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Billel Guelmame, Stéphane Junca, Didier Clamond

REGULARIZING EFFECT FOR CONSERVATION LAWS WITH A LIPSCHITZ CONVEX FLUX

BILLEL GUELMAME, STÉPHANE JUNCA AND DIDIER CLAMOND

Abstract. This paper studies the smoothing effect for entropy solutions of conservation laws with general nonlinear convex fluxes on $\mathbb{R}$. Beside convexity, no additional regularity is assumed on the flux. Thus, we generalize the well-known BV smoothing effect for C^2 uniformly convex fluxes discovered independently by P. D. Lax [23] and O. Oleinik [26], while in the present paper the flux is only locally Lipschitz. Therefore, the wave velocity can be dicontinuous and the one-sided Oleinik inequality is lost. This inequality is usually the fundamental tool to get a sharp regularizing effect for the entropy solution. We modify the wave velocity in order to get an Oleinik inequality useful for the wave front tracking algorithm. Then, we prove that the unique entropy solution belongs to a generalized BV space, BV^Φ.

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Key words: Scalar conservation laws, entropy solution, strictly convex flux, discontinuous velocity, wave front tracking, smoothing effect, BV^Φ spaces.

Contents

1. Introduction

This paper is about the regularization effect on the unique entropy solution of the scalar hyperbolic conservation law

$$u_t + f(u)_x = 0, \qquad u(0,x) = u_0(x), \qquad M = \|u_0\|_\infty. \tag{1}$$

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The initial datum u_0 belongs to $\mathrm{L}^\infty(\mathbb{R},\mathbb{R})$. In (1), f is a nonlinear convex flux on the whole real line, thence f is Lipschitz on $[-M,M]$. The regularity of u for positive time t depends on the nonlinearity of f on $[-M,M]$. (For a linear flux, the solution is nothing but a translation of the initial datum with a constant speed, so no regularity is enforced by the equation (1).) To obtain a smoothing effect, the following Tartar condition [30] is needed:

$$\textit{There are no non-trivial interval where } f \textit{ is affine.} \tag{2}$$

Here, the flux being nonlinear and convex on $\mathbb{R}$, it is strictly convex and thus it necessarily satisfies the condition (2).

In [23, 26], both Lax and Oleinik prove that for an uniform convex flux f such that $f'' > c > 0$ for some constant c (as, e.g., for the Burgers equation), the solution $u(t,\cdot)$ is in $\mathrm{BV}_{\mathrm{loc}}$, for all time $t > 0$. (Definitions of the various BV spaces, spaces of functions of bounded variation, can be found below and also in [1, 25].) This result is no longer true for flatter fluxes [12], such as $f(u) = |u|^3$ and $f(u) = u^4$. The solution regularities in SBV and Sobolev spaces are obtained in [19, 22]. To obtain more information on the regularity of u, generalized BV spaces, BV^s and BV^Φ, are needed. The regularity in those spaces implies the right regularity in Sobolev spaces, as well as the left and right traces for shock waves. For smooth fluxes with a polynomial degeneracy (e.g., $f(u) = |u|^3$, $f(u) = u^4$), the solution $u(t,\cdot)$ belongs to $\mathrm{BV}^s_{\mathrm{loc}}$ in space for $t > 0$ (see the last paragraph of section 2 and [3, 7], and see [2, 24] for non-convex fluxes). This kind of regularity is still true for a C^1 convex flux in a bigger generalized BV space, $u(t,\cdot) \in \mathrm{BV}^\Phi_{\mathrm{loc}}$, $t > 0$ with a convex function Φ depending on the nonlinearity of f [10, 24].

In this paper, we show that this last result [10] remains true for all convex fluxes f on $\mathbb{R}$ satisfying the condition (2), without requiring f to be in C^1. Such a flux can appear in applications, such as in traffic flow model [31] with a concave flux. If f is a strictly convex flux, then the (necessarily increasing) velocity

$$a(u) = f'(u)$$

exists almost everywhere. The set of discontinuity of a is countable, the left and right limits $a^-(u) \leqslant a^+(u)$ existing everywhere. Thanks to the maximum principle, the entropy solution u takes values only in $[-M,M]$, hence a is bounded on $[-M,M]$. In the case of C^1 convex fluxes, the simplest proof (see [7] after [26]) is based on the fundamental one-sided Oleinik inequality [20],

$$a(u(t,x)) - a(u(t,y)) \leqslant (x-y)/t \quad \text{a.e.} \quad x > y, \tag{3}$$

which implies that $a(u)$ is a BV function and then u belongs to a BV^Φ. Unfortunately, this inequality is no longer true for convex Lipschitz fluxes. Indeed, first, $a(u)$ is not well defined because a is not continuous and, second, the Oleinik inequality is not true almost everywhere, as shown in the example 3.1 below. To our knowledge, the loss of the Oleinik inequality appears in the classical literature of conservation laws only in [20], for a piecewise linear flux. Note that, though not always true, the Oleinik inequality is true on a large subset of $\mathbb{R}^+ \times \mathbb{R}$. We prove in this paper that this is enough to still obtain the smoothing effect in the right BV^Φ space with a modified wave velocity and a wave front tracking algorithm for scalar conservation laws [16].

The paper is organized as follows. In section 2, the function Φ is built to state the main theorem. The loss of the Oleinik inequality and the resulting difficulty to prove the main theorem 2.1 is discussed in section 3. The section 4 studies the approximate Riemann problem and a modified Oleinik inequality. The section 5 is devoted to obtaining a BV estimates on the modified velocity by the wave front tracking algorithm. The main result is proved in section 6.

2. The main result

In this section, definitions of weak entropy solutions and BV^Φ spaces are recalled and the function Φ related to the smoothing effect is built. Then, the smoothing effect is stated in Theorem 2.1.

Definition 2.1. *u is called a weak solution of (1), if for all smooth functions θ with a compact support, i.e., for $\theta \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R})$*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^+} [\, u(t,x)\, \theta_t(t,x) \;+\; f(u(t,x))\, \theta_x(t,x)\,]\, \mathrm{d}t\, \mathrm{d}x \;+\; \int_{\mathbb{R}} u_0(x)\, \theta(0,x)\, \mathrm{d}x \;=\; 0. \qquad (4)$$

For a given $u_0 \in \mathrm{L}^\infty$, the equation (4) has at least one weak solution [4, 15], the uniqueness being ensured by the Kruzkov entropy conditions:

Definition 2.2 (Kruzkov entropy solution)**.** *A weak solution of (4) is called an entropy solution if for all positive $\theta \in \mathcal{D}(\mathbb{R}^{+*} \times \mathbb{R})$ and for all convex functions $\eta \in \mathrm{C}^1$ and with $F \overset{def}{=} \int f'(u)\eta'(u)\, \mathrm{d}u$ (primes denoting the derivatives), the following inequality holds:*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^+} [\, \eta(u(t,x))\, \theta_t(t,x) \;+\; F(u(t,x))\, \theta_x(t,x)\,]\, \mathrm{d}t\, \mathrm{d}x \;\geqslant\; 0. \qquad (5)$$

In addition, u has to belong to $\mathrm{C}^0([0,+\infty[, \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}))$.

