A note on functional observability

Frédéric Rotella, Irène Zambettakis

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***Abstract*—In this note, we propose an alternative to characterize the functional observability for linear systems. The main feature is that we obtain a necessary and sufficient condition for the existence of a stable multi-functional observer of a time-invariant linear system. The proof of this condition is constructive and it leads to design a stable observer via a new procedure, neither based on the solution of a Sylvester equation nor on the use of canonical state space forms.**

***Index Terms*—Functional observer, linear systems, observer.**

I. Introduction

Since Luenberger's works [22]–[24] a significant amount of research has been devoted to the problem of observing a linear functional of the state of a linear time-invariant system. The main developments are detailed in [25] and, in the recent books [18], [32] and the reference therein. The problem can be formulated as follows. For the linear state-space model

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t)\end{aligned} \tag{1}$$

where, for every time t in $\mathbb{R}^+$, $x(t)$ is the n-dimensional state vector, $u(t)$ is the p-dimensional input, $y(t)$ is the m-dimensional measured output, and, A, B, and C are constant matrices of adapted dimensions, the objective is to get

$$v(t) = Lx(t) \tag{2}$$

where L is a constant $(l \times n)$ matrix. The observation of $v(t)$ can be carried out with the design of a Luenberger observer

$$\begin{aligned}\dot{z}(t) &= Fz(t) + Gu(t) + Hy(t)\\ w(t) &= Pz(t) + Vy(t)\end{aligned} \tag{3}$$

where $z(t)$ is the q-dimensional state vector. Constant matrices F, G, H, P, and V are determined such that

$$\lim_{t\to\infty} (v(t) - w(t)) = 0.$$

We know from [10] and [11] that the observable linear functional observer (3) exists if and only if there exists a $(q \times n)$ matrix T such that $G = TB$ and

$$TA - FT = HC \tag{4}$$

$$L = PT + VC \tag{5}$$

where F is a Hurwitz matrix. Namely, when all the real parts of the eigenvalues of F are strictly negative. When these conditions are fulfilled, we have $\lim_{t\to\infty}(z(t) - Tx(t)) = 0$. Moreover, it is well known, from [27] and [30], that the order q of the multi-functional observer is such that $q \geq \text{rank}(L)$ and, when the model (1) is detectable, $q < n - m$. Indeed, $n - m$ is the order of the reduced-order observer or Cumming-Gopinath observer [3], [14] which can be built to observe $x(t)$ and, consequently, $v(t)$. Among the observers, we can distinguish the minimum-order or Darouach observer [4] where $q = l$ and $P = I_q$. It has been shown in [29] that the minimum-order observer exists if and only if there exists a triplet (F, M, N) such that

$$LA = FL + MC + NCA$$

where F is a Hurwitz matrix.

In all the following we use the shorthand notation:

$$\mathcal{O}_{(M,N,k)} = \begin{bmatrix} N \\ NM \\ \vdots \\ NM^{k-1} \end{bmatrix}$$

where N and M are matrices with adapted dimensions and k is an integer.

Recently, to cope with the design problem of a minimal order functional observer, the interesting notion of functional observability of the triplet (A, C, L), which sums up the problem to solve, has been defined in [6]–[8].

Definition 1: The triplet (A, C, L) is functionally observable if there exists a matrix R such that a Darouach observer exists for the linear functional

$$\overline{v}(t) = \begin{bmatrix} R \\ L \end{bmatrix} x(t).$$

Some iterative procedures have been proposed in [6]–[8] to cope with the "*intriguing and challenging problem*" ([32]) to find R which leads to the minimum-order observer. A recent result based on matrix decompositions and canonical forms to design a minimal order observer with fixed eigenvalues at the outside is described in [9]. Nevertheless, it has been proposed in ([7]) that the triplet (A, C, L) is functionally observable if and only if

$$\text{rank}\left(\begin{bmatrix} \mathcal{O}_{(A,C,n)} \\ \mathcal{O}_{(A,L,n)} \end{bmatrix}\right) = \text{rank}\left(\mathcal{O}_{(A,C,n)}\right). \tag{6}$$

