Smooth projective horospherical varieties of Picard group Z^2

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Abstract: We classify all smooth projective horospherical varieties of Picard group $\mathbb{Z}^2$ and we give a first description of their geometry via the Log Minimal Model Program
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Contents

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1 Introduction

In this paper, varieties are algebraic varieties over $\mathbb{C}$ and groups are algebraic groups over $\mathbb{C}$.

Smooth projective horospherical varieties of Picard group $\mathbb{Z}$ are known since 2009 [Pas09] and give useful examples in various theories. For toric varieties there are only projective spaces. But for horospherical varieties, in addition to homogeneous spaces, there are 5 families of two-orbit varieties (two of them are infinite families).

Here we classify and give a first study of the geometry of smooth projective horospherical varieties of Picard group $\mathbb{Z}^2$. For toric varieties, there are only decomposable projective bundles over projective spaces [Kle88]. But for horospherical varieties, there are many other cases.

Indeed, in addition to homogeneous spaces, products of two varieties and decomposable projective bundles over projective spaces, we distinguish several types of other such horospherical varieties. We classify them in this paper, in particular by studying their Log MMP.

To write as nice as possible the classification of smooth projective horospherical varieties of Picard group $\mathbb{Z}^2$, we extend the notion of simple roots to the groups $\mathbb{C}^*$ and $\{1\}$. We first briefly recall the case of simple groups.

If G is a simple group, we fix a maximal torus contained in a Borel subgroup B of G, then it defines a root system and in particular a set of simple roots. To each simple root α is associated a fundamental weight denoted by ϖ_α and a fundamental G-module denoted by $V(\varpi_\alpha)$. More generally, if χ is a dominant weight (a non-negative sum of fundamental weights) we denote by $V(\chi)$ the G-module associated to χ: it is the unique irreducible G-module that contains a unique B-stable line where B acts with weight χ. A non-zero element of the B-stable line of $V(\chi)$ is called a highest weight vector (of weight χ) and the stabilizer of the B-stable line of $V(\chi)$ is denoted by $P(\chi)$ (it is a parabolic subgroup of G containing B).

In this paper, if $G = \mathbb{C}^*$, the simple root α of G denotes the identity endomorphism of $\mathbb{C}^*$ and we set $\varpi_\alpha = \alpha$. Then the natural $\mathbb{C}^*$-module $\mathbb{C}$ is denoted by $V(\varpi_\alpha)$ where α is the simple root of $\mathbb{C}^*$. And for any $n \in \mathbb{Z}$, $V(n\varpi_\alpha)$ is the $\mathbb{C}^*$-module $\mathbb{C}$ where $\mathbb{C}^*$ acts with weight $n\varpi_\alpha$; in particular, any character of $\mathbb{C}^*$ is dominant. Moreover, if $G = \{1\}$, the simple root α of G denotes the trivial morphism from G to $\mathbb{C}^*$ and we set $\varpi_\alpha = 0$. In these two cases a highest weight vector is any non-zero vector.

Suppose now that G is a product $G_0 \times \cdots \times G_t$ of simple groups, $\mathbb{C}^*$ and $\{1\}$. A simple root of G is a simple root of some G_i and it is said to be imaginary if it G_i is equal to $\mathbb{C}^*$ or $\{1\}$. Moreover if $\chi_0, \ldots, \chi_t$ are respectively dominant weights of $G_0, \ldots, G_t$, the G-module associated to $\chi = \chi_0 + \cdots + \chi_t$ is the tensor product $V(\chi_0) \otimes \cdots \otimes V(\chi_t)$ and a highest weight vector of this G-module is a decomposable tensor product of highest weight vectors.

We can now write the two main results of this paper.

Theorem 1.1. *Let X be a smooth projective horospherical variety with Picard group $\mathbb{Z}^2$. Suppose that X is not the product of two varieties. Then X is isomorphic to one of the following horospherical variety (which we still denote by X).*

Let $G = G_0 \times G_1 \times \cdots \times G_t$ be a product of simply connected simple groups, $\mathbb{C}^*$ and $\{1\}$.

Case (0): $t = 0$, G_0 *is simple and* X *is an homogeneous variety* G_0/P *where* P *is the intersection of two maximal (proper) parabolic subgroup of* G_0.

Case (1): $t \geq 0$ *and* X *is the closure of the* G*-orbit of a sum of highest weight vectors in*

$$\mathbb{P}\left(\bigoplus_{i=0}^{n} V(\varpi_{\alpha_i} + (1+a_i)\varpi_\beta)\right),$$

where $n \geq \max\{1, t\}$, β *is a simple root of* G_0, $\alpha_0, \ldots, \alpha_n$ *are distinct simple roots of* G *distinct from* β *and* $0 = a_0 \leq a_1 \leq \cdots \leq a_n$ *are integers, satisfying the following properties.*

Denote by R_0 *the maximal subset of* $\{\alpha_0, \ldots, \alpha_n\}$ *consisting of simple roots of* G_0. *The quadruple* (G_0, β, R_0, n) *is smooth (see Definition 4.2 or Proposition 6.1).*

If R_0 *is empty, then* G_0 *is the universal cover of the automorphism group of* $G/P(\varpi_\beta)$.

Let $0 \leq i_1 < \cdots < i_{t'} \leq n$ *such that* $R_0 = \{\alpha_i \mid i \notin \{i_1, \ldots, i_{t'}\}\}$ *(*$t \geq t' \geq 0$*). Then* $a_{i_1} < \cdots < a_{i_{t'}}$.

If $i < j$ *and* $a_i = a_j$ *then* $\alpha_j \in R_0$. *Moreover, if* α_i *and* α_j *are in* R_0, *we suppose them to be ordered with Bourbaki's notation as simple roots of* G_0.

And one of the three following cases occurs.

(a) $n = t = 1$, α_0 *and* α_1 *are both simple roots of* G_1 *so that the triple* $(G_1, \alpha_0, \alpha_1)$ *is smooth (see Definition 4.1); in particular,* $R_0 = \emptyset$, $t' = 2$, $i_1 = 0$ *and* $i_2 = 1$.

In the two next cases, $t = t'$, *and for any* $k \in \{1, \ldots, t\}$, *either* G_k *is isomorphic to some* SL_{d_k} *and* α_{i_k} *is the first simple root of* G_k, *or* G_k *is isomorphic to* $\mathbb{C}^*$ *or* $\{1\}$ *and* α_{i_k} *is the imaginary simple root of* G_k. *Moreover,* G_k *is isomorphic to* $\{1\}$ *if and only if* $k = 1$, $i_1 = 0$ *and* α_{i_1} *is imaginary.*

(b) *The simple root* α_n *is not imaginary or* $a_{n-1} = a_n$.

(c) *The simple root* α_n *is imaginary and* $a_{n-1} < a_n$.

Case (2): $t \geq 2$ *and* X *is the closure of the* G*-orbit of a sum of highest weight vectors in*

$$\mathbb{P}\left(\bigoplus_{i=0}^{n-1}\bigoplus_{b=0}^{1+a_i} V(\varpi_{\alpha_i} + b\varpi_{\alpha_n} + (1+a_i-b)\varpi_{\alpha_{n+1}})\right),$$

where $n \geq 2$, $0 = a_0 < a_1 < \cdots < a_{n-1}$ *are integers, and* $\alpha_0, \ldots, \alpha_{n+1}$ *are distinct simple roots of* G *satisfying the following properties.*

The triple $(G_t, \alpha_n, \alpha_{n+1})$ *is smooth of two-orbit type (see Definition 4.1; in particular,* α_n *and* α_{n+1} *are both simple roots of* G_t*) and* $\alpha_0, \ldots, \alpha_{n-1}$ *are simple roots of* $G_0 \times G_1 \times \cdots \times G_{t-1}$.

Moreover, for any $k \in \{0, \ldots, t\}$, G_k *is isomorphic to* $\{1\}$ *if and only if* $k = 0$ *and* α_0 *is imaginary.*

And one of the three following cases occurs.

(a) $n = 2$, $t = 1$ *and the triple* $(G_0, \alpha_0, \alpha_1)$ *is smooth (see Definition 4.1).*

In the two next cases: $t = n$ *and for any* $i \in \{1, \ldots, t\}$, *either* G_i *is isomorphic to some* SL_{d_i} *and* α_i *the first simple root of* G_i, *or* G_i *is isomorphic to* $\mathbb{C}^*$ *or* $\{1\}$ *and* α_i *is the imaginary simple root of* G_i.

(b) *The simple root α_{n-1} is not imaginary.*

(c) *The simple root α_{n-1} is imaginary.*

Remark 1.2. In Theorem 1.1, the decomposable projective bundles over projective spaces are the horospherical varieties X in Cases (1b) and (1c) with $R_0 = \emptyset$ and ϖ_β is the first simple root of $G_0 = \mathrm{SL}_{d_0}$ for some $d_0 \geq 2$ (and $0 < a_1 < \cdots < a_n$).

The horospherical varieties described in Theorem 1.1 are all distinct. This is a consequence of the following result.

Theorem 1.3. *Let X be one of the varieties described in Theorem 1.1. Then "the" Log MMP from X gives the following in each case.*

Case (0): There are two Mori fibrations from X, respectively into Y and Z, with (general) fibers respectively not isomorphic to Z and Y.

Case (1):

(a) *A "first" Log MMP consists of a Mori fibration from X to $G/P(\varpi_\beta)$ with general fibers not isomorphic to a projective space (but isomorphic to another homogeneous variety or to a two-orbit variety) and a "second" one consists of a flip from X followed by a fibration.*

(b) *A "first" Log MMP consists of a Mori fibration from X to $G/P(\varpi_\beta)$ with general fibers isomorphic to a projective space and a "second" one consists of a finite sequence (may be empty) of flips from X followed by a fibration.*

(c) *A "first" Log MMP consists of a Mori fibration from X to $G/P(\varpi_\beta)$ with general fibers isomorphic to a projective space and a "second" one consists of a finite sequence (may be empty) of flips from X followed by a divisorial contraction.*

Case (2): A "first" Log MMP consists of a fibration ψ to a two-orbit variety, the general fiber F_ψ of ψ and a "second" Log MMP are described as follows.

(a) *F_ψ is not isomorphic to a projective space (but isomorphic to another homogeneous variety or to a two-orbit varity) and a "second" Log MMP consists of a flip from X followed by a fibration.*

(b) *F_ψ is isomorphic to a projective space and a "second" Log MMP consists of a finite sequence (not empty) of flips from X followed by a fibration.*

(c) *F_ψ is isomorphic to a projective space and a "second" Log MMP consists of a finite sequence (may be empty) of flips from X followed by a divisorial contraction.*

Moreover, in every cases, up to reordering and up to symmetries of Dynkin diagrams, the data $a_1, \ldots, a_n$ (respectively $a_1, \ldots, a_r$), $G_0, \ldots, G_t$, $\alpha_0, \ldots, \alpha_n$, β (respectively $\alpha_0, \ldots, \alpha_{n+1}$) are invariants of the "two canonical ways" to realize the Log MMP from X (and then invariants of X).

Remark 1.4.

- In the paper (Proposition 3.3), we prove that for any smooth projective horospherical variety X with Picard group $\mathbb{Z}^2$, the nef cone of X is generated by the two elements of a basis of $\mathrm{Pic}(X)$, then this gives us two canonical ways to choose the log pair to compute Log MMP from X (see Section 5 for more details). Also, in Cases (1) and (2), one of the "two canonical" Log MMP is "naturally" defined (see Remark 3.2) and only consists of a fibration.

- In Case (1b), if the sequence of flips is empty, we get two fibrations from X. They could be both into homogeneous varieties. But one and only one of these fibrations has all its fibers isomorphic to each others. (On the contrary, in Case (0), each fibration has all their fibers isomorphic to each others.)

The paper is organized as follows.

We first recall in Section 2 the results on horospherical varieties that we use in the paper. Then, in Section 3, we easily describe a first (but not optimal) combinatorial classification, containing many repetitions. In Section 4, we give a first geometric description of all these latter cases in order to reduce the number of cases and prove Theorem 1.1. Then, in Section 5, we prove Theorem 1.3, by studying the Log MMP of all varieties of Theorem 1.1.

2 Some known results on horospherical varieties

2.1 First definitions, first properties of divisors, and smooth criterion

Let G be a connected reductive group. Fix a maximal torus T and a Borel subgroup B containing T. Denote by U the unipotent radical of B, by $\mathcal{S}$ the set of simple roots of (G, B, T), by $\mathfrak{X}(T)$ the lattice of characters of T (or B) and by $\mathfrak{X}(T)^+ \subset \mathfrak{X}(T)$ the cone of dominant characters.

For any lattice L we denote by $L_{\mathbb{Q}}$ the $\mathbb{Q}$-vector space $L \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 2.1. A horospherical variety X is a normal G-variety with an open orbit isomorphic to G/H where H is a subgroup of G containing U.

Then G/H is a torus fibration over the flag variety G/P where P is the parabolic subgroup of G containing B defined as the normalizer of H in G. The dimension of the torus is called the rank of G/H or the rank of X and it is denoted by n.

We denote by M the sublattice of $\mathfrak{X}(T)$ consisting of characters of P whose restrictions to H are trivial. Its dual is denoted by N. (The lattices M and N are of rank n.)

Let $\mathcal{R}$ be the subset of $\mathcal{S}$ consisting of simple roots that are not simple roots of P (ie, simple roots associated to fundamental weights that are characters of P).

For any simple root $\alpha \in \mathcal{R}$, the restriction of the coroot $\alpha^\vee$ to M is a point of N, which we denote by $\alpha^\vee_M$. We denote by σ the map $\alpha \longmapsto \alpha^\vee_M$ from $\mathcal{R}$ to N.

Definition 2.2. 1. A colored cone of $N_{\mathbb{Q}}$ is a couple $(\mathcal{C}, \mathcal{F})$ where $\mathcal{C}$ is a convex cone of $N_{\mathbb{Q}}$ and $\mathcal{F}$ is a subset of $\mathcal{R}$ (called the set of colors of the colored cone), such that

   (i) $\mathcal{C}$ is generated by finitely many elements of N and contains $\{\alpha^\vee_M \mid \alpha \in \mathcal{F}\}$,

   (ii) $\mathcal{C}$ does not contain any line and $\mathcal{F}$ does not contain any α such that $\alpha^\vee_M$ is zero.

2. A colored face of a colored cone $(\mathcal{C}, \mathcal{F})$ is a couple $(\mathcal{C}', \mathcal{F}')$ such that $\mathcal{C}'$ is a face of $\mathcal{C}$ and $\mathcal{F}'$ is the set of $\alpha \in \mathcal{F}$ satisfying $\alpha^\vee_M \in \mathcal{C}'$.

   A colored fan is a finite set $\mathbb{F}$ of colored cones such that

   (i) any colored face of a colored cone of $\mathbb{F}$ is in $\mathbb{F}$,

   (ii) and any element of $N_{\mathbb{Q}}$ is in the interior of at most one colored cone of $\mathbb{F}$.

The main result of Luna-Vust Theory of spherical embeddings is the following classification result (see for example [Kno91]).

Theorem 2.3. *(D. Luna-T. Vust) There is an explicit one-to-one correspondence between colored fans and isomorphic classes of horospherical G-varieties with open orbit G/H.*

Complete G/H-embeddings correspond to complete fans, ie, to fans such that $N_{\mathbb{Q}}$ is the union of their colored cones.

If $G = (\mathbb{C}^*)^n$ and $H = \{1\}$, we recover the well-known classification of toric varieties.

If X is a G/H-embedding, we denote by $\mathbb{F}_X$ the colored fan of X in $N_{\mathbb{Q}}$ and we denote by $\mathcal{F}_X$ the subset $\cup_{(\mathcal{C},\mathcal{F})\in\mathbb{F}_X}\mathcal{F}$ of $\mathcal{R}$, called the set of colors of X.

We now recall the characterization of Cartier, $\mathbb{Q}$-Cartier, globally generated and ample divisors of horospherical varieties, due to M. Brion in the more general case of spherical varieties ([Bri89]).

First, we describe the B-stable prime divisors of X. We denote by $X_1, \ldots, X_m$ the G-stable prime divisors of X. The valuations of $\mathbb{C}(X)$ defined by the zeros and poles along these divisors define primitive elements of N, denoted by $x_1, \ldots, x_m$ respectively.

And the B-stable but not G-stable prime divisors of X are the closures in X of B-stable prime divisors of G/H, which are the inverse images by the torus fibration $G/H \longrightarrow G/P$ of the Schubert divisors of the flag variety G/P. The Schubert divisors of G/P can be naturally indexed by the subset of simple roots $\mathcal{R}$. Hence, we denote the B-stable but not G-stable prime divisors of X by D_α with $\alpha \in \mathcal{R}$ (note that $\sigma(\alpha)$ is the element of N defined by the valuation of $\mathbb{C}(X)$ defined by the zeros and poles along the divisor D_α).

Theorem 2.4. *([Bri89, Section 3.3]) Every divisor of X is linearly equivalent to a linear combinaison of $X_1, \ldots, X_m$ and D_α with $\alpha \in \mathcal{R}$. Now, let $D = \sum_{i=1}^m a_i X_i + \sum_{\alpha\in\mathcal{R}} a_\alpha D_\alpha$ be a $\mathbb{Q}$-divisor of X.*

1. *D is $\mathbb{Q}$-Cartier if and only if there exists a piecewise linear function h_D, linear on each colored cone of $\mathbb{F}_X$, such that for any $i \in \{1, \ldots, m\}$, $h_D(x_i) = a_i$ and for any $\alpha \in \mathcal{F}_X$, $h_D(\alpha_M^\vee) = a_\alpha$.*

   And D is linearly equivalent to 0 if and only if h_D is linear on $N_{\mathbb{Q}}$.

   Moreover, if D is a divisor, D is Cartier if and only if it is $\mathbb{Q}$-Cartier and the linear functions defines as above can be identified to elements of M.

2. *Suppose that D is $\mathbb{Q}$-Cartier. Then D is globally generated (resp. ample) if and only if the piecewise linear function h_D is convex (resp. strictly convex) and for any $\alpha \in \mathcal{R}\backslash\mathcal{F}_X$, we have $h_D(\alpha_M^\vee) \leq a_\alpha$ (resp. $h_D(\alpha_M^\vee) < a_\alpha$).*

3. *Suppose that D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Let $\tilde{Q}_D$ be the polytope in $M_{\mathbb{Q}}$ (called pseudo-moment polytope) defined by the following inequalities, where $\chi \in M_{\mathbb{Q}}$: $(h_D)+\chi \geq 0$ and for any $\alpha \in \mathcal{R}\backslash\mathcal{F}_X$, $a_\alpha + \chi(\alpha_M^\vee) \geq 0$.*

   Let $v^0 := \sum_{\alpha\in\mathcal{R}} a_\alpha \varpi_\alpha$, then the polytope $v^0 + \tilde{Q}_D$ is called the moment polytope of D (or (X, D)).

4. *Suppose that D is a Cartier divisor. Note that the weight of the canonical section of D is v^0. Then the G-module $H^0(X, D)$ is the direct sum (with multiplicities one) of the irreducible G-modules of highest weights $\chi + v^0$ with χ in $\tilde{Q}_D \cap M$.*

From now on, a divisor of a horospherical variety is always supposed to be B-stable, ie, of the form $\sum_{i=1}^m a_i X_i + \sum_{\alpha\in\mathcal{R}} a_\alpha D_\alpha$.

Theorem 2.5. *([Pas06, Theorem 0.3]) Let X be a projective horospherical variety and let D be an ample Cartier divisor of X. Suppose that X is smooth.*

Then D is very ample.

Since $H \supset U$ and the unique U-stable lines of irreducible G-modules are the lines generated by highest weight vectors, we deduce from Theorems 2.4 and 2.5 the following result. (See also [Pas15, Remark 2.13] to explain why we can ignore duals.)

Corollary 2.6. *Let X be a smooth projective horospherical variety and let D be an ample Cartier divisor of X. Then X is isomorphic to the closure of the G-orbit of a sum of highest weight vectors in $\mathbb{P}(\oplus_{\chi\in\tilde{Q}_D\cap M}V(\chi+v^0))$.*

From Theorem 2.4, we can also deduce a locally factorial criterion.

Corollary 2.7. *A horospherical variety X is locally factorial if and only if for any colored cone $(\mathcal{C},\mathcal{F})$ of $\mathbb{F}_X$, $\mathcal{C}$ is generated by a basis of N and the map $\sigma : \alpha \longmapsto \alpha_M^\vee$ induces an injective map from $\mathcal{F}$ to this basis.*

In particular if X is locally factorial, the Picard number of X is given by the following formula

$$\rho_X = m + |\mathcal{R}| - n = (|\mathbb{F}_X(1)| - n) + |\mathcal{R}\backslash\mathcal{F}_X|,$$

where $\mathbb{F}_X(1)$ is the set of edges (one-dimensional colored cones) of $\mathbb{F}_X$.

To write the smooth criterion we need to give the following definition.

Definition 2.8. ([Pas06, Def. 2.4]) Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two disjoint subsets of $\mathcal{S}$. Let $\Gamma_{\mathcal{R}_1\cup\mathcal{R}_2}$ be the maximal subgraph of the Dynkin diagram of G whose vertices are in $\mathcal{R}_1 \cup \mathcal{R}_2$.

The couple $(\mathcal{R}_1,\mathcal{R}_2)$ is said to be smooth if, for any connected component Γ of $\Gamma_{\mathcal{R}_1\cup\mathcal{R}_2}$,

- there is at most one vertex of Γ in $\mathcal{R}_2$ and,
- if $\alpha \in \mathcal{R}_2$ is a vertex of Γ, then Γ is of type A or C and α is a short extremal simple root of Γ.

Theorem 2.9. *([Pas06, Theorem 2.6]) Let X be a locally factorial horospherical variety.*

Then X is smooth if and only if for any colored cone $(\mathcal{C},\mathcal{F})$ of $\mathbb{F}_X$, the couple $(\mathcal{S}\backslash\mathcal{R},\mathcal{F})$ is smooth.

Corollary 2.10. *([Pas06, Proposition 2.17]) Let X be a smooth horospherical variety. Any G-stable subvariety of X is a smooth horospherical variety.*

2.2 Log MMP via moment polytopes

The MMP [Pas15] and Log MMP [Pas17] of horospherical varieties can be completely computed and described by studying one-parameter families of polytopes. In this subsection, we recall the main results of this theory, as briefly as we can, in order to use them in Section 5.

From the previous section, to any horospherical variety X, is associated a parabolic subgroup P and a sublattice M of $\mathfrak{X}(P)$; and moreover, any ample B-stable $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D defines a pseudo-moment polytope $\tilde{Q}$ and a moment polytope Q. In fact, the map $(X,D) \longmapsto (P,M,Q,\tilde{Q})$ classifies polarized projective horospherical varieties in terms of quadruples $(P,M,Q,\tilde{Q})$.

Definition 2.11. A quadruple $(P,M,Q,\tilde{Q})$ is called admissible if it satisfies the following:

- P is a parabolic subgroup of G containing B, M is a sublattice of $\mathfrak{X}(P)$, Q is a polytope of $\mathfrak{X}(P)_\mathbb{Q}$ included in $\mathfrak{X}(P)^+_\mathbb{Q}$ and $\tilde{Q}$ is a polytope of $M_\mathbb{Q}$;

- there exists (a unique) $v^0 \in \mathfrak{X}(P)_{\mathbb{Q}}$ such that $Q = v^0 + \tilde{Q}$;
- the polytope $\tilde{Q}$ is of maximal dimension in $M_{\mathbb{Q}}$ (ie, its interior in $M_{\mathbb{Q}}$ is not empty);
- the polytope Q intersects the interior of $\mathfrak{X}(P)^+_{\mathbb{Q}}$.

Proposition 2.12. *1. The map $(X, D) \longmapsto (P, M, Q, \tilde{Q})$ is a bijection from the set of isomorphic classes of polarized projective horospherical varieties to the set of admissible quadruples.*

2. It induces a bijection between the set of G-orbits in X and the set of non-empty faces of Q (or $\tilde{Q}$), preserving the natural orders of both sets. Also, the G-orbit in X associated to a non-empty face $F = v^0 + \tilde{F}$ of Q is isomorphic to a horospherical homogenous space corresponding to (P_F, M_F) where P_F is the minimal parabolic subgroup of G containing P and M_F is the maximal sublattice of M such that $(P_F, M_F, F, \tilde{F})$ is an admissible quadruple. Moreover $(P_F, M_F, F, \tilde{F})$ is the quadruple associated to the (horospherical) closure in X of the G-orbit associated to F (polarized by some D_F we do not need to explicit here).

In particular, we easily get the following consequence.