The functional space BV^Φ [25] is defined as follow.

Definition 2.3. *Let be Φ a convex function such that $\Phi(0) = 0$ and $\Phi(h) > 0$ for $h > 0$, the total Φ-variation of v on $K \subset \mathbb{R}$ is*

$$\mathrm{TV}^\Phi v\ \{K\} = \sup_{p \in \mathcal{P}} \sum_{i=2}^{n} \Phi(|v(x_i) - v(x_{i-1})|) \qquad (6)$$

where $\mathcal{P} = \{\{x_1, \cdots, x_n\},\ x_1 < \cdots < x_n\}$ is the set of all subdivisions of K. The space BV^Φ is defined by $\mathrm{BV}^\Phi = \{v,\ \exists \lambda > 0,\ \mathrm{TV}^\Phi(\lambda v) < \infty\}$.

The BV^Φ space, a generalization of the BV space, is the space of functions with generalized bounded variations. Our goal here is to construct the best convex function Φ, such that u is in $\mathrm{BV}^\Phi_{\mathrm{loc}}$, that means choosing Φ to obtain the smallest space BV^Φ in order to characterize the regularity of the entropy solution.

The one-sided Oleinik inequality is directly linked with the increasing variation of an entropy solution. Such variation is defined with $y^+ = \max(y, 0)$ as follow with the same notations as in the previous definition 2.3.

$$\mathrm{TV}^{\Phi+}v\ \{K\} = \sup_{p\in\mathcal{P}} \sum_{i=2}^{n} \Phi((v(x_i) - v(x_{i-1}))^+) \tag{7}$$

In order to build the function Φ, a definition of the generalized inverse of non decreasing functions is needed

Definition 2.4. *Let be g a non decreasing function from $\mathbb{R}$ to $\mathbb{R}$, the generalized inverse of g is defined on $g([-M, M])$ as following*

$$g^{-1}(y) \ \stackrel{def}{=}\ \inf\{x \in \mathbb{R},\ y \leqslant g(x)\}, \tag{8}$$

Remark 2.1. *It's obvious from the definition that*

$$(g^{-1} \circ g)(x) \leqslant x,\ \forall x. \tag{9}$$

The usual properties of the generalized inverse can be found in [17].

Proposition 2.1 (see proposition 2.3 in [17])**.**

1- *If g is continuous then g^{-1} is a strictly increasing function and*

$$(g \circ g^{-1})(y) = y,\ \forall y \tag{10}$$

2- *If g is strictly increasing then g^{-1} is continuous and*

$$(g^{-1} \circ g)(x) = x,\ \forall x, y \tag{11}$$

Now, the function Φ will be built, as a consequence of the strict convexity of f, $a = f'$ is strictly increasing so its generalized inverse

$$b = \ a^{-1}, \tag{12}$$

is continuous and non decreasing. The function b is constant on $[a^-(u), a^+(u)]$ when a is discontinuous at u.

Let be $\omega[b]$ the modulus of continuity of b, i.e.,

$$\omega[b](h) \ \stackrel{\mathrm{def}}{=} \sup_{\substack{|x-y|\leqslant h \\ x,y\in a([-M,M])}} |b(x) - b(y)|, \tag{13}$$

and let be ϕ the generalized inverse of $\omega[b]$, i.e., $\phi(y) = \inf\{x \in \mathbb{R},\ y \leqslant \omega[b](x)\}$. We denote Φ the convex upper envelope of ϕ, that is related to the nonlinearity of the flux via the velocity. Let us write, in a concise way, the definition of Φ. This function is the key ingredient to define a suitable functional space describing the regularity of entropy solutions.

Definition 2.5 (Choice of Φ)**.** *Φ is the convex upper envelope of the generalized inverse of the modulus of continuity of the generalized inverse of the velocity,*

$$\Phi \ \stackrel{def}{=}\ \text{upper convex envelope of } \left(\omega[a^{-1}]\right)^{-1}. \tag{14}$$

Remark 2.2. *This definition generalizes the ones given in [3, 10] for a discontinuous velocity. This is the optimal choice for a flux with a power law degeneracy, as proved in e.g. [8, 18], for the convex power flux* $f(u) = |u|^{1+p}/(1+p)$, $p \geqslant 1$, $\Phi(u) = |u|^p = |a(u)|$. *Indeed, when the velocity is convex for* $u > 0$ *and is an odd function, then* $\Phi(u) = |a(u)|$ *[10].*

C_w denoting the space of continuous functions with modulus of continuity w, the goal of the paper is to prove the Theorem:

Theorem 2.1 (Regularising effect in BV^Φ)**.** *Let* f *be a strictly convex flux on* $\mathbb{R}$, $u_0 \in \mathrm{L}^\infty$ *and* u *being the unique entropy solution of* (1)*, then* $u(t,\cdot) \in \mathrm{BV}^\Phi_{\mathrm{loc}}$*, i.e., for all* $[\alpha, \beta] \subset \mathbb{R}$,

$$\mathrm{TV}^{\Phi+} u(t,\cdot)\{[\alpha,\beta]\} \;\leqslant\; (\beta-\alpha)\, t^{-1}, \tag{15}$$

$$\mathrm{TV}^{\Phi} u(t,\cdot)\{[\alpha,\beta]\} \;\leqslant\; 2\left(\|a(u_0)\|_\infty + (\beta-\alpha)\, t^{-1}\right), \tag{16}$$

Moreover, if u_0 *is compactly supported, then there exists* $C > 0$ *such that*

$$\mathrm{TV}^{\Phi} u(t,\cdot)\{\mathbb{R}\} \;\leqslant\; C\left(1 + t^{-1}\right), \tag{17}$$

and, in addition, for all $\tau > 0$

$$u \in \mathrm{C}_{w_\tau}\left(]\tau, +\infty[, \mathrm{L}^1_{\mathrm{loc}}\right) \qquad \text{where} \qquad w_\tau(y) = \Phi^{-1}(C\left(1+\tau^{-1}\right) y). \tag{18}$$

Inequality 15 is the natural way to recover the one-sided Oleinik inequality. When Φ is the identity function, so $\mathrm{TV}^\Phi u = \mathrm{TV} u$, the inequality 17 is the classical one for uniformly convex smooth flux. This regularity in time is proven in the last section of the present paper.

Theorem 2.1 covers all previous results (with a different proof) on the smoothing effect for a strictly convex flux, the C^2 case being considered in [3, 7, 9, 8, 22, 23, 24, 26] and the C^1 case being treated in [10]. All these proofs make use, directly or indirectly, of the Oleinik inequality (3). The proof of the theorem 2.1 here is necessarily more complicated due to the loss of the Oleinik inequality. This crucial point is discussed in details in the next section.

3. Notes on the Oleinik inequality for discontinuous wave speeds

The following example shows that the Oleinik inequality (3) is no longer true everywhere when $a = f'$ is not continuous on $[-M, M]$. The Oleinik inequality requires the velocity is defined everywhere. For this purpose, the velocity can be defined everywhere as the mean of its left and right limits with a weight $\lambda \in [0,1]$, i.e.,

$$\bar{a}(x) \;=\; \lambda\, a^+(x) \;+\; (1-\lambda)\, a^-(x).$$

Now, a key result about the Oleinik inequality for the proof of the theorem is stated with this mean velocity.

