Large deviations for the two-dimensional stochastic Navier-Stokes equation with vanishing noise correlation

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Abstract

We are dealing with the validity of a large deviation principle for the two-dimensional Navier-Stokes equation, with periodic boundary conditions, perturbed by a Gaussian random forcing. We are here interested in the regime where both the strength of the noise and its correlation are vanishing, on a length scale ϵ and $\delta(\epsilon)$, respectively, with $0 < \epsilon, \delta(\epsilon) << 1$. Depending on the relationship between ϵ and $\delta(\epsilon)$ we will prove the validity of the large deviation principle in different functional spaces.

1 Introduction

We are dealing here with the following randomly forced two-dimensional incompressible Navier-Stokes equation with periodic boundary conditions, defined on the domain $D = [0, 2\pi]^2$,

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) - (u(t,x)\cdot\nabla)u(t,x) + \nabla p(t,x) + \sqrt{\epsilon}\,\partial_t\,\xi^\delta(t,x), \qquad x \in D, \quad t \geq 0, \\ \\ \operatorname{div} u(t,x) = 0, \qquad x \in D, \quad t \geq 0, \qquad u(0,x) = u_0(x), \quad x \in D. \end{cases} \tag{1.1}$$

Here u denotes the velocity and p denotes the pressure of the fluid. Moreover, $\xi^\delta(t,x)$ denotes a Gaussian random forcing. We are interested in the regime where the noise is weak, that is its typical strength is of order $\sqrt{\epsilon} << 1$, and almost white in space, that is its correlation decays on a length-scale $\delta << 1$.

As well known, in order to have well posedness in $C([0,T];[L^2(D)]^2)$ for equation (1.1), the Gaussian noise ξ^δ cannot be white in space. In fact, white noise in space and time has been considered in [8], where the well-posedness of equation (1.1) has been studied in suitable Besov spaces of negative exponent, for μ_ϵ-almost every initial condition, where μ_ϵ is a suitable centered Gaussian measure, depending on $\epsilon > 0$. It turns out that, for different values of $\epsilon > 0$, the measures μ_ϵ are all singular, so that the result proved in [8] does not imply the

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well posedness of equation (1.1) for any initial datum in some subset of the Besov space that remains independent of $\epsilon > 0$.

In the present paper, we assume that for any fixed $\delta > 0$ the noise $\xi^\delta(t,x)$ is sufficiently smooth in the space variable $x \in D$ to guarantee that for any initial condition $u_0 \in [L^2(D)]^2$ there exists a unique generalized solution in $C([0,T];[L^2(D)]^2)$ (see Section 2 for all details). As a consequence of the contraction principle and of some continuity properties of the solution of equation (1.1), for any $\delta > 0$ fixed, the family $\{\mathcal{L}(u_{\epsilon,\delta})\}_{\epsilon>0}$, given by the solutions of equation (1.1), satisfies a large deviation principle in $C([0,T];[L^2(D)]^2)$, for any $T > 0$ fixed, with rate ϵ and action functional

$$I_T^\delta(f) = \frac{1}{2}\int_0^T |Q_\delta^{-1}\left(f'(t) - Af(t) - b(f(t))\right)|^2_{[L^2(D)]^2}\,dt,$$

where A is the Stokes operator, b is the Navier-Stokes nonlinearity and Q_δ is the square root of the covariance of the noise ξ^δ (see Section 2 for all definitions and notations and also [4]).

In [3], the limiting behaviors, as $\delta \downarrow 0$, for the large deviation action functional I_T^δ, as well as for the corresponding quasipotential V^δ have been studied. Namely it has been proven that if the operator Q_δ converges strongly to the identity operator, and a few other conditions are satisfied, then the operators I_T^δ and V^δ converge pointwise, as $\delta \downarrow 0$, to the operator

$$I_T(f) = \frac{1}{2}\int_0^T |f'(t) - Af(t) - b(f(t))|^2_{[L^2(D)]^2}\,dt, \tag{1.2}$$

and the operator

$$V(x) = |x|^2_{[H^1(D)]^2},$$

respectively. Notice that I_T and V would be the natural candidates for the large deviation action functional in $C([0,T];[L^2(D)]^2)$ and the quasi-potential in $[L^2(D)]^2$, in case equation (1.1), perturbed by space-time white noise, were well-posed in $[L^2(D)]^2$.

In [3] we have first taken the limit in ϵ and then in δ. In the present paper we describe what happens in the relevant case the parameter δ is a function of the parameter ϵ that describes the intensity of the noise, and

$$\lim_{\epsilon\to 0}\delta(\epsilon) = 0. \tag{1.3}$$

Namely, we show that in this case the family $\{u_{\epsilon,\delta(\epsilon)}\}_{\epsilon>0}$ satisfies a large deviation principle in the space $C([0,T];\mathcal{B}_p^\sigma(D))$, where $\mathcal{B}_p^\sigma(D)$ is a suitable Besov space of functions, with $\sigma < 0$ and $p \geq 2$. Moreover, in the case condition (1.3) is supplemented with the condition

$$\lim_{\epsilon\to 0}\epsilon\,\delta(\epsilon)^{-\eta} = 0, \tag{1.4}$$

for some $\eta > 0$, we prove that the family $\{u_{\epsilon,\delta(\epsilon)}\}_{\epsilon>0}$ satisfies a large deviation principle in the space $C([0,T];[L^2(D)]^2)$, where equation (1.1), corresponding to $\delta = 0$, is ill-posed. In both cases, the action functional that describes the large deviation principle is the operator I_T defined in (1.2).

We would like to mention the fact that in [12] Hairer and Weber have studied a similar problem for the stochastic reaction-diffusion equation

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + c\,u(t,\xi) - u^3(t,\xi) + \sqrt{\epsilon}\,\partial_t\,\xi^{\delta(\epsilon)}(t,x), \quad x \in D, \quad t \geq 0, \\ \\ u(0,x) = u_0(x), \quad x \in D, \end{cases} \tag{1.5}$$

where D is a bounded smooth domain, either in $\mathbb{R}^2$ or in $\mathbb{R}^3$. By using the recently developed theory of regularity structures, they study the validity of a large deviation principle for the solutions $\{u_{\epsilon,\delta(\epsilon)}\}_{\epsilon>0}$ of equation (1.5), in the case condition (1.3) is satisfied. Actually, they prove that if, in addition to (1.3), the following conditions hold

$$\lim_{\epsilon\to 0} \epsilon \, \log \delta(\epsilon)^{-1} = \lambda \in [0,\infty), \quad \text{for } d=2, \qquad \lim_{\epsilon\to 0} \epsilon \, \delta(\epsilon)^{-1} = \lambda \in [0,\infty), \quad \text{for } d=3,$$

then the family $\{u_{\epsilon,\delta(\epsilon)}\}_{\epsilon>0}$ satisfies a large deviation principle in $C([0,T], C^\eta(D))$, where $C^\eta(D)$ is some space of functions of negative regularity in space, with respect to the action functional

$$I_T^\lambda(f) = \frac{1}{2}\int_0^T |\partial_t f - \Delta f + c_\lambda \, f + f^3|^2_{[L^2(D)]^2} \, dt,$$

for some explicitly given constant c_λ, depending on λ and d and such that $c_0 = -c$.

Moreover, they also consider the renormalized equation

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + (c + 3\,\epsilon\, c^{(1)}_{\delta(\epsilon)} - 9\,\epsilon^2\, c^{(2)}_{\delta(\epsilon)})\, u(t,\xi) - u^3(t,\xi) + \sqrt{\epsilon}\, \partial_t\, \xi^{\delta(\epsilon)}(t,x), \\ \\ u(0,x) = u_0(x), \quad x \in D, \end{cases}$$

where $c^{(1)}_{\delta(\epsilon)}$ and $c^{(2)}_{\delta(\epsilon)}$ are the constants, depending on the dimension of the underlying space, arising from the renormalization procedure. They prove that, in this case, if (1.3) holds, then the family of solutions $\{u_{\epsilon,\delta(\epsilon)}\}_{\epsilon>0}$ satisfies a large deviation principle in $C([0,T], C^\eta(D))$, with action functional I_T^0.

Unlike Hairer and Weber, that use techniques from the theory of regularity structures to prove the validity of the large deviation principle, in this paper we use the weak convergence approach to large deviations, as developed in [5] for SPDEs (see also [13], [7] and [1] for some relevant applications of this method). The argument is simpler and gives a stronger result. In particular, we are able to prove that, when condition (1.4) is satisfied, then the family $\{u_{\epsilon,\delta(\epsilon)}\}_{\epsilon>0}$ satisfies a large deviation principle in the space of continuous trajectories with values in the space H itself and not in a functional space of negative regularity. Notice that in [6] we have studied an analogous problem for the Φ^{2n}_d- model

To this purpose, let $\{\varphi_\epsilon\}_{\epsilon>0}$ be any sequence of $\{\mathcal{F}_t\}_{t\geq 0}$-predictable processes, taking values in a ball of $L^2(0,T;[L^2(D)]^2)$, $\mathbb{P}$-almost surely, such that

$$\lim_{\epsilon\to 0} \varphi_\epsilon = \varphi \quad \text{weakly in } L^2(0,T;[L^2(D)]^2), \qquad \mathbb{P}-\text{a.s.}$$

for some $\{\mathcal{F}_t\}_{t\geq 0}$-predictable process φ taking values in the same ball of $L^2(0,T;[L^2(D)]^2)$. As we will explain in Section 3, in order to use the weak convergence approach to large deviations, we have to show that if $u_\epsilon^{\varphi_\epsilon}$ is the solution of the equation

$$du(t) = [Au(t) + b(u(t)) + Q_\epsilon\, \varphi_\epsilon(t)]\, dt + \sqrt{\epsilon}\, dw^{\delta(\epsilon)}(t), \qquad t \geq 0, \qquad u(0) = u_0,$$

then we have

$$\lim_{\epsilon\to 0} |u_\epsilon^{\varphi_\epsilon} - u^{\varphi}|_{\mathcal{E}} = 0 \qquad \mathbb{P}-\text{a.s.} \tag{1.6}$$

where u^φ the solution of the random problem

$$\frac{du}{dt}(t) = Au(t) + b(u(t)) + \varphi(t) \qquad t \geq 0, \qquad u(0) = u_0,$$

and $\mathcal{E}$ coincides with the space $C([0,T];[L^2(D)]^2)$ or $C([0,T];\mathcal{B}_p^\sigma(D))$, depending on whether condition (1.4) is satisfied or not. We would like to stress the fact that the proof of (1.6) is quite different, in the two different cases.

If (1.4) is satisfied, then we can work in a Hilbertian framework. For every $\alpha > 0$ and $\epsilon > 0$, we use the splitting

$$u_\epsilon^{\varphi_\epsilon} = v_\epsilon^{\alpha,\varphi_\epsilon} + z_\epsilon^\alpha + \Phi_\epsilon^{\varphi_\epsilon},$$

where

$$\Phi_\epsilon^{\varphi_\epsilon}(t) = \int_0^t e^{(t-s)A} Q_{\delta(\epsilon)}\, \varphi_\epsilon(s)\, ds, \quad t \geq 0,$$

and

$$z_\epsilon^\alpha(t) = \sqrt{\epsilon} \int_{-\infty}^t e^{(t-s)(A-\alpha)}\, d\bar{w}^{\delta(\epsilon)}(s), \qquad t \geq 0, \tag{1.7}$$

so that

$$u_\epsilon^{\varphi_\epsilon}(t) - u^\varphi(t) = [v_\epsilon^{\alpha,\varphi_\epsilon}(t) - v_0^{\alpha,\varphi}(t)] + z_\epsilon^\alpha(t) + [\Phi_\epsilon^{\varphi_\epsilon}(t) - \Phi_0^\varphi(t)]\,.$$

Our aim is to show that there exists a random family $\{\alpha_\epsilon\}_{\epsilon>0}$ such that the three terms on the right hand side above, corresponding to $\alpha = \alpha_\epsilon$, converge to zero in $L^p(\Omega; C([0,T];[L^2(D)]^2))$. In order to prove that we proceed with suitable energy estimates. Here, the key point is the fact that for every $p \geq 1$ there exist $\theta > 0$ and a random variable $K_\epsilon(p)$ such that

$$|z_\epsilon^\alpha|_{C([0,T];L^p(\mathcal{O}))} \leq (\alpha \vee 1)^{-\theta}\, c_p(T)\, K_\epsilon(p), \qquad \mathbb{P}-\text{a.s.}$$

and for any η small enough and any $p, q \geq 1$ there exist $c_{1,\eta}(p,q)$ and $c_{2,\eta}(p,q)$ such that

$$\mathbb{E}\,|K_\epsilon(p)|^q \leq c_{1,\eta}(p,q)\left(\epsilon\,\delta(\epsilon)^{-\eta}\right)^{c_{2,\eta}(p,q)}.$$

If (1.4) is not satisfied, we have to work with the mild formulation of the equation in the space $\mathcal{E}_T := C([0,T];\mathcal{B}_p^\sigma(D)) \cap L^\beta(0,T;\mathcal{B}_p^\alpha(D))$, where $\mathcal{B}_p^\sigma(D)$ and $\mathcal{B}_p^\alpha(D)$ are suitable Besov spaces, with $\sigma < 0 < \alpha$, $\beta \geq 1$ and $p \geq 2$ satisfying suitable conditions. Also in this case we proceed with a suitable splitting of the solution $u_\epsilon^{\varphi_\epsilon}$, but we cannot proceed with energy estimates. We consider the decomposition

$$u_\epsilon^{\varphi_\epsilon} - u^\varphi = [v_\epsilon(t) - u^\varphi(t)] + z_\epsilon(t),$$

where $z_\epsilon(t)$ is the process defined in (1.7), corresponding to $\alpha = 0$. In this case, one of the key points in order to prove (1.6) is showing that

$$\left|\int_0^\cdot e^{(\cdot-s)A} b(z_\epsilon(s))\, ds\right|_{\mathcal{E}_T} \leq c_2(t)\, \left|z_\epsilon \otimes z_\epsilon - \epsilon\,\vartheta_{\delta(\epsilon)} I\right|_{L^\rho(0,T;[H^{-\rho}(D)]^4)},$$

for a suitable $\rho > 1$ and a suitable constant $\vartheta_{\delta(\epsilon)}$ such that

$$\lim_{\epsilon\to 0} \mathbb{E}\,|z_\epsilon \otimes z_\epsilon - \epsilon\,\vartheta_{\delta(\epsilon)}\, I|^\kappa_{L^p(0,T;[H^\sigma(D)]^4)} = 0,$$

for any $\kappa, p \geq 1$ and $\sigma < 0$. This follows from arguments analogous to those used in [8]

2 Notations and preliminaries

We consider here the following incompressible Navier-Stokes equation with periodic boundary conditions on the two-dimensional domain $D = [0, 2\pi]^2$,

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) - (u(t,x)\cdot\nabla)u(t,x) + \nabla p(t,x) + \sqrt{\epsilon}\,\partial_t\,\xi^\delta(t,x), \quad x \in D, \quad t \geq 0, \\ \\ \operatorname{div} u(t,x) = 0, \quad x \in D, \quad t \geq 0, \qquad u(0,x) = u_0(x), \quad x \in D, \\ \\ u(t,x_1,0) = u(t,x_1,2\pi), \quad u(t,0,x_2) = u(t,2\pi,x_2), \quad (x_1,x_2) \in [0,2\pi]^2, \quad t \geq 0, \end{cases} \tag{2.1}$$

where $0 < \epsilon, \delta << 1$ are some small positive constants. Here $\xi^\delta(t,x)$ is a Wiener process on $[L^2(D)]^2$, with covariance Q_δ to be defined below.