Corollary 2.13. *Let (X, D) be a polarized projective horospherical variety and $(P, M, Q, \tilde{Q})$ be the corresponding admissible quadruple. Let F be a non-empty face of Q (or $\tilde{Q}$) and Ω be the corresponding G-orbit in X. Then*

$$\dim(\Omega) = \dim(G/P_F) + \operatorname{rank}(M_F) = \dim(G/P_F) + \dim(F).$$

We fix a basis of M (and consider the dual basis for N). Also we choose an order in $\{x_1, \dots, x_m\} \cup \{\alpha^\vee_M \mid \alpha \in \mathcal{R}\}$. Then we define $A \in \mathcal{M}_{m+|\mathcal{R}|,n}(\mathbb{Q})$ whose rows are the coordinates of the vectors of $\{x_1, \dots, x_m\} \cup \{\alpha^\vee_M \mid \alpha \in \mathcal{R}\}$ in the chosen basis.

Theorem 2.14. *Let X be a $\mathbb{Q}$-factorial projective horospherical variety and let Δ be a B-stable $\mathbb{Q}$-divisor of X. Then for any (general) choice of an ample B-stable $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D of X, a Log MMP from the pair (X, Δ) is described by the following one-parameter families of polytopes*

$$\tilde{Q}^\epsilon := \{x \in M_{\mathbb{Q}} \mid Ax \geq B + \epsilon C\} \text{ and } Q^\epsilon := v^\epsilon + \tilde{Q}^\epsilon$$

where B, C and $v^\epsilon = v^0 + \epsilon v^1$ are such that, for any $\epsilon \geq 0$ small enough, $\tilde{Q}^\epsilon$ and Q^ϵ are respectively the pseudo-moment and moment polytope of $(X, D + \epsilon(K_X + \Delta))$.

More precisely, there exist rational numbers

$$0 := \epsilon_{0,0} < \cdots < \epsilon_{0,k_0} < \epsilon_{0,k_0+1} = \epsilon_{1,0} < \cdots \\ \cdots < \epsilon_{1,k_1} < \epsilon_{1,k_1+1} = \epsilon_{2,0} < \cdots < \epsilon_{p,k_p} < \epsilon_{p,k_p+1} = \epsilon_{max}$$

(with $p \geq 1$, and for any $i \in \{0, \dots, p\}$, $k_i \geq 0$) such that, $(P, M, Q^\epsilon, \tilde{Q}^\epsilon)$ is an admissible quadruple if and only if $\epsilon \in [0, \epsilon_{max}[$, and for ϵ, $\eta \in [0, \epsilon_{max}[$ the following three assertions are equivalent:

- X^ϵ is isomorphic to X^η (where X^ϵ and X^η are the varieties associated to the admissible quadruples $(P, M, Q^\epsilon, \tilde{Q}^\epsilon)$ and $(P, M, Q^\eta, \tilde{Q}^\eta)$ respectively);

- the faces of Q^ϵ (or $\tilde{Q}^\epsilon$) and Q^η (or $\tilde{Q}^\eta$) are "the same": up to deleting inequalities corresponding to some x_j with $j \in \{1, \dots, m\}$ but without changing $\tilde{Q}^\epsilon$ and $\tilde{Q}^\eta$, we have for any set I of rows, the face of $\tilde{Q}^\epsilon$ corresponding to I (defined by replacing inequalities by equalities for the rows in I) is non empty if and only the face of $\tilde{Q}^\eta$ corresponding to I is non empty;

- there exists $i \in \{0, \dots, p\}$ such that ϵ and η are both in $[\epsilon_{i,0}, \epsilon_{i,1}[$, or both in $]\epsilon_{i,k}, \epsilon_{i,k+1}[$ with $k \in \{1, \dots, k_i\}$, or both equal to $\epsilon_{i,k}$ with $k \in \{1, \dots, k_i\}$.

Moreover, for any $i \in \{0, \dots, p\}$ and $k \in \{1, \dots, k_i\}$ there are morphisms from X^ϵ to $X^{\epsilon_{i,k}}$ with $\epsilon < \epsilon_{i,k}$ big enough and $\epsilon > \epsilon_{i,k}$ small enough, defining flips. For any $i \in \{1, \dots, p\}$, there are morphisms from X^ϵ to $X^{\epsilon_{i,0}}$ with $\epsilon < \epsilon_{i,0}$ big enough, defining divisorial contractions. Actually, divisorial contractions appear exactly when an inequality corresponding to some x_j with $j \in \{1, \dots, m\}$ becomes useless to define $\tilde{Q}^\epsilon$.

Also, there exists P' and M' such that $(P', M', Q^{\epsilon_{max}}, \tilde{Q}^{\epsilon_{max}})$ is an admissible quadruple associated to a variety $X^{\epsilon_{max}}$ and such that there is a fibration from X^ϵ to $X^{\epsilon_{max}}$ with $\epsilon < \epsilon_{max}$ big enough. Moreover, the general fibers of this fibration is a horospherical variety and can be described.

All morphisms above are G-equivariant and images of any G-orbit can be described as follows. To a face of Q^ϵ (or $\tilde{Q}^\epsilon$) we can associate the maximal set of rows for which equality holds for any element x of the face (in the inequalities $Ax \geq B + \epsilon C$). And similarly to a set of rows we can also naturally associate a face of Q^ϵ (may be empty). For any ϵ and $\epsilon_{i,k}$ as above, for any face F^ϵ of $\tilde{Q}^\epsilon$, we construct a face of $\tilde{Q}^{\epsilon_{i,k}}$ by taking the maximal set of rows associated to F^ϵ and then the face $F^{\epsilon_{i,k}}$ associated to these rows. Then, since there is a morphism ϕ from X^ϵ to $X^{\epsilon_{i,k}}$, the non-empty face $F^{\epsilon_{i,k}}$ corresponds to the G-orbit image by ϕ of the G-orbit corresponding to F^ϵ.

3 First combinatorial classification and first geometric description

3.1 Reduction to three cases

In this section, we only use Luna-Vust theory and Corollary 2.7 to reduce to the three main cases of Theorem 1.1.

Lemma 3.1. *Let X be a smooth projective horospherical variety with Picard group $\mathbb{Z}^2$. Then one the three following cases occurs (with notation of Section 2).*

Case (0): $n = 0$, $|\mathcal{R}| = 2$, $\mathcal{F}_X = \emptyset$, *and* $X = G/P$.

Case (1): $n \geq 1$, $\mathcal{R} = \mathcal{F}_X \sqcup \{\beta\}$, *there exist a basis* $(e_1, \dots, e_n)$ *of* N *and* n *integers* $0 \leq a_1 \leq \cdots \leq a_n$ *such that* σ *induces an injective map* $\tilde{\sigma}$ *from* $\mathcal{F}_X$ *to* $\{e_1, \dots, e_n, e_0 := -e_1 - \cdots - e_n\}$, $\sigma(\beta) = a_1 e_1 + \cdots + a_n e_n$ *and*

$$\mathbb{F}_X = \{(\mathcal{C}_I, \mathcal{F}_I) \mid I \subsetneq \{0, \dots, n\}\}$$

where $\mathcal{C}_I$ *is the cone generated by the* e_i*'s with* $i \in I$, *and* $\mathcal{F}_I = \tilde{\sigma}^{-1}(\{e_i \mid i \in I\})$.

Case (2): $n \geq 2$, $\mathcal{R} = \mathcal{F}_X$, *there exist integers* $r \geq 1$, $s \geq 1$, $0 \leq a_1 \leq \cdots \leq a_r$ *and a basis* $(u_1, \dots, u_r, v_1, \dots, v_s)$ *of* N *such that* σ *induces an injective map* $\tilde{\sigma}$ *from* $\mathcal{F}_X = \mathcal{R}$ *to*

$\{u_0, \dots, u_r, v_1, \dots, v_{s+1}\}$, *with* $u_0 := -u_1 - \cdots - u_r$ *and* $v_{s+1} := a_1u_1 + \cdots + a_ru_r - v_1 - \cdots - u_s$, *and*

$$\mathbb{F}_X = \{(\mathcal{C}_{I,J}, \mathcal{F}_{I,J}) \mid I \subsetneq \{0, \dots, r\} \text{ and } J \subsetneq \{1, \dots, s+1\}\}$$

where $\mathcal{C}_{I,J}$ *is the cone generated by the* u_i*'s with* $i \in I$ *and the* v_j*'s with* $j \in J$*, and* $\mathcal{F}_{I,J} = \tilde{\sigma}^{-1}(\{u_i \mid i \in I\} \cup \{v_j \mid j \in J\})$.

Proof. By Corollary 2.7, the map σ induces an injective map from $\mathcal{F}_X$ to $\mathbb{F}_X(1)$ and the Picard number of X is $\rho_X = (|\mathbb{F}_X(1)| - n) + |\mathcal{R}\backslash\mathcal{F}_X|$. But, since X and then $\mathbb{F}_X$ is complete, $|\mathbb{F}_X(1)| - n \geq 0$ with equality if and only if $n = 0$. (And $|\mathcal{R}\backslash\mathcal{F}_X| \geq 0$.) Thus, since $\rho_X = 2$ we distinguish three distinct cases:

Case (0): $n = 0$ and $|\mathcal{R}\backslash\mathcal{F}_X| = 2$;

Case (1): $|\mathbb{F}_X(1)| = n + 1$ and $|\mathcal{R}\backslash\mathcal{F}_X| = 1$;

Case (2): $|\mathbb{F}_X(1)| = n + 2$ and $|\mathcal{R}\backslash\mathcal{F}_X| = 0$.

We now detail each case.

Case (0): In the case where $n = 0$, X is the complete homogeneous variety G/P (and $\mathcal{F}_X = \emptyset$). And then $|\mathcal{R}| = 2$.

Case (1): Consider the fan $\tilde{\mathbb{F}} := \{\mathcal{C} \mid (\mathcal{C}, \mathcal{F}) \in \mathbb{F}_X\}$ associated to the colored fan $\mathbb{F}_X$ (in fact it is the fan of the toric fiber Y of the toroidal variety $\tilde{X} := G \times^P Y$ obtained from X by erasing all colors of X). Since X is locally factorial, the fan $\tilde{\mathbb{F}}$ is the fan of a smooth toric variety of Picard number 1 (because $|\tilde{\mathbb{F}}_X(1)| = n + 1$). Then it is well-known that such a fan is the fan of the projective space $\mathbb{P}^n$. In particular, there exists a basis $(e_1, \dots, e_n)$ of N such that $\tilde{\mathbb{F}} = \{\mathcal{C}_I \mid I \subsetneq \{0, \dots, n\}\}$ where $e_0 := -e_1 - \cdots - e_n$ and $\mathcal{C}_I$ is the cone generated by the e_i with $i \in I$.

Denote by β the unique element of $\mathcal{R}\backslash\mathcal{F}_X$. Then, up to reordering the e_i's (for $i \in \{0, \dots, n\}$), we can suppose that $\sigma(\beta)$ is in $\mathcal{C}_{\{1,\dots,n\}}$ and equals $a_1e_1 + \cdots + a_ne_n$ with $0 \leq a_1 \leq \cdots \leq a_n$.

Case (2): As above, consider the fan $\tilde{\mathbb{F}}$. Since X is locally factorial, it is the fan of a smooth toric variety of Picard number 2 (because $|\tilde{\mathbb{F}}_X(1)| = n+2$). Then, by [Kle88, Theorem 1], there exist integers $r \geq 1$, $s \geq 1$, $0 \leq a_1 \leq \cdots \leq a_r$ and a basis $(u_1, \dots, u_r, v_1, \dots, v_s)$ of N such that $\tilde{\mathbb{F}} = \{\mathcal{C}_{I,J} \mid I \subsetneq \{0, \dots, r\} \text{ and } J \subsetneq \{1, \dots, s+1\}\}$, where $u_0 := -u_1 - \cdots - u_r$, $v_{s+1} := a_1u_1 + \cdots + a_ru_r - v_1 - \cdots - u_s$ and $\mathcal{C}_{I,J}$ is the cone generated by the u_i's with $i \in I$ and the v_j's with $j \in J$.

We conclude by the following facts: for any $\alpha \in \mathcal{F}_X$ and for any $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}_X$, we have $\alpha \in \mathcal{F}$ if and only if $\sigma(\alpha) \in \mathcal{C}$; and for any $\alpha \in \mathcal{F}_X$, $\sigma(\alpha)$ is the primitive element of an edge of $\mathbb{F}_X$ (using again Corollary 2.7).

□

Remark 3.2. In section 5, we will use the MMP or the Log MMP to study and compare geometrically all these varieties X. We can already describe some Mori fibrations from these varieties.

Case (0): If X is a complete homogeneous variety G/P of Picard group $\mathbb{Z}^2$, then the MMP gives two Mori fibrations from X to the complete homogeneous varieties G/P_1 and G/P_2 of Picard group $\mathbb{Z}$, where P_1 and P_2 are the maximal proper parabolic subgroups of G containing B such that $P = P_1 \cap P_2$. Note moreover that G/P is a product if and only if $\mathrm{Aut}(G/P)$ is not simple.

Case (1): There exists a G-equivariant morphism ψ from X to $G/P(\varpi_\beta)$. Note that the general fiber of ψ is smooth of Picard group $\mathbb{Z}$ (in particular, it is homogeneous or one of the two-orbit varieties described in [Pas09]).

Case (2): Let P_Z be the parabolic subgroup containing B (and P) such that $\mathcal{R}_Z := \mathcal{R}_{P_Z} = \tilde{\sigma}^{-1}(\{v_j \mid j \in \{1,\dots,s+1\}\})$. Let M_Z be the sublattice of M orthogonal to $\mathbb{Z}u_1 \oplus \cdots \oplus \mathbb{Z}u_r \subset N$. The pair (P_Z, M_Z) corresponds to a horospherical homogeneous space G/H_Z with H_Z containing H. Also the dual lattice N_Z of M_Z is the image of the projection from N to $\mathbb{Z}u_1 \oplus \cdots \oplus \mathbb{Z}u_r$. We denote by $v_{1,Z}, \dots, v_{s+1,Z}$ the images of $v_1, \dots, v_{s+1}$ in N_Z, in particular $v_{s+1,Z} = -v_{1,Z} - \cdots - v_{s,Z}$. And finally we denote by $\mathbb{F}_Z$ the colored fan $\{\mathcal{C}_{J,Z}, \mathcal{F}_{J,Z}) \mid J \subsetneq \{1,\dots,s\}\}$ where $\mathcal{C}_{J,Z}$ is the cone generated by the $v_{j,Z}$ with $j \in J$, and $\mathcal{F}_{J,Z} = \tilde{\sigma}^{-1}(\{v_j \mid j \in J\})$. The colored fan $\mathbb{F}_Z$ corresponds to a G/H_Z-embedding Z. Moreover, we have a G-equivariant morphism ψ from X to Z, it is a Mori fibration. Note that Z and the general fiber of ψ are smooth horospherical varieties of Picard group $\mathbb{Z}$ (in particular, they are homogeneous or one of the two-orbit varieties described in [Pas09]).

3.2 Description via polytopes

We now describe X embedded in the projectivization of a G-module, by choosing the smallest ample Cartier divisor of X and by applying Corollary 2.6. We first study the nef cone of X, which is 2-dimensional.

Recall that any Cartier divisor of X is linearly equivalent to a B-stable divisor, and any prime G-stable divisor corresponds to an edge of $\mathbb{F}_X$ that is not generated by some $\sigma(\alpha)$ with $\alpha \in \mathcal{F}_X$, and any other prime B-stable divisor is a color of G/H. Then in Cases (1) and (2), we have $n+2$ prime B-stable divisors that we can denote naturally as follows:

Case (1): $D_{n+1} = D_\beta$; for any $i \in \{0,\dots,n\}$, D_i is the B-stable divisor corresponding to the edge generated by e_i (which equals D_α with $\alpha \in \mathcal{F}_X = \mathcal{R}\backslash\{\beta\}$ if and only if the edge is generated by $\sigma(\alpha)$, and which is G-stable if not).

Case (2): for any $i \in \{0,\dots,r\}$, D_i is the B-stable divisor corresponding to the edge generated by u_i; and for any $j \in \{1,\dots,s+1\}$, D_{j+r} is the B-stable divisor corresponding to the edge generated by v_i (which equals D_α with $\alpha \in \mathcal{F}_X = \mathcal{R}$ if and only if the edge is generated by $\sigma(\alpha)$, and which is G-stable if not).

Proposition 3.3. *In both cases (1) and (2), the nef cone of X is generated by D_0 and D_{n+1}. In particular, $D_0 + D_{n+1}$ is ample. Moreover (D_0, D_{n+1}) is a basis of* $\mathrm{Pic}(X)$.

Proof. By Theorem 2.4, we prove that D_0 and D_{n+1} are globally generated but not ample. We also check that for any a and b in $\mathbb{Q}$, $aD_0 + bD_{n+1}$ is Cartier if and only if a and b are integers. □

Before to apply Corollary 2.6, we reduce to the case where G is the product of simply connected simple groups and a torus, with the following lemma and remark.

Lemma 3.4. *[Pas06, proof of Proposition 3.10] We can suppose that G is the product of a semi-simple group with a torus by replacing G by the product of its semi-simple part $G' := [G,G]$ and the torus $\mathbb{T} = P/H$. In particular, P is the product of a parabolic subgroup of G' with $\mathbb{T}$, and the characters of P are sums of weights of the maximal torus of G' and characters of $\mathbb{T}$. Hence a basis of $M \simeq \mathfrak{X}(\mathbb{T})$ is of the form $(\chi_i + \theta_i)_{i\in\{1,\dots,n\}}$ such that $(\chi_i)_{i\in\{1,\dots,n\}}$ form a basis of $M = \mathfrak{X}(\mathbb{T})$, and the θ_i's are weights of the maximal torus of G'.*

Remark 3.5. We can moreover assume G' to be the product of simply connected simple groups by taking the universal cover of G'.

With these assumptions, we get the following result.

Lemma 3.6. *The embbedding of X given by the ample Cartier $D_0 + D_{n+1}$ is:*

Case (1):

$$X \hookrightarrow \mathbb{P}(\bigoplus_{i=0}^{n} V(\chi_i + \varpi_i + (1+a_i)\varpi_\beta)),$$

where $\chi_0 = 0$, $\chi_1, \dots, \chi_n$ are characters of $\mathbb{T}$, and for any $i \in \{0, \dots, n\}$, ϖ_i is either ϖ_α if $e_i = \sigma(\alpha)$ with $\alpha \in \mathcal{F}_X$ or 0 if not.

Case (2):

$$X \hookrightarrow \mathbb{P}(\bigoplus_{i, b_1, \dots, b_{s+1}} V(\chi_i + \varpi_i + \sum_{j=1}^{s+1} b_j(\chi_{r+j} + \varpi_{r+j})),$$

where $\chi_0 = \chi_{n+1} = 0$, $\chi_1, \dots, \chi_n$ are characters of $\mathbb{T}$, and for any $i \in \{0, \dots, n+1\}$, ϖ_i is either ϖ_α if u_i or v_{i-r} is $\sigma(\alpha)$ with $\alpha \in \mathcal{F}_X$ or 0 if not; and where the sum is taken over all $s+2$-uplets of non-negative integers $(i, b_1, \dots, b_{s+1})$ such that $0 \leq i \leq r$ and $\sum_{j=1}^{s+1} b_j = 1 + a_i$ (with $a_0 := 0$).

Proof. In each case, we describe the pseudo-moment polytope of $(X, D_0 + D_{n+1})$ in a particular basis of M and then the moment polytope of $(X, D_0 + D_{n+1})$. Then we use Corollary 2.6 to conclude.

Case (1): Consider the basis $(e_1^*, \dots, e_n^*)$ of M that is dual to the basis $(e_1, \dots, e_n)$ of N. By the previous lemma and the description of the images of colors, for any $i \in \{1, \dots, n\}$, the element e_i^* is of the form $\chi_i + \varpi_i - \varpi_0 + a_i\varpi_\beta$, where $\chi_1, \dots, \chi_n$ are characters of $\mathbb{T}$ and for any $i \in \{0, \dots, n\}$, ϖ_i is either ϖ_α if $e_i = \sigma(\alpha)$ with $\alpha \in \mathcal{F}_X$ or 0 if not.

The pseudo-moment polytope of $(X, D_0 + D_{n+1})$ is the simplex with vertices 0, $e_1^*, \dots, e_n^*$. The weight of the canonical section of $D_0 + D_{n+1}$ is $\varpi_0 + \varpi_\beta$, where ϖ_0 is either ϖ_α if $e_0 = \sigma(\alpha)$ with $\alpha \in \mathcal{F}_X$ or 0 if not.

Hence, the moment polytope of $(X, D_0 + D_{n+1})$ is the simplex with vertices $0 + \varpi_0 + \varpi_\beta = \chi_0 + \varpi_0 + (1+a_0)\varpi_\beta$ and $(\chi_i + \varpi_i - \varpi_0 + a_i\varpi_\beta) + (\varpi_0 + \varpi_\beta) = \chi_i + \varpi_i + (1+a_i)\varpi_\beta$ for any $i \in \{1, \dots, n\}$.

Case (2): Consider the basis $(u_1^*, \dots, u_r^*, v_1^*, \dots, v_s^*)$ of M that is dual to the basis $(u_1, \dots, u_r, v_1, \dots, v_s)$ of N. By the previous lemma and the description of the images of colors, for any $i \in \{1, \dots, r\}$ the element u_i^* is of the form $\chi_i + \varpi_i - \varpi_0 + a_i\varpi_{n+1}$ and for any $j \in \{1, \dots, s\}$ the element v_j^* is of the form $\chi_{r+j} + \varpi_{r+j} - \varpi_{n+1}$, where $\chi_1, \dots, \chi_n$ are characters of $\mathbb{T}$, and for any $i \in \{0, \dots, n+1\}$, ϖ_i is either ϖ_α if u_i (with $0 \leq i \leq r$) or v_{i-r} (with $r+1 \leq i \leq n+1$) is $\sigma(\alpha)$ with $\alpha \in \mathcal{F}_X$ or 0 if not.

The pseudo-moment polytope of $(X, D_0 + D_{n+1})$ is the polytope with the following vertices: 0, $u_1^*, \dots, u_r^*$, $v_1^*, \dots, v_s^*$ and $u_i^* + (a_i + 1)v_j^*$ for any $1 \leq i \leq r$ and for any $1 \leq j \leq s$. Note that the lattice points of this polytope are exactly 0, $v_1^*, \dots, v_s^*$ and for any $1 \leq i \leq r$ all the points $u_i^* + \sum_{j=1}^{s} b_j v_j^*$ where the b_j's are non-negative integers such that $\sum_{j=1}^{s} b_j \leq a_i + 1$. Moreover, the weight of the canonical section of $D_0 + D_{n+1}$ is $\varpi_0 + \varpi_{n+1}$, where ϖ_0 (respectively ϖ_{n+1}) is either ϖ_α if u_0 (respectively v_{s+1}) equals $\sigma(\alpha)$ with $\alpha \in \mathcal{F}_X$ or 0 if not.

Hence, the moment polytope of $(X, D_0 + D_{n+1})$ is the polytope with vertices $0 + \varpi_0 + \varpi_{n+1} = \chi_0 + \varpi_0 + (1+a_0)(\chi_{n+1} + \varpi_{n+1})$; for any $i \in \{1, \dots, r\}$, $\chi_i + \varpi_i + (a_i + 1)(\chi_{n+1} + \varpi_{n+1})$; for any $j \in \{1, \dots, s\}$, $\chi_0 + \varpi_0 + \chi_{r+j} + \varpi_{r+j}$; and for any $1 \leq i \leq r$, for any $1 \leq j \leq s$, $\chi_i + \varpi_i - \varpi_0 + a_i\varpi_{n+1} + (a_i+1)(\chi_{r+j} + \varpi_{r+j} - \varpi_{n+1}) + \varpi_0 + \varpi_{n+1} = \chi_i + \varpi_i + (a_i + 1)(\chi_{r+j} + \varpi_{r+j})$.

In particular, the lattice points of the pseudo-moment polytope translated by $\varpi_0 + \varpi_{n+1}$ are exactly the $\chi_i + \varpi_i + \sum_{j=1}^{s+1} b_j(\chi_{r+j} + \varpi_{r+j})$ where the sum is taken over all $s+2$-uplets of non-negative integers $(i, b_1, \dots, b_{s+1})$ such that $0 \leq i \leq r$ and $\sum_{j=1}^{s+1} b_j = 1 + a_i$.

□

Recall that, by lemma 3.4, $(\chi_1, \dots, \chi_n)$ is a basis of $\mathfrak{X}(\mathbb{T})$. Hence, there exists a subtorus $\mathbb{S}$ of $\mathbb{T}$ such that: $(\chi_{i|\mathbb{S}})_{i \in \{1,\dots,n\},\, \varpi_i = 0}$ is a basis of $\mathfrak{X}(\mathbb{S})$, and for any $i \in \{1, \dots, n\}$ such that $\varpi_i \neq 0$, we have $\chi_{i|\mathbb{S}} = 0$.