Obviously, when condition (6) is fulfilled, there exist matrices $L_0, L_1, \ldots, L_{n-1}$ such that

$$L = \sum_{i=0}^{n} L_i C A^i. \tag{7}$$

Thus

$$\text{rank}\left(\begin{bmatrix} \mathcal{O}_{(A,C,n)} \\ L \end{bmatrix}\right) = \text{rank}\left(\mathcal{O}_{(A,C,n)}\right). \tag{8}$$

Conversely, let us suppose that L can be written as (7). Thus, it is easy to prove by induction that, for every k in $\{1, \ldots, n-1\}$, there exist

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matrices $L_{k,i}$ such that

$$LA^k = \sum_{i=0}^{n} L_{k,i} C A^i.$$

These relationships lead to (6). So, we can claim the triplet (A, C, L) is functionally observable if and only if (8) is fulfilled.

From (7), we can relate the functional observability notion to other observability notions. Indeed, from (1), we get for $i = 0, 1, 2, \ldots$

$$y^{(i)}(t) = CA^i x(t) + \sum_{j=0}^{i-1} CA^j B u^{(i-1-j)}(t).$$

Thus, from (7), we can write

$$v(t) = \sum_{i=0}^{n-1} L_i y^{(i)}(t) - \sum_{i=0}^{n-1} L_i \sum_{j=0}^{i-1} CA^j B u^{(i-1-j)}(t).$$

Consequently, $v(t)$ is observable in the Fliess-Diop meaning [5].

Nevertheless, our aim is to propose another criterion to test functional observability of a triplet (A, C, L) which leads to a constructive procedure of functional observer. Consequently, the technical note is organized as follows. In a first part, we show that the existence of an integer ν, matrices $F_{L,0}, \ldots, F_{L,\nu-1}$ and matrices $F_{C,0}, \ldots, F_{C,\nu}$ such that

$$LA^\nu = \sum_{i=0}^{\nu-1} F_{L,i} LA^i + \sum_{i=0}^{\nu} F_{C,i} CA^i \tag{9}$$

leads, through realization theory, to the design of a candidate functional observer. The proof of the sufficiency of the condition (9) is completed with the exhibition of the analytical expression of the matrix T solution of the (4) and (5). Let us insist here that the determination of T is not a necessary step in the design of the observer. In a second part we show that this condition is necessary as well. A third part is devoted to a stability condition for the obtained observer structure. This condition states that a linear functional observer problem is equivalent to a static output stabilization problem. An example is proposed in a final section.

II. Sufficiency

Let us suppose here that (9) is fulfilled. As we have, for $k = 0, 1, \ldots$

$$v^{(k)}(t) = LA^k x(t) + \sum_{i=0}^{k-1} LA^{k-1-i} B u^{(i)}(t)$$

we can write

$$v^{(\nu)}(t) = \sum_{i=0}^{\nu-1} F_{L,i} LA^i x(t) + \sum_{i=0}^{\nu} F_{C,i} CA^i x(t) + \sum_{i=0}^{\nu-1} LA^{\nu-1-i} B u^{(i)}(t). \tag{10}$$

A. Observer Structure Design

Firstly, the elimination of $x(t)$ in (10) is carried out by means of, for $i = 1$ to $\nu - 1$

$$LA^i x(t) = v^{(i)}(t) - \sum_{j=0}^{i-1} LA^{i-1-j} B u^{(j)}(t) \tag{11}$$

and, for $i = 1$ to ν

$$CA^i x(t) = y^{(i)}(t) - \sum_{j=0}^{i-1} CA^{i-1-j} B u^{(j)}(t). \tag{12}$$

We get then

$$v^{(\nu)}(t) = \sum_{i=0}^{\nu-1} F_{L,i} v^{(i)}(t) + \sum_{i=0}^{\nu} F_{C,i} y^{(i)}(t) + \sum_{i=0}^{\nu-1} G_i u^{(i)}(t) \tag{13}$$

where the matrices G_i are given by $G_{\nu-1} = (L - F_{C,\nu} C)B$ and, for $\nu \geq 2$ and $j = 0$ to $\nu - 2$

$$G_j = \left(LA^{\nu-1-j} - \sum_{i=j+1}^{\nu-1} F_{L,i} LA^{i-1-j} - \sum_{i=j+1}^{\nu} F_{C,i} CA^{i-1-j} \right) B. \tag{14}$$

Remark 2: When $\nu = 0$, there exists a matrix Λ such that $L = \Lambda C$. So the functional observer becomes $w(t) = \Lambda y(t)$. The case $\nu = 1$ has been detailed in [28] and leads to $G = (L - F_{C,1} C)B$.