We assume that the initial condition u_0 and the noise ξ^δ have zero average in space, so that $u(t)$ remains with zero average for all time. It is not difficult to get rid of this assumption.

In what follows, we will denote by H the subspace of $[L^2(D)]^2$ consisting of periodic, divergence free and zero average functions, that is

$$H = \left\{ u \in [L^2(D)]^2 \,:\, \int_D u(x)\,dx = 0, \quad \operatorname{div} u = 0, \quad u \text{ is periodic in } D \right\}.$$

H turns out to be a Hilbert space, endowed with the standard scalar product $\langle\cdot,\cdot\rangle_H$ inherited from $[L^2(D)]^2$. Moreover, we will denote by P the Leray-Helmholtz projection of $[L^2(D)]^2$ onto H.

Now, for any $k = (k_1, k_2) \in \mathbb{Z}_0^2 = \mathbb{Z}^2 \setminus \{(0,0)\}$ we define

$$e_k(x) = \frac{1}{2\pi}\,\frac{k^\perp}{|k|}\,e^{i\,x\cdot k} = \frac{1}{2\pi}\,\frac{k^\perp}{|k|}\,e^{i\,(x_1k_1+x_2k_2)}, \quad x = (x_1,x_2) \in D, \quad k \in \mathbb{Z}_0,$$

where

$$k^\perp = (k_2, -k_1), \qquad |k| = \sqrt{k_1^2 + k_2^2}.$$

It turns out that the family $\{e_k\}_{k\in\mathbb{Z}_0^2}$ is a complete orthonormal system in $H_{\mathbb{C}}$, the complexification of the space H. For every $s \in \mathbb{R}$, we define

$$H^s(D) := \left\{ u : D \to \mathbb{R} \,:\, |u|^2_{H^s(D)} := \sum_{k\in\mathbb{Z}_0^2} |\langle u, e_k\rangle|^2 |k|^{2s} < \infty \right\}.$$

Next, for $q \in \mathbb{N}$, we set $\delta_q := \Pi_{2^q} - \Pi_{2^{q-1}}$, where Π_n denote the projection of H into $H_n := \operatorname{span}\{e_k\}_{|k|\leq n}$. Namely

$$\delta_q u = \sum_{2^{q-1}<|k|\leq 2^q} \langle u, e_k\rangle_H\, e_k, \qquad u \in \bigcup_{s\in\mathbb{R}} H^s(D).$$

For any $\sigma \in \mathbb{R}$ and $p \geq 1$, we define

$$\mathcal{B}^\sigma_p(D) := \left\{ u \in \bigcup_{s\in\mathbb{R}} H^s(D) \,:\, \sum_{q\in\mathbb{N}} 2^{pq\sigma} |\delta_q u|^p_{L^p(D)} < \infty \right\}.$$

$\mathcal{B}_p^\sigma(D)$ turns out to be a Banach space, endowed with the norm

$$|u|_{\mathcal{B}_p^\sigma(D)} := \left(\sum_{q \in \mathbb{N}} 2^{pq\sigma} |\delta_q u|^p_{L^p(D)} \right)^{\frac{1}{p}}.$$

Now, we define the Stokes operator

$$Au = P\Delta u, \qquad u \in D(A) = H \cap [H^2(D)]^2,$$

where P is the Helmodtz projection. It is immediate to check that for any $k \in \mathbb{Z}_0^2$

$$Ae_k = -|k|^2 e_k, \quad k \in \mathbb{Z}_0^2.$$

For any $r \in \mathbb{R}$, we denote by $(-A)^r$ the r-th fractional power of $-A$, defined on its domain $D((-A)^r)$. It is well known that $D((-A)^r)$ is the closure of the space spanned by $\{e_k\}_{k \in \mathbb{Z}_0^2}$ with respect to the norm in $[H^{2r}(D)]^2$ and the mapping

$$u \in D((-A)^r) \mapsto |(-A)^r u|_H \in [0, +\infty),$$

defines a norm on $D((-A)^r)$, equivalent to the usual norm in $[H^{2r}(D)]^2$. Moreover, we have that the Leray-Helmholtz projection P maps $[H^{2r}(D)]^2$ into $D((-A)^r)$, for every $r \in \mathbb{R}$.

Due to the incompressibility condition, the nonlinearity in equation (2.1) can be rewritten as

$$(u \cdot \nabla)v = \operatorname{div}(u \otimes v),$$

where

$$u \otimes v = \begin{pmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \end{pmatrix}.$$

In what follows, we shall set

$$b(u, v) = -P\operatorname{div}(u \otimes v), \quad b(u) = -P\operatorname{div}(u \otimes u). \tag{2.2}$$

We recall here that, whenever the quantities on the left-hand sides make sense, it holds

$$\langle b(u), u \rangle_H = 0, \quad \langle b(u), Au \rangle_H = 0, \tag{2.3}$$

(for a proof see e.g. [14]).

Finally, concerning the noisy perturbation $\xi^\delta(t, x)$ in equation (2.1), it is a Wiener process on $[L^2(D)]^2$ and has zero average. In what follows, we shall set

$$w^\delta(t) := P\xi^\delta(t), \quad t \geq 0.$$

$w^\delta(t)$ is now a Wiener process on H, and we assume it can be written as

$$w^\delta(t, x) = \sum_{k \in \mathbb{Z}_0^2} \lambda_k(\delta) e_k(x) \beta_k(t), \quad t \geq 0, \quad x \in D,$$

where $\{e_k\}_{k\in\mathbb{Z}_0^2}$ is the orthonormal basis that diagonalizes the operator A, $\{\beta_k(t)\}_{k\in\mathbb{Z}_0^2}$ is a sequence of independent Brownian motions defined on the stochastic basic $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$, and for any $\delta>0$

$$\lambda_k(\delta)=\left(1+\delta\,|k|^{2\gamma}\right)^{-\frac{1}{2}},\qquad k\in\mathbb{Z}_0^2,$$

for some fixed $\gamma>0$. In other words, w^δ is a Wiener process on H with covariance $Q_\delta=(I+\delta(-A)^\gamma)^{-1}$. We would like to stress that our result easily generalizes to more general covariance operators.

As we mentioned above, in the present paper we are interested in the asymptotic behavior of equation (2.1), as both ϵ and δ go to zero. In particular, we shall assume that δ is a function of ϵ, such that

$$\lim_{\epsilon\to 0}\delta(\epsilon)=0.$$

In what follows we shall denote by Q_ϵ the bounded linear operator in H defined by

$$Q_\epsilon e_k=\lambda_k(\delta(\epsilon))\,e_k,\qquad k\in\mathbb{N}.$$

Now, if we project equation (2.1) on H, with the notations we have just introduced, it can be rewritten as

$$du(t)=\left[Au(t)+b(u(t))\right]\,dt+\sqrt{\epsilon}\,dw^{\delta(\epsilon)}(t),\qquad t\geq 0,\qquad u(0)=u_0. \tag{2.4}$$

As proven e.g. in [11], equation (2.4) admits a unique *generalized* solution $u_\epsilon\in C([0,T];H)$. This means that u_ϵ is a progressively measurable process taking values in $C([0,T];H)$, such that $\mathbb{P}$-a.s. equation (2.4) is satisfied in the integral form

$$\langle u_\epsilon(t),\varphi\rangle_H=\langle u_0,\varphi\rangle_H+\int_0^t\langle u_\epsilon(s),A\varphi\rangle_H\,ds+\int_0^t\langle b(u_\epsilon(s),\varphi),u_\epsilon(s)\rangle_H\,ds+\sqrt{\epsilon}\left\langle w^{\delta(\epsilon)}(t),\varphi\right\rangle_H,$$

for every $t\in[0,T]$ and $\varphi\in D(A)$.

In what follows, for every $\alpha\geq 0$ and $\epsilon>0$, we consider the auxiliary Ornstein-Uhlenbeck problem

$$dz(t)=(A-\alpha)z(t)\,dt+\sqrt{\epsilon}\,dw^{\,\delta(\epsilon)}(t),\qquad t\geq 0, \tag{2.5}$$

whose unique stationary solution is given by

$$z_\epsilon^\alpha(t)=\sqrt{\epsilon}\int_{-\infty}^t e^{(t-s)(A-\alpha)}\,d\bar{w}^{\,\delta(\epsilon)}(s),\qquad t\in\mathbb{R}. \tag{2.6}$$

Notice that here $\bar{w}^{\,\delta(\epsilon)}(t)$ is a two sided cylindrical Wiener process, defined by

$$\bar{w}^{\,\delta(\epsilon)}(t,x)=\sum_{k\in\mathbb{Z}_0^2}\lambda_k(\delta(\epsilon))e_k(x)\,\bar{\beta}_k(t),\qquad (t,x)\in\mathbb{R}\times D,$$

where

$$\bar{\beta}_k(t)=\begin{cases}\beta_k(t), & \text{if } t\geq 0,\\ \tilde{\beta}_k(-t), & \text{if } t<0,\end{cases}$$

for some sequence of independent Brownian motions $\{\tilde{\beta}_k(t)\}_{k\in\mathbb{Z}_0^2}$, defined on the stochastic basis $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ and independent of the sequence $\{\beta_k(t)\}_{k\in\mathbb{Z}_0^2}$.

It is well known that for any fixed $\epsilon>0$ the process z_ϵ^α belongs to $L^p(\Omega;C([0,T];D((-A)^\beta))$, for any $T>0$, $p\geq 1$ and $\beta<\gamma/2$. In the case $\alpha=0$, we shall set

$$z_\epsilon(t):=z_\epsilon^0(t). \tag{2.7}$$

3 The problem and the method

We are here interested in the study of the validity of a large deviation principle, as $\epsilon\downarrow 0$, for the family $\{\mathcal{L}(u_\epsilon)\}_{\epsilon\in(0,1)}$, where u_ϵ is the solution of the equation

$$du(t)=[Au(t)+b(u(t))]\,dt+\sqrt{\epsilon}\,dw^{\delta(\epsilon)}(t),\qquad t\geq 0,\qquad u(0)=u_0. \tag{3.1}$$

Here and in what follows, $T>0$ is fixed and $\epsilon>0\mapsto\delta(\epsilon)>0$ is a function such that

$$\lim_{\epsilon\to 0}\delta(\epsilon)=0. \tag{3.2}$$

We will prove that depending on the scaling we assume between ϵ and $\delta(\epsilon)$, the family $\{\mathcal{L}(u_\epsilon)\}_{\epsilon\in(0,1)}$ satisfies a large deviation principle in $\mathcal{E}$, where $\mathcal{E}$ is a suitable space of trajectories on $[0,T]$, taking values in some space of functions defined on the domain D and containing H.

Theorem 3.1. *Let $\epsilon\mapsto\delta(\epsilon)$ be a function satisfying (3.2). Moreover, assume that there exists $\eta>0$ such that*

$$\lim_{\epsilon\to 0}\epsilon\,\delta(\epsilon)^{-\eta}=0. \tag{3.3}$$

Then, for any $u_0\in H$, the family $\{\mathcal{L}(u_\epsilon)\}_{\epsilon>0}$ satisfies a large deviation principle in $C([0,T];H)$, with action functional

$$I_T(f)=\frac{1}{2}\int_0^T|f'(t)-Af(t)-b(f(t))|_H^2\,dt. \tag{3.4}$$

Theorem 3.2. *Let $\epsilon\mapsto\delta(\epsilon)$ be a function satisfying (3.2). Moreover, let $\sigma<0$ and $p\geq 2$ be such that*

$$\sigma>-\frac{2}{p}\vee\left(\frac{2}{p}-1\right).$$

Then, for any $u_0\in H^\theta(D)$, with $\theta\geq\sigma+1-2/p$, the family $\{\mathcal{L}(u_\epsilon)\}_{\epsilon>0}$ satisfies a large deviation principle in $C([0,T];\mathcal{B}_p^\sigma(D))$, with the same action function I_T introduced in (3.4).