Lemma 3.7. *In both cases, X is also a horospherical $G' \times \mathbb{S}$-variety.*

Proof. Consider Case (1). For any $i \in \{1, \dots, n\}$ such that $\varpi_i \neq 0$, the G-orbit and the $G' \times \mathbb{S}$-orbit of the highest weight vector $v_{\chi_i + \varpi_i + (1+a_i)\varpi_\beta}$ in $V_G(\chi_i + \varpi_i + (1+a_i)\varpi_\beta) \simeq V_{G' \times \mathbb{S}}(\chi_i + \varpi_i + (1+a_i)\varpi_\beta) = V_{G' \times \mathbb{S}}(\varpi_i + (1+a_i)\varpi_\beta)$ are equal.

Case (2) is similar. □

We can replace $\chi_i + \varpi_i$ with ϖ_{α_i} such that

- if $\chi_{i|\mathbb{S}} = 0$ and $\varpi_i \neq 0$, α_i is a simple root of G' (that is supposed to be a product of simply connected simple groups);
- $\mathbb{S}$ is a product of $\mathbb{C}^*$'s whose imaginary simple roots are the α_i's with i such that $\chi_{i|\mathbb{S}} \neq 0$ and $\varpi_i = 0$;
- if $i = 0$ or $n+1$, $\chi_{i|\mathbb{S}} = 0$ and $\varpi_i = 0$, α_i is the imaginary root of $\{1\}$.

It finally gives the following proposition.

Proposition 3.8. *Let X be a smooth projective horospherical variety of Picard group $\mathbb{Z}^2$ as in Case (1) or (2). Then X is isomorphic to a smooth closure of a G-orbit of a sum of highest weight vectors as follows where G is the product $G_0 \times \cdots \times G_t$ of simply connected simple groups, $\mathbb{C}^*$ and $\{1\}$:*

Case (1):

$$\mathbb{P}(\bigoplus_{i=0}^{n} V(\varpi_{\alpha_i} + (1+a_i)\varpi_\beta)),$$

where $n \geq 1$;
β is a simple root of G_0;
$\alpha_0, \dots, \alpha_n$ are distinct simple roots (may be imaginary) of G distinct from β;
for any $k \in \{1, \dots, t\}$, $G_k = \{1\}$ if and only if $k = 1$ and α_0 is imaginary;
and $0 = a_0 \leq a_1 \leq \cdots \leq a_n$ are integers.

Case (2):

$$\mathbb{P}(\bigoplus_{i, b_1, \dots, b_{s+1}} V(\varpi_{\alpha_i} + \sum_{j=1}^{s+1} b_j(\varpi_{\alpha_{r+j}})),$$

where the sum is taken over all $s+2$-uplets of non-negative integers $(i, b_1, \dots, b_{s+1})$ such that $0 \leq i \leq r$ and $\sum_{j=1}^{s+1} b_j = 1 + a_i$ (with $a_0 := 0$);
$r \geq 1$, $s \geq 1$ and $r + s = n$;
$\alpha_0, \dots, \alpha_{n+1}$ are distinct simple roots (may be imaginary) of G;
for any $k \in \{0, \dots, t\}$, $G_k = \{1\}$ if and only if, $k = 0$ and α_0 is imaginary, or $k = t$ and α_{n+1} is imaginary;
and $0 = a_0 \leq a_1 \leq \cdots \leq a_r$ are integers.

Note that the two cases in Proposition 3.8, with $s = 1$ in Case (2), are similar to the ones of Theorem 1.1.

4 Reduction to the cases of Theorem 1.1

4.1 Smooth horospherical varieties and G-modules

To prove Theorem 1.1 from Proposition 3.8, we glue together G-modules as soon as we can, in order to enlarge the group G and reduce to "smaller" cases. For this, we first need to apply the smooth criterion to X (Theorem 2.9), which comes from the fact that smooth horospherical G-modules are the $\mathbb{C}^*$-modules $\mathbb{C}$, the SL_d-modules $V(\varpi_1) = \mathbb{C}^d$ and Sp_d-modules (with d even) $V(\varpi_1) = \mathbb{C}^d$. And then we use easy facts as "the $\mathrm{SL}_d \times \mathrm{SL}_e$-module $\mathbb{C}^d \oplus \mathbb{C}^e$ is isomorphic to the SL_{d+e}-module $\mathbb{C}^{d+e}$".

As in [Pas09, Theorem 1.7], the smooth criterion reveals 8 configurations including the 5 configurations that give the five families of horospherical two-orbit varieties corresponding to non-homogeneous smooth projective horospherical varieties of Picard group $\mathbb{Z}$. We recall these 8 configurations in the following definition.

Definition 4.1. Let K be a simple algebraic group over $\mathbb{C}$ and let γ, δ be two simple roots of K.

The triple (K, γ, δ) is said to be smooth if (type of K, γ, δ) is one of the following 8 cases, up to exchanging γ and δ (with the notation of Bourbaki [Bou75]).

1. $(A_m, \alpha_1, \alpha_m)$, with $m \geq 2$
2. $(A_m, \alpha_i, \alpha_{i+1})$, with $m \geq 3$ and $i \in \{1, \dots, m-1\}$
3. $(B_m, \alpha_{m-1}, \alpha_m)$, with $m \geq 3$
4. $(B_3, \alpha_1, \alpha_3)$
5. $(C_m, \alpha_i, \alpha_{i+1})$ with $m \geq 2$ and $i \in \{1, \dots, m-1\}$
6. $(D_m, \alpha_{m-1}, \alpha_m)$, with $m \geq 4$
7. $(F_4, \alpha_2, \alpha_3)$
8. $(G_2, \alpha_1, \alpha_2)$

We say that (type of K, γ, δ) is smooth of two-orbit type if it is one of the cases 3, 4, 5, 7 or 8 above.

Here we also need to introduce another "smooth object" (only used in Case (1)).

Definition 4.2. Let K be a simple algebraic group over $\mathbb{C}$ and let β be a simple root of K and let R be a subset of simple roots of K distinct from β. Let n be a non-negative integer. Denote by L a Levi subgroup of the maximal parabolic subgroup $P(\varpi_\beta)$, then the semi-simple part of L is a quotient by a finite central group of a product of simple groups $L^1, \dots, L^q$ (with $q \geq 0$).

The quadruple (K, β, R, n) is said to be smooth if

1. $n = 1$, $R = \{\gamma, \delta\}$ such that γ and δ are simple root of the same L^k so that the triple (L^k, γ, δ) is smooth;
2. or for any $k \in \{1, \dots, q\}$, at most one simple root of L^k is in R, and if $\gamma \in R$ is a simple root of L_k, then L_k is of type A or C and γ is a short extremal simple root of L_k.

We can list all smooth quadruple (K, β, R, n) (see the appendix). We remark, in particular, that R is at most of cardinality 3.

We obtain the following result by applying the smooth criterion to the smooth projective horospherical variety X with Picard group $\mathbb{Z}^2$ in both cases (1) and (2). Here, we suppose that X is as in Proposition 3.8, and in Case (1) we suppose that β is root of G_0.

Lemma 4.3.

Case (1): The quadruple (G_0, β, R_0, n) is smooth.

If there exist $0 \leq i < j \leq n$ such that α_i and α_j are simple roots of the same simple group G_k with $k \in \{1, \dots, t\}$ then $n = 1$, $i = 0$ and $j = 1$ (also $t = k = 1$). Moreover in that case, the triple $(G_k, \alpha_i, \alpha_j)$ is smooth.

If not, for any $i \in \{0, \dots, n\}$, the simple root α_i is either imaginary or in G_0 or the short extremal simple root of one of a simple group G_k with $k \in \{1, \dots, t\}$ that is of type A or C.

Case (2): If there exist $0 \leq i < j \leq n + 1$ such that α_i and α_j are simple roots of the same simple group G_k with $k \in \{0, \dots, t\}$ then either $r = 1$, $i = 0$ and $j = 1$, or $s = 1$, $i = n$ and $j = n + 1$. Moreover in that case, the triple $(G_k, \alpha_i, \alpha_j)$ is smooth.

For any $i \in \{0, \dots, n\}$, such that the simple root α_i is the unique α_j of a simple group G_k with $k \in \{0, \dots, t\}$. Then α_i is either imaginary or the short extremal simple root of one of G_k that is of type A or C.

Proof.

Case (1): With notation of Definition 4.2, suppose γ and δ are two simple roots of the same L^j. If $n > 1$, then there exists a maximal colored cone of $\mathbb{F}_X$ that contains $\gamma_M^\vee$ and $\delta_M^\vee$. By applying Theorem 2.9, we get a contradiction. Then $n = 1$ and applying Theorem 2.9 to the two one-dimensional colored cones of $\mathbb{F}_X$, we prove that the couples $(R_0 \backslash \{\beta, \delta\}, \gamma)$ and $(R_0 \backslash \{\beta, \gamma\}, \delta)$ are smooth, so that (L^j, γ, δ) is smooth.

Suppose that α is the unique simple root of L^j in R_0. By applying Theorem 2.9 to the colored cone $(\mathbb{Q}_{\geq 0}\alpha_M^\vee, \{\alpha\})$ we get that L^j is of type A or C and α is a short extremal simple root of L^j. It finishes the proof of the smoothness of (G_0, β, R_0, n).

If there exist $0 \leq i < j \leq n$ such that α_i and α_j are simple roots of the same simple group G_k with $k \in \{1, \dots, t\}$ then as above Theorem 2.9 implies that $n = 1$ and $(G_k, \alpha_i, \alpha_j)$ is smooth. The fact that $i = 0$, $j = 1$ and $t = k = 1$ is obvious.

Now, let $i \in \{0, \dots, n\}$ such that the simple root α_i is the unique α_j of a simple group G_k with $k \in \{1, \dots, t\}$ and suppose that α_i is not imaginary. Apply again Theorem 2.9 to the colored cone $(\mathbb{Q}_{\geq 0}\alpha_M^\vee, \{\alpha\})$ to get that α_i is the short extremal simple root G_k with $k \in \{1, \dots, t\}$ that is of type A or C. It finishes the proof of the lemma in Case (1).

Case (2): Suppose there exist $0 \leq i < j \leq n+1$ such that α_i and α_j are simple roots of the same simple group G_k with $k \in \{0, \dots, t\}$. Then Theorem 2.9 implies that $(G_k, \alpha_i, \alpha_j)$ is smooth. But it also gives a contradiction if there exists a maximal colored cone of $\mathbb{F}_X$ that contains $\alpha_{iM}^\vee$ and $\alpha_{jM}^\vee$. This contradiction occurs if and only if $0 \leq i \leq r$ and $r+1 \leq j \leq n+1$, or $0 \leq i, j \leq r$ and $r \geq 2$, or $r+1 \leq i, j \leq n+1$ and $s \geq 2$.

We conclude the proof of the lemma in Case (2) as in Case (1).

□

Now we list different ways to gather G-modules into a $\mathbb{G}$-module with $G \subset \mathbb{G}$.

Lemma 4.4. *Let $\tau \geq 1$. For $i \in \{1, \dots, \tau\}$, let G_i be $\mathbb{C}^*$, SL_{d_i} (with $d_i \geq 2$) or Sp_{d_i} (with $d_i \geq 2$ even). If $G_i = \mathbb{C}^*$ set $d_i = 1$ and ϖ_1^i the identity character of $\mathbb{C}^*$. If not, set ϖ_1^i the first fundamental weight of G_i. Let $G = G_1 \times \cdots \times G_\tau$.*

(a) Let $\mathbb{G} = \mathrm{SL}_d$ where $d = d_1 + \cdots + d_\tau$.
Then $V_{\mathbb{G}}(\varpi_1) = \bigoplus_{i=1}^\tau V_G(\varpi_1^i)$ and $G \cdot \left(\sum_{i=1}^\tau v_{\varpi_1}^i\right) \subset \mathbb{G} \cdot v_{\varpi_1}$.

(b) Let $\mathbb{G} = \mathrm{SL}_d$ where $d = d_1 + \cdots + d_\tau + 1$.
Then $V_{\mathbb{G}}(\varpi_1) = V_G(0) \oplus \bigoplus_{i=1}^\tau V_G(\varpi_1^i)$ and $G \cdot \left(1 + \sum_{i=1}^\tau v_{\varpi_1}^i\right) \subset \mathbb{G} \cdot v_{\varpi_1}$, where 1 is the unit in the trivial G-module $V_G(0) = \mathbb{C}$.

With notation of Bourbaki [Bou75] (we put primes to write differently fundamental weights of $\mathbb{G}$ from those of G).

(c) Let $G = \mathrm{SL}_d$ (with $d \geq 3$) and $\mathbb{G} = \mathrm{SO}_{2d}$. Then $V_{\mathbb{G}}(\varpi_1') = V_G(\varpi_1) \oplus V_G(\varpi_{d-1})$ and $G \cdot \left(v_{\varpi_1} + v_{\varpi_{d-1}}\right) \subset \mathbb{G} \cdot v_{\varpi_1'}$.

(d) Let $G = \mathrm{SL}_d$ (with $d \geq 4$), $\mathbb{G} = \mathrm{SL}_{d+1}$ and $1 \leq i \leq d-2$. Then $V_{\mathbb{G}}(\varpi_{i+1}') = V_G(\varpi_i) \oplus V_G(\varpi_{i+1})$ and $G \cdot \left(v_{\varpi_i} + v_{\varpi_{i+1}}\right) \subset \mathbb{G} \cdot v_{\varpi_{i+1}'}$.

(e) Let $G = \mathrm{Spin}_{2d}$ (with $d \geq 4$) and $\mathbb{G} = \mathrm{Spin}_{2d+1}$. Then $V_{\mathbb{G}}(\varpi_d') = V_G(\varpi_{d-1}) \oplus V_G(\varpi_d)$ and $G \cdot \left(v_{\varpi_{d-1}} + v_{\varpi_d}\right) \subset \mathbb{G} \cdot v_{\varpi_d'}$.

Moreover in each case, the projectivizations of the G-orbit and the $\mathbb{G}$-orbit have the same dimension, in particular the two projective varieties defined as the closure of these two orbits in the corresponding projective spaces are the same.

Remark 4.5. In the first case of Lemma 4.4, with $\tau = 1$ we have in particular that, for d even, $V_{\mathrm{Sp}_d}(\varpi_1) = V_{\mathrm{SL}_d}(\varpi_1)$. Note also that $\mathrm{Sp}_d / P(\varpi_1) = \mathrm{SL}_d / P(\varpi_1) (= \mathbb{P}^{d-1})$

Proof. The first two items are easy and left to the reader.
The last three items are given in [Pas09, Propositions 1.8, 1.9 and 1.10]. □

In Case (2), we need the following generalization of Lemma 4.4.

Lemma 4.6. *Let $a \in \mathbb{N}^*$.*

Let $\tau \geq 0$. For $i \in \{0, \dots, \tau\}$, let G_i be $\mathbb{C}^*$, SL_{d_i} (with $d_i \geq 2$) or Sp_{d_i} (with $d_i \geq 2$ even). If $G_i = \mathbb{C}^*$ set $d_i = 1$ and ϖ_1^i the identity character of $\mathbb{C}^*$. Else set ϖ_1^i the first fundamental weight of G_i. Let $G = G_0 \times \cdots \times G_\tau$.

(a) Let $\mathbb{G} = \mathrm{SL}_d$ where $d = d_0 + \cdots + d_\tau$. Then

$$V_{\mathbb{G}}(a\varpi_1) = \bigoplus_{b_0,\dots,b_\tau} V_G(\sum_{i=0}^{\tau} b_i \varpi_1^i),$$

where the sum is taken over all $(\tau+1)$-uplets of non-negative integers $(b_0, \dots, b_\tau)$ such that $\sum_{i=0}^{\tau} b_i = a$. And

$$G \cdot \left(\sum_{b_0,\dots,b_\tau} v_{\sum_{i=0}^{\tau} b_i \varpi_1^i} \right) \subset \mathbb{G} \cdot v_{a\varpi_1}.$$

(b) Let $\mathbb{G} = \mathrm{SL}_d$ where $d = d_0 + \cdots + d_\tau + 1$. Then

$$V_{\mathbb{G}}(a\varpi_1) = \bigoplus_{b_1,\dots,b_\tau} V_G(\sum_{i=0}^{\tau} b_i \varpi_1^i),$$

where the sum is taken over all $(\tau+1)$-uplets of non-negative integers $(b_0, \dots, b_\tau)$ such that $\sum_{i=0}^{\tau} b_i \leq a$. And

$$G \cdot \left(\sum_{b_1,\dots,b_\tau} v_{\sum_{i=0}^{\tau} b_i \varpi_1^i} \right) \subset \mathbb{G} \cdot v_{a\varpi_1}.$$

With notation of Bourbaki [Bou75] (we put primes to write differently fundamental weights of $\mathbb{G}$ from those of G).

(c) Let $G = \mathrm{SL}_d$ (with $d \geq 3$) and $\mathbb{G} = \mathrm{SO}_{2d}$. Then

$$V_{\mathbb{G}}(a\varpi_1') = \bigoplus_{b=0}^{a} V_G(b\varpi_1 + (a-b)\varpi_{d-1}) \text{ and } G \cdot \left(\sum_{b=0}^{a} v_{b\varpi_1 + (a-b)\varpi_{d-1}} \right) \subset \mathbb{G} \cdot v_{a\varpi_1'}.$$

(d) Let $G = \mathrm{SL}_d$ (with $d \geq 4$), $\mathbb{G} = \mathrm{SL}_{d+1}$ and $1 \leq i \leq d-2$. Then

$$V_{\mathbb{G}}(a\varpi_{i+1}') = \bigoplus_{b=0}^{a} V_G(b\varpi_i + (b-a)\varpi_{i+1}) \text{ and } G \cdot \left(\sum_{b=0}^{a} v_{b\varpi_i + (a-b)\varpi_{i+1}} \right) \subset \mathbb{G} \cdot v_{a\varpi_{i+1}'}.$$

(e) Let $G = \mathrm{Spin}_{2d}$ (with $d \geq 4$) and $\mathbb{G} = \mathrm{Spin}_{2d+1}$. Then

$$V_{\mathbb{G}}(a\varpi_d') = \bigoplus_{b=0}^{a} V_G(b\varpi_{d-1} + (b-a)\varpi_d) \text{ and } G \cdot \left(\sum_{b=0}^{a} v_{b\varpi_{d-1} + (b-a)\varpi_d} \right) \subset \mathbb{G} \cdot v_{a\varpi_d'}.$$

Moreover in each case, the projectivizations the G-orbit and the $\mathbb{G}$-orbit have the same dimension, in particular the two projective varieties defined as the closure of these two orbits in the corresponding projective spaces are the same.

Proof. Remark that for $a = 1$ the lemma is Lemma 4.4. For any $a \geq 1$, we denote by V_a the G-module that we consider in each case.

Consider the horospherical G-variety X defined as the closure of the G-orbit of a sum x_1 of highest weight vectors in $\mathbb{P}(V_1)$: it is a smooth projective variety with Picard group $\mathbb{Z}$ (it is isomorphic to $\mathbb{P}^{d-1}$, $\mathbb{P}^{d-1}$, Q^{2d-2}, $\mathrm{Gr}(i+1, d+1)$, $\mathrm{Spin}(2d+1)/P(\varpi_d)$ respectively). Moreover V_1^* is the G-module of global sections of $\mathcal{O}_X(1)$. And, for any $a \geq 1$, the G-module V_a^* is the set of global sections of $\mathcal{O}_X(a)$. But, in each case, X is also a homogeneous projective $\mathbb{G}$-variety $\mathbb{G}/P(\varpi)$ (with $\varpi = \varpi_1$ ϖ_1, ϖ_1', ϖ_{i+1}' and ϖ_d' respectively) by Lemma 4.4, then V_a is also the irreducible $\mathbb{G}$-module $V_{\mathbb{G}}(a\varpi)$.

Also, the image of x_1 in $\mathbb{P}(V_a)$ is the projectivization of a highest weight vector of weight in $V_{\mathbb{G}}(a\varpi)$ for a good choice of a Borel subgroup of $\mathbb{G}$ (because $\mathbb{G} \cdot x_1$ is the homogeneous projective $\mathbb{G}$-variety $\mathbb{G}/P(\varpi)$). □

4.2 Proof of Theorem 1.1 in Case (1)

A first part is already proved by Proposition 3.8 and Lemma 4.3, in particular X is embedded as the closure of the G-orbit of a sum of highest weight vectors in

$$\mathbb{P} := \mathbb{P}\left(\bigoplus_{i=0}^{n} V(\varpi_{\alpha_i} + (1+a_i)\varpi_\beta)\right).$$

It remains to prove that we can suppose that

- G_0 is the universal cover of the automorphism group of $G_0/P(\varpi_\beta)$ if R_0 is empty;
- if $i < j$ and $a_i = a_j$ then $\alpha_j \in R_0$;
- and some groups G_k of type C can be replaced by groups of type A.

•. If R_0 is empty and G_0 is not the universal cover of the automorphism group of $G_0/P(\varpi_\beta)$, then $G_0/P(\varpi_\beta)$ is isomorphic to $G_0'/P(\varpi_{\beta'})$ where G_0' is the universal cover of $\mathrm{Aut}(G_0/P(\varpi_\beta))$ and $(G_0, \beta, G_0', \beta')$ is one of the following: $(\mathrm{Sp}_{2m}, \varpi_1, \mathrm{SL}_{2m}, \varpi_1)$, $(G_2, \varpi_1, \mathrm{Spin}_7, \varpi_1)$, or $(\mathrm{Spin}_{2m+1}, \varpi_m, \mathrm{Spin}_{2m+2}, \varpi_m$ or $\varpi_{m+1})$. In any case, $V_{G_0}(\varpi_\beta) \simeq V_{G_0'}(\varpi_{\beta'})$ and $G_0 \cdot v_{\varpi_\beta} \simeq G_0' \cdot v_{\varpi_{\beta'}}$. Hence, the fact that R_0 is empty implies that $\bigoplus_{i=0}^{n} V_G(\varpi_{\alpha_i} + (1+a_i)\varpi_\beta) \simeq \bigoplus_{i=0}^{n} V_{\mathbb{G}}(\varpi_{\alpha_i} + (1+a_i)\varpi_{\beta'})$ where $\mathbb{G} = G_0' \times G_1 \times \cdots \times G_t$, and X is isomorphic to the closure of the $\mathbb{G}$-orbit of a sum of highest weight vectors in

$$\mathbb{P} := \mathbb{P}\left(\bigoplus_{i=0}^{n} V_{\mathbb{G}}(\varpi_{\alpha_i} + (1+a_i)\varpi_{\beta'})\right).$$

•. Suppose that there is $0 \leq i < j \leq n$ such that α_i and α_j are simple roots of the same simple group $G_1, \ldots, G_t$. Then by Lemma 4.3, we have $n = 1$, $i = 0$, $j = 1$ (also $t = 1$) and the triple $(G_1, \alpha_i, \alpha_j)$ is smooth. In particular, X is embedded as the closure of the G-orbit of a sum of highest weight vectors in

$$\mathbb{P}\left(V(\varpi_{\alpha_0} + \varpi_\beta) \oplus V(\varpi_{\alpha_1} + (1+a_1)\varpi_\beta)\right).$$

If $a_1 = 0$, the G-module $V(\varpi_{\alpha_0} + \varpi_\beta) \oplus V(\varpi_{\alpha_1} + (1+a_1)\varpi_\beta)$ is isomorphic to the tensor product of the G_0-module $V(\varpi_\beta)$ by the G_1-module $V(\varpi_{\alpha_0}) \oplus V(\varpi_{\alpha_1})$, so that X is the product of $G/P(\varpi_\beta)$ by the smooth projective horospherical variety of Picard group $\mathbb{Z}$ defined as the closure of the G_1-orbit of a sum of highest weight vectors in $\mathbb{P}(V(\varpi_{\alpha_0}) \oplus V(\varpi_{\alpha_1}))$.

We conclude that if X is not a product, X is as in Case (1a) (with $a_1 > 0$).

From now on, we suppose that there is no $0 \leq i < j \leq n$ such that α_i and α_j are simple roots of the same simple group $G_1, \ldots, G_t$.

•. Suppose that there exists $0 \leq i < j \leq n$ such that $a_i = a_j$ and both α_i and α_j are not simple roots of G_0.

Up to reordering, assume that α_i and α_j are simple roots of G_1 and G_2 ($t \geq 2$). Note that if $i = 0$ and α_0 is imaginary, $G_1 = \{1\}$. By Lemma 4.3, G_1 and G_2 are $\{1\}$, $\mathbb{C}^*$ ($d_k = 1$ in these two cases), SL_{d_k} (with $d_k \geq 2$) or Sp_{d_k} (with $d_k \geq 2$ even) and α_i, respectively α_j, is either an imaginary root or a short extremal root of G_1, respectively G_2.