Let us consider the Riemann problem, that is a Cauchy problem with a piecewise constant datum $u_0(x) = u_l$ for $x < 0$ and $u_0(x) = u_r$ for $x > 0$. The Oleinik inequality is clearly true for a shock wave, but it is not always valid for a rarefaction wave.

Proposition 3.1 (One-sided Oleinik inequality)**.** *For a Riemann problem producing a rarefaction wave — i.e., $u_l < u_r$ — if $x/t, y/t \in \bar{a}([-M, M])$ then the Oleinik inequality holds*

$$\bar{a}(u(t,x)) \;-\; \bar{a}(u(t,y)) \;\leqslant\; (x-y)/t \qquad \text{a.e.} \quad x \,>\, y. \tag{19}$$

The set $\bar{a}([-M, M])$ is not an interval since a is not continuous. This is a reason for the loss of the Oleinik inequality. Moreover, the solution is constant where a is not defined, as shown in the example 3.1 below. When the velocity is continuous this problem disappears, as proved in the next section.

Remark 3.1. *The Oleinik inequality is true a.e. with a velocity chosen as*

$$a^-(u(t,x)) \;-\; a^+(u(t,y)) \;\leqslant\; (x-y)/t \qquad \text{a.e.} \quad x \,>\, y. \tag{20}$$

But, it is less useful to get the BV *estimate for this velocity. This is the key point to prove the* BV^Φ *regularity of the entropy solution. It should also be noticed that the Oleinik inequality* (20) *can be invalid if the signs are exchanged.*

From now on, we denote $\xi = x/t$ and $\eta = y/t$ for brevity.

Example 3.1. *(The one-sided Oleinik inequality is not always valid) Consider*

$$f(u) \;=\; u^2 + |u|, \qquad a(u) \;=\; 2u + \operatorname{sign}(u) \tag{21}$$

and $u_0(x) = \operatorname{sign}(x)$. The entropy solution of (1) *is $u(t,x) = U(\xi)$ with*

$$U(\xi) \;=\; \begin{cases} -1 & \xi \leqslant -3, \\ \frac{1}{2}(\xi+1) & -3 \leqslant \xi \leqslant -1, \\ 0 & -1 \leqslant \xi \leqslant 1, \\ \frac{1}{2}(\xi-1) & 1 \leqslant \xi \leqslant 3, \\ 1 & 3 \leqslant \xi. \end{cases}$$

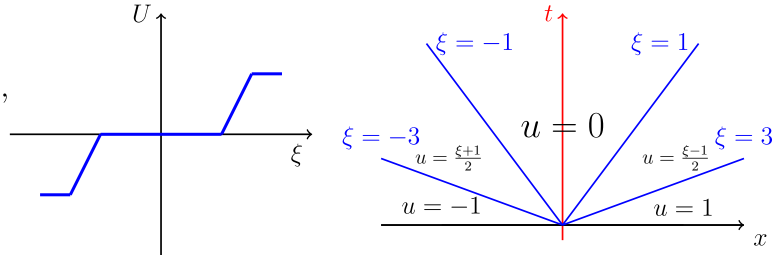

Fig. 1. *The solution of the problem.*

Considering $t > 0$, the Oleinik inequality is not satisfied and $\bar{a}(u(t,x)) - \bar{a}(u(t,y)) > (x-y)/t$ in the following cases

- *if $\lambda = 0$ and $-1 < \eta < 1 < \xi < 3$;*
- *if $\lambda = 1$ and $-3 < \eta < -1 < \xi < 1$;*
- *if $\lambda \in]0,1[$ and $(\; 2\lambda - 1 < \eta < 1 < \xi < 3 \text{ or } -3 < \eta < -1 < \xi < 1 - 2\lambda \;)$.*

Remark 3.2. *The function b is the generalized inverse of $\bar{a}$ for all λ, and $b(\bar{a}(u)) = u$.*

Hereinafter, we take $\lambda = 1/2$ for the sake of simplicity, thus

$$\bar{a}(x) \;=\; \tfrac{1}{2}\left[\, a^+(x) \,+\, a^-(x) \,\right]. \tag{22}$$

Remark 3.3. *If we change the flux f in the example 3.1 by $f(u) = u^2 + u + |u|$ and if $\alpha < 0 < \beta$, then $\bar{a}(u(t,\beta)) - \bar{a}(u(t,\alpha)) \geqslant 1$, where the inequality* (15) *remains true.*

Remark 3.4. *The converse of the proposition 3.1 is false in general. For instance, it is sufficient to take x, y in $]-t, 0[$ or in $]0, t[$ in the example 3.1.*

At this stage, it is important to outline the main difficulties for proving the theorem 2.1. These difficulties result from the discontinuity of the velocity $a(u)$ (yielding the loss of the Oleinik inequality). For instance, consider the case when $a(u)$ is discontinuous at $u = u^\#$ with the jump

$$[\![a]\!](u^\#) = a^+(u^\#) - a^-(u^\#) > 0.$$

Let be $u(x,t)$ a rarefaction wave, non-decreasing solution of the Riemann problem with initial data $u_0(x) = u_l$ for $x < 0$ and $u_0(x) = u_r$ for $x > 0$ with $u_l < u^\# < u_r$. The solution $u(x,t)$ is flat, with value $u = u^\#$ on the interval $x \in [a^-(u^\#) \times t; a^+(u^\#) \times t]$, see figure 1. The length of this interval, at time t, is exactly

$$\Delta x \ = \ t \times [\![a]\!](u^\#).$$

At first sight, it seems a good case where the Oleinik inequality is actually an equality. However, it is not the case for the two following reasons.

First, u being constant on the flat part (with $u \equiv u^\#$), there are of course no variations of u on this part, while there is a variation of a equals to the jump of a at $u^\#$. This shows that the variations of a are bad indicators of the variations of u. Usually, the total variation of u is controlled by the total variation of $a(u)$ [7].

Second, since u is constant on this part, the shock wave penetrates the flat part, reducing the length of this part, i.e.

$$\Delta x < t \times [\![a]\!](u^\#).^{1}$$

In other words, the jump of $a(u)$ at $u^\#$ does not represent well the size of the flat part, which is problematic as already mentioned. As a consequence, the total variation of $a(u)$ is not controlled in the present work. It is an important difference with the case of smooth fluxes, where $a(u)$ is known to be in BV [13], at least C^2, with precise assumptions and counter-examples given in [24]. Here, assuming only that the convex flux is Lipchitz, it is not clear whether $a(u)$ belongs to BV or not.

If the velocity has only one discontinuity, it is easy to overcome this difficulty. Consider

$$a(u) = \tilde{a}(u) + [\![a(u^\#)]\!] H(u - u^\#),$$

where H is the Heaviside function; that is to say $\tilde{a}$ is the continuous part of the velocity $a(u)$. The generalised inverses of a and $\tilde{a}$ having the same modulus of continuity, they define the same space BV^Φ. Moreover, $\tilde{a}$ is easy to estimate in BV. Thus, the BV^Φ regularity of u follows as in [7]. This simple case shows again that the variation of a through its discontinuity is useless to capture the regularity of u. Moreover, for the example 3.1 above, the regularity of u is simply BV since $\tilde{a}$ corresponds to a Burgers flux.

