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Random discretization of stationary continuous time processes

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Contents

1 Introduction

The long-range dependent time series have diverse applications in many fields, including hydrology, economics and telecommunications (see [1] ch.2). Most of the papers on this topic consider processes with discrete-time. However some models and estimation methods have been extended to continuous-time processes (see [11, 16, 5, 4]). In [11], Tsai and Chan introduced the continuous-time autoregressive fractionally integrated moving average (CARFIMA(p,H,q)) model. Under the long-range dependence condition $H \in (1/2, 1)$, they exhibit the autocovariance function of the stationay CARFIMA process and its spectral density function (see [12]). Theses properties are extended to the case $H \in (0, 1)$ in [14]. In [16], continuous-time fractional ARMA processes are constructed. Under some conditions, these processes are zero mean stationary Gaussian and they

give results on their spectral density and covariance function. Moreover, Viano *et al.* specify the asymptotic dependence of these processes. In [5], Comte and Renault presents a family of long memory models: the continuous time moving average fractional process. The statistical inference for continuous-time processes is generally constructed from the sampled process (see [11, 12, 3, 4]. Different scheme of sampling can be considered. In [11], the estimation method is based on the maximum likelihood estimation for irregularly spaced deterministic time series data. Under the assumption of identifiability, Chambers [3] considers the estimation of the long memory parameter of a continuous time fractional ARMA process with discrete time data using the low-frequency behaviour of the spectrum. Comte [4] studied two methods for the estimation with regularly spaced data : Whittle likelihood method and the semiparametric approach of Geweke and Porte-Hudak. We are interested in irregularly spaced data when the sampling intervals are independent and identically distributed positive random variables. In the light of previous results in discrete time, we there was an effect of the random sampling on the dependence structure of the process. Indeed [8] show that the intensity of the long memory is preserved when the law of sampling intervals is L^1, but they also show situations leading to a reduction of the long memory.

We start from $\mathbf{X} = (X_t)_{t\in\mathbb{R}^+}$, a second-order stationary continuous time process with auto-covariance function $\sigma_X(.)$ and from a random walk $(T_n)_{n\geq 0}$ independent of $\mathbf{X}$. The i.i.d. sampling intervals $T_{j+1}-T_j = \Delta_j$ have a common probability density function s supported by $\mathbb{R}^+$. Moreover, $T_0 = 0$. In the remainder of this paper, without loss of generality, we suppose that $\mathbf{X}$ is a zero-mean process.

We adopt the most usual definition of long memory. Namely, for a stationary process $\mathbf{U}$ having a covariance function σ_U

$$\int_{\mathbb{R}^+} |\sigma_U(x)|\ \mathrm{d}x = \infty \quad \text{in the continuous-time case,} \quad \sum_{h\geq 0} |\sigma_U(h)| = \infty \quad \text{in the discrete-time case.}$$

Two sampling methods, both based on the random walk $(T_n)_{n\geq 0}$, are investigated hereafter.

Sections 2, 3 and 4 are devoted to sampled processes at random instants. More precisely, we consider the discrete-time process $\mathbf{Y}$ defined by

$$Y_n = X_{T_n} \quad n = 0, 1 \ldots . \tag{1.1}$$

In section 2, we study the behaviour of the sampled process in general case. We prove that in the case of a gaussian initial process, the sampled Y is not gaussian, that under rather weak conditions on the covariance σ_X, sampling a short memory $\mathbf{X}$ process always produces a short memory $\mathbf{Y}$, and that when the sampling law satisfies $\mathbb{E}(T_1) < \infty$ the sampled process $\mathbf{Y}$ has long memory if it is the case for $\mathbf{X}$.

In section 3 we present the more specific situation of a regularly varying covariance where preservation or non-preservation of the memory can be quantified. In particular we prove that for heavy tailed sampling law, a long memory $\mathbf{X}$ can give raise to a short memory $\mathbf{Y}$.

Section 4 presents some cases of preservation of the existence of a spectral density when the spectrum of $\mathbf{X}$ is absolutely continuous.

Section 5 turns to a temporal aggregation scheme. The discrete-time sampled process $\mathbf{Z}$ is obtained by aggregating the process $\mathbf{X}$ over the intervals defined by the random walk

$$Z_h = \int_{T_h}^{T_{h+1}} X_t\ \mathrm{d}t.$$

We show that from the point of view of transmission of the memory, the results are similar to those obtained in the previous sections.

2 General results in time domain.

Let $\mathbf{Y}$ be the stationary discrete-time second order process defined by

$$Y_n = X_{T_n} \quad n = 0, 1 \ldots \tag{2.1}$$

with covariance sequence

$$\begin{cases} \sigma_Y(0) = \sigma_X(0) \\ \sigma_Y(h) = \mathbb{E}\left(\sigma_X(T_h)\right) \quad h \geq 1 \end{cases} \tag{2.2}$$

2.1 Distribution of the sampled process

Proposition 2.1. *Let* $\mathbf{X}$ *be a strictly stationary process. Then, the sampled process* $\mathbf{Y}$ *is a strictly stationary discrete-time process, ie the joint distribution of* $(Y_{k_1+p}, \ldots, Y_{k_n+p})$ *does not depend on* p *for any* $k_1, \ldots k_n \in \mathbb{N}$ *and* $n \geq 1$*.*

Proof. Let $n \geq 1$, $p \in \mathbb{N}$ and $0 \leq k_1 < \cdots < k_n$. For $(y_1, \ldots, y_n) \in \mathbb{R}^n$

$$\begin{aligned} P(Y_{k_1+p} &\leq y_1, \ldots, Y_{k_n+p} \leq y_n) \\ &= P(X_{T_{k_1+p}} \leq y_1, \ \ldots \ , X_{T_{k_n+p}} \leq y_n) \\ &= \mathbb{E}(P(X_{\Delta_0+\cdots+\Delta_{k_1+p-1}} \leq y_1, \ \ldots \ , X_{\Delta_0+\cdots+\Delta_{k_n+p-1}} \leq y_n | \Delta_0, \ \ldots \ \Delta_{k_n+p-1})) \\ &\qquad \text{by the strict stationarity of } \mathbf{X}, \text{ we get} \\ &= \mathbb{E}(P(X_{\Delta_p+\cdots+\Delta_{k_1+p-1}} \leq y_1, \ \ldots \ , X_{\Delta_p+\cdots+\Delta_{k_n+p-1}} \leq y_n | \Delta_0, \ \ldots \ \Delta_{k_n+p-1})) \\ &= P(X_{U_0+\cdots+U_{k_1-1}} \leq y_1, \ \ldots \ , X_{U_0+\cdots+U_{k_n-1}} \leq y_n) \\ &\qquad \text{where } U_i = \Delta_{i+p} \text{ are i.i.d with density } s \\ &= P(Y_{k_1} \leq y_1, \ \ldots \ , Y_{k_n} \leq y_n) \end{aligned}$$

□

This proposition gives a first result on the distribution of the sampled process: the normality of the process is not preserved by random sampling.

Proposition 2.2. *If* $\mathbf{X}$ *is a Gaussian process then the marginals of the sampled process* $\mathbf{Y}$ *are Gaussian. Furthermore, for* σ_X *not an almost everywhere constant function on the support of* s*,* $\mathbf{Y}$ *is a Gaussian process if and only if the sampling is deterministic.*

Proof. Let Φ_U denote the characteristic function of the random variable U.
We have

$$\Phi_{Y_k}(t) = \mathbb{E}(\mathbb{E}(e^{itX_{T_k}} | T_k)).$$

Given T_k, the conditional probability distribution of X_{T_k} is the centered normal distribution with variance $\sigma_X(0)$. Then

$$\Phi_{Y_k}(t) = e^{-\sigma_X(0)t^2/2}$$

and $Y_k \sim \mathcal{N}(0, \sigma_X(0))$.

It is clear that if $\Delta_i = h$ for all i then Y is Gaussian.
Suppose Y is a Gaussian process, then $Y_1 + Y_2$ is a Gaussian variable,

$$\Phi_{Y_1+Y_2}(t) = e^{-\mathrm{Var}(Y_1+Y_2)t^2/2} = e^{-\sigma(0)t^2} e^{-t^2\mathbb{E}(\sigma_X(T_2-T_1))}$$

and

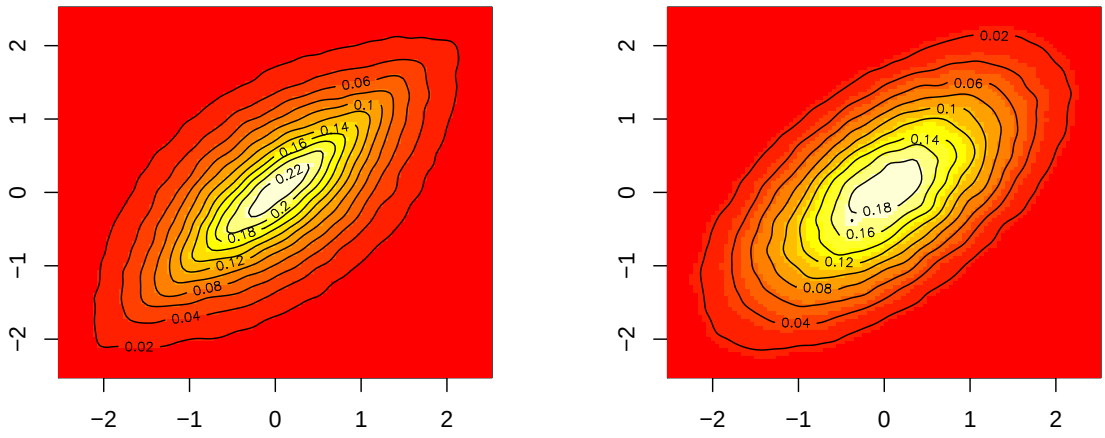
$$\begin{aligned} \Phi_{Y_1+Y_2}(t) &= \Phi_{X_{T_1}+X_{T_2}}(t) \\ &= \mathbb{E}\left(\exp\left[-\frac{t^2}{2}\begin{pmatrix}1\\1\end{pmatrix}^T \begin{pmatrix}\sigma_X(0) & \sigma_X(T_2-T_1)\\ \sigma_X(T_2-T_1) & \sigma_X(0)\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right]\right) \\ &= e^{-\sigma(0)t^2}\mathbb{E}(e^{-t^2\sigma_X(T_2-T_1)}). \end{aligned}$$

Then, for all t, $e^{\mathbb{E}(-t^2\sigma_X(T_2-T_1))} = \mathbb{E}(e^{-t^2\sigma_X(T_2-T_1)})$. It is the equality case in the Jensen inequality, then $\sigma_X(T_2 - T_1)$ is constant almost surely. □

In Figure 1, we illustrate the lack of normality for the sampled process. For $\mathbf{X}$ Gaussian process with autocovariance function $\sigma_X(t) = (1+t^{0.9})^{-1}$, we simulated in Figure (a) the joint probability density function of the centered couple (Y_1, Y_2) for intervals Δ_j having an exponential distribution with mean 1. To do that, we simulate first the time interval $T_2 - T_1$ according to an exponential distribution with mean 1. Then, for each simulated value of $T2-T1$, the couple (Y_1, Y_2) is Gaussian and is simulated using Cholesky decomposition of the variance matrix (see [9]). We simulate by this method $p = 200000$ realizations of (Y_1, Y_2) and find the bivariate density of (Y_1, Y_2) by kernel method. In (b), we represent the density of a centered Gaussian couple (W_1, W_2) with the same variance matrix as (Y_1, Y_2): $E(W_1^2) = E(W_2^2) = \sigma_X(0)$ and $E(W_1W_2) = \mathbb{E}(\sigma_X(\Delta_1))$ in order to compare the behaviour of the sampled process with the corresponding gaussian one. In this case, the bivariate density has an explicit form

$$f_{(W_1,W_2)}(w_1, w_2) = \left(2\pi\sqrt{\det(\Sigma_{1,2})}\right)^{-1} \exp\left(-\frac{1}{2}(w_1\ w_2)\Sigma_{1,2}^{-1}(w_1\ w_2)^T\right)$$

where $\Sigma_{1,2}$ is the common variance matrix. The form of the distribution of sampled process differs widely from Gaussian distribution.



(a) (b)

Figure 1: In Figure (a), the density of the centered couple (Y_1, Y_2) is represented for intervals Δ_j having an exponential distribution with mean 1 and gaussian initial process with autocovariance function $\sigma_X(t) = (1 + t^{0.9})^{-1}$. In (b), we represent the bivariate density of a centered Gaussian vector (W_1, W_2) with the same variance matrix as (Y_1, Y_2): $E(W_1^2) = E(W_2^2) = \sigma_X(0)$ and $E(W_1W_2) = \mathbb{E}(\sigma_X(\Delta_1))$.

