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November 30, 2018

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Abstract

We consider the motion of a rigid body in a viscoplastic material. This material is modeled by the 3D Bingham equations, and the Newton laws govern the displacement of the rigid body. Our main result is the existence of a weak solution for the corresponding system. The weak formulation is an inequality (due to the plasticity of the fluid), and it involves a free boundary (due to the motion of the rigid body). We approximate it by regularizing the convex terms in the Bingham fluid and by using a penalty method to take into account the presence of the rigid body.

Keywords: Bingham fluid, rigid body, fluid-structure interaction systems, weak solutions
2010 Mathematics Subject Classification. 35Q35, 76D03, 74F10

1 Introduction and main result

We analyse in this article a fluid-structure interaction system where the fluid is a Bingham viscoplastic liquid and where the structure is a rigid body. Viscoplastic liquids can model various natural and industrial fluids, for instance, mudflows, snow avalanche, volcanic lava flows, toothpaste, mayonnaise, etc. They behave as a liquid for high stresses and as solid for low stresses. The Bingham constitutive equation described below is one of the simplest models for a viscoplastic fluid. It was proposed by Bingham [3] in 1916. The corresponding system of partial differential equations has been studied in many works, for instance, in Duvaut and Lions [16, Chapter VI], where the existence of weak solutions for the Bingham fluid (without structures) is proved.

Let us describe our fluid-solid system: we consider $\Omega \subset \mathbb{R}^3$ an open, bounded and connected set containing a Bingham plastic fluid and a rigid body. We denote respectively by $S(t)$ and by $F(t)$ the domains of the structure and the fluid at instant t. We assume that the solid is a rigid body and its domain can be described from its initial configuration S_0: for $a \in \mathbb{R}^3$ and $Q \in SO(3)$ (the rotation group) we set

$$\widehat{S}(a,Q) := a + QS_0 \quad \text{and} \quad \widehat{F}(a,Q) := \Omega \setminus \overline{\widehat{S}(a,Q)}. \tag{1.1}$$

Then,

$$S(t) = \widehat{S}(h(t),R(t)) \quad \text{and} \quad F(t) = \widehat{F}(h(t),R(t)).$$

We assume in what follows that the center of mass of S_0 is located at the origin so that $h(t)$ is the position of the center of mass of the rigid body. We also suppose that S_0 (and thus $S(t)$) is open, bounded and connected and that $F_0 := \Omega \setminus \overline{S_0}$ (and thus $F(t)$ as long as the rigid body remains inside Ω) is connected.

We write the governing equations for the fluid flow by using the Cauchy momentum equation where the stress tensor is given by a subdifferential equation which represents the viscoplastic behavior of the Bingham fluid. The balance equations for linear and angular momentum govern the motion of the rigid body. The full system of equations modeling the motion of the fluid and the rigid body is:

$$\rho_f \left(\frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) - \operatorname{div} \sigma(u,p) = 0, \qquad x \in F(t), t \in (0,T), \tag{1.2}$$

$$\operatorname{div} u = 0, \qquad x \in F(t), t \in (0,T), \tag{1.3}$$

$$u = 0, \qquad x \in \partial\Omega, t \in (0,T), \tag{1.4}$$

$$u(t,x) = \ell(t) + \omega(t) \times (x - h(t)), \qquad x \in \partial S(t), t \in (0,T), \tag{1.5}$$

$$m\ell' = -\int_{\partial S} \sigma(u,p) n \, ds, \qquad t \in (0,T), \tag{1.6}$$

$$(J\omega)' = -\int_{\partial S} (x-h) \times \sigma(u,p) n \, ds, \qquad t \in (0,T), \tag{1.7}$$

$$R' = \mathbb{A}(\omega) R, \qquad t \in (0,T), \tag{1.8}$$

$$h' = \ell, \qquad t \in (0,T), \tag{1.9}$$

$$u(0,\cdot) = u_0, \qquad x \in F_0, \tag{1.10}$$

$$R(0) = I_3, \quad h(0) = 0, \tag{1.11}$$

$$\ell(0) = \ell_0, \quad \omega(0) = \omega_0. \tag{1.12}$$

In the above system the unknowns are $u(t,x)$ (velocity field of the fluid), $p(t,x)$ (pressure of the fluid), $h(t)$ and $\ell(t)$ (the position and the velocity of the center of mass of the rigid body), $R(t)$ and $\omega(t)$ (the orientation and the angular velocity of the rigid body). We have also denoted by n the outward normal to $F(t)$. For any $\omega \in \mathbb{R}^3$, $\mathbb{A}(\omega)$ is the skew-symmetric matrix:

$$\mathbb{A}(\omega) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

We assume that the densities ρ_f and ρ_s of the fluid and the solid are positive constants. In that case, the mass of the solid m is given by

$$m = \rho_s \left| S_0 \right|,$$

where $|S_0|$ is the volume of S_0, and the moment of inertia J is given by:

$$J(t) = \widehat{J}(h(t), R(t)),$$

where

$$\widehat{J}(a, Q) = \rho_s \int_{\widehat{S}(a,Q)} \left(|x-a|_2^2 \, I_3 - (x-a) \otimes (x-a) \right) dx.$$

We have denoted by $|a|_2 = \sqrt{a \cdot a}$ the Euclidean norm in $\mathbb{R}^3$.

We can check that

$$\widehat{J}(h, Q) = Q J_0 Q^*, \tag{1.13}$$

where we have denoted by M^* the transpose matrix of M and where

$$J_0 = \rho_s \int_{S_0} \left(|x|_2^2 \, I_3 - x \otimes x \right) dx.$$

In particular, $J(t)$ is symmetric and positive definite.

The Cauchy stress tensor is given by the constitutive equation for a Bingham fluid. To write this relation, first we decomposed the Cauchy stress tensor as follows:

$$\sigma(u, p) = -pI_3 + \sigma^d(D(u)), \quad D(u) = \frac{1}{2} \left(\nabla u + (\nabla u)^* \right), \tag{1.14}$$

where the function σ^d is given by the following subdifferential inclusion:

$$\sigma^d(D) \in \partial f(D) \tag{1.15}$$

with $f : \mathbb{M}^{3\times3} \to \mathbb{R}$ the convex function defined by:

$$f(D) = \mu \, |D|_2^2 + g \, |D|_2 \, . \tag{1.16}$$

We have denoted by $\mathbb{M}^{3\times3}$ the space of square matrices of order 3 and by $|D|_2 = \sqrt{D : D}$ the corresponding Frobenius norm. In the constitutive law given by f, the constant $g > 0$ is the yield stress and the constant $\mu > 0$ is the molecular viscosity.

By standard calculation, equation (1.15) is equivalent to:

$$\begin{cases} \left|\sigma^d(D)\right|_2 \leqslant g & \iff \quad D = 0, \\ \left|\sigma^d(D)\right|_2 > g & \iff \quad D \neq 0 \quad \text{and} \quad \sigma^d(D) = 2\mu D + g \dfrac{D}{|D|_2}. \end{cases} \tag{1.17}$$

Indeed, if $D \neq 0$, then (1.15) is equivalent to $\sigma^d(D) = 2\mu D + gD/\,|D|_2$. If we multiply $\sigma^d(D)$ by D, we notice that $|\sigma^d(D)|_2 > g$. If $D = 0$, (1.15) is equivalent to $|\sigma^d(D)|_2 \leqslant g$. The above representation says that a Bingham fluid behaves like a viscous fluid if $\left|\sigma^d(D(u))\right|_2 > g$, and as a rigid body otherwise. We note that if $g = 0$, we recover the Navier-Stokes equations coupled with the equations of the rigid body.

To write the weak formulation associated with system (1.2)-(1.12), we first introduce some notation. We denote by L^q and H^q the classical Lebesgue and Sobolev spaces. We also denote by $\mathcal{C}^q$ the space of q-times continuous differential functions. We write $\mathcal{C}_0^q$ the set of all functions in $\mathcal{C}^q$ with compact support.

We introduce the standard spaces in the study of the equations of fluid mechanics:

$$L^2_\sigma(\Omega) = \left\{ v \in L^2(\Omega) \ ; \ \operatorname{div}(v) = 0, \quad v \cdot n = 0 \quad \text{on } \partial\Omega \right\},$$

$$H^1_\sigma(\Omega) = L^2_\sigma(\Omega) \cap H^1_0(\Omega) \, .$$

We define the space of rigid velocities:

$$\mathcal{R} = \{ x \mapsto \ell + \omega \times x \ ; \ \ell, \omega \in \mathbb{R}^3 \} \tag{1.18}$$

and we introduce the following spaces due to the presence of the rigid body:

$$L^2_S(\Omega) = \left\{ v \in L^2_\sigma(\Omega) \ ; \ D(v) = 0 \quad \text{in } S \right\},$$

$$H^1_S(\Omega) = \left\{ v \in H^1_\sigma(\Omega) \ ; \ D(v) = 0 \quad \text{in } S \right\}.$$

We recall (see, for instance, [59, Lemma 1.1, p.18]) that

$$D(v) = 0 \quad \text{in } S \iff v_{|S} \in \mathcal{R}.$$

We extend the fluid velocity u to the whole domain Ω by

$$u(t,x) = \ell(t) + \omega(t) \times (x - h(t)) \quad x \in S(t) \tag{1.19}$$

and similarly,

$$u_0(x) = \ell_0 + \omega_0 \times x \quad x \in S_0. \tag{1.20}$$

In particular, $D(u) = 0$ in $S(t)$ and $D(u_0) = 0$ in S_0.

We also define a "global" density for the fluid-solid mixture as:

$$\rho(t,x) := \begin{cases} \rho_f & x \in F(t), \\ \rho_s & x \in S(t). \end{cases}$$

Then, we show in the next section the following result:

Proposition 1.1. *Assume that (u,p,ℓ,ω,h,R) is a regular function satisfying (1.2)-(1.12). Then the following inequality holds:*

$$\int_0^T \int_\Omega \rho \left(\frac{\partial v}{\partial t} + (u\cdot\nabla)v \right) \cdot (v-u) dxdt + 2\mu \int_0^T \int_\Omega D(u) : D(v-u) dxdt \\ + g \int_0^T \int_\Omega \left(|D(v)|_2 - |D(u)|_2 \right) dxdt \geqslant -\frac{1}{2} \int_\Omega \rho(0,x) \left|v(0,x) - u_0\right|_2^2 dx, \tag{1.21}$$

for any $v \in \mathcal{C}^1\left([0,T]; H^1_{S(t)}(\Omega)\right)$. Moreover, the following energy equality holds:

$$\frac{1}{2}\int_\Omega \rho(t,x) \left|u(t,x)\right|_2^2 dx + 2\mu \int_0^t \int_\Omega |D(u)|_2^2 dxdt + g \int_0^t \int_\Omega |D(u)|_2 dxdt = \frac{1}{2}\int_\Omega \rho(0,x)\left|u_0\right|_2^2 dx, \tag{1.22}$$

for all $t \in (0,T)$. On the other hand, if (u,p,ℓ,ω,h,R) is a regular function satisfying (1.8), (1.9), (1.11), (1.21) and (1.19), and if $|D(u)|_2 \neq 0$ in $F(t)$, then (u,p,ℓ,ω,h,R) satisfies (1.2)–(1.9).

Remark 1.2. *Since the potential energy f (defined in (1.16)) is not differentiable, the Bingham constitutive equation (1.15) leads us to the variational inequality (1.21). In this weak formulation, we also notice that the space of the test functions depends on the solution, which comes from the fact that we are working with a free boundary problem.*

The above proposition allows us to introduce the notion of weak solution of the system (1.2)-(1.12):

Definition 1.3 (Weak Solution)**.** *A weak solution of the system of equations (1.2)-(1.12) is a triplet (u,h,R) with the following properties:*

- *$(h,R) \in W^{1,\infty}(0,T;\mathbb{R}^3 \times SO(3))$ and satisfy (1.8), (1.9), (1.11).*
- *$u \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_\sigma(\Omega))$ and $u(t,x) = \ell(t) + \omega(t) \times (x - h(t))$ for $x \in S(t)$.*
- *Inequality (1.21) holds for any $v \in \mathcal{C}^1([0,T];H^1_{S(t)}(\Omega))$.*
- *The following energy inequality holds true*

$$\frac{1}{2}\int_\Omega \rho(t,x) \left|u(t,x)\right|_2^2 dx + 2\mu \int_0^t \int_\Omega |D(u)|_2^2 dxdt + g \int_0^t \int_\Omega |D(u)|_2 dxdt \leqslant \frac{1}{2}\int_\Omega \rho(0,x)\left|u_0\right|_2^2 dx, \tag{1.23}$$

a.e. in $(0,T)$.

The main result of this article is the following result.

