G$ and V it is obvious, for F_θ^ρ it follows from Proposition 2.9. For Y_θ , we get the following formula for the inverse:

$$Y_\theta^{-1} f := \int \widetilde{K_\theta(g_2, g_1)} \rho_{g_1} \otimes \lambda_{g_2} f d^\lambda(g_1) d^\lambda(g_2) ,$$

and the result follows again from Lemma 2.8. \square

3.3 The coproduct

In order to lighten the notations, from now on we will use the notation \star_θ instead of $\star_\theta \otimes \star_\theta$, to denote the doubly deformed product of $\mathcal{S}(G \times G)$ (that is the deformed product by the action $(\lambda \otimes \rho) \otimes (\lambda \otimes \rho)$ of $G^2 \times G^2$).

Let Δ be the ordinary coproduct of $L^\infty(G)$. Our task here is to define, for $f \in \mathcal{S}(G)$, the element Δf as a continuous multiplier of the Fréchet algebra $(\mathcal{S}(G \times G), \star_\theta)$. This turns out to be a delicate question. Indeed, even if, for $f \in \mathcal{S}(G)$, the function Δf is tempered on $G \times G$, for $f_1 \in \mathcal{S}(G)$ and $f_2 \in \mathcal{S}(G \times G)$ there is no direct way to give a meaning to the product $\Delta f_1 \star_\theta f_2$ in term of oscillatory integrals.

Not surprisingly, the easiest answer we found uses the deformed Kac-Takesaki operator V_θ . Indeed, since the constant unit function is the unit of the algebra of continuous multipliers of the Fréchet algebra $(\mathcal{S}(G), \star_\theta)$ (by Proposition 3.7), we may formally write, for $f_1, f_2, f_3 \in \mathcal{S}(G)$:

$$\begin{aligned} \Delta f_1 \star_\theta (f_2 \otimes f_3) &= \Delta f_1 \star_\theta (1 \otimes f_3) \star_\theta (f_2 \otimes 1) \\ &= [(g_1, g_2) \mapsto (\lambda_{g_1^{-1}} f_1) \star_\theta f_3(g_2)] \star_\theta (f_2 \otimes 1) = V_\theta(f_1 \otimes f_3) \star_\theta (f_2 \otimes 1). \end{aligned}$$

The point is that the RHS above now makes sense as an element of $\mathcal{S}(G \times G)$. Indeed, for $f \in \mathcal{S}(G)$, the function $f \otimes 1$ defines in an obvious way a continuous multiplier of $(\mathcal{S}(G \times G), \star_\theta)$ and, by Proposition 3.9, V_θ is continuous on $\mathcal{S}(G \times G)$. By the last item of Proposition 3.2, we get with the usual leg numbering notation:

$$V_\theta(f_1 \otimes f_3) \star_\theta (f_2 \otimes 1) = (\mu \otimes \text{Id})(F_\theta^\rho F_{-\theta}^\lambda \otimes \text{Id})(V_\theta)_{13}(f_1 \otimes f_2 \otimes f_3).$$

Hence, for $f_1 \in \mathcal{S}(G)$, the map

$$L^{\star_\theta}(\Delta f_1) : \mathcal{S}(G) \otimes \mathcal{S}(G) \rightarrow \mathcal{S}(G \times G), \quad f_2 \otimes f_3 \mapsto V_\theta(f_1 \otimes f_3) \star_\theta (f_2 \otimes 1),$$

extends uniquely as a continuous linear map on $\mathcal{S}(G \times G)$. The formula of the extension being given by

$$L^{\star_\theta}(\Delta f_1) f := (\mu \otimes \text{Id})(F_\theta^\rho F_{-\theta}^\lambda \otimes \text{Id})(V_\theta)_{13}(f_1 \otimes f), \quad f_1 \in \mathcal{S}(G), \quad f \in \mathcal{S}(G \times G).$$

Then, it is not difficult to see that the continuous map $L^{\star_\theta}(\Delta f)$, $f \in \mathcal{S}(G)$, is a left multiplier of the algebra $(\mathcal{S}(G \times G), \star_\theta)$. Indeed, let $f_1, \dots, f_5 \in \mathcal{S}(G)$ and, for fixed $g_2 \in G$, consider the function

$$\lambda_{\bullet^{-1}} f_1 \star_\theta f_3(g_2) := [g_1 \mapsto \lambda_{g_1^{-1}} f_1 \star_\theta f_3(g_2)] \in \mathcal{S}(G).$$

With this notation and, for $g_1, g_2 \in G$, we have by definition:

$$(L^{\star_\theta}(\Delta f_1)(f_2 \otimes f_3))(g_1, g_2) = (\lambda_{\bullet^{-1}} f_1 \star_\theta f_3(g_2)) \star_\theta f_2(g_1).$$

Hence we get

$$\begin{aligned} (L^{\star_\theta}(\Delta f_1)(f_2 \otimes f_3)) \star_\theta (f_4 \otimes f_5)(g_1, g_2) &= (\lambda_{\bullet^{-1}} f_1 \star_\theta f_3 \star f_5(g_2)) \star_\theta f_2 \star f_4(g_1) \\ &= L^{\star_\theta}(\Delta f_1)((f_2 \star_\theta f_4) \otimes (f_3 \star_\theta f_5)) \\ &= L^{\star_\theta}(\Delta f_1)((f_2 \otimes f_3) \star_\theta (f_4 \otimes f_5))(g_1, g_2). \end{aligned}$$

By density of $\mathcal{S}(G) \otimes \mathcal{S}(G)$ in $\mathcal{S}(G \times G)$, we get:

Proposition 3.12. *Let $f \in \mathcal{S}(G)$. Then the map*

$$L^{\star_\theta}(\Delta f_1) : f_2 \otimes f_3 \mapsto V_\theta(f_1 \otimes f_3) \star_\theta (f_2 \otimes 1),$$

sends $\mathcal{S}(G) \otimes \mathcal{S}(G)$ to $\mathcal{S}(G \times G)$ and extends uniquely to a continuous left multiplier of $(\mathcal{S}(G \times G), \star_\theta)$.

Of course, we can proceed similarly to give meaning to $R^{*\theta}(\Delta f_1) : \mathcal{S}(G \times G) \rightarrow \mathcal{S}(G \times G)$, $f_2 \mapsto f_2 \star_\theta \Delta f_1$ as a continuous right multiplier. But there is in fact a more direct way to do it, namely using the involution given by the complex conjugation:

$$R^{*\theta}(\Delta f_1) : \mathcal{S}(G \times G) \rightarrow \mathcal{S}(G \times G), \quad f_2 \mapsto \overline{L^{*\theta}(\Delta f_1) f_2}, \quad f_1 \in \mathcal{S}(G).$$

We are now ready to give the precise definition of the coproduct.

Definition 3.13. For $f \in \mathcal{S}(G)$, we let Δf be the element of the algebra of continuous multipliers of the Fréchet algebra $(\mathcal{S}(G \times G), \star_\theta)$ given by the above defined pair of left and right multipliers $(L^{*\theta}(\Delta f), R^{*\theta}(\Delta f))$.

To simplify notations, we will also denote by \star_θ the product in the algebra of continuous multipliers of $(\mathcal{S}(G \times G), \star_\theta)$. Of course, we can then express the deformed Kac-Takesaki operator in term of the coproduct:

Lemma 3.14. For $f_1, f_2 \in \mathcal{S}(G)$, the following relation

$$V_\theta(f_1 \otimes f_2) = \Delta f_1 \star_\theta (1 \otimes f_2),$$

holds true in the algebra of continuous multipliers of $(\mathcal{S}(G \times G), \star_\theta)$.

Proof. For $f_j \in \mathcal{S}(G)$, $j = 1, \dots, 4$, we have in $\mathcal{S}(G \times G)$:

$$\begin{aligned} (\Delta f_1 \star_\theta (1 \otimes f_2)) \star_\theta (f_3 \otimes f_4) &= (\Delta f_1 \star_\theta (f_3 \otimes f_2)) \star_\theta (1 \otimes f_4) \\ &= (V_\theta(f_1 \otimes f_2) \star_\theta (f_3 \otimes 1)) \star_\theta (1 \otimes f_4) = V_\theta(f_1 \otimes f_2) \star_\theta (f_3 \otimes f_4), \end{aligned}$$

and all the steps are justified by what precedes. \square

The last ingredient we need is the morphism property for the coproduct:

Proposition 3.15. For all $f_1, f_2 \in \mathcal{S}(G)$, we have in the algebra of continuous multipliers of $(\mathcal{S}(G \times G), \star_\theta)$:

$$\Delta f_1 \star_\theta \Delta f_2 = \Delta(f_1 \star_\theta f_2).$$

Proof. Note first that both sides of the equality we want to prove are well defined as continuous multipliers. But in fact, both sides are also well defined as tempered functions on $G \times G$, by which we mean functions that together with their left-(or right-)invariant derivatives, are bounded by a power of the modular weight $\mathfrak{D}_{G \times G}$. For the RHS this is obvious (since $f_1 \star_\theta f_2$ belongs to $\mathcal{S}(G)$). For the LHS this follows from Lemma 3.5. Indeed for fixed $g \in G$, the function

$$\psi_g := [(g_1, g_2) \mapsto (\lambda_{g_1^{-1}} f_1) \star_\theta (\lambda_{g_2^{-1}} f_2)(g)],$$

belongs to $\mathcal{S}(G \times G)$, so we may apply to it the continuous operator $F_{-\theta}^\lambda F_\theta^\rho : \mathcal{S}(G \times G) \rightarrow \mathcal{S}(G \times G)$ followed by the ordinary multiplication $\mu : \mathcal{S}(G \times G) \rightarrow \mathcal{S}(G)$. Hence, as a smooth function, we can define

$$\Delta f_1 \star_\theta \Delta f_2(g_1, g_2) := (F_{-\theta}^\lambda F_\theta^\rho \psi_{g_1})(g_2, g_2).$$

Rewriting the function ψ_g in term of the twists yields

$$\Delta f_1 \star_\theta \Delta f_2 = (\mu_{13} \mu_{24})((F_{-\theta}^\lambda)_{13} (F_\theta^\rho)_{13} (F_{-\theta}^\lambda)_{24} (F_\theta^\rho)_{24} (\Delta f_1 \otimes \Delta f_2)).$$

Since all these twists commute, we can use Lemma 2.12, to get (with the usual leg numbering notation):

$$\Delta f_1 \star_\theta \Delta f_2 = (\mu_{13} \mu_{24})((F_{-\theta}^\lambda)_{13} (F_\theta^\rho)_{24} (\Delta f_1 \otimes \Delta f_2)).$$

Using the facts that $F_{-\theta}^\lambda$ commutes with right translations and that F_θ^ρ commutes with left translations, we get the relation

$$(F_{-\theta}^\lambda)_{13} (F_\theta^\rho)_{24} (\Delta f_1 \otimes \Delta f_2) = (\Delta \otimes \Delta) F_{-\theta}^\lambda F_\theta^\rho (f_1 \otimes f_2),$$

to finally deduce

$$\Delta f_1 \star_\theta \Delta f_2 = (\mu_{13} \mu_{24}) ((\Delta \otimes \Delta) F_{-\theta}^\lambda F_\theta^\rho (f_1 \otimes f_2)) = \Delta (\mu (F_{-\theta}^\lambda F_\theta^\rho (f_1 \otimes f_2))) = \Delta (f_1 \star_\theta f_2),$$

which concludes the proof. \square

We will also need the morphism property between Δf and $\Delta \chi^\alpha$.

Proposition 3.16. *For $f \in \mathcal{S}(G)$ and $\alpha \in \mathbb{C}$, we have in the algebra of multipliers of $(\mathcal{S}(G \times G), \star_\theta)$:*

$$\Delta f \star_\theta \Delta \chi_G^\alpha = \Delta (f \star_\theta \chi_G^\alpha), \quad \Delta \chi_G^\alpha \star_\theta \Delta f = \Delta (\chi_G^\alpha \star_\theta f).$$

Proof. We only show the first equality, the second being similar. Recall that given a pair of tempered weights μ and ν , the space $\mathcal{B}_\lambda^\mu \cap \mathcal{B}_\rho^\nu(G)$ acts continuously by \star_θ -multiplication (on the left and on the right) on $\mathcal{S}(G)$ and that χ_G^α belongs to $\mathcal{B}_\lambda^{\Re(\alpha)} \cap \mathcal{B}_\rho^{\Re(\alpha)}(G)$. This implies that $\Delta \chi_G^\alpha = \chi_G^\alpha \otimes \chi_G^\alpha$ is a multiplier of $(\mathcal{S}(G \times G), \star_\theta)$. Consequently, both sides of the equality we have to prove are well defined as multipliers. (However, it is not true for general $F \in \mathcal{B}_\lambda^\mu \cap \mathcal{B}_\rho^\nu(G)$, that $\Delta(F)$ defines as a multiplier.)