Secondly, the differential (13) is realized through the well-known Ruffini-Horner procedure [17]. Namely, we write (13) as

$$\begin{aligned} v(t) = {} & F_{C,\nu} y(t) \\ & + p^{-1} \left[F_{L,\nu-1} v(t) + F_{C,\nu-1} y(t) + G_{\nu-1} u(t) \right. \\ & + \vdots \\ & \quad p^{-1} \left[F_{L,1} v(t) + F_{C,1} y(t) + G_1 u(t) \right. \\ & \quad + p^{-1} \left[F_{L,0} v(t) + F_{C,0} y(t) + G_0 u(t) \right] \cdots \left. \right] \end{aligned}$$

where p stands for the continuous-time derivative operator and p^{-1} for the continuous-time integrator. With $z_0(t) = p^{-1}[F_{L,0} v(t) + F_{C,0} y(t) + G_0 u(t)]$

$$z_i(t) = p^{-1} \left[F_{L,i} v(t) + F_{C,i} y(t) + G_i u(t) + z_{i-1}(t) \right]$$

for $i = 1$ to $\nu - 1$, and, $v(t) = z_{\nu-1}(t) + F_{C,\nu} y(t)$, we obtain

$$\begin{aligned} \dot{z}_0(t) &= F_{L,0} z_{\nu-1}(t) + H_{C,0} y(t) + G_0 u(t) \\ \dot{z}_1(t) &= F_{L,1} z_{\nu-1}(t) + H_{C,1} y(t) + G_1 u(t) + z_0(t) \\ &\vdots \\ \dot{z}_{\nu-1}(t) &= F_{L,\nu-1} z_{\nu-1}(t) + H_{C,\nu-1} y(t) \\ &\quad + G_{\nu-1} u(t) + z_{\nu-2}(t) \end{aligned}$$

where, for $i = 0$ to $\nu - 1$, $H_{C,i} = F_{C,i} + F_{L,i} F_{C,\nu}$. The vector

$$z(t) = \begin{bmatrix} z_0^\top(t) & \cdots & z_{\nu-1}^\top(t) \end{bmatrix}^\top$$

is the state of the Luenberger observer structure (3) with

$$F = \begin{bmatrix} & & & F_{L,0} \\ I_l & & & F_{L,1} \\ & \ddots & & \vdots \\ & & I_l & F_{L,\nu-2} \\ & & & I_l \quad F_{L,\nu-1} \end{bmatrix}, \; G = \begin{bmatrix} G_0 \\ G_1 \\ \vdots \\ G_{\nu-2} \\ G_{\nu-1} \end{bmatrix}$$

$$H = \begin{bmatrix} F_{C,0} \\ F_{C,1} \\ \vdots \\ F_{C,\nu-2} \\ F_{C,\nu-1} \end{bmatrix} + \begin{bmatrix} F_{L,0} \\ F_{L,1} \\ \vdots \\ F_{L,\nu-2} \\ F_{L,\nu-1} \end{bmatrix} F_{C,\nu}$$

$$P = [0 \quad \cdots \quad 0 \quad I_l], \; V = F_{C,\nu}. \tag{15}$$

Remark 3: Notice that the realization (15) is observable.

Remark 4: In the case $\nu = 1$, the Darouach-Luenberger observer structure is given by [29]

$$F = F_{L,0},\ G = (L - F_{C,1}C)B,\ V = F_{C,1}$$
$$P = I_l,\ H = F_{C,0} + F_{L,0}F_{C,1}. \tag{16}$$

B. An Expression for T

In order to complete the proof we obtain here the expression of the matrix T. Let us begin with the case $\nu = 1$. We claim $T = L - F_{C,1}C$. Indeed, from (16), we have

$$\begin{aligned} TA - FT &= LA - F_{C,1}CA - F_{L,0}L + F_{L,0}F_{C,1}C \\ &= (F_{C,0} + F_{L,0}F_{C,1})C = HC \end{aligned}$$

and, $L = T + F_{C,1}C = PT + VC$.