Theorem 1.4. *Assume $S_0 \Subset \Omega$, $\partial\Omega$ and ∂S_0 are of class $\mathcal{C}^2$, $u_0 \in L^2_\sigma(\Omega)$, with $u_0(x) = \ell_0 + \omega_0 \times x$ for $x \in S_0$. Then, there exists a weak solution of the system (1.2)–(1.12) defined on a maximal time interval $(0,T)$, and one of the following alternatives holds true:*

1. *$T = +\infty$;*
2. *$\lim_{t\to T} \operatorname{dist}(S(t), \partial\Omega) = 0$.*

Remark 1.5. *One can write a bidimensional version of system* (1.2)-(1.12) *and following the proof of the above theorem, it is possible to obtain the same existence result for the corresponding system. Let us mention that even in dimension 2 in space, the uniqueness of weak solutions can be a delicate question. For a Bingham fluid alone (without rigid bodies), it is done in [16, p.301]. However, for the system composed by a rigid body and a fluid governed by the Navier-Stokes system, this issue has been solved only recently (see [7], [31] and also [10] for a weak-strong uniqueness property).*

Remark 1.6. *One of the difficulties to prove Theorem 1.4 comes from the fact that we are working with a free boundary problem. Such a difficulty is classical in the study of fluid-structure interaction systems and a standard method consists in using a penalization method. There exist a least two different penalization approaches: a "L^2" penalization (see for instance [11, 29]), and a "H^1" penalization (see for instance [52]). We follow the first method (see (3.10)), but it could also be possible to consider a H^1 penalization. In that case, we would have to consider a variable viscosity in the approximation problems of Section 3, with a viscosity that goes to infinity in the solid domain. With such an approach, one would need to consider arguments from [52], whereas here we have used or adapted results both from [29] and [52]. These two penalization methods can be used in numerical schemes to simulate the motion of rigid bodies in a fluid, but the drawback of the H^1 penalization method is that the solid can change its shape (it is approximated by a very viscous fluid). We refer for instance to [4] for the analysis of a numerical scheme base on the L^2 penalization method and also [51], [32], [34], [43], etc. for some other works on the numerical study of fluid-rigid body systems.*

Remark 1.7. *Let us note that the pressure of the fluid p does not appear in the weak formulation* (1.21) *due to the property of the test functions. One could also work with a mixed formulation where we keep the pressure and where the test functions do not satisfy the free divergence condition. The corresponding study is more complicated since we need to obtain estimates of the pressure during the proof of existence. Such an approach is made for the Bingham system without structures in [8] but the authors need to consider some slip boundary conditions to obtain their results. A method to obtain the pressure for a non-Newtonian fluid with Dirichlet boundary conditions is developed by Wolf [60]. This pressure is called by the author "local" pressure and is the sum of a regular pressure and of the time derivative of an harmonic function. We refer the reader to [20] where, the case of a non-Newtonian fluid with a power law and rigid body interaction is treated. Part of this work is devoted to the study of the "local" pressure where the authors manage to pass to the limit in the nonlinearity associated with the stress tensor taking advantage of the more regular structure of the stress tensor.*

Remark 1.8. *The interesting problem of obtaining some information on the set where the Bingham fluid behaves as a solid (where $D(u) = 0$), and also to know how this set interacts with the rigid body, is entirely open from the theoretical point of view, even without any rigid body. However, tackling these questions in a numerical study is possible. Lots of works have been done to solve numerically the Bingham fluid. We refer the reader to the book [53] and the review paper [12].*

Remark 1.9. *Let us point out that several problems can be addressed on the system* (1.2)-(1.12)*: behavior as $t \to \infty$ or as the distance between $S(t)$ and $\partial\Omega$ goes to* 0*. We refer the reader to [17], [37], [38] for some works done in the case of the Navier-Stokes system instead of the Bingham equations. Another problem that can be considered corresponds to the limits as $g \to 0$ or as $g \to \infty$. The study of $g \to 0$ is done in the case of a 2D Bingham fluid alone in [16, pp.306–310], and the authors obtain for the limit the Navier-Stokes equations. One can consider a similar problem in the case of system* (1.2)-(1.12) *(in the 2D case), however the study would be more complicated since we deal here with a moving domain for the fluid. The case $g \to \infty$ is simpler in the case of a fluid alone (and is done in [16, pp.306–310]) and the proof can be adapted in our case. We give below the corresponding statement.*

Corollary 1.10. *Under the hypotheses of Theorem 1.4, there exists a time T independent of $g > 0$ such that the weak solutions (u_g, h_g, R_g) of the system* (1.2)–(1.12) *exist on $(0, T)$. Moreover,*

$$u_g \overset{*}{\rightharpoonup} 0 \text{ weak star in } L^\infty(0,T; L^2_\sigma(\Omega)), \tag{1.24}$$

$$u_g \rightharpoonup 0 \text{ weakly in } L^2(0,T; H^1_\sigma(\Omega)), \tag{1.25}$$

$$D(u_g) \to 0 \quad \text{in } L^1(0,T; L^1(\Omega)), \tag{1.26}$$

and

$$(h_g, R_g) \to (0, I_3) \quad \text{in } \mathcal{C}([0,T]; \mathbb{R}^3 \times SO(3)). \tag{1.27}$$

The proof of this result is done in Section 6.

The mathematical study of fluid-structure interaction systems has been the subject of an intensive research since around 2000. A large part of the articles devoted to this study concern the case of rigid bodies moving into a viscous incompressible fluid modeled by the Navier-Stokes system. We can quote for instance [11, 13, 19, 24, 26, 35, 36, 52, 55, 56, 57], etc. Some works deal with different fluids [30, 39, 48] (incompressible perfect fluid), [5, 6, 14, 18] (viscous compressible fluid), [21] (viscous multipolar fluid), [20, 27] (incompressible non-Newtonian fluid). Let us also mention some results for the Navier-Stokes system but with other types of boundary conditions: [1, 9, 29]).

Up to our knowledge, the case of a Bingham fluid has not been treated yet. The first studies on Bingham fluid were done by Oldroyd [47] and Prager [49]. The works of Mosolov and Miasnikov in [45, 46] present a variational method and give some well-posedness results. We can also quote [50] where the authors consider the case of a stationary Bingham fluid around a rigid body. They consider a weak formulation and analyse the case where the motion of the rigid body is given. In [22], the authors provide a relation between the yield number and an eigenvalue problem.

A strong motivation to study multiphase problems involving rigid structures and non-Newtonian fluids is in the pursuit of a better understanding of the granular matter. According to [33], a granular flow is a collection of solid particles immersed in a fluid that can be water or air. The modeling and understanding of granular materials represent a significant purpose of human activities since a broad range of materials can be considered as a granular media. According to [28], measured in tons, the first material manipulated on earth is water; the second is granular matter. Several examples of granular materials can be found in the industry such as mine tailings, pharmaceutical tablets and capsules; and in nature such as landslides, debris avalanches, pyroclastic flows, rice, and sand.

A comprehensive view of the mechanical and thermodynamical properties of materials is needed it to write constitutive equations. In particular, granular materials reveals various mechanical behaviors, similar to elastoplastic solids in the case of a quasi-static regime to dense gazes in the cases of strong agitation [44]. Then, the properties of a granular material are somewhere between those of a liquid and those of a real solid. Even at rest, granular material can sustain some shearing stress but only an amount proportional to the average stress. This yielding property is dominant in dense regimes and, several authors have proposed constitutive equations resembling a viscoplastic material. The most remarkable ones are the Drucker-Prager [15], that is an extension of the Mohr-Coulomb yield criterion, and more recently the $\mu(I)$-rheology [41]. Both models are an extension of the Bingham constitutive equation where the yield stress is no more constant but pressure dependent. However, these models face the lack of good mathematical properties and accurate numerical methods. For example, [54] proved that the Mohr-Coulomb constitutive equation is ill-posed in all two-dimensional contexts and all realistic three-dimensional contexts. However, the research of granular materials using the $\mu(I)$-rheology is promising. For example, [2] proved that the $\mu(I)$-rheology is well posed under certain conditions on a parameter called inertial number. In the numerical front, [40] obtained accurate results using an augmented Lagrangian method to simulate the collapse of a granular wall.

On the other hand, a multiphase approach where the phases have a well-defined constitutive behavior can also be applied to the modeling of granular matter. A multiphase model uses more simple constitutive equations but adds the problem of how the different parts of a material interact. In this line, the Bingham fluid model is the simplest constitutive equation that possesses the yielding property. Then, a Bingham fluid-rigid body system can be useful to understand and shed some light about granular materials.

Let us describe the outline of the paper. In Section 2, we introduce some additional notation and we prove Proposition 1.1. We also give some technical results proved in [29] but that we state differently and that we prove for the sake of completeness. In Section 3, we introduce some approximations of the variational inequality (1.21). More precisely, we use a Galerkin method (of dimension M) where the plastic term is regularized (with a parameter ε) and where the free-boundary is replaced by a penalization term (with a parameter k). Section 4 is devoted to passing to the limit in M and ε. Finally, in Section 5, we prove the main result by passing to the limit in k. The last section corresponds to the proof of Corollary 1.10 (that is $g \to \infty$).

2 Notation and preliminary results

Assume $(a, Q) \in \mathbb{R}^3 \times SO(3)$ and set

$$S = \widehat{S}(a, Q).$$

We denote by P_S the orthogonal projection of $L^2(S)$ onto $\mathcal{R}$. By standard calculation, if

$$\ell + \omega \times (x-a) = P_S u,$$

then ℓ and ω are given by:

$$\ell = \frac{1}{m} \int_{\widehat{S}(a,Q)} \rho_s u \, dx \tag{2.1}$$

and

$$\omega = \widehat{J}(a,Q)^{-1} \int_{\widehat{S}(a,Q)} \rho_s (x-a) \times u \, dx. \tag{2.2}$$

We define the global density by

$$\widehat{\rho}_{a,Q} := \rho_f \mathbb{1}_{\widehat{F}(a,Q)} + \rho_s \mathbb{1}_{\widehat{S}(a,Q)}. \tag{2.3}$$

In what follows, we also need the following notation: for any set $\Omega_1 \subset \mathbb{R}^3$,

$$(\Omega_1)^\delta := \{x \in \mathbb{R}^3 \ ; \ \operatorname{dist}(x, \Omega_1) < \delta\} \tag{2.4}$$

and

$$(\Omega_1)_\delta := \{x \in \Omega_1 \ ; \ \operatorname{dist}(x, \partial\Omega_1) > \delta\}.$$

Given $(a,Q) \in \mathbb{R}^3 \times SO(3)$ we define two operators of $L^2_{loc}(\mathbb{R}^3)$ as follows: assume $v \in L^2_{loc}(\mathbb{R}^3)$, then

$$\Phi_{a,Q}(v)(y) := Q^* v\left(a + Qy\right), \quad y \in \mathbb{R}^3 \tag{2.5}$$

and

$$\overline{\Phi}_{a,Q}(v)(x) := Qv\left(Q^*(x-a)\right), \quad x \in \mathbb{R}^3. \tag{2.6}$$

Let us notice the relation

$$\overline{\Phi}_{a,Q} \circ P_{S_0} \circ \Phi_{a,Q} = P_{\widehat{S}(a,Q)}. \tag{2.7}$$

We will need the following result.

Lemma 2.1. *Assume $(h_n, R_n) \to (h, R)$ in $\mathbb{R}^3 \times SO(3)$. Then,*

$$\mathbb{1}_{S(h_n,R_n)} \to \mathbb{1}_{S(h,R)} \text{ in } L^p(\Omega) \quad \forall p \in [1,\infty). \tag{2.8}$$

Similarly, if $(h_n, R_n) \to (h, R)$ strongly in $\mathcal{C}([0,T]; \mathbb{R}^3 \times SO(3))$, then

$$\mathbb{1}_{S(h_n,R_n)} \to \mathbb{1}_{S(h,R)} \text{ strongly in } \mathcal{C}([0,T]; L^p(\Omega)) \quad \forall p \in [1,\infty). \tag{2.9}$$

The proof of this lemma is standard and is based on the approximation of $\mathbb{1}_{S_0}$ by a smooth function with compact support.

2.1 Weak form and energy inequality

In this section, we first prove Proposition 1.1:

Proof of Proposition 1.1. Let $v \in \mathcal{C}^1\left([0,T]; H^1_{S(t)}(\Omega)\right)$. Using the results in Section 1, there exist two $\mathcal{C}^1$ functions, ℓ_v and ω_v, such that $v(t,x) = \ell_v(t) + \omega_v(t) \times (x - h(t))$ for $x \in S(t)$.