We proceed similarly to Proposition 3.15 but to use the cancellation property of Lemma 2.12, we need to show that $\Delta f \star_\theta \Delta \chi^\alpha$ is expressible as $(\mu_{13} \mu_{24}) ((F_{-\theta}^\lambda)_{13} (F_\theta^\rho)_{13} (F_{-\theta}^\lambda)_{24} (F_\theta^\rho)_{24} (\Delta f \otimes \Delta \chi^\alpha))$. For that, and for fixed $g \in G$, we start to consider the two-variables function:

$$[(g_1, g_2) \mapsto (\lambda_{g_1^{-1}} f) \star_\theta (\lambda_{g_2^{-1}} \chi^\alpha)(g)].$$

Since χ is a character on G , this function coincides with

$$[(g_1, g_2) \mapsto (\lambda_{g_1^{-1}} f) \star_\theta \chi^\alpha(g) \chi^\alpha(g_2)].$$

Hence, what we need to do is to define the doubly deformed product \star_θ between $[g_1 \mapsto (\lambda_{g_1^{-1}} f) \star_\theta \chi^\alpha(g)]$ and χ^α . Since $\mathcal{B}_\lambda^\mu \cap \mathcal{B}_\rho^\nu(G)$ acts continuously by \star_θ -multiplication on $\mathcal{S}(G)$, it is sufficient to prove that the function $[g_1 \mapsto (\lambda_{g_1^{-1}} f) \star_\theta \chi^\alpha(g)]$ belongs, for fixed $g \in G$, to $\mathcal{S}(G)$. What we know is that for fixed $g_1 \in G$, the five-variables function

$$\Xi_{g_1} := \left[(h_1, h'_1, h_2, h'_2) \mapsto [g \mapsto \lambda_{h_1} \rho_{h'_1} (\lambda_{g_1^{-1}} f)(g) \lambda_{h_2} \rho_{h'_2} (\chi^\alpha)(g)] \right],$$

belongs to $\mathcal{B}_\lambda^\mu(G^4, \mathcal{S}(G))$. In term of oscillatory integrals (see Theorem 2.2), we then have

$$(\lambda_{g_1^{-1}} f) \star_\theta \chi^\alpha = \int \widetilde{K_{-\theta} \otimes K_\theta}(h_1, h'_1, h_2, h'_2) \Xi_{g_1}(h_1, h'_1, h_2, h'_2) d^\lambda(h_1) d^\lambda(h_2) d^\lambda(h'_1) d^\lambda(h'_2).$$

Now, we need to permute the roles of the variables g and g_1 . Observe that

$$\Xi_{g_1}(h_1, h'_1, h_2, h'_2, g) = \Theta_g(h_1, h'_1, h_2, h'_2, g_1),$$

if, for fixed $g \in G$, we define Θ_g to be the five-variables function given by:

$$\Theta_g = \chi^\alpha(g) \left[(h_1, h'_1, h_2, h'_2) \mapsto \chi^\alpha(h_2) \chi^\alpha(h'_2) [g_1 \mapsto \lambda_{h_1} \rho_{h'_1} (\rho_g f)(g_1)] \right].$$

Clearly, $\Theta_g \in \mathcal{B}_\lambda^\mu(G^4, \mathcal{S}(G))$ and since

$$[g_1 \mapsto (\lambda_{g_1^{-1}} f) \star_\theta \chi^\alpha(g)] = \int \widetilde{K_{-\theta} \otimes K_\theta}(h_1, h'_1, h_2, h'_2) \Theta_g(h_1, h'_1, h_2, h'_2) d^\lambda(h_1) d^\lambda(h_2) d^\lambda(h'_1) d^\lambda(h'_2),$$

Theorem 2.2 shows that $[g_1 \mapsto (\lambda_{g_1^{-1}} f) \star_\theta \chi^\alpha(g)] \in \mathcal{S}(G)$ as needed. \square

3.4 The invariant weight

In this subsection we are going to prove that the Haar integrals (left and right) define continuous positive invariant functionals on the Fréchet algebra $(\mathcal{S}(G), \star_\theta)$. Here positivity underlies the involution given by the complex conjugation and invariance underlies the undeformed coproduct. The situation is completely left/right symmetric but since we have chosen to deform V and not W , see (2.3), we shall mainly work with the right-invariant weight.

From the discussion just before Definition 1.6, from Lemma 1.21 and from Proposition 3.10 of [6], one knows that complex powers of the modular function act continuously by $\tilde{\star}_\theta$ -multiplication on $\mathcal{S}(G)$. In fact, it defines a continuous multiplier of the Fréchet algebra $(\mathcal{S}(G), \tilde{\star}_\theta)$. Hence, we may consider the following continuous operator:

$$\tilde{T}_\theta : \mathcal{S}(G) \rightarrow \mathcal{S}(G), \quad f \mapsto \chi_G^{-1/2}(f \tilde{\star}_\theta \chi_G^{1/2}). \quad (3.5)$$

This operator played a key role in [8]. Here, we also need the right invariant version of it, namely the operator

$$\underline{T}_\theta := S \tilde{T}_\theta S,$$

which, obviously, is also continuous on $\mathcal{S}(G)$. In what follows, for $\alpha \in \mathbb{C}$, we denote by $M(\chi_G^\alpha)$ the operator of pointwise multiplication by χ_G^α , by $L^{\tilde{\star}_\theta}(\chi_G^\alpha)$ the operator of $\tilde{\star}_\theta$ -multiplication on the left by χ_G^α and by $R^{\star_\theta}(\chi_G^\alpha)$ the operator of \star_θ -multiplication on the right by χ_G^α . For instance, we then have $\tilde{T}_\theta = M(\chi_G^{-1/2})L^{\tilde{\star}_\theta}(\chi_G^{1/2})$. The operators $L^{\pm_\theta}(\chi_G^\alpha)$, $R^{\pm_\theta}(\chi_G^\alpha)$, $L^{\star_\theta}(\chi_G^\alpha)$ and $R^{\star_\theta}(\chi_G^\alpha)$ are defined in a similar way. By equation (3.3), it follows that $L^{\pm_\theta}(\chi_G^\alpha)$ and $R^{\pm_\theta}(\chi_G^\alpha)$ acts continuously on $\mathcal{S}(G)$. For $L^{\star_\theta}(\chi_G^\alpha)$ and $R^{\star_\theta}(\chi_G^\alpha)$ the same holds true too by Proposition 3.7.

We now list the main properties of the operators \tilde{T}_θ and \underline{T}_θ , properties coming essentially from [8, Lemma 2.1].

Lemma 3.17. *Let $\theta \in \mathbb{R}$. Then*

1. \tilde{T}_θ and \underline{T}_θ are homeomorphisms of $\mathcal{S}(G)$ and they satisfy

$$\tilde{T}_\theta^{-1} f = \tilde{T}_{-\theta} f = \overline{\tilde{T}_\theta f} \quad \text{and} \quad \underline{T}_\theta^{-1} f = \underline{T}_{-\theta} f = \overline{\underline{T}_\theta f}, \quad \forall f \in \mathcal{S}(G),$$

2. \tilde{T}_θ commutes with left translations and \underline{T}_θ commutes with right translations,
3. As operators on $\mathcal{S}(G)$, \tilde{T}_θ , \underline{T}_θ , $M(\chi_G)$, $L^{\tilde{\star}_\theta}(\chi_G)$, $R^{\star_\theta}(\chi_G)$, $L^{\pm_\theta}(\chi_G)$ and $R^{\pm_\theta}(\chi_G)$ commute pairwise,
4. For $\alpha \in \mathbb{C}$, we have

$$\tilde{T}_\theta^\alpha = L^{\tilde{\star}_\theta}(\chi_G^{-\alpha/4}) R^{\tilde{\star}_\theta}(\chi_G^{\alpha/4}), \quad \underline{T}_\theta^\alpha = L^{\pm_\theta}(\chi_G^{-\alpha/4}) R^{\pm_\theta}(\chi_G^{\alpha/4}) \quad \text{and} \quad \tilde{T}_\theta^\alpha \underline{T}_\theta^\alpha = L^{\star_\theta}(\chi_G^{-\alpha/4}) R^{\star_\theta}(\chi_G^{\alpha/4}),$$

5. Given with the initial domain $\mathcal{S}(G)$, \tilde{T}_θ and \underline{T}_θ are essentially selfadjoint both on $L_\rho^2(G)$ and on $L_\lambda^2(G)$.

Proof. Consider the family of numerical functions

$$f_\theta(x) := (1 + \pi^2 \theta^2 x^2)^{1/2} + \pi \theta x, \quad \theta \in \mathbb{R}. \quad (3.6)$$

Let E be the element of the Lie algebra of \mathbb{S} characterized by the relation $\exp\{tE\} = (0, 0, t) \in \mathbb{S}$. In the case where G is elementary, that is $G = \mathbb{S}$, it is shown in [8, Lemma 2.1] that, with \tilde{E} the left invariant vector field associated to E , we have:

$$\tilde{T}_\theta = f_\theta(i\tilde{E})^{\dim(\mathbb{S})/4}.$$

In the general case, according to the semidirect product decomposition (2.4) of G , we define E_j as the element of the Lie algebra of G which coincide with the element E as defined above, for each factor \mathbb{S}_j . Observe that E_1, \dots, E_N generates an Abelian subalgebra of the Lie algebra of G . Now, a direct generalization of the computations done in [8, Lemma 2.1] yields:

$$\tilde{T}_\theta = \prod_{j=1}^N f_\theta(i\tilde{E}_j)^{\dim(\mathbb{S}_j)/4}. \quad (3.7)$$

Since the (classical) antipode S intertwines λ and ρ , we get $S\tilde{E}_j S = \underline{E}_j$, and thus

$$\underline{T}_\theta = \prod_{j=1}^N f_\theta(i\underline{E}_j)^{\dim(\mathbb{S}_j)/4}. \quad (3.8)$$

Then, the first item follows from the fact $f_\theta(-x) = f_{-\theta}(x) = f_\theta(x)^{-1}$.

The second item follows just from the facts that $\tilde{\star}_\theta$ is equivariant under left translations while \star_θ is equivariant under right translations and that χ_G is a character on G .

We come to the third item. Note first that \tilde{T}_θ and \underline{T}_θ commute. Indeed, \tilde{T}_θ is of the form $\rho(\Phi)$ while \underline{T}_θ is of the form $\lambda(\Phi)$ for Φ a distribution on G . Then, note that \tilde{T}_θ and \underline{T}_θ both commute with $M(\chi_G)$. Indeed, working in the coordinate system (2.5) for each \mathbb{S}_j and under the parametrization $g = g_1 \cdots g_N$ of $G = (\mathbb{S}_N \times \dots) \times \mathbb{S}_1$, we have $\chi_G(g) = e^{\sum_{j=1}^N \dim(\mathbb{S}_j) a_j}$, while $\tilde{E}_j = \partial_{t_j}$ and $\underline{E}_j = -e^{-2a_j} \partial_{t_j}$. Now, when G is elementary, we can deduce from [8, Lemma 2.1] that $L^{\tilde{\star}_\theta}(\chi_G)$, $R^{\tilde{\star}_\theta}(\chi_G)$, $L^{\star_\theta}(\chi_G)$ and $R^{\star_\theta}(\chi_G)$ are combinations of \tilde{T}_θ , \underline{T}_θ and $M(\chi_G)$. But the computations made in [8, Lemma 2.1] (and thus the relations between all these operators) extend directly for general negatively curved Kählerian Lie group G . Hence the result.

The first relation of the fourth item come again from [8, Lemma 2.1] (extended to general negatively curved Kählerian Lie groups). The second relation can be deduced from the first using Equation (3.3) and the fact that $S(\chi_G^\alpha) = \chi_G^{-\alpha}$. For the third relation, we proceed with formal computations which, however, can be rendered rigorous working with oscillatory integrals. Let $f \in \mathcal{S}(G)$ and $\alpha \in \mathbb{C}$. We have:

$$\begin{aligned} L^{\star_\theta}(\chi_G^\alpha) f &= \chi_G \star_\theta f = \int_{G^4} K_\theta(g_1, g_2) K_{-\theta}(g_3, g_4) (\rho_{g_1} \lambda_{g_3} \chi_G^\alpha) (\rho_{g_2} \lambda_{g_4} f) d^\lambda(g_1) d^\lambda(g_2) d^\lambda(g_3) d^\lambda(g_4) \\ &= \chi_G^{-\alpha} \int_{G^2} K_\theta(g_1, g_2) (\rho_{g_1} \chi_G^\alpha) \rho_{g_2} \left(\int_{G^2} K_{-\theta}(g_3, g_4) (\lambda_{g_3} \chi_G^\alpha) (\lambda_{g_4} f) d^\lambda(g_3) d^\lambda(g_4) \right) d^\lambda(g_1) d^\lambda(g_2) \\ &= \chi_G^{-\alpha} \int_{G^2} K_\theta(g_1, g_2) (\rho_{g_1} \chi_G^\alpha) (\rho_{g_2} (\chi_G^\alpha \tilde{\star}_\theta f)) d^\lambda(g_1) d^\lambda(g_2) = \chi_G^{-\alpha} (\chi_G^\alpha \tilde{\star}_\theta (\chi_G^\alpha \star_\theta f)). \end{aligned}$$

Hence we get $L^{\star_\theta}(\chi_G^\alpha) = M(\chi_G^{-\alpha}) L^{\tilde{\star}_\theta}(\chi_G^\alpha) L^{\star_\theta}(\chi_G^\alpha)$. Since the complex conjugation is an involution for all these deformed products, we get

$$R^{\star_\theta}(\chi_G^{-\alpha}) f = \overline{L^{\star_\theta}(\chi_G^{-\alpha}) \bar{f}} = \overline{M(\chi_G^\alpha) L^{\tilde{\star}_\theta}(\chi_G^{-\alpha}) L^{\star_\theta}(\chi_G^{-\alpha}) \bar{f}} = M(\chi_G^\alpha) R^{\tilde{\star}_\theta}(\chi_G^{-\alpha}) R^{\star_\theta}(\chi_G^{-\alpha}) f.$$

From this and the first two relations, we get

$$L^{\star_\theta}(\chi_G^\alpha) R^{\star_\theta}(\chi_G^{-\alpha}) = L^{\tilde{\star}_\theta}(\chi_G^\alpha) R^{\tilde{\star}_\theta}(\chi_G^{-\alpha}) L^{\star_\theta}(\chi_G^\alpha) R^{\star_\theta}(\chi_G^{-\alpha}) = \tilde{T}_\theta^{4\alpha} \underline{T}_\theta^{4\alpha}.$$

The statement in the last item comes from the fact that in the coordinate system (2.5) we have $\tilde{E}_j = \partial_{t_j}$ and $\underline{E}_j = -e^{-2a_j} \partial_{t_j}$ while the left and right Haar measures read $d^\lambda(g) = da_1 dv_1 dt_1 \cdots da_N dv_N dt_N$ and $d^\rho(g) = e^{\sum_{j=1}^N \dim(\mathbb{S}_j) a_j} da_1 dv_1 dt_1 \cdots da_N dv_N dt_N$. \square

Remark 3.18. The fourth item of Lemma 3.17 above implies that $\widetilde{T}_\theta^\alpha$ (resp. $\underline{T}_\theta^\alpha$) is an inner automorphism (in the sense of multipliers) of the product $\widetilde{\star}_\theta$ (resp. $\underline{\star}_\theta$). But it will follow from modular theory that \widetilde{T}_θ^4 and \underline{T}_θ^4 are also automorphisms of the product \star_θ . However, they are probably not inner.