Removing the discontinuity of a can be done only if a has finitely many points of discontinuity. This is not possible in general. Consider the example with the following velocity

$$a(u) = \sum_n 2^{-n} H(u - r_n),$$

[1] Note that it is the converse of the Oleinik inequality.

where the sequence (r_n) takes all the values of rational numbers. This velocity corresponds to a strictly convex flux where the second derivatives has only an atomic part. Thus, u is regularised in some BV^{Φ}. Removing all the discontinuities of a is not a good idea here since the corresponding flux is flat and does not correspond to any smoothing effect.

The way we solve this difficulty is to keep the Oleinik inequality by introducing a new velocity, called χ below. The total variation of χ can be estimate geometrically by the mean of characteristics through a wave front tracking algorithm. Moreover, the variation of χ corresponds exactly to the generalised variation of the entropy solution u.

In order to prove Theorem 2.1, the wave front tracking algorithm will be used. To do so, u_0 is approximated by a sequence of step functions and thus the Riemann problem for each sequence can be solved [16, Lemma 3.1]. A Riemann problem with a discontinuous velocity is expounded in the next section.

4. Riemann problem

This section is devoted to the Riemann problem. The shock wave is solved as usual, but solving the rarefaction wave is more complicated. For this purpose, the flux f is approached by piecewise quadratic $\mathcal{C}^1$ fluxes, in order to show that, for a rarefaction wave, the solution is given by $b\,(x/t)$; this is the classic formula for smooth fluxes where b is the inverse of the velocity. We extend this formula when b is the generalized inverse of a discontinuous velocity. The second part deals with a piecewise linear flux f_ε [16], which gives a modified Oleinik inequality up to a small error.

4.1. **The exact solution of the Riemann problem.** The Riemann problem consists in solving $u_t + f(u)_x = 0$ with the initial condition

$$u(0,x) \;=\; u_l \quad \text{for} \quad x<0 \qquad \text{and} \qquad u(0,x) \;=\; u_r \quad \text{for} \quad x>0. \tag{23}$$

If $u_l > u_r$, the solution generates a shock with a speed given by the Rankine-Hugoniot relation $s = [\![f(u)]\!]/[\![u]\!]$, where $[\![u]\!] = u_r - u_l$ is the jump of u. The entropy solution is

$$u(t,x) \;=\; u_l \quad \text{for} \quad x<st \qquad \text{and} \qquad u(t,x) \;=\; u_r \quad \text{for} \quad x>st.$$

The interesting case is $u_l < u_r$ because the Oleinik inequality is not always true in this case. The solution has a non decreasing rarefaction wave between $x = a^+(u_l)t$ and $x = a^-(u_r)t$, given by the following proposition:

Proposition 4.1. *Let be u the entropy solution of the Riemann problem with $u_l < u_r$. For $\xi = x/t$ the solution is*

$$u(t,x) = \begin{cases} u_l & \xi < a^-(u_l), \\ b\,(\xi) & a^-(u_l) < \xi < a^+(u_r), \\ u_r & \xi > a^+(u_r). \end{cases} \tag{24}$$

Remark 4.1. *A similar formula for systems with Lipschitz fluxes is given in [14, Th. 3.3, p. 279]. Another formula is proposed by Bressan in [4, Problem 3, p. 120]. In Bressan's book, the result is given for all Lipschitz fluxes. One can take the upper convex envelop of the flux instead of the flux in the same formula, thanks to the Oleinik criteria for entropy solutions with general fluxes. Moreover, if the flux is strictly convex, then the solution is defined everywhere by formula (24). Otherwise, it is defined a.e in [4]. To be self contained, the proposition 4.1 has been added here with a short proof (see also [5, 6]).*

For $n \in \mathbb{N}^*$, let be $v_i = u_l + (i/n)(u_r - u_l)$ $(i = 0, 1, \cdots, n)$ and let be a_n the sequence of functions such that $\forall i$

$$a_n(v_i) \;=\; \bar{a}(v_i),$$

a_n being linear on $[v_i, v_{i+1}]$, so $f_n(u) = f(u_l) + \int_{u_l}^{u} a_n(v)\,\mathrm{d}v$. For proving the proposition 4.1, we need the lemma 4.1:

Lemma 4.1. *For all $(v, \xi) \in [-M, M] \times a([-M, M])$, the sequences $f_n(v)$ and $b_n(\xi)$ converge, respectively, to $f(v)$ and $b(\xi)$, and $a_n(v) \to a(v)$ when $n \to \infty$ if a is continuous at v. Moreover, the sequence (b_n) converges uniformly towards b on any bounded set.*

Proof.

- For a given v such that a is continuous at v, $v_i = u_l + i(u_r - u_l)/n$ and v_{i+1} are chosen such that $v \in [v_i, v_{i+1}]$ (note that v_i depends on n). By definition, $a_n(v) = n\frac{\bar{a}(v_{i+1}) - \bar{a}(v_i)}{u_r - u_l}(v - v_i) + \bar{a}(v_i)$. Since $|v - v_i| \leqslant \frac{u_r - u_l}{n}$ and since a is continuous at v, then $\lim\limits_{n\to\infty} a_n(v) = a(v)$. Therefore, $\lim\limits_{n\to\infty} f_n(v) = f(v)$ follows at once from the definition of f_n.
- For $n \in \mathbb{N}^*$ and $\xi \in a([-M, M])$, there exists i such that $b(\xi) \in [v_i, v_{i+1}]$. Moreover, b_n being linear on $[a_n(v_i), a_n(v_{i+1})]$, $|b_n(\xi) - b(\xi)| \leqslant v_{i+1} - v_i = \frac{u_r - u_l}{n}$. In addition, b being continuous, a Dini's Lemma yields the uniform convergence of b_n to b as $n \to \infty$. □

We are now able to prove the Proposition 4.1.

Proof. By definition of a_n, $a_n(u_l) = \bar{a}(u_l)$ and $a_n(u_r) = \bar{a}(u_r)$. Since $f_n \in \mathcal{C}^1$, then, from [4, 15], the exact entropy solution of $u_t + f_n(u)_x = 0$ with the initial condition (23) is

$$u_n(t, x) \;=\; \begin{cases} u_l & \xi < \bar{a}(u_l), \\ b_n(\xi) & \bar{a}(u_l) < \xi < \bar{a}(u_r), \\ u_r & \bar{a}(u_r) < \xi. \end{cases} \tag{25}$$

Taking the limit $n \to \infty$, thanks to Lemma 4.1, u has an explicit continuous formula. Thus, u is a weak entropy solution satisfying (4) and (5). Noting that $b(\xi) = u_l$ $\forall \xi \in [a^-(u_l), a^+(u_l)]$, $\bar{a}(u_l)$ in (25) can be replaced by $a^-(u_l)$ or by $a^+(u_l)$. Similarly, $\bar{a}(u_r)$ can be replaced by $a^-(u_r)$ or $a^+(u_r)$, thus concluding the proof. □

Finally, the proposition 3.1 can be proved:

Proof. Since $x/t, y/t \in \bar{a}([-M, M])$ and since b is the generalized inverse of a, then

$$\bar{a}\left(u(t,x)\right) = \bar{a}\left(b(x/t)\right) = x/t, \quad \bar{a}\left(u(t,y)\right) = \bar{a}\left(b(y/t)\right) = y/t.$$

Therefore, the Oleinik inequality is true. □

4.2. **Approximate Riemann solver.** In the section 5 below, the wave front tracking algorithm is used. Therefore, a suitable Oleinik inequality is needed for the approximate solutions. In order to get this inequality, the flux is replaced by a suitable piecewise linear approximation.