In order to prove Theorems 3.1 and 3.2, we follow the weak convergence approach, as developed in [5]. To this purpose, we need to introduce some notations. For any $T>0$, we denote by $\mathcal{P}_T$ the set of predictable processes in $L^2(\Omega\times[0,T];H)$, and for any $\gamma>0$, we define the sets

$$\mathcal{S}_T^\gamma=\left\{f\in L^2(0,T;H)\ :\ \int_0^T|f(t)|_H^2\,dt\leq\gamma\right\},$$

and

$$\mathcal{A}_T^\gamma = \left\{ u \in \mathcal{P}_T \ : \ u \in \mathcal{S}_T^\gamma, \ \ \mathbb{P} - \text{a.s.} \right\}.$$

Next, for any predictable process $\varphi(t)$ taking values in $L^2([0,T];H)$, we denote by u^φ the solution of the problem

$$\frac{du}{dt}(t) = Au(t) + b(u(t)) + \varphi(t) \qquad t \geq 0, \qquad u(0) = u_0. \tag{3.5}$$

Moreover, for every $\epsilon > 0$ we denote by $u_\epsilon^\varphi(t)$ the generalized solution of the problem

$$du(t) = [Au(t) + b(u(t)) + Q_\epsilon\, \varphi(t)]\ dt + \sqrt{\epsilon}\, dw^{\delta(\epsilon)}(t), \qquad t \geq 0, \qquad u(0) = u_0. \tag{3.6}$$

Notice that $w^{\delta(\epsilon)}(t) = Q_{\delta(\epsilon)} w(t)$, where

$$w(t,x) = \sum_{k=1}^{\infty} e_k(x) \beta_k(t).$$

Then, by using the notations introduced in [5], we have

$$u_\epsilon^\varphi = \mathcal{G}_\epsilon \left(\sqrt{\epsilon}\, w + \int_0^{\cdot} \varphi(t)\, dt \right),$$

where $\mathcal{G}_\epsilon(\psi)$ denotes the solution f of the problem

$$df(t) = [Af(t) + b(f(t))]\ dt + Q_{\delta(\epsilon)} d\psi(t), \qquad f(0) = u_0.$$

As for equation (2.4), for any fixed $\epsilon \geq 0$ and for any $T > 0$ and $\kappa \geq 1$, equation (3.6) admits a unique generalized solution u_ϵ^φ in $L^\kappa(\Omega; C([0,T];H))$. As a particular case ($\epsilon = 0$), we have also well-posedness for equation (3.5).

By proceeding as in [5], the following result can be proven.

Theorem 3.3. *Let $\mathcal{E}$ be a Polish space of trajectories defined on $[0,T]$ with values in a space of functions defined on the domain D and containing the space H, and let I_T be the functional defined in* (3.4). *Assume that*

1. *the level sets $\{I_T(f) \leq r\}$ are compact in $\mathcal{E}$, for every $r \geq 0$;*
2. *for every family $\{\varphi_\epsilon\}_{\epsilon>0} \subset \mathcal{A}_T^\gamma$ that converges in distribution, as $\epsilon \downarrow 0$, to some $\varphi \in \mathcal{A}_T^\gamma$, in the space $L^2(0,T;H)$, endowed with the weak topology, the family $\{u_\epsilon^{\varphi_\epsilon}\}_{\epsilon>0}$ converges in distribution to u^φ, as $\epsilon \downarrow 0$, in $\mathcal{E}$.*

Then the family $\{\mathcal{L}(u_\epsilon)\}_{\epsilon>0}$ satisfies a large deviation principle in $\mathcal{E}$, with action functional I_T.

Actually, as shown in [5], the convergence of $u_\epsilon^{\varphi_\epsilon}$ to u^φ implies the validity of the Laplace principle in $\mathcal{E}$ with rate functional I_T. This means that, for any continuous mapping $\Gamma : \mathcal{E} \to \mathbb{R}$ it holds

$$\lim_{\epsilon \to 0} -\epsilon \log \mathbb{E} \exp\left(-\frac{1}{\epsilon}\, \Gamma(u_\epsilon) \right) = \inf_{f \in \mathcal{E}} \left(\Gamma(f) + I_T(f) \right). \tag{3.7}$$

And, once one has shown that the level sets of I_T are compact in $\mathcal{E}$, the validity of the Laplace principle as in (3.7) is equivalent to say that the family $\{\mathcal{L}(u_\epsilon)\}_{\epsilon>0}$ satisfies a large deviation principle in $\mathcal{E}$, with action functional I_T.

The proof of condition 1 in Theorem 3.3 is obtained once we show that, when the space $L^2(0,T;H)$ is endowed with the topology of weak convergence, the mapping

$$\varphi \in L^2(0,T;H) \mapsto u^\varphi \in \mathcal{E},$$

is continuous. More precisely, condition 1 will follow if we can prove that for any sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ in $L^2(0,T;H)$, weakly convergent to some $\varphi \in L^2(0,T;H)$, it holds

$$\lim_{n\to\infty} |u^{\varphi_n} - u^\varphi|_{\mathcal{E}} = 0.$$

As for condition 2, we will use Skorohod theorem and rephrase such a condition in the following way. Let $(\bar\Omega, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ be a probability space and let $\{\bar w^{\delta(\epsilon)}(t)\}_{t\geq 0}$ be a Wiener process, with covariance Q_δ, defined on such a probability space and corresponding to the filtration $\{\bar{\mathcal{F}}_t\}_{t\geq 0}$. Moreover, let $\{\bar\varphi_\epsilon\}_{\epsilon>0}$ and $\bar\varphi$ be $\{\bar{\mathcal{F}}_t\}_{t\geq 0}$-predictable processes taking values in $\mathcal{S}_T^\gamma$, $\bar{\mathbb{P}}$ almost surely, such that the distribution of $(\bar\varphi_\epsilon, \bar\varphi, \bar w^{\,\delta(\epsilon)})$ coincides with the distribution of $(\varphi_\epsilon, \varphi, w^{\,\delta(\epsilon)})$ and

$$\lim_{\epsilon\to 0} \bar\varphi_\epsilon = \bar\varphi \quad \text{weakly in } L^2(0,T;H), \qquad \bar{\mathbb{P}} - \text{a.s.}$$

Then, if $\bar u_\epsilon^{\bar\varphi_\epsilon}$ is the solution of an equation analogous to (3.6), with φ_ϵ and $w^{\,\delta(\epsilon)}$ replaced respectively by $\bar\varphi_\epsilon$ and $\bar w^{\,\delta(\epsilon)}$, we have that

$$\lim_{\epsilon\to 0} \bar u_\epsilon^{\bar\varphi_\epsilon} = \bar u^{\,\bar\varphi} \quad \text{in } \mathcal{E}, \quad \overline{\mathbb{P}} - \text{a.s.} \tag{3.8}$$

We would like to stress that condition 1 in Theorem 3.3 follows from condition 2. Actually, if we take in equation (3.6) $\sqrt{\epsilon} = 0$ and $\{\varphi_\epsilon\}_{\epsilon>0} = \{\varphi_n\}_{n\in\mathbb{N}}$ and φ deterministic, then condition 1 is a particular case of condition 2.

4 Proof of Theorem 3.1

In what follows, $\{\varphi_\epsilon\}_{\epsilon\in(0,1)}$ and φ are predictable processes in $\mathcal{A}_T^\gamma$, for some $\gamma > 0$ fixed, such that φ_ϵ converges to φ, $\mathbb{P}$ almost surely, in the weak topology of $L^2(0,T;H)$.

For any $\alpha \geq 0$ and $\epsilon > 0$, we introduce the random equation

$$\frac{dv}{dt}(t) = Av(t) + b(v(t) + z_\epsilon^\alpha(t) + \Phi_\epsilon(t)) + \alpha\, z_\epsilon^\alpha(t), \qquad v(0) = u_0 - z_\epsilon^\alpha(0), \tag{4.1}$$

where z_ϵ^α is the process introduced in (2.6), solution of the linear equation (2.5), and

$$\Phi_\epsilon(t) = \int_0^t e^{(t-s)A} Q_\epsilon\, \varphi_\epsilon(s)\, ds, \quad t \geq 0,$$

is the solution of the problem

$$\frac{d\Phi_\epsilon}{dt}(t) = A\Phi_\epsilon(t) + Q_\epsilon\, \varphi_\epsilon(t), \qquad \Phi_\epsilon(0) = 0.$$

Notice that if $\varphi_\epsilon \in \mathcal{A}_T^\gamma$, for some $\gamma > 0$, then

$$|\Phi_\epsilon(t)|_{L^p(D)} \leq c \int_0^t (t-s)^{-\frac{p-2}{2p}} |\varphi_\epsilon(s)|_H \, ds,$$

so that

$$|\Phi_\epsilon|^p_{L^p(0,T;L^p(D))} \leq c \int_0^T \left(\int_0^t (t-s)^{-\frac{p-2}{2p}} |\varphi_\epsilon(s)|_H \, ds \right)^p dt$$

$$\leq c_T \, |\varphi_\epsilon|^p_{L^2(0,T;H)} \left(\int_0^T s^{-\frac{p-2}{2p}} \, ds \right)^{\frac{p+2}{2}}.$$

This implies that

$$|\Phi_\epsilon|_{L^p(0,T;L^p(D))} \leq c_{T,p} \sqrt{\gamma}, \quad \mathbb{P}-\text{a.s.} \tag{4.2}$$

As shown e.g. in [11], equation (4.1) admits a unique solution

$$v_\epsilon^\alpha \in C([0,T];H) \cap L^2(0,T;V), \tag{4.3}$$

and the unique generalized solution u_ϵ^α of equation

$$du(t) = [Au(t) + b(u(t)) + Q_\epsilon \, \varphi_\epsilon(t)] \, dt + \sqrt{\epsilon} \, dw^{\delta(\epsilon)}(t), \qquad t \geq 0, \qquad u(0) = u_0, \tag{4.4}$$

can be decomposed as

$$u_\epsilon^\alpha(t) = v_\epsilon^\alpha(t) + z_\epsilon^\alpha(t) + \Phi_\epsilon(t), \qquad t \in [0,T].$$

Lemma 4.1. *Assume that $\{\varphi_\epsilon\}_{\epsilon>0} \subset \mathcal{A}_T^\gamma$, for some fixed $\gamma > 0$. Then, there exists $c_{T,\gamma} > 0$ such that for every $\epsilon > 0$ and $t \in [0,T]$*

$$\begin{aligned} &|v_\epsilon^\alpha(t)|_H^2 + \int_0^t |v_\epsilon^\alpha(s)|_V^2 \, ds \\ &\leq c_{T,\gamma} \exp\left(c \, |z_\epsilon^\alpha|^4_{L^4(0,t;L^4(D))} \right) \left(|u_0|_H^2 + |z_\epsilon^\alpha(0)|_H^2 + (\alpha^2+1) \, |z_\epsilon^\alpha|^4_{L^4(0,t;L^4(D))} + 1 \right). \end{aligned} \tag{4.5}$$

Moreover, we have

$$\begin{aligned} &|v_\epsilon^\alpha|^4_{L^4(0,T;L^4(D))} \\ &\leq c_{T,\gamma} \exp\left(c \, |z_\epsilon^\alpha|^4_{L^4(0,t;L^4(D))} \right) \left(|u_0|_H^2 + |z_\epsilon^\alpha(0)|_H^2 + (\alpha^2+1) \, |z_\epsilon^\alpha|^4_{L^4(0,t;L^4(D))} + 1 \right)^2. \end{aligned} \tag{4.6}$$

Proof. Let v_ϵ^α be the solution of problem (4.1), having the regularity specified in (4.3). Due to the first identity in (2.3), we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} |v_\epsilon^\alpha(t)|_H^2 + |v_\epsilon^\alpha(t)|_V^2 \\ &= \langle b(z_\epsilon^\alpha(t) + \Phi_\epsilon(t)), v_\epsilon^\alpha(t) \rangle_H + \langle b(v_\epsilon^\alpha(t), z_\epsilon^\alpha(t) + \Phi_\epsilon(t)), v_\epsilon^\alpha(t) \rangle_H + \alpha \, \langle z_\epsilon^\alpha(t), v_\epsilon^\alpha(t) \rangle_H. \end{aligned}$$

For every $\eta > 0$, we have

$$|\langle b(z_\epsilon^\alpha(t) + \Phi_\epsilon(t)), v_\epsilon^\alpha(t)\rangle_H| = |\langle b(z_\epsilon^\alpha(t) + \Phi_\epsilon(t), v_\epsilon^\alpha(t)), z_\epsilon^\alpha(t) + \Phi_\epsilon(t)\rangle_H|$$

$$\leq |v_\epsilon^\alpha(t)|_V \, |z_\epsilon^\alpha(t) + \Phi_\epsilon(t)|^2_{L^4(D)} \leq \eta \, |v_\epsilon^\alpha(t)|^2_V + c_\eta \left(|z_\epsilon^\alpha(t)|^4_{L^4(D)} + |\Phi_\epsilon(t)|^4_{L^4(D)} \right).$$

As $H^{1/2}(D) \hookrightarrow L^4(D)$, by interpolation, we have

$$|\langle b(v_\epsilon^\alpha(t), z_\epsilon^\alpha(t) + \Phi_\epsilon(t)), v_\epsilon^\alpha(t)\rangle_H| = |\langle b(v_\epsilon^\alpha(t)), z_\epsilon^\alpha(t) + \Phi_\epsilon(t)\rangle_H|$$

$$\leq c|v_\epsilon^\alpha(t)|_V \, |v_\epsilon^\alpha(t)|_{H^{1/2}} |z_\epsilon^\alpha(t) + \Phi_\epsilon(t)|_{L^4(D)} \leq c|v_\epsilon^\alpha(t)|^{3/2}_V \, |v_\epsilon^\alpha(t)|^{1/2}_H |z_\epsilon^\alpha(t) + \Phi_\epsilon(t)|_{L^4(D)}$$

$$\leq \eta \, |v_\epsilon^\alpha(t)|^2_V + c_\eta \, |v_\epsilon^\alpha(t)|^2_H \left(|z_\epsilon^\alpha(t)|^4_{L^4(D)} + |\Phi_\epsilon(t)|^4_{L^4(D)} \right).$$

Moreover, we have

$$\alpha| \, \langle z_\epsilon^\alpha(t), v_\epsilon^\alpha(t)\rangle_H \, | \leq \eta \, |v_\epsilon^\alpha(t)|^2_V + c_\eta \, \alpha^2 \, |z_\epsilon^\alpha(t)|^2_{H^{-1}}.$$

Therefore, if we pick $\eta = 1/6$, we get

$$\frac{d}{dt}|v_\epsilon^\alpha(t)|^2_H + |v_\epsilon^\alpha(t)|^2_V$$

$$\leq c \, |v_\epsilon^\alpha(t)|^2_H \left(|z_\epsilon^\alpha(t)|^4_{L^4(D)} + |\Phi_\epsilon(t)|^4_{L^4(D)} \right) + c \, (\alpha^2 + 1) \, |z_\epsilon^\alpha(t)|^4_{L^4(D)} + c \, |\Phi_\epsilon(t)|^4_{L^4(D)}.$$

Due to (4.2), by using the Gronwall lemma this yields (4.5).