Let $\mathbb{G} = G_0 \times G_3 \times \cdots \times G_t \times \mathrm{SL}_{d_1+d_2}$. By Lemma 4.4 ((a) if $i > 0$ or α_0 is not imaginary and (b) if not), the G-module $V(\varpi_{\alpha_i} + (1+a_i)\varpi_\beta) \oplus V(\varpi_{\alpha_j} + (1+a_j)\varpi_\beta)$ is isomorphic to the $\mathbb{G}$-module $V((1+a_i)\varpi_\beta) \otimes \mathbb{C}^{d_1+d_2}$. And X is a subvariety of the closure $\mathbb{X}$ of the $\mathbb{G}$-orbit of a sum of highest weight vectors in $\mathbb{P}$ under the action of $\mathbb{G}$.

We can now compare the dimension of the open $\mathbb{G}$-orbit $\Omega_{\mathbb{X}}$ of $\mathbb{X}$ with the dimension of the open G-orbit of X. Indeed $\Omega_{\mathbb{X}}$ is isomorphic to a horospherical homogeneous space of rank $n-1$ over $((G_0 \times G_3 \times \cdots \times G_t)/(P \cap G_0 G_3 \times \times \cdots \times G_t)) \times (\mathrm{SL}_{d_1+d_2}/P(\varpi_1))$, while G/H is of rank n over $((G_0 \times G_3 \times \cdots \times G_t)/P \cap (G_0 \times G_3 \times \cdots \times G_t)) \times ((G_1 \times G_2)/P \cap (G_1 \times G_2))$. But the dimension of $\mathrm{SL}_{d_1+d_2}/P(\varpi_1)$ is $d_1 + d_2 - 1$ while the dimension of $(G_1 \times G_2)/P \cap (G_1 \times G_2)$ is $(d_1 - 1) + (d_2 - 1)$. Hence $\Omega_{\mathbb{X}}$ and G/H have the same dimension, so that $X = \mathbb{X}$.

Then we can replace, without changing X, the product of the two simple groups corresponding to two simple roots α_i and α_j with $a_i = a_j$, with a unique simple group of type A. Note that n decrease by this change. (Also note that, if $i = 0$ and α_0 is imaginary then the new α_0 is not imaginary any more.)

With similar arguments, we can also replace any group $G_1, \ldots, G_t$, of type C and that contains a unique simple root α_i, by a group of type A.

• What we did just above also works in the cases where $n = 1$, $a_1 = 0$, α_0 and α_1 are simple roots of G_1 and G_2 (and $t = 2$). In that case, it proves that X is the closure of the $\mathrm{SL}_d \times G_0$-orbit of a highest weight vector in $\mathbb{P}\left(\mathbb{C}^d \bigotimes V(\varpi_\beta)\right)$. Hence, in that case, X is isomorphic to $\mathbb{P}^{d-1} \times G_0/P(\varpi_\beta)$.

Hence, we conclude the proof by iteration.

4.3 Proof of Theorem 1.1 in Case (2)

A first part is already proved by Proposition 3.8 and Lemma 4.3, in particular X is embedded as the closure of the G-orbit of a sum of highest weight vectors in

$$\mathbb{P} := \mathbb{P}\left(\bigoplus_{i,b_1,\ldots,b_{s+1}} V(\varpi_{\alpha_i} + \sum_{j=1}^{s+1} b_j \varpi_{\alpha_{r+j}})\right),$$

where the sum is taken over all $s+2$-uplets of non-negative integers $(i, b_1, \ldots, b_{s+1})$ such that $0 \leq i \leq r$ and $\sum_{j=1}^{s+1} b_j = 1 + a_i$.

It remains to prove that we can suppose that

- $s = 1$, α_n, α_{n+1} are both simple roots of G_t and $(G_t, \alpha_n, \alpha_{n+1})$ is smooth of two-orbit type;

– $0 < a_1 < \cdots < a_r$;

– and some groups G_k of type C can be replaced by groups of type A.

• Suppose first that $s > 1$, or $s = 1$ and α_n, α_{n+1} are not simple roots of the same simple group G_k. Up to reordering and applying Lemma 4.3, for any $j \in \{1, \dots, s\}$, α_{r+j} is either an imaginary root of G_{t-s+j} that is $\mathbb{C}^*$ or $\{1\}$, or a short extremal simple root of G_{t-s+j} that is of type A or C. Moreover, the simple groups $G_{t-s+1}, \dots, G_t$ contain no other α_i with $i \in \{0, \dots, r\}$. Also, $G_{t-s+j} = \{1\}$ if and only if $j = s$ and α_{r+s} is imaginary.

We now apply Lemma 4.6 ((a) if α_{r+s} is not imaginary and (b) if not). Hence, there exists $d \leq 2$ such that, with $G \subset \mathbb{G} := G_0 \times \cdots \times G_{t-s} \times \mathrm{SL}_d$, we have

$$\mathbb{P} = \mathbb{P}\left(\bigoplus_{i,b_1,\dots,b_{s+1}} V(\varpi_{\alpha_i}) \otimes V(\sum_{j=1}^{s+1} b_j \varpi_{\alpha_{r+j}})\right) = \mathbb{P}\left(\bigoplus_{i=0}^{r} V_{\mathbb{G}}(\varpi_{\alpha_i}) \otimes V_{\mathbb{G}}((1+a_i)\varpi_1)\right),$$

X is a subvariety of the closure $\mathbb{X}$ of the $\mathbb{G}$-orbit $\Omega_{\mathbb{X}}$ of a sum of highest weight vectors in $\mathbb{P}$, and $\dim((G_{t+1-s} \times \cdots \times G_t)/P \cap (G_{t+1-s} \times \cdots \times G_t) = d - s - 1$. In particular the dimension of $\Omega_{\mathbb{X}}$ (which is horospherical of rank r) equals the dimension of G/H. Hence, $X = \mathbb{X}$. Now remark that $\mathbb{X}$ is a horospherical variety as in Case (1) (case that we previously deal with).

• From now on, we suppose that $s = 1$ (and $n = r + 1$), and that α_n, α_{n+1} are both simple roots of G_t (up to reordering). In particular, X is embedded as the closure of the G-orbit of a sum of highest weight vectors in

$$\mathbb{P}\left(\bigoplus_{i=0}^{n-1}\bigoplus_{b=0}^{1+a_i} V(\varpi_{\alpha_i} + b\varpi_{\alpha_{r+1}} + (1 + a_i - b)\varpi_{\alpha_{r+2}})\right).$$

Note now that for any $k \in \{0, \dots, t\}$, $G_k = \{1\}$ if and only if $k = 0$ and α_0 is imaginary.

Recall that, by Lemma 4.3, $\alpha_0, \dots, \alpha_r$ are not simple roots of G_t and the triple $(G_t, \alpha_n, \alpha_{n+1})$ is smooth. Then X is embedded as the closure of the G-orbit of a sum of highest weight vectors in

$$\mathbb{P} := \mathbb{P}\left(\bigoplus_{i=0}^{n-1}\bigoplus_{b=0}^{1+a_i} V(\varpi_{\alpha_i}) \otimes V(b\varpi_{\alpha_{r+1}} + (1 + a_i - b)\varpi_{\alpha_{r+2}})\right).$$

If $(G_t, \alpha_n, \alpha_{n+1})$ is not of two-orbit type, we can apply Lemma 4.6 ((c), (d) or (e)) to get $G \subset \mathbb{G} := G_0 \times \cdots \times G_{t-1} \times \mathbb{G}_t$ such that $\mathbb{P} = \mathbb{P}\left(\bigoplus_{i=0}^{r} V_{\mathbb{G}}(\varpi_{\alpha_i}) \otimes V_{\mathbb{G}}((1+a_i)\varpi)\right)$, X is a subvariety of the closure $\mathbb{X}$ of the $\mathbb{G}$-orbit $\Omega_{\mathbb{X}}$ of a sum of highest weight vectors in $\mathbb{P}$, and $\dim(G_t/P \cap G_t) + 1 = \dim(\mathbb{G}_t/P(\varpi))$. In particular the dimension of $\Omega_{\mathbb{X}}$ (which is horospherical of rank r) equals the dimension of G/H. Hence, $X = \mathbb{X}$. And remark that $\mathbb{X}$ is a horospherical variety as in Case (1).

• Now suppose that $r > 1$, or $r = 1$ and α_0, α_1 are not simple roots of the same simple group.

Let $i \neq i'$ in $\{0, \dots, r\}$ such that $a_i = a_{i'}$. Up to reordering and applying Lemma. 4.3, α_i and $\alpha_{i'}$ are, imaginary or short extremal, simple roots respectively of G_0 and G_1 that are $\mathbb{C}^*$, $\{1\}$ or simple groups of type A or C. Moreover G_0 and G_1 contain no other α_k's.

We can apply Lemma 4.4 ((a) if $i > 0$ or α_0 is imaginary and (b) if not) to get $G \subset \mathbb{G} := \mathrm{SL}_d \times G_2 \cdots \times G_t$ such that

$$\mathbb{P} = \mathbb{P}\left(\left(\bigoplus_{k\neq i,\, i'} \bigoplus_{b=0}^{1+a_k} V_{\mathbb{G}}(\varpi_{\alpha_k}) \otimes V_{\mathbb{G}}(b\varpi_{\alpha_{r+1}} + (1+a_k-b)\varpi_{\alpha_{r+2}})\right) \oplus \left(\bigoplus_{b=0}^{1+a_i} V_{\mathbb{G}}(\varpi) \otimes V_{\mathbb{G}}(b\varpi_{\alpha_{r+1}} + (1+a_i-b)\varpi_{\alpha_{r+2}})\right)\right),$$

X is a subvariety of the closure $\mathbb{X}$ of the $\mathbb{G}$-orbit $\Omega_{\mathbb{X}}$ of a sum of highest weight vectors in $\mathbb{P}$, and $\dim((G_0 \times G_1)/P \cap (G_0 \times G_1)) + 1 = d - 1$. In particular the dimension of $\Omega_{\mathbb{X}}$ (which is horospherical of rank $(r-1)+1$) equals the dimension of G/H. Hence, $X = \mathbb{X}$. Now remark that $\mathbb{X}$ is either a horospherical variety as in Case (2) of rank one less than X, or a horospherical variety as in Case (1) if $r = 1$.

With similar argument, we can also replace any group $G_0, \ldots, G_{t-1}$, of type C and that contains a unique simple root α_i, by a group of type A.

• By iteration of the above process, we can now assume that $0 < a_1 < \cdots < a_r$, or that $r = 1$ (and $t = 1$) and α_0, α_1 are two simple roots of G_0. In the second case, note that by Lemma. 4.3, the triple $(G_0, \alpha_0, \alpha_1)$ is smooth.

Suppose $r = 1$, α_0, α_1 are two simple roots of G_0 and that $a_1 = a_0 = 0$. Then, X is the closure of the $G_0 \times G_1$-orbit of a sum of highest weight vectors in

$$\mathbb{P} = \mathbb{P}\left((V_{G_0}(\varpi_0) \oplus V_{G_0}(\varpi_1)) \otimes (V_{G_1}(\varpi_{\alpha_2}) \oplus V_{G_1}(\varpi_{\alpha_3}))\right).$$

Hence in that case, X is the product of two varieties: the closure of the G_0-orbit of a sum of highest weight vectors in $\mathbb{P}\left((V_{G_0}(\varpi_0) \oplus V_{G_0}(\varpi_1))\right)$ and the closure of the G_1-orbit of a sum of highest weight vectors in $\mathbb{P}\left((V_{G_1}(\varpi_2) \oplus V_{G_1}(\varpi_3))\right)$.

Hence, in any case we can assume that $0 < a_1 < \cdots < a_r$. This finish the proof of Theorem 1.1.

5 The MMP and Log MMP for smooth projective horospherical varieties of Picard group $\mathbb{Z}^2$

The main goal of this section is to prove Theorem 1.3.

5.1 Generalities

Let X be a smooth projective horospherical variety with Picard group $\mathbb{Z}^2$. Here, we suppose that X is as in Case (1) or (2) of Lemma 3.1 (or Theorem 1.1).

By Proposition 3.3, up to linear equivalence, the ample Cartier divisors of X are the $D = d_0 D_0 + d_{n+1} D_{n+1}$ with positive integers d_0 and d_{n+1}.

We can apply [Pas15] to the polarized variety (X, D) and obtain a description of the MMP from X, via moment polytopes (if X is Fano, we obtain two different paths of the program depending on the choice of d_0 and d_{n+1}; if X is not Fano, we obtain a unique path of the program).

Moreover, we can also choose a B-stable $\mathbb{Q}$-divisor Δ of X and apply [Pas17] to the polarized pair $((X, D), \Delta)$ and obtain a description of the Log MMP from (X, Δ), via moment polytopes as described in Section 2.2. To get a uniform Log MMP for any smooth

projective horospherical variety with Picard group $\mathbb{Z}^2$, we choose $D = D_0 + D_{n+1}$ and $\Delta = D_i - K_X$ for $i \in \{0, n+1\}$.

Remark 5.1. In Case (1), an anticanonical divisor of X is (see for example [Pas08, Proposition 3.1])

$$-K_X = \sum_{i=0}^{n} b_i D_i + b_\beta D_\beta \sim (\sum_{i=0}^{n} b_i) D_0 + (b_\beta - \sum_{i=0}^{n} a_i b_i) D_{n+1},$$

where $b_i = 1$ if D_i is G-stable, $b_i = b_{\alpha_i} \geq 2$ if D_i is the color D_{α_i} and $b_\beta \geq 2$ (recall that $D_\beta = D_{n+1}$). In particular, X is Fano (i.e., $-K_X$ ample) if and only if $b_\beta > \sum_{i=1}^{n} a_i b_i$.

To describe the MMP from X we could choose the ample divisor $D = (\sum_{i=0}^{n} b_i) D_1 + (b_\beta + 1) D_\beta$, so that $D + \epsilon K_X$ is ample for any $\epsilon \in [0, 1[$ and $D + K_X \sim (\sum_{i=0}^{n} a_i b_i + 1) D_\beta$ is not ample but globally generated. Then, for that choice of D, the MMP from X consists of the Mori fibration to $G/P(\varpi_\beta)$ described in Remark 3.2.
Moreover, this Mori fibration is also the unique contraction of the Log MMP obtained with the choices $D = D_0 + D_{n+1}$ and $\Delta = D_0 - K_X$.

In Case (2), an anticanonical divisor of X is

$$-K_X = \sum_{i=0}^{r} b_i D_i + \sum_{j=1}^{s+1} b_{r+j} D_{r+j} \sim (\sum_{i=0}^{r} b_i) D_0 + (\sum_{j=1}^{s+1} b_{r+j} - \sum_{i=0}^{r} a_i b_i) D_{n+1},$$

where $b_i = 1$ (respectively b_{r+j}) if D_i (respectively D_{r+j}) is G-stable and $b_i = b_{\alpha_i} \geq 2$ (respectively $b_{r+j} = b_{\alpha_{r+j}} \geq 2$) if D_i is the color D_{α_i} (respectively D_{r+j} is the color $D_{\alpha_{r+j}}$). In particular, X is Fano if and only if $\sum_{j=1}^{s+1} b_{r+j} > \sum_{i=0}^{r} a_i b_i$.

To describe the MMP from X we could choose the ample divisor $D = (\sum_{i=0}^{r} b_i) D_0 + (1 + \sum_{j=1}^{s+1} b_{r+j}) D_{n+1}$, so that $D + \epsilon K_X$ is ample for any $\epsilon \in [0, 1[$ and $D + K_X \sim (1 + \sum_{i=0}^{r} a_i b_i) D_{n+1}$ is not ample but globally generated. Then, for that choice of D, the MMP from X consists of the Mori fibration ψ from X to Z described in Remark 3.2.
Moreover, this Mori fibration is also the unique contraction of the Log MMP obtained with the choices $D = D_0 + D_{n+1}$ and $\Delta = D_0 - K_X$.

Hence, in both cases, we will describe the Log MMP obtained with the choices $D = D_0 + D_{n+1}$ and $\Delta = D_{n+1} - K_X$.

In the next four subsections, X is one the varieties of Theorem 1.1 in Case (1) or (2). We begin by constructing the families of polytopes for the log pairs $(X, \Delta = D_{n+1} - K_X)$ with the choice of ample divisor $D = D_0 + D_{n+1}$, and then we describe in detail the Log MMP's obtained with these families.

5.2 Case (1): the "second" Log MMP via moment polytopes

To describe the one-parameter family $(\tilde{Q}^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$ defined in Theorem 2.14, we consider the basis $(e_i^*)_{i \in \{1,\dots,n\}}$ of M, where for any $i \in \{1, \dots, n\}$, $e_i^* = \varpi_{\alpha_i} - \varpi_{\alpha_0} + a_i \varpi_\beta$, and we

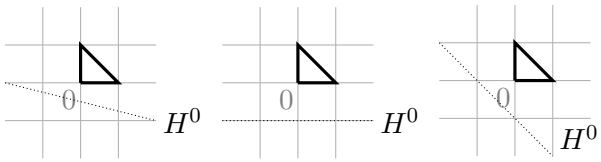

Figure 1: The polytopes $\tilde{Q}^0$ in the cases where $a_1 = 1$ and $a_2 = 2$, $a_1 = 0$ and $a_2 = 1$ and $a_1 = a_2 = 1$ respectively

define the matrices A, B and C as follows

$$A = \begin{pmatrix} -1 & \cdots & \cdots & -1 \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ a_1 & \cdots & \cdots & a_n \end{pmatrix}, \; B = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ -1 \end{pmatrix} \text{ and } \; C = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then $\tilde{Q}^\epsilon = \{x \in M_{\mathbb{Q}} \mid Ax \geq B + \epsilon C\}$ is the set of $x = (x_1, \ldots, x_n)$ such that $x_1, \ldots, x_n$ are non-negative, $x_1 + \cdots + x_n \leq 1$ and $a_1 x_1 + \cdots + a_n x_n \geq \epsilon - 1$.

Example 5.2. If $n = 2$ we are in one of the following situations:

1. $a_2 > a_1 > 0$ and α_2 is not imaginary;
2. $a_2 > a_1 > 0$ and α_2 is imaginary;
3. $a_2 > a_1 = 0$ and α_2 is not imaginary;
4. $a_2 > a_1 = 0$ and α_2 is imaginary;
5. $a_2 = a_1 > 0$;
6. $a_2 = a_1 = 0$.

We draw, in Figure 1, these polytopes for $\epsilon = 0$ in different cases with the hyperplane $H^0 := \{x \in M_{\mathbb{Q}} \mid a_1 x_1 + a_2 x_2 = -1\}$. Note that there is no such hyperplane if $a_2 = a_1 = 0$.

• If $a_n = 0$, $\tilde{Q}^\epsilon = \tilde{Q}^0$ for any $\epsilon \in [0, 1]$ and it is empty if $\epsilon > 1$. Moreover, for any $\epsilon \in [0, 1]$, Q^ϵ intersects the interior of $\mathfrak{X}(P)^+_{\mathbb{Q}}$ if and only if $\epsilon < 1$. In that case, the Log MMP described by the family $(Q^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$ consists of a fibration $\phi_0 : X \longrightarrow Y^0$.

The fibers of this fibration can be easily computed because the faces of Q^0 are "the same" as the faces of Q^1 and then the fibration induces a bijection between the sets of G-orbits of X and Y^0. More precisely, we deduce the fibers of ϕ_0 from the description of G-orbits of X and Y^0 given in Section 2.2: they are isomorphic to the homogeneous projective spaces $(\bigcap_{i \in I} P(\varpi_{\alpha_i}))/(P(\varpi_\beta) \cap \bigcap_{i \in I} P(\varpi_{\alpha_i}))$ (of Picard group $\mathbb{Z}$), with $\emptyset \neq I \subset \{0, \ldots, n\}$. Here, we use the following notation: if α_i is imaginary, $P(\varpi_{\alpha_i}) = G$ (and if not, it is the (proper) maximal parabolic subgroup of G associated to α_i).

In particular, the general fiber of the fibration is $(\bigcap_{i=0}^n P(\varpi_{\alpha_i}))/(P(\varpi_\beta) \cap \bigcap_{i=0}^n P(\varpi_{\alpha_i}))$ and the smallest fibers are the $P(\varpi_{\alpha_i})/(P(\varpi_\beta) \cap P(\varpi_{\alpha_i}))$ with $i \in \{0, \ldots, n\}$. Then we

deduce that $\alpha_0 \notin R_0$ if and only if there exists a fiber isomorphic to $G/P(\varpi_\beta)$.

• Suppose now that $a_n \neq 0$, then $\tilde{Q}^\epsilon$ is the intersection of the simplex $\tilde{Q} = \mathrm{Conv}(e_0^*, e_1^*, \dots, e_n^*)$ with the closed half-space $H_+^\epsilon := \{x \in M_{\mathbb{Q}} \mid a_1x_1 + \cdots + a_nx_n \geq \epsilon - 1\}$, where $e_0^* := 0$. We denote by H_{++}^ϵ the interior of H_+^ϵ and by H^ϵ the hyperplane $H_+^\epsilon \backslash H_{++}^\epsilon$.

In the next proposition, we give a description of the non-empty faces of $\tilde{Q}^\epsilon$ by distinguishing whether a face is in the hyperplane H^ϵ or not.

Note first that the non-empty faces of the simplex $\tilde{Q}$ are the $F_I := \mathrm{Conv}(e_i^* \mid i \in \{0, \dots, n\} \backslash I)$, with $I \subsetneq \{0, \dots, n\}$. In particular, the facets of $\tilde{Q}$ are the $F_i := F_{\{i\}}$ and for any $I \subsetneq \{0, \dots, n\}$, $F_I = \bigcap_{i \in I} F_i$.

Then, for any $I \subsetneq \{0, \dots, n\}$, we define $F_I^\epsilon := F_I \cap H_+^\epsilon$ and $F_{I,\beta}^\epsilon := F_I \cap H^\epsilon$. They are faces (may be empty and not distinct) of $\tilde{Q}^\epsilon$.

(Recall that $a_0 = 0$ and that $a_n \neq 0$ here.)

Proposition 5.3. *The polytope $\tilde{Q}^\epsilon$ is of dimension n if and only if $\epsilon < \max_{i=0}^n (1 + a_i) = 1 + a_n$.*

Suppose now that $\epsilon < 1 + a_n$. The non-empty faces of $\tilde{Q}^\epsilon$ are the distinct following F_I^ϵ and $F_{I,\beta}^\epsilon$, with $I \subsetneq \{0, \dots, n\}$:

- *F_I^ϵ (of codimension $|I|$) if $\epsilon < \max_{i \notin I}(1 + a_i)$;*
- *$F_{I,\beta}^\epsilon$ (of codimension $|I|+1$ or $|I|$ respectively) if $\min_{i \notin I}(1+a_i) < \epsilon < \max_{i \notin I}(1+a_i)$ or $\epsilon = \min_{i \notin I}(1 + a_i) = \max_{i \notin I}(1 + a_i)$.*

In particular, the facets of $\tilde{Q}^\epsilon$ are: F_i^ϵ with $i \in \{0, \dots, n-1\}$ (for any $\epsilon < 1 + a_n$), F_n^ϵ if $\epsilon < 1 + a_{n-1}$, $F_{\emptyset,\beta}^\epsilon$ if $\epsilon > 1$, and $F_{n,\beta}^\epsilon$ if $\epsilon = 1$ and $a_{n-1} = 0$.

Moreover, for any $I \subsetneq \{0, \dots, n\}$ such that $\epsilon < \max_{i \notin I}(1 + a_i)$, $F_I^\epsilon = \bigcap_{i \in I} F_i^\epsilon$.
For any $I \subsetneq \{0, \dots, n\}$ such that $\min_{i \notin I}(1+a_i) < \epsilon < \max_{i \notin I}(1+a_i)$, $F_{I,\beta}^\epsilon = F_{\emptyset,\beta}^\epsilon \cap \bigcap_{i \in I} F_i^\epsilon$.
For any $I \subsetneq \{0, \dots, n\}$ such that $\epsilon = \min_{i \notin I}(1+a_i) = \max_{i \notin I}(1+a_i)$, $F_{I,\beta}^\epsilon = F_{n,\beta}^\epsilon \cap \bigcap_{i \in I} F_i^\epsilon$ if $\epsilon = 1$, $n \in I$ and $a_{n-1} = 0$ or $F_{I,\beta}^\epsilon = \bigcap_{i \in I} F_i^\epsilon$ if $\epsilon \neq 1$, $n \notin I$ or $a_{n-1} \neq 0$.

Proof. The polytope $\tilde{Q}^\epsilon$ is of dimension n if and only if $\tilde{Q}$ intersects H_{++}^ϵ if and only if there exists $i \in \{0, \dots, n\}$ such that $e_i^* \in H_{++}^\epsilon$ if and only if there exists $i \in \{0, \dots, n\}$ such that $a_i > \epsilon - 1$ if and only if $a_n > \epsilon - 1$ (because $0 = a_0 \leq \cdots \leq a_n$). This proves the first statement of the proposition.