Proposition 3.19. *The right Haar integral τ_ρ is a continuous, positive and faithful linear functional of the involutive (for the complex conjugation) Fréchet algebra $(\mathcal{S}(G), \star_\theta)$. Indeed, we have for all $f \in \mathcal{S}(G)$:*

$$\tau_\rho(\overline{f} \star_\theta f) = \tau_\rho(|\widetilde{T}_\theta(f)|^2).$$

Proof. Continuity is obvious. It remains to prove positivity and non-degeneracy. By [6, Proposition 5.10], we have for $f \in \mathcal{S}(G)$:

$$\overline{f} \star_\theta f = \widetilde{\int_{G \times G}} K_\theta(g_1, g_2) (\rho_{g_1} \overline{f}) \underline{\star}_\theta (\rho_{g_2} f) d^\lambda(g_1) d^\lambda(g_2),$$

Since $\tau_\rho : \mathcal{S}(G) \rightarrow \mathbb{C}$ is continuous, we can use [6, Lemma 1.37], to get:

$$\tau_\rho(\overline{f} \star_\theta f) = \widetilde{\int_{G \times G}} K_\theta(g_1, g_2) \tau_\rho((\rho_{g_1} \overline{f}) \underline{\star}_\theta (\rho_{g_2} f)) d^\lambda(g_1) d^\lambda(g_2).$$

The right invariant version of [8, Lemma 2.6] (obtained by intertwining everything with the undeformed antipode S) gives $\tau_\rho(f_1 \underline{\star}_\theta f_2) = \tau_\rho(f_1 f_2)$ for all $f_1, f_2 \in \mathcal{S}(G)$, and thus

$$\tau_\rho(\overline{f} \star_\theta f) = \widetilde{\int_{G \times G}} K_\theta(g_1, g_2) \tau_\rho((\rho_{g_1} \overline{f}) (\rho_{g_2} f)) d^\lambda(g_1) d^\lambda(g_2).$$

Using [6, Lemma 1.37] backwards, one deduces

$$\tau_\rho(\overline{f} \star_\theta f) = \tau_\rho\left(\widetilde{\int_{G \times G}} K_\theta(g_1, g_2) (\rho_{g_1} \overline{f}) (\rho_{g_2} f) d^\lambda(g_1) d^\lambda(g_2)\right) = \tau_\rho(\overline{f} \widetilde{\star}_\theta f).$$

Since $\chi_G = \chi_G^{1/2} \widetilde{\star}_\theta \chi_G^{1/2}$ (which follows from a direct computation – see also [8, Remark 2.2]) we get

$$\tau_\rho(\overline{f} \widetilde{\star}_\theta f) = \int_G \overline{f} \widetilde{\star}_\theta f(g) \chi_G(g) d^\lambda(g) = \int_G \overline{f} \widetilde{\star}_\theta f(g) \chi_G^{1/2} \widetilde{\star}_\theta \chi_G^{1/2}(g) d^\lambda(g).$$

Then, using a bounded approximate unit argument (see [6, Proposition 4.19]) coupled with [8, Lemma 2.6], one can show that

$$\int_G \overline{f} \widetilde{\star}_\theta f(g) \chi_G^{1/2} \widetilde{\star}_\theta \chi_G^{1/2}(g) d^\lambda(g) = \int_G |f \widetilde{\star}_\theta \chi_G^{1/2}|^2(g) d^\lambda(g) = \int_G |(f \widetilde{\star}_\theta \chi_G^{1/2}) \chi_G^{-1/2}|^2(g) d^\lambda(g) = \tau_\rho(|\widetilde{T}_\theta(f)|^2),$$

which implies positivity and non-degeneracy since \widetilde{T}_θ is invertible on $\mathcal{S}(G)$. \square

We now come to right-invariance of the right Haar integral on $(\mathcal{S}(G), \star_\theta)$. We first establish an invariance property for the deformed Kac-Takesaki operator.

Proposition 3.20. *As continuous linear maps from $\mathcal{S}(G \times G)$ to $\mathcal{S}(G)$, we have*

$$(\tau_\rho \otimes \text{Id}) V_\theta = \tau_\rho \otimes \text{Id}.$$

More generally, for $f \in \mathcal{S}(G)$, then we have

$$(\tau_\rho \otimes L^{\star_\theta}(f)) V_\theta = \tau_\rho \otimes L^{\star_\theta}(f).$$

Proof. For the first equality, we need to prove that for all $f_1, f_2 \in \mathcal{S}(G)$, we have

$$\int_G (\lambda_{g^{-1}} f_1) \star_\theta f_2 \, d^\rho(g) = \left(\int_G f_1(g) \, d^\rho(g) \right) f_2 .$$

For the second equality, we need to prove that for all $f_1, f_2, f_3 \in \mathcal{S}(G)$, we have

$$\int_G f_1 \star_\theta (\lambda_{g^{-1}} f_2) \star_\theta f_3 \, d^\rho(g) = \left(\int_G f_2(g) \, d^\rho(g) \right) f_1 \star_\theta f_3 .$$

Algebraically speaking, both equalities are obvious since by invariance under right translations we have

$$\int_G \lambda_{g^{-1}} f \, d^\rho(g) = \int_G f(g) \, d^\rho(g), \quad \forall f \in \mathcal{S}(G),$$

and since the constant unit function is the unit for \star_θ (in the sense of multipliers – see Proposition 3.7). But this formal observation can be made rigorous by coming back to the definition of the oscillatory integral and using the ordinary Fubini theorem to commute the integrals. \square

Since $(\text{Id} \otimes L^{\star_\theta}(f_1)) V_\theta(f_2 \otimes f_3) = (1 \otimes f_1) \star_\theta \Delta(f_2) \star_\theta (1 \otimes f_3)$, we deduce:

Corollary 3.21. *The right Haar integral τ_ρ is right-invariant on $(\mathcal{S}(G), \star_\theta, \Delta)$:*

$$\tau_\rho \otimes \tau_\rho((1 \otimes f_1) \star_\theta \Delta(f_2) \star_\theta (1 \otimes f_3)) = \tau_\rho(f_1 \star_\theta f_3) \tau_\rho(f_2), \quad \forall f_1, f_2, f_3 \in \mathcal{S}(G).$$

Definition 3.22. *We let \mathcal{H}_θ be the Hilbert space completion of $\mathcal{S}(G)$ with respect to the inner product:*

$$\langle f_1, f_2 \rangle_\theta := \tau_\rho(\overline{f_1} \star_\theta f_2) .$$

Endowed with the inner product $\langle \cdot, \cdot \rangle_\theta$ and the involution given by the complex conjugation, we may be tempted to see $(\mathcal{S}(G), \star_\theta)$ as a left Hilbert algebra. Indeed, by Proposition 3.8 one knows that $\mathcal{S}(G) \star_\theta \mathcal{S}(G)$ is dense in $\mathcal{S}(G)$ with respect to its Fréchet topology, topology that clearly dominates the Hilbert space topology of \mathcal{H}_θ . Also we have for $f_1, f_2, f_3 \in \mathcal{S}(G)$:

$$\langle f_1 \star_\theta f_2, f_3 \rangle_\theta = \tau_\rho(\overline{f_2} \star_\theta \overline{f_1} \star_\theta f_3) = \langle f_2, \overline{f_1} \star_\theta f_3 \rangle_\theta . \quad (3.9)$$

Moreover, it is not difficult to prove that the complex conjugation is preclosed on \mathcal{H}_θ . But what is missing at this stage is the fact that $\mathcal{S}(G)$ acts by \star_θ -multiplication on the left of \mathcal{H}_θ by bounded operators. However, we can determine by hand the “modular datas”:

Lemma 3.23. *Set $J_\theta : \mathcal{S}(G) \rightarrow \mathcal{S}(G)$, $f \mapsto \widetilde{T}_\theta^{-2}(\overline{f})$ and $I_\theta : \mathcal{S}(G) \rightarrow \mathcal{S}(G)$, $f \mapsto \widetilde{T}_\theta^{-4}(\overline{f})$. Then I_θ defines a continuous involution of the Fréchet algebra $(\mathcal{S}(G), \star_\theta)$ and, moreover, we have for all $f_1, f_2, f_3 \in \mathcal{S}(G)$:*

$$\langle f_1, f_2 \rangle_\theta = \langle J_\theta f_2, J_\theta f_1 \rangle_\theta, \quad \langle \overline{f_1}, f_2 \rangle_\theta = \langle I_\theta f_2, f_1 \rangle_\theta \quad \text{and} \quad \langle f_1 \star_\theta f_2, f_3 \rangle_\theta = \langle f_1, f_3 \star_\theta I_\theta f_2 \rangle_\theta .$$

Proof. We observe that for $f_1, f_2 \in \mathcal{S}(G)$, we have by Lemma 3.17 and Proposition 3.19:

$$\langle f_1, f_2 \rangle_\theta = \tau_\rho(\overline{\widetilde{T}_\theta f_1} \widetilde{T}_\theta f_2) = \tau_\rho(\overline{\widetilde{T}_{-\theta} f_2} \widetilde{T}_{-\theta} f_1) = \langle \widetilde{T}_{-\theta}^2 f_2, \widetilde{T}_{-\theta}^2 f_1 \rangle_\theta = \langle \widetilde{T}_\theta^{-2} f_2, \widetilde{T}_\theta^{-2} f_1 \rangle_\theta ,$$

and

$$\langle \overline{f_1}, f_2 \rangle_\theta = \tau_\rho(\overline{\widetilde{T}_\theta f_1} \widetilde{T}_\theta f_2) = \tau_\rho(\overline{\widetilde{T}_\theta^{-1} f_1} \widetilde{T}_\theta^3 f_2) = \tau_\rho(\overline{\widetilde{T}_\theta^{-3} f_2} \widetilde{T}_\theta f_1) = \langle \widetilde{T}_\theta^{-4} f_2, f_1 \rangle_\theta ,$$

which are the first two relations we have to prove. Next we prove that I_θ is an involution. First, items 1. and 4. of Lemma 3.17 entails that $I_\theta^2 = \text{Id}$. Then, from the second identity, we have:

$$\langle I_\theta(f_1 \star_\theta f_2), f_3 \rangle_\theta = \langle \overline{f_3}, f_1 \star_\theta f_2 \rangle_\theta = \tau_\rho(f_3 \star_\theta f_1 \star_\theta f_2) .$$

On the other hand we get since the complex conjugation is an involution:

$$\begin{aligned} \langle I_\theta f_2 \star_\theta I_\theta f_1, f_3 \rangle_\theta &= \langle I_\theta f_1, \overline{I_\theta f_2} \star_\theta f_3 \rangle_\theta = \langle \overline{f_3} \star_\theta I_\theta f_2, f_1 \rangle_\theta \\ &= \langle I_\theta f_2, f_3 \star_\theta f_1 \rangle_\theta = \langle \overline{f_3 \star_\theta f_1}, f_2 \rangle_\theta = \tau_\rho(f_3 \star_\theta f_1 \star_\theta f_2) . \end{aligned}$$

Thus, $\langle I_\theta(f_1 \star_\theta f_2), f_3 \rangle_\theta = \langle I_\theta f_2 \star_\theta I_\theta f_1, f_3 \rangle_\theta$ for all $f_3 \in \mathcal{S}(G)$ and hence we deduce by density of $\mathcal{S}(G)$ in \mathcal{H}_θ that $I_\theta(f_1 \star_\theta f_2) = I_\theta f_2 \star_\theta I_\theta f_1$ as needed. Last we have to prove the formula for the (formal) adjoint of the operator of \star_θ -multiplication on the right. From what precedes we deduce:

$$\langle f_1 \star_\theta f_2, f_3 \rangle_\theta = \langle \overline{f_3}, I_\theta f_2 \star_\theta I_\theta f_1 \rangle_\theta = \langle \overline{I_\theta f_2} \star_\theta \overline{f_3}, I_\theta f_1 \rangle_\theta = \langle \overline{f_3 \star_\theta I_\theta f_2}, I_\theta f_1 \rangle_\theta = \langle f_1, f_3 \star_\theta I_\theta f_2 \rangle_\theta ,$$

which concludes the proof. \square

Remark 3.24. By analogy with Tomita-Takesaki theory, we call J_θ the modular conjugation and $N_\theta := \tilde{T}_\theta^{-4}$ the modular operator. Observe also that the canonical antilinear isomorphism from \mathcal{H}_θ to its conjugate Hilbert space is then given on $\mathcal{S}(G)$ by $f \mapsto J_\theta f$.

Since the composition of two involutions is an automorphism, we deduce:

Corollary 3.25. *The operator \tilde{T}_θ^4 is an automorphism of the algebra $(\mathcal{S}(G), \star_\theta)$.*

Remark 3.26. We believe that the automorphism \tilde{T}_θ^4 is outer. Since, $\underline{T}_\theta = S\tilde{T}_\theta S$, we deduce by Proposition 3.6 that \underline{T}_θ^4 is also an automorphism of $(\mathcal{S}(G), \star_\theta)$. We believe that it is outer too. Note last that in the proof of Proposition 3.19, we have obtained

$$\tau_\rho(\overline{f} \star_\theta f) = \tau_\rho(\overline{f} \tilde{\star}_\theta f) = \tau_\rho(|\tilde{T}_\theta(f)|^2), \quad \forall f \in \mathcal{S}(G),$$

we deduce that \tilde{T}_θ^4 is an automorphism of the algebra $(\mathcal{S}(G), \star_\theta)$. Composing with the antipode S , we get that \underline{T}_θ^4 is an automorphism of the algebra $(\mathcal{S}(G), \tilde{\star}_\theta)$. We also believe that they are outer.

3.5 Unitarity and multiplicativity

We come to the main result of this section, which is now a straightforward application of what we have already proven.