In order to prove (4.6), we notice that, as $H^{1/2}(D) \hookrightarrow L^4(D)$, by interpolation we have

$$|v_\epsilon^\alpha|^4_{L^4(0,T;L^4(D))} \leq c \int_0^T |v_\epsilon^\alpha(s)|^2_V \, |v_\epsilon^\alpha(s)|^2_H \, ds \leq |v_\epsilon^\alpha|^2_{L^\infty(0,T;H)} |v_\epsilon^\alpha|^2_{L^2(0,T;V)}.$$

Therefore, (4.6) follows immediately from (4.5).

□

Remark 4.2. 1. Due to (A.8), there exist $\bar\kappa \geq 1$ and $c(T) > 0$ such that for any $\epsilon > 0$

$$\alpha_\epsilon := c(T) \, |K_\epsilon(4, \beta_\eta)|^{\bar\kappa} \vee 1 \Longrightarrow |z_\epsilon^{\alpha_\epsilon}|_{L^4(0,T;L^4(D))} \leq 1 \text{ and } |z_\epsilon^{\alpha_\epsilon}(0)|_H \leq 1. \tag{4.7}$$

Thanks to (4.6), this implies that

$$|v_\epsilon^{\alpha_\epsilon}|_{L^4(0,T;L^4(D))} \leq c_{T,\gamma} \left(|u_0|_H + \alpha_\epsilon^2 + 1 \right), \quad \mathbb{P} - \text{a.s.} \tag{4.8}$$

and in view of (A.9), we can conclude that if (3.3) holds, then

$$\mathbb{E} \, |v_\epsilon^{\alpha_\epsilon}|^\kappa_{L^4(0,T;L^4(D))} \leq c_\gamma(T, \kappa) \left(|u_0|^\kappa_H + 1 \right), \quad \kappa \geq 1. \tag{4.9}$$

2. As a consequence of (4.5), if $\varphi \in \mathcal{A}_T^\gamma$ and v^φ is a solution to the problem

$$\frac{dv}{dt}(t) = Av(t) + b(v(t) + \Gamma(\varphi)(t)), \qquad v(0) = u_0,$$

where

$$\Gamma(\varphi)(t) := \int_0^t e^{(t-s)A}\varphi(s)\,ds,$$

we have

$$|v^\varphi(t)|_H^2 + \int_0^t |v^\varphi(s)|_H^2\,ds \le c_{T,\gamma}\left(1 + |u_0|_H^2\right). \tag{4.10}$$

Moreover, by interpolation,

$$|v^\varphi|_{L^4(0,T;L^4(D))} \le c_{T,\gamma}(1 + |u_0|_H). \tag{4.11}$$

In the next lemma we investigate the continuity properties of the operator Γ and we prove the convergence of Φ_ϵ to $\Gamma(\varphi)$ in case the sequence $\{\varphi_\epsilon\}_{\epsilon>0}$ is weakly convergent to φ.

Lemma 4.3. *For every $\rho < 1$, there exist $\theta_\rho > 0$ and $c_\theta > 0$ such that*

$$|\Gamma(\varphi)|_{C^{\theta_\rho}([0,T];H^\rho(D))} \le c_\rho\,|\varphi|_{L^2(0,T;H)}, \quad \mathbb{P}-a.s. \tag{4.12}$$

for every $\varphi \in L^2(0,T;H)$. In particular, if $\{\varphi_\epsilon\}_{\epsilon>0}$ is a family in $\mathcal{A}_T^\gamma$, weakly convergent in $L^2(0,T;H)$ to some $\varphi \in \mathcal{A}_T^\gamma$, for every $\rho < 1$ we have

$$\lim_{\epsilon\to 0} |\Phi_\epsilon - \Gamma(\varphi)|_{C([0,T];H^\rho(D))} = 0, \qquad \mathbb{P}-a.s. \tag{4.13}$$

Proof. For every $\beta \in (0,1)$, we have

$$\Gamma(\varphi)(t) = c_\beta \int_0^t (t-s)^{-\beta+1} e^{(t-s)A} Y_\beta(\varphi)(s)\,ds,$$

where

$$Y_\beta(\varphi)(s) = \int_0^s (s-\sigma)^{-\beta} e^{(s-\sigma)A}\varphi(\sigma)\,d\sigma.$$

Due to the Young inequality, we get

$$|Y_\beta(\varphi)|_{L^p(0,T;H)}^p = \int_0^T \left(\int_0^s (s-\sigma)^{-\beta}|\varphi(\sigma)|_H\,d\sigma\right)^p ds \le |\varphi|_{L^2(0,T;H)}^p \left(\int_0^T s^{-\frac{2\beta p}{p+2}}\,ds\right)^{\frac{p+2}{2}},$$

and hence, if $\beta < 1/2 + 1/p$, we have

$$|Y_\beta(\varphi)|_{L^p(0,T;H)} \le c_p(T)\,|\varphi|_{L^2(0,T;H)}.$$

Now, as shown e.g. in [10], if $\beta > \rho/2 + 1/p$ we have that the mapping

$$Y \in L^p(0,T;H) \mapsto \int_0^t (t-s)^{-\beta+1} e^{(t-s)A} Y(s)\,ds \in C^{\beta-\frac{\rho}{2}-\frac{1}{p}}([0,T];H^\rho(D)),$$

is continuous. Therefore, we can conclude that

$$|\Gamma(\varphi)|_{C^{\beta-\frac{\rho}{2}-\frac{1}{p}}([0,T];H^\rho(D))} \leq c_{\rho,\beta}(T)\,|\varphi|_{L^2(0,T;H)}, \qquad \mathbb{P}-\text{a.s.}$$

if $\rho/2+1/p<\beta<1/2+1/p$, and this implies (4.12).

Now, in order to prove (4.13), we notice that

$$\Phi_\epsilon-\Gamma(\varphi)=\Gamma(Q_\epsilon(\varphi_\epsilon-\varphi))+\Gamma(Q_\epsilon\varphi-\varphi).$$

Since $Q_\epsilon(\varphi_\epsilon-\varphi)\in\mathcal{A}_T^\gamma$ and $Q_\epsilon(\varphi_\epsilon-\varphi)\rightharpoonup 0$, as $\epsilon\downarrow 0$, weakly in $L^2(0,T;H)$, due to the compactness of the immersion of $C^{\theta\rho_1}([0,T];H^{\rho_1}(D))$ into $C([0,T];H^{\rho_2}(D))$, for every $\rho_1>\rho_2$, from (4.12) we conclude that

$$\lim_{\epsilon\to 0}|\Gamma(Q_\epsilon(\varphi_\epsilon-\varphi))|_{C([0,T];H^\rho(D))}=0, \qquad \mathbb{P}-\text{a.s.} \tag{4.14}$$

for every $\rho<1$. Moreover, thanks again to (4.12),

$$|\Gamma(Q_\epsilon\,\varphi-\varphi)|^p_{C([0,T];H^\rho(D))}\leq c_\rho|Q_\epsilon\,\varphi-\varphi|_{L^2(0,T;H)}\to 0, \qquad \mathbb{P}-\text{a.s.}$$

as $\epsilon\to 0$, and together with (4.14), this implies (4.13). □

Remark 4.4. Notice that, as the sequence $\{\varphi_\epsilon\}_{\epsilon>0}$ and the process φ are in $\mathcal{A}_T^\gamma$, we can conclude that the convergence in (4.13) is in $L^p(\Omega)$, for any $p\geq 1$.

In what follows, we shall denote

$$\rho_\epsilon^\alpha(t):=v_\epsilon^\alpha(t)-v^\varphi(t), \qquad t\geq 0.$$

It is immediate to check that ρ_ϵ^α is a solution to the problem

$$\frac{d\rho_\epsilon^\alpha}{dt}(t)=A\rho_\epsilon^\alpha(t)+b(v_\epsilon^\alpha(t)+z_\epsilon^\alpha(t)+\Phi_\epsilon(t))-b(v^\varphi(t)+\Gamma(\varphi)(t))+\alpha\,z_\epsilon^\alpha(t), \qquad \rho_\epsilon^\alpha(0)=-z_\epsilon^\alpha(0). \tag{4.15}$$

Lemma 4.5. *If $\{\varphi_\epsilon\}_{\epsilon>0}\subset\mathcal{A}_T^\gamma$ and $\varphi\in\mathcal{A}_T^\gamma$, for every $\alpha\geq 0$ we have*

$$\begin{aligned}&\sup_{t\in[0,T]}|\rho_\epsilon^\alpha(t)|_H^2+\int_0^T|\rho_\epsilon^\alpha(t)|_V^2\,dt\leq c_\gamma(T)\,\exp\left(|u_0|_H^4+1\right)\\&\left(|z_\epsilon^\alpha(0)|_H^2+|z_\epsilon^\alpha|^2_{L^4(0,T;L^4(D))}\left(|v_\epsilon^\alpha|^2_{L^4(0,T;L^4(D))}+1+\alpha^2\right)+|z_\epsilon^\alpha|^4_{L^4(0,T;L^4(D))}\right.\\&\left.+|\Phi_\epsilon-\Gamma(\varphi)|^2_{L^4(0,T;L^4(D))}\left(1+|u_0|_H^2+|v_\epsilon^\alpha|^2_{L^4(0,T;L^4(D))}\right)\right).\end{aligned} \tag{4.16}$$

Proof. Taking into account of the first identity in (2.3), we have

$$\frac{1}{2}\frac{d}{dt}|\rho_\epsilon^\alpha(t)|_H^2 + |\rho_\epsilon^\alpha(t)|_V^2 = \langle b(v_\epsilon^\alpha(t)) - b(v^\varphi(t)), \rho_\epsilon^\alpha(t)\rangle_H + \langle b(\Phi_\epsilon(t)) - b(\Gamma(\varphi)(t)), \rho_\epsilon^\alpha(t)\rangle_H$$

$$+ \langle b(z_\epsilon^\alpha(t)), \rho_\epsilon^\alpha(t)\rangle_H + \langle b(v_\epsilon^\alpha(t), z_\epsilon^\alpha(t)) + b(z_\epsilon^\alpha(t), v_\epsilon^\alpha(t)), \rho_\epsilon^\alpha(t)\rangle_H$$

$$+ \langle b(z_\epsilon^\alpha(t), \Phi_\epsilon(t)) + b(\Phi_\epsilon(t), z_\epsilon^\alpha(t)), \rho_\epsilon^\alpha(t)\rangle_H + \langle b(\rho_\epsilon^\alpha(t), \Phi_\epsilon(t)), \rho_\epsilon^\alpha(t)\rangle_H$$

$$+ \langle b(v^\varphi(t), \Phi_\epsilon(t) - \Gamma(\varphi)(t)) + b(\Phi_\epsilon(t) - \Gamma(\varphi)(t), v_\epsilon^\alpha(t)), \rho_\epsilon^\alpha(t)\rangle_H + \alpha\, \langle z_\epsilon^\alpha(t), \rho_\epsilon^\alpha(t)\rangle_H$$

$$:= \sum_{j=1}^{8} I_{\epsilon,j}^\alpha(t).$$

Now, we are going to estimate each one of the terms $I_{\epsilon,j}^\alpha(t)$, for $j = 1, \dots, 8$. We have

$$I_{\epsilon,1}^\alpha(t) = \langle b(\rho_\epsilon^\alpha(t), v^\varphi(t), \rho_\epsilon^\alpha(t)\rangle_H = -\langle b(\rho_\epsilon^\alpha(t)), v^\varphi(t)\rangle_H,$$

so that, by interpolation, for any $\eta > 0$,

$$|I_{\epsilon,1}^\alpha(t)| \le |\rho_\epsilon^\alpha(t)|_V |\rho_\epsilon^\alpha(t)|_{L^4(D)} |v^\varphi(t)|_{L^4(D)} \le \eta\, |\rho_\epsilon^\alpha(t)|_V^2 + c_\eta\, |\rho_\epsilon^\alpha(t)|_H^2\, |v^\varphi(t)|_{L^4(D)}^4. \tag{4.17}$$

For $I_{\epsilon,2}^\alpha(t)$ we have

$$I_{\epsilon,2}^\alpha(t) = \langle b(\Phi_\epsilon(t), \Phi_\epsilon(t) - \Gamma(\varphi)(t)) + b(\Phi_\epsilon(t) - \Gamma(\varphi)(t), \Gamma(\varphi)(t)), \rho_\epsilon^\alpha(t)\rangle_H,$$

and, by proceeding as for $I_{\epsilon,1}^\alpha(t)$, we have

$$|I_{\epsilon,2}^\alpha(t)| \le \eta\, |\rho_\epsilon^\alpha(t)|_V^2 + c_\eta \left(|\Phi_\epsilon(t)|_{L^4(D)}^2 + |\Gamma(\varphi)(t)|_{L^4(D)}^2 \right) |\Phi_\epsilon(t) - \Gamma(\varphi)(t)|_{L^4(D)}^2. \tag{4.18}$$

For $I_{\epsilon,3}^\alpha(t)$, we have

$$|I_{\epsilon,3}^\alpha(t)| = |\langle b(z_\epsilon^\alpha(t)), \rho_\epsilon^\alpha(t)\rangle_H| \le \eta\, |\rho_\epsilon^\alpha(t)|_V^2 + c_\eta\, |z_\epsilon^\alpha(t)|_{L^4(D)}^4, \tag{4.19}$$

and, in an analogous way,

$$|I_{\epsilon,4}^\alpha(t)| + |I_{\epsilon,5}^\alpha(t)| \le \eta\, |\rho_\epsilon^\alpha(t)|_V^2 + c_\eta\, |z_\epsilon^\alpha(t)|_{L^4(D)}^2 \left(|v_\epsilon^\alpha(t)|_{L^4(D)}^2 + |\Phi_\epsilon(t)|_{L^4(D)}^2 \right). \tag{4.20}$$