2.2 Memory of the sampled process

Proposition 2.3. *Let $p \geq 1$. If there is a positive bounded function $\sigma_*(.)$, non increasing on $\mathbb{R}^+$, such that*

1. $|\sigma_X(t)| \leq \sigma_*(t), \quad \forall t \in \mathbb{R}^+$
2. $\int_{\mathbb{R}^+} \sigma_*^p(t)dt < \infty$

then, the sampled process Y has a covariance function in ℓ^p, i.e $\sum_{h\geq 0} |\sigma_Y(h)|^p < \infty$.

Remark 1. The proposition confirms an intuitive claim: random sampling cannot produce long-memory from short memory. The particular case $p = 1$ implies that if $\mathbf{X}$ has short memory then, the sampled process $\mathbf{Y}$ has short memory too.

Proof. It is clearly enough to prove that

$$\sum_{h\geq 1} \mathbb{E}\left(\sigma_*^p(T_h)\right) < \infty \tag{2.3}$$

We use inequality

$$\Delta_h \sigma_*^p(T_h + \Delta_h) = (T_{h+1} - T_h)\sigma_*^p(T_{h+1}) \leq \int_{T_h}^{T_{h+1}} \sigma_*^p(t)dt, \quad \forall h \geq 0. \tag{2.4}$$

Taking expectation of the left member and noticing that Δ_h and T_h are independent gives, for every $a > 0$,

$$\begin{aligned}
& \mathbb{E}\left(\Delta_h \sigma_*^p(T_h + \Delta_h)\right) = \int_{\mathbb{R}^+} u\mathbb{E}\left(\sigma_*^p(T_h + u)\right) dS(u) \\
= & \int_0^a u\mathbb{E}\left(\sigma_*^p(T_h + u)\right) dS(u) + \int_a^{+\infty} u\mathbb{E}\left(\sigma_*^p(T_h + u)\right) dS(u) \\
\geq & \int_0^a u\mathbb{E}\left(\sigma_*^p(T_h + u)\right) dS(u) + a\int_a^{+\infty} \mathbb{E}\left(\sigma_*^p(T_h + u)\right) dS(u) \\
= & \int_0^a u\mathbb{E}\left(\sigma_*^p(T_h + u)\right) dS(u) + a\left(\int_{\mathbb{R}^+} \mathbb{E}\left(\sigma_*^p(T_h + u)\right) dS(u) - \int_0^a \mathbb{E}\left(\sigma_*^p(T_h + u)\right) dS(u)\right) \\
= & \int_0^a (u - a)\mathbb{E}\left(\sigma_*^p(T_h + u)\right) dS(u) + a\mathbb{E}\left(\sigma_*^p(T_{h+1})\right)
\end{aligned}$$

Now, $\sigma_*^p(T_h + u) \leq \sigma_*^p(T_h)$ and $u - a \leq 0$ for $u \in [0, a]$, leading to

$$\mathbb{E}\left(\Delta_h \sigma_*^p(T_h + \Delta_h)\right) \geq \left(\int_{[0,a[} (u - a)dS(u)\right) \mathbb{E}\left(\sigma_*^p(T_h)\right) + a\mathbb{E}\left(\sigma_*^p(T_{h+1})\right). \tag{2.5}$$

It is possible to choose a such that $S([0, a]) < 1$. For such a choice we obtain

$$0 \leq -\int_{[0,a[} (u - a)dS(u) =: \ell(a) \leq aS([0, a]) < a.$$

Summing up inequalities (2.5) gives, for every $K \geq 0$

$$\begin{aligned}
\mathbb{E}\left[\sum_{h=1}^{\infty} \Delta_h \sigma_*^p(T_{h+1})\right] & \geq \sum_{h=1}^{K} \left[-\ell(a)\mathbb{E}(\sigma_*^p(T_h)) + a\mathbb{E}(\sigma_*^p(T_{h+1}))\right] \\
& = a\left[\mathbb{E}(\sigma_*^p(T_{K+1})) - \mathbb{E}(\sigma_*^p(T_1))\right] + (a - \ell(a))\sum_{h=1}^{K} \mathbb{E}\left(\sigma_*^p(T_h)\right) \\
& \geq -a\sigma_*^p(0) + (a - \ell(a))\sum_{h=1}^{K} \mathbb{E}\left(\sigma_*^p(T_h)\right),
\end{aligned}$$

which implies

$$\mathbb{E}\left[\sum_{h=1}^{\infty} \Delta_h \sigma_*^p(T_{h+1})\right] \geq -a\sigma_*^p(0) + (a - \ell(a))\sum_{h\geq 1} \mathbb{E}\left(\sigma_*^p(T_h)\right)$$

Now, using (2.4)

$$\mathbb{E}\left(\sum_{h\geq 1}\Delta_h\sigma_*^p(T_{h+1})\right)\leq\mathbb{E}\left(\sum_{h\geq 1}\int_{T_h}^{T_{h+1}}\sigma_*^p(t)dt\right)\leq\int_{\mathbb{R}^+}\sigma_*^p(t)dt<\infty$$

and consequently, as $a-\ell(a)>0$

$$\sum_{h=1}^{\infty}\mathbb{E}\left(\sigma_*^p(T_h)\right)<\infty. \tag{2.6}$$

□

Proposition 2.4. *When $\sigma_X(.)$ is ultimately positive and non-increasing on $\mathbb{R}^+$, i.e there exists $t_0\geq 0$ such that $\sigma_X(.)$ is positive and non increasing on the interval $[t_0,\infty)$, and $\mathbb{E}(T_1)<\infty$, then, if $\mathbf{X}$ has long-memory, so has the sampled process $\mathbf{Y}$.*

Proof. Let h_0 be the (random) first index such that $T_{h_0}\geq t_0$. For every $h\geq h_0$,

$$\int_{T_h}^{T_{h+1}}\sigma_X(t)dt\leq(T_{h+1}-T_h)\sigma_X(T_h). \tag{2.7}$$

Summing up gives

$$\sum_{h\geq 1}\mathbb{I}_{h\geq h_0}\int_{T_h}^{T_{h+1}}\sigma_X(t)dt\quad\leq\quad\sum_{h\geq 1}\mathbb{I}_{h\geq h_0}\Delta_h\sigma_X(T_h).$$

Now, taking expectations, and noticing that, since $\mathbb{E}(T_1)=\mathbb{E}(\Delta_1)>0$, the law of large numbers implies that $T_h\xrightarrow{a.s.}\infty$, and in particular $h_0<\infty$ a.s., whence

$$\mathbb{E}\left(\int_{T_{h_0}}^{\infty}\sigma_X(t)dt\right)\quad\leq\quad\mathbb{E}\left[\sum_{h=1}^{\infty}\Delta_h\sigma_X(T_h)\,\mathbb{I}_{h_0\leq h}\right].$$

In this inequality, the left hand side is $+\infty$, and, Δ_h being independent of $\sigma_X(T_h)\,\mathbb{I}_{h_0\leq h}$, the right hand side is $\mathbb{E}(T_1)\sum_{h\geq 1}\mathbb{E}(\sigma_X(T_h)\,\mathbb{I}_{h_0\leq h})$. Consequently, since $\mathbb{E}(T_1)<\infty$,

$$\sum_{h\geq 1}\mathbb{E}(\sigma_X(T_h)\,\mathbb{I}_{h_0\leq h})=\infty. \tag{2.8}$$

It remains to notice that $\mathbb{E}(h_0)<\infty$ (see for example [6] p.185), which implies

$$\sum_{h\geq 1}\mathbb{E}(|\sigma_X(T_h)|\,\mathbb{I}_{h_0>h})\leq\sigma_X(0)\sum_{h\geq 1}P(h_0\geq h)\leq\sigma_X(0)\mathbb{E}(h_0)<\infty,$$

leading, via (2.8) to $\sum_{h\geq 1}|\mathbb{E}(\sigma_X(T_h))|=\infty$.

□

3 Long memory processes

In this section, we consider long memory processes with specific form of covariance function. The assumption is satisified, for example by CARFIMA models.

Definition 3.1. • *A function L on $[0,\infty)$ is said to be slowly varying at infinity if L is positive on $[t_0,\infty)$ for some $t_0>0$ and*

$$\lim_{x\to+\infty}\frac{L(ax)}{L(x)}=1,\qquad\forall a>0$$

- *A function f on $[0,\infty)$ is said to be regularly varying at infinity with index $\delta \in \mathbb{R}$, if f is positive on $[t_0,\infty)$ for some $t_0 > 0$ and*

$$\lim_{x\to+\infty} \frac{f(ax)}{f(x)} = a^{\delta}, \qquad \forall a > 0$$

Now, the auto-covariance σ_X is regularly varying function at infinity of the form

$$\sigma_X(t) = t^{-1+2d}L(t), \qquad \forall t \geq 1 \tag{3.1}$$

where L is slowly varying at infinity and ultimately non-increasing. This means that the parameter $0 < d < 1/2$ summarizes the memory of $\mathbf{X}$ and in order to be more specific about results of Section 2, we describe what can happen to d after sampling.

3.1 Preservation of the memory parameter when $\mathbb{E}(T_1) < \infty$

Theorem 3.2. *Under hypothesis (3.1), if $0 < \mathbb{E}(T_1) < \infty$, the discrete time process $\mathbf{Y}$ has a long memory and its covariance behaves as*

$$\sigma_Y(h) \sim (h\mathbb{E}(T_1))^{-1+2d}L(h) \qquad h \to \infty$$

Remark 2. We can write

$$\sigma_Y(h) = (h\mathbb{E}(T_1))^{-1+2d}\tilde{L}(h)$$

where $\tilde{L}$ is slowly varying at infinity and $\tilde{L}(h) \sim L(h)$ as $h \to \infty$.

Proof. • We show first that

$$\liminf_{h\to\infty} \frac{\sigma_Y(h)}{(h\mathbb{E}(T_1))^{-1+2d}L(h)} \geq 1$$

Let $0 < c < \mathbb{E}(T_1)$, and $h \in \mathbb{N}$ such that $ch \geq 1$,

$$\sigma_Y(h) \quad \geq \quad \mathbb{E}\left(\sigma_X(T_h)\,\mathbb{I}_{T_h>ch}\right) \geq \inf_{t>ch}\{L(t)t^{2d}\}\mathbb{E}\left(\frac{\mathbb{I}_{T_h>ch}}{T_h}\right)$$

Thanks to Hölder inequality,

$$\left(P(T_h > ch)\right)^2 \leq \mathbb{E}(T_h)\mathbb{E}\left(\frac{\mathbb{I}_{T_h>ch}}{T_h}\right),$$

that is

$$\mathbb{E}\left(\frac{\mathbb{I}_{T_h>ch}}{T_h}\right) \geq \frac{\left(P(T_h > ch)\right)^2}{h\mathbb{E}(T_1)}.$$

Summarizing,

$$\sigma_Y(h) \geq \inf_{t>ch}\{L(t)t^{2d}\}\frac{\left(P(T_h > ch)\right)^2}{h\mathbb{E}(T_1)}$$

$$\frac{\sigma_Y(h)}{(h\mathbb{E}(T_1))^{-1+2d}L(h)} \geq \inf_{t>ch}\{L(t)t^{2d}\}\frac{\left(P(T_h > ch)\right)^2}{(h\mathbb{E}(T_1))^{2d}L(h)} \tag{3.2}$$

Using [2] (Th 1.5.3, p23), we obtain, since $d > 0$

$$\inf_{t\geq ch}\{L(t)t^{2d}\} \sim L(ch)(ch)^{2d} \quad as \quad h \to \infty. \tag{3.3}$$

The law of large numbers implies that $T_h/h \xrightarrow{a.s.} \mathbb{E}(T_1)$. As $c < \mathbb{E}(T_1)$, we have $P(T_h > ch) \to 1$ and the r.h.s. of (3.2) tends to $\left(c/\mathbb{E}(T_1)\right)^{2d}$ as $h \to \infty$. Finally, for all $c < \mathbb{E}(T_1)$,

$$\liminf_{h\to\infty} \frac{\sigma_Y(h)}{(h\mathbb{E}(T_1))^{-1+2d}L(h)} \geq \left(\frac{c}{\mathbb{E}(T_1)}\right)^{2d}$$

Taking the limit as $c \to \mathbb{E}(T_1)$, we get the lower bound.