Suppose now that $\epsilon < 1 + a_n$. For any non-empty face F of $\tilde{Q}^\epsilon$, either $F \not\subset H^\epsilon$ and F is the intersection of a non-empty face of $\tilde{Q}$ with H_+^ϵ, or $F \subset H^\epsilon$ and F is the intersection of a non-empty face of $\tilde{Q}$ with H^ϵ.

Let $I \subsetneq \{0, \dots, n\}$. The set F_I^ϵ is not empty if and only if there exists $i \notin I$ such that $e_i^* \in H_+^\epsilon$ if and only if there exists $i \notin I$ such that $a_i \geq \epsilon - 1$ if and only if $\epsilon \leq \max_{i \notin I}(1 + a_i)$. Moreover, F_I^ϵ is not empty and not included in H^ϵ if and only if it intersects H_{++}^ϵ if and only if there exists $i \notin I$ such that $e_i^* \in H_{++}^\epsilon$ if and only if there exists $i \notin I$ such that $a_i > \epsilon - 1$ if and only if $\epsilon < \max_{i \notin I}(1 + a_i)$. Also, in that latter case, the dimension of F_I^ϵ is the same as the dimension of F_I; in particular the non-empty F_I^ϵ that are not included in H^ϵ are all distinct.

Similarly, $F_{I,\beta}^\epsilon$ is not empty if and only if there exist i and j not in I (may be equal) such that $e_i^* \in H_+^\epsilon$ and $e_j^* \notin H_{++}^\epsilon$ (ie, $a_i \geq \epsilon - 1$ and $a_j \leq \epsilon - 1$). Then $F_{I,\beta}^\epsilon$ is not empty if and only $\min_{i \notin I}(1 + a_i) \leq \epsilon \leq \max_{i \notin I}(1 + a_i)$. Moreover, $F_{I,\beta}^\epsilon$ is not empty and included in no proper face of F_I (ie, H^ϵ intersects the relative interior of F_I) if and only if there exist $i \neq j$ not in I such that $e_i^* \in H_{++}^\epsilon$ and $e_j^* \notin H_+^\epsilon$ (ie, $a_i > \epsilon - 1$ and

$a_j < \epsilon - 1$) or for any $i \notin I$ we have $e_i^* \in H^\epsilon$ (ie, $a_i = \epsilon - 1$). Then $F^\epsilon_{I,\beta}$ is not empty and included in no proper face of F_I if and only $\min_{i\notin I}(1+a_i) < \epsilon < \max_{i\notin I}(1+a_i)$ or $\epsilon = \min_{i\notin I}(1+a_i) = \max_{i\notin I}(1+a_i)$. Note also that the non-empty $F^\epsilon_{I,\beta}$ that are not included in a proper face of F_I are all distinct and describe all non-empty faces of $\tilde{Q}^\epsilon$ included in H^ϵ. This finishes the proof of the second statement of the proposition.

To describe the facets, it is sufficient to find the F_i^ϵ with $\epsilon < \max_{j\neq i}(1+a_j)$, the $F^\epsilon_{i,\beta}$ with $\epsilon = \min_{j\neq i}(1+a_j) = \max_{j\neq i}(1+a_j)$ and $F^\epsilon_{\emptyset,\beta}$ with $1 = \min_{i=0}^n(1+a_i) < \epsilon < \max_{i=0}^n(1+a_i) = 1+a_n$. We easily find the F_i^ϵ with $i \in \{0,\dots,n-1\}$ for any $\epsilon < 1+a_n$, and F_n^ϵ for any $\epsilon < 1+a_{n-1}$. We conclude by noticing that, for any $i \in \{0,\dots,n\}$, we have $\epsilon = \min_{j\neq i}(1+a_j) = \max_{j\neq i}(1+a_j) < 1+a_n$ if and only if $i = n$ and $0 = a_0 = \cdots = a_{n-1}$ (and in particular, $\epsilon = 1$).

To get the last statement, apply the fact that any face of a polytope is the intersection of the facets containing it. □

From Proposition 5.3, we deduce the following result with the following notation. Let $i_0 := 0, i_1, \dots, i_k, i_{k+1} := n+1$ be positive integers so that $0 = a_{i_0} = \cdots = a_{i_1-1} < a_{i_1} = \cdots = a_{i_1-1} < \cdots < a_{i_l} = \cdots = a_{i_l-1} < \cdots < a_{i_k} = \cdots = a_n$. (Note that $0 < i_1 < \cdots < i_k < n+1$ are the integers defined in Theorem 1.1 with $k = t'$.)

Corollary 5.4. *The isomorphic classes of the horospherical varieties X^ϵ associated to the polytopes in the family $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ are given by the following subsets of $\mathbb{Q}_{\geq 0}$:*

- $[0,1[$*;*
- $]1+a_{i_l}, 1+a_{i_{l+1}}[$ *for any* $l \in \{0,\dots,k-2\}$*;*
- $\{1+a_{i_l}\}$ *for any* $l \in \{0,\dots,k-2\}$*;*
- $]1+a_{i_{k-1}}, 1+a_{i_k}[$ *and* $\{1+a_{i_{k-1}}\}$ *if* $i_k \neq n$ *(ie, if* $a_{n-1} = a_n$*) or the simple root* α_n *is not imaginary (ie, when* X *is as in Case (1b) of Theorem 1.1);*
- $[1+a_{i_{k-1}}, 1+a_{i_k}[$ *if* $i_k = n$ *(ie, if* $a_{n-1} < a_n$*) and the simple root* α_n *is imaginary (ie, when* X *is as in Case (1c) of Theorem 1.1).*

Proof. We apply the theory described in Section 2.2, in particular the fact that the isomorphic classes of the varieties X^ϵ are obtained with looking at the ϵ's for which "the faces of Q^ϵ change".

Note first that, by Proposition 5.3, $(P, M, Q^\epsilon, \tilde{Q}^\epsilon)$ is an admissible quadruple if and only if $\epsilon < 1+a_n$.

Also, the facets of $\tilde{Q}^\epsilon$ are: F_i^ϵ with $i \in \{0,\dots,n-1\}$, F_n^ϵ if $\epsilon < 1+a_{n-1}$, $F^\epsilon_{\emptyset,\beta}$ if $\epsilon > 1$, and $F^\epsilon_{n,\beta}$ (orthogonal to $\alpha^\vee_{nM}$) if $\epsilon = 1$ and $a_{n-1} = 0$. In particular, for any $\epsilon, \eta \in [0, 1+a_n[$, if $a_{n-1} \neq 0$, the facets of Q^ϵ and Q^η are "the same" if and only if ϵ and η are both in $[0,1]$ or $]1, 1+a_{n-1}[$ or $[1+a_{n-1}, 1+a_n[$ (which may be empty). And if $a_{n-1} = 0$, the facets of Q^ϵ and Q^η are "the same" for any $\epsilon, \eta \in [0, 1+a_n[$ (indeed, in that case, the facets F_n^ϵ if $\epsilon < 1$, $F^\epsilon_{\emptyset,\beta}$ if $\epsilon > 1$, and $F^\epsilon_{n,\beta}$ if $\epsilon = 1$ are "the same", in particular all orthogonal to $\beta^\vee_M = a_n\alpha^\vee_{nM}$).

We now use a consequence of the proof of Proposition 5.3: for any $I \subsetneq \{0,\dots,n\}$, $\bigcap_{i\in I} F_i^\epsilon$ is not empty if and only if $\epsilon \leq \max_{i\notin I}(1+a_i)$, $F^\epsilon_{\emptyset,\beta} \cap \bigcap_{i\in I} F_i^\epsilon$ is not empty if and only if $\min_{i\notin I}(1+a_i) \leq \epsilon \leq \max_{i\notin I}(1+a_i)$ and $F^\epsilon_{n,\beta} \cap \bigcap_{i\in I} F_i^\epsilon$ is not empty if and only if $\min_{i\notin I}(1+a_i) = \epsilon = \max_{i\notin I}(1+a_i)$. In particular for any $l \in \{0,\dots,k-2\}$, suppose that for $I = \{i_{l+1},\dots,n\}$ and that $\bigcap_{i\in I} F_i^\epsilon$ is not empty; suppose also that for $I = \{0,\dots,i_l-1\}$ and that $F^\epsilon_{\emptyset,\beta} \cap \bigcap_{i\in I} F_i^\epsilon$ is not empty; then $\epsilon = 1+a_{i_l}$. Similarly for

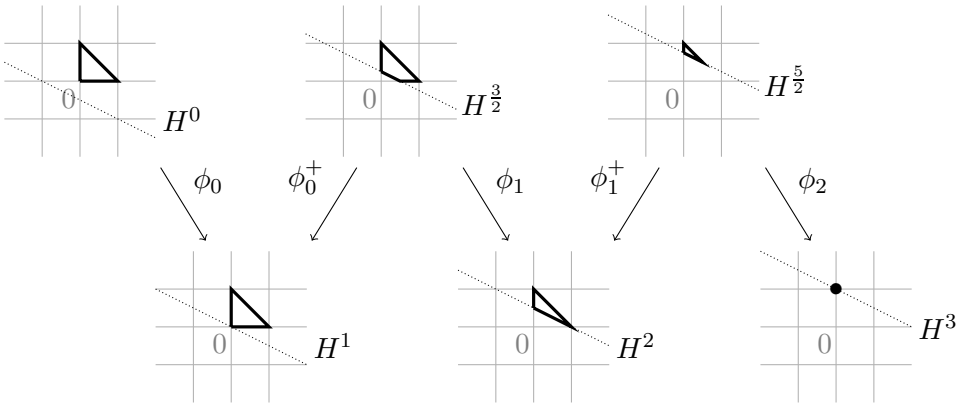

Figure 2: The Log MMP described by the polytopes $\tilde{Q}^\epsilon$ in the case where $n = 2$, $a_1 = 1$, $a_2 = 2$ and α_2 is not imaginary.

any $l \in \{0, \dots, k-2\}$, suppose that that for $I = \{i_{l+1} - 1, \dots, n\}$ and $\bigcap_{i\in I} F_i^\epsilon$ is not empty; suppose also that for $I = \{0, \dots, i_l - 1\}$ and that $F^\epsilon_{\emptyset,\beta} \cap \bigcap_{i\in I} F_i^\epsilon$ is not empty; then $\epsilon \in [1 + a_{i_l}, 1 + a_{i_{l+1}}]$. If $i_k \neq n$, F_n^ϵ is still a facet of Q^ϵ and what we did above with $l \in \{0, \dots, k-2\}$ can be done as well with $l = k - 1$.

Hence, it proves that if the two varieties X^ϵ and X^η are isomorphic then ϵ and η are a one of the subsets described in the corollary.

To conclude, we have to prove that the two varieties X^ϵ and X^η are isomorphic when ϵ and η are in one of these subsets. It is obvious with Proposition 5.3 except in the case where $i_k = n$ and the simple root α_n is imaginary. But in that case, all polytopes Q^ϵ with $\epsilon \in [1 + a_{n-1}, 1 + a_n[= [1 + a_{i_{k-1}}, 1 + a_{i_k}[$ are simplexes with facets F_i^ϵ for $i \in \{0, \dots, n-1\}$ and $F^\epsilon_{\emptyset,\beta}$ or $F^\epsilon_{n,\beta}$ if $\epsilon = 1 + a_{n-1} = 1$, ie, they could be defined with deleting the row corresponding to the simple root α_n that is imaginary, so that their faces are "the same". □

We can reformulate this corollary as follows, and get the first statement of Theorem 1.3 in Case (1). We denote $X^0 = X$ and for any $l \in \{1, \cdots, k\}$, $X^l := X^\epsilon$ with $\epsilon \in]1+a_{i_{l-1}}, 1+a_{i_l}[$, and for any $l \in \{0, \cdots, k\}$, $Y^l := X^{1+a_{i_l}}$.

Corollary 5.5. *The family $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ describes a Log MMP from X as follows:*

- *k flips $\phi_l : X^l \longrightarrow Y^l \longleftarrow X^{l+1} : \phi_l^+$ for any $l \in \{0, \cdots, k-1\}$ and a fibration $\phi_k : X^k \longrightarrow Y^k$, if $i_k \neq n$ or the simple root α_n is not imaginary;*
- *$k-1$ flips $\phi_l : X^l \longrightarrow Y^l \longleftarrow X^{l+1} : \phi_l^+$ for any $l \in \{0, \cdots, k-2\}$, a divisorial contraction $\phi_{k-1} : X^{k-1} \longrightarrow Y^{k-1} \simeq X^k$ and a fibration $X^k \longrightarrow Y^k \simeq \mathrm{pt}$, if $i_k = n$ and the simple root α_n is imaginary.*

Example 5.6. In the fives different cases with $n = 2$ and $a_2 \neq 0$, we illustrate this corollary in terms of polytopes in Figures 2, 3, 4, 5 and 6

5.3 Proof of the last statement of Theorem 1.3 in Case (1)

The previous section proves that $a_{i_1}, \dots, a_{i_k}$ are invariants of X. To finish the proof of Theorem 1.3 in Case (1), we have to prove that $G_0, \dots, G_t$, $\alpha_0, \dots, \alpha_n, \beta$ and $i_1, \dots, i_k$ are also invariants of X. For this, we have to describe some exceptional loci and some fibers

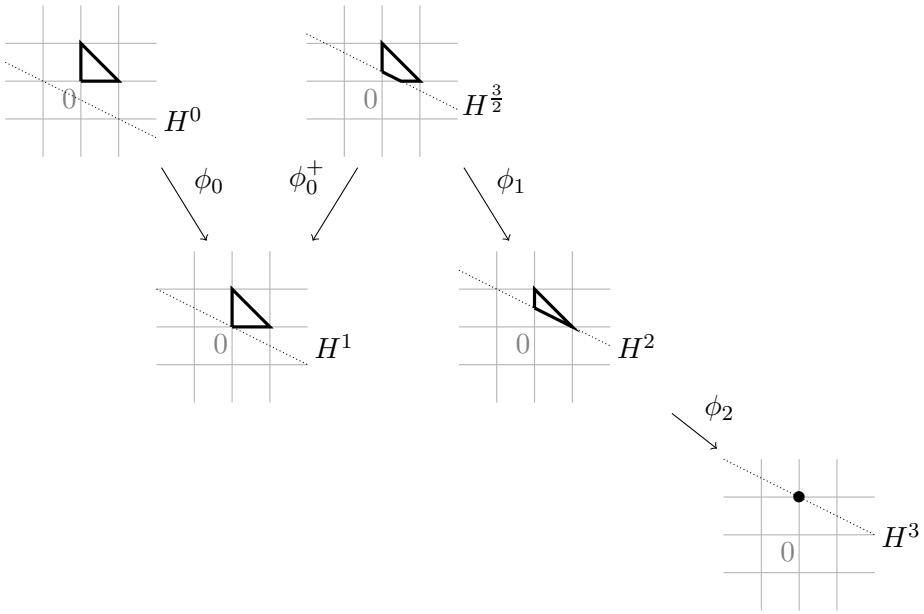

Figure 3: The Log MMP described by the polytopes $\tilde{Q}^\epsilon$ in the case where $n = 2$, $a_1 = 1$, $a_2 = 2$ and α_2 is imaginary.

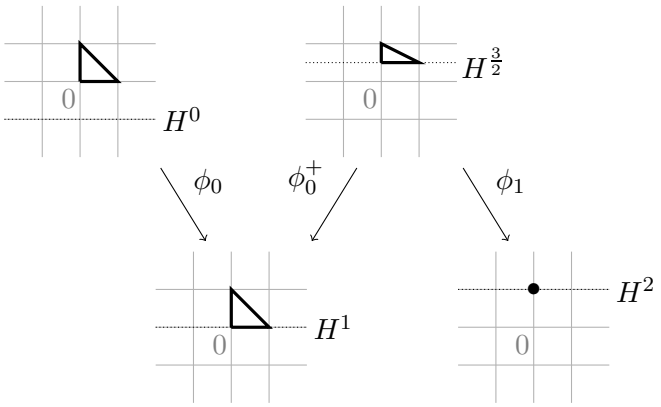

Figure 4: The Log MMP described by the polytopes $\tilde{Q}^\epsilon$ in the case where $n = 2$, $a_1 = 0$, $a_2 = 1$ and α_2 is not imaginary.

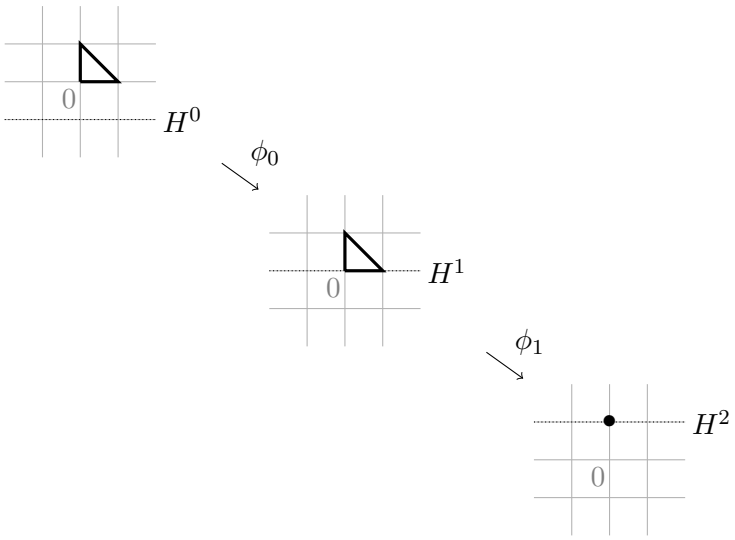

Figure 5: The Log MMP described by the polytopes $\tilde{Q}^\epsilon$ in the case where $n = 2$, $a_1 = 0$, $a_2 = 1$ and α_2 is imaginary.

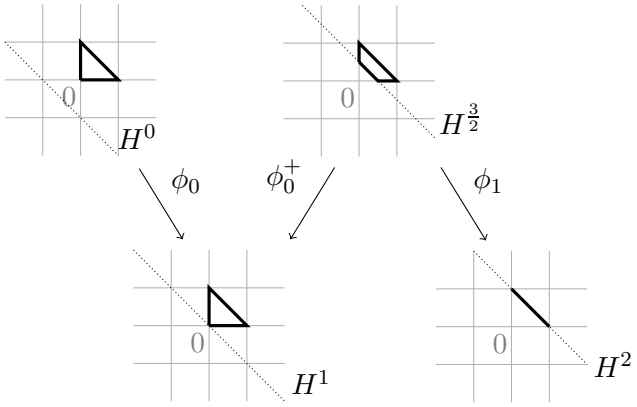

Figure 6: The Log MMP described by the polytopes $\tilde{Q}^\epsilon$ in the case where $n = 2$ and $a_1 = a_2 = 1$.

of the different morphisms of the Log MMP.

We first distinguish two cases by the following result.

Proposition 5.7. *Define the simple subgroups of $P(\varpi_\beta)$ as in Definition 4.2.*

- *Suppose that $n = 1$ and that α_0 and α_1 are two simple roots of the same simple subgroup of $P(\varpi_\beta)$.*

  Then, the fiber of $\psi : X \longrightarrow G/P(\varpi_\beta)$ is either a homogeneous variety different from a projective space (a quadric Q^{2m} with $m \geq 2$, a Grassmannian $\mathrm{Gr}(i, m)$ with $p \geq 5$ and $2 \leq i \leq m-2$, or a spinor variety $\mathrm{Spin}(2m+1)/P(\varpi_m)$ with $m \geq 4$), or a two-orbit variety as in [Pas09].

- *Suppose that $n > 1$ or that α_0 and α_1 are not two simple roots of the same simple subgroup of $P(\varpi_\beta)$.*

  Then, the fiber of $\psi : X \longrightarrow G/P(\varpi_\beta)$ is a projective space.

Proof. The fiber of $\psi : X \longrightarrow G/P(\varpi_\beta)$ is the smooth projective $P(\varpi_\beta)$-variety of Picard group $\mathbb{Z}$ isomorphic to the closure of the $P(\varpi_\beta)$-orbit of a sum of highest weight vectors in $\mathbb{P} := \mathbb{P}(V(\varpi_{\alpha_0}) \oplus \cdots \oplus V(\varpi_{\alpha_n}))$. Hence, the proposition is a consequence of [Pas09, Section 1]. □

• In the case where $n = 1$ and that α_0 and α_1 are two simple roots of the same simple subgroup of $P(\varpi_\beta)$, $G = G_0$, the Log MMP described by Corollary 5.5 consists of a fibration if $a_1 = 0$, or a flip and a fibration if $a_1 > 0$.

If $a_1 = 0$, up to exchanging α_0 and α_1, we can suppose that α_1 is between α_0 and β in the Dynkin diagram of G_0. Since $X \subset \mathbb{P}(V(\varpi_{\alpha_0} + \varpi_\beta) \oplus V(\varpi_{\alpha_1} + \varpi_\beta))$ and $Y^0 \subset \mathbb{P}(V(\varpi_{\alpha_0}) \oplus V(\varpi_{\alpha_1}))$, we easily compute that the fibration $\phi_0 : X \longrightarrow Y^0$ has two different types of fibers: one isomorphic to $P(\varpi_{\alpha_0})/(P(\varpi_{\alpha_0}) \cap P(\varpi_\beta))$ over a G-orbit isomorphic to $G/P(\varpi_{\alpha_0})$ and another one of smaller dimension isomorphic to $P(\varpi_{\alpha_1})/(P(\varpi_{\alpha_1}) \cap P(\varpi_\beta))$.

In particular, $G/P(\varpi_{\alpha_0})$ (as $G/P(\varpi_\beta)$) is an invariant of X. Then if G_0 is not the universal cover of the automorphism group of $G/P(\varpi_\beta)$ it must be the universal cover of the automorphism group of $G/P(\varpi_{\alpha_0})$, so that G_0 is an invariant of X. And then β is also an invariant of X up to symmetries of the Dynkin diagram of G_0. The description of the fiber of $\psi : X \longrightarrow G/P(\varpi_\beta)$ implies that α_0 and α_1 are also invariants of X unless may be if two simple subgroups of $P(\varpi_\beta)$ have the same type (and rank ≥ 2). This could happens if and only if: G_0 is of type A_m with $m \geq 5$ odd and $\varpi_\beta = \varpi_{\frac{m+1}{2}}$, or G_0 is of type E_6 and $\varpi_\beta = \varpi_3$. In these two cases α_0 and α_1 are invariants of X up to symmetries.

If $a_1 > 0$, $X \subset \mathbb{P}(V(\varpi_{\alpha_0} + \varpi_\beta) \oplus V(\varpi_{\alpha_1} + (1+a_1)\varpi_\beta))$, $Y^0 \subset \mathbb{P}(V(\varpi_{\alpha_0}) \oplus V(\varpi_{\alpha_1} + a_1\varpi_\beta))$, $X^1 \subset \mathbb{P}(V(\varpi_{\alpha_0} + \varpi_{\alpha_1}) \oplus V(2\varpi_{\alpha_1} + a_1\varpi_\beta))$ and $Y^1 \simeq G/P(\varpi_{\alpha_1}) \subset \mathbb{P}(V(\varpi_{\alpha_1}))$. In particular X, Y^0 and X^1 have two closed G-orbits and one open G-orbit so that we easily compute exceptional locus and fibers as follows.

For example, the exceptional locus of $\phi_0 : X \longrightarrow Y^0$ is the G-orbit of X isomorphic to $G/(P(\varpi_{\alpha_0}) \cap P(\varpi_\beta))$. Then the universal cover of its automorphism group G_0 is an invariant of X. And then β is also an invariant of X up to symmetries of the Dynkin diagram of G_0. As for the case where $a_1 = 0$, the (same) description of the fiber of $\psi : X \longrightarrow G/P(\varpi_\beta)$ implies that the pair (α_0, α_1) is an invariant of X (up to symmetries). Note now that the exceptional locus of ϕ_0 is sent to the G-orbit of Y^0 isomorphic

to $G/P(\varpi_{\alpha_0})$, so that the couple (α_0, α_1) is an invariant of X (still up to symmetries).

• Now we suppose that $n > 1$ or that α_0 and α_1 are not two simple roots of the same simple subgroup of $P(\varpi_\beta)$.

We define different exceptional loci in X as follows. Let $l \in \{0, \dots, k-1\}$, define E_l to be the closure in X of the set of points $x \in X$ such that x is in the open isomorphic set of the first l contractions and x is in the exceptional locus of ϕ_l.