Concerning $I_{\epsilon,6}^\alpha(t)$, by interpolation we get

$$|I_{\epsilon,6}^\alpha(t)| \le \eta\, |\rho_\epsilon^\alpha(t)|_V^2 + c_\eta\, |\Phi_\epsilon(t)|_{L^4(D)}^4\, |\rho_\epsilon^\alpha(t)|_H^2. \tag{4.21}$$

Finally, with the same arguments used for $I_{\epsilon,3}^\alpha$, and also for $I_{\epsilon,4}^\alpha$ and $I_{\epsilon,5}^\alpha$, we get

$$|I_{\epsilon,7}^\alpha(t)| \le \eta\, |\rho_\epsilon^\alpha(t)|_V^2 + c_\eta \left(|v^\varphi|_{L^4(D)}^2 + |v_\epsilon^\alpha(t)|_{L^4(D)}^2 \right) |\Phi_\epsilon(t) - \Gamma(\varphi)|_{L^4(D)}^2. \tag{4.22}$$

For the last term, we have

$$|I_{\epsilon,8}^\alpha(t)| \le \eta\, |\rho_\epsilon^\alpha(t)|_V^2 + c_\eta\, \alpha^2\, |z_\epsilon^\alpha(t)|_{H^{-1}}^2. \tag{4.23}$$

Therefore, if we take $\eta = 1/14$, we obtain

$$\frac{d}{dt}|\rho_\epsilon^\alpha(t)|_H^2 + |\rho_\epsilon^\alpha(t)|_V^2 \le c\,|\rho_\epsilon^\alpha(t)|_H^2 \left(|v^\varphi(t)|_{L^4(D)}^4 + |\Phi_\epsilon(t)|_{L^4(D)}^4\right)$$

$$+ \left(|\Phi_\epsilon(t)|_{L^4(D)}^2 + |\Gamma(\varphi)(t)|_{L^4(D)}^2 + |v^\varphi(t)|_{L^4(D)}^2 + |v_\epsilon^\alpha(t)|_{L^4(D)}^2\right) |\Phi_\epsilon(t) - \Gamma(\varphi)(t)|_{L^4(D)}^2$$

$$+c\,|z_\epsilon^\alpha(t)|_{L^4(D)}^2 \left(|v_\epsilon^\alpha(t)|_{L^4(D)}^2 + |\Phi_\epsilon(t)|_{L^4(D)}^2 + \alpha^2\right) + c\,|z_\epsilon^\alpha(t)|_{L^4(D)}^4.$$

Recalling that

$$\varphi \in \mathcal{A}_T^\gamma \Longrightarrow |\Gamma(\varphi)|_{L^p(0,T;L^p(D))} \le c_p(T)\,\gamma, \quad \mathbb{P}-\text{a.s.}$$

as a consequence of the Gronwall lemma, this implies that

$$\sup_{t\in[0,T]} |\rho_\epsilon^\alpha(t)|_H^2 + \int_0^T |\rho_\epsilon^\alpha(t)|_V^2\,dt \le c_\gamma(T)\,\exp\left(|v^\varphi|_{L^4(0,T;L^4(D))}^4\right)$$

$$\left(|z_\epsilon^\alpha(0)|_H^2 + |z_\epsilon^\alpha|_{L^4(0,T;L^4(D))}^2 \left(|v_\epsilon^\alpha|_{L^4(0,T;L^4(D))}^2 + 1 + \alpha^2\right) + |z_\epsilon^\alpha|_{L^4(0,T;L^4(D))}^4\right.$$

$$\left.+|\Phi_\epsilon - \Gamma(\varphi)|_{L^4(0,T;L^4(D))}^2 \left(1 + |v^\varphi|_{L^4(0,T;L^4(D))}^2 + |v_\epsilon^\alpha|_{L^4(0,T;L^4(D))}^2\right)\right).$$

Thanks to (4.11), we conclude that (4.16) holds.

□

4.1 Conclusion of the proof of Theorem 3.1

We have already seen that, if α is any given non-negative constant and $v_\epsilon^\alpha(t)$ is the solution to problem (4.1), then it holds

$$u_\epsilon^{\varphi_\epsilon}(t) = v_\epsilon^\alpha(t) + z_\epsilon^\alpha(t) + \Phi_\epsilon(t), \quad t \ge 0.$$

Since $u^\varphi(t) = v^\varphi(t) + \Gamma(\varphi)(t)$, this implies that we can write

$$u_\epsilon^{\varphi_\epsilon}(t) - u^\varphi(t) = \left[v_\epsilon^{\alpha_\epsilon}(t) - v^\varphi(t)\right] + z_\epsilon^{\alpha_\epsilon}(t) + \left[\Phi_\epsilon(t) - \Gamma(\varphi)(t)\right], \quad t \ge 0,$$

where α_ϵ is the random constant defined in (4.7).

Due to (4.8) and (4.16), it is immediate to check that

$$|v_\epsilon^{\alpha_\epsilon}(t) - v^\varphi(t)|_H^2 \le c_\gamma(T, |u_0|_H) \left[|z_\epsilon^{\alpha_\epsilon}(0)|_H^2 + |z_\epsilon^{\alpha_\epsilon}|_{L^4(0,T;L^4(D))}^4\right.$$

$$\left.+ \left(|z_\epsilon^{\alpha_\epsilon}|_{L^4(0,T;L^4(D))}^2 + |\Phi_\epsilon - \Gamma(\varphi)|_{L^4(0,T;L^4(D))}\right)\left(1 + \alpha_\epsilon^2\right)\right].$$

Now, in view of (A.8), for any $\beta \in (0, 1/4)$ there exists $c_\beta(T)$ such that for every $\alpha > 0$

$$|z_\epsilon^\alpha|_{C([0,T];L^4(D))} \le c_\beta(T)\,K_\epsilon(4,\beta), \quad \mathbb{P}-\text{a.s.}$$

This implies that, if we fix any $\eta \in (0, 1/2\gamma)$ satisfying (3.3) and $\beta_\eta \in (0, 1/4)$ so that (A.9) holds, we get

$$|v_\epsilon^{\alpha_\epsilon}(t) - v^\varphi(t)|_H^2 \leq c_{\gamma,\eta}(T, |u_0|_H) \left(K_\epsilon^4(4, \beta_\eta) + K_\epsilon^2(2, \beta) + |\Phi_\epsilon - \Gamma(\varphi)|_{L^4(0,T;L^4(D))}\right) \left(1 + \alpha_\epsilon^2\right). \tag{4.24}$$

As a consequence of (A.9) and assumption (3.3), we have

$$\sup_{\epsilon \in (0,1)} \mathbb{E}\, \alpha_\epsilon^\kappa < \infty, \qquad \kappa \geq 1.$$

Then, thanks again to (A.9), from (4.24) we can conclude that for any $\kappa \geq 1$

$$\mathbb{E}\, |v_\epsilon^{\alpha_\epsilon} - v^\varphi|_{C([0,T];H)}^\kappa \leq c_{\gamma,\eta,\kappa}(T, |u_0|_H) \left[\left(\epsilon\, \delta(\epsilon)^{-\eta}\right)^{c_\kappa} + \left(\mathbb{E}\, |\Phi_\epsilon - \Gamma(\varphi)|_{L^4(0,T;L^4(D))}^\kappa\right)^{\frac{1}{2}}\right].$$

Because of (3.3), (4.12) and (4.13), this implies that

$$\lim_{\epsilon \to 0} \epsilon\, \delta(\epsilon)^{-\eta} = 0 \Longrightarrow \lim_{\epsilon \to 0} \mathbb{E}\, |v_\epsilon^{\alpha_\epsilon} - v^\varphi|_{C([0,T];H)}^\kappa = 0, \qquad \kappa \geq 1. \tag{4.25}$$

Since

$$|u_\epsilon^{\varphi_\epsilon} - u^\varphi|_{C([0,T];H)} \leq |v_\epsilon^{\alpha_\epsilon} - v^\varphi|_{C([0,T];H)} + |z_\epsilon^{\alpha_\epsilon}|_{C([0,T];H)} + |\Phi_\epsilon - \Gamma(\varphi)|_{C([0,T];H)},$$

(4.25), together once more with (4.12) and (4.13), implies that

$$\lim_{\epsilon \to 0} \epsilon\, \delta(\epsilon)^{-\eta} = 0 \Longrightarrow \lim_{\epsilon \to 0} \mathbb{E}\, |u_\epsilon^{\varphi_\epsilon} - u^\varphi|_{C([0,T];H)}^\kappa = 0, \qquad \kappa \geq 1. \tag{4.26}$$

In view of Theorem 3.3 and all comments in Section 3 after Theorem 3.3, we can conclude that Theorem 3.1 is proved.

5 Proof of Theorem 3.2

In what follows, we fix any $\sigma < 0$ and $p \geq 2$ such that

$$\sigma > -\frac{2}{p} \vee \left(\frac{2}{p} - 1\right).$$

Because of such a condition, we can fix two real constants α and β such that

$$\frac{2}{p} > \alpha > -\sigma > 0, \quad p \geq 2, \quad \beta \geq 2, \quad -\frac{1}{2} + \frac{1}{p} < \frac{\alpha}{2} - \frac{1}{\beta} < \frac{\sigma}{2}. \tag{5.1}$$

Once fixed α, σ, p and β, for any $0 \leq s < t$ we denote

$$\mathcal{E}_{s,t} := C([s,t]; \mathcal{B}_p^\sigma(D)) \cap L^\beta(s,t; \mathcal{B}_p^\alpha(D)).$$

$\mathcal{E}_{s,t}$ turns out to be a Banach space, endowed with the norm

$$|v|_{\mathcal{E}_{s,t}} := \sup_{r \in [s,t]} |v(r)|_{\mathcal{B}_p^\sigma(D)} + |v|_{L^p(s,t;\mathcal{B}_p^\alpha(D)}.$$

In the case $s = 0$, we shall set $\mathcal{E}_{0,t} = \mathcal{E}_t$.

Our purpose here is to show that under condition (3.2) the family $\{u_\epsilon\}_{\epsilon\in(0,1)}$ satisfies a large deviation principle in $C([0,T];\mathcal{B}_p^\sigma(D))$, with action functional I_T, as defined in (3.4). In view of Theorem 3.3 and the arguments in Section 3, this follows once we prove that for any sequence $\{\varphi_\epsilon\}_{\epsilon>0} \subset \mathcal{A}_T^\gamma$, $\mathbb{P}$-almost surely convergent to some $\varphi \in \mathcal{A}_T^\gamma$, with respect to the topology of weak covergence in $L^2(0,T;H)$, the sequence $\{u_\epsilon^{\varphi_\epsilon}\}_{\epsilon>0}$ converges $\mathbb{P}$-almost surely to u^φ in $C([0,T];\mathcal{B}_p^\sigma(D))$.

For any $\epsilon > 0$, we introduce the random equation

$$\frac{dv_\epsilon}{dt}(t) = Av_\epsilon(t) + b(v_\epsilon(t) + z_\epsilon(t)) + Q_\epsilon\,\varphi_\epsilon, \qquad v_\epsilon(0) = u_0 - z_\epsilon(0), \tag{5.2}$$

where $z_\epsilon(t) = z_\epsilon^0(t)$ is the process introduced in (2.7). In particular, we have

$$u_\epsilon^{\varphi_\epsilon}(t) - u^\varphi(t) = [v_\epsilon(t) - u^\varphi(t)] + z_\epsilon(t) =: \rho_\epsilon(t) + z_\epsilon(t), \qquad t \geq 0,$$

Since

$$\frac{d\rho_\epsilon}{dt}(t) = A\rho_\epsilon(t) + b(v_\epsilon(t) + z_\epsilon(t)) - b(u^\varphi(t)) + Q_\epsilon\,\varphi_\epsilon(t) - \varphi(t), \qquad \rho_\epsilon(0) = -z_\epsilon(0),$$

we have that $\rho_\epsilon(t)$ solves the following integral equation

$$\rho_\epsilon(t) = -e^{tA}z_\epsilon(0) + \int_0^t e^{(t-s)A}\left(b(v_\epsilon(s)) - b(u^\varphi(s))\right)\,ds + \int_0^t e^{(t-s)A}b(z_\epsilon(s))\,ds$$

$$+ \int_0^t e^{(t-s)A}\left(b(\rho_\epsilon(s), z_\epsilon(s)) + b(z_\epsilon(s), \rho_\epsilon(s))\right)\,ds$$

$$+ \int_0^t e^{(t-s)A}\left(b(u^\varphi(s), z_\epsilon(s)) + b(z_\epsilon(s), u^\varphi(s))\right)\,ds + [\Phi_\epsilon(t) - \Gamma(\varphi)(t)] =: \sum_{i=1}^6 I_{\epsilon,i}(t).$$

Our first goal here is to estimate the norm of each term $I_{\epsilon,i}$ in the space $\mathcal{E}_t$, for every $t \leq T$, and prove a uniform bound for ρ_ϵ in $\mathcal{E}_T$. To this purpose, we first prove a suitable bound for u^φ in $H^\theta(D)$, with $\theta \in (0,1)$.