- Let us now prove

$$\limsup_{h\to\infty} \frac{\sigma_Y(h)}{(h\mathbb{E}(T_1))^{-1+2d}L(h)} \leq 1$$

We use a similar proof to that presented in [10] (Thm 1). We denote for $h \geq 1$ and $0 < s < 1$,

$$\begin{aligned}
\mu_h &= \mathbb{E}(T_h) = h\mathbb{E}(T_1) \\
T_{h,s} &= \sum_{j=0}^{h-1} \Delta_j \, \mathbb{I}_{\Delta_j \leq \mu_h^s/\sqrt{h}} \\
\mu_{h,s} &= \mathbb{E}\left(T_{h,s}\right) = h\mathbb{E}\left(\Delta_0 \, \mathbb{I}_{\Delta_0 \leq \mu_h^s/\sqrt{h}}\right)
\end{aligned}$$

Since $\mathbb{E}(T_1) < \infty$, we have that for $1/2 < s < 1$, $\mu_{h,s} \sim \mu_h$ as $h \to \infty$.
Let $\frac{1}{2} < s < \tau < 1$, t_0 such that $L(.)$ is nonincreasing on $[t_0, \infty)$ and h such that $\mu_{h,s} - \mu_{h,s}^\tau \geq t_0$,

$$\begin{aligned}
\sigma_Y(h) &= \mathbb{E}\left[T_h^{-1+2d} L(T_h) \, \mathbb{I}_{T_{h,s} \geq \mu_{h,s} - \mu_{h,s}^\tau}\right] + \mathbb{E}\left[T_h^{-1+2d} L(T_h) \, \mathbb{I}_{T_{h,s} < \mu_{h,s} - \mu_{h,s}^\tau}\right] \\
&= M_1 + M_2
\end{aligned}$$

$$\begin{aligned}
M_1 &\leq \mathbb{E}\left[T_{h,s}^{-1+2d} L(T_{h,s}) \, \mathbb{I}_{T_{h,s} \geq \mu_{h,s} - \mu_{h,s}^\tau}\right] \leq \left(\mu_{h,s} - \mu_{h,s}^\tau\right)^{-1+2d} L(\mu_{h,s} - \mu_{h,s}^\tau) \\
&= (h\mathbb{E}(T_1))^{-1+2d} L(h) \left(\frac{\mu_{h,s} - \mu_{h,s}^\tau}{h\mathbb{E}(T_1)}\right)^{-1+2d} \frac{L(\mu_{h,s} - \mu_{h,s}^\tau)}{L(h)} \qquad (3.4)
\end{aligned}$$

As $\tau < 1$ and $1/2 < s < 1$, $\left(\frac{\mu_{h,s} - \mu_{h,s}^\tau}{h\mathbb{E}(T_1)}\right)^{-1+2d} \to 1$ as $h \to \infty$. Then,

$$\frac{L(\mu_{h,s} - \mu_{h,s}^\tau)}{L(h)} = \frac{L\left(h\mathbb{E}(T_1)\frac{\mu_{h,s} - \mu_{h,s}^\tau}{h\mathbb{E}(T_1)}\right)}{L(h\mathbb{E}(T_1))} \frac{L(h\mathbb{E}(T_1))}{L(h)}$$

As we have uniform convergence of $\frac{L(h\mathbb{E}(T_1).)}{L(h\mathbb{E}(T_1))}$ to 1 in each interval $[a, b]$ and as $\frac{\mu_{h,s} - \mu_{h,s}^\tau}{h\mathbb{E}(T_1)} \to 1$, we get

$$\frac{L(\mu_{h,s} - \mu_{h,s}^\tau)}{L(h)} \to 1$$

as $h \to \infty$. We obtain

$$M_1 \leq \left(\mu_{h,s} - \mu_{h,s}^\tau\right)^{-1+2d} L(\mu_{h,s} - \mu_{h,s}^\tau) \sim (h\mathbb{E}(T_1))^{-1+2d} L(h) \qquad (3.5)$$

Since $\sup_{t\in\mathbb{R}^+} |\sigma_X(t)| = \sigma_X(0) < \infty$, we have

$$M_2 \leq \sigma_X(0) P\left(T_{h,s} < \mu_{h,s} - \mu_{h,s}^\tau\right) = \sigma_X(0) P\left(-T_{h,s} + \mathbb{E}(T_{h,s}) > \mu_{h,s}^\tau\right)$$

We apply Hoeffding inequality to variables $Z_j = -\Delta_j \, \mathbb{I}_{\Delta_j \leq \mu_h^s/\sqrt{h}}$ which are a.s in $[-\frac{\mu_h^s}{\sqrt{h}}, 0]$ to get,

$$M_2 \leq \sigma_X(0) \exp\left(-2\left(\frac{\mu_{h,s}^\tau}{\mu_h^s}\right)^2\right)$$

and $\left(\frac{\mu_{h,s}^\tau}{\mu_h^s}\right)^2 \sim (h\mathbb{E}(T_1))^{2(\tau-s)}$. Finally

$$M_2 = o((h\mathbb{E}(T_1))^{-1+2d} L(h)) \qquad (3.6)$$

With (3.5) and (3.6), we get the upper bound.

□

Lemma 3.3. *If* $\mathbf{X}$ *is a Gaussian process with regularly varying covariance function* $\sigma_X(t) = L(t)t^{-1+2d}$*, with* $0 < d < 1/2$ *and* L *is slowly varying at infinity and ultimately non-increasing.*
Then, if $\mathbb{E}(T_1) < \infty$*,*

$$L(n)^{-1}n^{-1-2d}\mathrm{Var}(X_{T_1} + \cdots + X_{T_n}|T_1\ , \ldots,\ T_n) \xrightarrow[n\to\infty]{p}\ \gamma_d \tag{3.7}$$

where $\gamma_d := \frac{\mathbb{E}(T_1)^{-1+2d}}{d(1+2d)}$

Proof. See Appendix 6.1. □

Theorem 3.4. *Let* $S_n(\tau) = \sum_{j=1}^{[n\tau]} Y_j$*,* $0 \le \tau \le 1$*, be the partial-sum process. If* $\mathbf{X}$ *is a Gaussian process with regularly varying covariance function* $\sigma_X(t) = L(t)t^{-1+2d}$*, with* $0 < d < 1/2$ *and* L *slowly varying at infinity and ultimately non increasing. Then, if* $\mathbb{E}(T_1) < \infty$*, we get*

$$\gamma_d^{-1/2}L(n)^{-1/2}n^{-1/2-d}S_n(.) \Rightarrow B_{1/2+d}(.), \quad \text{in } \mathcal{D}[0,1] \text{ with the uniform metric.} \tag{3.8}$$

where $B_{1/2+d}$ *is a fractional Brownian motion with parameter* $1/2 + d$ *and* $\gamma_d := \frac{\mathbb{E}(T_1)^{-1+2d}}{d(1+2d)}$*.*

Proof. We first prove the weak convergence in finite-dimensional distributions of

$$\gamma_d^{-1/2}L(n)^{-1/2}n^{-1/2-d}S_n(.)$$

to the corresponding finite-dimensional distributions of $B_{1/2+d}(.)$.
It suffices to show that for every $k \ge 1$, $b_1,\ \ldots\ , b_k \in \mathbb{R}$, $0 \le t_1, \ldots, t_k \le 1$,

$$A_n := \gamma_d^{-1/2}L(n)^{-1/2}n^{-1/2-d}(b_1S_n(t_1) + \cdots + b_kS_n(t_k))$$

satisfies $A_n \xrightarrow{d} b_1B_{1/2+d}(t_1) + \cdots + b_kB_{1/2+d}(t_k)$.
Denote $T^{(n)} = (T_1, \ldots, T_{[n\max_i(t_i)]})$, let calculate the characteristic function of A_n

$$\Phi_{A_n}(t) = \mathbb{E}(e^{itA_n}) = \mathbb{E}(e^{-\frac{t^2}{2}\mathrm{Var}(A_n|T^{(n)})})$$

and

$$\begin{aligned}
&\mathrm{Var}(A_n|T^{(n)}) \\
&= \sum_{i,j=1}^{k} b_ib_j\gamma_d^{-1}L(n)^{-1}n^{-1-2d}\mathbb{E}(S_n(t_i)S_n(t_j)|T^{(n)}) \\
&= \sum_{i,j=1}^{k} \frac{b_ib_j\gamma_d^{-1}L(n)^{-1}n^{-1-2d}}{2}\left[\mathrm{Var}(S_n(t_i)|T^{(n)}) + \mathrm{Var}(S_n(t_j)|T^{(n)}) - \mathrm{Var}(S_n(t_i) - S_n(t_j)|T^{(n)})\right]
\end{aligned}$$

By Lemma 3.3,

$$L(n)^{-1}n^{-1-2d}\mathrm{Var}(Y_1 + \cdots + Y_n|T_1\ , \ldots,\ T_n) \xrightarrow[n\to\infty]{p} \gamma_d.$$

then

$$\gamma_d^{-1}L(n)^{-1}n^{-1-2d}\mathrm{Var}(S_n(t_i)|T^{(n)}) \xrightarrow[n\to\infty]{p} t_i^{1+2d}$$

for $t_i > t_j$

$$\begin{aligned}
\gamma_d^{-1}L(n)^{-1}n^{-1-2d}\mathrm{Var}(S_n(t_i) - S_n(t_j)|T^{(n)}) &= \gamma_d^{-1}L(n)^{-1}n^{-1-2d}\mathrm{Var}(Y_{[nt_i]+1} + \cdots + Y_{[nt_j]}|T^{(n)}) \\
&\xrightarrow[n\to\infty]{p} (t_i - t_j)^{1+2d}.
\end{aligned}$$

Finally,

$$\mathrm{Var}(A_n|T^{(n)}) \xrightarrow[n\to\infty]{p} \sum_{i,j=1}^{k} b_i b_j r_{1/2+d}(t_i,t_j)$$

where $r_{1/2+d}$ is the covariance function of a fractional Brownian motion.
By continuous mapping theorem, for $t \in \mathbb{R}$ fixed, $e^{-\frac{t^2}{2}\mathrm{Var}(A_n|T^{(n)})} \xrightarrow[n\to\infty]{p} e^{-\frac{t^2}{2}\sum_{i,j=1}^{k} b_i b_j r_{1/2+d}(t_i,t_j)}$ and

$$P\left(|e^{-\frac{t^2}{2}\mathrm{Var}(A_n|T^{(n)})}|\right) \leq 1) = 1,$$

then by dominated convergence theorem,

$$e^{-\frac{t^2}{2}\mathrm{Var}(A_n|T^{(n)})} \xrightarrow[n\to\infty]{L^1} e^{-\frac{t^2}{2}\sum_{i,j=1}^{k} b_i b_j r_{1/2+d}(t_i,t_j)}.$$

In particular,

$$\Phi_{A_n}(t) \xrightarrow[n\to\infty]{} e^{-\frac{t^2}{2}\sum_{i,j=1}^{k} b_i b_j r_{1/2+d}(t_i,t_j)}.$$

The sequence of partial-sum processes $\gamma_d^{-1/2} L(n)^{-1/2} n^{-1/2-d} S_n(.)$ is tight with respect to the uniform norm (for the proof of the tightness, see [7] Prop 4.4.2 p78) and then we get the convergence in $\mathcal{D}[0,1]$ with the uniform metric. □

As a consequence of this limit theorem, we valide the nonparametric estimation method based on the re-scaled range (R/S) statistics to estimate d from the sampled process Y. The R/S statistic is defined as follows

$$R_n := \max_{1\leq k\leq n} \sum_{j=1}^{k} (Y_j - \overline{Y_n}) - \min_{1\leq k\leq n} \sum_{j=1}^{k} (Y_j - \overline{Y_n})$$

and

$$S_n := \left(\frac{1}{n} \sum_{j=1}^{n} (Y_j - \overline{Y_n})^2 \right)^{1/2}.$$

Proposition 3.5. *Under the same assumptions as Theorem 3.4, we have*

$$\frac{1}{L(n)^{1/2} n^{1/2+d}} \frac{R_n}{S_n} \xrightarrow[n\to\infty]{d} \mathcal{R}(1) := \sqrt{\frac{\gamma_d}{\sigma_X(0)}} \left(\max_{0\leq t\leq 1} B^0_{1/2+d}(t) - \min_{0\leq t\leq 1} B^0_{1/2+d}(t) \right)$$

where $B^0_{1/2+d}(t) = B_{1/2+d}(t) - tB_{1/2+d}(1)$ is a fractional Brownian bridge and γ_d is defined in Lemma 3.3.