Theorem 3.27. *The deformed Kac-Takesaki operator V_θ extends to a multiplicative unitary on \mathcal{H}_θ .*

Proof. Unitarity : From Proposition 3.15, we get for $f_1, f_2 \in \mathcal{S}(G)$:

$$\overline{V_\theta(f_1 \otimes f_2)} \star_\theta V_\theta(f_1 \otimes f_2) = (1 \otimes \overline{f_2}) \star_\theta \Delta \overline{f_1} \star_\theta \Delta f_1 \star_\theta (1 \otimes f_2) = (1 \otimes \overline{f_2}) \star_\theta \Delta(\overline{f_1} \star_\theta f_1) \star_\theta (1 \otimes f_2) .$$

Applying $\tau_\rho \otimes \tau_\rho$ to both sides, Corollary 3.21 shows that V_θ extends to an isometry on \mathcal{H}_θ . By Corollary 3.11, V_θ is invertible on $\mathcal{S}(G)$, hence its extension to \mathcal{H}_θ is surjective and, therefore, V_θ is unitary on $\mathcal{H}_\theta \bar{\otimes} \mathcal{H}_\theta$.

Multiplicativity : Remark that, as V_θ can be written (Lemma 3.14)

$$V_\theta(f_1 \otimes f_2) = \Delta f_1 \star_\theta (1 \otimes f_2), \quad \text{for } f_1, f_2 \in \mathcal{S}(G),$$

in the algebra of continuous multipliers of $(\mathcal{S}(G \times G), \star_\theta)$, and as \star_θ is associative, Δ is coassociative and a \star_θ -homomorphism (Lemma 3.15), the pentagonal equation (which characterizes multiplicativity) is automatically fulfilled (see [23, proof of theorem 1.7.4]). Then, by density of $\mathcal{S}(G) \otimes \mathcal{S}(G) \otimes \mathcal{S}(G)$ in $\mathcal{H}_\theta \bar{\otimes} \mathcal{H}_\theta \bar{\otimes} \mathcal{H}_\theta$, the multiplicativity holds, as needed, in $\mathcal{H}_\theta \bar{\otimes} \mathcal{H}_\theta \bar{\otimes} \mathcal{H}_\theta$. \square

4 Properties of the multiplicative unitary

4.1 The legs of V_θ

The left and right legs, $\widehat{A}(V_\theta)$ and $A(V_\theta)$, of a multiplicative unitary are defined as the norm closures of the vector subspaces of $\mathcal{B}(\mathcal{H}_\theta)$ given by³

$$\widehat{A}_0(V_\theta) := \{\text{Id} \otimes \omega(V_\theta) : \omega \in \mathcal{B}(\mathcal{H}_\theta)_*\} \quad \text{and} \quad A_0(V_\theta) := \{\omega \otimes \text{Id}(V_\theta) : \omega \in \mathcal{B}(\mathcal{H}_\theta)_*\} .$$

In general, $A(V_\theta)$ and $\widehat{A}(V_\theta)$ are subalgebras of $\mathcal{B}(\mathcal{H}_\theta)$ but need not to be $*$ -subalgebras. We will prove that it is indeed the case here. For $f_1, f_2 \in \mathcal{H}_\theta$, we let $\omega_{f_1, f_2} \in \mathcal{B}(\mathcal{H}_\theta)_*$ be, as usual, the normal functional given for $A \in \mathcal{B}(\mathcal{H}_\theta)$ by $\omega_{f_1, f_2}(A) := \langle f_1, Af_2 \rangle_\theta$.

For $g \in G$, we define

$$\rho_\theta(g) : \mathcal{S}(G) \rightarrow \mathcal{S}(G), \quad f \mapsto \widetilde{T}_\theta^{-1} \rho_g \widetilde{T}_\theta f . \quad (4.1)$$

By Proposition 3.19 we see that ρ_θ extends to a unitary representation of G on \mathcal{H}_θ . Therefore, it yields a representation (still denoted by ρ_θ) of the convolution algebra $(L_\lambda^1(G), *)$ on the Hilbert space \mathcal{H}_θ . Since G is solvable, it is amenable and thus the norm closure of $\rho_\theta(L_\lambda^1(G), *)$ is isomorphic as C^* -algebra to $C^*(G)$, the group C^* -algebra of G . Since the coproduct Δ has not been deformed, it is natural to guess that $\widehat{A}(V_\theta)$ is isomorphic to $C^*(G)$. To prove this, we need some preparatory materials.

Lemma 4.1. *Let $f \in \mathcal{S}(G)$. Then we have the equality of tempered functions:*

$$(\widetilde{T}_\theta \otimes \text{Id})\Delta f = (\text{Id} \otimes \underline{T}_\theta^{-1})\Delta f .$$

Proof. For $f \in \mathcal{S}(G)$, we have $(\rho_g \otimes \text{Id})\Delta f = (\text{Id} \otimes \lambda_{g^{-1}})\Delta f$, which (with the notations of the proof of Lemma 3.17) implies that

$$(\widetilde{E}_j \otimes \text{Id})\Delta f = (\text{Id} \otimes (-\underline{E}_j))\Delta f, \quad \forall j = 1 \dots N .$$

Therefore, using (3.7) and (3.8) together with the fact that $f_\theta(-x) = f_\theta(x)^{-1}$, we get

$$\begin{aligned} (\widetilde{T}_\theta \otimes \text{Id})\Delta f &= \left(\prod_{j=1}^N f_\theta(i\widetilde{E}_j)^{\dim(\mathbb{S}_j)/4} \otimes \text{Id} \right) \Delta f \\ &= \left(\text{Id} \otimes \prod_{j=1}^N f_\theta(-i\underline{E}_j)^{\dim(\mathbb{S}_j)/4} \right) \Delta f = \left(\text{Id} \otimes \prod_{j=1}^N f_\theta(i\underline{E}_j)^{-\dim(\mathbb{S}_j)/4} \right) \Delta f = (\text{Id} \otimes \underline{T}_\theta^{-1})\Delta f , \end{aligned}$$

which completes the proof. \square

Remember that for $f \in \mathcal{S}(G)$, $\rho(f)$ is the continuous operator on $\mathcal{S}(G)$ given by $\rho(f) := \int_G f(g) \rho_g d^\lambda(g)$. Similarly, we define $\rho_\theta(f) := \int_G f(g) \rho_\theta(g) d^\lambda(g)$, where $\rho_\theta(g)$ is defined in (4.1). (Note also that the representation ρ of G is not unitary on \mathcal{H}_θ but ρ_θ is unitary on \mathcal{H}_θ .)

Lemma 4.2. *Let $f \in \mathcal{S}(G)$ and set $Q_\theta := \widetilde{T}_\theta^{-2} \underline{T}_\theta^{-2}$. As continuous operators on $\mathcal{S}(G)$, we have*

$$\rho(f) = \rho_\theta(Q_\theta^{-1/2} f) .$$

³We use the definition of Baaž and Skandalis [1] for the left leg and not the one of Woronowicz [33]. But since we will prove stability under adjunction, both definitions coincide here.

Proof. Let $f_1, f_2 \in \mathcal{S}(G)$. Then

$$\rho_\theta(f_1)f_2(g') = \int_G f_1(g) \tilde{T}_\theta^{-1} \rho_g \tilde{T}_\theta f_2(g') d^\lambda(g) = \int_G f_1(g) (\tilde{T}_\theta^{-1} \otimes \text{Id}) \Delta(\tilde{T}_\theta f_2)(g', g) d^\lambda(g).$$

Hence, Lemma 4.1 gives

$$\rho_\theta(f_1)f_2(g') = \int_G f_1(g) (\text{Id} \otimes \underline{T}_\theta) \Delta(\tilde{T}_\theta f_2)(g', g) d^\lambda(g).$$

Now, Lemma 3.17 implies that \underline{T}_θ is symmetric on $L_\lambda^2(G)$ and $\overline{\underline{T}_\theta(f)} = \underline{T}_\theta^{-1}(\bar{f})$ for all $f \in \mathcal{S}(G)$. Thus,

$$\rho_\theta(f_1)f_2(g') = \int_G (\underline{T}_\theta^{-1} f_1)(g) \Delta(\tilde{T}_\theta f_2)(g', g) d^\lambda(g) = \int_G (\underline{T}_\theta^{-1} f_1)(g) (\lambda_{g'^{-1}} \tilde{T}_\theta f_2)(g) d^\lambda(g).$$

Since \tilde{T}_θ commutes with left translations, is symmetric on $L_\lambda^2(G)$ and satisfies $\overline{\tilde{T}_\theta(f)} = \tilde{T}_\theta^{-1}(\bar{f})$, we finally get

$$\rho_\theta(f_1)f_2(g') = \int_G (\tilde{T}_\theta^{-1} \underline{T}_\theta^{-1} f_1)(g) (\lambda_{g'^{-1}} f_2)(g) d^\lambda(g) = \int_G (\tilde{T}_\theta^{-1} \underline{T}_\theta^{-1} f_1)(g) (\rho_g f_2)(g') d^\lambda(g).$$

Therefore $\rho_\theta(f_1) = \rho(\tilde{T}_\theta^{-1} \underline{T}_\theta^{-1} f_1)$ which implies that $\rho(f_1) = \rho_\theta(\tilde{T}_\theta \underline{T}_\theta f_1)$ as needed. \square

Proposition 4.3. *For all $f_1, f_2 \in \mathcal{S}(G)$, we have*

$$\text{Id} \otimes \omega_{f_1, f_2}(V_\theta) = \rho_\theta(\chi_G \tilde{T}_\theta^2 Q_\theta^{-1/2}(f_2 \star_\theta \tilde{T}_\theta^{-2} J_\theta f_1)),$$

Proof. Let $f_1, f_2, f_3, f_4 \in \mathcal{S}(G)$. Then, we have:

$$\begin{aligned} \langle f_3, \text{Id} \otimes \omega_{f_1, f_2}(V_\theta) f_4 \rangle_\theta &= \langle f_3 \otimes f_1, V_\theta(f_4 \otimes f_2) \rangle_\theta \\ &= \langle f_3 \otimes f_1, \Delta f_4 \star_\theta (1 \otimes f_2) \rangle_\theta = \langle f_3, [g \mapsto \langle f_1, (\lambda_{g^{-1}} f_4) \star_\theta f_2 \rangle_\theta] \rangle_\theta, \end{aligned}$$

where the last equality follows by Fubini. Then, with $\langle \cdot, \cdot \rangle$ the usual inner product of $L_\rho^2(G)$, Proposition 3.19 and the last item of Lemma 3.17 give

$$\langle f_1, f_2 \rangle_\theta = \langle \tilde{T}_\theta^2 f_1, f_2 \rangle.$$

With that relation in mind, we observe:

$$\begin{aligned} \langle f_1, (\lambda_{g^{-1}} f_4) \star_\theta f_2 \rangle_\theta &= \langle \tilde{T}_\theta^2 f_1, (\lambda_{g^{-1}} f_4) \star_\theta f_2 \rangle = \langle \bar{f}_2 \star_\theta (\lambda_{g^{-1}} \bar{f}_4), \overline{\tilde{T}_\theta^2 f_1} \rangle = \langle \bar{f}_2 \star_\theta (\lambda_{g^{-1}} \bar{f}_4), \tilde{T}_\theta^{-4} \bar{f}_1 \rangle_\theta \\ &= \langle \lambda_{g^{-1}} \bar{f}_4, f_2 \star_\theta \tilde{T}_\theta^{-4} \bar{f}_1 \rangle_\theta = \langle \lambda_{g^{-1}} \bar{f}_4, \tilde{T}_\theta^2 (f_2 \star_\theta \tilde{T}_\theta^{-4} \bar{f}_1) \rangle \\ &= \int (\lambda_{g^{-1}} f_4)(g') \tilde{T}_\theta^2 (f_2 \star_\theta \tilde{T}_\theta^{-4} \bar{f}_1)(g') d^\rho(g') \\ &= \int \chi_G(g') \tilde{T}_\theta^2 (f_2 \star_\theta \tilde{T}_\theta^{-4} \bar{f}_1)(g') (\rho_{g'} f_4)(g) d^\lambda(g') = \rho(\chi_G \tilde{T}_\theta^2 (f_2 \star_\theta \tilde{T}_\theta^{-4} \bar{f}_1)) f_4(g). \end{aligned}$$

Hence,

$$\text{Id} \otimes \omega_{f_1, f_2}(V_\theta) = \rho(\chi_G \tilde{T}_\theta^2 (f_2 \star_\theta \tilde{T}_\theta^{-4} \bar{f}_1)),$$

which by Lemma 4.2 and the fact that $\tilde{T}_\theta \underline{T}_\theta$ commutes with the operator of pointwise multiplication by χ_G gives the result. \square

By Proposition we know that $\mathcal{S}(G) \star_\theta \mathcal{S}(G)$ is dense in $\mathcal{S}(G)$, hence dense in $L_\lambda^1(G)$ too. This implies that the norm closure of $\{\text{Id} \otimes \omega_{f_1, f_2}(V_\theta) : f_1, f_2 \in \mathcal{S}(G)\}$ contains $\rho_\theta(L_\lambda^1(G))$ and therefore we deduce (remember that G is solvable hence amenable and thus the reduced and full group C^* -algebras coincide):

Corollary 4.4. *The left leg $\widehat{A}(V_\theta)$ of the multiplicative unitary V_θ is isomorphic to $C^*(G)$ as C^* -algebras.*

We next go to the right leg of V_θ . Remember that $L^{*\theta}(f)$ is defined as the continuous operator on $\mathcal{S}(G)$ of left \star_θ -multiplication by f .