Lemma 5.1. *Assume that $u_0 \in H^\theta(D)$, for some $\theta \in [0,1)$. Then, for any $\varphi \in L^2(0,T;H)$ we have*

$$\sup_{t\in[0,T]} |u^\varphi(t)|^2_{H^\theta(D)} + \int_0^T |u^\varphi(s)|^2_{H^{\theta+1}(D)}\,ds \leq c\left(|u_0|_{H^\theta(D)}, |\varphi|_{L^2(0,T;H)}\right). \tag{5.3}$$

Proof. Since

$$\frac{1}{2}\frac{d}{dt}|u^\varphi(t)|^2_H + |u^\varphi(t)|^2_V = \langle\varphi(t), u^\varphi(t)\rangle_H,$$

we immediately have

$$|u^\varphi(t)|^2_H + \int_0^t |u^\varphi(s)|^2_V\,ds \leq |u_0|^2_H + \frac{\lambda_1}{2}\int_0^t |\varphi(s)|^2_H\,ds. \tag{5.4}$$

For every $\theta \geq 0$, we have

$$\frac{1}{2}\frac{d}{dt}|u^\varphi(t)|^2_{H^\theta(D)} + |u^\varphi(t)|^2_{H^{\theta+1}(D)} = \langle b(u^\varphi(t)), (-A)^\theta u^\varphi(t)\rangle_H + \langle \varphi(t), (-A)^\theta u^\varphi(t)\rangle_H.$$

Now, if we assume $\theta < 1$ and set $q_1 = 2/(1-\theta)$ and $q_2 = 2/\theta$, we have

$$\left|\langle b(u^\varphi(t)), (-A)^\theta u^\varphi(t)\rangle_H\right| \leq |u^\varphi(t)|_{L^{q_1}(D)}|(-A)^\theta u^\varphi(t)|_{L^{q_2}(D)}|u^\varphi(t)|_V.$$

As

$$W^{\theta,2}(D) \hookrightarrow L^{q_1}(D), \quad W^{1-\theta,2}(D) \hookrightarrow L^{q_2}(D),$$

this implies that

$$\left|\langle b(u^\varphi(t)), (-A)^\theta u^\varphi(t)\rangle_H\right| \leq |u^\varphi(t)|_{H^\theta(D)}|u^\varphi(t)|_{H^{1+\theta}(D)}|u^\varphi(t)|_V$$

$$\leq \frac{1}{4}|u^\varphi(t)|^2_{H^{1+\theta}(D)} + c\,|u^\varphi(t)|^2_{H^\theta(D)}|u^\varphi(t)|^2_V.$$

Therefore, as

$$\left|\langle \varphi(t), (-A)^\theta u^\varphi(t)\rangle_H\right| \leq |\varphi(t)|_H\,|u^\varphi(t)|_{H^{2\theta}(D)} \leq \frac{1}{4}|u^\varphi(t)|^2_{H^{1+\theta}(D)} + c\,|\varphi|^2_H,$$

we conclude that

$$\frac{d}{dt}|u^\varphi(t)|^2_{H^\theta(D)} + |u^\varphi(t)|^2_{H^{\theta+1}(D)} \leq c\,|u^\varphi(t)|^2_{H^\theta(D)}|u^\varphi(t)|^2_V + c\,|\varphi|^2_H.$$

Thanks to (5.4), this implies

$$|u^\varphi(t)|^2_{H^\theta(D)} \leq \exp\left(c\int_0^T |u^\varphi(s)|^2_V\,ds\right)\left(|u_0|^2_{H^\theta(D)} + c\,|\varphi|^2_{L^2(0,T;H)}\right)$$

$$\leq \exp\left(c|u_0|^2_H + c\,|\varphi|^2_{L^2(0,T;H)}\right)\left(|u_0|^2_{H^\theta(D)} + c\,|\varphi|^2_{L^2(0,T;H)}\right),$$

and (5.3) easily follows. □

Now, let us estimate each term $I_{\epsilon,i}$, for $i = 1, \dots, 6$. Since

$$|I_{\epsilon,1}(t)|_{\mathcal{E}_t} = \sup_{s\in[0,t]}|e^{sA}z_\epsilon(0)|_{\mathcal{B}^\sigma_p(D)} + \left(\int_0^t |e^{sA}z_\epsilon(0)|^\beta_{\mathcal{B}^\alpha_p(D)}\,ds\right)^{\frac{1}{\beta}},$$

according to (5.1), for any $t \leq T$ we have

$$|I_{\epsilon,1}(t)|_{\mathcal{E}_t} \leq c\,|z_\epsilon(0)|_{\mathcal{B}^\sigma_p(D)} + c\left(\int_0^t s^{-\frac{1}{2}(\alpha-\sigma)\beta}\,ds\right)^{\frac{1}{\beta}}|z_\epsilon(0)|_{\mathcal{B}^\sigma_p(D)} \leq c_T\,|z_\epsilon(0)|_{\mathcal{B}^\sigma_p(D)}. \tag{5.5}$$

Now, for any two processes $u(t)$ and $v(t)$, we define

$$\Lambda(u,v)(t) := \int_0^t e^{(t-s)A}b(u(s), v(s))\,ds, \quad t \geq 0.$$

By proceeding as in [8, proof of Lemma 6.3], it is possible to show that if v_1 and v_2 are measurable mappings defined on $[0,T]$, with values in $\mathcal{B}^\alpha_p(D)$ and $\mathcal{B}^\sigma_p(D)$, respectively, then

$$|\Lambda(v_i,v_j)(t)|_{\mathcal{B}^\sigma_p(D)} \le c\int_0^t (t-s)^{-\frac{1}{2}(1+\frac{2}{p}-\alpha)}|v_1(s)|_{\mathcal{B}^\alpha_p(D)}|v_2(s)|_{\mathcal{B}^\sigma_p(D)}\,ds, \quad t\le T, \tag{5.6}$$

and

$$|\Lambda(v_i,v_j)(t)|_{\mathcal{B}^\alpha_p(D)} \le c\int_0^t (t-s)^{-\frac{1}{2}(1+\frac{2}{p}-\sigma)}|v_1(s)|_{\mathcal{B}^\alpha_p(D)}|v_2(s)|_{\mathcal{B}^\sigma_p(D)}\,ds, \quad t\le T, \tag{5.7}$$

both for $(i,j)=(1,2)$ and for $(i,j)=(2,1)$.

It is immediate to check that

$$b(v_\epsilon(t)) - b(u^\varphi(t)) = b(\rho_\epsilon(t)) + b(\rho_\epsilon(t),u^\varphi(t)) + b(u^\varphi(t),\rho_\epsilon(t)), \quad t\ge 0,$$

so that, thanks to (5.1), from (5.6) and (5.7) we get

$$|I_{\epsilon,2}|_{\mathcal{E}_t} \le c_1(t)|\rho_\epsilon|_{L^\beta(0,t;\mathcal{B}^\alpha_p(D))}\left(|\rho_\epsilon|_{C([0,t];\mathcal{B}^\sigma_p(D))} + |u^\varphi|_{C([0,t];\mathcal{B}^\sigma_p(D))}\right), \quad t\ge 0,$$

for some continuous increasing function $c_1(t)$, such that $c_1(0)=0$. Since we are assuming that $\theta \ge \sigma + 1 - 2/p$, we have that $H^\theta(D)\hookrightarrow \mathcal{B}^\sigma_p(D)$, so that from (5.3) we obtain

$$|I_{\epsilon,2}|_{\mathcal{E}_t} \le c_1(t)\,c_\gamma(|u_0|_{H^\theta(D)})|\rho_\epsilon|_{\mathcal{E}_t}\left(|\rho_\epsilon|_{C([0,t];\mathcal{B}^\sigma_p(D))}+1\right). \tag{5.8}$$

Concerning $I_{\epsilon,3}(t)$, we first notice that

$$b(z_\epsilon(t)) = \text{div}\,(z_\epsilon(t)\otimes z_\epsilon(t)) = \text{div}\,\left(z_\epsilon(t)\otimes z_\epsilon(t) - \epsilon\,\vartheta_{\delta(\epsilon)}I\right), \quad t\ge 0,$$

where $\vartheta_{\delta(\epsilon)}$ is the constant defined in (A.11), for $\delta=\delta(\epsilon)$. Then, since for every $\rho\ge -1$, $\eta\ge 0$ and $p\ge 2$ we have

$$|e^{tA}x|_{\mathcal{B}^\rho_p(D)} \le c\,t^{-(1+\frac{\rho}{2}-\frac{1}{p}+\frac{\eta}{2})}|x|_{H^{-(1+\eta)}(D)}, \quad t>0,$$

from (2.2) we get

$$|I_{\epsilon,3}(t)|_{\mathcal{B}^\sigma_p(D)} \le c\int_0^t (t-s)^{-(1+\frac{\sigma}{2}-\frac{1}{p}+\frac{\eta}{2})}\left|\text{div}(z_\epsilon(s)\otimes z_\epsilon(s)-\epsilon\,\vartheta_{\delta(\epsilon)}I)\right|_{[H^{-(1+\eta)}(D)]^4}\,ds$$

$$\le c\int_0^t (t-s)^{-(1+\frac{\sigma}{2}-\frac{1}{p}+\frac{\eta}{2})}\left|z_\epsilon(s)\otimes z_\epsilon(s)-\epsilon\,\vartheta_{\delta(\epsilon)}I)\right|_{[H^{-\eta}(D)]^4}\,ds.$$

In the same way, we have

$$|I_{\epsilon,3}(t)|_{\mathcal{B}^\alpha_p(D)} \le c\int_0^t (t-s)^{-(1+\frac{\alpha}{2}-\frac{1}{p}+\frac{\eta}{2})}\left|z_\epsilon(s)\otimes z_\epsilon(s)-\epsilon\,\vartheta_{\delta(\epsilon)}I)\right|_{[H^{-\eta}(D)]^4}\,ds.$$

Due to (5.1), this implies that we can find $\eta>0$ and $\rho\ge 1$ such that

$$|I_{\epsilon,3}|_{\mathcal{E}_t} \le c_2(t)\,\left|z_\epsilon\otimes z_\epsilon - \epsilon\,\vartheta_{\delta(\epsilon)}I\right|_{L^\rho(0,T;[H^{-(1+\gamma)}(D)]^4)}. \tag{5.9}$$

For $I_{\epsilon,4}(t)$, by using again (5.6) and (5.7), we have

$$|I_{\epsilon,4}(t)|_{\mathcal{B}^\sigma_p(D)} \le c\int_0^t (t-s)^{-\frac{1}{2}(1+\frac{2}{p}-\alpha)}|\rho_\epsilon(s)|_{\mathcal{B}^\alpha_p(D)}|z_\epsilon(s)|_{\mathcal{B}^\sigma_p(D)}\,ds,$$

and

$$|I_{\epsilon,4}(t)|_{\mathcal{B}^\alpha_p(D)} \le c\int_0^t (t-s)^{-\frac{1}{2}(1+\frac{2}{p}-\sigma)}|\rho_\epsilon(s)|_{\mathcal{B}^\alpha_p(D)}|z_\epsilon(s)|_{\mathcal{B}^\sigma_p(D)}\,ds,$$

and then, according to (5.1), we can find $\rho \ge 1$ such that

$$|I_{\epsilon,4}|_{\mathcal{E}_t} \le c_3(t)\,|\rho_\epsilon|_{L^\beta(0,t;\mathcal{B}^\alpha_p(D))}|z_\epsilon|_{L^\rho(0,T;\mathcal{B}^\sigma_p(D))} \le c_3(t)\,|\rho_\epsilon|_{\mathcal{E}_t}|z_\epsilon|_{L^\rho(0,T;\mathcal{B}^\sigma_p(D))}. \tag{5.10}$$

As for $I_{\epsilon,4}(t)$, for $I_{\epsilon,5}(t)$ we have

$$|I_{\epsilon,5}(t)|_{\mathcal{B}^\sigma_p(D)} \le c\int_0^t (t-s)^{-\frac{1}{2}(1+\frac{2}{p}-\alpha)}|u^\varphi(s)|_{\mathcal{B}^\alpha_p(D)}|z_\epsilon(s)|_{\mathcal{B}^\sigma_p(D)}\,ds,$$

and

$$|I_{\epsilon,5}(t)|_{\mathcal{B}^\alpha_p(D)} \le c\int_0^t (t-s)^{-\frac{1}{2}(1+\frac{2}{p}-\sigma)}|u^\varphi(s)|_{\mathcal{B}^\alpha_p(D)}|z_\epsilon(s)|_{\mathcal{B}^\sigma_p(D)}\,ds.$$

As we are assuming $\theta \ge \sigma + 1 - 2/p$, we have that $\theta > \alpha - 2/p$, so that for any $\eta > 0$ such that $\theta - \eta > \alpha - 2/p$, we have $H^{1+\theta-\eta}(D) \hookrightarrow \mathcal{B}^\alpha_p(D)$. By interpolation, this implies

$$|x|_{\mathcal{B}^\alpha_p(D)} \le c_\eta\,|x|_{H^{1+\theta-\eta}(D)} \le c_\eta\,|x|^{1-\eta}_{H^{1+\theta}(D)}|x|^{\eta}_{H^\theta(D)},$$

so that

$$|x|^{\frac{2}{1-\eta}}_{\mathcal{B}^\alpha_p(D)} \le c_\eta\,|x|^2_{H^{1+\theta}(D)}|x|^{\frac{2\eta}{1-\eta}}_{H^\theta(D)}.$$

According to (5.3), this implies that $u^\varphi \in L^{\frac{2}{1-\eta}}(0,T;\mathcal{B}^\alpha_p(D))$ and

$$|u^\varphi|_{L^{\frac{2}{1-\eta}}(0,T;\mathcal{B}^\alpha_p(D))} \le c_{\gamma,\eta}(|u_0|_{H^\theta(D)}). \tag{5.11}$$

Due to condition (5.1), since $\theta \ge \sigma + 1 - 2/p$, we can find $\eta \in (0,1)$ such that

$$1 - \frac{2}{\beta} < \eta < \theta + \frac{2}{p} - \alpha.$$

For such $\eta > 0$ we have

$$|I_{\epsilon,5}(t)|_{\mathcal{B}^\sigma_p(D)} \le c\left(\int_0^t s^{-\frac{1}{2}(1+\frac{2}{p}-\alpha)\frac{\beta}{\beta-1}}\,ds\right)^{\frac{\beta-1}{\beta}}|u^\varphi|_{L^{\frac{2}{1-\eta}}(0,t;\mathcal{B}^\alpha_p(D))}|z_\epsilon|_{L^\kappa(0,t;\mathcal{B}^\sigma_p(D))}, \tag{5.12}$$

where

$$\frac{1}{\kappa} = 1 - \left[\frac{1-\eta}{2} + \frac{\beta-1}{\beta}\right] = \frac{1}{\beta} - \frac{1-\eta}{2}.$$

Analogously, if we pick $\eta > 1 - 2/p$, we get

$$\int_0^t |I_{\epsilon,5}(s)|^\beta_{\mathcal{B}^\alpha_p(D)}\,ds \le c\left(\int_0^t s^{-\frac{1}{2}(1+\frac{2}{p}-\sigma)}\,ds\right)^\beta |u^\varphi|^\beta_{L^{\frac{2}{1-\eta}}(0,t;\mathcal{B}^\alpha_p(D))}|z_\epsilon|_{L^{\frac{2\beta}{2-\beta(1-\eta)}}(0,t;\mathcal{B}^\sigma_p(D))}.$$

Thanks to (5.11), this, together with (5.12), implies that there exists some $\rho \geq 1$ such that

$$|I_{\epsilon,5}|_{\mathcal{E}_T} \leq c_4(t)\, c_\gamma(|u_0|_{H^\theta(D)})|z_\epsilon|_{L^\rho(0,t;\mathcal{B}_p^\sigma(D))}. \tag{5.13}$$

Collecting together (5.5), (5.8), (5.9), (5.10) and (5.13), we conclude that

$$|\rho_\epsilon|_{\mathcal{E}_t} \leq c(t)\, c_\gamma(|u_0|_{H^\theta(D)})|\rho_\epsilon|_{\mathcal{E}_t}\, \Big(|\rho_\epsilon|_{C([0,t];\mathcal{B}_p^\sigma(D))} + |z_\epsilon|_{L^\rho(0,T;\mathcal{B}_p^\sigma(D))} + 1\Big) + c_T\, |z_\epsilon(0)|_{\mathcal{B}_p^\sigma(D)}$$

$$+c\,(t)\, c_\gamma(|u_0|_{H^\theta(D)})\, \Big(|z_\epsilon|_{L^\rho(0,T;\mathcal{B}_p^\sigma(D))} + \big|z_\epsilon \otimes z_\epsilon - \epsilon\, \vartheta_{\delta(\epsilon)} I\big|_{L^\rho(0,T;[H^{-\gamma}(D)]^4)}\Big) + |\Phi_\epsilon - \Gamma(\varphi)|_{\mathcal{E}_T}\,,$$

for some continuous increasing function $c(t)$ such that $c(0) = 0$.