Proof. Using the equality

$$\sum_{j=1}^{k} (Y_j - \overline{Y_n}) = \sum_{j=1}^{k} Y_j - \frac{k}{n} \sum_{j=1}^{n} Y_j = S_n\left(\frac{k}{n}\right) - \frac{k}{n} S_n(1)$$

and the convergence of the partial-sum process in Theorem 3.4, we get that

$$\frac{R_n}{L(n)^{1/2} n^{1/2+d}} \xrightarrow[n\to\infty]{d} \sqrt{\gamma_d} \left(\max_{0\leq t\leq 1} B^0_{1/2+d}(t) - \min_{0\leq t\leq 1} B^0_{1/2+d}(t) \right).$$

Then, we study the convergence in probability of

$$S_n^2 = \frac{1}{n} \sum_{j=1}^{n} Y_j^2 - \left(\frac{1}{n} \sum_{j=1}^{n} Y_j \right)^2.$$

Since $\mathrm{Var}\left(\sum_{j=1}^{n} Y_j\right) \sim Cn^{1+2d}$, we have for $\varepsilon > 0$

$$P\left(\left|\frac{1}{n}\sum_{j=1}^{n} Y_j\right| > \varepsilon\right) \leq \frac{1}{n^2\varepsilon^2}\mathrm{Var}\left(\sum_{j=1}^{n} Y_j\right) \xrightarrow[n\to\infty]{} 0$$

and

$$\begin{aligned} P\left(\left|\frac{1}{n}\sum_{j=1}^{n} Y_j^2 - \sigma_X(0)\right| > \varepsilon\right) \leq & \frac{1}{n^2\varepsilon^2}\mathrm{Var}\left(\sum_{j=1}^{n} Y_j^2\right) \\ &= \frac{1}{n^2\varepsilon^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\mathrm{Cov}\left(Y_j^2, Y_k^2\right) \\ &= \frac{1}{n^2\varepsilon^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\mathbb{E}\left[\mathbb{E}[X_{T_j}^2 X_{T_k}^2|T_j, T_k]\right] - \sigma_X(0)^2\right) \end{aligned}$$

For $(s,t) \in (\mathbb{R}^+)^2$, we decompose X_s^2 and X_t^2 in the complete orthogonal system of Hermite polynomials $(H_k)_{k\geq 0}$, $\left(\frac{X_s}{\sqrt{\sigma_X(0)}}\right)^2 = H_0\left(\frac{X_s}{\sqrt{\sigma_X(0)}}\right) + H_2\left(\frac{X_s}{\sqrt{\sigma_X(0)}}\right)$, then

$$\begin{aligned} \frac{\mathbb{E}[X_s^2 X_t^2]}{\sigma_X(0)^2} = & \mathbb{E}\left[H_0\left(\frac{X_s}{\sqrt{\sigma_X(0)}}\right)H_0\left(\frac{X_t}{\sqrt{\sigma_X(0)}}\right)\right] + \mathbb{E}\left[H_2\left(\frac{X_s}{\sqrt{\sigma_X(0)}}\right)H_0\left(\frac{X_t}{\sqrt{\sigma_X(0)}}\right)\right] \\ & + \mathbb{E}\left[H_0\left(\frac{X_s}{\sqrt{\sigma_X(0)}}\right)H_2\left(\frac{X_t}{\sqrt{\sigma_X(0)}}\right)\right] + \mathbb{E}\left[H_2\left(\frac{X_s}{\sqrt{\sigma_X(0)}}\right)H_2\left(\frac{X_t}{\sqrt{\sigma_X(0)}}\right)\right] \end{aligned}$$

Using the orthogonality property of Hermite polynomials for a bivariate normal density with unit variances (see for instance Prop 2.4.1 of [7]), we obtain

$$\begin{aligned} \mathbb{E}[X_s^2 X_t^2] &= \sigma_X^2(0)\left[1 + 2\mathrm{Cov}^2\left(\frac{X_s}{\sqrt{\sigma_X(0)}}, \frac{X_t}{\sqrt{\sigma_X(0)}}\right)\right] \\ &= \sigma_X^2(0) + 2\sigma_X^2(t-s) \end{aligned}$$

Finally,

$$\begin{aligned} P\left(\left|\frac{1}{n}\sum_{j=1}^{n} Y_j^2 - \sigma_X(0)\right| > \varepsilon\right) \leq & \frac{2}{n^2\varepsilon^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\mathbb{E}\left[\sigma_X^2(T_j - T_k)\right] \\ &= \frac{4}{n^2\varepsilon^2}\sum_{j=0}^{n-1}(n-j)\mathbb{E}\left[\sigma_X^2(T_j)\right] \end{aligned}$$

If $0 \leq d \leq 1/4$, $\frac{1}{n^2}\sum_{j=0}^{n-1}(n-j)\mathbb{E}\left[\sigma_X^2(T_j)\right] \leq \frac{1}{n}\sum_{j=0}^{\infty}\mathbb{E}\left[\sigma_X^2(T_j)\right] \xrightarrow[n\to\infty]{} 0$ using Proposition 2.3 with $p = 1$ and the function σ_X^2. If $1/4 < d < 1/2$, we use Theorem 3.2 to have

$$\frac{1}{n^2}\sum_{j=0}^{n-1}(n-j)\mathbb{E}\left[\sigma_X^2(T_j)\right] \sim Cn^{-2+4d}$$

and we get in both cases that

$$P\left(\left|\frac{1}{n}\sum_{j=1}^{n} Y_j^2 - \sigma_X(0)\right| > \varepsilon\right) \xrightarrow[n\to\infty]{} 0$$

We conclude that $S_n \xrightarrow[n\to\infty]{p} \sqrt{\sigma_X(0)}$ and

$$\frac{1}{L(n)^{1/2}n^{1/2+d}}\frac{R_n}{S_n} \xrightarrow[n\to\infty]{d} \mathcal{R}(1) := \sqrt{\frac{\gamma_d}{\sigma_X(0)}}\left(\max_{0\le t\le 1} B^0_{1/2+d}(t) - \min_{0\le t\le 1} B^0_{1/2+d}(t)\right)$$

□

In the case $L(t) = c > 0$ for all $t > t_0$, taking logarithms of both sizes, we obtain a heuristic identity

$$\log\left(\frac{R_n}{S_n}\right) \sim (1/2+d)\log(n) + \log(\sqrt{c}\mathcal{R}(1))$$

We estimate the slope of the regression line of $(\log(n), \log(R_n/S_n))$ which provides an R/S estimate of d. Remark that for the more general case with L slowly varying at infinity and ultimately non increasing, we have $\log\left(\frac{R_n}{S_n}\right) \sim (1/2+d)\log(n)+\log(L(n))/2+\log(\mathcal{R}(1))$ and $\log(L(n))$ is negligible compared to $\log(n)$.

3.2 Decrease of memory when $\mathbb{E}(T_1) = \infty$

The phenomenon is the same as in the discrete case (see [8]) : starting from a long memory process, an heavy tailed sampling law can lead to a short memory process.

Proposition 3.6. *Assume that the covariance of* $\mathbf{X}$ *satisfies*

$$|\sigma_X(t)| \le c\min(1, t^{-1+2d}) \qquad \forall t \in \mathbb{R}^+ \tag{3.9}$$

where $0 < d < 1/2$. *If for some* $\beta \in (0,1)$

$$\liminf_{x\to\infty}\left(x^\beta P(T_1 > x)\right) > 0 \tag{3.10}$$

(implying $\mathbb{E}(T_1^\beta) = \infty$*) then*

$$|\sigma_Y(h)| \le Ch^{\frac{-1+2d}{\beta}}. \tag{3.11}$$

Proof. From hypothesis (3.9),

$$|\sigma_Y(h)| \le \mathbb{E}(|\sigma_X(T_h)|) \le c\mathbb{E}(\min\{1, T_h^{-1+2d}\})$$

Then, denoting S^{*h} the distribution function of T_h and integrating by parts,

$$\begin{aligned}
\mathbb{E}(\min\{1, T_h^{-1+2d}\}) &= \int_0^1 dS^{*h}(x) + \int_1^\infty x^{-1+2d}dS^{*h}(x) \\
&= S^{*h}(1) + (1-2d)\int_1^\infty x^{-2+2d}S^{*h}(x)dx - S^{*h}(1) \\
&= (1-2d)\int_1^\infty x^{-2+2d}S^{*h}(x)dx.
\end{aligned} \tag{3.12}$$

From hypothesis (3.10) on the tail of the sampling law, it follows that, there exists $C > 0$ and $x_0 \ge 1$ such that

$$\forall x \ge x_0, \quad P(T_1 > x) \ge Cx^{-\beta}.$$

Furthermore for $x \in [1, x_0]$,

$$x^\beta P(T_1 > x) \ge P(T_1 > x_0) \ge Cx_0^{-\beta}.$$

We obtain that $\forall x \ge 1,\ P(T_1 > x) \ge \tilde{C}x^{-\beta}$ with $\tilde{C} = Cx_0^{-\beta}$.

$$\begin{aligned}
S^{*h}(x) &= P(T_h \le x) \le P\left(\max_{0\le l\le h-1}\Delta_l \le x\right) = P\left(T_1 \le x\right)^h \\
&\le \left(1 - \tilde{C}x^{-\beta}\right)^h \le e^{-\frac{\tilde{C}h}{x^\beta}}.
\end{aligned} \tag{3.13}$$

Gathering (3.12) and (3.13) then gives

$$\begin{aligned}\mathbb{E}(\min\{1, T_h^{-1+2d}\}) &\leq (1-2d)\int_1^\infty x^{-2+2d} e^{-\frac{\tilde{C}h}{x^\beta}} dx \\ &= \frac{1-2d}{\beta} h^{-(1-2d)/\beta} \int_0^h u^{(1-2d)/\beta - 1} e^{-\tilde{C}u} du\end{aligned}$$

and the result follows since

$$\int_0^h u^{(1-2d)/\beta-1} e^{-\tilde{C}u} du \xrightarrow{h\to\infty} \int_0^\infty u^{(1-2d)/\beta-1} e^{-\tilde{C}u} du.$$

□

Next proposition states that the bound in Proposition 3.6 is sharp under some additional hypotheses.

Proposition 3.7. *Assume that*

$$\sigma_X(t) = t^{-1+2d} L(t)$$

where $0 < d < 1/2$ and where L is slowly varying at infinity and ultimately monotone.

If

$$\beta =: \sup\{\gamma : \mathbb{E}(T_1^\gamma) < \infty\} \in]0,1[\tag{3.14}$$

then, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\sigma_Y(h) \geq C_\varepsilon h^{-\frac{1-2d}{\beta} - \varepsilon}, \qquad \forall h \geq 1. \tag{3.15}$$

Proof. Let $\varepsilon > 0$. We have

$$\frac{\sigma_X(T_h)}{h^{-\frac{1-2d}{\beta}-\varepsilon}} = \frac{T_h^{-1+2d}}{h^{-\frac{1-2d}{\beta}-\varepsilon}} L(T_h) = \frac{T_h^{-1+2d-\frac{\beta\varepsilon}{2}}}{h^{-\frac{1-2d}{\beta}-\varepsilon}} T_h^{\frac{\beta\varepsilon}{2}} L(T_h) = \left(\frac{T_h}{h^\delta}\right)^{-1+2d-\frac{\beta\varepsilon}{2}} T_h^{\frac{\beta\varepsilon}{2}} L(T_h)$$

where

$$\delta = \frac{(1-2d)/\beta + \varepsilon}{1-2d+\frac{\beta\varepsilon}{2}} = \frac{1}{\beta}\left(\frac{1-2d+\beta\varepsilon}{1-2d+\beta\varepsilon/2}\right).$$

Using Proposition 1.3.6 in [2],

$$T_h^{\frac{\beta\varepsilon}{2}} L(T_h) \xrightarrow{h\to\infty} +\infty \quad \text{a.s}$$

Moreover $\delta > \frac{1}{\beta}$. From (3.14), this implies $\mathbb{E}(T_1^{1/\delta}) < \infty$. Then, the law of large numbers of Marcinkiewicz-Zygmund (see [15] Theorem 3.2.3) yields

$$\frac{T_h}{h^\delta} \xrightarrow{a.s.} 0 \quad \text{as} \quad h \to \infty. \tag{3.16}$$

Therefore by applying Fatou's Lemma

$$\frac{\sigma_Y(h)}{h^{-\frac{1-2d}{\beta}-\varepsilon}} \xrightarrow{h\to\infty} \infty.$$

□

4 Results in frequency domain

Many estimates for long memory processes are based on the behavior of the spectral density (see [7] for a review). In this section we study the existence of the spectral density of sampled process Y. In the next proposition we establish the relation between the spectral densities of X and Y.