Proposition 4.5. *For all $f_1, f_2 \in \mathcal{S}(G)$, we have*

$$\omega_{f_1, f_2} \otimes \text{Id}(V_\theta) = L^{*\theta}([g \mapsto \langle f_1, \rho_g f_2 \rangle_\theta]).$$

Proof. Let $f_1, f_2, f_3, f_4 \in \mathcal{S}(G)$. Then, we have

$$\begin{aligned} \langle f_3, \omega_{f_1, f_2} \otimes \text{Id}(V_\theta) f_4 \rangle_\theta &= \langle f_1 \otimes f_3, V_\theta(f_2 \otimes f_4) \rangle_\theta \\ &= \langle f_1 \otimes f_3, \Delta f_2 \star_\theta (1 \otimes f_4) \rangle_\theta = \langle f_3, [g \mapsto \langle f_1, \rho_g f_2 \rangle_\theta] \star_\theta f_4 \rangle_\theta, \end{aligned}$$

which is the formula we were looking for. \square

Our next task is to show that $A(V_\theta)$ is a C^* -algebra. We will do this by obtaining an explicit formula for the adjoint of $\omega_{f_1, f_2} \otimes \text{Id}(V_\theta)$. This formula will also be fundamental to prove manageability of V_θ .

Proposition 4.6. *For all $f_1, f_2 \in \mathcal{S}(G)$, we have*

$$\omega_{f_1, f_2} \otimes \text{Id}(V_\theta)^* = \omega_{\widetilde{T}_\theta^{-2} J_\theta f_1, \widetilde{T}_\theta^2 J_\theta f_2} \otimes \text{Id}(V_\theta).$$

Hence the adjoints of the elements in $A_0(V_\theta)$ still belong to $A_0(V_\theta)$. So, the right leg of the multiplicative unitary V_θ , $A(V_\theta)$, is a C^ -algebra.*

Proof. We have by Proposition 4.5 and equation (3.9):

$$\omega_{f_1, f_2} \otimes \text{Id}(V_\theta)^* = L^{*\theta}([g \mapsto \langle f_1, \rho_g f_2 \rangle_\theta])^* = L^{*\theta}([g \mapsto \overline{\langle f_1, \rho_g f_2 \rangle_\theta}]).$$

Hence, we deduce by Lemma 3.23:

$$\overline{\langle f_1, \rho_g f_2 \rangle_\theta} = \langle \rho_g f_2, f_1 \rangle_\theta = \langle J_\theta f_1, J_\theta \rho_g f_2 \rangle_\theta = \langle \widetilde{T}_\theta^{-2} J_\theta f_1, \rho_g \overline{f_2} \rangle_\theta = \langle \widetilde{T}_\theta^{-2} J_\theta f_1, \rho_g \widetilde{T}_\theta^2 J_\theta f_2 \rangle_\theta,$$

which completes the proof. \square

4.2 Manageability and the antipode

For an Hilbert space \mathcal{H} we denote by $\overline{\mathcal{H}}$ the conjugate Hilbert space and by $\mathcal{H} \rightarrow \overline{\mathcal{H}}$, $\eta \mapsto \overline{\eta}$, the canonical antilinear isomorphism. Recall that a multiplicative unitary V on \mathcal{H} is manageable in the sense of Woronowicz [33] if there exist a unitary operator \widetilde{V} on $\overline{\mathcal{H}} \otimes \mathcal{H}$ and a densely defined positive self-adjoint operator Q on \mathcal{H} with densely defined inverse Q^{-1} , such that for all $\eta_1, \eta_2 \in \mathcal{H}$ and all $\xi_1 \in \text{Dom}(Q)$, $\xi_2 \in \text{Dom}(Q^{-1})$, we have

$$\langle \eta_1 \otimes \xi_1, V(\eta_2 \otimes \xi_2) \rangle = \langle \overline{\eta_2} \otimes Q\xi_1, \widetilde{V}(\overline{\eta_1} \otimes Q^{-1}\xi_2) \rangle \quad \text{and} \quad V^*(Q \otimes Q)V = (Q \otimes Q).$$

Note that in our context, the antilinear isomorphism $\mathcal{H}_\theta \rightarrow \overline{\mathcal{H}}_\theta$ is implemented by the modular conjugation J_θ . We will prove manageability with $Q = \widetilde{T}_\theta^{-2} \underline{T}_\theta^{-2}$ and $\widetilde{V} = V_\theta^*$. To this end, we need preparatory lemmas.

Lemma 4.7. *In the algebra of continuous operators on $\mathcal{S}(G \times G)$, V_θ commutes with $\underline{T}_\theta^\alpha \otimes \text{Id}$, $\alpha \in \mathbb{C}$.*

Proof. Let $f_1, f_2 \in \mathcal{S}(G)$. Since $\underline{T}_\theta^2 1 = 1$, we have

$$(\underline{T}_\theta^\alpha \otimes \text{Id})V_\theta(f_1 \otimes f_2) = (\underline{T}_\theta^\alpha \otimes \text{Id})(\Delta f_1 \star_\theta (1 \otimes f_2)) = ((\underline{T}_\theta^\alpha \otimes \text{Id})\Delta f_1) \star_\theta (1 \otimes f_2).$$

Since $\underline{T}_\theta^\alpha$ commutes with right translations, we have $(\underline{T}_\theta^\alpha \otimes \text{Id})\Delta f_1 = \Delta(\underline{T}_\theta^\alpha f_1)$ from which the result follows. \square

Lemma 4.8. *Given with the domain $\mathcal{S}(G)$, the operators $Q_\theta = \tilde{T}_\theta^{-2} \underline{T}_\theta^{-2}$ and $Q_\theta^{-1} = \tilde{T}_\theta^2 \underline{T}_\theta^2$ are essentially selfadjoint and positive on \mathcal{H}_θ .*

Proof. By Proposition 3.19, \tilde{T}_θ defines a unitary operator from \mathcal{H}_θ to $L_\rho^2(G)$. Hence, an operator A on \mathcal{H}_θ is essentially selfadjoint if and only if $\tilde{T}_\theta A \tilde{T}_\theta^{-1}$ is essentially selfadjoint on $L_\rho^2(G)$. But since \tilde{T}_θ commutes with \underline{T}_θ , we get $\tilde{T}_\theta Q_\theta \tilde{T}_\theta^{-1} = Q_\theta$. By Lemma 3.17, Q_θ is the product of two commuting, domain preserving, essentially selfadjoint operators on $L_\rho^2(G)$ with initial domain $\mathcal{S}(G)$. Thus Q_θ is essentially selfadjoint on $L_\rho^2(G)$ and so on \mathcal{H}_θ too. Positivity is clear. \square

Lemma 4.9. *In the algebra of continuous operators on $\mathcal{S}(G \times G)$, V_θ commutes with $Q_\theta^\alpha \otimes Q_\theta^\alpha$, $\alpha \in \mathbb{C}$.*

Proof. By Lemma 3.17, we have $Q_\theta^\alpha = L^{\star\theta}(\chi_G^{\alpha/2})R^{\star\theta}(\chi_G^{-\alpha/2})$ on $\mathcal{S}(G)$. Take $f_1, f_2 \in \mathcal{S}(G)$, we then have:

$$\begin{aligned} (Q_\theta^\alpha \otimes Q_\theta^\alpha)V_\theta(f_1 \otimes f_2) &= (\chi_G^{\alpha/2} \otimes \chi_G^{\alpha/2}) \star_\theta \Delta f_1 \star_\theta (1 \otimes f_2) \star_\theta (\chi_G^{-\alpha/2} \otimes \chi_G^{-\alpha/2}) \\ &= \Delta \chi_G^{\alpha/2} \star_\theta \Delta f_1 \star_\theta (\chi_G^{-\alpha/2} \otimes f_2 \star_\theta \chi_G^{-\alpha/2}) \\ &= \Delta \chi_G^{\alpha/2} \star_\theta \Delta f_1 \star_\theta \Delta \chi_G^{-\alpha/2} \star_\theta (1 \otimes \chi_G^{\alpha/2} \star_\theta f_2 \star_\theta \chi_G^{-\alpha/2}) \\ &= \Delta (\chi_G^{\alpha/2} \star_\theta f_1 \star_\theta \chi_G^{-\alpha/2}) \star_\theta (1 \otimes \chi_G^{\alpha/2} \star_\theta f_2 \star_\theta \chi_G^{-\alpha/2}) \\ &= \Delta (Q_\theta^\alpha f_1) \star_\theta (1 \otimes Q_\theta^\alpha f_2) = V_\theta(Q_\theta^\alpha \otimes Q_\theta^\alpha)(f_1 \otimes f_2), \end{aligned}$$

where we used Proposition 3.16 in the forth equality. \square

Theorem 4.10. *For all $f_1, f_2, f_3, f_4 \in \mathcal{S}(G)$, we have*

$$\langle f_1 \otimes f_2, V_\theta(f_3 \otimes f_4) \rangle_\theta = \langle J_\theta f_3 \otimes Q_\theta f_2, V_\theta^*(J_\theta f_1 \otimes Q_\theta^{-1} f_4) \rangle_\theta.$$

Hence, the multiplicative unitary V_θ is manageable in the sense of Woronowicz.

Proof. We have by Proposition 4.6:

$$\begin{aligned} \langle f_1 \otimes f_2, V_\theta(f_3 \otimes f_4) \rangle_\theta &= \langle f_2, \omega_{f_1, f_3} \otimes \text{Id}(V_\theta) f_4 \rangle_\theta = \langle \omega_{f_1, f_3} \otimes \text{Id}(V_\theta)^* f_2, f_4 \rangle_\theta \\ &= \langle \omega_{\tilde{T}_\theta^{-2} J_\theta f_1, \tilde{T}_\theta^2 J_\theta f_3} \otimes \text{Id}(V_\theta) f_2, f_4 \rangle_\theta = \langle V_\theta(\tilde{T}_\theta^2 J_\theta f_3 \otimes f_2), \tilde{T}_\theta^{-2} J_\theta f_1 \otimes f_4 \rangle_\theta \\ &= \langle \tilde{T}_\theta^2 J_\theta f_3 \otimes f_2, V_\theta^*(\tilde{T}_\theta^{-2} J_\theta f_1 \otimes f_4) \rangle_\theta = \langle J_\theta f_3 \otimes f_2, (\tilde{T}_\theta^2 \otimes \text{Id}) V_\theta^*(\tilde{T}_\theta^{-2} \otimes \text{Id})(J_\theta f_1 \otimes f_4) \rangle_\theta. \end{aligned}$$

Now, Lemma 4.9 gives $(Q_\theta^{-1} \otimes Q_\theta^{-1})V_\theta(Q_\theta \otimes Q_\theta) = V_\theta$ on $\mathcal{S}(G \times G)$. Passing to the adjoints we get $(Q_\theta \otimes Q_\theta)V_\theta^*(Q_\theta^{-1} \otimes Q_\theta^{-1}) = V_\theta^*$. Hence, we deduce the equality of operators on $\mathcal{S}(G \times G)$:

$$\begin{aligned} (\tilde{T}_\theta^2 \otimes \text{Id}) V_\theta^*(\tilde{T}_\theta^{-2} \otimes \text{Id}) &= (\tilde{T}_\theta^2 \otimes \text{Id})(Q_\theta \otimes Q_\theta) V_\theta^*(Q_\theta^{-1} \otimes Q_\theta^{-1})(\tilde{T}_\theta^{-2} \otimes \text{Id}) \\ &= (\underline{T}_\theta^{-2} \otimes Q_\theta) V_\theta^*(\underline{T}_\theta^2 \otimes Q_\theta^{-1}) = (\text{Id} \otimes Q_\theta) V_\theta^*(\text{Id} \otimes Q_\theta^{-1}), \end{aligned}$$

where the last equality follows from Lemma 4.7 (passed to the adjoint). This concludes the proof. \square

For a manageable multiplicative unitary, there is a well defined notion of antipode $A_0(V_\theta) \rightarrow A(V_\theta)$, unitary antipode $A(V_\theta) \rightarrow A(V_\theta)$ and an associated one-parameter group of automorphisms on $A(V_\theta)$ [33, 32]. We are going now to describe these objects at the level of the Fréchet algebra $(\mathcal{S}(G), \star_\theta)$.

Recall that the antipode is defined as the linear map

$$A_0(V_\theta) \rightarrow A(V_\theta), \quad \omega \otimes \text{Id}(V_\theta) \mapsto \omega \otimes \text{Id}(V_\theta^*).$$

Let $f_1, f_2 \in \mathcal{S}(G)$. Then, since $\omega_{f_1, f_2} \otimes \text{Id}(V_\theta^*) = \omega_{f_2, f_1} \otimes \text{Id}(V_\theta)^*$ we deduce from Proposition 4.6 that on such normal functionals, the antipode coincides with the map

$$\omega_{f_1, f_2} \otimes \text{Id}(V_\theta) \mapsto \omega_{\tilde{T}_\theta^2 J_\theta f_2, \tilde{T}_\theta^{-2} J_\theta f_1} \otimes \text{Id}(V_\theta).$$

Since the bilinear map

$$\mathcal{S}(G) \times \mathcal{S}(G) \rightarrow \mathcal{S}(G) \times \mathcal{S}(G), \quad (f_1, f_2) \mapsto (\tilde{T}_\theta^2 J_\theta f_2, \tilde{T}_\theta^{-2} J_\theta f_1),$$

squares to $\tilde{T}_\theta^4 \times \tilde{T}_\theta^{-4}$, we immediately see that in our situation, the antipode cannot square to the identity. More precisely, we have:

Proposition 4.11. *At the level of the Fréchet algebra $(\mathcal{S}(G), \star_\theta)$, the antipode S_θ coincides with $\tilde{T}_\theta^{-2} S \tilde{T}_\theta^{-2}$, the unitary antipode R_θ coincides with the undeformed antipode S and the analytic generator τ_i^θ of the associated automorphisms group is Q_θ^{-1} .*

Proof. We just give the formula for S_θ since the formulas for R_θ and τ_i^θ follow by polar decomposition. By Proposition 4.5, we have

$$\omega_{f_1, f_2} \otimes \text{Id}(V_\theta) = L^{\star_\theta}([g \mapsto \langle f_1, \rho_g f_2 \rangle_\theta]).$$

Hence, we need to show that

$$\tilde{T}_\theta^{-2} S \tilde{T}_\theta^{-2}[g \mapsto \langle f_1, \rho_g f_2 \rangle_\theta] = [g \mapsto \langle \tilde{T}_\theta^2 J_\theta f_2, \rho_g \tilde{T}_\theta^{-2} J_\theta f_1 \rangle_\theta].$$