Proposition 5.8. *For any $l \in \{0, \dots, k\}$ the exceptional locus E_l is the closure in X of the G-orbit associated to the non-empty face F_{I_l} of Q with $I_l := \{i_{l+1}, \dots, n\}$. In particular E_l is isomorphic to the closure of the G-orbit of a sum of highest weight vectors in*

$$\mathbb{P} := \mathbb{P}(\bigoplus_{i=0}^{i_{l+1}-1} V(\varpi_{\alpha_i} + (1+a_i)\varpi_\beta)),$$

and E_l is a smooth projective horospherical of Picard group $\mathbb{Z}^2$ as in Case (1), unless $l = 0$, $i_1 = 1$ so that E_l is homogeneous (projective of Picard group $\mathbb{Z}$ or $\mathbb{Z}^2$).

Note that for $l = k$, $I_k = \emptyset$ and $E_k = X$.

Proof. Let $l \in \{0, \dots, k\}$ and $\epsilon_l \in \mathbb{Q}_{\geq 0}$ such that $X^l = X^{\epsilon_l}$.

We denote by Ω^l_I and $\Omega^l_{I,\beta}$ the G-orbits of X^l associated to the non-empty faces $F^{\epsilon_l}_I$ and $F^{\epsilon_l}_{I,\beta}$ of the polytope $\tilde{Q}^{\epsilon_l}$. We denote by ω^l_I and $\omega^l_{I,\beta}$ the G-orbits of $Y^l = X^{1+a_{i_l}}$ associated to the non-empty faces $F^{1+a_{i_l}}_I$ and $F^{1+a_{i_l}}_{I,\beta}$ of the polytope $\tilde{Q}^{1+a_{i_l}}$. Recall that, for any $\epsilon \in \mathbb{Q}_{\geq 0}$, we have an order on the G-orbits of X^ϵ compatible with the order on the non-empty faces of $\tilde{Q}^\epsilon$: in particular $\Omega^l_I \subset \overline{\Omega^l_{I'}}$ and $\Omega^l_{I,\beta} \subset \overline{\Omega^l_{I',\beta}}$ respectively if and only if $I' \subset I$, and $\Omega^l_{I,\beta} \subset \overline{\Omega^l_I}$ (as soon as these orbits are defined, ie, as soon as the corresponding faces are non-empty).

For any $I \subsetneq \{0, \dots, n\}$ such that there exists $i \geq i_l$ not in I (ie, such that Ω^l_I is defined), $\phi_l(\Omega^l_I) = \omega^l_I$ if there exists $i \geq i_{l+1}$ not in I, and $\phi_l(\Omega^l_I) = \omega^l_{I\cup\{0,\dots i_l-1\},\beta}$ if for any $i \geq i_{l+1}$, $i \in I$. Indeed $I \cup \{0, \dots i_l - 1\}$ is the minimal subset of $\{0, \dots, n\}$ containing I such that $\omega^l_{I\cup\{0,\dots i_l-1\},\beta}$ is defined and there is no I' containing I such that $\omega^l_{I'}$ is defined. And for any $I \subsetneq \{0, \dots, n\}$ such that there exist $i \geq i_l$ and $i' < i_l$ not in I (ie, such that $\Omega^l_{I,\beta}$ is defined), $\phi_l(\Omega^l_{I,\beta}) = \omega^l_{I,\beta}$ if there exists $i \geq i_{l+1}$ not in I, and $\phi_l(\Omega^l_{I,\beta}) = \omega^l_{I\cup\{0,\dots i_l-1\},\beta}$ if for any $i \geq i_{l+1}$, $i \in I$. Indeed $I \cup \{0, \dots i_l - 1\}$ is the minimal subset of $\{0, \dots, n\}$ containing I such that $\omega^l_{I\cup\{0,\dots i_l-1\},\beta}$ is defined.

In particular, we have $\phi_l(\Omega^l_{I_l}) = \omega^l_{I_l\cup\{0,\dots i_l-1\},\beta}$ (which is also $\phi_l(\Omega^l_{I_l,\beta})$ if $l \geq 1$). But $\Omega^l_{I_l}$ and $\omega^l_{I_l\cup\{0,\dots i_l-1\},\beta}$ are not isomorphic horospherical homogeneous spaces by Proposition 2.12, so that $\Omega^l_{I_l}$ is in the exceptional locus of ϕ_l. Moreover, if Ω is a G-orbit of X^{ϵ_l} not contained in $\overline{\Omega^l_{I_l}}$, it is of the form Ω^l_I or $\Omega^l_{I,\beta}$ where $I_l \not\subset I$. Hence, in that case $\phi_l(\Omega) = \Omega$. And then the exceptional locus of ϕ_l is $\overline{\Omega^l_{I_l}}$. Note that $\Omega^0_{I_l}, \dots, \Omega^{l-1}_{I_l}$ are not in the exceptional locus of $\phi_0, \dots, \phi_{l-1}$ respectively, to conclude that $E_l = \overline{\Omega^0_{I_l}}$.

We use again Proposition 2.12 to see that $E_l = \overline{\Omega^0_{I_l}}$ corresponds to the admissible quadruple $(P_F, M_F, F, \tilde{F})$ with $F = F^0_{I_l}$ (and with some ample divisor of E_l). Then we conclude by Corollaries 2.6 and 2.10. □

The Log MMP now defines, by restriction, fibrations $\tilde{\phi_l} : E_l \backslash E_{l-1} \longrightarrow E'_l := \overline{\omega^l_{I_l\cup\{0,\dots i_l-1\},\beta}}$, for any $l \in \{0, \dots, k\}$.

Definition 5.9. We say that the fibers of $\tilde{\phi}_l$ are locally maximal over $\omega \subset E'_l$ if the dimensions of the fibers of $\tilde{\phi}_l$ over any point of ω are the same and bigger than the fibers of $\tilde{\phi}_l$ over any point of a neighborhood of ω that is not in ω.

We say that the fibers of $\tilde{\phi}_l$ are locally almost maximal over $\omega \subset E'_l$ if there exists $\omega' \subsetneq \omega$ such that the fibers of $\tilde{\phi}_l$ are locally maximal over ω' and the fibers of $\tilde{\phi}_{l_{|\tilde{\phi}_l^{-1}(E'_l \backslash \omega')}}$ are locally maximal over $\omega \backslash \omega' \subset E'_l \backslash \omega'$

We now prove the following result, which implies in particular that $i_1, \dots, i_k$ are invariant of X.

Proposition 5.10. *(We still are in the case where $n > 1$ or that α_0 and α_1 are not two simple roots of the same simple subgroup of $P(\varpi_\beta)$.)*

Let $l \in \{0, \dots, k\}$.

The map $\tilde{\phi}_l$ is surjective and we distinguish four distinct cases.

1. *$i_{l+1} - i_l = 1$ and α_{i_l} is not a simple root of G_0. The fibers of $\tilde{\phi}_l$ are locally maximal over E'_l and $\dim E_l - \dim E_{l-1} = 1 + \dim E'_l$ (here we set $\dim E_{-1} := \dim G/P(\varpi_\beta) - 1$ so that it stays true for $l = 0$). Moreover, E'_l is homogeneous isomorphic to $G/P(\varpi_{\alpha_{i_l}})$ (which is a point if α_{i_l} is imaginary).*

2. *$i_{l+1} - i_l = 1$ and α_{i_l} is a simple root of G_0. The fibers of $\tilde{\phi}_l$ are locally maximal over E'_l and $\dim E_l - \dim E_{l-1} \neq 1 + \dim E'_l$ (here also $\dim E_{-1} := \dim G/P(\varpi_\beta) - 1$ so that it stays true for $l = 0$). Moreover, E'_l is homogeneous isomorphic to $G/P(\varpi_{\alpha_{i_l}})$.*

3. *$i_{l+1} - i_l > 1$ and α_{i_l} is not a simple root of G_0. The fibers of $\tilde{\phi}_l$ are locally maximal over a unique proper subset of E'_l, which is a closed G-orbit ω' of E'_l isomorphic to $G/P(\varpi_{\alpha_{i_l}})$. Also the fibers of $\tilde{\phi}_l$ are locally almost maximal over exactly $i_{l+1} - i_l - 1 (> 0)$ subsets of E'_l containing ω', respectively of dimensions $\dim G/P(\varpi_{\alpha_{i_l}}) + \dim G/P(\varpi_{\alpha_j}) + 1$ with $j \in \{i_l + 1, \dots, i_{l+1} - 1\}$.*

4. *$i_{l+1} - i_l > 1$ and α_{i_l} is a simple root of G_0. The fibers of $\tilde{\phi}_l$ are locally maximal over $i_{l+1} - i_l$ closed G-orbits, which are respectively isomorphic to $G/P(\varpi_{\alpha_j})$ with $j \in \{i_l, \dots, i_{l+1} - 1\}$.*

Moreover, in the four cases, we can compute with Corollary 2.13 the dimensions of the fibers over all pointed subsets of E'_l.

1. *The dimension of fibers of $\tilde{\phi}_l$ is $1 + \dim E_{l-1}$ (in particular $\dim G/P(\varpi_\beta)$ if $l = 0$).*

2. *The dimension of fibers of $\tilde{\phi}_l$ is*

$$d_{i_l} := i_l + \dim \left(P(\varpi_{\alpha_{i_l}}) / (P(\varpi_\beta) \cap \bigcap_{i=0}^{i_l} P(\varpi_{\alpha_i})) \right).$$

3. *The dimension of the locally maximal fibers of $\tilde{\phi}_l$ is $1 + \dim E_{l-1}$ (in particular $\dim G/P(\varpi_\beta)$ if $l = 0$). And for any $j \in \{i_l + 1, \dots, i_{l+1} - 1\}$, the dimension of locally almost maximal fibers of $\tilde{\phi}_l$ over of the subset of E'_l of dimension $\dim G/P(\varpi_{\alpha_{i_l}}) + \dim G/P(\varpi_{\alpha_j}) + 1$ is*

$$d_j := i_l + \dim \left(P(\varpi_{\alpha_j}) / (P(\varpi_\beta) \cap \bigcap_{i=0}^{i_l - 1} P(\varpi_{\alpha_i}) \cap P(\varpi_{\alpha_j})) \right).$$

4. *For any $j \in \{i_l, \dots, i_{l+1}-1\}$, the dimension of locally maximal fibers of $\tilde{\phi}_l$ over of the closed G-orbit isomorphic to $G/P(\varpi_{\alpha_j})$ is*

$$d_j := i_l + \dim\left(P(\varpi_{\alpha_j})/(P(\varpi_\beta) \cap \bigcap_{i=0}^{i_l-1} P(\varpi_{\alpha_i}) \cap P(\varpi_{\alpha_j}))\right).$$

Proof. We keep the notation of the proof of Proposition 5.8.

Let ω be a G-orbit of Y^l in $\overline{\omega^l_{I_l\cup\{0,\dots i_l-1\},\beta}}$. Then there exists $I \subsetneq \{0,\dots,n\}$ containing $I_l \cup \{0,\dots i_l-1\}$ such that $\omega = \omega^l_{I,\beta}$. Then $\tilde{\phi_l}^{-1}(\omega) = \bigsqcup_J \Omega^l_J$ where the union is taken over all J such that $J \cap I_{l-1} = I \cap I_{l-1}$. In particular, $\tilde{\phi}_l$ is surjective and $\overline{\tilde{\phi_l}^{-1}(\omega)} = \overline{\Omega^l_{I\cap I_{l-1}}}$. We then compute $\dim(\omega) = \dim(F^l_{I,\beta}) + \dim(G/\bigcap_{i\notin I} P(\varpi_{\alpha_i}))$, and $\dim(\Omega^l_{I\cap I_{l-1}}) = \dim(F_{I\cap I_{l-1}}) + \dim(G/P(\varpi_\beta) \cap \bigcap_{i\notin I\cap I_{l-1}} P(\varpi_{\alpha_i}))$, so that the dimension of a fiber of $\tilde{\phi}_l$ over ω is

$$\begin{aligned}
&\dim(F_{I\cap I_{l-1}}) - \dim(F^l_{I,\beta})) + (\dim(G/(P(\varpi_\beta) \cap \textstyle\bigcap_{i\notin I\cap I_{l-1}} P(\varpi_{\alpha_i})) - \dim(G/\textstyle\bigcap_{i\notin I} P(\varpi_{\alpha_i})))\\
&= i_l + \dim(\textstyle\bigcap_{i\notin I} P(\varpi_{\alpha_i})/(P(\varpi_\beta) \cap \bigcap_{i\notin I\cap I_{l-1}} P(\varpi_{\alpha_i}))\\
&= i_l + \dim(\textstyle\bigcap_{i\notin I} P(\varpi_{\alpha_i})/(\bigcap_{i\notin I} P(\varpi_{\alpha_i}) \cap \bigcap_{i=0}^{i_l-1} P(\varpi_{\alpha_i}) \cap P(\varpi_\beta)).
\end{aligned}$$

These dimensions are the biggest when I is the biggest (in particular when $I = \{0,\dots,n\}$, which is not allowed to define ω). Moreover, if we remove to I some i, the dimension changes if and only if j is such that α_i is in G_0 (ie, α_i is not imaginary and not the only simple root α_j in a simple group of G different from G_0, by hypothesis). From this, we will deduce the different following cases.

If α_{i_l} is not a simple root of G_0, then the locus in $\overline{\omega^l_{I_l\cup\{0,\dots i_l-1\},\beta}}$ where the fibers of $\tilde{\phi}_l$ are maximal is the unique closed G-orbit $\omega' := \omega^l_{\{0,\dots n\}\setminus\{i_l\},\beta}$ isomorphic to $G/P(\varpi_{\alpha_{i_l}})$. This gives the first case of the proposition if $i_{l+1} - i_l = 1$. And if $i_{l+1} - i_l > 1$ the locus in $\overline{\omega^l_{I_l\cup\{0,\dots i_l-1\},\beta}}$ where the fiber of $\tilde{\phi}_l$ is almost maximal is the union of the subsets $\omega^l_{\{0,\dots n\}\setminus\{i_l,j\},\beta} \cup \omega'$ with $j \in \{i_l+1,\dots,i_{l+1}-1\}$, which are affine cones over $G/P(\varpi_{\alpha_i})$. This gives the third case of the proposition.

Now, if α_{i_l} is a simple root of G_0 (ie, for any $j \in \{i_l,\dots,i_{l+1}-1\}$, α_j is a simple root of G_0), then the locus in $\overline{\omega^l_{I_l\cup\{0,\dots i_l-1\},\beta}}$ where the fiber of $\tilde{\phi}_l$ is maximal is the (disjoint) union of the $i_{l+1}-i_l$ closed G-orbits $\omega^l_{\{0,\dots n\}\setminus\{j\},\beta}$ of $\overline{\omega^l_{I_l\cup\{0,\dots i_l-1\},\beta}}$, which are respectively isomorphic to $G/P(\varpi_{\alpha_j})$ for any $j \in \{i_l,\dots,i_{l+1}-1\}$. This gives the second case of the proposition if $i_{l+1}-i_l = 1$ and the fourth case if $i_{l+1}-i_l > 1$.

□

We easily deduce the following.

Corollary 5.11. *With the notation of Proposition 5.10: for any $j \in \{0,\dots,n\}$,*

$$\dim G/P(\varpi_\beta) + d_j - \dim E_{l-1} - 1 = \dim P(\varpi_{\alpha_j})/(P(\varpi_\beta) \cap P(\varpi_{\alpha_j}))$$

and

$$\dim G/P(\varpi_{\alpha_j}) + d_j - \dim E_{l-1} - 1 = \dim P(\varpi_\beta)/(P(\varpi_\beta) \cap P(\varpi_{\alpha_j})).$$

In particular, for any $l \in \{0,\dots,k\}$, the sets

$$\{(\dim P(\varpi_{\alpha_j})/(P(\varpi_\beta) \cap P(\varpi_{\alpha_j})), \dim P(\varpi_\beta)/(P(\varpi_\beta) \cap P(\varpi_{\alpha_j}))) \mid j \in \{i_l,\dots,i_{l+1}-1\}\}$$

are invariants of X.

And then we conclude the proof of Case (1) of Theorem 1.3 (ie, that G_0, β, $\alpha_0, \dots, \alpha_n$ are invariants of X) by the following lemma (still in the case where $n > 1$ or that α_0 and α_1 are not two simple roots of the same simple subgroup of $P(\varpi_\beta)$).

Lemma 5.12. *Let G, G' be two products of simply connected simple groups and $\mathbb{C}^*$'s. Let β, β' be two simple roots of two simple subgroups G_0 and G'_0 of G and G' respectively. And let $\alpha_0, \dots, \alpha_n$, respectively $\alpha'_0, \dots, \alpha'_n$ be simple roots of G, G' both as in Case (1) of Theroem 1.1 (with the same integers k and $i_1, \dots, i_k$).*

Suppose that

$$G/P(\varpi_\beta) \simeq G'/P(\varpi_{\beta'})$$

and for any $l \in \{0, \dots, k\}$,

$$\{(\dim P(\varpi_{\alpha_j})/(P(\varpi_\beta) \cap P(\varpi_{\alpha_j})), \dim P(\varpi_\beta)/(P(\varpi_\beta) \cap P(\varpi_{\alpha_j}))) \mid j \in \{i_l, \dots, i_{l+1} - 1\}\} =$$
$$\{(\dim P(\varpi_{\alpha'_j})/(P(\varpi_{\beta'}) \cap P(\varpi_{\alpha'_j})), \dim P(\varpi_{\beta'})/(P(\varpi_{\beta'}) \cap P(\varpi_{\alpha'_j}))) \mid j \in \{i_l, \dots, i_{l+1} - 1\}\}.$$

Then $G = G'$, $\beta = \beta'$ and for any $i \in \{0, \cdots, n\}$, $\alpha_i = \alpha'_i$ up to reordering the α_i's and α'_i's inside the sets $\{i_l, \dots, i_{l+1} - 1\}$.

Proof. Step 1: for any $l \in \{0, \dots, k\}$, $\alpha_{i_l} \notin R_0$ if and only if $\alpha_{i'_l} \notin R'_0$, and in that case, α_{i_l} and $\alpha_{i'_l}$ are both extremal simple roots of SL_{m+1} with $m = \dim P(\varpi_\beta)/(P(\varpi_\beta) \cap P(\varpi_{\alpha_j})) = \dim P(\varpi_{\beta'})/(P(\varpi_{\beta'}) \cap P(\varpi_{\alpha'_j}))$.

Indeed, $\alpha_{i_l} \notin R_0$ if and only if $\dim P(\varpi_{\alpha_{i_l}})/(P(\varpi_\beta) \cap P(\varpi_{\alpha_{i_l}})) = \dim G/P(\varpi_\beta) = \dim G/P(\varpi_{\beta'}) = \dim P(\varpi_{\alpha'_{i_l}})/(P(\varpi_{\beta'}) \cap P(\varpi_{\alpha'_{i_l}}))$ if and only if $\alpha_{i'_l} \notin R'_0$. The second statement is obvious from the hypothesis on the α_i's and α'_i's. Note that $\alpha_{i_l+1}, \dots, \alpha_{i_{l+1}-1}$ are in R_0 by hypothesis.

Step 2: $G_0 = G'_0$ and $\beta = \beta'$ up to symmetries of the Dynkin diagram. If not, R_0 and R'_0 are not empty and $\{(G_0, \varpi_\beta), (G'_0, \varpi_{\beta'})\}$ is one of the three following sets up to symmetries of the Dynkin diagram (by [Akh95, Section 3.3]): $\{(\mathrm{Sp}_{2m}, \varpi_1), (\mathrm{SL}_{2m}, \varpi_1)\}$, $\{(\mathrm{Spin}_{2m+1}, \varpi_m), (\mathrm{Spin}_{2m+2}, \varpi_{m+1})\}$ or $\{(G_2, \varpi_1), (\mathrm{Spin}_7, \varpi_1)\}$. Let $\alpha_j \in R_0$, there exists $l \in \{0, \dots, k\}$ such that $j \in \{i_l, \dots, i_{l+1} - 1\}$. By Step 1, $\alpha'_j \in R'_0$ and up to reordering α_i's and α'_i's in $\{i_l, \dots, i_{l+1} - 1\}$ we can suppose that $\dim P(\varpi_{\alpha_j})/(P(\varpi_\beta) \cap P(\varpi_{\alpha_j})) = \dim P(\varpi_{\alpha'_j})/(P(\varpi'_\beta) \cap P(\varpi_{\alpha'_j}))$ and $\dim P(\varpi_\beta)/(P(\varpi_\beta) \cap P(\varpi_{\alpha_j})) = \dim P(\varpi_{\beta'})/(P(\varpi_{\beta'}) \cap P(\varpi_{\alpha'_j}))$. We have to check that this is not possible in the three cases.

If $((G_0, \varpi_\beta), (G'_0, \varpi_{\beta'}))$ is $((\mathrm{Sp}_{2m}, \varpi_1), (\mathrm{SL}_{2m}, \varpi_1))$ then ϖ_{α_j} is the fundamental weight ϖ_2 of Sp_{2m} (by the smooth condition) so that $\dim P(\varpi_{\alpha_j})/(P(\varpi_\beta) \cap P(\varpi_{\alpha_j})) = 1$ and $\varpi_{\alpha'_j}$ has to be the fundamental weight ϖ_2 (by the smooth condition and because $\dim P(\varpi_{\alpha'_j})/(P(\varpi_{\beta'}) \cap P(\varpi_{\alpha'_j})) = 1$). But then $\dim P(\varpi_\beta)/(P(\varpi_\beta) \cap P(\varpi_{\alpha_j})) = 2m-3 < 2m-2 = \dim P(\varpi_{\beta'})/(P(\varpi_{\beta'}) \cap P(\varpi_{\alpha'_j}))$.

If $((G_0, \varpi_\beta), (G'_0, \varpi_{\beta'}))$ is $((\mathrm{Spin}_{2m+1}, \varpi_m), (\mathrm{Spin}_{2m+2}, \varpi_{m+1}))$ then ϖ_{α_j} is the fundamental weight ϖ_1 or ϖ_{m-1} of Spin_{2m+1}. In both cases, $\dim P(\varpi_\beta)/(P(\varpi_\beta) \cap P(\varpi_{\alpha_j})) = m - 1$. But $\varpi_{\alpha'_j}$ is the fundamental weight ϖ_1 or ϖ_m of Spin_{2m+2} so that $\dim P(\varpi_{\beta'})/(P(\varpi_{\beta'}) \cap P(\varpi_{\alpha'_j})) = m$.

If $((G_0, \varpi_\beta), (G'_0, \varpi_{\beta'}))$ is $((G_2, \varpi_1), (\mathrm{Spin}_7, \varpi_1))$, then ϖ_{α_j} is the fundamental weight ϖ_2 of G_2 and $\varpi_{\alpha'_j}$ is the fundamental weight ϖ_3 of Spin_7. But then $\dim P(\varpi_\beta)/(P(\varpi_\beta) \cap P(\varpi_{\alpha_j})) = 1 < 3 = \dim P(\varpi_{\beta'})/(P(\varpi_{\beta'}) \cap P(\varpi_{\alpha'_j}))$.

We can now assume that $G_0 = G'_0$ and $\beta = \beta'$. There are at most three simple subgroups of $P(\varpi_\beta)$ (their Dynkin diagram could be obtained from the Dynkin diagram of

G_0 by removing β).

Step 3: let $\alpha_j \in R_0$ and $\alpha'_j \in R'_0$ such that $\dim P(\varpi_\beta)/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j})) = \dim P(\varpi_\beta)/(P(\varpi_\beta)\cap P(\varpi_{\alpha'_j}))$. By the smooth condition, α_j and α'_j are extremal short simple roots of a simple subgroup of $P(\varpi_\beta)$ of type A or C. If the type is A_p then $\dim P(\varpi_\beta)/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j})) = p$. If the type is C_p then $\dim P(\varpi_\beta)/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j})) = 2p-1$. Hence, we have two cases: they are extremal short simple roots of simple subgroups of $P(\varpi_\beta)$ both of type A_p, or they are extremal short simple roots of simple subgroups of $P(\varpi_\beta)$ of types A_{2p-1} and C_p.

Step 4: Suppose moreover that $\dim P(\varpi_{\alpha_j})/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j})) = \dim P(\varpi_{\alpha'_j})/(P(\varpi_\beta)\cap P(\varpi_{\alpha'_j}))$, then we prove that $\alpha_j = \alpha'_j$ up to symmetries, by studying all cases up to symmetries, where $P(\varpi_\beta)$ has at least two simple subgroups of types A_p and A_p with $p \geq 1$, or A_{2p-1} and C_p with $p \geq 2$.