Proposition 4.1. *Under hypothesis of Proposition 2.3 with $p \in [1,2]$, both processes $\mathbf{X}$ and $\mathbf{Y}$ have spectral densities and we have the relations*

$$\sigma_Y(k) = \int_{\mathbb{R}} \widehat{S}(x)^k f_X(x)dx, \qquad k \in \mathbb{N} \tag{4.1}$$

$$f_Y(\lambda) = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \left(e^{-i\lambda}\widehat{S}(x)\right)^k f_X(x)dx, \qquad \lambda \in [-\pi,\pi] \tag{4.2}$$

where

$$\hat{S}(x) = E(e^{ixT_1}) = \rho_x e^{i\tau_x} \tag{4.3}$$

is the characteristic function of S. The functions ρ and τ are defined as the absolute value and the argument of the characteristic function.

Proof. We have in this case

$$\int_{\mathbb{R}^+} |\sigma_X(t)|^2 dt \le \sigma_X(0)^{2-p} \int_{\mathbb{R}^+} \sigma_*^p(t)dt < \infty$$

and by Proposition 2.3, $\sum_{h\ge 0} |\sigma_Y(h)|^p < \infty$. Similarly,

$$\sum_{h\ge 0} |\sigma_Y(h)|^2 \le \sigma_X(0)^{2-p} \sum_{h\ge 0} |\sigma_Y(h)|^p < \infty.$$

X and Y have spectral densities $f_X \in L^2(\mathbb{R})$ and $f_Y \in L^2([-\pi,\pi])$. For $\lambda \in [-\pi,\pi]$,

$$\begin{aligned} f_Y(\lambda) &= \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} e^{-ik\lambda} \mathbb{E}(\sigma_X(T_k)) \\ &= \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} e^{-ik\lambda} \mathbb{E}\left(\int_{\mathbb{R}} e^{ixT_k} f_X(x)dx\right) \end{aligned}$$

As $\mathbb{E}\left(\int_{\mathbb{R}} |e^{ixT_k} f_X(x)|dx\right) = \int_{\mathbb{R}} f_X(x)dx = E[X_0^2] < \infty$, we deduce by Fubini's theorem

$$\begin{aligned} f_Y(\lambda) &= \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{-ik\lambda} \mathbb{E}\left(e^{ixT_k}\right) f_X(x)dx \\ &= \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{-ik\lambda} \widehat{S}(x)^k f_X(x)dx \end{aligned}$$

where the series converges in $L^2([-\pi,\pi])$. When $p \neq 1$, the covariance is square summable without being summable. □

Lemma 4.2. *Assume that X has a spectral density denoted by f_X. Then, for all $k \ge 0$*

$$\sigma_Y(k) = \lim_{r\to 1^-} \int_{-\pi}^{\pi} e^{ikx} g(r,x)dx, \tag{4.4}$$

where

$$g(r,x) = \frac{1}{4} \int_{\mathbb{R}} f_X(\lambda) \left(\frac{1}{\pi} + P_{r\rho_\lambda}(\tau_\lambda - x) + P_{r\rho_\lambda}(\tau_\lambda + x)\right) d\lambda \tag{4.5}$$

and where $P_s(t)$ is the Poisson kernel (see Appendix 6.2), and where $\widehat{S}(\lambda) = \rho_\lambda e^{i\tau_\lambda}$.

Proof. See Appendix 6.3. □

Proposition 4.3. *Assume that* $\mathbf{X}$ *has a spectral density denoted by* f_X *which has the form*

$$f_X(\lambda) = |\lambda|^{-2d}\phi(\lambda) \tag{4.6}$$

where ϕ *is non negative, integrable and bounded in a neighbourhood of zero, and where* $0 \leq d < 1/2$ *If one of these conditions holds*

- ***(C1)*** $d = 0$*, i.e* f_X *is bounded in a neighbourhood of zero*
- ***(C2)*** $\mathbb{E}(T_1) < \infty$
- ***(C3)*** *the density of* T_1 *satisfies*

$$s(x) \sim cx^{-\gamma} \text{ when } x \to \infty \tag{4.7}$$

where $1 < \gamma < 2$ *and* $c > 0$ *(in particular* $\mathbb{E}(T_1) = \infty$*),*

then the sampled process has a spectral density f_Y *defined on* $[-\pi, \pi]$ *(except eventually at the point* $x = 0$*) by*

$$f_Y(x) = \lim_{r \to 1^-} g(r, x), \tag{4.8}$$

where g *is defined in (4.5).*

Remark 3. It the process $\mathbf{X}$ has a short memory in the sense that $\mathbf{X}$ has a continuous bounded spectral density f_X, then the result of Proposition 4.3 is still valid .

Proof. See Appendix 6.4 for the proofs. □

5 Comparison between subsampling and temporal aggregation

We start from $\mathbf{X} = (X_t)_{t \in \mathbb{R}^+}$, a zero-mean second-order stationary continuous time second order process with auto-covariance function $\sigma_X(.)$ and from a random walk $(T_n)_{n \geq 0}$ independent of $\mathbf{X}$. The i.i.d. sampling intervals $T_{j+1} - T_j = \Delta_j$ have a common distribution S supported by $\mathbb{R}^+$, with $P(\Delta_0 = 0) = 0$. Moreover, $T_0 = 0$. We define the aggregated process as

$$Z_h = \int_{T_h}^{T_{h+1}} X_t \, \mathrm{d}t \ , \qquad h \in \mathbb{N} \tag{5.1}$$

In [13], the process is aggregated on the interval of the form $[h\Delta, (h+1)\Delta]$ where Δ is a deterministic step. They study the correlation structure of the r-differenced series $\left\{ \bigtriangledown^r \left(\int_{h\Delta}^{(h+1)\Delta} L(u)du \right) \right\}_{h \in \mathbb{N}}$ under the assumption that the rth derivative process of L is a CARFIMA(p, H, q) model. In particular, they give results on the behaviour of the correlation structure when $\Delta \to \infty$.

In this part we compare the auto covariance functions of aggregated and sampled processes.

Lemma 5.1. *The aggregated process defined in* (5.1) *is a stationary zero mean process and its auto covariance function is given by*

$$\sigma_Z(h) = \begin{cases} 2 \iint_{0 \leq s < t} \sigma_X(t-s) P(t < T_1) \, \mathrm{d}t \, \mathrm{d}s & \text{if } h = 0, \\ \iint_{0 \leq s < t} \sigma_X(t-s) P(0 \leq s < T_1, \ T_h \leq t < T_{h+1}) \, \mathrm{d}t \, \mathrm{d}s, & \text{for } h \geq 1 \ . \end{cases}$$

Proof. We have

$$\mathbb{E}(Z_h) = \mathbb{E}\left(\int_{T_h}^{T_{h+1}} X_t \, \mathrm{d}t\right) = \int \mathbb{E}(X_t \, \mathbb{I}_{T_h \le t < T_{h+1}}) \, \mathrm{d}t$$
$$= \int \mathbb{E}(X_t)\mathbb{E}(\, \mathbb{I}_{T_h \le t < T_{h+1}}) \, \mathrm{d}t = 0$$

since X and T are independent.
Let $p \in \mathbb{N}$ and $h \ge 1$,

$$\begin{aligned}
&\mathbb{E}(Z_{h+p} Z_p) \qquad (5.2)\\
&= \mathbb{E}\left(\int_{T_p}^{T_{p+1}} X_s \, \mathrm{d}s \int_{T_{h+p}}^{T_{h+p+1}} X_t \, \mathrm{d}t\right)\\
&= \iint \mathbb{E}(X_s X_t \, \mathbb{I}_{T_p \le s < T_{p+1},\ T_{h+p} \le t < T_{h+p+1}}) \, \mathrm{d}t \, \mathrm{d}s\\
&= \iint_{s<t} \sigma_X(t-s) P(T_p \le s < T_{p+1},\ T_{h+p} \le t < T_{h+p+1}) \, \mathrm{d}t \, \mathrm{d}s\\
&= \mathbb{E}\left(\mathbb{E}\left(\iint_{s<t} \sigma_X(t-s) \, \mathbb{I}_{0 \le s - T_p < \Delta_p,\ \Delta_p + \cdots + \Delta_{p+h-1} \le t - T_p < \Delta_p + \cdots + \Delta_{p+h}} \, \mathrm{d}u \, \mathrm{d}v | T_p\right)\right)\\
&= \mathbb{E}\left(\mathbb{E}\left(\iint_{u<v} \sigma_X(v-u) \, \mathbb{I}_{0 \le u < \Delta_p,\ \Delta_p + \cdots + \Delta_{p+h-1} \le v < \Delta_p + \cdots + \Delta_{p+h}} \, \mathrm{d}t \, \mathrm{d}s | T_p\right)\right)\\
&= \iint_{u<v} \sigma_X(v-u) P(0 \le u < \Delta_p,\ \Delta_p + \cdots + \Delta_{p+h-1} \le v < \Delta_p + \cdots + \Delta_{p+h}) \, \mathrm{d}u \, \mathrm{d}v\\
&= \iint_{u<v} \sigma_X(v-u) P(0 \le u < T_1,\ T_h \le v < T_{h+1}) \, \mathrm{d}u \, \mathrm{d}v \qquad (5.3)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(Z_p^2) &= \mathbb{E}\left(\int_{T_p}^{T_{p+1}} X_s \, \mathrm{d}s \int_{T_p}^{T_{p+1}} X_t \, \mathrm{d}t\right)\\
&= 2 \iint_{s<t} \mathbb{E}(X_s X_t \, \mathbb{I}_{T_p \le s < t < T_{h+p+1}}) \, \mathrm{d}t \, \mathrm{d}s\\
&= 2 \iint_{0 \le u < v} \sigma_X(v-u) P(v < T_{h+1}) \, \mathrm{d}u \, \mathrm{d}v \qquad (5.4)
\end{aligned}$$

As (5.3) and (5.4) do not depend on p the stationarity of the process is proved.

□

Example 1. *Poisson sampling.*
In the case of Poisson sampling, the i.i.d. sampling intervals $T_{j+1} - T_j = \Delta_j$ have a common exponential distribution. Let N be the counting process associated with the random walk T :

$$N(t) = \sum_{j \ge 1} \mathbb{I}_{T_j \le t}.$$

The process N has stationary and independent increments and, for every fixed t, $N(t)$ has Poisson distribution with parameter $t\ell$ where $\ell = \mathbb{E}(T_1)^{-1}$. We denote $p(t,k) = P(N(t) = k)$.

For $h \geq 1$,

$$\begin{aligned}
\sigma_Z(h) &= \iint_{0\leq s<t} \sigma_X(t-s) P(N(s)=0,\ N(t)-N(s)=h)\ \mathrm{d}t\ \mathrm{d}s \\
&= \iint_{0\leq s<t} \sigma_X(t-s) p(s,0) p(t-s,h)\ \mathrm{d}t\ \mathrm{d}s \\
&= \iint_{\mathbb{R}_+\times\mathbb{R}_+} \sigma_X(x) p(y,0) p(x,h)\ \mathrm{d}x\ \mathrm{d}y \\
&= \int_{\mathbb{R}_+} \sigma_X(x) p(x,h)\ \mathrm{d}x \int_{\mathbb{R}_+} p(y,0)\ \mathrm{d}y
\end{aligned}$$

Moreover

$$\int_{\mathbb{R}_+} p(y,0)\ \mathrm{d}y = \int_{\mathbb{R}_+} P(T_1 > y)\ \mathrm{d}y = \mathbb{E}(T_1) = \ell^{-1}$$

Thus we have

$$\sigma_Z(h) = \ell^{-1} \int_{\mathbb{R}_+} \sigma_X(x) e^{-\ell x} \ell^h x^h \frac{1}{h!}\ \mathrm{d}x$$

Since the distribution of T_{h+1} is the gamma distribution with parameter $(h+1,\lambda)$ we have

$$\sigma_Z(h) = \ell^{-2} \mathbb{E}(\sigma_X(T_{h+1}))$$

leading to an explicit form of the autocovariance function for the aggregated process with a Poisson sampling

$$\sigma_Z(h) = \mathbb{E}(T_1)^2 \mathbb{E}(\sigma_X(T_{h+1})) = \mathbb{E}(T_1)^2 \sigma_Y(h+1), \qquad \text{for } h \geq 1 \tag{5.5}$$

and with the same calculation

$$\sigma_Z(0) = 2\mathbb{E}(T_1)^2 \mathbb{E}(\sigma_X(T_1)) = 2\mathbb{E}(T_1)^2 \sigma_Y(1) \tag{5.6}$$

Lemma 5.2. *If σ_X is non-increasing on $\mathbb{R}^+$, then for $h \geq 1$*

$$\sigma_Z(h) \leq \mathbb{E}(T_1)^2\ \sigma_Y(h-1) \tag{5.7}$$

Proof. In the r.h.s. of the relation $\sigma_Z(h) = \mathbb{E}(\int_{s<t} \sigma_X(t-s)\, \mathbb{I}_{0\leq s<T_1\leq T_h\leq t<T_{h+1}}\ \mathrm{d}t\ \mathrm{d}s)$,

$$T_h - T_1 \leq t - s \leq T_{h+1}$$

from which follows

$$\sigma_Z(h) \leq \mathbb{E}\left(T_1 (T_{h+1} - T_h) \sigma_X(T_h - T_1)\right).$$

Now, $T_{h+1} - T_h$ is independent of (T_1, T_h) and $T_h - T_1$ is independent of T_1. The result follows. □

Thanks to Lemma 5.2, some properties of the sampled process can be directly transposed to the aggregation scheme. The following proposition brings them together.