Now, consider the case $\nu \geq 2$. Firstly, let us remark that the relationship $G = TB$, with (14), leads to the induction

$$T = \begin{bmatrix} T_1^\top & \cdots & T_\nu^\top \end{bmatrix}^\top$$

where, for $j = 1$ to $\nu - 1$

$$T_j = LA^{\nu-j} - \sum_{i=j}^{\nu-1} F_{L,i}LA^{i-j} - \sum_{i=j}^{\nu} F_{C,i}CA^{i-j}$$

and, $T_\nu = L - F_{C,\nu}C$. In the following, we state that this matrix T is a solution of $TA - FT = HC$ where F and H are defined in (15). Let us denote

$$TA = \begin{bmatrix} (TA)_1 \\ \vdots \\ (TA)_\nu \end{bmatrix} \text{ and } FT = \begin{bmatrix} (FT)_1 \\ \vdots \\ (FT)_\nu \end{bmatrix}$$

where, for $j = 1$ to ν, the blocks $(TA)_j$ and $(FT)_j$ have l rows. On the one hand, we have $(TA)_\nu = LA - F_{C,\nu}CA$, and, for $j = 1$ to $\nu - 1$

$$(TA)_j = LA^{\nu-j+1} - \sum_{i=j}^{\nu-1} F_{L,i}LA^{i-j+1} - \sum_{i=j}^{\nu} F_{C,i}CA^{i-j+1}.$$

On the other hand, we have $(FT)_1 = F_{L,0}L - F_{L,0}F_{C,\nu}C$, and, for $j = 2$ to ν

$$\begin{aligned} (FT)_j =& T_{j-1} + F_{L,j-1}T_\nu \\ =& LA^{\nu+1-j} - \sum_{i=j-1}^{\nu-1} F_{L,i}LA^{i+1-j} \\ & - \sum_{i=j-1}^{\nu} F_{C,i}CA^{i+1-j} + F_{L,j-1}L - F_{L,j-1}F_{C,\nu}C. \end{aligned} \tag{17}$$

For $j = 2$ to $\nu - 1$, (17) can be written as

$$\begin{aligned} (FT)_j = LA^{\nu+1-j} - \sum_{i=j}^{\nu-1} F_{L,i}LA^{i+1-j} - \sum_{i=j}^{\nu} F_{C,i}CA^{i+1-j} \\ - (F_{C,j-1} + F_{L,j-1}F_{C,\nu})C \end{aligned}$$

and, for $j = \nu$

$$(FT)_\nu = LA - F_{C,\nu}CA - (F_{C,\nu-1} + F_{L,\nu-1}F_{C,\nu})C.$$

Let us remark that (9) leads to write

$$\begin{aligned} & LA^\nu - \sum_{i=1}^{\nu-1} F_{L,i}LA^i - \sum_{i=1}^{\nu} F_{C,i}CA^i \\ &= \sum_{i=0}^{\nu-1} F_{L,i}LA^i + \sum_{i=0}^{\nu} F_{C,i}CA^i \\ &\quad - \sum_{i=1}^{\nu-1} F_{L,i}LA^i - \sum_{i=1}^{\nu} F_{C,i}CA^i \\ &= F_{L,0}L + F_{C,0}C. \end{aligned}$$

Thus, after some calculations, $(TA)_j - (FT)_j$ can be read, for $j = 1$ to ν, as $(F_{C,j-1} + F_{L,j-1}F_{C,\nu})C$. Taking into account that, for $j = 0$ to $\nu - 1$, $H_{C,j} = F_{C,j} + F_{L,j}F_{C,\nu}$, we deduce that T fulfills the Sylvester equation $TA - FT = HC$.

Moreover, as $P = [0\ \cdots\ 0\ I_l]$ and $V = F_{C,\nu}$, we are led to

$$PT + VC = L - F_{C,\nu}C + F_{C,\nu}C = L$$

which ends the proof.

We can then deduce the following lemma.