Type of G_0	ϖ_β	ϖ_{α_j}	$\dim P(\varpi_{\alpha_j})/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j}))$
A_m, $m \geq 5$ m odd	$\varpi_{\frac{m+1}{2}}$	ϖ_1 or $\varpi_{\frac{m+3}{2}}$	$\frac{(m+1)(m-1)}{4}$ or $\frac{m+1}{2}$, and $\frac{(m+1)(m-1)}{4} = \frac{m+1}{2}\frac{m-1}{2} \geq 2\frac{m+1}{2}$
B_3	ϖ_2	ϖ_1 or ϖ_3	2 or 3
B_6	ϖ_4	ϖ_1 or ϖ_6	18 or 8
B_6	ϖ_4	ϖ_3 or ϖ_6	5 or 8
C_m, $m \geq 3$ m multiple of 3	ϖ_i $i = \frac{2}{3}m$	ϖ_1 or ϖ_{i+1}	$\frac{(4m-3i)(i-1)}{2} = \frac{3i(i-1)}{2}$ or i, and $\frac{3i(i-1)}{2} > i$ because $i \geq 2$
C_m, $m \geq 3$ m multiple of 3	ϖ_i $i = \frac{2}{3}m$	ϖ_{i-1} or ϖ_{i+1}	$2m-2i-1 = i-1$ or i,
D_7	ϖ_4	ϖ_1 or ϖ_7	21 or 12
D_7	ϖ_4	ϖ_3 or ϖ_7	6 or 12
E_6	ϖ_4	ϖ_1 or ϖ_5	15 or 6

□

5.4 Case (2): the "second" Log MMP via moment polytopes

To describe the one-parameter family $(\tilde{Q}^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ defined in Theorem 2.14, we consider the basis $(u_i^*)_{i\in\{1,\dots,r\}} \cup (v_1^*)$ of M, where for any $i \in \{1,\dots,r\}$, $u_i^* = \varpi_{\alpha_i} - \varpi_{\alpha_0} + a_i\varpi_{\alpha_{r+2}}$ and $v_1^* = \varpi_{\alpha_{r+1}} - \varpi_{\alpha_{r+2}}$ and we define the matrices A, B and C as follows

$$A = \begin{pmatrix} -1 & \cdots & -1 & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ a_1 & \cdots & a_r & -1 \end{pmatrix},\ B = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ -1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Then $\tilde{Q}^\epsilon = \{x \in M_\mathbb{Q} \mid Ax \geq B + \epsilon C\}$ is the set of $x = (x_1,\dots,x_n)$ such that $x_1,\dots,x_n$ are non-negative, $x_1 + \cdots + x_r \leq 1$ and $a_1x_1 + \cdots + a_rx_r - x_{r+1} - \cdots - x_n \geq \epsilon - 1$.

In particular, $\tilde{Q}^\epsilon$ is the intersection of $\tilde{Q}^0$ with the closed half-space $H_+^\epsilon := \{x \in M_\mathbb{Q} \mid a_1x_1 + \cdots + a_rx_r - x_{r+1} \geq \epsilon - 1\}$. We denote by H_{++}^ϵ the interior of H_+^ϵ and by H^ϵ the hyperplane $H_+^\epsilon \backslash H_{++}^\epsilon$.

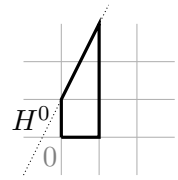

Figure 7: The polytope $\tilde{Q}^0$ in the case where $a_1 = 2$

Example 5.13. If $n = 2$ (ie, $r = s = 1$) we have $a_1 > 0$, and either α_1 is imaginary or not.

We draw, in Figure 7, such a polytope for $\epsilon = 0$ with the hyperplane $H^0 := \{x \in M_{\mathbb{Q}} \mid a_1 x_1 - x_2 = -1\}$.

Note that $\tilde{Q}^0$ is a polytope with vertices $u_0^* := 0$, $u_1^*, \dots, u_r^*$, $u_0^* + (1+a_0)v_0^*, \dots, u_r^* + (1+a_r)v_1^*$ (recall that $a_0 = 0$) and facets $F_I := \mathrm{Conv}((u_i^* \mid i \notin I) \cup (u_i^* + (1+a_i)v_1^* \mid i \notin I))$, $F_{I,1} := \mathrm{Conv}(u_i^* \mid i \notin I)$ and $F_{I,2} := \mathrm{Conv}(u_i^* + (1+a_i)v_1^* \mid i \notin I)$ with $I \subsetneq \{0, \dots, r\}$. In particular, the facets of $\tilde{Q}^0$ are the $F_i := F_{\{i\}}$ with $i \in \{0, \dots, r\}$, $F_{\emptyset,1}$ and $F_{\emptyset,2}$. Moreover for any $I \subsetneq \{0, \dots, n\}$, $F_I = \bigcap_{i \in I} F_i$, $F_{I,1} = \bigcap_{i \in I} F_i \cap F_{\emptyset,1}$ and $F_{I,2} = \bigcap_{i \in I} F_i \cap F_{\emptyset,2}$.

Then, for any $I \subsetneq \{0, \dots, r\}$, we define $F_I^\epsilon := F_I \cap H_+^\epsilon$, $F_{I,1}^\epsilon := F_{I,1} \cap H_+^\epsilon$, $F_{I,2}^\epsilon := F_I \cap H^\epsilon$ and $F_{I,1,2}^\epsilon := F_{I,1} \cap H^\epsilon$. They are faces (may be empty and not distinct) of $\tilde{Q}^\epsilon$. (Recall $0 = a_0 < a_1 < \cdots < a_r$ and $n = r + 1$.)

Proposition 5.14. *The polytope $\tilde{Q}^\epsilon$ is of dimension n if and only if $\epsilon < 1 + a_r$.*

Suppose now that $\epsilon < 1 + a_r$. The non-empty faces of $\tilde{Q}^\epsilon$ are the distinct following F_I^ϵ, $F_{I,1}^\epsilon$, $F_{I,2}^\epsilon$ and $F_{I,1,2}^\epsilon$ with $I \subsetneq \{0, \dots, r\}$:

- *F_I^ϵ (of codimension $|I|$) if $\epsilon < \max_{i \notin I}(1 + a_i)$;*
- *$F_{I,1}^\epsilon$ (of codimension $|I| + 1$) if $\epsilon < \max_{i \notin I}(1 + a_i)$;*
- *$F_{I,2}^\epsilon$ (of codimension $|I| + 1$) if $\epsilon < \max_{i \notin I}(1 + a_i)$;*
- *$F_{I,1,2}^\epsilon$ (of codimension $|I|+2$ or $|I|+1$ respectively) if $\min_{i \notin I}(1+a_i) < \epsilon < \max_{i \notin I}(1+a_i)$ or $\epsilon = \min_{i \notin I}(1 + a_i) = \max_{i \notin I}(1 + a_i)$.*

In particular, the facets of $\tilde{Q}^\epsilon$ are: F_i^ϵ with $i \in \{0, \dots, r-1\}$, F_r^ϵ if $\epsilon < 1 + a_{r-1}$, $F_{\emptyset,1}^\epsilon$ and $F_{\emptyset,2}^\epsilon$.

Moreover, for any $I \subsetneq \{0, \dots, r\}$ such that $\epsilon < \max_{i \notin I}(1 + a_i)$, $F_I^\epsilon = \bigcap_{i \in I} F_i^\epsilon$.
For any $I \subsetneq \{0, \dots, r\}$ such that $\epsilon < \max_{i \notin I}(1 + a_i)$, $F_{I,1}^\epsilon = \bigcap_{i \in I} F_i^\epsilon \cap F_{\emptyset,1}^\epsilon$.
For any $I \subsetneq \{0, \dots, r\}$ such that $\epsilon < \max_{i \notin I}(1 + a_i)$, $F_{I,2}^\epsilon = \bigcap_{i \in I} F_i^\epsilon \cap F_{\emptyset,2}^\epsilon$.
For any $I \subsetneq \{0, \dots, r\}$ such that $\min_{i \notin I}(1 + a_i) \leq \epsilon \leq \max_{i \notin I}(1 + a_i)$, $F_{I,1,2}^\epsilon = \bigcap_{i \in I} F_i^\epsilon \cap F_{\emptyset,1}^\epsilon \cap F_{\emptyset,2}^\epsilon$.

Remark that, if $\epsilon = \min_{i \notin I}(1 + a_i) = \max_{i \notin I}(1 + a_i)$, then $I = \{0, \dots, r\} \backslash \{i\}$ where i is such that $\epsilon = 1 + a_i$.

Note also that $\tilde{Q}^{1+a_r}$ is the point u_r^* so that Q^{1+a_r} is the point ϖ_{α_r}.

Proof. For any $\epsilon \geq 0$, the polytope $\tilde{Q}^\epsilon$ is of dimension n if and only if $\tilde{Q}^0$ intersects H_{++}^ϵ if and only if there exists $i \in \{0, \dots, r\}$ such that u_i^* (or $u_i^* + (1 + a_i)v_1^*$) is in H_{++}^ϵ if and only if there exists $i \in \{0, \dots, r\}$ such that $a_i > \epsilon - 1$ (or $-1 > \epsilon - 1$) if and only if $a_r > \epsilon - 1$. This proves the first statement of the proposition.

Suppose now that $\epsilon < 1 + a_r$. A non-empty face of $\tilde{Q}^\epsilon$ is either the intersection with H^ϵ_+ of a non-empty face of $\tilde{Q}^0$ that intersects H^ϵ_{++}, or the intersection of a non-empty face of $\tilde{Q}^0$ with H^ϵ.

Let $I \subsetneq \{0, \dots, r\}$. The set F^ϵ_I is not empty if and only if there exists $i \notin I$ such that u^*_i (or $u^*_i + (1 + a_i)v^*_1$) is in H^ϵ_+ if and only if there exists $i \notin I$ such that $a_i \geq \epsilon - 1$ (or $-1 \geq \epsilon - 1$) if and only if $\epsilon \leq \max_{i \notin I}(1 + a_i)$. Moreover with the same argument, F^ϵ_I is not empty and intersects H^ϵ_{++} if and only if $\epsilon < \max_{i \notin I}(1 + a_i)$. Also, in that case, the dimension of F^ϵ_I is the same as the dimension of F_I; in particular the non-empty F^ϵ_I that intersect H^ϵ_{++} are all distinct.

Similarly, $F^\epsilon_{I,1}$ is not empty if and only if there exists $i \notin I$ such that $u^*_i \in H^\epsilon_+$ if and only if there exists $i \notin I$ such that $a_i \geq \epsilon - 1$ if and only if $\epsilon \leq \max_{i \notin I}(1 + a_i)$. Also, $F^\epsilon_{I,1}$ is not empty and intersects H^ϵ_{++} if and only if $\epsilon < \max_{i \notin I}(1 + a_i)$. In that case, the dimension of $F^\epsilon_{I,1}$ is the same as the dimension of $F_{I,1}$; in particular the non-empty $F^\epsilon_{I,1}$ that intersect H^ϵ_{++} are all distinct and also distinct from the non-empty F^ϵ_I.

Let $I \subsetneq \{0, \dots, r\}$. Note that for any $\epsilon \geq 0$ (respectively $\epsilon > 0$) and for any $i \in \{0, \dots, r\}$, $u^*_i + (1 + a_i)v^*_1 \notin H^\epsilon_{++}$ (respectively $u^*_i + (1 + a_i)v^*_1 \notin H^\epsilon_+$). Then the set $F^\epsilon_{I,2}$ is not empty if and only if there exists $i \notin I$ such that $u^*_i \in H^\epsilon_+$ if and only if there exists $i \notin I$ such that $a_i \geq \epsilon - 1$ if and only if $\epsilon \leq \max_{i \notin I}(1 + a_i)$. Moreover, $F^\epsilon_{I,2}$ is not empty and H^ϵ intersects F_I in its relative interior if and only if there exists $i \notin I$ such that $a_i > \epsilon - 1$ if and only if $\epsilon < \max_{i \notin I}(1 + a_i)$. Hence, the dimension of $F^\epsilon_{I,2}$ is the dimension of F_I minus 1 if $\epsilon < \max_{i \notin I}(1 + a_i)$ and it equals the dimension of F_I if $\epsilon = \max_{i \notin I}(1 + a_i)$. In the first case, the $F^\epsilon_{I,2}$ are all distinct and describe all non-empty faces of $\tilde{Q}^\epsilon$ included in H^ϵ but not in $F_{\emptyset,1}$. In the second case, $F^\epsilon_{I,2} = F^\epsilon_{I,1,2}$.

Now, the set $F^\epsilon_{I,1,2}$ is not empty if and only if there exist i and j not in I (may be equal) such that $u^*_i \in H^\epsilon_+$ and $u^*_j \notin H^\epsilon_{++}$ if and only if there exist i and j not in I such that $a_i \geq \epsilon - 1$ and $a_j \leq \epsilon - 1$ if and only if $\min_{i \notin I}(1 + a_i) \leq \epsilon \leq \max_{i \notin I}(1 + a_i)$. Moreover, $F^\epsilon_{I,1,2}$ is not empty and included in no proper face of $F_{I,1}$ if and only if there exist i and j not in I such that $u^*_i \in H^\epsilon_{++}$ and $u^*_j \notin H^\epsilon_+$ if and only if there exist i and j not in I such that $a_i > \epsilon - 1$ and $a_j < \epsilon - 1$ (ie, $a_i < \epsilon - 1$ and $a_j > \epsilon - 1$) or for any $i \notin I$ we have $u^*_i \in H^\epsilon$ (ie, $a_i = \epsilon - 1$). Then $F^\epsilon_{I,1,2}$ is not empty and included in no proper face of $F_{I,1}$ if and only if $\min_{i \notin I}(1 + a_i) < \epsilon < \max_{i \notin I}(1 + a_i)$ or $\epsilon = \min_{i \notin I}(1 + a_i) = \max_{i \notin I}(1 + a_i)$. In particular, the dimension of $F^\epsilon_{I,1,2}$ is the dimension of $F_{I,1}$ minus 1 if $\min_{i \notin I}(1 + a_i) < \epsilon < \max_{i \notin I}(1 + a_i)$ and it equals the dimension of $F_{I,1}$ if $\epsilon = \min_{i \notin I}(1 + a_i) = \max_{i \notin I}(1 + a_i)$. Note also that the non-empty $F^\epsilon_{I,1,2}$ that are not included in a proper face of $F_{I,1}$ are all distinct and describe all non-empty faces of $\tilde{Q}^\epsilon$ included in $H^\epsilon \cap F_{\emptyset,1}$. This finishes the proof of the second statement of the proposition.

To get the last statements, apply that a facet is a face of codimension 1 and that any face of a polytope is the intersection of the facets containing it. □

From Proposition 5.14, we deduce the following result.

Corollary 5.15. *The isomorphic classes of the horospherical varieties X^ϵ associated to the polytopes in the family $(Q^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$ are given by the following subsets of $\mathbb{Q}_{\geq 0}$:*

- $[0, 1[$;
- $]1 + a_i, 1 + a_{i+1}[$ *for any* $i \in \{0, \dots, r - 2\}$;
- $\{1 + a_i\}$ *for any* $i \in \{0, \dots, r - 2\}$;
- $]1 + a_{r-1}, 1 + a_r[$ *and* $\{1 + a_{r-1}\}$ *if the simple root α_r is not imaginary (ie, when X is as in Case (2b) of Theorem 1.1);*

- $[1+a_{r-1}, 1+a_r[$ *if the simple root* α_r *is imaginary (ie, when* X *is as in Case (2c) of Theorem 1.1).*

Proof. We apply the theory described in Section 2.2, in particular the fact that the isomorphic classes of the varieties X^ϵ are obtained with looking at the ϵ's for which "the faces of Q^ϵ change".

Note first that, by Proposition 5.14, $(P, M, Q^\epsilon, \tilde{Q}^\epsilon)$ is an admissible quadruple if and only if $\epsilon < 1 + a_r$.

Also, the facets of $\tilde{Q}^\epsilon$ are: F_i^ϵ (orthogonal to $\alpha_{iM}^\vee$) with $i \in \{0, \dots, r-1\}$, F_r^ϵ (orthogonal to $\alpha_{rM}^\vee$) if $\epsilon < 1 + a_{r-1}$, $F_{\emptyset,1}^\epsilon$ (orthogonal to $\alpha_{(r+1)M}^\vee$) and $F_{\emptyset,2}^\epsilon$ (orthogonal to $\alpha_{(r+2)M}^\vee$). In particular, for any ϵ, $\eta \in [0, 1 + a_r[$, the facets of Q^ϵ and Q^η are "the same" if and only if ϵ and η are both in $[0, 1 + a_{r-1}[$ or $[1 + a_{r-1}, 1 + a_r[$.

We now use a consequence of the proof of Proposition 5.14: for any $I \subsetneq \{0, \dots, r\}$, $\bigcap_{i\in I} F_i^\epsilon$ is not empty if and only if $\epsilon \leq \max_{i\notin I}(1 + a_i)$, $F_{\emptyset,1}^\epsilon \cap \bigcap_{i\in I} F_i^\epsilon$ is not empty if and only if $\epsilon \leq \max_{i\notin I}(1 + a_i)$, $F_{\emptyset,2}^\epsilon \cap \bigcap_{i\in I} F_i^\epsilon$ is not empty if and only if $\epsilon \leq \max_{i\notin I}(1 + a_i)$, and $F_{\emptyset,1,2}^\epsilon \cap \bigcap_{i\in I} F_i^\epsilon$ is not empty if and only if $\min_{i\notin I}(1 + a_i) \leq \epsilon \leq \max_{i\notin I}(1 + a_i)$. In particular, for any $i \in \{0, \dots, r-2\}$, suppose that for $I = \{i+1, \dots, r\}$ and that $\bigcap_{i\in I} F_i^\epsilon$ is not empty; suppose also that for $I = \{0, \dots, i-1\}$ and that $F_{\emptyset,1,2}^\epsilon \cap \bigcap_{i\in I} F_i^\epsilon$ is not empty; then $\epsilon = 1 + a_i$. Similarly for any $i \in \{0, \dots, r-2\}$, suppose that for $I = \{i+2, \dots, n\}$ and that $\bigcap_{i\in I} F_i^\epsilon$ is not empty; suppose also that for $I = \{0, \dots, i-1\}$ and that $F_{\emptyset,1,2}^\epsilon \cap \bigcap_{i\in I} F_i^\epsilon$ is not empty; then $\epsilon \in [1 + a_i, 1 + a_{i+1}]$.

Hence, it proves that if two varieties X^ϵ and X^η are isomorphic then ϵ and η are a one of the subsets described in the corollary.

To conclude, we have to prove that the two varieties X^ϵ and X^η are isomorphic when ϵ and η are in one of these subsets. It is obvious with Proposition 5.14 except in the case where the simple root α_n is imaginary. But in that case, all polytopes Q^ϵ with $\epsilon \in [1 + a_{r-1}, 1 + a_r[$ could be defined with deleting the row corresponding to the simple root α_r that is imaginary, so that their faces are "the same" (they are simplexes with facets F_i^ϵ for $i \in \{0, \dots, r-1\}$, $F_{\emptyset,1}^\epsilon$ and $F_{\emptyset,2}^\epsilon$). □

We can reformulate this corollary as follows, and get the first statement of Theorem 1.3 in Case (2). We denote $X_0 = X$ and for any $i \in \{1, \cdots, r\}$, $X^i := X^\epsilon$ with $\epsilon \in]1+a_{i-1}, 1+a_i[$ and for any $i \in \{0, \cdots, r\}$, $Y^i := X^{1+a_i}$.

Corollary 5.16. *The family* $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ *describes a Log MMP from* X *as follows:*

- r *flips* $\phi_i : X^i \longrightarrow Y^i \longleftarrow X^{i+1} : \phi_i^+$ *for any* $i \in \{0, \cdots, r-1\}$ *and a fibration* $\phi_r : X^r \longrightarrow Y^r$*, if the simple root* α_r *is not imaginary;*
- $r-1$ *flips* $\phi_i : X^i \longrightarrow Y^i \longleftarrow X^{i+1} : \phi_i^+$ *for any* $i \in \{0, \cdots, k-2\}$*, a divisorial contraction* $\phi_{r-1} : X^{r-1} \longrightarrow Y^{r-1} \simeq X^r$ *and a fibration* $X^r \longrightarrow Y^r \simeq \mathrm{pt}$*, if the simple root* α_n *is imaginary.*

Example 5.17. In the two different cases with $n = 2$ and $a_1 = 2$, we illustrate this corollary in terms of polytopes in Figures 8 and 9.

5.5 Proof of the last statement of Theorem 1.3 in Case (2)

The previous section proves that $a_1, \dots, a_r$ are invariants of X. To finish the proof of Theorem 1.3 in Case (2), we have to prove that $G_0, \dots, G_t$ and $\alpha_0, \dots, \alpha_{r+2}$ are also invariants. Since the "first" Log MMP consists of a fibration $\psi : X \longrightarrow Z$ where Z is a two-orbit variety embedded in $\mathbb{P}(V(\varpi_{\alpha_{r+1}}) \oplus V(\varpi_{\alpha_{r+2}}))$ as in [Pas09], G_t, α_{r+1} and α_{r+2}

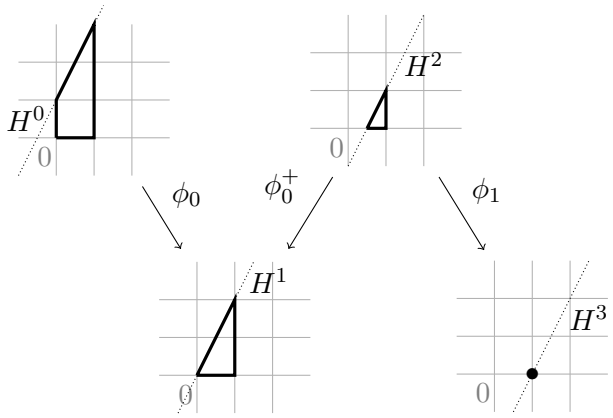

Figure 8: The Log MMP described by the polytopes $\tilde{Q}^\epsilon$ in the case where $n = 2$, $a_1 = 2$ and α_1 is not imaginary.

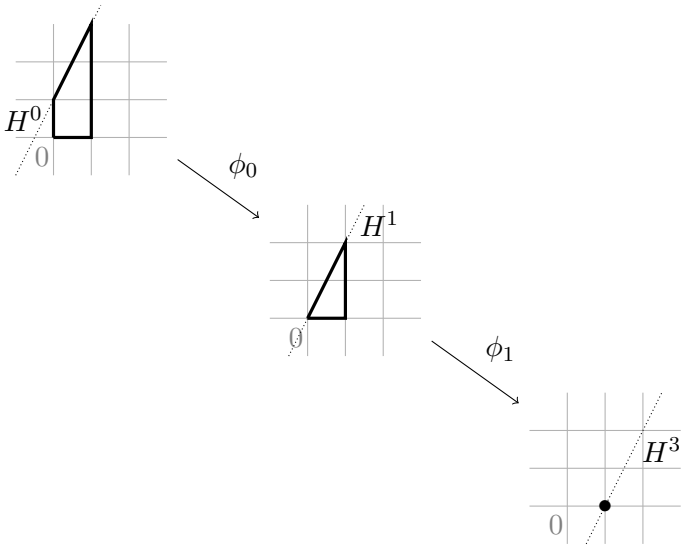

Figure 9: The Log MMP described by the polytopes $\tilde{Q}^\epsilon$ in the case where $n = 2$, $a_1 = 2$ and α_1 is imaginary.

are invariants of X. As in Case (1), we will describe some exceptional loci and some fibers of different morphisms of the Log MMP, but we first distinguish two cases by the following result.

Proposition 5.18. • *Suppose that $r = 1$ and that α_0 and α_1 are two simple roots of G_0 (and then $t = 1$).*

Then, the general fiber of $\psi : X \longrightarrow Z$ is either a homogeneous variety different from a projective space (a quadric Q^{2m} with $m \geq 2$, a Grassmannian $\mathrm{Gr}(i, m)$ with $m \geq 5$ and $2 \leq i \leq m-2$, or a spinor variety $\mathrm{Spin}(2m+1)/P(\varpi_m)$ with $m \geq 4$), or a two-orbit variety as in [Pas09].

- *Suppose that $r > 1$ or that α_0 and α_1 are simple roots of G_0 and G_1 respectively.*

  Then, the general fiber of $\psi : X \longrightarrow Z$ is a projective space.

Proof. The general fiber of $\psi : X \longrightarrow Z$ is the smooth projective horospherical $G_0 \times \cdots \times G_{t-1}$-variety of Picard group $\mathbb{Z}$ isomorphic to the closure of the $G_0 \times \cdots \times G_{t-1}$-orbit of a sum of highest weight vectors in $\mathbb{P} := \mathbb{P}(V(\varpi_{\alpha_0}) \oplus \cdots \oplus V(\varpi_{\alpha_r}))$. Hence, the proposition is a consequence of [Pas09, Section 1]. □

• In the case where $r = 1$ and that α_0 and α_1 are two simple roots of G_0, $G = G_0 \times G_1$ and the description of the general fiber of $\psi : X \longrightarrow G/P(\varpi_\beta)$ implies that G_0, α_0 and α_1 are invariants of X.