For this purpose, a piecewise constant approximation of the velocity is used as in [16]. A key point is to choose a discrete set of the value of the approximate solution u^ε, taking into account the discontinuities of the velocity. Let be $\varepsilon > 0$ and let be

$$\mathfrak{B} = \{c_0 = -M, c_1, \cdots, c_p = M\} \tag{26}$$

a subdivision of the interval $[-M, M]$ including a too large jump of the velocity. For this purpose the subdivision is chosen such that $c_i < c_{i+1}$, $\forall i$, and

$$a^-(c_{i+1}) \; - \; a^+(c_i) \; \leqslant \; \tfrac{1}{4}\,\varepsilon. \tag{27}$$

That means that the variation of a is small on $]c_i, c_{i+1}[$, $\forall i$, thus the big jumps of a are located at c_i.

Remark 4.2. *Due to the jumps of the velocity a which are not expected on a uniform grid, the subdivision $2^{-n}\mathbb{Z}$ cannot be used as in [4].*

Remark 4.3. $\frac{a^+(u)-a^-(u)}{2} = \frac{[\![a]\!](u)}{2}$ *can be bigger than $\varepsilon/4$, thus, the condition (27) cannot be replaced by $\bar{a}(c_{i+1}) - \bar{a}(c_i) \leqslant \frac{1}{4}\varepsilon$. Notice that in this case u necessarily belongs to $\mathfrak{B}$, because the velocity has a big discontinuity at u.*

Remark 4.4. *The condition (27) is enough to show that if $\varepsilon \to 0$, then $c_{i+1} - c_i \to 0$, $\forall i$, since*

$$c_{i+1} - c_i = b\left(a^-(c_{i+1})\right) - b\left(a^+(c_i)\right) \leqslant \omega[b]\left(a^-(c_{i+1}) \; - \; a^+(c_i)\right) \leqslant \omega[b](\tfrac{1}{4}\,\varepsilon). \tag{28}$$

And $\omega[b]$ is continuous at 0, thanks to the Heine theorem.

As in [16], the flux f is approximated by a continuous and piecewise linear flux. For this purpose, the approximate flux f_ε is chosen as the continuous piecewise linear interpolation of f on the subdivision $\mathfrak{B}$, $f_\varepsilon(c_i) = f(c_i)$ $\forall i$ and f_ε is linear on $[c_i, c_{i+1}]$. Its derivative $a_\varepsilon = f'_\varepsilon$ is piecewise constant.

Now, the approximate Riemann solver is expounded. Let be $u_l, u_r \in \mathfrak{B}$, and let be u^ε the entropy solution of the Riemann problem $u_t + f_\varepsilon(u)_x = 0$, with the initial data (23).

If $u_l > u_r$, as the previous subsection, the solution generates a shock with the Rankine-Hungoniot relation $s = [\![f_\varepsilon(u)]\!]/[\![u]\!] = [\![f(u)]\!]/[\![u]\!]$ (Notice that $f = f_\varepsilon$ on $\mathfrak{B}$).

If $u_r > u_l$, let $u_l = c_k$, $u_r = c_{k'}$, for a fixed $t > 0$, $u^\varepsilon(t, \cdot)$ is non decreasing and piecewise constant. Defining

$$s_i \;=\; \frac{f_\varepsilon(c_i) - f_\varepsilon(c_{i-1})}{c_i - c_{i-1}} \;=\; \frac{f(c_i) - f(c_{i-1})}{c_i - c_{i-1}} = a_\varepsilon(c) \text{ on } (c_{i-1}, c_i). \tag{29}$$

The solution is given by $u^\varepsilon(t,x) = c_i$ for $x/t \in]s_i, s_{i+1}[$ as in [16]. The curves of discontinuity in this case are called contact discontinuities. In fact those curves represent an approximation of a rarefaction wave, so in this paper we call them rarefaction curves.

Since $s_i \leqslant \bar{a}(c_i) \leqslant s_{i+1}$, then (29) implies

$$u(t,ts_i) \;=\; b(s_i) \;\leqslant\; b(\bar{a}(c_i)) \;=\; c_i \;\leqslant\; b(s_{i+1}) \;=\; u(t,ts_{i+1}). \tag{30}$$

For i with $k < i < k'$, the equation $u(t,\tilde{x}_i) = c_i$ has at least one solution since the exact solution u is non decreasing and continuous. Adding the condition $\tilde{\xi}_i = \tilde{x}_i/t \in \bar{a}([-M,M])$, this solution is unique, and $\tilde{x}_i = \bar{a}(c_i)t$ so $\tilde{\xi}_i = \bar{a}(c_i)$.

Let be $\tilde{\xi}_k$, $\tilde{\xi}_k^+$, $\tilde{\xi}_{k'}$, $\tilde{\xi}_{k'}^-$ defined as

- $\tilde{\xi}_k = \bar{a}(u_l)$,
- $\tilde{\xi}_k^+ = a_\varepsilon^+(u_l)$,
- $\tilde{\xi}_{k'} = \bar{a}(u_r)$,
- $\tilde{\xi}_{k'}^- = a_\varepsilon^-(u_r)$.

Now, $\tilde{x}_k, \tilde{x}_k^+, \tilde{x}_{k'}, \tilde{x}_{k'}^-$ are defined by the relation $x = \xi\, t$.

By construction, the approximate solution at the point $(t,\tilde{x}_i)$ equals the exact solution at the same point, i.e., $u^\varepsilon(t,\tilde{x}_i) = u(t,\tilde{x}_i)$ $\forall i = k+1,\dots,k'-1$. Since $\tilde{\xi}_i \in \bar{a}([-M,M])$, then

$$\bar{a}(u^\varepsilon(t,\tilde{x}_{i+1})) - \bar{a}(u^\varepsilon(t,\tilde{x}_i)) \;=\; \bar{a}(b(\tilde{x}_{i+1}/t)) - \bar{a}(b(\tilde{x}_i/t)) \;=\; (\tilde{x}_{i+1}-\tilde{x}_i)/t,\ \forall i = k+1,\dots,k'-2.$$

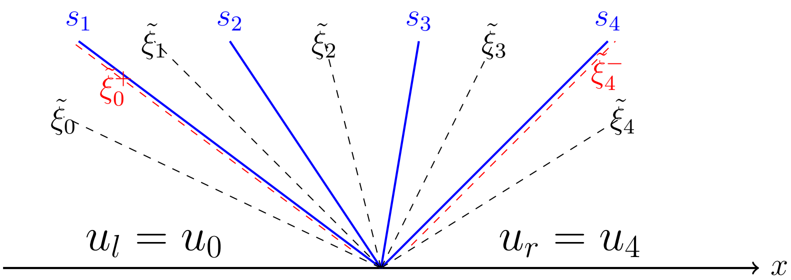

Fig. 2. An example of a rarefaction wave with $k = 0$, $k' = 4$.