Lemma 5: If there exist an integer ν and matrices $F_{L,0}, \ldots, F_{L,\nu-1}$ and $F_{C,0}, \ldots, F_{C,\nu}$ such that

$$LA^\nu = \sum_{i=0}^{v-1} F_{L,i}LA^i + \sum_{i=0}^{\nu} F_{C,i}CA^i$$

then a solution (T, F, H, P, V) of the equations $TA - FT = HC$ and $PT + VC = L$ is given by (15) and

$$T = -\begin{bmatrix} F_{L,1} & F_{L,2} & \cdots & F_{L,\nu-1} & -I_l \\ F_{L,2} & & \iddots & \iddots & \\ \vdots & \iddots & \iddots & & \\ F_{L,\nu-1} & -I_l & & & \\ -I_l & & & & \end{bmatrix} \mathcal{O}_{(A,L,\nu)}$$
$$-\begin{bmatrix} F_{C,1} & F_{C,2} & \cdots & F_{C,\nu-1} & F_{C,\nu} \\ F_{C,2} & & \iddots & \iddots & \\ \vdots & \iddots & \iddots & & \\ F_{C,\nu-1} & F_{C,\nu} & & & \\ F_{C,\nu} & & & & \end{bmatrix} \mathcal{O}_{(A,C,\nu)}.$$

III. Necessity

Lemma 6: Let us suppose that the q-order asymptotic observer (3) of $Lx(t)$ for the system (1) is observable then, there exist matrices $F_{L,i}$ and $F_{C,i}$, $i = 0$ to $q - 1$, and $F_{C,q}$ such that

$$LA^q = \sum_{i=0}^{q-1} F_{L,i}LA^i + \sum_{i=0}^{q} F_{C,i}CA^i. \tag{18}$$

Proof: From [10] and [11], when the linear multi-functional observer (3) of $Lx(t)$ exists, then, there exists T such that (4) and (5) are fulfilled. On the one hand, from (5), we can write, for $k \in \mathbb{N}$

$$LA^k = PTA^k + VCA^k.$$

On the other hand, writing (4) as $TA = FT + HC$, we can easily deduce by induction that, for $k \in \mathbb{N} - \{0\}$

$$TA^k = F^kT + \sum_{i=0}^{k-1} F^iHCA^{k-1-i}.$$

Consequently, we obtain, for $k \in \mathbb{N} - \{0\}$

$$LA^k = PF^kT + \sum_{i=0}^{k-1} PF^i HCA^{k-1-i} + VCA^k. \quad (19)$$

Gathering (5) and the previous expressions for $k = 1$ to $q-1$, we are led to

$$\mathcal{O}_{(A,L,q)} = \mathcal{O}_{(F,P,q)}T + \Pi\mathcal{O}_{(A,C,q)} \quad (20)$$

where Π is the matrix

$$\Pi = \left\{ (I_q \otimes V) + \begin{bmatrix} 0 & & & & \\ P & \ddots & & & \\ PF & P & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ PF^{q-2} & \cdots & PF & P & 0 \end{bmatrix} (I_q \otimes H) \right\}$$

and, $\otimes$ stands for the Kronecker product of two matrices [15], [21].

As the observer (3) is observable we have rank$\mathcal{O}_{(F,P)} = q$. Thus, the matrix T defined by (20) is unique and is given by

$$T = \mathcal{O}^{[1]}_{(F,P,q)} \left\{ \mathcal{O}_{(A,L,q)} - \Pi\mathcal{O}_{(A,C,q)} \right\}$$

where ${\mathcal{O}_{(F,P,q)}}^{[1]}$ stands for an arbitrary generalized inverse of the observability matrix, namely [1]

$$\mathcal{O}^{[1]}_{(F,P,q)} \in \left\{ X,\ \mathcal{O}_{(F,P,q)} X \mathcal{O}_{(F,P,q)} = \mathcal{O}_{(F,P,q)} \right\}.$$

Consequently, there exist matrices $T_{L,i}$ and $T_{C,i}$, $i = 0$ to $q-1$, such that

$$T = \sum_{i=0}^{q-1} T_{L,i} LA^i + \sum_{i=0}^{q-1} T_{C,i} CA^i.$$

Let us remark that (19) gives, for $k = q$

$$\begin{aligned} LA^q &= PF^qT + \sum_{i=0}^{q-1} PF^i HCA^{q-1-i} + VCA^q \\ &= PF^q \left\{ \sum_{i=0}^{q-1} T_{L,i} LA^i + \sum_{i=0}^{q-1} T_{C,i} CA^i \right\} \\ &\quad + \sum_{i=0}^{q-1} PF^i HCA^{q-1-i} + VCA^q. \end{aligned}$$

Thus, there exist matrices $F_{L,i}$ and $F_{C,i}$, $i = 0$ to $q-1$, and $F_{C,q}$ such that

$$LA^q = \sum_{i=0}^{q-1} F_{L,i} LA^i + \sum_{i=0}^{q} F_{C,i} CA^i$$

which concludes the proof. □

IV. A Stability Condition

The previous sections concern the design of a candidate observer for the linear functional (2). The final step consists in finding stability conditions for F defined in (15) to ensure an asymptotic observation.