Proposition 5.3. *1. If σ_X is dominated by some positive bounded integrable and non increasing function and if $\mathbb{E}(T_1) < \infty$ then $\sum_{h\geq 0} |\sigma_Z(h)| < \infty$.*

2. If $\mathbb{E}(T_1) < \infty$ and $\sigma_X(t) = t^{-1+2d} L(t)$ where L is non-increasing slowing varying at infinity and $0 < d < 1/2$ then,

$$\sigma_Z(h) \leq \sigma_1(h) \sim \mathbb{E}(T_1)^{1+2d} h^{-1+2d} L(h) \qquad h \to \infty$$

3. Suppose $\liminf_{x\to\infty} \left[x^\beta P(T_1 > x)\right] > 0$ for some $\beta \in (0,1)$ (implying $\mathbb{E}(T_1^\beta) = \infty$).
If σ_X is non-increasing on $\mathbb{R}^+$ and $|\sigma_X(t)| \leq c \min(1, t^{-1+2d})$ with $0 < d < 1/2$, then

$$|\sigma_Z(h)| \leq C h^{-(1-2d)/\beta}.$$

Proof. These results are immediat consequences of Lemma 5.2 and respectively Proposition 2.3 with $p = 1$, Theorem 3.2 and Proposition 3.2. □

6 Appendix

6.1 Proof of Lemma 3.3

For the proof of Lemma 3.3, we need the following intermediate result:

Lemma 6.1. *If* $\mathbb{E}(T_1) < \infty$ *and* $\mathbf{X}$ *has a regularly varying covariance function*

$$\sigma_X(t) = L(t)t^{-1+2d}$$

with $0 < d < 1/2$ *and* L *slowly varying at infinity and ultimately nonincreasing. Then,*

$$\mathrm{Var}(\sigma_X(T_h)) = \circ(L(h)^2 h^{-2+4d}) \qquad \text{as } h \to \infty \tag{6.1}$$

Proof. By theorem 3.2, we have $\mathbb{E}(\sigma_X(T_h)) \underset{h\to\infty}{\sim} L(h)(h\mathbb{E}(T_1))^{-1+2d}$. To get the result, it is enough to prove that

$$\mathbb{E}(\sigma_X(T_h)^2) \underset{h\to\infty}{\sim} L(h)^2(h\mathbb{E}(T_1))^{-2+4d}.$$

To prove the asymptotic behaviour of $\mathbb{E}(\sigma_X(T_h)^2)$, we will follow a similar proof as theorem 3.2:

• Let $0 < c < \mathbb{E}(T_1)$, and $h \in \mathbb{N}$ such that $ch \geq 1$,

$$\mathbb{E}(\sigma_X(T_h)^2) \quad \geq \quad \mathbb{E}\left(\sigma_X(T_h)^2 \mathbb{I}_{T_h>ch}\right) \geq \mathbb{E}\left(L(T_h)^2 T_h^{-2+4d} \mathbb{I}_{T_h>ch}\right) \geq \inf_{t>ch}\{L(t)^2 t^{4d}\}\mathbb{E}\left(\frac{\mathbb{I}_{T_h>ch}}{T_h^2}\right)$$

Thanks to Jensen and Hölder inequalities,

$$\mathbb{E}\left(\frac{\mathbb{I}_{T_h>ch}}{T_h^2}\right) \geq \mathbb{E}\left(\frac{\mathbb{I}_{T_h>ch}}{T_h}\right)^2 \text{ and } P(T_h > ch)^2 \leq \mathbb{E}(T_h)\mathbb{E}\left(\frac{\mathbb{I}_{T_h>ch}}{T_h}\right),$$

that is

$$\mathbb{E}\left(\frac{\mathbb{I}_{T_h>ch}}{T_h^2}\right) \geq \frac{P(T_h > ch)^4}{\mathbb{E}(T_h)^2}.$$

Summarizing,

$$\frac{\mathbb{E}(\sigma_X(T_h)^2)}{L(h)^2(h\mathbb{E}(T_1))^{-2+4d}} \geq \frac{\inf_{t>ch}\{L(t)^2t^{4d}\}}{L(h)^2h^{4d}\mathbb{E}(T_1)^{4d}} P(T_h > ch)^4 \tag{6.2}$$

Then, for $c < \mathbb{E}(T_1)$, we have $P(T_h > ch) \to 1$ and $\inf_{t>ch}\{L(t)^2t^{4d}\} \sim L(ch)^2(ch)^{4d}$. Finally, for all $c < \mathbb{E}(T_1)$,

$$\liminf_{h\to\infty} \frac{\mathbb{E}(\sigma_X(T_h)^2)}{L(h)^2(h\mathbb{E}(T_1))^{-2+4d}} \geq \left(\frac{c}{\mathbb{E}(T_1)}\right)^{4d}$$

Taking the limit as $c \to \mathbb{E}(T_1)$, we get

$$\liminf_{h\to\infty} \frac{\mathbb{E}(\sigma_X(T_h)^2)}{L(h)^2(h\mathbb{E}(T_1))^{-2+4d}} \geq 1$$

• Let $\frac{1}{2} < s < \tau < 1$, t_0 such that $L(.)$ is nonincreasing and positive on $[t_0, \infty)$ and h such that $\mu_{h,s} - \mu_{h,s}^\tau \geq t_0$, with the same notation as Theorem 3.2,

$$\begin{aligned}
\mathbb{E}(\sigma_X(T_h)^2) &= \mathbb{E}\left[L(T_h)^2 T_h^{-2+4d} \mathbb{I}_{T_{h,s} \geq \mu_{h,s} - \mu_{h,s}^\tau}\right] + \mathbb{E}\left[\sigma(T_h)^2 \mathbb{I}_{T_{h,s} < \mu_{h,s} - \mu_{h,s}^\tau}\right] \\
&\leq L(\mu_{h,s} - \mu_{h,s}^\tau)^2 \left(\mu_{h,s} - \mu_{h,s}^\tau\right)^{-2+4d} + \sigma_X(0)^2 P\left(T_{h,s} < \mu_{h,s} - \mu_{h,s}^\tau\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\mathbb{E}(\sigma_X(T_h)^2)}{L(h)^2(h\mathbb{E}(T_1))^{-2+4d}} \leq \left(\frac{L(\mu_{h,s} - \mu_{h,s}^\tau)}{L(h)}\right)^2 \left(\frac{\mu_{h,s} - \mu_{h,s}^\tau}{h\mathbb{E}(T_1)}\right)^{-2+4d} \\
+ \sigma_X(0)^2 \frac{P\left(T_{h,s} < \mu_{h,s} - \mu_{h,s}^\tau\right)}{L(h)^2(h\mathbb{E}(T_1))^{-2+4d}}
\end{aligned}$$

Finally

$$\limsup_{h\to\infty} \frac{\mathbb{E}(\sigma_X(T_h)^2)}{L(h)^2(h\mathbb{E}(T_1))^{-2+4d}} \leq 1$$

□

Let return to the proof of Lemma 3.3:
Denote

$$W_n = L(n)^{-1}n^{-1-2d}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_X(T_j - T_i) = L(n)^{-1}n^{-1-2d}\mathrm{Var}(X_{T_1} + \cdots + X_{T_n}|T_1\ ,\ldots,\ T_n)$$

we want to prove that W_n converges in probability to γ_d. For that, we will show that $\mathbb{E}(W_n) \xrightarrow[n\to\infty]{} \gamma_d$ and $\mathrm{Var}(W_n) \xrightarrow[n\to\infty]{} 0$.
• As $\mathbf{X}$ is a centered process $E(W_n) = L(n)^{-1}n^{-1-2d}\mathrm{Var}(Y_1 + \cdots + Y_n)$. By theorem 3.2, we have

$$\sigma_Y(h) \sim L(h)(h\mathbb{E}(T_1))^{-1+2d} \qquad h \to \infty,$$

then

$$L(n)^{-1}n^{-1-2d}\mathrm{Var}(Y_1 + \cdots + Y_n) \xrightarrow[n\to\infty]{} \gamma_d \tag{6.3}$$

(see for instance [7] Prop 3.3.1 p.43).
and we obtain

$$E(W_n) \xrightarrow[n\to\infty]{} \gamma_d.$$

• Furthermore,

$$\begin{aligned}
\mathrm{Var}(W_n) &= L(n)^{-2}n^{-2-4d}\mathrm{Var}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_X(T_j - T_i)\right) \\
&\leq L(n)^{-2}n^{-2-4d}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{\mathrm{Var}(\sigma_X(T_j - T_i))}\right)^2 \\
&= \left(2n^{-1-2d}L(n)^{-1}\sum_{h=1}^{n}(n-h)\sqrt{\mathrm{Var}(\sigma_X(T_h))}\right)^2
\end{aligned}$$

Then, by Lemma 6.1, $\sqrt{\mathrm{Var}(\sigma_X(T_h))} = \circ(L(h)h^{-1+2d})$ and $2\sum_{h=1}^{n}(n-h)L(h)h^{-1+2d} \sim \frac{L(n)n^{1+2d}}{d(1+2d)}$.
We get

$$2\sum_{h=1}^{n}(n-h)\sqrt{\mathrm{Var}(\sigma_X(T_h))} = \circ(L(n)n^{1+2d})$$

Finally, $\mathrm{Var}(W_n) = \circ(1)$ which means that $\mathrm{Var}(W_n) \xrightarrow[n\to\infty]{} 0$. We obtain

$$W_n \xrightarrow[n\to\infty]{L^2,\ p} \gamma_d.$$

6.2 Poisson kernel

We recall some properties of the Poisson kernel used in the proof of Appendix.

$$P_s(t) = \frac{1}{2\pi}\left(\frac{1-s^2}{1-2s\cos(t)+s^2}\right)$$

for $s \in [0,1)$, and some of its properties (see for instance [8] for the proofs) :

$$\bullet\ \forall 0 \le s < 1-\eta < 1,\ t \in \mathbb{R}, \qquad 0 \le 2\pi P_s(t) \le 2/\eta \tag{6.4}$$

$$\bullet\ \forall x \in (-\pi,\pi),\ r \in [0,1), \quad \frac{1}{4}\int_{-\pi}^{\pi}\left(\frac{1}{\pi} + P_{r\rho_\lambda}(\tau_\lambda - x) + P_{r\rho_\lambda}(\tau_\lambda + x)\right) d\lambda = 1 \tag{6.5}$$

$$\bullet\ \text{if } 0 < \delta < |t| \le \pi, \text{ then} \quad P_s(t) < P_s(\delta) \tag{6.6}$$

$$\bullet\ 2\pi \sup_{0<s<1} P_s(t) = \begin{cases} 1/|\sin(t)| & \text{if } t \in (-\pi/2, \pi/2) \\ 1 & \text{if } t \in (-\pi, -\pi/2] \cup [\pi/2, \pi) \end{cases} \tag{6.7}$$

where ρ_λ and τ_λ are defined by $\hat{S}(\lambda) = E(e^{i\lambda T_1}) = \rho_\lambda e^{i\tau_\lambda}$.