Since \tilde{T}_θ^{-2} is a right convolution operator (see Lemma 4.13 in the next subsection for the explicit form), we have

$$\tilde{T}_\theta^{-2}[g \mapsto \langle f_1, \rho_g f_2 \rangle_\theta] = [g \mapsto \langle f_1, \rho_g \tilde{T}_\theta^{-2} f_2 \rangle_\theta],$$

and thus

$$S \tilde{T}_\theta^{-2}[g \mapsto \langle f_1, \rho_g f_2 \rangle_\theta] = [g \mapsto \langle f_1, \rho_{g^{-1}} \tilde{T}_\theta^{-2} f_2 \rangle_\theta].$$

In term of the unitary representation ρ_θ of G on \mathcal{H}_θ given in (4.1), we get

$$\begin{aligned} \langle f_1, \rho_{g^{-1}} \tilde{T}_\theta^{-2} f_2 \rangle_\theta &= \langle f_1, \tilde{T}_\theta \rho_\theta(g^{-1}) \tilde{T}_\theta^{-3} f_2 \rangle_\theta = \langle \tilde{T}_\theta f_1, \rho_\theta(g^{-1}) \tilde{T}_\theta^{-3} f_2 \rangle_\theta \\ &= \langle \rho_\theta(g) \tilde{T}_\theta f_1, \tilde{T}_\theta^{-3} f_2 \rangle_\theta = \langle \rho_g \tilde{T}_\theta^2 f_1, \tilde{T}_\theta^{-4} f_2 \rangle_\theta. \end{aligned}$$

By Lemma 3.17, we have $\overline{\tilde{T}_\theta^{-2} f} = \tilde{T}_\theta^2 \bar{f}$, thus

$$\tilde{T}_\theta^{-2} S \tilde{T}_\theta^{-2}[g \mapsto \langle f_1, \rho_g f_2 \rangle_\theta] = [g \mapsto \langle \rho_g \tilde{T}_\theta^4 f_1, \tilde{T}_\theta^{-4} f_2 \rangle_\theta].$$

To conclude, we use Lemma 3.23, to get

$$\langle \rho_g \tilde{T}_\theta^4 f_1, \tilde{T}_\theta^{-4} f_2 \rangle_\theta = \langle J_\theta \tilde{T}_\theta^{-4} f_2, J_\theta \rho_g \tilde{T}_\theta^4 f_1 \rangle_\theta = \langle \tilde{T}_\theta^4 J_\theta f_2, \tilde{T}_\theta^{-2} \rho_g \tilde{T}_\theta^{-4} \bar{f}_1 \rangle_\theta = \langle \tilde{T}_\theta^2 J_\theta f_2, \rho_g \tilde{T}_\theta^{-2} J_\theta f_1 \rangle_\theta,$$

which concludes the proof. \square

4.3 Equivalence with De Commer's approach

Near the end of section 2.4, we have mentioned that the adjoint of the twist F_θ^λ defines a unitary dual 2-cocycle for G (see [24] for more details). In the notations of [24] and of [8], $F_\theta^{\lambda*}$ is Ω_θ . Using De Commer's machinery [10], one can therefore define a locally compact quantum group G_θ , deforming an arbitrary negatively curved Kählerian Lie group G . The most difficult part in De Commer's work is to construct the invariant weights of G_θ . In [8, Proposition 2.8], we were able to determine the modular conjugation of De Commer's deformed weight and therefore we were able to obtain an explicit formula for \widehat{W}_θ , the multiplicative unitary of De Commer's locally compact quantum group G_θ [8, Equation 2.12]. Explicitly, \widehat{W}_θ is the unitary operator acting on $L_\lambda^2(G \times G)$ given by

$$\widehat{W}_\theta = (J \otimes \widehat{J}) F_\theta^{\lambda*} \widehat{W}^* (J \otimes \widehat{J}) F_\theta^\lambda, \quad (4.2)$$

where J and \widehat{J} are the modular conjugations of the group G and of its dual (quantum group) \widehat{G} given in (2.2) and $\widehat{W} = \Sigma W^* \Sigma \in W^*(G) \otimes L^\infty(G)$ is the multiplicative unitary of the dual quantum group of G :

$$\widehat{W}f(g_1, g_2) = f(g_2 g_1, g_2) .$$

Let $U_\theta := S \widetilde{T}_\theta$. Then U_θ defines a unitary operator from \mathcal{H}_θ onto $L^2_\lambda(G)$. We will spend the remaining of this section to prove the following unitary equivalence result:

Theorem 4.12. *With the notations displayed above, we have the equality of unitary operators on $L^2_\lambda(G \times G)$:*

$$(U_\theta \otimes U_\theta) V_\theta (U_\theta^* \otimes U_\theta^*) = \widehat{W}_\theta .$$

Before giving the proof of Theorem 4.12, we need two technical results. The first one follows easily from Equation (3.7) (see also the computation in [8, Lemma 2.1]).

Lemma 4.13. *For $\alpha \in \mathbb{R}$, define Φ_α to be the distribution on G given by the relation $\widetilde{T}_\theta^\alpha =: \rho(\Phi_\alpha)$. Then, under the parametrization $g = g_1 \cdots g_N$ of $G = (\mathbb{S}_N \times \dots) \times \mathbb{S}_1$ and in the coordinate system (2.5) for each \mathbb{S}_j , we have with \mathcal{F} the ordinary Fourier transform⁴ on \mathbb{R} and f_θ the function given in (3.6):*

$$\Phi_\alpha(g) = \prod_{j=1}^N \delta_0(a_j) \delta_0(v_j) \mathcal{F}(f_\theta^{\alpha \dim(\mathbb{S}_j)/4})(t_j) .$$

Consider the three-point kernel $K_\theta^3 : G^3 \rightarrow \mathbb{C}$, $(g, g', g'') \mapsto K_\theta(g^{-1}g', g^{-1}g'')$. It is obvious that K_θ^3 is invariant under diagonal left-translations and it follows from the definition of the two-point kernel K_θ that K_θ^3 is invariant under cyclic permutations. The following distributional identity is the key step to prove Theorem 4.12:

Lemma 4.14. *For $\alpha, \beta \in \mathbb{R}$ and $g', g'' \in G$, we have in the sense of distributions:*

$$\int_G \chi_G^\beta(g) \Phi_\alpha(g) K_\theta^3(g, g', g'') d^\lambda(g) = \chi_G^{-\alpha/2}(g' g''^{-1}) K_\theta(g', g'') .$$

Proof. From Lemma 4.13, we get

$$\int_G \chi_G^\beta(g) \Phi_\alpha(g) K_\theta^3(g, g', g'') d^\lambda(g) = \int_{\mathbb{R}^N} \left(\prod_{j=1}^N \mathcal{F}(f_\theta^{\alpha \dim(\mathbb{S}_j)/4})(t_j) \right) K_\theta^3((0, 0, t_1) \cdots (0, 0, t_N), g', g'') dt_1 \cdots dt_N .$$

Now, by Equation (2.8) (and the formulas that precedes it), we have

$$K_\theta^3((0, 0, t_1) \cdots (0, 0, t_N), g', g'') = K_\theta(g', g'') \prod_{j=1}^N e^{\frac{2i}{\theta} t_j \sinh(2a'_j - 2a''_j)} ,$$

and therefore

$$\int_G \chi_G^\beta(g) \Phi_\alpha(g) K_\theta^3(g, g', g'') d^\lambda(g) = K_\theta(g', g'') \prod_{j=1}^N \left(\int_{\mathbb{R}} \mathcal{F}(f_\theta^{\alpha \dim(\mathbb{S}_j)/4})(t_j) e^{\frac{2i}{\theta} t_j \sinh(2a'_j - 2a''_j)} dt_j \right) .$$

Then, since $f_\theta(x) = e^{\arcsinh(\pi\theta x)}$, we get

$$\int_{\mathbb{R}} \mathcal{F}(f_\theta^{\alpha \dim(\mathbb{S}_j)/4})(t_j) e^{\frac{2i}{\theta} t_j \sinh(2a'_j - 2a''_j)} dt_j = f_\theta^{\alpha \dim(\mathbb{S}_j)/4} \left(-(\pi\theta)^{-1} \sinh(2a'_j - 2a''_j) \right)$$

⁴We normalize the Fourier transform by $\mathcal{F}f(t) = \int e^{2i\pi\xi t} f(\xi) d\xi$.

$$= e^{-\frac{\alpha}{2} \dim(\mathbb{S}_j)(a'_j - a''_j)} = \chi_{\mathbb{S}_j}^{-\alpha/2}(g'_j) \chi_{\mathbb{S}_j}^{\alpha/2}(g''_j),$$

so finally

$$\int_G \chi_G^\beta(g_1) \Phi_\alpha(g) K_\theta^3(g_1, g_2, g_3) d^\lambda(g_1) = K_\theta(g', g'') \prod_{j=1}^N \chi_{\mathbb{S}_j}^{-\alpha/2}(g'_j) \chi_{\mathbb{S}_j}^{\alpha/2}(g''_j) = K_\theta(g', g'') \chi_G^{-\alpha/2}(g') \chi_G^{\alpha/2}(g''),$$

and we are done. \square

Proof of Theorem 4.12. To prove the unitary equivalence between V_θ and \widehat{W}_θ , we proceed by formal computations which, however, can be rigorously justified working with oscillatory integrals. So, take $f_1, f_2 \in \mathcal{S}(G)$. Then we have

$$\begin{aligned} \widehat{W}_\theta(f_1 \otimes f_2)(g_1, g_2) &= (J \otimes \widehat{J}) F_\theta \widehat{W}^* (J \otimes \widehat{J}) F_\theta^*(f_1 \otimes f_2)(g_1, g_2) \\ &= \chi_G^{1/2}(g_2) \overline{F_\theta \widehat{W}^* (J \otimes \widehat{J}) F_\theta^*(f_1 \otimes f_2)}(g_1, g_2^{-1}) \\ &= \chi_G^{1/2}(g_2) \int_{G^2} K_\theta(g_3, g_4) \overline{\widehat{W}^* (J \otimes \widehat{J}) F_\theta^*(f_1 \otimes f_2)}(g_3 g_1, g_4 g_2^{-1}) d^\lambda(g_3) d^\lambda(g_4) \\ &= \chi_G^{1/2}(g_2) \int_{G^2} K_\theta(g_3, g_4) \overline{(J \otimes \widehat{J}) F_\theta^*(f_1 \otimes f_2)}(g_2 g_4^{-1} g_3 g_1, g_4 g_2^{-1}) d^\lambda(g_3) d^\lambda(g_4) \\ &= \chi_G^{1/2}(g_2) \int_{G^2} K_\theta(g_3, g_4) \chi_G^{1/2}(g_4 g_2^{-1}) \overline{F_\theta^*(f_1 \otimes f_2)}(g_2 g_4^{-1} g_3 g_1, g_2 g_4^{-1}) d^\lambda(g_3) d^\lambda(g_4) \\ &= \int_{G^4} K_\theta(g_3, g_4) K_\theta(g_5, g_6) \chi_G^{1/2}(g_4) f_1(g_5^{-1} g_2 g_4^{-1} g_3 g_1) f_2(g_6^{-1} g_2 g_4^{-1}) d^\lambda(g_3) d^\lambda(g_4) d^\lambda(g_5) d^\lambda(g_6). \end{aligned} \quad (4.3)$$

In the other hand, we have by Lemmas 4.7 and 4.9

$$\begin{aligned} (U_\theta \otimes U_\theta) V_\theta (U_\theta^* \otimes U_\theta^*) &= (S \widetilde{T}_\theta \otimes S \widetilde{T}_\theta) V_\theta (\widetilde{T}_\theta^{-1} S \otimes \widetilde{T}_\theta^{-1} S) \\ &= (S \widetilde{T}_\theta Q_\theta^{1/2} \otimes S \widetilde{T}_\theta Q_\theta^{1/2}) V_\theta (Q_\theta^{-1/2} \widetilde{T}_\theta^{-1} S \otimes Q_\theta^{-1/2} \widetilde{T}_\theta^{-1} S) \\ &= (S \underline{T}_\theta^{-1} \otimes S \underline{T}_\theta^{-1}) V_\theta (\underline{T}_\theta S \otimes \underline{T}_\theta S) = (S \otimes S \underline{T}_\theta^{-1}) V_\theta (S \otimes \underline{T}_\theta S). \end{aligned}$$

Since $S \widetilde{T}_\theta = \underline{T}_\theta S$, we get finally

$$(U_\theta \otimes U_\theta) V_\theta (U_\theta^* \otimes U_\theta^*) = (\text{Id} \otimes \widetilde{T}_\theta^{-1})(S \otimes S) V_\theta (S \otimes S) (\text{Id} \otimes \widetilde{T}_\theta).$$

Next we observe that, by Equation (3.3), we have:

$$\begin{aligned} (S \otimes S) V_\theta (S \otimes S) (f_1 \otimes f_2)(g_1, g_2) &= V_\theta (S f_1 \otimes S f_2)(g_1^{-1}, g_2^{-1}) \\ &= (\lambda_{g_1} S f_1) \star_\theta S f_2(g_2^{-1}) = (\rho_{g_1} f_1) \star_{-\theta} f_2(g_2), \end{aligned}$$