A. The Solution Set

Let us consider the matrix

$$\Sigma_\nu = \begin{bmatrix} \mathcal{O}_{(A,L,\nu)} \\ \mathcal{O}_{(A,C,\nu+1)} \end{bmatrix}. \quad (21)$$

The existence condition of an integer ν and matrices $F_{L,0}, \ldots, F_{L,\nu-1}$ and $F_{C,0}, \ldots, F_{C,\nu}$, such that (9) is fulfilled, is equivalent to the consistency condition of the linear equation

$$LA^\nu = \Phi\Sigma_\nu. \quad (22)$$

Namely, the integer ν is such that

$$\text{rank}\left(\begin{bmatrix} LA^\nu \\ \Sigma_\nu \end{bmatrix} \right) = \text{rank}(\Sigma_\nu).$$

From [1], when this rank condition is verified, the solution set for the (22) can be written

$$\begin{aligned} &[F_{L,0} \quad F_{L,1} \quad \cdots \quad F_{L,\nu-1} \quad F_{C,0} \quad F_{C,1} \quad \cdots \quad F_{C,\nu}] \\ &\qquad = LA^\nu \Sigma_\nu^{[1]} + \Omega \left(I_\rho - \Sigma_\nu \Sigma_\nu^{[1]} \right) \end{aligned} \quad (23)$$

where $\rho = m + \nu(m+l)$, Ω is an arbitrary $(l \times \rho)$ matrix and $\Sigma_\nu^{[1]}$ is an arbitrary generalized inverse of Σ_ν.

Remark 7: If rank$(\Sigma_\nu) = \rho$, the solution $LA^\nu\Sigma_\nu^{[1]}$ is unique and independent of a particular choice for $\Sigma_\nu^{[1]}$.

Remark 8: In the case where rank$(\Sigma_\nu) = r < \rho$, the number of degrees of freedom for the design is reduced to the dimension of the co-rank of the matrix Σ_ν, namely $\rho - r$.

B. A Stabilizability Condition

In the case where rank$(\Sigma_\nu) = \rho$, we can test if F is a Hurwitz matrix through an eigenvalues inspection or by using the Routh-Hurwitz criterion. Let us suppose now that rank$(\Sigma_\nu) = r < \rho$, and, consider the SVD decomposition [12], [13] of Σ_ν

$$\Sigma_\nu = U_\nu S_\nu V_\nu^\top$$

where $U_\nu(\rho \times \rho)$ and $V_\nu(n \times n)$ are unitary matrices, and S_ν is the $(\rho \times n)$-sized diagonal matrix of the ordered singular values, $\sigma_1 \geq \cdots \geq \sigma_r > 0$

$$S_\nu = \text{diag}\{\sigma_1, \ldots, \sigma_r, 0, \ldots, 0\}.$$

A particular choice for $\Sigma_\nu^{[1]}$ can be

$$\Sigma_\nu^{[1]} = \Sigma_\nu^\dagger = V_\nu S_\nu^{-\top} U_\nu^\top \quad (24)$$

where $S_\nu^{-\top} = \text{diag}\{\sigma_1^{-1}, \ldots, \sigma_r^{-1}, 0, \ldots, 0\}$. Thus, we are led to

$$I_\rho - \Sigma_\nu\Sigma_\nu^\dagger = I_\rho - U_\nu \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U_\nu^\top = U_\nu \begin{bmatrix} 0 & 0 \\ 0 & I_{\rho-r} \end{bmatrix} U_\nu^\top.$$

Remark 9: Two reasons motivate the proposed choice (24) for $\Sigma_\nu^{[1]}$. Firstly, the pseudo-inverse $\Sigma_\nu^{[1]} = \Sigma_\nu^\dagger$ of Σ_ν is unique. Secondly, the SVD decomposition is numerically robust.