6.3 Proof of Lemma 4.2

Let us consider the two z-transforms of the bounded sequence $(\sigma_Y(j))_j$:

$$\sigma_Y^-(z) = \sum_{j=0}^{\infty} z^j \sigma_Y(j), \qquad |z| < 1 \tag{6.8}$$

$$\widehat{\sigma}_Y^+(z) = \sum_{j=0}^{\infty} z^{-j} \sigma_Y(j), \qquad |z| > 1 \tag{6.9}$$

On the first hand, as $\mathbb{E}\left(\int_{\mathbb{R}} |e^{ixT_j} f_X(x)|dx\right) = \int_{\mathbb{R}} f_X(x)dx = E[X_0^2] < \infty$, we deduce by Fubini's theorem that

$$\begin{aligned} \sigma_Y(j) &= \mathbb{E}(\sigma_X(T_j)) \\ &= \mathbb{E}\left(\int_{\mathbb{R}} e^{i\lambda T_j} f_X(\lambda) d\lambda\right) \\ &= \int_{\mathbb{R}} \widehat{S}(x)^j f_X(\lambda) d\lambda \end{aligned} \tag{6.10}$$

Gathering (6.10) with (6.8) and (6.9), we have as

$$\sum_{j=0}^{\infty} \int_{\mathbb{R}} |z^j \widehat{S}(\lambda)^j f_X(\lambda)| d\lambda \le \sum_{j=0}^{\infty} |z|^j \int_{\mathbb{R}} f_X(\lambda) d\lambda < \infty, \qquad \text{for } |z| < 1$$

$$\sum_{j=0}^{\infty} \int_{\mathbb{R}} |z^{-j} \widehat{S}(\lambda)^j f_X(\lambda)| d\lambda \le \sum_{j=0}^{\infty} |z|^{-j} \int_{\mathbb{R}} f_X(\lambda) d\lambda < \infty, \qquad \text{for } |z| > 1$$

that,

$$\widehat{\sigma}_Y^-(z) = \int_{\mathbb{R}} \frac{1}{1 - z\widehat{S}(\lambda)} f_X(\lambda) d\lambda, \qquad |z| < 1$$

$$\widehat{\sigma}_Y^+(z) = \int_{\mathbb{R}} \frac{1}{1 - \widehat{S}(\lambda)/z} f_X(\lambda) d\lambda, \qquad |z| > 1$$

On the second hand, let C_r be the circle $|z| = r$. If $0 < r < 1$, for all $j \ge 0$

$$\begin{aligned} \frac{1}{2i\pi}\int_{C_r} \widehat{\sigma}_Y^-(z) z^{-k-1} dz &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (re^{ix})^{-k} \widehat{\sigma}_Y^-(re^{ix}) dx \\ &= \frac{1}{2\pi}\sum_{j=0}^{\infty} r^{j-k}\sigma_Y(j)\int_{-\pi}^{\pi} e^{i(j-k)x} dx \\ &= \sigma_Y(k) \end{aligned}$$

and, similarly, if $r > 1$

$$\begin{aligned}\frac{1}{2i\pi}\int_{C_r}\widehat{\sigma}_Y^+(z)z^{k-1}dz &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(re^{ix})^k\widehat{\sigma}_Y^+(re^{ix})dx\\ &= \frac{1}{2\pi}\sum_{j=0}^{\infty}r^{k-j}\sigma_Y(j)\int_{-\pi}^{\pi}e^{i(k-j)x}dx\\ &= \sigma_Y(k)\end{aligned}$$

We finally have

$$\sigma_Y(k) = \begin{cases}\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}e^{-ikx}\left(r^{-k}\int_{\mathbb{R}}\frac{1}{1-re^{ix}\widehat{S}(\lambda)}f_X(\lambda)d\lambda\right)dx & \text{if } r<1,\\ \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}e^{ikx}\left(r^{k}\int_{\mathbb{R}}\frac{1}{1-e^{-ix}\widehat{S}(\lambda)/r}f_X(\lambda)d\lambda\right)dx & \text{if } r>1,\end{cases} \tag{6.11}$$

$$\sigma_Y(k) = \begin{cases}r^{-k}\displaystyle\int_{-\pi}^{\pi}e^{iky}\left(\frac{1}{2\pi}\int_{\mathbb{R}}\frac{1}{1-re^{-iy}\widehat{S}(\lambda)}f_X(\lambda)d\lambda\right)dy & \text{if } r<1,\\ (1/r)^{-k}\displaystyle\int_{-\pi}^{\pi}e^{ikx}\left(\frac{1}{2\pi}\int_{\mathbb{R}}\frac{1}{1-(1/r)e^{-ix}\widehat{S}(-\rho)}f_X(-\rho)d\rho\right)dx & \text{if } 1/r<1,\end{cases} \tag{6.12}$$

As f is an even function and changing r for $1/r$ when $r > 1$ leads to

$$\sigma_Y(k) = r^{-k}\int_{-\pi}^{\pi}e^{ikx}\tilde{g}(r,x)dx, \qquad \forall r\in[0,1)$$

where

$$\tilde{g}(r,x) = \frac{1}{4\pi}\int_{\mathbb{R}}f_X(\lambda)\left(\frac{1}{1-re^{-ix}\widehat{S}(\lambda)}+\frac{1}{1-re^{-ix}\widehat{S}(-\lambda)}\right)d\lambda \tag{6.13}$$

As the first member does not depend on r, we have

$$\sigma_Y(k) = \lim_{r\to1^-}\int_{-\pi}^{\pi}e^{ikx}\tilde{g}(r,x)dx, \qquad \forall r\in[0,1)$$

To conclude the proof we show that the last integrand can be expressed from the Poisson Kernel. Firstly,

$$\begin{aligned}&\text{Im}\quad\left(\frac{1}{1-re^{-ix}\widehat{S}(\lambda)}+\frac{1}{1-re^{-ix}\widehat{S}(-\lambda)}\right)\\ =\quad&\frac{1}{2i}\left(\frac{1}{1-re^{-ix}\widehat{S}(\lambda)}+\frac{1}{1-re^{-ix}\widehat{S}(-\lambda)}-\frac{1}{1-re^{ix}\widehat{S}(-\lambda)}-\frac{1}{1-re^{ix}\widehat{S}(\lambda)}\right)\end{aligned}$$

is an odd function of λ. Hence the imaginary part of the integrand in (4.4) disappears after integration. Secondly, let $z = 1-re^{-ix}\widehat{S}(\lambda)$ and $z' = 1-re^{-ix}\widehat{S}(-\lambda)$

$$Re\left(\frac{1}{z}+\frac{1}{z'}\right) = \frac{1}{2}\left(\frac{1}{z}+\frac{1}{z'}+\frac{1}{\overline{z}}+\frac{1}{\overline{z'}}\right) = \frac{1}{2}\left(\frac{z+\overline{z}}{|z|^2}+\frac{z'+\overline{z'}}{|z'|^2}\right)$$

and

$$\begin{aligned}z+\overline{z} &= 1-re^{-ix}\widehat{S}(\lambda)+1-re^{ix}\widehat{S}(-\lambda)\\ &= 1+(1-re^{-ix}\widehat{S}(\lambda))(1-re^{ix}\widehat{S}(-\lambda))-r^2|\widehat{S}(\lambda)|^2\\ &= 1+|z|^2-(r\rho_\lambda)^2\end{aligned}$$

$$
\begin{aligned}
|z|^2 &= |1 - re^{-ix}\widehat{S}(\lambda)|^2 \\
&= (1 - r\rho_\lambda \cos(\tau_\lambda - x))^2 + (r\rho_\lambda \sin(\tau_\lambda - x))^2 \\
&= 1 - 2r\rho_\lambda \cos(\tau_\lambda - x) + (r\rho_\lambda)^2
\end{aligned}
$$

Finally

$$
Re\left(\frac{1}{1 - re^{-ix}\widehat{S}(\lambda)} + \frac{1}{1 - re^{-ix}\widehat{S}(-\lambda)}\right) = \frac{1}{2}\left(\frac{1 - (r\rho_\lambda)^2}{|1 - re^{-ix}\widehat{S}(\lambda)|^2} + \frac{1 - (r\rho_\lambda)^2}{|1 - re^{-ix}\widehat{S}(-\lambda)|^2} + 2\right) \tag{6.14}
$$

and the proof is over.

6.4 Proof of Proposition 4.3

Firstly, if $x \neq 0$, $g(r, x)$ has a limit as $r \to 1^-$. Hence, we have in (4.4) that

$$
\sigma_Y(k) = \lim_{r \to 1^-} \int_{-\pi}^{\pi} e^{ikx} g(r, x) dx,
$$

the proof of the proposition simply consists in exchanging the limit and integration in order to show that

$$
\sigma_Y(k) = \int_{-\pi}^{\pi} e^{ikx} \lim_{r \to 1^-} g(r, x) dx.
$$

implying that

$$
f_Y(x) = \lim_{r \to 1^-} g(r, x).
$$

Now we prove that conditions of Lebesgue's theorem hold for (4.4).

6.4.1 Under the condition (C1)

We have

$$
|\hat{S}(\lambda)| < 1, \quad \forall \lambda > 0 \qquad \text{and} \qquad |\hat{S}(\lambda)| \to 0 \quad \text{as } |\lambda| \to \infty
$$

(see [6] p.501 and p.514). Hence, thanks to the continuity of $|\hat{S}(\lambda)|$,

$$
\sup_{|\lambda| > \varepsilon} |\hat{S}(\lambda)| < 1, \quad \forall \varepsilon > 0.
$$

The integral (6.13) is split in two parts: choosing $\varepsilon \in [0, \pi]$ such that f_X is bounded on $[-\varepsilon, \varepsilon]$ and using the fact that the real part of the integrand in (6.13) is positive (see (6.14)).

$$
\begin{aligned}
&\int_{-\varepsilon}^{\varepsilon} f_X(\lambda) Re\left(\frac{1}{1 - re^{-ix}\hat{S}(\lambda)} + \frac{1}{1 - re^{-ix}\hat{S}(-\lambda)}\right) d\lambda \\
\leq \quad &\sup_{y \in [-\varepsilon, \varepsilon]} [f_X(y)] \int_{-\varepsilon}^{\varepsilon} Re\left(\frac{1}{1 - re^{-ix}\hat{S}(\lambda)} + \frac{1}{1 - re^{-ix}\hat{S}(-\lambda)}\right) d\lambda
\end{aligned}
$$

which leads, thanks to the property (6.5) and the equation (6.14) between the integrand and the Poisson kernel, to

$$
\begin{aligned}
&\int_{-\varepsilon}^{\varepsilon} f_X(\lambda) Re\left(\frac{1}{1 - re^{-ix}\hat{S}(\lambda)} + \frac{1}{1 - re^{-ix}\hat{S}(-\lambda)}\right) d\lambda \\
&\qquad \leq 4\pi \sup_{y \in [-\varepsilon, \varepsilon]} [f_X(y)] \frac{1}{4} \int_{-\pi}^{\pi} \left(\frac{1}{\pi} + P_{r\rho_\lambda}(\tau_\lambda - x) + P_{r\rho_\lambda}(\tau_\lambda + x)\right) d\lambda \\
&\qquad = 4\pi \sup_{y \in [-\varepsilon, \varepsilon]} [f_X(y)]\,.
\end{aligned} \tag{6.15}
$$

Now,

$$\begin{aligned}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \quad & f_X(\lambda) Re\left(\frac{1}{1-re^{-ix}\hat{S}(\lambda)}+\frac{1}{1-re^{-ix}\hat{S}(-\lambda)}\right)d\lambda \\ &= \pi\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} f_X(\lambda)\left(\frac{1}{\pi}+P_{r\rho_\lambda}(\tau_\lambda-x)+P_{r\rho_\lambda}(\tau_\lambda+x)\right)d\lambda.\end{aligned}$$

Applying the property (6.4) of the Poisson kernel with

$$s=r\rho_\lambda<\rho_\lambda, \quad t=\tau_\lambda\pm x, \quad \text{and} \quad \eta_\varepsilon=1-\sup_{|\lambda|>\varepsilon}|\hat{S}(\lambda)|$$

yields,

$$\begin{aligned}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} f_X(\lambda) Re\left(\frac{1}{1-re^{-ix}\hat{S}(\lambda)}+\frac{1}{1-re^{-ix}\hat{S}(-\lambda)}\right)d\lambda \leq & \quad \left(1+\frac{2}{\eta_\varepsilon}\right)\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} f_X(\lambda)d\lambda \\ \leq & \quad \left(1+\frac{2}{\eta_\varepsilon}\right)\int_{\mathbb{R}} f_X(\lambda)d\lambda. \quad (6.16)\end{aligned}$$

Gathering (6.15) and (6.16) leads to

$$|e^{ikx}g(r,x)|\leq 4\pi\sup_{y\in[-\varepsilon,\varepsilon]}[f_X(y)]+\left(1+\frac{2}{\eta_\varepsilon}\right)\int_{\mathbb{R}} f_X(\lambda)d\lambda, \qquad \forall r\in[0,1),\ \forall x\in(-\pi,\pi)$$

and the proof is complete via Lebesgue's theorem.