We multiply equation (1.2) by $(v-u)$ and we integrate in $F(t)$ and in $[0,T]$

$$\int_0^T \int_{F(t)} \rho_f \left(\frac{\partial u}{\partial t} + (u \cdot \nabla) u\right) \cdot (v-u) \, dxdt = \int_0^T \int_{F(t)} \operatorname{div}(\sigma(u,p)) \cdot (v-u) \, dxdt. \tag{2.10}$$

By the divergence theorem,

$$\int_{F(t)} \operatorname{div}(\sigma(u,p)) \cdot (v-u) \, dx = -\int_{F(t)} \sigma(u,p) : \nabla(v-u) \, dx + \int_{\partial F(t)} \sigma(u,p) n \cdot (v-u) \, ds. \tag{2.11}$$

Using that $\operatorname{div}(v-u) = 0$, the boundary conditions of $v-u$ (see (1.4) and (1.5)) and the fact that $\sigma^d(D(u))$ is a symmetric matrix, we deduce from (2.11)

$$\int_{F(t)} \operatorname{div} \sigma(u,p) \cdot (v-u) dx = -\int_{F(t)} \sigma^d(D(u)) : D(v-u) dx - m\ell' \cdot (\ell_v - \ell) - (J\omega)' \cdot (\omega_v - \omega). \tag{2.12}$$

Since $D \in \mathbb{M}^{3\times 3} \mapsto |D|_2^2$ is differentiable, $\sigma^d(D(u)) \in \partial f(D(u))$ implies that

$$g\,|D(v)|_2 - g\,|D(u)|_2 \geqslant \Big(\sigma^d(D(u)) - 2\mu D(u)\Big) : (D(v) - D(u)).$$

Combining the above relation with (2.12) yields

$$\int_{F(t)} \operatorname{div}\sigma(u,p)\cdot(v-u)\;dx + 2\mu\int_{F(t)} D(u) : D(v-u)\;dx \\ + g\int_{F(t)} |D(v)|_2 - |D(u)|_2\;dx + m\ell'\cdot(\ell_v - \ell) + (J\omega)'\cdot(\omega_v - \omega) \geqslant 0. \quad (2.13)$$

On the other hand, using the Reynolds transport theorem and standard calculation we deduce that:

$$\int_0^T \int_{F(t)} \rho_f\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right)\cdot(v-u)\;dxdt + m\ell'\cdot(\ell_v-\ell) + (J\omega)'\cdot(\omega_v-\omega) \\ = \int_0^T\int_\Omega \rho\left(\frac{\partial v}{\partial t} + (u\cdot\nabla)v\right)\cdot(v-u)\;dxdt \\ + \frac{1}{2}\int_\Omega \rho(0,\cdot)\,|u(0,\cdot)-v(0,\cdot)|_2^2 - \rho(T,\cdot)\,|u(T,\cdot)-v(T,\cdot)|_2^2\,dx. \quad (2.14)$$

Then, using that $D(u) = D(v) = 0$ in $S(t)$, and $|u(T,\cdot) - v(T,\cdot)|_2^2$ is non negative, gathering (2.10), (2.13) and (2.14) we arrive to (1.21).

To obtain the energy equality, we multiply equation (1.2) by u and we integrate in $F(s)$ and in $[0,t]$. Following the above calculations, this yields

$$\int_0^t\left(\int_{F(s)} \rho_f\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right)\cdot u\;dx + \int_{F(s)} 2\mu|D(u)|_2^2 + g\,|D(u)|_2\;dx + m\ell'\cdot\ell + (J\omega)'\cdot\omega\right)\;ds = 0.$$

Using again the Reynolds transport theorem and standard calculation we deduce (1.22).

We end the proof by showing that if (u,h,R) is a regular function satisfying (1.8), (1.9), (1.11) and (1.21), and if $|D(u)|_2 \neq 0$ in $F(t)$, then (u,h,R) satisfies (1.2)–(1.9). To do this we follow the arguments in [16, pp. 287–288]. We start with (1.21) and using that $D(u) \neq 0$ with the arguments in [16, pp. 287–288], we obtain

$$\int_0^T\int_\Omega \rho\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right)\cdot v\;dxdt + 2\mu\int_0^T\int_\Omega D(u) : D(v)\;dxdt \\ + g\int_0^T\int_{F(t)} \frac{D(u):D(v)}{|D(u)|_2}\;dxdt = -\frac{1}{2}\int_\Omega \rho(0,x)\,|u(0,x) - u_0(x)|_2^2\,dx, \quad (2.15)$$

for any $v \in \mathcal{C}^1([0,T]; H^1_{S(t)}(\Omega))$ such that $v(T,\cdot) = v(0,\cdot) = 0$. Taking $v = 0$ we recover the initial conditions and taking v such that $\ell_v = 0$ and $\omega_v = 0$ we obtain

$$\int_0^T\int_{F(t)}\left(\rho_f\frac{\partial u}{\partial t} + \rho_f(u\cdot\nabla)u - \operatorname{div}\sigma^d(D(u))\right)\cdot v\;dxdt = 0$$

Then, we recover the pressure p using Lemma III.1.1 in [25] and we obtain equation (1.2). Finally, combining (1.2) with (2.15), integrating by parts and using that $\operatorname{div} v = 0$, we obtain:

$$\int_0^T\left(m\ell' + \int_{\partial S}\sigma(u,p)n\;ds\right)\cdot\ell_v\;dt + \int_0^T\left((J\omega)' + \int_{\partial S}(x-h)\times\sigma(u,p)n\;ds\right)\cdot\omega_v\;dt = 0.$$

Since the above equation holds for all ℓ_v and ω_v in $\mathcal{C}^1([0,T];\mathbb{R}^3)$ with $\ell_v(T) = \omega_v(T) = 0$, we recover the equations (1.6) and (1.7). $\square$

2.2 Junction of solenoidal fields

Here we state some technical results obtained and proved in [29]. The statements used in this article are slightly different and we thus recall the main steps of the proofs.

Theorem 2.2. *Assume that $\delta_1 > \delta_2 > 0$. Then, there exists a family of bounded operators*

$$\Lambda^{\delta_1,\delta_2} : H^1_\sigma\left(\mathbb{R}^3\right) \times \mathcal{R} \to H^1_\sigma(\mathbb{R}^3)$$

such that, for all $(u_1, u_2) \in H^1_\sigma(\mathbb{R}^3) \times \mathcal{R}$ we have that:

$$\Lambda^{\delta_1,\delta_2}(u_1, u_2) = u_2 \qquad \text{in } S_0 \tag{2.16}$$

$$\Lambda^{\delta_1,\delta_2}(u_1, u_2) = u_1 \quad \text{in } \mathbb{R}^3 \setminus S_0^{\delta_1}, \tag{2.17}$$

where $S_0^{\delta_1}$ is defined by (2.4), and the following inequality holds:

$$\left\|\Lambda^{\delta_1,\delta_2}(u_1, u_2) - u_1\right\|_{L^p(S_0^{\delta_1}\setminus S_0)} \leqslant C_{\delta_1,S_0}\left(\delta_2^{\frac{1}{p}-\frac{1}{6}}\left\|u_1 - u_2\right\|_{H^1(S_0^{\delta_1}\setminus S_0)} + \left\|(u_1 - u_2)\cdot n\right\|_{L^p(\partial S_0)}\right), \tag{2.18}$$

for $p \in [2, 6]$.

Proof. We consider an orthogonal curvilinear coordinate system (s_1, s_2, z) defined around ∂S_0 such that $\partial S_0 = \{z = 0\}$. For δ_1 small enough we have $\partial S_0^{\delta_1} = \{z = \delta_1\}$. We consider $\varphi \in \mathcal{C}_0^\infty([0, 1); [0, 1])$ such that $\varphi(0) = 1$ and we define the function $\varphi_\delta(z) := \varphi(\frac{z}{\delta})$. Notice that:

$$\|\varphi_\delta\|_{L^a(\mathbb{R})} = \delta^{\frac{1}{a}}\,\|\varphi\|_{L^a(\mathbb{R})}. \tag{2.19}$$

We define $\Lambda^{\delta_1,\delta_2}(u_1, u_2)$ in $S_0^{\delta_1} \setminus S_0$ as follows:

$$\Lambda^{\delta_1,\delta_2}(u_1, u_2) := V_1 + V_2 + V_3$$

where

$$V_1 = (1 - \varphi_{\delta_2})u_1 + \varphi_{\delta_2}\left(u_2 - ((u_2 - u_1)\cdot e_z)e_z\right),$$

$$V_2 = \{(u_2 - u_1)\cdot e_z\}\Big|_{z=0}\varphi_{\delta_1} e_z,$$

and V_3 solution of the system

$$\begin{cases} \operatorname{div} V_3 = -\operatorname{div}(V_1 + V_2) & \text{in } S_0^{\delta_1} \setminus S_0 \\ V_3 = 0. & \text{on } \partial S_0^{\delta_1} \cup \partial S_0. \end{cases} \tag{2.20}$$

From [25, Theorem III.3.1, p.171], the above system admits a solution since the compatibility condition holds. We can also check that

$$\Lambda^{\delta_1,\delta_2}(u_1, u_2) = u_2 \quad \text{on } \partial S_0, \quad \Lambda^{\delta_1,\delta_2}(u_1, u_2) = u_1 \quad \text{on } \partial S_0^{\delta_1}.$$

Moreover we have the following properties:

$$|V_1 - u_1|_2 \leqslant \varphi_{\delta_2}\,|u_2 - u_1|_2\,, \quad |V_2|_2 \leqslant \left|(u_2 - u_1)\cdot n\big|_{z=0}\right|, \tag{2.21}$$

$$\operatorname{div} V_1 = -\varphi_{\delta_2}\operatorname{div}([(u_1 - u_2)\cdot e_z]e_z), \tag{2.22}$$

$$\operatorname{div} V_2 = ((u_1 - u_2)\cdot e_z)\big|_{z=0}\left(\varphi_{\delta_1}\operatorname{div} e_z + e_z\cdot\nabla\varphi_{\delta_1}\right). \tag{2.23}$$

Combining (2.21), (2.19) and a Sobolev embedding we deduce for any $p \in [1, 6]$

$$\|V_1 - u_1\|_{L^p(S_0^{\delta_1}\setminus S_0)} \leqslant C\delta_2^{\frac{1}{p}-\frac{1}{6}}\,\|u_2 - u_1\|_{H^1(S_0^{\delta_1}\setminus S_0)} \tag{2.24}$$

and

$$\|V_2\|_{L^p(S_0^{\delta_1}\setminus S_0)} \leqslant C_{\delta_1,S_0}\,\|(u_2 - u_1)\cdot n\|_{L^p(\partial S_0)}. \tag{2.25}$$

Assume $q \leqslant 2$. From (2.22) and (2.23) we deduce that:

$$\|\operatorname{div} V_1\|_{L^q(S_0^{\delta_1}\setminus S_0)} \leqslant C_{\delta_1,S_0}\delta_2^{\frac{1}{q}-\frac{1}{2}}\,\|u_1 - u_2\|_{H^1\left(S_0^{\delta_1}\setminus S_0\right)}, \tag{2.26}$$

$$\|\mathrm{div}\, V_2\|_{L^q(S_0^{\delta_1}\setminus S_0)} \leqslant C_{\delta_1,S_0} \|(u_2-u_1)\cdot n\|_{L^q(\partial S_0)}. \tag{2.27}$$

Using [25, Theorem III.3.1, p.171] and a Sobolev embedding, we conclude that if $\frac{1}{p}=\frac{1}{q}-\frac{1}{3}$,

$$\|V_3\|_{L^p(S_0^{\delta_1}\setminus S_0)} \leqslant C_{\delta_1,S_0} \|V_3\|_{W^{1,q}(S_0^{\delta_1}\setminus S_0)}$$
$$\leqslant C_{\delta_1,S_0}\left(\delta_2^{\frac{1}{p}-\frac{1}{6}} \|u_1-u_2\|_{H^1\left(S^{\delta_1}\setminus S_0\right)} + \|(u_1-u_2)\cdot n\|_{L^q(\partial S_0)}\right). \tag{2.28}$$

Gathering (2.26), (2.25) and (2.28) yields (2.18). □

Definition 2.3. *Assume $(a,Q)\in\mathbb{R}^3\times SO(3)$ and assume $\delta_1>\delta_2>0$. We define the operator $Q_{a,Q}^{\delta_1,\delta_2}\in\mathcal{L}(H^1_\sigma(\Omega),H^1_\sigma(\mathbb{R}^3))$ as follows:*

$$Q_{a,Q}^{\delta_1,\delta_2}(u) := \overline{\Phi}_{a,Q}\left(\Lambda^{\delta_1,\delta_2}\left(\Phi_{a,Q}(u), P_{S_0}\Phi_{a,Q}(u)\right)\right) \quad (u\in H^1_\sigma(\Omega)), \tag{2.29}$$

where Φ and $\overline{\Phi}$ are defined in (2.5) and (2.6). If $\mathrm{dist}(\widehat{S}(a,Q),\partial\Omega)>\delta_1$, then $Q_{a,Q}^{\delta_1,\delta_2}\in\mathcal{L}(H^1_\sigma(\Omega))$.

Using (2.7), we can check that

$$Q_{a,Q}^{\delta_1,\delta_2}(u) = \begin{cases} u & \text{in } \Omega\setminus\widehat{S}(a,Q)^{\delta_1}, \\ P_{\widehat{S}(a,Q)}u & \text{in } \widehat{S}(a,Q). \end{cases} \tag{2.30}$$

Moreover, if $(h,R)\in L^\infty(0,T;\mathbb{R}^3\times SO(3))$ and $\mathrm{dist}(\widehat{S}(h,R),\partial\Omega)>\delta_1$ a.e in $(0,T)$, then we deduce from (2.29) that $Q_{h,R}^{\delta_1,\delta_2}$ is a linear bounded operator in $L^2(0,T;H^1_\sigma(\Omega))$ into itself.