Let us define $U_{2,\nu}^\top$ as the matrix built with the $\rho - r$ last rows of $U_\nu^\top$

$$\begin{aligned} U_{2,\nu}^\top &= [U_\nu(:, r+1:\rho)]^\top \\ &= [\Upsilon_{L,0} \quad \Upsilon_{L,1} \quad \cdots \quad \Upsilon_{L,\nu-1} \quad \Upsilon_{C,0} \quad \cdots \quad \Upsilon_{C,\nu}] \end{aligned}$$

Γ_2, the $\rho - r$ last columns of the arbitrary matrix $\Gamma = \Omega U_\nu$, and

$$\begin{aligned} \Phi^b &= \left[F^b_{L,0} \quad F^b_{L,1} \quad \cdots \quad F^b_{L,\nu-1} \quad F^b_{C,0} \quad \cdots \quad F^b_{C,\nu} \right] \\ &= LA^\nu \Sigma_\nu^\dagger. \end{aligned}$$

The previous partitions lead to the structure of F defined in (15) where, for $i = 0$ to $\nu - 1$, $F_{L,i} = F^b_{L,i} + \Gamma_2 \Upsilon_{L,i}$. Due to commutativity, with respect to the block-column partition of the matrix F, its eigenvalues are identical to the eigenvalues of the matrix

$$F^* = \begin{bmatrix} & I_l & & & \\ & & \ddots & & \\ & & & I_l & \\ & & & & I_l \\ F_{L,0} & F_{L,1} & \cdots & F_{L,\nu-2} & F_{L,\nu-1} \end{bmatrix}.$$

The interest in considering F^* instead of F is that we have the following decomposition:

$$F^* = \begin{bmatrix} & I_l & & & \\ & & \ddots & & \\ & & & I_l & \\ & & & & I_l \\ F^b_{L,0} & F^b_{L,1} & \cdots & F^b_{L,\nu-2} & F^b_{L,\nu-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_l \end{bmatrix} \Gamma_2 [\Upsilon_{L,0} \quad \Upsilon_{L,1} \quad \cdots \quad \Upsilon_{L,\nu-2} \quad \Upsilon_{L,\nu-1}].$$

We can now state the following test.

Lemma 10: There exists a matrix Ω such that F defined in (15) is an Hurwitz matrix if and only if the system

$$\begin{aligned} \dot{\eta}(t) &= \begin{bmatrix} & I_l & & \\ & & \ddots & \\ & & & I_l \\ F^b_{L,0} & F^b_{L,1} & \cdots & F^b_{L,\nu-1} \end{bmatrix} \eta(t) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_l \end{bmatrix} \varpi(t) \\ \varsigma(t) &= [\Upsilon_{L,0} \quad \Upsilon_{L,1} \quad \cdots \quad \Upsilon_{L,\nu-1}] \eta(t) \end{aligned} \tag{25}$$

is static output feedback stabilizable.

Consequently, any well-known static output feedback stabilizability criteria (see [2], [19], [31]) can be used here. Moreover, when the system (25) is stabilizable with a static output feedback, we can apply, for instance, LMI based methods [26], [33], [34] or software built-in procedures to get a matrix Γ_2 which solves the problem. In the opposite, when such a matrix cannot be found, ν has to be increased up to a value such that the static output stabilizability problem can be solved. Taking into account the specific form of (25), we propose, in the following section, a simple method to get a possible Γ_2.

V. Illustrative Example

Let us consider the observation problem (1), (2) where

$$A = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The first step deals with the determination of ν. Denoting $r_\nu = \text{rank}\begin{pmatrix} \begin{bmatrix} LA^\nu \\ \Sigma_\nu \end{bmatrix} \end{pmatrix} - \text{rank}(\Sigma_\nu)$, we obtain $r_0 = r_1 = 2$, and $r_2 = 0$. Thus, $\nu = 2$. Secondly, with

$$\Sigma_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -4 & 0 & 1 & 1 \\ -2 & 3 & 0 & 5 & 2 & 0 & -2 \end{bmatrix}$$

and $LA^2 = \begin{bmatrix} 3 & 0 & 3 & -3 & 4 & -1 & -1 \\ 1 & 1 & 0 & 9 & 1 & -4 & -5 \end{bmatrix}$, the singular value decomposition of Σ_2 gives