6.4.2 Under the condition (C2)

The proof consists in finding an integrable function $g(x)$ such that

$$|g(r,x)|\leq g(x) \quad \forall r\in[0,1), x\in(-\pi,\pi). \tag{6.17}$$

For that purpose, we need the following estimation of $\hat{S}(\lambda)$ near zero.

$$\begin{aligned}|1-\hat{S}(\lambda)| &= \left|(1-e^{i\lambda})\int_{\mathbb{R}+}\frac{1-e^{ix\lambda}}{1-e^{i\lambda}}dS(x)\right| = |1-e^{i\lambda}|\,\left|\int_{\mathbb{R}+}e^{\frac{i(x-1)\lambda}{2}}\frac{\sin(x\lambda/2)}{\sin(\lambda/2)}dS(x)\right| \\ &\leq |1-e^{i\lambda}|\int_{\mathbb{R}+}(1+x)dS(x) \\ &= \left|2i\sin(\lambda/2)e^{i\lambda/2}\right|(1+\mathbb{E}(T_1)) = 2\,|\sin(\lambda/2)|\,(1+\mathbb{E}(T_1)) \leq |\lambda|(1+\mathbb{E}(T_1)) \\ &= C|\lambda|. \end{aligned} \tag{6.18}$$

Now we use the fact that for $|u|\leq u_0\leq 1$, we have

$$|1-u|\geq 1-u_0, \quad |\sin(\arg(1-u))|\leq u_0 \quad \text{and} \quad |\arg(1-u)|\leq \pi u_0/2.$$

From this and inequality (6.18) we obtain

$$|\lambda|\leq\frac{1}{C}\Longrightarrow|\tau_\lambda|\leq\frac{\pi C|\lambda|}{2} \tag{6.19}$$

Recall equation (4.5):

$$g(r,x)=\frac{1}{4}\int_{\mathbb{R}} f_X(\lambda)\left(\frac{1}{\pi}+P_{r\rho_\lambda}(\tau_\lambda-x)+P_{r\rho_\lambda}(\tau_\lambda+x)\right)d\lambda$$

Let λ_0 be such that ϕ is bounded on $[-\lambda_0, \lambda_0]$. For a fixed $x \in (0, \pi)$, denoting

$$b(x) = \min\left\{\lambda_0, \frac{1}{C}, \frac{x}{\pi C}, \frac{2(\pi - x)}{\pi C}\right\},$$

we separate $\mathbb{R}$ into four intervals:

$$(-\infty, -b(x)),\ [-b(x), 0),\ (0, b(x)],\ ]b(x), +\infty).$$

Let us treat the two last intervals (the proof for the other ones is similar) and concerning the integrand in (4.5) we only treat the part $P_{r\rho_\lambda}(\tau_\lambda - x)$:

$$I_1(x) = \int_0^{b(x)} f_X(\lambda) P_{r\rho_\lambda}(\tau_\lambda - x) d\lambda \quad \text{and} \quad I_2(x) = \int_{b(x)}^{+\infty} f_X(\lambda) P_{r\rho_\lambda}(\tau_\lambda - x) d\lambda.$$

• Bounding I_1:
From (6.19), since $|\lambda| \leq \frac{1}{C}$ for λ in $(0, b(x)]$, we have $|\tau_\lambda| \leq \frac{\pi C|\lambda|}{2} \leq \frac{\pi C b(x)}{2} \leq x/2$ which implies

$$\frac{x}{2} \leq |\tau_\lambda - x|$$

and $|\tau_\lambda - x| \leq |\tau_\lambda| + x \leq \frac{\pi C b(x)}{2} + x \leq \pi$.
Via properties (6.6) and (6.7), this leads to

$$P_{r\rho_\lambda}(\tau_\lambda - x) \leq P_{r\rho_\lambda}(x/2) \leq \frac{1}{2\pi} \frac{1}{\sin(x/2)} \leq \frac{1}{2x}.$$

Consequently

$$I_1(x) \leq \frac{\sup_{[0,b(x)]} \phi(.)}{2x} \int_0^{b(x)} \lambda^{-2d} d\lambda = \frac{\sup_{[0,b(x)]} \phi(.)}{2x} \ \frac{b(x)^{-2d+1}}{-2d+1} \leq C_1 x^{-2d}$$

since $b(x) \leq x/(\pi C)$ and $-2d + 1 > 0$.

• Bounding I_2:
When $\lambda > b(x)$, we have $\lambda^{-2d} \leq C_2 \max\{x^{-2d}, 1\}$ for some constant C_2. Hence

$$I_2(x) \leq C_2 \max\{x^{-2d}, 1\} \int_{\mathbb{R}} \phi(\lambda) P_{r\rho_\lambda}(\tau_\lambda - x) d\lambda. \tag{6.20}$$

Since ϕ is bounded in a neighbourhood of zero, the arguments used to prove Proposition 4.3 with condition **(C1)** show that the integral in (6.20) is bounded by a constant.

Finally $I_1 + I_2$ is bounded by an integrable function $g(x)$ and the proposition is proved.

6.4.3 Under the condition (C3)

• We first give the local behaviour of $\hat{S}(\lambda)$ under assumption (4.7).

$$1 - \hat{S}(\lambda) = \int_{\mathbb{R}^+} (1 - \cos(\lambda x)) s(x) dx + i \int_{\mathbb{R}^+} \sin(\lambda x) s(x) dx$$

From the assumption on s,

$$\begin{aligned} \int_{\mathbb{R}^+} (1 - \cos(\lambda x)) s(x) dx &\underset{\lambda \to 0}{\sim} c \int_{\mathbb{R}^+} (1 - \cos(\lambda x)) x^{-\gamma} dx \\ &= c|\lambda|^{\gamma - 1} \int_{\mathbb{R}^+} (1 - \cos(y)) y^{-\gamma} dy \\ &=: c|\lambda|^{\gamma - 1} c_\gamma \end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^+} \sin(\lambda x) s(x) dx &\underset{\lambda\to 0}{\sim} c \int_{\mathbb{R}^+} \sin(\lambda x) x^{-\gamma} dx \\ &= c|\lambda|^{\gamma-1} \int_{\mathbb{R}^+} \sin(y) y^{-\gamma} dy \\ &=: c|\lambda|^{\gamma-1} d_\gamma.\end{aligned}$$

It is clear that $c_\gamma > 0$, and $d_\gamma > 0$. Then,

$$|\lambda|^{1-\gamma}(1-\hat{S}(\lambda)) \underset{\lambda\to 0}{\longrightarrow} Z, \text{ where } Re(Z) > 0 \text{ and } Im(Z) > 0. \tag{6.21}$$

• In the sequel we take $\lambda > 0$. If λ is small enough (say $0 \le \lambda \le \lambda_0$),

$$c_3\lambda^{\gamma-1} \le \quad \tau_\lambda \quad \le c_3'\lambda^{\gamma-1} \qquad \text{where } c_3 \text{ and } c_3' \text{ are negative constants} \tag{6.22}$$

and

$$1 - c_4\lambda^{\gamma-1} \le \quad \rho_\lambda \quad \le 1 - c_4'\lambda^{\gamma-1} \qquad \text{where } c_4 \text{ and } c_4' \text{ are positive constants.} \tag{6.23}$$

Let λ_1 such that ϕ is bounded on $[0, \lambda_1]$ and for a fixed $x \in (0, \pi)$, define

$$c(x) = \min\left\{\lambda_0, \lambda_1, \left(\frac{\pi - x}{-c_3}\right)^{1/(\gamma-1)}, \left(\frac{x}{-c_3'}\right)^{1/(\gamma-1)}\right\}.$$

we deduce from (6.22) for $\lambda \in (0, c(x))$

$$0 < x - c_3'\lambda^{\gamma-1} \le x - \tau_\lambda \le x - c_3\lambda^{\gamma-1} \le \pi \tag{6.24}$$

Then we split $\mathbb{R}$ into four intervals

$$(-\infty, -c(x)),\ [-c(x), 0),\ (0, c(x)],\ (c(x), +\infty).$$

We only consider the integral on the two last domains and the part $P_{r\rho_\lambda}(\tau_\lambda - x)$ of the integrand in (4.5).

• When $\lambda \in (0, c(x)]$, inequality (6.24) and properties (6.6) and (6.7) of the Poisson kernel lead to

$$\begin{gathered}P_{r\rho_\lambda}(\tau_\lambda - x) \le P_{r\rho_\lambda}(x - c_3'\lambda^{\gamma-1}) \\ P_{r\rho_\lambda}(x - c_3'\lambda^{\gamma-1}) \le \begin{cases} \frac{1}{2\pi \sin(x - c_3'\lambda^{\gamma-1})} \le \frac{1}{4(x - c_3'\lambda^{\gamma-1})} & \text{if } x - c_3'\lambda^{\gamma-1} \in [0, \pi/2] \\ \frac{1}{2\pi} \le \frac{1}{2(x - c_3'\lambda^{\gamma-1})} & \text{if } x - c_3'\lambda^{\gamma-1} \in [\pi/2, \pi] \end{cases}\end{gathered} \tag{6.25}$$

$$\begin{aligned}I_1(x) &= \int_0^{c(x)} f_X(\lambda) P_{r\rho_\lambda}(\tau_\lambda - x) d\lambda \le \sup_{[0,\lambda_1]} \phi(.) \int_0^{c(x)} \frac{C_1\lambda^{-2d}}{x - c_3'\lambda^{\gamma-1}} d\lambda \\ &\le C_1' x^{-1} \int_0^{c(x)} \frac{\lambda^{-2d}}{1 - c_3' x^{-1}\lambda^{\gamma-1}} d\lambda \\ &\le C_2' x^{\frac{-2d+1}{\gamma-1}-1} \int_0^1 \frac{u^{\frac{-2d+1}{\gamma-1}-1}}{1+u} du.\end{aligned}$$

Since $\frac{-2d+1}{\gamma-1} - 1 > -1$, the last integral is finite, implying

$$I_1(x) \le C_3' x^{\frac{-2d+1}{\gamma-1}-1},$$

which is an integrable function of x.

- Finally,

$$I_2(x) = \int_{c(x)}^{+\infty} f_X(\lambda) P_{r\rho_\lambda}(\tau_\lambda - x) d\lambda \leq c(x)^{-2d} \int_{\mathbb{R}} \phi(\lambda) P_{r\rho_\lambda}(\tau_\lambda - x) d\lambda$$

which has already been treated in Proposition 4.3 with condition **(C1)** since ϕ is bounded near zero:

$$\int_{\mathbb{R}} \phi(\lambda) P_{r\rho_\lambda}(\tau_\lambda - x) d\lambda \leq C_4$$

and $x \to c(x)^{-2d}$ is an integrable function since $\frac{-2d}{\gamma - 1} > -1$.
Gathering the above results on I_1 and I_2 completes the proof.

References

[1] Beran, J., Feng, Y., Ghosh, S., and Kulik, R. (2013). *Long memory processes: Probabilistic properties and statistical methods*. Springer London.

[2] Bingham N. H., Goldie C. M. and Teugels J. L. (1987). *Regular variation*. Encyclopedia of Mathematics. Cambridge U. Press.

[3] Chambers, M.J. (1996). The estimation of continuous parameter long-memory time series models. Econometric Theory 12 374-390.

[4] Comte, F. (1996). Simulation and estimation of long memory continuous-time models. J. Time Ser. Anal. 17 19-36.

[5] Comte, F. and Renault, E. (1996). Long memory continuous-time models. J. Econometrics 73 101-149.

[6] Feller W. (1971). *An introduction to probability theory and its applications*. Vol. II. Wiley.

[7] Giraitis, L., Koul, H. L., Surgailis, D. (2012). *Large sample inference for long memory processes.* Imperial College Press.

[8] Philippe A., Viano M.-C. (2009). Random sampling of long-memory stationary processes. Journal of Statistical Planning and Inference. Vol 140 pp. 1110-1124

[9] Ripley B. D. (1987). *Stochastic simulation.* Wiley Series in Probability and Statistics.

[10] Shi X., Wu Y., Liu Y. (2010). A note on asymptotic approximations of inverse moments of nonnegative random variables. Stat. Prob. Letters. 80. 1260–1264

[11] Tsai, H. and Chan, K. S. (2005a). Maximum likelihood estimation of linear continuous time long memory processes with discrete time data. J. Roy. Statist. Soc. Ser. B **67**, 703-716.

[12] Tsai, H. and Chan, K. S. (2005b). Quasi-maximum likelihood estimation for a class of continuous-time long-memory processes. J. Time Ser. Anal. **26**, 691-713.

[13] Tsai, H. and Chan, K. S. (2005c). Temporal aggregation of stationary and nonstationary continuous-time processes. Scand. J. Statist. **32**, 583-597.

[14] Tsai, H. (2009) On continuous-time autoregressive fractionally integrated moving average processes. Bernoulli **15** 178–194

[15] Stout W. F. (1974). Almost sure convergence. Academic Press New York.

[16] Viano, M.C., Deniau, C. and Oppenheim, G. (1994). Continuous-time fractional ARMA processes. Statist. Probab. Lett. **21**, 323-336.