Now, we are going to show that for any sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ converging to zero, there exists a subsequence $\{\epsilon_{n_k}\}_{k\in\mathbb{N}} \subset \{\epsilon_n\}_{n\in\mathbb{N}}$, such that

$$\lim_{k\to\infty} |\rho_{\epsilon_{n_k}}|_{\mathcal{E}_T} = 0, \quad \mathbb{P}-\text{a.s.} \tag{5.14}$$

and this clearly implies that

$$\lim_{\epsilon\to 0} |\rho_\epsilon|_{\mathcal{E}_T} = 0, \quad \mathbb{P}-\text{a.s.}$$

As $u_\epsilon^{\varphi_\epsilon}(t) - u^\varphi(t) = \rho_\epsilon(t) + z_\epsilon(t)$, for $t \in [0,T]$, according to (A.1) we can conclude that

$$\lim_{\epsilon\to 0} \sup_{t\in[0,T]} |u_\epsilon^{\varphi_\epsilon}(t) - u^\varphi(t)|_{\mathcal{B}_p^\sigma(D)} = 0, \quad \mathbb{P}-\text{a.s.} \tag{5.15}$$

Let $\{\epsilon_n\}_{n\in\mathbb{N}}$ be a sequence converging to zero. As we are assuming that $\alpha < 2/p$, there exists $\rho < 1$ such that $H^\rho(D) \hookrightarrow \mathcal{B}_p^\alpha(D)$, so that, due to (4.13) we have

$$\lim_{\epsilon\to 0} |\Phi_\epsilon - \Gamma(\varphi)|_{\mathcal{E}_T} = 0, \quad \mathbb{P}-\text{a.s.} \tag{5.16}$$

Then, as a consequence of (A.1), (A.13) and (5.16), we have that there exists a subsequence of $\{\epsilon_n\}_{n\in\mathbb{N}}$, that for simplicity of notations we are still denoting by $\{\epsilon_n\}_{n\in\mathbb{N}}$, and a set $\Omega' \subseteq \Omega$ with $\mathbb{P}(\Omega') = 1$, such that

$$\begin{aligned}\lim_{n\to\infty} \quad & \Big(|z_{\epsilon_n}(\omega)|_{C([0,T];\mathcal{B}_p^\sigma(D))} + \big|z_\epsilon(\omega)\otimes z_\epsilon(\omega) - \epsilon\,\vartheta_{\delta(\epsilon)} I\big|_{L^\rho(0,T;[H^{-\gamma}(D)]^4)} \\ & + |\Phi_{\epsilon_n}(\omega) - \Gamma(\varphi)(\omega)|_{\mathcal{E}_T}\Big) = 0, \quad \omega \in \Omega'.\end{aligned} \tag{5.17}$$

Next, for any $\epsilon > 0$ we denote

$$\tau_\epsilon := \inf\Big\{\, t \geq 0 \;:\; |\rho_\epsilon(t)|_{\mathcal{B}_p^\sigma(D)} \geq 1 \,\Big\}.$$

If we fix any $\omega \in \Omega'$, in view of (A.1) there exists some $n_0 = n_0(\omega) \in \mathbb{N}$ such that for any $n \geq n_0$ and $t \leq \tau_{\epsilon_n}(\omega)$

$$|\rho_\epsilon(\omega)|_{\mathcal{E}_t} \leq 3\, c(t)\, c_\gamma(|u_0|_{H^\theta(D)})|\rho_\epsilon(\omega)|_{\mathcal{E}_t} + c_T\, |z_\epsilon(0)|_{\mathcal{B}_p^\sigma(D))} + |\Phi_\epsilon(\omega) - \Gamma(\varphi)(\omega)|_{\mathcal{E}_T}$$

$$+c\,(t)\, c_\gamma(|u_0|_{H^\theta(D)})\, \Big(|z_\epsilon(\omega)|_{L^\rho(0,T;\mathcal{B}_p^\sigma(D))} + \big|z_\epsilon(\omega)\otimes z_\epsilon(\omega) - \epsilon\,\vartheta_{\delta(\epsilon)} I\big|_{L^\rho(0,T;[H^{-\gamma}(D)]^4)}\Big),$$

This implies that if we take $t_0 > 0$ such that

$$3\, c(t_0)\, c_\gamma(|u_0|_{H^\theta(D)}) \leq \frac{1}{2},$$

for any $n \geq n_0$ and $t \leq \tau_{\epsilon_n}(\omega) \wedge t_0$

$$|\rho_{\epsilon_n}(\omega)|_{\mathcal{E}_t} \leq c\, |\Phi_{\epsilon_n}(\omega) - \Gamma(\varphi)(\omega)|_{\mathcal{E}_T}$$

$$+c_T \left(|z_{\epsilon_n}(\omega)|_{C([0,T];\mathcal{B}_p^\sigma(D))} + \left| z_{\epsilon_n}(\omega) \otimes z_{\epsilon_n}(\omega) - \epsilon_n\, \vartheta_{\delta(\epsilon)} I \right|_{L^\rho(0,T;[H^{-\gamma}(D)]^4)} \right).$$

As a consequence of (5.17), there exists $n_1 = n_1(\omega) \geq n_0$ such that

$$c_T \left(|z_{\epsilon_n}(\omega)|_{C([0,T];\mathcal{B}_p^\sigma(D))} + \left| z_{\epsilon_n}(\omega) \otimes z_{\epsilon_n}(\omega) - \epsilon_n\, \vartheta_{\delta(\epsilon)} I \right|_{L^\rho(0,T;[H^{-\gamma}(D)]^4)} \right)$$

$$+c\, |\Phi_{\epsilon_n}(\omega) - \Gamma(\varphi)(\omega)|_{\mathcal{E}_T} \leq \frac{1}{2}, \quad n \geq n_1,$$

so that $\tau_{\epsilon_n}(\omega) \wedge t_0 = t_0$, for $n \geq n_1$, and

$$|\rho_{\epsilon_n}(\omega)|_{\mathcal{E}_{t_0}} \leq c\, |\Phi_{\epsilon_n}(\omega) - \Gamma(\varphi)(\omega)|_{\mathcal{E}_T}$$

$$+c_T \left(|z_{\epsilon_n}(\omega)|_{C([0,T];\mathcal{B}_p^\sigma(D))} + \left| z_{\epsilon_n}(\omega) \otimes z_{\epsilon_n}(\omega) - \epsilon_n\, \vartheta_{\delta(\epsilon)} I \right|_{L^\rho(0,T;[H^{-\gamma}(D)]^4)} \right).$$

Now, we can repeat the same argument in the intervals $[(i-1)t_0, it_0]$, for $i = 0, \ldots, i_T$, where i_T is the smallest integer such that $i_T t_0 \geq T$, and we find

$$\begin{aligned} &|\rho_{\epsilon_n}(\omega)|_{\mathcal{E}_{(i-1)t_0, it_0}} \leq i\, c\, |\Phi_{\epsilon_n}(\omega) - \Gamma(\varphi)(\omega)|_{\mathcal{E}_T} \\ &+i\, c_T \left(|z_{\epsilon_n}(\omega)|_{C([0,T];\mathcal{B}_p^\sigma(D))} + \left| z_{\epsilon_n}(\omega) \otimes z_{\epsilon_n}(\omega) - \epsilon_n\, \vartheta_{\delta(\epsilon)} I \right|_{L^\rho(0,T;[H^{-\gamma}(D)]^4)} \right), \end{aligned} \tag{5.18}$$

for every $n \geq n_i = n_i(\omega)$, where $n_i(\omega) \geq n_{i-1}(\omega)$ is such that

$$c_T \left(|z_{\epsilon_n}(\omega)|_{C([0,T];\mathcal{B}_p^\sigma(D))} + \left| z_{\epsilon_n}(\omega) \otimes z_{\epsilon_n}(\omega) - \epsilon_n\, \vartheta_{\delta(\epsilon)} I \right|_{L^\rho(0,T;[H^{-\gamma}(D)]^4)} \right)$$

$$+c\, |\Phi_{\epsilon_n}(\omega) - \Gamma(\varphi)(\omega)|_{\mathcal{E}_T} \leq \frac{1}{2i}, \quad n \geq n_i.$$

Therefore, from (5.18) we obtain that for any $n \geq n_{i_T}(\omega)$

$$|\rho_{\epsilon_n}(\omega)|_{\mathcal{E}_T} \leq i\, c\, |\Phi_{\epsilon_n}(\omega) - \Gamma(\varphi)(\omega)|_{\mathcal{E}_T}$$

$$+i\, c_T \left(|z_{\epsilon_n}(\omega)|_{C([0,T];\mathcal{B}_p^\sigma(D))} + \left| z_{\epsilon_n}(\omega) \otimes z_{\epsilon_n}(\omega) - \epsilon_n\, \vartheta_{\delta(\epsilon)} I \right|_{L^\rho(0,T;[H^{-\gamma}(D)]^4)} \right),$$

and due to (5.17) we can conclude that

$$\lim_{n \to \infty} |\rho_{\epsilon_n}(\omega)|_{\mathcal{E}_T} = 0.$$

A Appendix

Here we describe and prove some properties of the solution of the linear problem. As in Section 2, for every $\alpha \geq 0$ and $\epsilon > 0$ we denote by $z_\epsilon^\alpha(t)$ the stationary solution of the linear problem

$$dz(t) = (A - \alpha)z(t)\,dt + \sqrt{\epsilon}\,dw^{\delta(\epsilon)}(t), \qquad t \geq 0.$$

The process $z_\epsilon^\alpha(t)$ is given by

$$z_\epsilon^\alpha(t) = \sqrt{\epsilon}\int_{-\infty}^t e^{(t-s)(A-\alpha)}\,d\bar{w}^{\delta(\epsilon)}(s), \qquad t \geq 0.$$

As we already mentioned in Section 2, for any fixed $\epsilon > 0$ the process z_ϵ^α belongs to the space $L^p(\Omega; C([0,T]; D((-A)^\beta))$, for any $T > 0$, $p \geq 1$ and $\beta < \gamma/2$.

We first want to estimate the norm of z_ϵ^α in Besov spaces of negative exponent.

Lemma A.1. *For any $\alpha \geq 0$ and $\epsilon > 0$ and for any $p, \kappa \geq 1$ and $\sigma < \sigma' < 0$ it holds*

$$\mathbb{E}\sup_{t\in[0,T]}|z_\epsilon^\alpha(t)|^\kappa_{\mathcal{B}_p^\sigma(D)} \leq c_{\kappa,p}\left(\epsilon\sum_{k\in\mathbb{Z}_0^2}|k|^{2(\sigma'-1)}\right)^{\frac{\kappa}{2}}. \tag{A.1}$$

Proof. Since $z_\epsilon^\alpha(t) = (-A)^{-\frac{\sigma}{2}}(-A)^{\frac{\sigma}{2}}z_\epsilon^\alpha(t)$, we have

$$|z_\epsilon^\alpha(t)|_{\mathcal{B}_p^\sigma(D)} \leq |(-A)^{\frac{\sigma}{2}}z_\epsilon^\alpha(t)|_{L^p(D)}. \tag{A.2}$$

By using stochastic factorization, for any $\beta \in (0,1)$ we have

$$(-A)^{\frac{\sigma}{2}}z_\epsilon^\alpha(t) = \frac{\sin\pi\beta}{\pi}\int_{-\infty}^t (t-s)^{\beta-1}e^{(t-s)A}Y_{\epsilon,\beta}(s)\,ds,$$

where

$$Y_{\epsilon,\beta}(s) = \int_{-\infty}^s (s-\rho)^{-\beta}e^{(s-\rho)A}(-A)^{\frac{\sigma}{2}}dw^{\delta(\epsilon)}(\rho).$$

Therefore, if we take $p \geq 1/\beta$, we get

$$|(-A)^{\frac{\sigma}{2}}z_\epsilon^\alpha(t)|^p_{L^p(D)} \leq c_{\beta,p}\left(\int_{-\infty}^t s^{-\frac{(1-\beta)p}{p-1}}e^{-\frac{p}{p-1}s}\,ds\right)^{p-1}\int_{-\infty}^t |Y_{\epsilon,\beta}(s)|^p_{L^p(D)}\,ds. \tag{A.3}$$