Summing up for all i, since

$$a^-(u_{k'}) - a_\varepsilon^-(u_{k'}) \leqslant a^-(u_{k'}) - a^+(u_{k'-1}) \leqslant \varepsilon/4,$$

and

$$a_\varepsilon^+(u_k) - a^+(u_k) \leqslant a^-(u_{k+1}) - a^+(u_k) \leqslant \varepsilon/4,$$

the error is smaller than $\varepsilon/4$ on each boundary term. Thus, the modified Oleinik inequality holds also on the whole interval as

$$\bar{a}(u^\varepsilon(t,\tilde{x}_{k'})) \;-\; \bar{a}(u^\varepsilon(t,\tilde{x}_k)) \;=\; (\tilde{x}_{k'} - \tilde{x}_k)/t \tag{31a}$$

$$a^-(u^\varepsilon(t,\tilde{x}^-_{k'})) \;-\; \bar{a}(u^\varepsilon(t,\tilde{x}_k)) \;\leqslant\; (\tilde{x}^-_{k'} - \tilde{x}_k)/t \;+\; \varepsilon/4, \tag{31b}$$

$$\bar{a}(u^\varepsilon(t,\tilde{x}_{k'})) \;-\; a^+(u^\varepsilon(t,\tilde{x}^+_k)) \;\leqslant\; (\tilde{x}_{k'} - \tilde{x}^+_k)/t \;+\; \varepsilon/4, \tag{31c}$$

$$a^-(u^\varepsilon(t,\tilde{x}^-_{k'})) \;-\; a^+(u^\varepsilon(t,\tilde{x}^+_k)) \;\leqslant\; (\tilde{x}^-_{k'} - \tilde{x}^+_k)/t \;+\; \varepsilon/2. \tag{31d}$$

The value of $u^\varepsilon(t,\tilde{x}^-_{k'})$ is considered on the right of the curve of discontinuity, and the value of $u^\varepsilon(t,\tilde{x}^+_k)$ is considered on its left (see figure 2), i.e., $u^\varepsilon(t,\tilde{x}^-_{k'}) = u_{k'} = u_r$, $u^\varepsilon(t,\tilde{x}^+_k) = u_k = u_l$. This proves the approximated Oleinik inequality for rarefaction waves. □

Here, the rarefaction wave has been approached by a sequence of step functions and satisfies the approximate Oleinik inequality (31). Usually, the Oleinik inequality gives that $a(u)$ is in BV and then u in BV^Φ [7]. Unfortunately, $a(u)$ is not well defined. Moreover, the modified Oleinik inequality (31) does not imply that $\bar{a}(u)$ is BV. The next section is devoted to define another velocity $\chi \cong a(u^\varepsilon)$ which can be controlled in BV with the wave front tracking algorithm and the restricted Oleinik inequality (31).

5. Wave front tracking algorithm

This section deals with the BV estimate of a velocity χ defined below. For that purpose, a BV^+ estimate is used. With $(x)^+ \stackrel{\text{def}}{=} \max\{x,0\}$, the BV^+ space is defined by $\mathrm{BV}^+ \stackrel{\text{def}}{=} \{u,\ \mathrm{TV}^+u < \infty\}$, TV^+ being the *positive total variation*

$$\mathrm{TV}^+ v \;\stackrel{\text{def}}{=}\; \sup_{p\in\mathcal{P}} \sum_{i=2}^{n} \left(v(x_i) - v(x_{i-1})\right)^+, \tag{32}$$

where $\mathcal{P} = \{\{x_1, x_2, \cdots, x_n\},\ x_1 < \cdots < x_n,\ 1 \leqslant n\}$ is the set of all subdivisions of $\mathbb{R}$.

The function u_0 being bounded, we can assume that u_0 has a compact support to prove Theorem 2.1 (thanks to the finite speed of waves propagation).

Let be A positive such that $\mathrm{supp}(u_0) \subset [-A, A]$. Let be $h = A/2m$ and $x_i = -A + hi$ with $i = 0, 1, \cdots, m$. The initial datum u_0 is approached by a sequence of step functions $(u_{0,m})_m$ taking values in $\mathfrak{B}$ (see remark 4.4) and constant in $]x_i, x_{i+1}[$ as in [15, Chapter XIV]. Consider the initial value problem

$$u_t \;+\; f_\varepsilon(u)_x \;=\; 0, \qquad u(0,x) \;=\; u_{0,m}. \tag{33}$$

The entropy solution of (33), u^ε_m, is piecewise constant [4, 15]. The problem (33) requires to solve $m+1$ Riemann problems and the need to study the waves interactions.

Note that in the special case of figure 2, if there is a shock on the right of the rarefaction wave, that has the values u_4 on its left and $u_5 < u_3$ on its right, then the distance between the shock and the rarefaction becomes very small. There, the total variation of $\bar{a}(u)$ is bigger than $[\![a]\!](u_4)$, that is problematic because the total variation of $\bar{a}(u)$ can not be controlled by the distance between the rarefaction wave and the shock wave. To avoid this problem, the velocity on the part $u = u_4$ is replaced by $a^-(u_4)$ instead of $\bar{a}(u_4)$.

In the general case, a new velocity, χ_m^ε, is introduced. This velocity is defined by removing the jumps of $a(u)$ on the boundaries of the rarefaction wave, if this wave is close to a shock. Consider $t > 0$ a fixed time and $x \in \mathbb{R}$. If there is a shock on the left of the point (t, x) and a rarefaction on its right, then

$$\chi_m^\varepsilon(t,x) = a^+(u_m^\varepsilon(t,x)).$$

If there is a shock on the right of the point (t, x) and a rarefaction of its left, then

$$\chi_m^\varepsilon(t,x) = a^-(u_m^\varepsilon(t,x)).$$

Otherwise

$$\chi_m^\varepsilon(t,x) = \bar{a}(u_m^\varepsilon(t,x)).$$

This definition avoids the problem mentioned above for the special case of figure 2.

Note that in all the three cases, the solution u_m^ε can be obtained by $u_m^\varepsilon(t,x) = b(\chi_m^\varepsilon(t,x))$, which is a key point to take the limit in section 6. Note also that if $f \in \mathcal{C}^1$, then $\chi_m^\varepsilon = a(u_m^\varepsilon)$.

The choice of the inequality (31) depends on the following cases:

- If two shocks appear on both sides of the rarefaction wave, then the inequality (31d) is used;
- If a shock appears only on the left of the rarefaction wave, then the inequality (31c) is used;
- If a shock appears only on the right of the rarefaction wave, then the inequality (31b) is used;
- Else, the inequality (31a) is used.