Lemma 2.4. *Assume $\delta_1>\delta_2>0$ and*

$$(h_M,R_M)\overset{*}{\rightharpoonup}(h,R) \quad \text{weak star in } W^{1,\infty}(0,T;\mathbb{R}^3\times SO(3)),$$
$$(h_M,R_M)\to(h,R) \quad \text{strongly in } \mathcal{C}([0,T];\mathbb{R}^3\times SO(3)).$$

We define $S_M:=\widehat{S}(h_M,R_M)$ and $S:=\widehat{S}(h,R)$. We also assume

$$u_M\rightharpoonup u \quad \text{weakly in } L^2(0,T;H^1_\sigma(\Omega)),$$
$$u_M\to u \quad \text{strongly in } L^2(0,T;L^2(\Omega)).$$

Then we have that

$$Q_{S_M}^{\delta_1,\delta_2}(u_M)\rightharpoonup Q_S^{\delta_1,\delta_2}(u) \text{ weakly in } L^2(0,T;H^1(\Omega)) \tag{2.31}$$

and

$$Q_{S_M}^{\delta_1,\delta_2}(u_M)\to Q_S^{\delta_1,\delta_2}(u) \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{2.32}$$

Proof. The proof of (2.31) and (2.32) are similar, so we only proof (2.31). We set

$$U_M:=\Phi_{h_M,R_M}(u_M) \quad U:=\Phi_{h,R}(u).$$

Using Lemma A.2 of [29] we deduce that

$$U_M\rightharpoonup U \quad \text{weakly in } L^2(0,T;H^1_\sigma(\mathbb{R}^3)),$$

and thus

$$\Lambda^{\delta_1,\delta_2}(U_M,P_{S_0}U_M)\rightharpoonup\Lambda^{\delta_1,\delta_2}(U,P_{S_0}U) \text{ weakly in } L^2(0,T;H^1_\sigma(\mathbb{R}^3)).$$

Then, using again Lemma A.2 of [29] we conclude (2.31). □

The second type of junction we consider here is given by the following result. It corresponds to Lemma 5.3 of [29].

Theorem 2.5. *Assume $\delta_1 > 2\delta_2 > 0$ and $s < \frac{1}{3}$. Then, there exists a family of bounded operators*

$$\widehat{\Lambda}^{\delta_1,\delta_2} : H^1_\sigma\left(\mathbb{R}^3\right) \times \mathcal{R} \to H^s_\sigma(\mathbb{R}^3)$$

such that, for all $(u_1, u_2) \in H^1_\sigma(\mathbb{R}^3) \times \mathcal{R}$ we have that:

$$\widehat{\Lambda}^{\delta_1,\delta_2}(u_1, u_2) = u_2 \qquad \text{in } S_0^{\delta_2} \tag{2.33}$$

$$\Lambda^{\delta_1,\delta_2}(u_1, u_2) = u_1 \qquad \text{in } \mathbb{R}^3 \setminus S_0^{\delta_1}, \tag{2.34}$$

where $S_0^{\delta_1}$ and $S_0^{\delta_2}$ are defined in (2.4), and the following inequality holds:

$$\left\|\widehat{\Lambda}^{\delta_1,\delta_2}(u_1, u_2) - u_1\right\|_{H^s(\Omega\setminus S_0^{\delta_2})} \leqslant C_{\delta_1,S_0}\left(\delta_2^{\frac{1}{3}-s}(\|u_1\|_{H^1(\Omega)} + \|u_2\|_{H^1(\Omega)}) + \|(u_1-u_2)\cdot n\|_{L^2(\partial S_0)}\right). \tag{2.35}$$

Proof. The proof of this theorem is similar to the proof of Theorem 2.2. We use the same notation for the orthogonal curvilinear coordinate system (s_1, s_2, z) and for the functions φ_δ. We define $\widehat{\Lambda}^{\delta_1,\delta_2}(u_1, u_2)$ in $S_0^{\delta_1} \setminus S_0^{\delta_2}$ as follows:

$$\widehat{\Lambda}^{\delta_1,\delta_2}(u_1, u_2) = V_1 + V_2 + V_3$$

where

$$V_1 = (1 - \varphi_{\delta_2}(z - \delta_2))u_1 + \varphi_{\delta_2}(z - \delta_2)\left(u_2 - ((u_2 - u_1)\cdot e_z)e_z\right),$$

V_2 is solution of the equation

$$\operatorname{div} V_2 = -\operatorname{div} V_1 \text{ in } S_0^{\delta_1} \setminus S_0^{\delta_2}, \tag{2.36}$$

$$V_2 = 0 \text{ in } \partial S_0^{\delta_1} \cup \partial S_0^{\delta_2}, \tag{2.37}$$

and $V_3 = \nabla Y_3$ where

$$\Delta Y_3 = 0 \text{ in } S_0^{\delta_1} \setminus S_0^{\delta_2}, \tag{2.38}$$

$$\frac{\partial Y_3}{\partial n} = 0 \text{ in } \partial S_0^{\delta_1}, \tag{2.39}$$

$$\frac{\partial Y_3}{\partial n} = (u_2 - u_1)\cdot e_z \text{ in } \partial S_0^{\delta_2}. \tag{2.40}$$

One can check that the compatibility conditions are satisfied so that (2.36)-(2.37) and (2.38)-(2.40) are well-posed with the estimates

$$\|V_2\|_{H^1(S_0^{\delta_1}\setminus S_0^{\delta_2})} \leqslant C_{\delta_1,S_0}(\|u_1\|_{H^1(\Omega)} + \|u_2\|_{H^1(\Omega)}), \tag{2.41}$$

$$\|V_2\|_{L^2(S_0^{\delta_1}\setminus S_0^{\delta_2})} \leqslant C_{\delta_1,S_0}\|V_2\|_{W^{1,6/5}(S_0^{\delta_1}\setminus S_0^{\delta_2})} \leqslant C_{\delta_1,S_0}\delta_2^{1/3}(\|u_1\|_{H^1(\Omega)} + \|u_2\|_{H^1(\Omega)}), \tag{2.42}$$

and

$$\|V_3\|_{H^{1/2}(S_0^{\delta_1}\setminus S_0^{\delta_2})} \leqslant C_{\delta_1,S_0}\left\|(u_2 - u_1)\cdot n\right\|_{L^2(\partial S_0^{\delta_2})}.$$

Using Lemma 5.10 of [23], the above estimate yields

$$\|V_3\|_{H^{1/2}(S_0^{\delta_1}\setminus S_0^{\delta_2})} \leqslant C_{\delta_1,S_0}\left(\delta_2^{1/2}\left\|u_2 - u_1\right\|_{H^1(\Omega)} + \left\|(u_2 - u_1)\cdot n\right\|_{L^2(\partial S_0)}\right). \tag{2.43}$$

We also remark that

$$(V_1 + V_2 + V_3)\cdot n = u_2\cdot n \quad \text{on} \quad \partial S_0^{\delta_2},$$

$$(V_1 + V_2 + V_3)\cdot n = u_1\cdot n \quad \text{on} \quad \partial S_0^{\delta_1}.$$

Using the definition of V_1 and (2.19) we deduce that:

$$\|V_1 - u_1\|_{L^2(S_0^{\delta_1}\setminus S_0^{\delta_2})} \leqslant C_{\delta_1,S_0}\delta_2^{\frac{1}{3}}\left\|u_1 - u_2\right\|_{H^1(\Omega)},$$

$$\|\nabla(V_1 - u_1)\|_{L^2(S_0^{\delta_1}\setminus S_0^{\delta_2}))} \leqslant C_{\delta_1,S_0}\delta_2^{-\frac{2}{3}}\left\|u_1 - u_2\right\|_{H^1(\Omega)},$$

so that

$$\|V_1 - u_1\|_{H^s(S_0^{\delta_1}\setminus S_0^{\delta_2})} \leqslant C_{\delta_1,S_0}\delta_2^{\frac{1}{3}-s}\|u_1 - u_2\|_{H^1(\Omega)}. \tag{2.44}$$

Then combining (2.41), (2.42), (2.43) and the above estimate, we deduce (2.35). $\square$

We define the space

$$H_S^s(\Omega) := \{v \in H_\sigma^s(\Omega)\ ;\ D(v) = 0 \text{ in } S\} \tag{2.45}$$

and we denote by

$$\mathcal{P}_S^s : H_\sigma^s(\Omega) \mapsto H_S^s(\Omega) \tag{2.46}$$

the orthogonal projection.

As consequence of the above theorem, we obtain the following result on the orthogonal projection defined above

Corollary 2.6. *Let $u \in H^1(\Omega)$ and $(h,R) \in \mathbb{R}^3 \times SO(3)$ such that $\operatorname{dist}(\widehat{S}(h,R),\partial\Omega) \geqslant \delta_1 > 0$. Then for all $d < \frac{\delta_1}{2}$ and $s \in (0,1/3)$, we have that*

$$\left\|u - \mathcal{P}^s_{(\widehat{S}(h,R))^d}u\right\|_{H^s(\Omega)} \leqslant C_{\delta_1,S_0}\left(d^{\frac{1}{3}-s}\|u\|_{H^1(\Omega)} + \|u\|_{H^1(\Omega)}^{1/2}\left\|u - P_{\widehat{S}(h,R)}u\right\|_{L^2(\widehat{S}(h,R))}^{1/2}\right). \tag{2.47}$$

Proof. We set

$$v := \overline{\Phi}_{h,R}\left(\widehat{\Lambda}^{\delta_1,d}\left(\Phi_{h,R}(u), P_{S_0}\Phi_{h,R}(u)\right)\right)$$

where Φ and $\overline{\Phi}$ are defined in (2.5) and (2.6). Then, by Theorem 2.5, we have $v = P_{\widehat{S}(h,R)}u$ in $\widehat{S}(h,R)^d$ $v = u$ in $\Omega \setminus \widehat{S}(h,R)^{\delta_1}$, and

$$\|v - u\|_{H^s(\Omega\setminus\widehat{S}(h,R)^d)} \leqslant C\left(d^{\frac{1}{3}-s}\|u\|_{H^1(\Omega)} + \left\|(u - P_{\widehat{S}(h,R)}u)\cdot n\right\|_{L^2(\partial\widehat{S}(h,R))}\right). \tag{2.48}$$

We deduce that

$$\begin{aligned}\|v - u\|_{H^s(\Omega)} \leqslant C\Bigg(& d^{\frac{1}{3}-s}\|u\|_{H^1(\Omega)} + \left\|(u - P_{\widehat{S}(h,R)}u)\cdot n\right\|_{L^2(\partial\widehat{S}(h,R))} \\ &+ \left\|u - P_{\widehat{S}(h,R)}u\right\|_{H^s(\widehat{S}(h,R)^d\setminus\widehat{S}(h,R))} + \left\|u - P_{\widehat{S}(h,R)}u\right\|_{H^s(\widehat{S}(h,R))}\Bigg).\end{aligned} \tag{2.49}$$

Now we have the following relations

$$\left\|(u - P_{\widehat{S}(h,R)}u)\cdot n\right\|_{L^2(\partial\widehat{S}(h,R))} \leqslant C\|u\|_{H^1(\Omega)}^{1/2}\left\|u - P_{\widehat{S}(h,R)}u\right\|_{L^2(\widehat{S}(h,R))}^{1/2},$$

$$\left\|u - P_{\widehat{S}(h,R)}u\right\|_{H^s(\widehat{S}(h,R)^d\setminus\widehat{S}(h,R))} \leqslant Cd^{1/3(1-s)}\|u\|_{H^1(\Omega)}$$

and

$$\left\|u - P_{\widehat{S}(h,R)}u\right\|_{H^s(\widehat{S}(h,R))} \leqslant C\|u\|_{H^1(\Omega)}^s\left\|u - P_{\widehat{S}(h,R)}u\right\|_{L^2(\widehat{S}(h,R))}^{1-s}.$$

Combining these relations with (2.49), we deduce the result. $\square$

3 Approximated Problems

To prove the existence of weak solutions of the system (1.2)-(1.12), we consider some approximations of (1.2)-(1.12). More precisely, we introduce 3 parameters:

- ε corresponds to the approximation of the plastic term,
- M corresponds to the dimension in the Galerkin method,
- k corresponds to the penalization term used to deal with the free boundary problem.

More precisely, we replace $j : \mathbb{M}^{3\times 3} \to \mathbb{R},\ D \mapsto |D|_2$ by the $\mathcal{C}^1$ convex functions

$$j_\varepsilon : \mathbb{M}^{3\times 3} \to \mathbb{R}, \quad D \mapsto \frac{1}{1+\varepsilon} |D|_2^{1+\varepsilon}. \tag{3.1}$$

The gradient of j_ε is given by

$$\nabla j_\varepsilon(D) = |D|_2^{\varepsilon-1} D \tag{3.2}$$

and satisfies

$$|\nabla j_\varepsilon(D)|_2 = |D|_2^{\varepsilon} \leqslant 1 + |D|_2 \tag{3.3}$$

if $\varepsilon \leqslant 1$.

Since $H^1_\sigma(\Omega)$ is a separable Hilbert space and $\mathcal{C}^\infty_0(\Omega) \cap H^1_\sigma(\Omega)$ is dense in $H^1_\sigma(\Omega)$, there exists an orthonormal basis $\{v_q\}_{q\in\mathbb{N}^*}$ of $H^1_\sigma(\Omega)$ such that $v_q \in \mathcal{C}^\infty_0(\Omega)$ for all $q \geqslant 1$. We define

$$V_M = \operatorname{span}\{v_1, \dots, v_M\}$$

and we look for an approximated velocity in V_M.

This subspace does not impose that the velocity is rigid in the solid domain. That is why we add in the weak formulation a penalization term of the form

$$k \int_S (u - P_S(u)) \cdot (v - P_S(v)) dx,$$

with $k \to \infty$.