$$\begin{aligned} \Upsilon_{L,0} &= U_2^\top(1:2:, 8:10) = \begin{bmatrix} 0 & -0.6 \\ 0 & 0.3 \\ 0 & 0.22 \end{bmatrix} \\ \Upsilon_{L,1} &= U_2^\top(3:4:, 8:10) = \begin{bmatrix} 0.03 & -0.02 \\ -0.15 & -0.49 \\ -0.41 & 0.45 \end{bmatrix} \\ F^b_{L,0} &= LA^\nu \Sigma_\nu^\dagger(1:2, 1:2) = \begin{bmatrix} -1 & -1.3 \\ 0 & -2.27 \end{bmatrix} \\ F^b_{L,1} &= LA^\nu \Sigma_\nu^\dagger(1:2, 3:4) = \begin{bmatrix} -0.56 & -0.44 \\ -1.13 & -2.8 \end{bmatrix}. \end{aligned}$$

The third step consists in detecting the stabilizability of (25). Let us consider the permutation matrix

$$\Pi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We obtain

$$\Pi F^* \Pi = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & \varphi_{22} & \varphi_{23} & \varphi_{24} \\ 0 & 0 & 0 & 1 \\ 0 & \varphi_{42} & \varphi_{43} & \varphi_{44} \end{bmatrix}$$

where, denoting γ_1 and γ_2 the rows of Γ_2, we obtain $[\varphi_{22} + 0.56 \quad \varphi_{23} + 1.3 \quad \varphi_{24} + 0.44] = \gamma_1 M$ and $[\varphi_{42} + 1.13 \quad \varphi_{43} + 2.27 \quad \varphi_{44} + 2.8] = \gamma_2 M$ where M is the nonsingular matrix

$$M = \begin{bmatrix} 0.03 & -0.6 & -0.02 \\ -0.15 & 0.3 & -0.49 \\ -0.48 & 0.22 & 0.45 \end{bmatrix}.$$

Consequently, we can choose any value for the φ_{ij} coefficients. Using usual methods for pole assignment [20], a particular but interesting choice is $\varphi_{23} = \varphi_{24} = \varphi_{42} = 0$. In this case, it is possible to fix φ_{22}, φ_{43} and φ_{44}, and therefore γ_1 and γ_2, to obtain a static output feedback that stabilizes the system (25). The linear functional $Lx(t)$ is functionally observable for the system (1) with a fourth-order Luenberger observer.

VI. Conclusion

All these results can be summed up in the following theorem which provides a test of functional observability of a linear functional with respect to a given linear time-invariant system.

Theorem 11: The triplet (A, C, L) is functionally observable if and only if there exists an integer ν such that

$$\text{rank}\left(\begin{bmatrix} LA^{\nu} \\ \Sigma_{\nu} \end{bmatrix}\right) = \text{rank}(\Sigma_{\nu})$$

where

$$\Sigma_{\nu} = \begin{bmatrix} \mathcal{O}_{(A,L,\nu)} \\ \mathcal{O}_{(A,C,\nu+1)} \end{bmatrix}$$

has the singular value decomposition $\Sigma_{\nu} = U_{\nu} S_{\nu} V_{\nu}^{\top}$, and, the system (25) where the essential matrices are defined by

$$\begin{aligned} &[\Upsilon_{L,0} \quad \Upsilon_{L,1} \quad \cdots \quad \Upsilon_{L,\nu-1} \quad \Upsilon_{C,0} \quad \Upsilon_{C,1} \quad \cdots \quad \Upsilon_{C,\nu}] \\ &\quad = U_{\nu}^{\top}(r+1:\rho,:) \\ &[F_{L,0}^{b} \quad F_{L,1}^{b} \quad \cdots \quad F_{L,\nu-1}^{b} \quad F_{C,0}^{b} \quad F_{C,1}^{b} \quad \cdots \quad F_{C,\nu}^{b}] \\ &\quad = LA^{\nu}\Sigma_{\nu}^{\dagger} \end{aligned}$$

is static output feedback stabilizable.

When this theorem is fulfilled, the previous sections indicate a design procedure which leads to a $l\nu$-order stable Luenberger observer. When ν is minimal and $\text{rank}(T) = q < l\nu$, keeping in T the linearly independent rows and eliminating the corresponding components in the state of the observer, the order of the observer can be reduced to q. Indeed, another particular feature of the presented work is the closed form of the matrix T solution of the Sylvester equation (4).

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