For $t > 0$, the positive total variation of χ_m^ε over a rarefaction is smaller than the length of the rarefaction divided by t, plus a small error (31). For a shock, the positive total variation is equal to zero.

Here, the positive total variation is estimated after wave interactions, as in Bressan's book [4, Chap. 6, Prob. 6]. Let be u_1, u_2 and u_3 the values of u_m^ε from the left to the right. The speed of the left jump is $s_1 = \frac{f_\varepsilon(u_2)-f_\varepsilon(u_1)}{u_2-u_1}$ and the speed of the right jump is $s_2 = \frac{f_\varepsilon(u_2)-f_\varepsilon(u_3)}{u_2-u_3}$. The rarefaction wave is replaced by a contact discontinuity since the flux is piecewise affine. All the possibilities for wave interactions are listed below.

(SS) Shock–shock interaction: $u_3 < u_2 < u_1$. When two shocks collide they generate a new shock, and the positive total variation is always equal to zero.

(RS) Rarefaction–shock interaction: $u_3 \leqslant u_1 < u_2$. the values u_1 and u_2 are consecutive in I, which implies that $u_3 \leqslant u_1$. We consider the two cases:

- $u_1 = u_3$. After the interaction, the curves of discontinuity will disappear, and the positive total variation will be zero.
- $u_3 < u_1$. After the interaction, a shock will appear (see figure), the rarefaction fan will be smaller, and it will lose the curve of discontinuity on the right (between u_1 and u_2). The value of χ_m^ε will be changed from $\bar{a}(u_m^\varepsilon(u_1))$ to $a^-(u_m^\varepsilon(u_1))$ (see the definition of χ_m^ε), and also the choice of the points will be changed from $\tilde{\xi}_1$ to $\tilde{\xi}_1^-$, which makes the inequality (31) holds after the interaction.

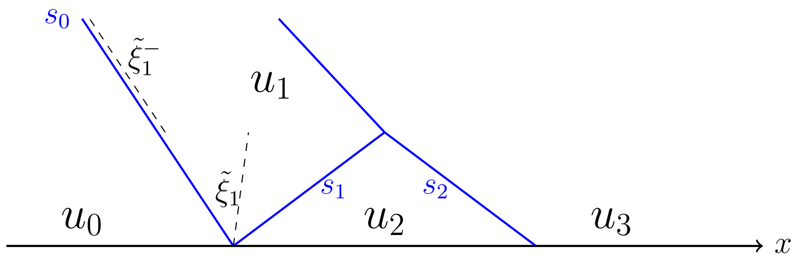

Fig. 3. Interaction (RS).

(SR) Shock–rarefaction interaction: $u_2 < u_3 \leqslant u_1$. This case can be treated exactly like the case Rarefaction–shock.

(RR) Rarefaction–rarefaction interaction: $u_1 < u_2 < u_3$. Two rarefactions cannot collide (even the points $\tilde{x}_i$). This case is impossible, because the convexity of f_ε implies that $s_1 < s_2$.

Remark 5.1. *In the case (RS) the new shock that appears can be very close to the rarefaction, which means that if the jump of the velocity a on u_1 is big enough, then, the positive total variation of $\bar{a}(u_m^\varepsilon)$ cannot be controlled by the length of the rarefaction wave. That is the reason of using the function χ_m^ε.*

In summary, the positive total variation of χ_m^ε and the number of rarefaction waves do not increase. Also, $\mathrm{TV}^+\chi_m^\varepsilon(t,\cdot)$ is bounded by summing up the positive variation of all the rarefaction waves, thanks to the modified one-sided Oleinik inequality (31). For each rarefaction wave, this variation is related to the length of the interval at time $t > 0$ up to a small error $\varepsilon/2$. The sum of all lengths of the intervals cannot exceed the size of the support of the solution. Since the number of rarefaction waves is less than m, there are only m error terms of size less than $\varepsilon/2$. Thence

$$\mathrm{TV}^+\chi_m^\varepsilon(t,\cdot) \;\leqslant\; L(t)/t \;+\; m\,\varepsilon/2, \tag{34}$$

where $L(t) = 2A(t)$ and $\operatorname{supp} u_m^\varepsilon(t,\cdot) \subset [-A(t), A(t)]$. Recall that $\mathrm{BV}^+ \cap \mathrm{L}^\infty = \mathrm{BV}$ since

$$\mathrm{TV}\chi \;\leqslant\; 2\,(\mathrm{TV}^+\chi + \|\chi\|_\infty).$$

The boundedness of the propagation velocity yields $L(t) \leqslant 2A + 2t\|a(u)\|_\infty$. Then, taking the constant $C = C(u_0, f) = \max(4A, 6\|a(u)\|_\infty) > 0$, which doesn't depend on ε and m, gives

$$\mathrm{TV}\chi_m^\varepsilon(t,\cdot)) \;\leqslant\; C\,(1 + 1/t) \;+\; m\,\varepsilon. \tag{35}$$

6. Compactness and regularity

This section is devoted to the proof of Theorem 2.1. For this purpose, an uniform estimate of the velocity χ_m^ε is obtained, which gives the compactness of the sequence (χ_m^ε). To take the limit in (35) as $m \to \infty$ and $\varepsilon \to 0$, the parameters m can be chosen such that

$$\lim_{\varepsilon \to 0} \, m_\varepsilon\, \varepsilon \;=\; 0. \tag{36}$$

We begin with an estimate of the velocity.

The BV estimate of the velocity χ_m^ε proved in the previous section yields to $\mathrm{Lip}_t\mathrm{L}^1_x$ estimates. First, let t a fixed time belonging to $[T_1, T_2] \subset]0, +\infty[$. We have

$$\int_{\mathbb{R}} \Big|\chi_m^\varepsilon(t, x+h) - \chi_m^\varepsilon(t,x)\Big| \mathrm{d}x \ \leqslant \ TV\chi_m^\varepsilon(t,\cdot)|h| \ \leqslant \ \left[C\left(1+\frac{1}{T_1}\right) + m\,\varepsilon\right]|h|. \tag{37}$$

Second, consider two different times $T_1 < T_2$. χ_m^ε is piecewise constant and it has exactly the same curves of discontinuity of u_m^ε. These curves are Lipchitz and the speed of any curve cannot exceed $k = \|a(u_0)\|_\infty$. We suppose at first that there is no wave interaction between T_1 and T_2. Then the domain $[T_1, T_2] \times \mathbb{R}$ can be divided as the following

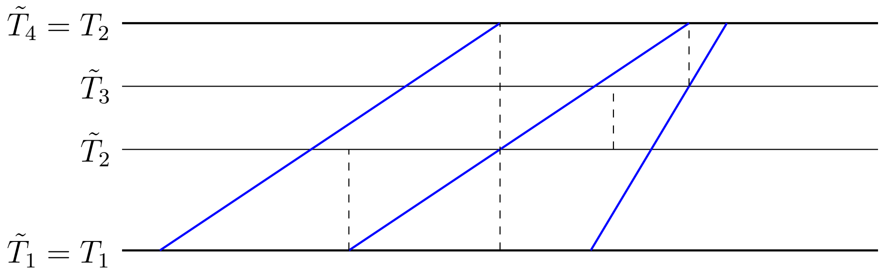

Fig. 4. Decomposition of the domain, the blue lines are the curves of discontinuity.