Notation 3.1. *To simplify the notation, in this section we write*

$$n = (\varepsilon, k, M),$$

for instance u_n means $u_{\varepsilon,k,M}$.

Then, the approximated problem is defined as follows: to find

$$h_n \in \mathcal{C}^1([0,T];\mathbb{R}^3), \quad R_n \in \mathcal{C}^1([0,T]; SO(3)) \quad \alpha_n \in \mathcal{C}^1([0,T];\mathbb{R}^M) \tag{3.4}$$

satisfying the following properties:

$$S_n(t) := \widehat{S}(h_n(t), R_n(t)), \quad F_n(t) := \widehat{F}(h_n(t), R_n(t)), \tag{3.5}$$

where $\widehat{S}$ and $\widehat{F}$ are defined in (1.1);

$$u_n := \sum_{j=1}^M \alpha_{n,j} v_j, \tag{3.6}$$

$$\ell_n + \omega_n \times (x - h_n) := P_{S_n}(u_n), \tag{3.7}$$

where P_{S_n} is the projection defined in Section 2,

$$h_n'(t) = \ell_n(t), \quad h_n(0) = 0, \tag{3.8}$$

$$R_n'(t) = \mathbb{A}(\omega_n) R_n(t), \quad R_n(0) = I_3, , \tag{3.9}$$

$$\begin{aligned}
&\int_\Omega \rho_n \frac{\partial u_n}{\partial t} \cdot v_j dx + \int_\Omega \rho_n (Q_{S_n}(u_n) \cdot \nabla) u_n \cdot v_j dx + 2\mu \int_\Omega D(u_n) : D(v_j) dx \\
&+ g \int_\Omega \nabla j_\varepsilon(D(u_n)) : D(v_j) dx + k \int_{S_n} (u_n - P_{S_n}(u_n)) \cdot (v_j - P_{S_n}(v_j)) dx = 0 \quad (j \in \{1, \cdots, M\}),
\end{aligned} \tag{3.10}$$

and

$$u_n(0,\cdot) = \mathbb{P}_{V_M}(u_0) := \sum_{j=1}^M \alpha_{n,0,j} v_j. \tag{3.11}$$

The operator $Q_{S_n} := Q^{\delta, \frac{\delta}{k}}_{h^n, R^n}$ is given in Definition 2.3, where δ is a (small) positive constant so that

$$\operatorname{dist}(S_0, \partial\Omega) > 3\delta. \tag{3.12}$$

We also consider the following condition in our definition of approximated solutions

$$S_n(t) \Subset \Omega, \quad \operatorname{dist}(S_n(t), \partial\Omega) \geqslant 2\delta. \tag{3.13}$$

We recall that, with the above condition, $Q_{S_n}(u_n)$ satisfies the following relation (see (2.30)):

$$Q_{S_n}(u_n) = \begin{cases} u_n & \text{in } \Omega \setminus (S_n)^\delta \\ P_{S_n} u_n & \text{in } S_n \end{cases} \quad \text{and} \quad \operatorname{div} Q_{S_n}(u_n) = 0. \tag{3.14}$$

The operator $\mathbb{P}_{V_M}$ is the L^2 orthogonal projection of $L^2(\Omega)$ onto V_M. The global density is defined by $\rho_n = \widehat{\rho}_{h_n,R_n}$ where $\widehat{\rho}$ is defined by (2.3).

Using the above properties of $Q_{S_n}(u_n)$, we can show that (3.10) implies

$$\begin{aligned} -\int_0^T \int_\Omega \rho_n \left(\frac{\partial v}{\partial t} + (Q_{S_n}(u_n) \cdot \nabla) v \right) \cdot u_n dx dt + \int_0^T \int_\Omega (2\mu D(u_n) + g \nabla j_\varepsilon(D(u_n))) : D(v) dx dt \\ + k \int_0^T \int_{S_n} (u_n - P_{S_n}(u_n)) \cdot (v - P_{S_n}(v)) dx dt \\ = \int_\Omega \left[\rho_n(0,\cdot) u_n(0,\cdot) \cdot v(0,\cdot) - \rho_n(T,\cdot) u_n(T,\cdot) \cdot v(T,\cdot) \right] \, dx, \end{aligned} \tag{3.15}$$

for any $v \in \mathcal{C}^1([0,T]; V_M)$.

In the following proposition, we prove the existence of a solution of the approximated problems.

Proposition 3.2. *There exists a time T, depending on $\|u_0\|_{L^2(\Omega)}$ and on $\operatorname{dist}(S_0, \partial\Omega) - \delta$ such that for any $M \in \mathbb{N}^*$, $k, \varepsilon > 0$, we have the following property: there exists a solution (h_n, R_n, α_n) of the system (3.4)–(3.11), (3.13) on a time interval $[0,T]$. Moreover, we have the energy equality for all $t \in [0,T]$:*

$$\begin{aligned} \frac{1}{2} \int_\Omega \rho_n(t,\cdot) \left| u_n(t,\cdot) \right|_2^2 \, dx + 2\mu \int_0^t \int_\Omega \left| D(u_n) \right|_2^2 \, dx dt + g \int_0^t \int_\Omega \nabla j_n(D(u_n)) : D(u_n) \, dx dt \\ + k \int_0^t \int_{S_n} \left| u_n - P_{S_n}(u_n) \right|_2^2 \, dx dt = \frac{1}{2} \int_\Omega \rho_0 \left| \mathbb{P}_{V_M} u_0 \right|_2^2 \, dx. \end{aligned} \tag{3.16}$$

Proof. We write (3.5)–(3.11) as a Cauchy problem

$$\frac{d}{dt} \begin{pmatrix} h_n \\ R_n \\ \alpha_n \end{pmatrix} = F\left(\begin{pmatrix} h_n \\ R_n \\ \alpha_n \end{pmatrix} \right), \quad \begin{pmatrix} h_n \\ R_n \\ \alpha_n \end{pmatrix}(0) = \begin{pmatrix} 0 \\ I_3 \\ \alpha_{n,0} \end{pmatrix} \tag{3.17}$$

where $F = (F_1, F_2, F_3)$ depends on n and can be expressed by using (3.5)–(3.11), (2.1)–(2.2) and (1.13):

$$F_1(a, Q, \beta) = \frac{\rho_s}{m} \sum_{i=1}^M \beta_i \int_\Omega \mathbb{1}_{\widehat{S}(a,Q)} v_i \, dx,$$

$$F_2(a, Q, \beta) = \rho_s \sum_{i=1}^M \beta_i \mathbb{A} \left(Q J_0^{-1} Q^* \int_\Omega \mathbb{1}_{\widehat{S}(a,Q)} (x - a) \times v_i(x) \, dx \right) Q,$$

and

$$F_3(a, Q, \beta) = C(a, Q)^{-1} G(a, Q, \beta),$$

where

$$C(a, Q)_{i,j} = \int_\Omega \widehat{\rho}_{a,Q} v_i \cdot v_j \, dx \quad (i, j \in \{1, \ldots, M\})$$

and

$$\begin{aligned} G(a, Q, \beta)_j = -2\mu \sum_{i=1}^M \beta_i \int_\Omega D(v_i) : D(v_j) dx \\ - \sum_{i=1}^M \beta_i \int_\Omega \widehat{\rho}_{a,Q} \left(Q_{a,Q}^{\delta,\delta/k} \left(\sum_{l=1}^M \beta_l v_l \right) \cdot \nabla \right) v_i \cdot v_j dx - g \int_\Omega \nabla j_\varepsilon \left(\sum_{i=1}^M \beta_i D(v_i) \right) : D(v_j) dx \\ - k \sum_{i=1}^M \beta_i \int_\Omega \mathbb{1}_{\widehat{S}(a,Q)} (v_i - P_{\widehat{S}(a,Q)} v_i) \cdot (v_j - P_{\widehat{S}(a,Q)} v_j) dx \end{aligned} \tag{3.18}$$

for $j \in \{1, \dots, M\}$.

By Lemma 2.1 and (1.1) we have that

$$\mathbb{R}^3 \times SO(3) \to L^1(\mathbb{R}^3), \quad (a, Q) \mapsto \mathbb{1}_{\widehat{S}(a,Q)}$$

is continuous and thus F_1, F_2 and C are continuous functions. For the continuity of G, we gather the following arguments:

- Since j_ε is $\mathcal{C}^1$, then
$$(a, Q, \beta) \mapsto \int_\Omega \nabla j_\varepsilon \left(\sum_{i=1}^M \beta_i D(v_i) \right) : D(v_j)\ dx$$
is continuous.
- Using (2.1), (2.2) and (1.13), we have that
$$(a, Q) \mapsto P_{\widehat{S}(a,Q)} v_i \in \mathcal{R}$$
is continuous.
- Using the definition (2.29) and the continuity of
$$(a, Q) \in \mathbb{R}^3 \times SO(3) \mapsto \Phi_{a,Q} \in \mathcal{L}(H^1(\mathbb{R}^3)), \quad (a, Q) \in \mathbb{R}^3 \times SO(3) \mapsto \overline{\Phi}_{a,Q} \in \mathcal{L}(H^1(\mathbb{R}^3)),$$
we deduce
$$(a, Q, \beta) \mapsto \int_\Omega \widehat{\rho}_{a,Q} \left(Q_{a,Q}^{\delta,\delta/k} \left(\sum_{l=1}^M \beta_l v_l \right) \cdot \nabla \right) v_i \cdot v_j\ dx$$
is continuous.

Consequently, in (3.17), we have that F is continuous. As a consequence, we can apply the Peano theorem and deduce the existence of a solution (h_n, R_n, α_n) of (3.5)–(3.11) on some time interval. By continuity of (h_n, R_n) and from (3.12), there exists a time $T_n > 0$ such that (h_n, R_n, α_n) satisfies also (3.13) on $[0, T_n]$. It remains to prove that we can choose a time interval independent of n with the same properties.

As long as the solution (h_n, R_n, α_n) of (3.5)–(3.11) exists and satisfies (3.13), we can show, by a standard calculation, the relation (3.16). Using the definition of $P_{S_n}(u_n)$, such a relation yields the existence of a constant C independent of n such that

$$|\ell_n| + |\omega_n| \leqslant C \|u_0\|_{L^2(\Omega)}.$$

In particular, from (3.8) and (3.9), there exists $T > 0$ depending only on $\|u_0\|_{L^2(\Omega)}$ and on $\operatorname{dist}(S_0, \partial\Omega) - \delta$ such that for all $t \in [0, T]$, (3.13) holds true. Using a proof by contradiction, we deduce that as long as the solution, if $t \leqslant T$, then (3.13) holds true.

Moreover, we note that relation (3.16) yields also a bound of the form

$$|(h_n, R_n, \alpha_n)| \leqslant \kappa. \tag{3.19}$$

Let us consider a bound $\zeta > 0$ of $|F|$ on the closed ball $\overline{B(0, 2\kappa)}$. Since

$$\overline{B((h_n, R_n, \alpha_n)(0), \kappa)} \subset \overline{B(0, 2\kappa)},$$

the Peano theorem gives the existence of a solution of (3.5)–(3.11) for a time $\tau = \kappa/\zeta > 0$. If $\tau > T$, then we obtain the result. Else, using (3.13), we deduce (3.16) and thus (3.19) in $[0, \tau]$. In particular,

$$\overline{B((h_n, R_n, \alpha_n)(\tau), \kappa)} \subset \overline{B(0, 2\kappa)},$$

we can use again Peano theorem with initial condition $(h_n, R_n, \alpha_n)(\tau)$ and on the time interval $[\tau, 2\tau]$. This solution satisfies (3.19) in $[\tau, 2\tau]$ and we can use it to extend our solution on $[0, \tau]$ on the time interval $[0, 2\tau]$. Then, by (3.17), we conclude that $(h_n, R_n, \alpha_n) \in \mathcal{C}^1([0, 2\tau])$.

By induction, we deduce the existence of a solution of the system (3.4)–(3.11) and (3.13) on the interval $[0, T]$. □

4 Passing to the limit $M \to \infty$ and $\varepsilon \to 0$

This section aims to pass to the limit for the parameters M and ε:

$$M \to \infty, \quad \varepsilon \to 0.$$

We take

$$\varepsilon = \frac{1}{M}$$

so that $n = (1/M, k, M)$. Again to simplify the notation, we write in this section the index (k, M) instead of $(1/M, k, M)$. For instance $u_{k,M}$ means $u_{1/M,k,M}$.