We then compute:

$$\begin{aligned} (\rho_{g_1} f_1) \star_{-\theta} f_2(g_2) &= \int_{G^4} K_{-\theta}(g_3, g_4) K_\theta(g_5, g_6) f_1(g_5^{-1} g_2 g_3 g_1) f_2(g_6^{-1} g_2 g_4) d^\lambda(g_3) d^\lambda(g_4) d^\lambda(g_5) d^\lambda(g_6) \\ &= \int_{G^4} K_{-\theta}(g_3, g_4^{-1}) K_\theta(g_5, g_6) \chi_G(g_4) f_1(g_5^{-1} g_2 g_3 g_1) f_2(g_6^{-1} g_2 g_4^{-1}) d^\lambda(g_3) d^\lambda(g_4) d^\lambda(g_5) d^\lambda(g_6) \\ &= \int_{G^4} K_{-\theta}(g_4^{-1} g_3, g_4^{-1}) K_\theta(g_5, g_6) \chi_G(g_4) f_1(g_5^{-1} g_2 g_4^{-1} g_3 g_1) f_2(g_6^{-1} g_2 g_4^{-1}) d^\lambda(g_3) d^\lambda(g_4) d^\lambda(g_5) d^\lambda(g_6) \\ &= \int_{G^4} K_\theta(g_3, g_4) K_\theta(g_5, g_6) \chi_G(g_4) f_1(g_5^{-1} g_2 g_4^{-1} g_3 g_1) f_2(g_6^{-1} g_2 g_4^{-1}) d^\lambda(g_3) d^\lambda(g_4) d^\lambda(g_5) d^\lambda(g_6), \end{aligned} \quad (4.4)$$

where the last equality comes from the fact that $K_{-\theta}(g_4^{-1} g_3, g_4^{-1}) = K_{-\theta}(g_4, g_3) = K_\theta(g_3, g_4)$ (see for instance [8, Equations 2.2 & 2.3]). The striking point is that the only difference between Equations (4.3)

and (4.4) is that on the first one there is $\chi_G^{1/2}(g_4)$ while on the second one it is $\chi_G(g_4)$. This really looks like a computational mistake but fortunately it is not: This is the conjugation by $\text{Id} \otimes \widetilde{T}_\theta$ that will make the exponents matching! To see this, for $\alpha \in \mathbb{C}$ we set $\widetilde{T}_\theta^\alpha = \rho(\Phi_\alpha)$ where Φ_α is the distribution on G given in Lemma 4.13. Then we have

$$\begin{aligned} (U_\theta \otimes U_\theta)V_\theta (U_\theta^* \otimes U_\theta^*)(f_1 \otimes f_2)(g_1, g_2) &= \widetilde{T}_\theta^{-1}((\rho_{g_1} f_1) \star_{-\theta} \widetilde{T}_\theta f_2)(g_2) \\ &= \int_{G^6} K_\theta(g_3, g_4) K_\theta(g_5, g_6) \chi_G(g_4) \Phi_{-1}(g_7) \Phi_1(g_8) f_1(g_5^{-1} g_2 g_7 g_4^{-1} g_3 g_1) f_2(g_6^{-1} g_2 g_7 g_4^{-1} g_8) \\ &\quad \times d^\lambda(g_3) d^\lambda(g_4) d^\lambda(g_5) d^\lambda(g_6) d^\lambda(g_7) d^\lambda(g_8) \\ &= \int_{G^6} K_\theta(g_8^{-1} g_3, g_8^{-1} g_4 g_7) K_\theta(g_5, g_6) \chi_G(g_4) \chi_G(g_7) \chi_G(g_8) \Phi_{-1}(g_7) \Phi_1(g_8^{-1}) f_1(g_5^{-1} g_2 g_4^{-1} g_3 g_1) f_2(g_6^{-1} g_2 g_4^{-1} g_8) \\ &\quad \times d^\lambda(g_3) d^\lambda(g_4) d^\lambda(g_5) d^\lambda(g_6) d^\lambda(g_7) d^\lambda(g_8), \end{aligned}$$

where, to go from the second to the last line, we used the successive transformations $g_4 \mapsto g_8 g_4 g_7$, $g_3 \mapsto g_8 g_3$, $g_8 \mapsto g_8^{-1}$. Hence, we are left to evaluate (in the sense of distributions) the following integral:

$$\int_{G^2} K_\theta(g_8^{-1} g_3, g_8^{-1} g_4 g_7) \chi_G(g_7) \chi_G(g_8) \Phi_{-1}(g_7) \Phi_1(g_8^{-1}) d^\lambda(g_7) d^\lambda(g_8).$$

Note first that $f_\theta^{\alpha \dim(\mathbb{S}_j)/4}$ is real valued, so that $\mathcal{F}(f_\theta^{\alpha \dim(\mathbb{S}_j)/4})$ is even and by Lemma 4.13 one deduces that the distribution Φ_α is invariant under the group inversion. Hence, what we really need to evaluate is the integral:

$$\int_{G^2} K_\theta(g_8^{-1} g_3, g_8^{-1} g_4 g_7) \chi_G(g_7) \chi_G(g_8) \Phi_{-1}(g_7) \Phi_1(g_8) d^\lambda(g_7) d^\lambda(g_8).$$

Since moreover K_θ^3 is invariant by cyclic permutation and by left diagonal action of G , we get

$$K_\theta(g_8^{-1} g_3, g_8^{-1} g_4 g_7) = K_\theta^3(g_8, g_3, g_4 g_7) = K_\theta^3(g_7, g_4^{-1} g_8, g_4^{-1} g_3),$$

and thus Lemma 4.14 gives

$$\begin{aligned} &\int_{G^2} K_\theta(g_8^{-1} g_3, g_8^{-1} g_4 g_7) \chi_G(g_7) \chi_G(g_8) \Phi_{-1}(g_7) \Phi_1(g_8^{-1}) d^\lambda(g_7) d^\lambda(g_8) \\ &= \int_G \left(\int_G K_\theta^3(g_7, g_4^{-1} g_8, g_4^{-1} g_3) \chi_G(g_7) \Phi_{-1}(g_7) d^\lambda(g_7) \right) \chi_G(g_8) \Phi_1(g_8^{-1}) d^\lambda(g_8) \\ &= \chi_G^{-1/2}(g_3) \int_G K_\theta(g_4^{-1} g_8, g_4^{-1} g_3) \chi_G^{3/2}(g_8) \Phi_1(g_8^{-1}) d^\lambda(g_8). \end{aligned}$$

Similarly, we have $K_\theta(g_4^{-1} g_8, g_4^{-1} g_3) = K_\theta^3(g_8, g_3, g_4)$ and thus

$$\chi_G^{-1/2}(g_3) \int_G K_\theta(g_4^{-1} g_8, g_4^{-1} g_3) \chi_G^{3/2}(g_8) \Phi_1(g_8^{-1}) d^\lambda(g_8) = \chi_G^{-1/2}(g_4).$$

This implies that

$$\begin{aligned} &(U_\theta \otimes U_\theta)V_\theta (U_\theta^* \otimes U_\theta^*)(f_1 \otimes f_2)(g_1, g_2) \\ &= \int_{G^4} K_\theta(g_3, g_4) K_\theta(g_5, g_6) \chi_G^{1/2}(g_4) f_1(g_5^{-1} g_2 g_4^{-1} g_3 g_1) f_2(g_6^{-1} g_2 g_4^{-1}) d^\lambda(g_3) d^\lambda(g_4) d^\lambda(g_5) d^\lambda(g_6), \end{aligned}$$

which is exactly (4.3). This completes the proof of Theorem 4.12. \square

This unitary equivalence between V_θ and \widehat{W}_θ combined with the main result of [8] has an important consequence:

Corollary 4.15. *Let $\theta \in \mathbb{R}$. Then, the right leg $A(V_\theta)$ of the multiplicative unitary V_θ is isomorphic to $C_0(G)_{-\theta, \theta}$, the deformation in the sense of [6] of the C^* -algebra $C_0(G)$ for the action $\lambda \otimes \rho$ of $G \times G$ and parameters $(-\theta, \theta) \in \mathbb{R}^2$.*

Proof. By Theorem 4.12 (with the notations given at the beginning of subsection 4.1), we have $A(V_\theta) \simeq A(\widehat{W}_\theta)$. Now, \widehat{W}_θ is the multiplicative unitary of the locally compact quantum group \widehat{G}_θ deforming the quantum group \widehat{G} (the dual of the group G) by the unitary 2-cocycle $F_\theta^{\lambda^*}$ via De Commer's method [10]. By duality, $A(\widehat{W}_\theta)$ is isomorphic to $\widehat{A}(W_\theta)$, where W_θ is the multiplicative unitary of the locally compact quantum group G_θ (the dual of \widehat{G}_θ). It follows then by [24, Example 3.11 iii] that $\widehat{A}(W_\theta)$ (which is denoted by $C_0(G_\theta)$ there) is isomorphic to the deformation of $C_0(G)$ in the sense of [24] by the unitary dual 2-cocycle $F_{-\theta}^{\lambda^*} \otimes F_\theta^{\lambda^*}$ of $G \times G$ for the action $\lambda \otimes \rho$. But by [8, Theorem 3.4], the deformation in the sense of [6] is isomorphic to the deformation in the sense of [24], hence the result. \square

It now makes sense to talk about modular element and scaling constant. It is clear that the modular element is given by $L^{*\theta}(\chi_G)$. But since that operator commutes with Q_θ , it commutes with the automorphism group of the right Haar weight. Hence we get:

Proposition 4.16. *The scaling constant of the non-compact LCQG G_θ is trivial.*

A Proof of Theorem 2.5

In [6], two of us constructed an oscillatory integral for admissible tempered groups, a construction that we first review. A **left tempered group** is a pair (G, \mathbf{S}) where G is a connected and simply connected Lie group with Lie algebra \mathfrak{g} , and \mathbf{S} is a real-valued smooth function on G , satisfying the following temperedness conditions:

(i) The map

$$G \rightarrow \mathfrak{g}^* : x \mapsto [\mathfrak{g} \rightarrow \mathbb{R} : X \mapsto \widetilde{X}\mathbf{S}(x)] , \quad (\text{A.1})$$

is a global diffeomorphism.

(ii) In these (dual-Lie-algebra-)coordinates, the multiplication and inverse of G are *tempered* (or slowly increasing) functions.

To introduce the notion of admissibility for tempered groups, we need more notations. First, to a vector space decomposition

$$\mathfrak{g} = \bigoplus_{n=0}^N V_n , \quad (\text{A.2})$$

and, for every $n = 0, \dots, N$, an ordered basis $\{e_{i_n}^n\}_{i_n=1, \dots, \dim V_n}$ of V_n , we can associate coordinates on G :

$$x_n^{i_n} := \widetilde{e_{i_n}^n} \mathbf{S}(x) , \quad n = 0, \dots, N , \quad i_n = 1, \dots, \dim V_n . \quad (\text{A.3})$$

Identifying the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ with the symmetric algebra $\mathfrak{S}(\mathfrak{g})$ of \mathfrak{g} through the Poincaré-Birkhoff-Witt linear isomorphism, we may view $\mathfrak{S}(V_n)$ as a linear subspace of $\mathcal{U}(\mathfrak{g})$. Then, a left tempered pair (G, \mathbf{S}) is called **left admissible** if there exists a decomposition (A.2) with associated coordinate system (A.3) such that for every $n = 0, \dots, N$, there exists an element $X_n \in \mathfrak{S}(V_n) \subset \mathcal{U}(\mathfrak{g})$ whose associated multiplier $\alpha_n := e^{-i\mathbf{S}} \widetilde{X}_n e^{i\mathbf{S}}$, satisfies the following properties:

(i) There exists a positive constants C, ρ such that:

$$|\alpha_n|^{-1} \leq (1 + |x_n|)^{-\rho}, \quad x_n := (x_n^1, \dots, x_n^{\dim V_n}),$$

(ii) There exists a tempered function $\mu_n > 0$ such that:

(a) For every $A \in \mathfrak{S}(\bigoplus_{k=0}^n V_k) \subset \mathcal{U}(\mathfrak{g})$ there exists $C_A > 0$ such that, denoting \tilde{A}^* the formal adjoint on $L_\lambda^2(G)$ of the left invariant differential operator \tilde{A} , we have:

$$|\tilde{A}^* \alpha_n| < C_A |\alpha_n| \mu_n, \quad (\text{A.4})$$

(b) The function μ_n is independent of the variables $\{x_r^{j_r}\}_{j_r=1, \dots, \dim V_r}$ with $r \leq n$:

$$\frac{\partial \mu_n}{\partial x_r^{j_r}} = 0, \quad \forall r \leq n, \forall i_r = 1, \dots, \dim(V_r). \quad (\text{A.5})$$

Within the previous notations, we now define the operators on $C^\infty(G)$:

$$D_n^\lambda \Phi := \tilde{X}_n^* \left(\frac{\Phi}{\alpha_n} \right).$$

and, for every $(N+1)$ -tuple of integers $\vec{r} = (r_0, \dots, r_N) \in \mathbb{N}^{N+1}$, we set

$$\mathbf{D}_{\vec{r}}^\lambda := (D_0^\lambda)^{r_0} (D_1^\lambda)^{r_1} \dots (D_N^\lambda)^{r_N}. \quad (\text{A.6})$$

Of course, we have

$$\mathbf{D}_{\vec{r}}^{\lambda*} e^{i\mathbf{S}} = e^{i\mathbf{S}}, \quad \forall \vec{r} \in \mathbb{N}^{N+1}.$$

But what is really remarkable (see [6, Proposition 2.28]), is that for any $\vec{R} \in \mathbb{N}^{N+1}$, there exist $\vec{r} \in \mathbb{N}^{N+1}$, $k \in \mathbb{N}$ and $C > 0$ such that for every $\Phi \in C^\infty(G)$, we have the estimate

$$|\mathbf{D}_{\vec{r}}^\lambda \Phi| \leq \frac{C}{(1 + |x_0|)^{R_0} \dots (1 + |x_N|)^{R_N}} \sup_{X \in S_k(\mathfrak{g})} |\tilde{X} \cdot \Phi|, \quad (\text{A.7})$$

where the unit sphere $S_k(\mathfrak{g})$ is defined just after Equation (2.10). Consider now \mathcal{E} a Fréchet space, realised as a countable projective limit of Banach spaces $\prod_{j \in J} \mathcal{E}_j$, and take any family $\underline{\mu} = \{\mu_j\}_{j \in J}$ of tempered weights. It then follows from (A.7) that for any $j \in J$ there exists $\vec{r}_j \in \mathbb{N}^{N+1}$ such that $\mathbf{D}_{\vec{r}_j}^\lambda$ sends $\mathcal{B}_{\lambda}^\mu(G, \mathcal{E})$ to $L_\lambda^1(G, \mathcal{E}_j)$ continuously. Hence, we have a well defined oscillatory integral, given by the continuous linear mapping

$$\mathcal{B}_{\lambda}^\mu(G, \mathcal{E}) \rightarrow \mathcal{E}, \quad f \mapsto \int_G \widetilde{e^{i\mathbf{S}(g)} F(g)} d^\lambda(g) := \left\{ \int_G e^{i\mathbf{S}(g)} (\mathbf{D}_{\vec{r}_j}^\lambda F)(g) d^\lambda(g) \right\}_{j \in J}.$$