Using (37) within each small rectangle, gives

$$\int_{\mathbb{R}} \Big|\chi_m^\varepsilon(\tilde{T}_j, x) - \chi_m^\varepsilon(\tilde{T}_{j+1}, x)\Big| \mathrm{d}x \leqslant K\left[C\left(1+\frac{1}{T_1}\right) + m\,\varepsilon\right]\Big|\tilde{T}_{j+1} - \tilde{T}_j\Big|.$$

In general, there are many interactions, so $[T_1, T_2] = \cup_{j=0}^{J}[t_j, t_{j+1}]$, where $t_0 = T_1$, $t_J = T_2$ and the points t_j, $j = 1, \cdots, J-1$ are the instants of the interactions. Let be $0 < \delta < \frac{1}{2}\inf_j(t_{j+1} - t_j)$. The inequality holds true for $t \in [t_j + \delta, t_{j+1} - \delta]$. Taking $\delta \longrightarrow 0$ and using that $\chi_m^\varepsilon(\cdot, x)$ is continuous at t_j for almost all x, the inequality

$$\int_{\mathbb{R}} \Big|\chi_m^\varepsilon(T_1, x) - \chi_m^\varepsilon(T_2, x)\Big| \mathrm{d}x \leqslant K\left[C\left(1+\frac{1}{T_1}\right) + m\,\varepsilon\right]\Big|T_1 - T_2\Big|, \tag{38}$$

follows. Notice that in the estimates (37) and (38), the term $m\,\varepsilon$ is bounded by a constant, thanks to (36). These two inequalities and the uniform boundedness of χ_m^ε by k are the conditions of the classical compactness theorem A.8 in [21]. Hence, the sequence χ_m^ε converges up to a sub-sequence (if necessary) to some function χ in $\mathcal{C}([T_1, T2], \mathrm{L}^1_{\mathrm{loc}})$

$$\chi = \lim_{\varepsilon \to 0} \ \chi_m^\varepsilon. \tag{39}$$

Due to the lower semi-continuity of the total variation, we have $\chi \in BV$ and Lipchitz in time with value in $\mathrm{L}^1_{\mathrm{loc}}$ in space. Thus, for any $0 < t$, χ satisfies

$$\mathrm{TV}\,\chi(t,\cdot) \leqslant C\left(1+\frac{1}{t}\right), \tag{40}$$

$$\int_{\mathbb{R}} \Big|\chi(T_1,x)-\chi(T_2,x)\Big|\mathrm{d}x \leqslant KC\left(1+\frac{1}{T_1}\right)|T_1-T_2|\,. \tag{41}$$

Using that $u_m^\varepsilon = b(\chi_m^\varepsilon)$, which also provides the compactness of the sequence of approximate solutions. Taking the limit $m \to \infty$ in (4) and (5), gives that

$$u = b(\chi) \tag{42}$$

is an entropy solution of (1). The main theorem of [11] (see also [27, 28]) ensures that the initial datum is recovered. Then, u is the unique Kruzkov entropy solution with the initial datum u_0.

Remark 6.1. *Equality (42) means that $\chi = a(u)$ for a smooth velocity. Here, it is not necessarily true almost everywhere since the velocity $a(u)$ can be discontinuous where u is constant.*

Now, the BV^Φ regularity of the entropy solution is proven. Let us first check that Φ is positive. The function b is not constant on the whole interval $[-M,M]$. Then, $\omega(h) > 0$ for $h > 0$. Thanks to Heine theorem, ω is continuous at 0, ensuring that $\phi(y) > 0$ for $y > 0$. Φ the convex envelope of ϕ is then also strictly positive [10].

The BV^Φ regularity of u is a direct consequence of the BV regularity of χ and the definition of Φ, which yield with (42), (13), and remark 2.1 to the following inequality for almost all t_1, t_2, x, y

$$\begin{aligned}
\Phi(|u(t_1,x)-u(t_2,y)|) &= \Phi(|b(\chi(t_1,x))-b(\chi(t_2,y))|) \\
&\leqslant \phi(|b(\chi(t_1,x))-b(\chi(t_2,y))|) \\
&\leqslant \phi\Big(\omega\big(|\chi(t_1,x)-\chi(t_2,y)|\big)\Big) \\
&\leqslant \big|\chi(t_1,x)-\chi(t_2,y)\big|.
\end{aligned} \tag{43}$$

The BV regularity of χ (40) and the inequality (43) show that $u \in BV^\Phi$. The $\mathrm{Lip}_t\mathrm{L}^1_x$ regularity of χ (41) and inequality (43) again implies that

$$\int_{\mathbb{R}} \Phi\Big(\big|u(t_1,x)-u(t_2,x)\big|\Big)\mathrm{d}x \leqslant KC(1+\tau^{-1})|t_1-t_2|. \tag{44}$$

This is an estimate in the Orlicz space L^Φ, i.e., $\mathrm{L}^\Phi(\mathbb{R})$ denotes the set of measurable functions f such that $\int_{\mathbb{R}} \Phi(|f(x)|)\mathrm{d}x < \infty$ [29].

If $\Phi(u) = |u|^p$ then the $\mathrm{Lip}^{\frac{1}{p}}(]\tau,+\infty[,\mathrm{L}^p_{\mathrm{loc}})$ estimate in [3] is recovered. In general, Jensen inequality gives $u \in \mathcal{C}_w(]\tau,+\infty[,\mathrm{L}^1_{\mathrm{loc}})$.

Proof of the inequality (15).

The proof of Theorem 2.1 is done for u_0 compactly supported. In the general case, to prove the inequality (15), the segment $[\alpha, \beta]$ is divided by a subdivision $\mathfrak{S} = \{\alpha_0 = \alpha, \alpha_1, \cdots, \alpha_l = \beta\}$ in order to separate rarefaction waves and shocks. Inequality (31) implies that

$$a^-(u(t, \alpha_{i+1})) - a^+(u(t, \alpha_i)) \leqslant (\alpha_{i+1} - \alpha_i)\, t^{-1} + \varepsilon/4 . \tag{45}$$

Replacing $\chi(t_1, x) - \chi(t_2, y)$ by $a^-(u(t_1, x)) - a^+(u(t_2, y))$ in the inequality (43), the result follows by restarting a similar proof, summing up for all i and taking the limit $m \to +\infty$.

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(**Billel Guelmame**) Université Côte d'Azur, CNRS UMR 7351, LJAD & Inria, Parc Valrose, F-06108 Nice cedex 2, France.
E-mail address: `billel.guelmame@unice.fr`

(**Stéphane Junca**) Université Côte d'Azur, CNRS UMR 7351, LJAD & Inria, Parc Valrose, F-06108 Nice cedex 2, France.
E-mail address: stephane.junca@unice.fr

(**Didier Clamond**) Université Côte d'Azur, CNRS UMR 7351, LJAD, Parc Valrose, F-06108 Nice cedex 2, France.
E-mail address: didier.clamond@unice.fr