4.1 Weak convergences

Using (3.16) and that

$$\frac{1}{2} \int_\Omega \rho_0 \left| \mathbb{P}_{V_M} u_0 \right|_2^2 \ dx \leqslant C \|u_0\|^2_{L^2(\Omega)}, \tag{4.1}$$

we deduce that

$$\{u_{k,M}\}_{k,M} \quad \text{is bounded in} \quad L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_\sigma(\Omega)). \tag{4.2}$$

Therefore, there exists a subsequence of $\{u_{k,M}\}_{k,M}$ (still denoted $\{u_{k,M}\}_{k,M}$), and a function

$$u_k \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_\sigma(\Omega))$$

such that:

$$u_{k,M} \overset{*}{\rightharpoonup} u_k \text{ weak star in } L^\infty(0,T;L^2_\sigma(\Omega)) \tag{4.3}$$

and

$$u_{k,M} \rightharpoonup u_k \text{ weakly in } L^2(0,T;H^1_\sigma(\Omega)). \tag{4.4}$$

We also deduce from (4.2):

$$(h_{k,M}, R_{k,M}) \overset{*}{\rightharpoonup} (h_k, R_k) \quad \text{weak star in } W^{1,\infty}(0,T;\mathbb{R}^3 \times SO(3)), \tag{4.5}$$

and

$$(h_{k,M}, R_{k,M}) \to (h_k, R_k) \quad \text{strongly in } \mathcal{C}([0,T];\mathbb{R}^3 \times SO(3)). \tag{4.6}$$

We write

$$S_k := \widehat{S}(h_k, R_k)$$

and

$$J_{S_k} := \widehat{J}(h_k, R_k).$$

From (3.13) and (4.6), we deduce

$$S_k(t) \Subset \Omega, \quad \operatorname{dist}(S_k(t), \partial\Omega) \geqslant 2\delta \quad (t \in [0,T]). \tag{4.7}$$

By Lemma 2.1 we have that

$$\mathbb{1}_{S_{k,M}} \to \mathbb{1}_{S_k} \text{ strongly in } \mathcal{C}([0,T];L^p(\Omega)) \quad \forall p \in [1,\infty) \tag{4.8}$$

and thus

$$\rho_{k,M} \to \rho_k := \widehat{\rho}_{h_k,R_k} \text{ strongly in } \mathcal{C}([0,T];L^p(\Omega)) \quad \forall p \in [1,\infty). \tag{4.9}$$

Using (2.1) and (2.2), we deduce

$$P_{S_{k,M}} u_{k,M} \overset{*}{\rightharpoonup} P_{S_k} u_k \text{ weakly star in } L^\infty(0,T,\mathcal{R}). \tag{4.10}$$

4.2 Strong convergence of the velocity

As usual in the Navier Stokes equations, we require the strong convergence of the velocity to pass to the limit the convective term. In the case of a Bingham fluid we also have to deal with the plastic term $\nabla j_{\frac{1}{k}}(D(u_{k,M}))$ which does not converge directly to $\nabla j_{\frac{1}{k}}(D(u_k))$ since the convergence of $\{D(u_{k,M})\}_{k,M}$ is only weak. We start by proving the strong convergence of $\{u_{k,M}\}_{k,M}$.

By (4.3) and (4.9) we have that

$$\rho_{k,M}u_{k,M} \to \rho_k u_k \text{ weak star in } L^\infty(0,T;L^2(\Omega)). \tag{4.11}$$

Let us fix $i \geqslant 1$ and take $M \geqslant i$. We recall that $\mathbb{P}_{V_i} : L^2(\Omega) \to V_i$ the orthogonal projection onto V_i. We can write (3.15) as follows:

$$\frac{\partial}{\partial t}\mathbb{P}_{V_i}(\rho_{k,M}u_{k,M}) + \mathbb{P}_{V_i}A_{k,M} = 0,$$

in $(\mathcal{C}_0^\infty([0,T];H^1_\sigma(\Omega)))'$, where $A_{k,M}$ is defined by

$$\begin{aligned}\langle A_{k,M}, v\rangle &:= \int_0^T \int_\Omega \rho_{k,M}(Q_{S_{k,M}}(u_{k,M})\cdot\nabla)v\cdot u_{k,M}dxdt - 2\mu\int_0^T\int_\Omega D(u_{k,M}):D(v)dxdt \\ &- g\int_0^T\int_\Omega \nabla j_{\frac{1}{M}}(D(u_{k,M})):D(v)dxdt - k\int_0^T\int_{S_{k,M}}(u_{k,M}-P_{S_{k,M}}(u_{k,M}))\cdot(v-P_{S_{k,M}}(v))dxdt\end{aligned} \tag{4.12}$$

for all $v \in L^\infty(0,T;H^1_\sigma(\Omega))$. The next step is to prove that $\{A_{k,M}\}_M$ is bounded in $L^{4/3}(0,T;(H^1_\sigma(\Omega))')$. Using (3.3), we deduce

$$\left|\int_0^T\int_\Omega \nabla j_{\frac{1}{M}}(D(u_{k,M})):D(v)\ dxdt\right| \leqslant ((T\,|\Omega|)^{1/2} + \|u_{k,M}\|_{L^2(0,T;H^1_\sigma(\Omega))})\,\|v\|_{L^2(0,T;H^1_\sigma(\Omega))},$$

and, by using the property of the operator $Q^{\delta_1,\delta_2}_{a,Q}$ (see Definition 2.3), we have that:

$$\|Q_{S_{k,M}}(u_{k,M})\|_{L^2(0,T;H^1_\sigma(\Omega))} \leqslant C\,\|u_{k,M}\|_{L^2(0,T;H^1_\sigma(\Omega))}$$

with a constant $C = C(k) > 0$ and thus

$$\begin{aligned}&\left|\int_0^T\int_\Omega \rho_{k,M}(Q_{S_{k,M}}(u_{k,M})\cdot\nabla)v\cdot u_{k,M}dxdt\right| \\ &\qquad\leqslant C\,\|u_{k,M}\|^{3/2}_{L^2(0,T;H^1_\sigma(\Omega))}\,\|u_{k,M}\|^{1/2}_{L^\infty(0,T;L^2_\sigma(\Omega))}\,\|v\|_{L^4(0,T;H^1_\sigma(\Omega))}.\end{aligned}$$

The other terms in (4.12) can be estimated in a standard way and by using (4.2), this implies that $\frac{\partial}{\partial t}\mathbb{P}_{V_i}(\rho_{k,M}u_{k,M})$ is bounded in $L^{4/3}(0,T;(H^1_\sigma(\Omega))')$. Using (4.11), we can apply the Aubin-Lions compactness result and we deduce that

$$\mathbb{P}_{V_i}(\rho_{k,M}u_{k,M}) \to \mathbb{P}_{V_i}(\rho_k u_k) \text{ strongly } L^2(0,T;(H^1_\sigma(\Omega))'). \tag{4.13}$$

Let us denote by $\mathbb{P} : L^2(\Omega) \to L^2_\sigma(\Omega)$ the orthogonal projection (the Leray projection). For any $z \in L^2(\Omega)$,

$$\|\mathbb{P}(z) - \mathbb{P}_{V_i}(z)\|_{(H^1_\sigma(\Omega))'} \leqslant \|z\|_{L^2(\Omega)} \sup_{\varphi\in H^1_\sigma(\Omega),\|\varphi\|_{H^1_\sigma(\Omega)}=1} \|\varphi - \mathbb{P}_{V_i}(\varphi)\|_{L^2(\Omega)}.$$

Using the compactness of the embedding $H^1_\sigma(\Omega) \subset L^2_\sigma(\Omega)$ and that $\{v_q\}$ is an orthonormal basis of $H^1_\sigma(\Omega)$,

$$\sup_{\varphi\in H^1_\sigma(\Omega),\|\varphi\|_{H^1_\sigma(\Omega)}=1} \|\varphi - \mathbb{P}_{V_i}(\varphi)\|_{L^2(\Omega)} \to 0 \text{ as } i\to\infty.$$

Combining this with (4.13) and with the fact that $\{\rho_{k,M}u_{k,M}\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$ we deduce

$$\mathbb{P}(\rho_{k,M}u_{k,M}) \to \mathbb{P}(\rho_k u_k) \text{ strongly in } L^2(0,T;(H^1_\sigma(\Omega))'). \tag{4.14}$$

Now we follow an argument given in [42, p.47]: using (4.14) and (4.4), we first have

$$\int_0^T \int_\Omega \rho_{k,M} |u_{k,M}|_2^2 \ dxdt = \int_0^T \langle \mathbb{P}(\rho_{k,M} u_{k,M}), u_{k,M} \rangle_{(H^1_\sigma(\Omega))', H^1_\sigma(\Omega)} \ dt$$
$$\to \int_0^T \langle \mathbb{P}(\rho_k u_k), u_k \rangle_{(H^1_\sigma(\Omega))', H^1_\sigma(\Omega)} \ dt = \int_0^T \int_\Omega \rho_k |u_k|_2^2 \ dxdt. \quad (4.15)$$

This yields

$$\sqrt{\rho_{k,M}} u_{k,M} \to \sqrt{\rho_k} u_k \text{ strongly in } L^2(0,T;L^2(\Omega)). \quad (4.16)$$

From (4.9) we have that

$$\frac{1}{\sqrt{\rho_{k,M}}} \to \frac{1}{\sqrt{\rho_k}} \text{ strongly in } \mathcal{C}([0,T];L^3(\Omega)).$$

The above convergence and (4.16) imply

$$u_{k,M} \to u_k \text{ strongly in } L^2(0,T;L^{\frac{6}{5}}(\Omega)). \quad (4.17)$$

From (4.2), we have that $\{u_{k,M}\}$ is bounded in $L^2(0,T;L^6(\Omega))$ and thus

$$u_{k,M} \to u_k \text{ strongly in } L^2(0,T;L^p(\Omega)) \quad (p < 6). \quad (4.18)$$

4.3 A monotonicity argument

In this section, we pass to the limit as $M \to \infty$ (and thus as $\varepsilon \to 0$), by using a monotonicity argument. This type of technique is used to prove the existence of a weak solution of a Bingham fluid without the solid part, see [16, pp.296-297].

Let $\varphi \in \mathcal{C}^1([0,T];H^1_\sigma(\Omega))$. We denote by $\mathbb{P}^1_{V_M} : H^1_\sigma(\Omega) \to V_M$ the orthogonal projection and we define

$$\varphi_M := \mathbb{P}^1_{V_M} \varphi. \quad (4.19)$$

Then,

$$\varphi_M \to \varphi \text{ strongly in } \mathcal{C}^1([0,T];H^1_\sigma(\Omega)). \quad (4.20)$$

We set:

$$Z_M = \int_0^T \int_\Omega \rho_{k,M} \frac{\partial}{\partial t}(\varphi_M - u_{k,M}) \cdot (\varphi_M - u_{k,M}) dxdt + \frac{1}{2} \int_\Omega \rho(0,x) |u_{k,M}(0,x) - \varphi_M(0,x)|_2^2 \, dx.$$
$$+ \int_0^T \int_\Omega \rho_{k,M} (Q_{S_{k,M}}(u_{k,M}) \cdot \nabla)(\varphi_M - u_{k,M}) \cdot (\varphi_M - u_{k,M}) dxdt$$
$$+ g \int_0^T \int_\Omega j_{\frac{1}{M}}(D(\varphi_M)) - j_{\frac{1}{M}}(D(u_{k,M})) - \nabla j_{\frac{1}{M}}(D(u_{k,M})) : D(\varphi_M - u_{k,M}) \ dxdt. \quad (4.21)$$

By the Reynolds transport theorem and the convexity of $j_{\frac{1}{M}}$, we have that:

$$Z_M \geqslant 0.$$

Then, using equation (3.10) with the test function $\varphi_M - u_{k,M}$, Z_M can be written as follows:

$$Z_M = \int_0^T \int_\Omega \rho_{k,M} \frac{\partial \varphi_M}{\partial t} \cdot (\varphi_M - u_{k,M}) dxdt + \int_0^T \int_\Omega \rho_{k,M} (Q_{S_{k,M}}(u_{k,M}) \cdot \nabla) \varphi_M \cdot (\varphi_M - u_{k,M}) dxdt$$
$$+ 2\mu \int_0^T \int_\Omega D(u_{k,M}) : D(\varphi_M - u_{k,M}) dxdt + g \int_0^T \int_\Omega j_{\frac{1}{M}}(D(\varphi_M)) - j_{\frac{1}{M}}(D(u_{k,M})) dxdt$$
$$- k \int_0^T \int_{S_{k,M}} |u_{k,M} - P_{S_{k,M}}(u_{k,M})|_2^2 \, dxdt$$
$$+ k \int_0^T \int_{S_{k,M}} (u_{k,M} - P_{S_{k,M}}(u_{k,M})) \cdot (\varphi_M - P_{S_{k,M}}(\varphi_M))) dxdt + \frac{1}{2} \int_\Omega \rho(0,x) |u_{k,M}(0,x) - \varphi_M(0,x)|_2^2 \, dx.$$

Since $Z_M \geqslant 0$ and, by (3.16),

$$0 \leqslant \left\| u_{k,M} - P_{S_{k,M}} u_{k,M} \right\|_{L^2(0,T;L^2(S_{k,M}))} \leqslant \frac{C}{\sqrt{k}},$$

we deduce the following inequality:

$$\begin{aligned}\int_0^T \int_\Omega \rho_{k,M} \left(\frac{\partial \varphi_M}{\partial t} + (Q_{S_{k,M}}(u_{k,M}) \cdot \nabla) \varphi_M \right) \cdot (\varphi_M - u_{k,M}) dxdt \\ + 2\mu \int_0^T \int_\Omega D(u_{k,M}) : D(\varphi_M) dxdt + g \int_0^T \int_\Omega j_{\frac{1}{M}}(D(\varphi_M)) dxdt \\ + C\sqrt{k} \left\| \varphi_M - P_{S_{k,M}}(\varphi) \right\|_{L^2(0,T;L^2(S_{k,M}))} \geqslant -\frac{1}{2} \int_\Omega \rho(0,x) \left| u_{k,M}(0,x) - \varphi_M(0,x) \right|_2^2 dx \\ + g \int_0^T \int_\Omega j_{\frac{1}{M}}(D(u_{k,M})) dxdt + 2\mu \int_0^T \int_\Omega |D(u_{k,M})|_2^2 dxdt. \end{aligned} \tag{4.22}$$