• Now we suppose that $r > 1$ or that α_0 and α_1 are not two simple roots of the same simple subgroup of $P(\varpi_\beta)$.

We define different exceptional loci in X as follows. Let $i \in \{0, \ldots, r\}$, define E_i to be the closure in X of the set of points $x \in X$ such that x is in the open isomorphic set of the first i contractions and x is in the exceptional locus of ϕ_i.

Proposition 5.19. *For any $i \in \{0, \ldots, r\}$ the exceptional loci E_i is the closure in X of the G-orbit associated to the non-empty face F_{I_i} with $I_i := \{i+1, \ldots, r\}$. In particular E_i is isomorphic to the closure of the G-orbit of a sum of highest weight vectors in*

$$\mathbb{P} := \mathbb{P}\left(\bigoplus_{j=0}^{i}\bigoplus_{b=0}^{1+a_j} V(\varpi_{\alpha_j} + b\varpi_{\alpha_{r+1}} + (1 + a_j - b)\varpi_{\alpha_{r+2}})\right),$$

hence for $i \in \{1, \ldots, r\}$, E_i is a smooth projective horospherical of Picard group $\mathbb{Z}^2$ as in Case (2), and E_0 is the product a two-orbit variety with a homogeneous (projective of Picard group $\mathbb{Z}$) variety.

Note that $E_r = X$ and that in any case, the rank of the horospherical G-variety E_i is $i + 1$.

Proof. Let $i \in \{0, \ldots, r\}$ and $\epsilon_i \in \mathbb{Q}_{\geq 0}$ such that $X^i = X^{\epsilon_i}$.

We denote by Ω^i_I, $\Omega^i_{I,1}$, $\Omega^i_{I,2}$ and $\Omega^i_{I,1,2}$ the G-orbits of X^i associated to the non empty faces $F^{\epsilon_i}_I$, $F^{\epsilon_i}_{I,1}$, $F^{\epsilon_i}_{I,2}$ and $F^{\epsilon_i}_{I,1,2}$ of the polytope $\tilde{Q}^{\epsilon_i}$. We denote by ω^i_I, $\omega^i_{I,1}$, $\omega^i_{I,2}$ and $\omega^i_{I,1,2}$ the G-orbits of $Y^i = X^{1+a_i}$ associated to the non-empty faces $F^{1+a_i}_I$, $F^{1+a_i}_{I,1}$, $F^{1+a_i}_{I,2}$ and $F^{1+a_i}_{I,1,2}$ of the polytope $\tilde{Q}^{1+a_i}$. Recall that, for any $\epsilon \in \mathbb{Q}_{\geq 0}$, we have an order on the G-orbits of X^ϵ compatible with the order on the non-empty faces of $\tilde{Q}^\epsilon$: in particular

$\Omega^i_I \subset \overline{\Omega^i_{I'}}$, $\Omega^i_{I,1} \subset \overline{\Omega^i_{I',1}}$, $\Omega^i_{I,2} \subset \overline{\Omega^i_{I',2}}$ and $\Omega^i_{I,1,2} \subset \overline{\Omega^i_{I',1,2}}$ respectively if and only if $I' \subset I$, and $\Omega^i_{I,1} \subset \overline{\Omega^i_I}$, $\Omega^i_{I,2} \subset \overline{\Omega^i_I}$, $\Omega^i_{I,1,2} \subset \overline{\Omega^i_{I,1}}$ and $\Omega^i_{I,1,2} \subset \overline{\Omega^i_{I,2}}$ (as soon as these orbits are defined, ie, as soon as the corresponding faces are non-empty).

For any $I \subsetneq \{0,\dots,r\}$ such that there exists $j \geq i$ not in I (ie, such that Ω^i_I is defined), $\phi_i(\Omega^i_I) = \omega^i_I$ if there exists $j \geq i+1$ not in I, and $\phi_i(\Omega^i_I) = \omega^i_{I\setminus\{i\},1,2}$ if for any $j \geq i+1$, $j \in I$. Indeed $I \cup \{0,\dots i-1\} = I\setminus\{i\}$ is the minimal subset of $\{0,\dots,r\}$ containing I such that $\omega^i_{I\setminus\{i\},1,2}$ is defined and there is no I' containing I such that $\omega^i_{I'}$, $\omega^i_{I',1}$ or $\omega^i_{I',2}$ is defined. Similarly, with $k=1$ or 2, for any $I \subsetneq \{0,\dots,r\}$ such that there exists $j \geq i$ not in I (ie, such that $\Omega^i_{I,k}$ is defined), $\phi_i(\Omega^i_{I,k}) = \omega^i_{I,k}$ if there exists $j \geq i+1$ not in I, and $\phi_i(\Omega^i_{I,k}) = \omega^i_{\{0,\dots,r\}\setminus\{i\},1,2}$ if for any $j \geq i+1$, $j \in I$. Indeed $I \cup \{0,\dots i-1\} = \{0,\dots,r\}\setminus\{i\}$ is the minimal subset of $\{0,\dots,r\}$ containing I such that $\omega^i_{\{0,\dots,r\}\setminus\{i\},1,2}$ is defined and there is no I' containing I such that $\omega^i_{I',k}$ is defined.

And for any $I \subsetneq \{0,\dots,r\}$ such that there exist $j \geq i$ and $j' < i$ not in I (ie, such that $\Omega^i_{I,1,2}$ is defined), $\phi_i(\Omega^l_{I,1,2}) = \omega^i_{I,1,2}$ if there exists $i \geq i+1$ not in I, and $\phi_i(\Omega^i_{I,1,2}) = \omega^i_{\{0,\dots,r\}\setminus\{i\},\beta}$ if for any $j \geq i+1$, $j \in I$. Indeed $\{0,\dots,r\}\setminus\{i\} = I \cup \{0,\dots i_l-1\}$ is the minimal subset of $\{0,\dots,n\}$ containing I such that $\omega^i_{\{0,\dots,r\}\setminus\{i\},1,2}$ is defined.

In particular, we have $\phi_i(\Omega^i_{I_i}) = \omega^i_{I\setminus\{i\},1,2}$. But $\Omega^i_{I_i}$ and $\omega^i_{\{0,\dots,r\}\setminus\{i\},1,2}$ are not isomorphic horospherical homogeneous spaces by Proposition 2.12, so that $\Omega^i_{I_i}$ is in the exceptional locus of ϕ_l. Moreover, if Ω is a G-orbit of X^i not contained in $\overline{\Omega^i_{I_i}}$, it is of the form Ω^i_I, $\Omega^i_{I,1}$, $\Omega^i_{I,2}$ or $\Omega^i_{I,1,2}$ where $I_i \not\subset I$. Hence, in that case $\phi_i(\Omega) = \Omega$. And then the exceptional locus of ϕ_i is $\overline{\Omega^i_{I_i}}$. Note that $\Omega^0_{I_i},\dots,\Omega^{l-1}_{I_i}$ are not in the exceptional locus of $\phi_0,\dots,\phi_{i-1}$ respectively, to conclude that $E_i = \overline{\Omega^0_{I_i}}$.

We use again Proposition 2.12 to see that $E_i = \overline{\Omega^0_{I_i}}$ corresponds to the admissible quadruple $(P_F, M_F, F, \tilde{F})$ with $F = F^0_{I_i}$ (and with some ample divisor of E_i). Then we conclude by Corollaries 2.6 and 2.10. □

The Log MMP now defines, by restriction, fibrations $\tilde{\phi_i} : E_i\setminus E_{i-1} \longrightarrow E'_i := \overline{\omega^i_{\{0,\dots,r\}\setminus\{i\},1,2}}$, for any $i \in \{0,\dots,i\}$.

Proposition 5.20. *For any $i \in \{0,\dots,r\}$, E'_i is a closed G-orbit of Y^i isomorphic to $G/P(\varpi_{\alpha_i})$ (which is a point if α_i is imaginary). In particular, the map $\tilde{\phi_i}$ is surjective.*

Moreover, the dimension of fibers of $\tilde{\phi_i}$ is

$$i+1+\dim P(\varpi_{\alpha_i})/(P(\varpi_{\alpha_{r+1}}) \cap P(\varpi_{\alpha_{r+2}}) \cap \bigcap_{j=0}^{i} P(\varpi_{\alpha_j})).$$

Proof. Let $i \in \{0,\dots,r\}$. The face $F^{1+a_i}_{\{0,\dots,r\}\setminus\{i\},1,2}$ of $\tilde{Q}^{1+a_i}$ is the vertex u^*_i and then the corresponding face of Q^{1+a_i} is the vertex ϖ_{α_i}. In particular, the G-orbit $\omega^i_{\{0,\dots,r\}\setminus\{i\},1,2}$ is closed and isomorphic to $G/P(\varpi_{\alpha_i})$.

Now, since $\tilde{\phi_i}$ is G-equivariant, it must be surjective.

Moreover, the dimension of the fibers of $\tilde{\phi_i}$ is

$$\dim E_i - \dim E'_i = (i+1+\dim G/(P(\varpi_{\alpha_{r+1}}) \cap P(\varpi_{\alpha_{r+2}}) \cap \bigcap_{j=0}^{i} P(\varpi_{\alpha_j}))) - \dim G/P(\varpi_{\alpha_i})$$

that is $i+1+\dim P(\varpi_{\alpha_i})/(P(\varpi_{\alpha_{r+1}}) \cap P(\varpi_{\alpha_{r+2}}) \cap \bigcap_{j=0}^{i} P(\varpi_{\alpha_j}))$. □

Corollary 5.21. *The dimension of the fibers of $\tilde{\phi}_i$ is*

$$i+1+\dim G/(P(\varpi_{\alpha_{r+1}})\cap P(\varpi_{\alpha_{r+2}}))+\sum_{j=0}^{i-1}\dim G/P(\varpi_{\alpha_j}).$$

In particular the dimensions d_j of the $G/P(\varpi_{\alpha_j})$'s, which are projective space under $G_i=\mathrm{SL}_{d_j+1}$, are invariants of X.

Proof. Since $r>1$, or $r=1$ and α_0, α_1 are not two simple roots of the same simple subgroups of G, the simple roots $\alpha_0,\dots,\alpha_r$ are respectively the first simple roots of $G_0,\dots,G_r$ that are of type A. (And α_{r+1}, α_{r+2} are simple roots of G_{r+1}.) Then the corollary can be easily deduced from the proposition. □

6 Appendix

Proposition 6.1. *Let (K,β,R,n) be a smooth quadruple. Then we are in one of the following cases, up to symmetries.*

1. *$n=1$ and one of the following case occurs..*

   - *K is of type A_m ($m\geq 3$). Then, $\beta=\alpha_k$ with $3\leq k\leq m$ and $R=\{\alpha_1,\alpha_{k-1}\}$; or $\beta=\alpha_k$ with $4\leq k\leq m$ and $R=\{\alpha_i,\alpha_{i+1}\}$ with $1\leq i\leq k-2$.*
   - *K is of type B_m ($m\geq 3$). Then, $\beta=\alpha_k$ with $3\leq k\leq m$ and $R=\{\alpha_1,\alpha_{k-1}\}$ or $R=\{\alpha_i,\alpha_{i+1}\}$ with $1\leq i\leq k-2$; or $\beta=\alpha_k$ with $1\leq k\leq m-2$ and $R=\{\alpha_{m-1},\alpha_m\}$; or $\beta=\alpha_{m-3}$ and $R=\{\alpha_{m-2},\alpha_m\}$.*
   - *K is of type C_m ($m\geq 3$). Then, $\beta=\alpha_k$ with $3\leq k\leq m$ and $R=\{\alpha_1,\alpha_{k-1}\}$; or $\beta=\alpha_k$ with $4\leq k\leq m$ and $R=\{\alpha_i,\alpha_{i+1}\}$ with $1\leq i\leq k-2$; $\beta=\alpha_k$ with $1\leq k\leq m-2$ and $R=\{\alpha_i,\alpha_{i+1}\}$ with $1\leq i\leq k-2$.*
   - *K is of type D_m ($m\geq 4$). Then, $\beta=\alpha_k$ with $3\leq k\leq m-2$ or $k=m$ and $R=\{\alpha_1,\alpha_{k-1}\}$; or $\beta=\alpha_k$ with $4\leq k\leq m-2$ or $k=m$ and $R=\{\alpha_i,\alpha_{i+1}\}$ with $1\leq i\leq k-2$; $\beta=\alpha_k$ with $1\leq k\leq m-4$ and $R=\{\alpha_{m-1},\alpha_m\}$; or $m\geq 5$, $\beta=\alpha_{m-3}$ and R is any subset of cardinality 2 of $\{\alpha_{m-2},\alpha_{m-1},\alpha_m\}$; or $m\geq 5$, $\beta=\alpha_{m-2}$ and $R=\{\alpha_{m-1},\alpha_m\}$; all modulo symmetries.*
   - *K is of type E_6. Then $\beta=\alpha_1$ and $R=\{\alpha_2,\alpha_3\}$; or $\beta=\alpha_2$ and $R=\{\alpha_1,\alpha_6\}$, $\{\alpha_1,\alpha_3\}$ or $\{\alpha_3,\alpha_4\}$; or $\beta=\alpha_3$ and $R=\{\alpha_2,\alpha_6\}$, $\{\alpha_2,\alpha_4\}$, $\{\alpha_4,\alpha_5\}$ or $\{\alpha_5,\alpha_6\}$; or $\beta=\alpha_4$ and $R=\{\alpha_1,\alpha_3\}$.*
   - *K is of type E_7. Then $\beta=\alpha_1$ and $R=\{\alpha_2,\alpha_3\}$; or $\beta=\alpha_2$ and $R=\{\alpha_1,\alpha_7\}$, $\{\alpha_1,\alpha_3\}$, $R=\{\alpha_3,\alpha_4\}$, $\{\alpha_4,\alpha_5\}$, $\{\alpha_5,\alpha_6\}$ or $\{\alpha_6,\alpha_7\}$; or $\beta=\alpha_3$ and $R=\{\alpha_2,\alpha_7\}$, $\{\alpha_2,\alpha_4\}$, $\{\alpha_4,\alpha_5\}$, $\{\alpha_5,\alpha_6\}$ or $\{\alpha_6,\alpha_7\}$; or $\beta=\alpha_4$ and $R=\{\alpha_1,\alpha_3\}$, $\{\alpha_5,\alpha_7\}$, $\{\alpha_5,\alpha_6\}$ or $\{\alpha_6,\alpha_7\}$; or $\beta=\alpha_5$ and $R=\{\alpha_1,\alpha_2\}$, $\{\alpha_1,\alpha_3\}$, $\{\alpha_3,\alpha_4\}$, $\{\alpha_2,\alpha_4\}$ or $\{\alpha_6,\alpha_7\}$; or $\beta=\alpha_6$ and $R=\{\alpha_2,\alpha_5\}$.*
   - *K is of type E_8. Then $\beta=\alpha_1$ and $R=\{\alpha_2,\alpha_3\}$; or $\beta=\alpha_2$ and $R=\{\alpha_1,\alpha_8\}$, $\{\alpha_1,\alpha_3\}$, $R=\{\alpha_3,\alpha_4\}$, $\{\alpha_4,\alpha_5\}$, $\{\alpha_5,\alpha_6\}$, $\{\alpha_6,\alpha_7\}$ or $\{\alpha_7,\alpha_8\}$; or $\beta=\alpha_3$ and $R=\{\alpha_2,\alpha_8\}$, $\{\alpha_2,\alpha_4\}$, $\{\alpha_4,\alpha_5\}$, $\{\alpha_5,\alpha_6\}$, $\{\alpha_6,\alpha_7\}$ or $\{\alpha_7,\alpha_8\}$; or $\beta=\alpha_4$ and $R=\{\alpha_1,\alpha_3\}$, $\{\alpha_5,\alpha_8\}$, $\{\alpha_5,\alpha_6\}$, $\{\alpha_6,\alpha_7\}$ or $\{\alpha_7,\alpha_8\}$; or $\beta=\alpha_5$ and $R=\{\alpha_1,\alpha_2\}$, $\{\alpha_1,\alpha_3\}$, $\{\alpha_3,\alpha_4\}$, $\{\alpha_2,\alpha_4\}$, $\{\alpha_6,\alpha_8\}$, $\{\alpha_6,\alpha_7\}$ or $\{\alpha_7,\alpha_8\}$; or $\beta=\alpha_6$ and $R=\{\alpha_2,\alpha_5\}$ or $\{\alpha_7,\alpha_8\}$.*
   - *K is of type F_4. Then $\beta=\alpha_1$ and $R=\{\alpha_3,\alpha_4\}$ or $\{\alpha_2,\alpha_3\}$; $\beta=\alpha_2$ and $R=\{\alpha_3,\alpha_4\}$; $\beta=\alpha_3$ and $R=\{\alpha_1,\alpha_2\}$; $\beta=\alpha_4$ and $R=\{\alpha_2,\alpha_3\}$ or $\{\alpha_1,\alpha_3\}$.*

2. *R is empty or one of the following case occurs.*

   - *K is of type A_m ($m \geq 2$). Then, $\beta = \alpha_1$ and R is $\{\alpha_2\}$ or $\{\alpha_m\}$ (if $m \geq 3$); $\beta = \alpha_k$ with $2 \leq k \leq \frac{m}{2}$ and R is a subset of $\{\alpha_1, \alpha_{k+1}\}$, $\{\alpha_1, \alpha_m\}$, $\alpha_{k-1}, \alpha_{k+1}\}$ (if $k \geq 3$) or $\alpha_{k-1}, \alpha_m\}$ (if $k \geq 3$); or $\beta = \alpha_{\frac{m+1}{2}}$ (if m is odd) and R is a subset of $\{\alpha_1, \alpha_m\}$ or $R = \{\alpha_{k-1}\}$, $\{alpha_1, \alpha_{k+1}\}$ or $\alpha_{k-1}, \alpha_{k+1}\}$ (if $m \geq 5$).*
   - *K is of type B_m ($m \geq 3$). Then, $m = 3$, $\beta = \alpha_1$ and R is $\{\alpha_3\}$; $\beta = \alpha_k$ with $2 \leq k \leq m-3$ and R is $\{\alpha_1\}$ or $\{\alpha_{k-1}\}$ (if $k \geq 3$); or $\beta = \alpha_{m-2}$ ($m \geq 4$) and R is a subset of $\{\alpha_1, \alpha_m\}$ or $\{\alpha_{m-3}, \alpha_m\}$ (if $m \geq 5$); or $\beta = \alpha_m - 1$ and R is a subset of $\{\alpha_1, \alpha_m\}$ or R is $\{\alpha_{m-2}\}$ (if $m \geq 4$) or $\{\alpha_{m-2}, \alpha_m\}$ (if $m \geq 5$); or $\beta = \alpha_m$ and R is $\{\alpha_1\}$ or $\{\alpha_{m-1}\}$.*
   - *K is of type C_m ($m \geq 2$). Then, $\beta = \alpha_1$ and R is $\{\alpha_2\}$; or $\beta = \alpha_k$ with $2 \leq k \leq m-1$ ($m \geq 3$) and R is a subset of $\{\alpha_1, \alpha_{k+1}\}$ or $\{\alpha_{k-1}, \alpha_{k+1}\}$ (if $k \geq 3$ and $m \geq 4$); or $\beta = \alpha_m$ and $R = \{\alpha_1\}$ or $\{\alpha_{m-1}\}$ (if $m \geq 3$).*
   - *K is of type D_m ($m \geq 4$). Then, $\beta = \alpha_k$ with $2 \leq k \leq m-4$ ($m \geq 6$) and R is $\{\alpha_1\}$ or $\{\alpha_{k-1}\}$ (if $k \geq 3$ and $m \geq 7$); or $\beta = \alpha_{m-3}$ and R is $\{\alpha_{m-1}\}$, or a subset of $\{\alpha_1, \alpha_{m-1}\}$ (if $m \geq 5$) or $\{\alpha_{m-4}, \alpha_{m-1}\}$ (if $m \geq 6$); or $\beta = \alpha_{m-2}$ and R is $\{\alpha_1\}$, $\{\alpha_1, \alpha_{m-1}\}$ or $\{\alpha_1, \alpha_{m-1}, \alpha_m\}$, or R is a subset of $\{\alpha_{m-3}, \alpha_{m-1}\}$ (if $m \geq 5$), R is $\{\alpha_{m-3}, \alpha_{m-1}, \alpha_m\}$ (if $m \geq 5$); or $\beta = \alpha_m$ and R is $\{\alpha_1\}$ or $\{\alpha_{m-1}\}$.*
   - *K is of type E_6. Then $\beta = \alpha_2$ and $R = \{\alpha_1\}$; or $\beta = \alpha_3$ and R is a subset of $\{\alpha_1, \alpha_2\}$ or $\{\alpha_1, \alpha_6\}$; or $\beta = \alpha_4$ and R is subset of $\{\alpha_2, \alpha_i, \alpha_j\}$ with $i = 1$ or 3 and $j = 5$ or 6 modulo symmetries.*
   - *K is of type E_7. Then $\beta = \alpha_2$ and $R = \{\alpha_1\}$ or $\{\alpha_7\}$; or $\beta = \alpha_3$ and R is a subset of $\{\alpha_1, \alpha_2\}$ or $\{\alpha_1, \alpha_7\}$; or $\beta = \alpha_4$ and R is subset of $\{\alpha_2, \alpha_i, \alpha_j\}$ with $i = 1$ or 3 and $j = 5$ or 7; or $\beta = \alpha_5$ and R is a subset of $\{\alpha_i, \alpha_j\}$ with $i = 1$ or 2 and $j = 6$ or 7; or $\beta = \alpha_6$ and $R = \{\alpha_7\}$.*
   - *K is of type E_8. Then $\beta = \alpha_2$ and $R = \{\alpha_1\}$ or $\{\alpha_8\}$; or $\beta = \alpha_3$ and R is a subset of $\{\alpha_1, \alpha_2\}$ or $\{\alpha_1, \alpha_8\}$; or $\beta = \alpha_4$ and R is subset of $\{\alpha_2, \alpha_i, \alpha_j\}$ with $i = 1$ or 3 and $j = 5$ or 8; or $\beta = \alpha_5$ and R is a subset of $\{\alpha_i, \alpha_j\}$ with $i = 1$ or 2 and $j = 6$ or 8; or $\beta = \alpha_6$ and R is α_7 or α_8; or $\beta = \alpha_7$ and $R = \{\alpha_8\}$.*
   - *K is of type F_4. Then $\beta = \alpha_1$ and $R = \{\alpha_4\}$; $\beta = \alpha_2$ and R is a subset of $\{\alpha_1, \alpha_3\}$ or $\{\alpha_1, \alpha_4\}$; $\beta = \alpha_3$ and R is a subset of $\{\alpha_1, \alpha_4\}$ or $\{\alpha_2, \alpha_4\}$.*
   - *K is of type G_2. Then $\beta = \alpha_1$ and $R = \{\alpha_2\}$; or $\beta = \alpha_2$ and $R = \{\alpha_1\}$*

References

[Akh95] Dmitri N. Akhiezer, *Lie group actions in complex analysis*, Aspects of Mathematics, E27, Friedr. Vieweg & Sohn, Braunschweig, 1995. MR 1334091

[Bou75] N. Bourbaki, *Éléments de mathématique*, Hermann, Paris, 1975, Fasc. XXXVIII: Groupes et algèbres de Lie. Chapitre VII: Sous-algèbres de Cartan, éléments réguliers. Chapitre VIII: Algèbres de Lie semi-simples déployées, Actualités Scientifiques et Industrielles, No. 1364.

[Bri89] Michel Brion, *Groupe de picard et nombres caractéristiques des variétés sphériques*, Duke Math. J. **58** (1989), no. 2, 397–424.

[Kle88] Peter Kleinschmidt, *A classification of toric varieties with few generators*, Aequationes Math. **35** (1988), no. 2-3, 254–266.

[Kno91] Friedrich Knop, *The Luna-Vust theory of spherical embeddings*, Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989) (Madras), Manoj Prakashan, 1991, pp. 225–249.

[Pas06] Boris Pasquier, *Variétés horosphériques de Fano*, Ph.D. thesis, Université Joseph Fourier, Grenoble 1, available at http://tel.archives-ouvertes.fr/tel-00111912, 2006.

[Pas08] ———, *Variétés horosphériques de Fano*, Bull. Soc. Math. France **136** (2008), no. 2, 195–225.

[Pas09] ———, *On some smooth projective two-orbit varieties with Picard number 1*, Math. Ann. **344** (2009), no. 4, 963–987.

[Pas15] ———, *An approach of the minimal model program for horospherical varieties via moment polytopes*, J. Reine Angew. Math. **708** (2015), 173–212.

[Pas17] ———, *The Log Minimal Model Program for horospherical varieties via moment polytopes*, preprint available at arXiv:1612.03695, to appear in Acta Mathematica Sinica (2017).