Using right invariant vector fields instead left invariant ones everywhere, one defines the notions of **right temperedness** and **right admissibility**. In fact, it is not difficult to see that the notions of left temperedness and right temperedness are equivalent. Indeed, fix $\{X_j\}_{j=1, \dots, \dim(G)}$ a basis of \mathfrak{g} and consider the two coordinates systems:

$$\tilde{x}_i := \tilde{X}_i S(x) \quad \text{and} \quad \underline{x}_i := X_i S(x).$$

Let then $A(\tilde{x})$ be the matrix of $\text{Ad}_{x^{-1}}(\tilde{x})$ expressed in the \tilde{x} -coordinates. Suppose that (G, \mathbf{S}) is left-tempered. Since the multiplication and inversion are tempered maps (in the \tilde{x} -coordinates), it follows that the matrix entries $A_j^i(\tilde{x})$ are tempered functions too. Since moreover we have

$$\underline{x}_j = \sum_{i=1}^{\dim(G)} A_j^i(\tilde{x}) \tilde{x}_i,$$

it follows that (G, \mathbf{S}) is right-tempered as well. Similarly, right-temperedness implies left-tempered. However, the notions of left and right admissibility may differ. Nevertheless, for a right admissible right tempered pair one can repeat the the arguments of [6, Proposition 2.29] with the operators

$$\mathbf{D}_{\vec{r}}^\rho := (D_0^\rho)^{r_0} (D_1^\rho)^{r_1} \cdots (D_N^\rho)^{r_N} \quad \text{where} \quad D_n^\rho \Phi := \underline{X}_n^* \left(\frac{\Phi}{\alpha_n} \right),$$

to get the right-handed version of [6, Proposition 2.29]:

Proposition A.1. *Let (G, \mathbf{S}) be a right admissible tempered pair, \mathcal{E} be a complex Fréchet space with seminorms $\{\|\cdot\|_j\}_{j \in J}$ and let $\underline{\mu} = \{\mu_j\}_{j \in J}$ be an associated family of tempered weights. Then for all $j \in J$, there exist $\vec{r}_j \in \mathbb{N}^{N+1}$, $C_j > 0$ and $k_j \in \mathbb{N}$, such that for every element $F \in \mathcal{B}_\rho^\mu(G, \mathcal{E})$, we have*

$$\int_G \|\mathbf{D}_{\vec{r}_j}^\rho F(g)\|_j d^\lambda(g) \leq C_j \|F\|_{j, k_j, \underline{\mu}}^\rho.$$

From this, it follows oscillatory integral mapping makes perfect good sense on $\mathcal{B}_\rho^\mu(G, \mathcal{E})$ too:

$$\mathcal{B}_\rho^\mu(G, \mathcal{E}) \rightarrow \mathcal{E}, \quad f \mapsto \left\{ \int_G e^{i\mathbf{S}(g)} (\mathbf{D}_{\vec{r}_j}^\rho F)(g) d^\lambda(g) \right\}_{j \in J}.$$

Let now G be a negatively curved Kählerian Lie group and $S_G \in C^\infty(G \times G, \mathbb{R})$ be the phase function of the kernel (2.8). It is proven in [6, Chapter 3] that $(G \times G, S_G)$ it left tempered (hence right too) and left admissible. We will now prove that $(G \times G, S_G)$ right admissibility too. Therefore, the proof of Theorem 2.5 will be an immediate consequence of the Proposition A.1.

Proposition A.2. *Let G be a negatively curved Kählerian Lie group. Then, the tempered pair $(G \times G, S_G)$ is right admissible.*

Proof. Using the same induction argument (over the number of elementary factors of G in its Pyatetskii-Shapiro decomposition (2.4) and based on the fact that the extension homomorphisms are tempered and take values in the linear symplectic group), it suffices to treat the case where G is elementary, that is $G = \mathbb{S}$ within our notations.

We fix a symplectic basis $\{e_j, f_j\}$ of the symplectic vector space V , i.e. it satisfies $\omega(e_i, f_j) = \delta_{ij}$ and equals zero everywhere else. According to the associated Lagrangian decomposition we let $v = (n, m) \in V$. An easy computation shows that we have the following expressions for the right-invariant vector fields on the group \mathbb{S} (which are all skew-adjoint with respect to the left Haar measure):

$$\underline{H} = -\partial_a; \quad \underline{e}_j = -e^{-a} \partial_{e_j} - \frac{1}{2} e^{-a} m_j \partial_t; \quad \underline{f}_j = -e^{-a} \partial_{f_j} + \frac{1}{2} e^{-a} n_j \partial_t; \quad \underline{E} = -e^{-2a} \partial_t. \quad (\text{A.8})$$

Now, we consider the following associated basis of $\mathfrak{s} \oplus \mathfrak{s}$:

$$\begin{aligned} H_1 &:= H \oplus \{0\}, & H_2 &:= \{0\} \oplus H, & f_j^1 &:= f_j \oplus \{0\}, & f_j^2 &:= \{0\} \oplus f_j, \\ e_j^1 &:= e_j \oplus \{0\}, & e_j^2 &:= \{0\} \oplus e_j, & E_1 &:= E \oplus \{0\}, & E_2 &:= \{0\} \oplus E, \end{aligned}$$

and we define for $\ell = 1, 2$:

$$V_{\ell 0} := \mathbb{R}H_\ell; \quad V_{\ell 1} := \text{span}\{f_j^\ell\}; \quad V_{\ell 2} := \text{span}\{e_j^\ell\}; \quad V_{\ell 3} := \mathbb{R}E_\ell,$$

and set

$$\mathfrak{V}_k := V_{1k} \oplus V_{2k}, \quad k = 0, 1, 2, 3.$$

Accordingly, we consider the coordinates:

$$x_{\ell 0} := \underline{H}_\ell S_G, \quad x_{\ell 1}^j := \underline{f}_j^\ell S_G, \quad x_{\ell 2}^j := \underline{e}_j^\ell S_G, \quad x_{\ell 3} := \underline{E}_\ell S_G, \quad \ell = 1, 2, \quad (\text{A.9})$$

that we combine as vectors:

$$\begin{aligned} \vec{x}_0 &:= (x_{1,0}, x_{2,0}) \in \mathbb{R}^2, & \vec{x}_1 &:= ((x_{1,1}^j)_{j=1}^d, (x_{2,1}^j)_{j=1}^d) \in \mathbb{R}^{2d}, \\ \vec{x}_2 &:= ((x_{1,2}^j)_{j=1}^d, (x_{2,2}^j)_{j=1}^d) \in \mathbb{R}^{2d}, & \vec{x}_3 &:= (x_{1,3}, x_{2,3}) \in \mathbb{R}^2. \end{aligned}$$

Set

$$A = \begin{pmatrix} -\cosh(2a_2) & 0 \\ 0 & \cosh(2a_1) \end{pmatrix}, \quad B = \begin{pmatrix} e^{-a_1} \cosh(a_2) & 0 \\ 0 & e^{-a_2} \cosh(a_1) \end{pmatrix} \begin{pmatrix} \sinh(a_2) & -\cosh(a_1) \\ -\cosh(a_2) & \sinh(a_1) \end{pmatrix},$$

and

$$\vec{\gamma} = (\cosh(a_1) \sinh(a_2), \cosh(a_2) \sinh(a_1)), \quad \vec{\delta} = (e^{-2a_1} \sinh(2a_2), -e^{-2a_2} \sinh(2a_1)).$$

Straightforward computations then lead to the following expressions for the coordinates (A.9):

$$\vec{x}_3 = \vec{\delta}, \quad \vec{x}_2 = B \cdot \vec{n}, \quad \vec{x}_1 = -B \cdot \vec{m}, \quad \vec{x}_0 = -2A \cdot \vec{t} - \omega(v_1, v_2) \vec{\gamma}, \quad (\text{A.10})$$

where $\vec{t} = (t_1, t_2)$, $\vec{n} = (n_1, n_2)$, $\vec{m} = (m_1, m_2)$ (the last two notations underly the Lagrangian decomposition $v = (n, m) \in V$). Cyclicity of the derivatives of the hyperbolic functions yields the following observations:

1. There exist finitely many matrices $B_r \in M_2(\mathbb{R}[e^{a_1}, a^{a_2}])$ such that for all integers N_1 and N_2 , the element $\underline{H}_1^{N_1} \underline{H}_2^{N_2} B$ consists in a linear combination of the B_r 's.
2. There exist finitely many matrices $A_r \in M_2(\mathbb{R}[e^{a_1}, a^{a_2}])$ such that for all integers N_1 and N_2 , the element $\underline{H}_1^{N_1} \underline{H}_2^{N_2} A$ consists in a linear combination of the A_r 's.
3. There exist finitely many vectors $\gamma_r \in \mathbb{R}^2[e^{a_1}, a^{a_2}]$ such that for all integers N_1 and N_2 , the element $\underline{H}_1^{N_1} \underline{H}_2^{N_2} \vec{\gamma}$ consists in a linear combination of the γ_r 's.

Also, the expressions (A.8) for the invariant vector fields imply that for every $X \in \mathfrak{Y}_1 \oplus \mathfrak{Y}_2$:

$$\underline{X} \vec{x}_1, \underline{X} \vec{x}_2 \in \mathbb{R}[e^{a_1}, a^{a_2}],$$

which yields in particular that for all $X \in \mathfrak{S}^{\geq 2}(\mathfrak{Y}_1 \oplus \mathfrak{Y}_2)$, we have $\underline{X} \vec{x}_1 = \underline{X} \vec{x}_2 = 0$.

Note that from the expressions of $x_{\ell 3}$, $\ell = 1, 2$, one easily deduces that $e^{\pm a_\ell}$ is a tempered function of the $x_{\ell 3}$'s. Therefore the above discussion implies that there exist finitely many tempered functions $\mathbf{m}_{2,r}$ depending on the variables $x_{\ell 3}$ only, $\ell = 1, 2$, such that, for every $X \in \mathfrak{S}(\mathfrak{Y}_0 \oplus \mathfrak{Y}_1 \oplus \mathfrak{Y}_2)$, the elements $\underline{X} \vec{x}_k$ ($k = 1, 2$) belong to the space spanned by the $\mathbf{m}_{2,r}$'s.

Observing that

$$\vec{t} = -\frac{1}{2} A^{-1} (\vec{x}_0 + \omega(v_1, v_2) \vec{\gamma}),$$

the above observation 3 then yields:

4. There exist finitely many matrices $M_r \in M_2(\mathbb{R}[e^{a_1}, a^{a_2}])$ and finitely many vectors $v_s \in \mathbb{R}^2[e^{a_1}, a^{a_2}]$ such that for all integers N_1 and N_2 , one has

$$\underline{H}_1^{N_1} \underline{H}_2^{N_2} \vec{x}_0 = M_{N_1, N_2} \vec{x}_0 + \omega(v_1, v_2) v_{N_1, N_2},$$

with

$$M_{N_1, N_2} \in \text{span}\{M_r\} \quad \text{and} \quad v_{N_1, N_2} \in \text{span}\{v_s\}.$$

We may therefore summarize the above discussion by

5. For every $k = 0, \dots, 3$, there exists a tempered function \mathbf{m}_k , with $\partial_{x_{\ell i}} \mathbf{m}_k = 0$ for every $i \leq k$, $\ell = 1, 2$ and such that for every $X \in \mathfrak{S}(\bigoplus_{i=0}^k \mathfrak{V}_i)$, there exists $C_X > 0$ with

$$|\underline{X} \vec{x}_k| \leq C_X |\mathbf{m}^k| (1 + |\vec{x}_k|).$$

Defining

$$X_0 := 1 - H_1^2 - H_2^2, \quad X_1 := 1 - \sum_j ((f_j^1)^2 + (f_j^2)^2), \quad X_2 := 1 - \sum_j ((e_j^1)^2 + (e_j^2)^2), \quad X_3 := 1 - E_1^2 - E_2^2,$$

the corresponding multipliers $\alpha_k := e^{-iS_G} \underline{X}_k e^{iS_G}$ yields right admissibility for the tempered pair $(G \times G, S_G)$. Indeed, we start by observing the following expression of the multiplier:

$$\alpha_k = 1 + |\vec{x}_k|^2 - i\beta \quad \text{where} \quad \beta := \underline{X}_{1k} x_{1k} + \underline{X}_{2k} x_{2k}.$$

Then:

$$\frac{1}{|\alpha_k|} = \frac{|1 + |\vec{x}_k|^2 + i\beta|}{(1 + |\vec{x}_k|^2)^2 + \beta^2} \leq \frac{1 + |\vec{x}_k|^2 + |\beta|}{(1 + |\vec{x}_k|^2)^2} \leq \frac{1 + |\vec{x}_k|^2 + C|\mathbf{m}_k|(1 + |\vec{x}_k|)}{(1 + |\vec{x}_k|^2)^2} \leq C' \frac{1 + C|\mathbf{m}_k|}{1 + |\vec{x}_k|^2}.$$

Let now $X \in \mathfrak{S}(\bigoplus_{i=0}^k \mathfrak{V}_i)$, then observation 5 above combined with the Leibniz rule yields:

$$|\underline{X} \alpha_k| \leq C_1 |\vec{x}_k| |\mathbf{m}_k| (1 + |\vec{x}_k|) + C_2 |\mathbf{m}_k| (1 + |\vec{x}_k|) \leq C_3 |\alpha_k| |\mathbf{m}_k|.$$

This completes the proof of Proposition A.1 and thus the proof of Theorem 2.5 too. \square

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