To conclude, we need to pass to the limit the terms in the above inequality as $M \to \infty$:

- Combining (4.9), (4.18) and (4.20), we deduce

$$\int_0^T \int_\Omega \rho_{k,M} \frac{\partial \varphi_M}{\partial t} \cdot (\varphi_M - u_{k,M}) \; dxdt \to \int_0^T \int_\Omega \rho_k \frac{\partial \varphi}{\partial t} \cdot (\varphi - u_k) \; dxdt. \tag{4.23}$$

- By Lemma 2.4 and (4.18), we deduce that:

$$Q_{S_{k,M}}(u_{k,M}) \to Q_{S_k}(u_k) \text{ strongly in } L^2(0,T;L^5(\Omega)). \tag{4.24}$$

  Combining this with (4.9), (4.18) and (4.20) yields

$$\begin{aligned}\int_0^T \int_\Omega \rho_{k,M} (Q_{S_{k,M}}(u_{k,M}) \cdot \nabla) \varphi_M \cdot (\varphi_M - u_{k,M}) \; dxdt \\ \to \int_0^T \int_\Omega \rho_k (Q_{S_k}(u_k) \cdot \nabla) \varphi \cdot (\varphi - u_k) \; dxdt. \end{aligned} \tag{4.25}$$

- From (4.20) and (4.8), we obtain

$$\mathbb{1}_{S_{k,M}} \varphi_{k,M} \to \mathbb{1}_{S_k} \varphi \quad \text{in } L^2(0,T;L^2(\Omega))$$

  and thus (with (2.1) and (2.2))

$$\mathbb{1}_{S_{k,M}} P_{S_{k,M}} \varphi_{k,M} \to \mathbb{1}_{S_k} P_{S_k} \varphi \quad \text{in } L^2(0,T;L^2(\Omega)).$$

  Consequently,

$$\left\| \varphi_M - P_{S_{k,M}} \varphi_M \right\|_{L^2(0,T;L^2(S_{k,M}))} \to \left\| \varphi - P_{S_k} \varphi \right\|_{L^2(0,T;L^2(S_k))}. \tag{4.26}$$

  Similarly, since $u_{k,M} \to u_k$ strongly in $L^2(0,T;L^2(\Omega))$ and $\{u_{k,M}\}$ is bounded in $L^2(0,T;L^6(\Omega))$, we deduce that

$$\left\| u_{k,M} - P_{S_{k,M}} u_{k,M} \right\|_{L^2(0,T;L^2(S_{k,M}))} \to \left\| u_k - P_{S_k} u_k \right\|_{L^2(0,T;L^2(S_k))}. \tag{4.27}$$

- From (4.20) and (4.4), we have that:

$$\int_0^T \int_\Omega D(u_{k,M}) : D(\varphi_M) dxdt \to \int_0^T \int_\Omega D(u_k) : D(\varphi) \; dxdt. \tag{4.28}$$

- From (4.4), we also have that

$$\liminf_{M \to \infty} \int_0^T \int_\Omega |D(u_{k,M})|_2^2 \; dxdt \geqslant \int_0^T \int_\Omega |D(u_k)|_2^2 \; dxdt. \tag{4.29}$$

- From the definition (3.1) of $j_{\frac{1}{M}}$,

$$\int_0^T \int_\Omega j_{\frac{1}{M}}(D(\varphi_M))\ dxdt = \frac{M}{M+1}\int_0^T \int_\Omega |D(\varphi_M)|_2^{\frac{1}{M}+1}\ dxdt.$$

Using (4.20) and the dominated convergence theorem, we deduce

$$\int_0^T \int_\Omega j_{\frac{1}{M}}(D(\varphi_M))\ dxdt \to \int_0^T \int_\Omega |D(\varphi)|_2\ dxdt. \tag{4.30}$$

- Following the argument of [16, p.298], we are going now to prove

$$\liminf_{M\to\infty} \int_0^T \int_\Omega j_{\frac{1}{M}}(D(u_{k,M}))dxdt \geqslant \int_0^T \int_\Omega |D(u_k)|_2\, dxdt. \tag{4.31}$$

First, by Hölder's inequality we have that:

$$\int_0^T \int_\Omega |D(u_{k,M})|_2\, dxdt \leqslant \left(\int_0^T \int_\Omega |D(u_{k,M})|_2^{1+\frac{1}{M}}\ dxdt\right)^{\frac{M}{1+M}} (T|\Omega|)^{\frac{1}{M+1}}$$

and thus

$$\int_0^T \int_\Omega j_{\frac{1}{M}}(D(u_{k,M}))dxdt \geqslant \frac{M}{(1+M)(T|\Omega|)^{\frac{1}{M}}} \left(\int_0^T \int_\Omega |D(u_{k,M})|_2\, dxdt\right)^{1+\frac{1}{M}}.$$

Since $D(u_{k,M})$ is bounded in $L^1(0,T;L^1(\Omega))$, we have

$$\liminf_{M\to\infty} \int_0^T \int_\Omega j_{\frac{1}{M}}(D(u_{k,M}))dxdt \geqslant \liminf_{M\to\infty} \int_0^T \int_\Omega |D(u_{k,M})|_2\, dxdt$$

and since the application $v \mapsto \int_0^T \int_\Omega |D(v)|_2\, dxdt$ is continuous and convex on $L^2(0,T;H^1_\sigma(\Omega))$, it is lower semi-continuous for the weak topology. Using this with (4.4) yields (4.31).

- Using (3.11) and (4.20), we deduce that

$$\int_\Omega \rho_0 \left|u_{k,M}(0,\cdot) - \varphi_M(0,\cdot)\right|_2^2\ dx \to \int_\Omega \rho_0 \left|u_0 - \varphi(0,\cdot)\right|_2^2\ dx. \tag{4.32}$$

Gathering (4.22), (4.23), (4.25), (4.26), (4.28), (4.29), (4.30), (4.31), (4.32), we deduce the following inequality:

$$\begin{aligned}\int_0^T \int_\Omega \rho \left(\frac{\partial\varphi}{\partial t} + (Q_{S_k}(u_k)\cdot\nabla)\varphi\right)\cdot(\varphi - u_k)dxdt + 2\mu \int_0^T \int_\Omega D(u_k) : D(\varphi)dxdt \\ + g\int_0^T \int_\Omega |D(\varphi)|_2\, dxdt + C\sqrt{k}\,\|\varphi - P_{S_k}(\varphi)\|_{L^2(0,T;L^2(S_k))} \geqslant -\frac{1}{2}\int_\Omega \rho(0,x)\left|u_0 - \varphi(0,x)\right|_2^2 dx \\ + g\int_0^T \int_\Omega |D(u_k)|_2\, dxdt + 2\mu \int_0^T \int_\Omega |D(u_k)|_2^2\ dxdt\end{aligned} \tag{4.33}$$

for any $\varphi \in \mathcal{C}^1([0,T];H^1_\sigma(\Omega))$. Using standard techniques, see for example [58, pp. 290-291], by (4.3), (4.27), (4.31) and (4.29) we deduce the following energy estimate for a.e. $t \in (0,T)$:

$$\begin{aligned}\frac{1}{2}\int_\Omega \rho_k(t,x)\left|u_k(t,x)\right|_2^2 dx + 2\mu \int_0^t \int_\Omega |D(u_k)|_2^2\, dxdt + g\int_0^t \int_\Omega |D(u_k)|_2\, dxdt \\ + k\int_0^t \int_{S_k} |u_k - P_{S_k}u_k|_2^2\, dxdt \leqslant \frac{1}{2}\int_\Omega \rho_0 \left|u_0\right|_2^2 dx.\end{aligned} \tag{4.34}$$

5 Passing to the limit $k \to \infty$

The aim of this section is to finish the proof of Theorem 1.4. From (4.34), we deduce that there exist

$$u \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_\sigma(\Omega)) \quad (h,R) \in W^{1,\infty}(0,T;\mathbb{R}^3 \times SO(3))$$

such that:

$$u_k \overset{*}{\rightharpoonup} u \text{ weak star in } L^\infty(0,T;L^2_\sigma(\Omega)), \tag{5.1}$$

$$u_k \rightharpoonup u \text{ weakly in } L^2(0,T;H^1_\sigma(\Omega)), \tag{5.2}$$

$$(h_k,R_k) \overset{*}{\rightharpoonup} (h,R) \text{ weak star in } W^{1,\infty}(0,T;\mathbb{R}^3 \times SO(3)) \tag{5.3}$$

and

$$(h_k,R_k) \to (h,R) \text{ strongly in } \mathcal{C}([0,T];\mathbb{R}^3 \times SO(3)). \tag{5.4}$$

We write

$$S := \widehat{S}(h,R)$$

and

$$J := \widehat{J}(h,R).$$

From (4.7) and (5.4), we deduce

$$S(t) \Subset \Omega, \quad \operatorname{dist}(S(t),\partial\Omega) \geqslant 2\delta \quad (t \in [0,T]). \tag{5.5}$$

By Lemma 2.1 we have that

$$\mathbb{1}_{S_k} \to \mathbb{1}_S \text{ strongly in } \mathcal{C}([0,T];L^p(\Omega)) \quad \forall p \in [1,\infty) \tag{5.6}$$

and thus

$$\rho_k \to \rho := \widehat{\rho}_{h,R} \text{ strongly in } \mathcal{C}([0,T];L^p(\Omega)) \quad \forall p \in [1,\infty). \tag{5.7}$$

Using (2.1) and (2.2), we deduce

$$P_{S_k} u_k \overset{*}{\rightharpoonup} P_S u \text{ weakly star in } L^\infty(0,T,\mathcal{R}). \tag{5.8}$$

We write

$$P_S u =: \ell + \omega \times (x-h) \quad \text{in } (0,T). \tag{5.9}$$

By the energy estimate (4.34) we deduce that:

$$\|u_k - P_{S_k} u_k\|_{L^2(0,T;L^2(S_k))} \leqslant \frac{C}{\sqrt{k}}. \tag{5.10}$$

Then, taking $k \to \infty$, we deduce that $u = P_S u$ in S. Therefore, we conclude that $u(t,\cdot) \in H^1_{S(t)}(\Omega)$ a.e. in $(0,T)$.

5.1 Strong Convergence of the velocity field

As in the limit in M, we require the strong convergence of the velocity as $k \to \infty$ to show the convergence of the convective term. To do this we follow the main steps of Section 7 of [52] (see also Section 5.5 of [29]).

We recall that $H^s_S(\Omega)$ is defined by (2.45) and that $\mathcal{P}^s_S$ is defined by (2.46).

First of all, we need another relation than (4.33) and (4.34). Let us take φ_M given by (4.19) with $\varphi \in \mathcal{C}^1_0((0,T);H^1_\sigma(\Omega))$ as a test function in (3.15).

$$-\int_0^T \int_\Omega \rho_{k,M} \left(\frac{\partial \varphi_M}{\partial t} + (Q_{S_{k,M}}(u_{k,M}) \cdot \nabla)\varphi_M \right) \cdot u_{k,M} \, dxdt + \int_0^T \int_\Omega (2\mu D(u_{k,M}) + g\nabla j_{\frac{1}{M}}(D(u_{k,M}))) : D(\varphi_M) \, dxdt + k \int_0^T \int_{S_{k,M}} (u_{k,M} - P_{S_{k,M}}(u_{k,M})) \cdot (\varphi_M - P_{S_{k,M}}(\varphi_M)) \, dxdt = 0. \tag{5.11}$$

Using (3.3), (3.16) and (4.1), we deduce that

$$\left\|\nabla j_{\frac{1}{M}}(D(u_{k,M}))\right\|_{L^2(0,T;L^2(\Omega))} \leqslant C,$$

with a constant C independent of the solution and of k. Therefore, for any k, there exists an element $\chi_k \in L^2(0,T;L^2(\Omega))$ such that:

$$\nabla j_{\frac{1}{M}}(D(u_{k,M})) \rightharpoonup \chi_k \text{ weakly in } L^2(0,T;L^2(\Omega)) \tag{5.12}$$

and

$$\|\chi_k\|^2_{L^2(0,T;L^2(\Omega))} \leqslant C. \tag{5.13}$$

Then taking $M \to \infty$ in equation (5.11) and using (4.23), (4.25), (4.26), (4.27), (4.28), (5.12), we obtain the following equation:

$$-\int_0^T \int_\Omega \rho_k \left(\frac{\partial \varphi}{\partial t} + (Q_{S_k}(u_k)\cdot\nabla)\varphi\right)\cdot u_k dxdt + \int_0^T \int_\Omega (2\mu D(u_k) + g\chi_k) : D(\varphi) dxdt \\ + k\int_0^T \int_{S_k} (u_k - P_{S_k}(u_k))\cdot(\varphi - P_{S_k}(\varphi)) dxdt = 0. \tag{5.14}$$