Now, for any $t \in \mathbb{R}$ and $x \in D$

$$\mathbb{E}\,|Y_{\epsilon,\beta}(t,x)|^p = c_p\,\epsilon^{\frac{p}{2}}\mathbb{E}\left(\sum_{k\in\mathbb{Z}_0^2}\int_{-\infty}^t \lambda_k(\delta(\epsilon))|k|^\sigma(t-s)^{-\beta}\,e^{-(t-s)(|k|^2+\alpha)}e_k(x)\,d\beta_k(s)\right)^p$$

$$\leq c_p\,\epsilon^{\frac{p}{2}}\left(\sum_{k\in\mathbb{Z}_0^2}\int_{-\infty}^t \lambda_k(\delta(\epsilon))^2|k|^{2\sigma}(t-s)^{-2\beta}\,e^{-2(t-s)(|k|^2+\alpha)}|e_k(x)|^2\,ds\right)^{\frac{p}{2}}$$

$$\leq c_p\,\epsilon^{\frac{p}{2}}\left(\sum_{k\in\mathbb{Z}_0^2}|k|^{2\sigma+4\beta-2}\right)^{\frac{p}{2}},$$

so that, integrating with respect to $x \in D$, for any $\beta < -\sigma/2$, and hence $p \geq -2/\sigma$,

$$\mathbb{E}\,|Y_{\epsilon,\beta}(t)|^p_{L^p(D)} \leq c_p \left(\epsilon \sum_{k\in\mathbb{Z}^2_0} |k|^{2(\sigma+2\beta-1)} \right)^{\frac{p}{2}}.$$

Therefore, thanks to (A.2) and (A.3), for any $\kappa \geq p \geq 2/\sigma$ this yields

$$\mathbb{E} \sup_{t\in[0,T]} |z^\alpha_\epsilon(t)|^\kappa_{\mathcal{B}^\sigma_p(D)} \leq \mathbb{E} \sup_{t\in[0,T]} |(-A)^{\frac{\sigma}{2}} z^\alpha_\epsilon(t)|^\kappa_{L^p(D)} \leq c_{\kappa,p}\, \mathbb{E} \sup_{t\in[0,T]} |(-A)^{\frac{\sigma}{2}} z^\alpha_\epsilon(t)|^\kappa_{L^k(D)}$$

$$\leq c_{\kappa,p} \left(\epsilon \sum_{k\in\mathbb{Z}^2_0} |k|^{2(\sigma+2\beta-1)} \right)^{\frac{\kappa}{2}}.$$

The general case follows from the Hölder inequality. □

Next, we estimate the norm of z^α_ϵ in $L^p(D)$-spaces. This Lemma could be proved using stochastic calculus in Banach spaces and the notion of γ-radonfying operators (see [2]). We give here an elementary proof.

Lemma A.2. *For every $\alpha \geq 0$ and $\epsilon > 0$ and for every $p \geq 1$ it holds*

$$\mathbb{E}\,|z^\alpha_\epsilon(t)|^p_{L^p(D)} \leq c_p(T) \left(\epsilon \log\left(\frac{1+\delta(\epsilon)}{\delta(\epsilon)} \right) \right)^{\frac{p}{2}}, \qquad t \in [0,T]. \tag{A.4}$$

Proof. For every $p \geq 1$ we have

$$\mathbb{E}\,|z^\alpha_\epsilon(t)|^p_{L^p(D)} = \epsilon^{\frac{p}{2}}\, \mathbb{E} \int_D \left| \sum_{k\in\mathbb{Z}^2_0} \int_{-\infty}^t e^{-(t-s)(|k|^2+\alpha)} \lambda_k(\delta(\epsilon)) e_k(x)\, d\bar\beta_k(s) \right|^p dx$$

$$\leq \epsilon^{\frac{p}{2}} \int_D \left(\sum_{k\in\mathbb{Z}^2_0} e^{-2(t-s)(|k|^2+\alpha)} \lambda_k(\delta(\epsilon))^2\, |e_k(x)|^2\, ds \right)^{\frac{p}{2}} dx$$

$$\leq |D|\, \epsilon^{\frac{p}{2}} \left(\sum_{k\in\mathbb{Z}^2_0} \frac{1}{|k|^2(1+\delta(\epsilon)|k|^{2\gamma})} \right)^{\frac{p}{2}}.$$

Since we have

$$\sum_{k\in\mathbb{Z}^2_0} \frac{1}{|k|^2(1+\delta(\epsilon)|k|^{2\gamma})} \sim \int_1^{+\infty} \frac{1}{x(1+\delta(\epsilon)x^\gamma)}\, dx$$

$$= \frac{1}{\gamma} \int_{\delta(\epsilon)}^\infty \frac{1}{x(1+x)}\, dx = \frac{1}{\gamma} \left(\log(1+\delta(\epsilon)) + \log\frac{1}{\delta(\epsilon)} \right),$$

this implies that (A.4) holds. □

Now, by proceeding as in the proof of [9, Proposition 2.1], it is possible to show that for any $p \geq 1$ and $\beta \in (0, 1/4)$ there exist $\theta = \theta(p, \beta) \in (0, 1/4)$ and $\rho = \rho(p, \beta) \in (0, 1)$, and a random variable $K_\epsilon(p, \beta)$ such that for any $\alpha \geq 0$ and $\epsilon > 0$

$$|z_\epsilon^\alpha(t)|_{L^p(D)} \leq (\alpha \vee 1)^{-\theta}(1 + t^\rho)\, K_\epsilon(p, \beta), \qquad \mathbb{P} - \text{a.s.}, \tag{A.5}$$

where

$$K_\epsilon(p, \beta) = c_{p,\beta} \left(\int_{-\infty}^{+\infty} (1 + \sigma^2)^{-1} |Y_\epsilon(\sigma)|_{L^p(D)}^m \, d\sigma \right)^{1/m}, \tag{A.6}$$

for some $m = m(p, \beta) \geq 1$, and where

$$Y_\epsilon(\sigma) = \sqrt{\epsilon} \int_{-\infty}^{\sigma} (\sigma - s)^{-\beta} e^{(\sigma - s)A} \, dw^{\delta(\epsilon)}(s). \tag{A.7}$$

In particular, we have

$$|z_\epsilon^\alpha|_{C([0,T];L^p(D))} \leq (\alpha \vee 1)^{-\theta}\, c_p(T)\, K_\epsilon(p, \beta), \qquad \mathbb{P} - \text{a.s.} \tag{A.8}$$

In what follows, it will be important that the random variable $K_\epsilon(p, \beta)$ has all moments finite, with an uniform bound with respect to $\epsilon > 0$.

Lemma A.3. *Let $p, q \geq 1$ and $\epsilon > 0$ be fixed. Then, for any $\eta \in (0, 1/2\gamma)$ there exists $\beta_\eta \in (0, 1/4)$ such that*

$$\mathbb{E}\, |K_\epsilon(p, \beta_\eta)|^q \leq c_{p,\beta_\eta,q} \left(\epsilon\, \delta(\epsilon)^{-\eta} \right)^{c_{q,p}}, \tag{A.9}$$

Proof. It is immediate to check that, for any $q \geq m$, we have

$$\mathbb{E}|K_\epsilon(p, \beta)|^q \leq c_{p,\beta,q} \int_{-\infty}^{+\infty} (1 + \sigma^2)^{-1} \mathbb{E}\, |Y_\epsilon(\sigma)|_{L^p(D)}^q \, d\sigma.$$

Now, since

$$Y_\epsilon(\sigma, x) = \sqrt{\epsilon} \sum_{k \in \mathbb{Z}_0^2} \int_{-\infty}^{\sigma} (\sigma - s)^{-\beta} \lambda_k(\delta(\epsilon)) e^{-|k|^2(\sigma - s)} e_k(x) \, d\bar{\beta}_k(s),$$

we have

$$\mathbb{E}\ |Y_\epsilon(\sigma, x)|^p \leq c_p\, \epsilon^{p/2} \left(\sum_{k \in \mathbb{Z}_0^2} |e_k|_{L^\infty(D)}^2 \int_0^\infty s^{-2\beta} \lambda_k(\delta(\epsilon))^2 e^{-|k|^2 s} \, ds \right)^{\frac{p}{2}}$$

$$\leq c_p \left(\epsilon \sum_{k \in \mathbb{Z}_0^2} |k|^{-2(1-2\beta)} (1 + \delta(\epsilon)|k|^{2\gamma})^{-1} \right)^{\frac{p}{2}} =: c_p\, \Lambda_\beta(\epsilon)^{\frac{p}{2}}.$$

This implies that for any $p, q \geq 1$

$$\mathbb{E}\, |Y_\epsilon(\sigma)|_{L^p(D)}^q \leq c_1(q, p)\, \Lambda_\beta(\epsilon)^{c_2(q,p)},$$

for some positive constants $c_1(q,p)$ and $c_2(q,p)$. Now, we have

$$\Lambda_\beta(\epsilon) \sim \epsilon \int_1^{+\infty} \frac{1}{x^{1-2\beta}(1+\delta(\epsilon)x^\gamma)}\,dx = \epsilon\,\frac{1}{\gamma}\left(\frac{1}{\delta(\epsilon)}\right)^{\frac{2\beta}{\gamma}} \int_{\delta(\epsilon)}^{+\infty} y^{\frac{2\beta}{\gamma}-1}\frac{1}{1+y}\,dy.$$

Therefore, if we pick any $\eta \in (0, 1/2\gamma)$ and define $\beta_\eta := \eta\,\gamma/2$, we have

$$\Lambda_{\beta_\eta}(\epsilon) \le c\,\epsilon\,\delta(\epsilon)^{-\eta},$$

and this implies (A.9).

□

In what follows, we shall denote $\mathcal{H} := \mathbb{R}^{\mathbb{Z}^2_0}$ and $\mu := \mathcal{N}(0, (-A)^{-1}/2)$. The Gaussian measure μ is defined on $\mathcal{H}$, but in fact $\mu(H^\sigma(D)) = 1$, if $\sigma < 0$, so that the support of μ is contained in $H^\sigma(D)$, for every $\sigma < 0$.

Now, for any $h \in \mathcal{H}$ and $\delta > 0$, we define

$$h_\delta := \sum_{k\in\mathbb{Z}^2_0} \langle h, e_k\rangle \lambda_k(\delta)\, e_k,$$

where we recall that, for any $k \in \mathbb{Z}^2_0$ and $\delta > 0$,

$$\lambda_k(\delta) = \frac{1}{\sqrt{1+\delta\,|k|^{2\gamma}}}.$$

Next, for $i = 1, 2$ we define

$$:(h^i_\delta)^2:(x) = \sqrt{2}\,\left[(h^i_\delta)^2(x) - \vartheta_\delta\right], \quad x \in D, \quad \delta > 0, \tag{A.10}$$

where

$$\vartheta_\delta = \frac{1}{2(2\pi)^2}\sum_{k\in\mathbb{Z}^2_0}\frac{k_1^2}{|k|^4}\lambda_k(\delta)^2 = \frac{1}{2(2\pi)^2}\sum_{k\in\mathbb{Z}^2_0}\frac{k_2^2}{|k|^4}\lambda_k(\delta)^2. \tag{A.11}$$

By proceeding as in [8, Appendix] it is possible to prove that for $i = 1, 2$

$$\exists \lim_{\delta\to 0} :(h^i_\delta)^2: \quad \text{in } L^\kappa(\mathcal{H},\mu; H^\sigma(D)),$$

and

$$\exists \lim_{\delta\to 0} h^1_\delta\, h^2_\delta \quad \text{in } L^\kappa(\mathcal{H},\mu; H^\sigma(D)),$$

for every $\kappa \ge 1$ and $\sigma < 0$. In particular, due to definition (A.10), this implies that

$$\exists \lim_{\delta\to 0} \left(h_\delta \otimes h_\delta - \vartheta_\delta\, I_{\mathbb{R}^2}\right), \quad \text{in } L^\kappa(\mathcal{H},\mu; [H^\sigma(D)]^4). \tag{A.12}$$

Lemma A.4. *For every $\epsilon > 0$, let us denote $z_\epsilon(t) := z^0_\epsilon(t)$. Then, for $\sigma < 0$ and $\kappa, p \ge 1$ we have*

$$\lim_{\epsilon\to 0} \mathbb{E}\,|z_\epsilon \otimes z_\epsilon - \epsilon\,\vartheta_{\delta(\epsilon)}\, I|^\kappa_{L^p(0,T;[H^\sigma(D)]^4)} = 0. \tag{A.13}$$

Proof. It is immediate to check that

$$z_\epsilon(t) = \sqrt{\epsilon}\, Q_\epsilon z(t), \quad t \in \mathbb{R},$$

where

$$z(t) = \int_{-\infty}^t e^{(t-s)A} dw(t) = \sum_{k \in \mathbb{Z}_0^2} \int_{-\infty}^t e^{-(t-s)|k|^2}\, d\beta_k(s).$$

The process $z(t)$ is stationary Gaussian and $\mathcal{L}(z(t)) = \mu$, for every $t \in \mathbb{R}$. This means that for any $p \geq 1$

$$\mathbb{E}\, |z_\epsilon \otimes z_\epsilon - \epsilon\, \vartheta_{\delta(\epsilon)}\, I|^p_{L^p(0,T;[H^\sigma(D)]^4)} = \mathbb{E} \int_0^T |z_\epsilon(t) \otimes z_\epsilon(t) - \epsilon\, \vartheta_{\delta(\epsilon)}\, I|^p_{[H^\sigma(D)]^4}\, dt$$

$$= \epsilon^p\, T \int_{\mathcal{H}} |Q_\epsilon h \otimes Q_\epsilon h - \vartheta_{\delta(\epsilon)} I|^p_{[H^\sigma(D)]^4} \mu(dh) = \epsilon^p\, T \int_{\mathcal{H}} |h_{\delta(\epsilon)} \otimes h_{\delta(\epsilon)} - \vartheta_{\delta(\epsilon)} I|^p_{[H^\sigma(D)]^4}\, \mu(dh).$$

Because of (A.12), this implies (A.13) in the case $\kappa = p \geq 1$. The case $\kappa, p \geq 1$ follows from the Hölder inequality and the fact that $L^p(D) \subset L^q(D)$, if $p \geq q$.

□

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