We use this new relation to obtain some compactness that will imply the strong convergence of the velocity (as $k \to \infty$). Using (5.4), we deduce that for all $d > 0$, there exists k_0 such that for all $k \geqslant k_0$,

$$S_k(t) \subset (S(t))^{\frac{d}{2}} \quad \forall t \in [0,T]. \tag{5.15}$$

Moreover, using the Heine theorem, there exists $N(d) > 0$ such that if

$$\tau := T/N \quad \text{and} \quad I_j := [j\tau, (j+1)\tau]$$

then

$$(S(t))^{\frac{d}{2}} \subset (S(j\tau))^d \subset (S(t))^{2d} \quad (t \in I_j).$$

Then, we consider a test function $\varphi \in C_0^\infty((0,T), H^1_\sigma(\Omega))$ such that $D(\varphi(t,\cdot)) = 0$ in $(S(j\tau))^d$ and $\varphi(t,\cdot) = 0$ if $t \notin I_j$. With such a test function in (5.14), the integral related to the penalization term vanishes, and we obtain the following estimate:

$$\left|\int_{I_j}\int_\Omega \rho_k u_k \cdot \frac{\partial\varphi}{\partial t} dxdt\right| \leqslant C\Big(\|Q_{S_k}(u_k)\|_{L^2(0,T;L^4(\Omega))} \|u_k\|^{1/4}_{L^\infty(0,T;L^2_\sigma(\Omega))} \|u_k\|^{3/4}_{L^2(0,T;H^1_\sigma(\Omega))} \\ + \|u_k\|_{L^2(0,T;H^1_\sigma(\Omega))} + \|\chi_k\|_{L^2(0,T;H^1_\sigma(\Omega))}\Big) \|\varphi\|_{L^8(I_j;H^1_\sigma(\Omega))}. \tag{5.16}$$

From (2.18) and (2.29), we have

$$\|Q_{S_k}(u_k) - u_k\|_{L^p(F_k)} \leqslant C\left(\left(\frac{1}{k}\right)^{\frac{1}{p}-\frac{1}{6}} \|u_k\|_{H^1(F_k)} + \|(u_k - P_{S_k}u_k)\cdot n\|_{L^p(\partial S_k)}\right), \tag{5.17}$$

for $p \in [2,6]$. Moreover, using a Sobolev embedding, a trace theorem and an interpolation result, we can check that for $p \in [2,4]$,

$$\|(u_k - P_{S_k}u_k)\cdot n\|_{L^p(\partial S_k)} \leqslant C\|u_k - P_{S_k}u_k\|^{2/p-1/2}_{L^2(S_k)} \|u_k - P_{S_k}u_k\|^{3/2-2/p}_{H^1(S_k)}.$$

Combining this with (5.10), we deduce

$$\|(u_k - P_{S_k}u_k)\cdot n\|_{L^p(\partial S_k)} \leqslant C\left(\frac{1}{k}\right)^{1/p-1/4} \|u_k\|^{3/2-2/p}_{H^1(S_k)}. \tag{5.18}$$

In particular,

$$\{Q_{S_k}(u_k)\} \quad \text{is bounded in } L^2(0,T;L^4(\Omega)).$$

Using the above estimate, (4.34) and (5.13) in (5.16), we deduce that

$$\left\{\frac{\partial}{\partial t}\mathcal{P}^0_{(S(j\tau))^d}(\rho_k u_k)\right\}_k \quad \text{is bounded in} \quad L^{8/7}(I_j;(H^1_{(S(j\tau))^d}(\Omega))').$$

Using the Aubin-Lions lemma we deduce

$$\mathcal{P}^0_{(S(j\tau))^d}(\rho_k u_k) \to \mathcal{P}^0_{(S(j\tau))^d}(\rho u) \text{ strongly in } L^2(I_j;(H^s_{(S(j\tau))^d}(\Omega))') \quad (s \in (0,1]). \tag{5.19}$$

Then using the relation

$$\mathcal{P}^0_{(S(j\tau))^d}\mathcal{P}^s_{(S(t))^{2d}} = \mathcal{P}^s_{(S(t))^{2d}} \quad \forall t \in I_j,$$

we deduce for any $s \in (0,1]$,

$$\lim_{k\to\infty}\int_0^T\int_\Omega \rho_k u_k \cdot \mathcal{P}^s_{(S(t))^{2d}}(u_k)\ dxdt = \int_0^T\int_\Omega \rho u \cdot \mathcal{P}^s_{(S(t))^{2d}}(u)\ dxdt.$$

Then, using Corollary 2.6 and (5.10), we have for $s \in (0,1/3)$

$$\int_0^T \left\|u_k(t,\cdot) - \mathcal{P}^s_{(S_n(t))^d}u_k(t,\cdot)\right\|^2_{H^s(\Omega)} dt \leqslant C(d^{2(1/3-s)} + k^{-1/2}) \tag{5.20}$$

and

$$\int_0^T \left\|u(t,\cdot) - \mathcal{P}^s_{(S(t))^d}u(t,\cdot)\right\|^2_{H^s(\Omega)} dt \leqslant Cd^{2(1/3-s)}. \tag{5.21}$$

so that

$$\lim_{k\to\infty}\int_0^T\int_\Omega \rho_k|u_k|^2\ dxdt = \int_0^T\int_\Omega \rho|u|^2\ dxdt$$

and, by the same arguments than the ones of the end of Section 4.2, this allows us to deduce that

$$u_k \to u \text{ strongly in } L^2(0,T;L^p(\Omega)) \quad (p<6). \tag{5.22}$$

5.2 Passing to the limit in the velocity inequality

Assume $v \in \mathcal{C}^1([0,T];H^1_{S(t)}(\Omega))$ with $\operatorname{supp} v \subset \Omega_\eta$, $\eta > 0$. We set

$$v_k := \overline{\Phi}_{h_k,R_k} \circ \Phi_{h,R}(v), \tag{5.23}$$

where Φ and $\overline{\Phi}$ are defined in (2.5) and (2.6). Then, for k large enough we have

$$v_k \in \mathcal{C}([0,T];H^1_\sigma(\Omega)). \tag{5.24}$$

Moreover,

$$D(v_k) = 0 \quad \text{in } S_k. \tag{5.25}$$

Using Lemma A.2 in [29] and (5.4), we deduce

$$v_k \to v \text{ strongly in } \mathcal{C}([0,T];H^1_\sigma(\Omega)). \tag{5.26}$$

Similarly, deriving (5.23), we obtain

$$\frac{\partial v_k}{\partial t} + (P_{S_k}u_k\cdot\nabla)\,v_k - \omega_k \times v_k \to \frac{\partial v}{\partial t} + (P_S u\cdot\nabla)\,v - \omega\times v \text{ strongly in } L^\infty(0,T;L^2(\Omega)). \tag{5.27}$$

On the other hand, from (5.22) and (2.1), (2.2), we have

$$P_{S_k}u_k \to P_S u \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{5.28}$$

Finally, combining (5.27) with (5.26) and with (5.28), we conclude

$$\frac{\partial v_k}{\partial t} \to \frac{\partial v}{\partial t} \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{5.29}$$

Taking $\varphi = v_k$ in (4.33), we obtain

$$\int_0^T \int_\Omega \rho \left(\frac{\partial v_k}{\partial t} + (Q_{S_k}(u_k)\cdot\nabla)v_k \right)\cdot (v_k - u_k)\ dxdt + 2\mu \int_0^T \int_\Omega D(u_k) : D(v_k)\ dxdt$$
$$+ g \int_0^T \int_\Omega |D(v_k)|_2\ dxdt \geqslant -\frac{1}{2}\int_\Omega \rho(0,x)\,|u_0 - v_k(0,x)|_2^2\,dx$$
$$+ g\int_0^T \int_\Omega |D(u_k)|_2\,dxdt + 2\mu \int_0^T \int_\Omega |D(u_k)|_2^2\,dxdt. \quad (5.30)$$

We can pass to the limit as in Section 4, the only term that needs more details is the first term:

$$\int_0^T \int_\Omega \rho_k \left(\frac{\partial v_k}{\partial t} + (Q_{S_k}u_k\cdot\nabla)\,v_k \right)\cdot (v_k - u_k)dxdt = \rho_f \int_0^T \int_\Omega \mathbb{1}_{F_k} \frac{\partial v_k}{\partial t}\cdot (v_k - u_k)dxdt$$
$$+ \rho_f \int_0^T \int_\Omega \mathbb{1}_{F_k}\,(Q_{S_k}u_k\cdot\nabla)\,v_k\cdot (v_k - u_k)dxdt$$
$$+ \rho_s \int_0^T \int_\Omega \mathbb{1}_{S_k} \left(\frac{\partial v_k}{\partial t} + (P_{S_k}u_k\cdot\nabla)\,v_k \right)\cdot (v_k - u_k)dxdt. \quad (5.31)$$

Combining (5.6), (5.22), (5.26) and (5.29) we deduce that

$$\rho_f \int_0^T \int_\Omega \mathbb{1}_{F_k} \frac{\partial v_k}{\partial t}\cdot (v_k - u_k)\ dxdt \to \rho_f \int_0^T \int_\Omega \mathbb{1}_F \frac{\partial v}{\partial t}\cdot (v-u)\ dxdt.$$

Relation (5.18), (5.17) and (4.34) imply

$$\mathbb{1}_{F_k}(Q_{S_k}(u_k) - u_k) \to 0 \quad \text{strongly in } L^2(0,T;L^p(\Omega))$$

for $p < 4$. Gathering the above limit with (5.22) and (5.6), we deduce

$$\mathbb{1}_{F_k} Q_{S_k}(u_k) \to \mathbb{1}_F u \text{ strongly in } L^2(0,T;L^p(\Omega))$$

for $p < 4$. Combining this with (5.22) and (5.26), we obtain

$$\rho_f \int_0^T \int_{F_k} (Q_{S_k}(u_k)\cdot\nabla)v_k\cdot (v_k - u_k)\ dxdt \to \rho_f \int_0^T \int_F (u\cdot\nabla)v\cdot (v-u)\ dxdt.$$

Finally, from (5.6), (5.22), (5.26) and (5.27),

$$\int_0^T \int_\Omega \mathbb{1}_{S_k} \left(\frac{\partial v_k}{\partial t} + (P_{S_k}u_k\cdot\nabla)\,v_k \right)\cdot (v_k - u_k)\ dxdt \to \int_0^T \int_\Omega \mathbb{1}_S \left(\frac{\partial v}{\partial t} + (P_S u\cdot\nabla)\,v \right)\cdot (v-u)\ dxdt.$$

We thus conclude that u satisfies inequality (1.21). We can also pass to the limit in (4.34) we deduce (1.23). We deduce the existence of a weak solution of the system (1.2)–(1.12) in the sense of Definition 1.3 and on the interval $(0,T)$. To finish the proof of Theorem 1.4, it remains to obtain that one of the alternatives stated there holds true. This is standard and the proof can be found for instance in [11] or in [19, Lemma 2.2].

6 Limit as $g \to \infty$

Here we prove Corollary 1.10.

Proof of Corollary 1.10. We write the energy estimate (1.23) for all $g > 0$:

$$\frac{1}{2}\int_\Omega \rho_g(t,x)\,|u_g(t,x)|_2^2\ dx + 2\mu \int_0^t \int_\Omega |D(u_g)|_2^2\ dxdt + g\int_0^t \int_\Omega |D(u_g)|_2\ dxdt \leqslant \frac{1}{2}\int_\Omega \rho(0,x)\,|u_0|_2^2\,dx, \quad (6.1)$$

a.e. in $(0,T_g)$ where T_g is the time of existence of the weak solution for all g. This shows that (ℓ_g,ω_g) is bounded uniformly with respect to g, and using (1.8), (1.9), (1.11), we deduce that there exists a uniform time $T > 0$ for all g such that the solution (u_g,h_g,R_g) exists in $(0,T)$.

Using (6.1) and following the proof of the above sections ($M \to \infty$ or $k \to \infty$), we deduce the existence of

$$u \in L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; H^1_\sigma(\Omega)), \quad (h,R) \in W^{1,\infty}(0,T; \mathbb{R}^3 \times SO(3))$$

such that (up to a subsequence)

$$u_g \overset{*}{\rightharpoonup} u \text{ weak star in } L^\infty(0,T; L^2_\sigma(\Omega)), \tag{6.2}$$

$$u_g \rightharpoonup u \text{ weakly in } L^2(0,T; H^1_\sigma(\Omega)), \tag{6.3}$$

$$D(u_g) \to 0 \quad \text{in } L^1(0,T; L^1(\Omega)), \tag{6.4}$$

and

$$(h_g, R_g) \to (h,R) \quad \text{strongly in } \mathcal{C}([0,T]; \mathbb{R}^3 \times SO(3)). \tag{6.5}$$

In particular, $D(u) = 0$ and using Korn's inequality, we obtain $u = 0$. We deduce that (h,R) satisfies (1.8), (1.9), (1.11) with $(\ell, \omega) = 0$ and this concludes the proof of the corollary. $\square$

Acknowledgments

T. Takahashi was partially supported by the ANR research project IFSMACS (ANR-15-CE40-0010). B. Obando was also partially supported by CONICYT-PCHA/Doctorado Nacional/2014.

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