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OPTIMAL CONTROL OF PANIC SPREADING IN COMPLEX NETWORKS OF DYNAMICAL SYSTEMS FOR A BEHAVIOURAL MODEL

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Abstract. After a sudden catastrophic event occurring in a population of individuals, panic can spread, persist and become more problematic than the catastrophe itself. In this paper, we propose to investigate the possible actions in order to limit panic at a low level, using a recent behavioural model given by a system of ordinary differential equations, and optimal control theory applied to complex networks. We establish the existence of a solution to the optimal control problem for a wide class of abstract complex networks of dynamical systems, and show how it can be used to reach synchronisation under control in the network. Finally, we explore numerically different scenarios of panic spreading corresponding to a tsunami on the Mediterranean coast.

Keywords. Optimal control, dynamical system, complex network, panic, synchronisation.

§ 1. Introduction

Hippocrates, widely referred as the “Father of Medicine” [16], used to classify psychic disorders and describe their symptoms; he proposed to treat those psychic troubles, among them panic (associated to the god Pan), by ritual dances accompanied with the sound of music in Phrygian mode. Nowadays, far from those ancient considerations, although aware of, researchers are trying to better understand panic, which is not only a disorder troubling single individuals, but also a sneaky phenomenon that can quickly spread within a population.

The aim of this paper is to explore the possibility to control the panic spreading in the particular situation when individuals are facing a catastrophic event, with a mathematical approach, using a recent behavioural modelling called the Panic-Control-Reflex system (PCR system) [3, 10]. This modelling is given by a set of ordinary differential equations (see Equation (1) below), and reproduces the behavioural process from reflex to control behaviour, with the eventuality to transit through panic, and possibly to exhibit a persistence of panic. The geographical background of the areas impacted by catastrophic events naturally leads to studying complex networks of PCR systems [4, 2], that is, geographical networks whose nodes are coupled with multiple instances of a PCR system, with connexions between those nodes, corresponding to physical displacements. In [2], it is proved that the evacuation of high risk zones towards refuge zones is a necessary and sufficient condition for the whole population in the network to return to a daily behaviour, and to avoid a persistence of panic (see Theorem 2 below). But this necessary evacuation can be awkward in some particular places, or even impossible. Thus it is natural to ask how to act in that case, in order to limit the persistence of panic at a reasonable level. Here, we propose to model two different types of external actions to remedy this problem. The first type is a preventive treatment which is expected to facilitate the behavioural evolution from panic to control behaviour, whereas the second type

corresponds to a curative action of rescue services on the connexions in the network, targeted on the dangerous zones. Our goal is to investigate whether this optimal control problem is well-posed, and to discuss the effect of each control on the panic level.

Our paper is organised as follows. In the next section, we present the modelling ingredients which motivate the PCR system, and recall the main theoretical results about its dynamics. We also show how to construct complex networks with multiple instances of a PCR system, and state the optimal control problem ¹. In the third section, we prove our main result which states that the optimal control problem for the complex network admits a solution. We demonstrate this theorem for a class of abstract complex networks built with non-identical instances of a given dynamical system, intending to generalize our study, so that it can be used for application to other problems, since existing works on the subject of optimal control in complex networks are quite rare (see however [14]). We explain how control can be used to achieve synchronisation, and give a new definition of *synchronisation under control*, which extends the classical definition of synchronisation in complex networks [1, 8]. Finally, we propose a numerical exploration of different scenarios for a concrete network corresponding to a tsunami on the Mediterranean coast.

§ 2. Problem statement

2.1. The Panic-Control-Reflex system

The Panic-Control-Reflex system (PCR system) is a mathematical model for human behaviours during catastrophic events, developed with the collaboration of geographers in order to better understand, predict and control the behavioural reactions of individuals facing a brutal disaster [3, 10]. It is given by the following system of ordinary differential equations

$$\begin{cases} \dot{r} = \gamma(t)q(r_m - r) - Br + F(r, c)rc + G(r, p)rp \\ \dot{c} = B_1r - C_2c + C_1p - F(r, c)rc + H(c, p)cp - \varphi(t)c(b_m - b) \\ \dot{p} = B_2r + C_2c - C_1p - G(r, p)rp - H(c, p)cp \\ \dot{q} = -\gamma(t)q(r_m - r) \\ \dot{b} = +\varphi(t)c(b_m - b), \end{cases} \quad (1)$$

where the unknowns r, c, p, q, b are real-valued functions defined on \mathbb{R} , which model the numbers of individuals in *reflex*, *control*, *panic*, *daily* and *back to daily* behaviours respectively. The parameters $B_1 > 0, B_2 > 0, B = B_1 + B_2, C_1 \geq 0, C_2 \geq 0, r_m > 0$, and $b_m > 0$ are real coefficients, γ, φ are smooth functions of t with positive values, F, G, H smooth functions defined on \mathbb{R}^2 with values in \mathbb{R} .

Let us briefly describe the behavioural process modelled by the PCR system. First, when the considered catastrophic event occurs, individuals are brought to the reflex behaviour; this evolution is modelled by the non-linear term $\gamma(t)q(r_m - r)$, in which $\gamma(t)$ corresponds to the impact of the catastrophe. Next, individuals are naturally subject to behavioural evolutions towards control behaviour or panic; those evolutions are modelled by the linear terms B_1r and B_2r . Additionally, contagion phenomena can act in parallel between the 3 main behavioural subgroups (reflex, control behaviour, panic); those contagion phenomena are modelled by the non-linear terms $F(r, c)rc, G(r, p)rp$ and $H(c, p)cp$. It is worth noting that the weight functions F, G and H have been designed to change their signs according to the values of the proportions $\frac{r}{c}, \frac{r}{p}$ and $\frac{c}{p}$: more precisely, if the number of individuals in panic is widely greater than the number of individuals in control behaviour, then $H(c, p) < 0$, which means that the contagion brings individuals in control behaviour to imitate individuals in panic. We

¹In order to avoid misunderstandings of our work, we emphasize that it is worth distinguishing *control behaviour* and *control theory*.

refer the reader to [3] or [10] for more details about those imitation functions. In the mean time, evolutions between panic and control behaviour are modelled by the linear terms C_1p , C_2c . We will show below the decisive role of the parameter C_1 which models the evolution from panic towards control behaviour. Finally, the return to daily behaviour operates from the control behaviour; it is modelled by the non-linear term $\varphi(t)c(b_m - b)$. The whole behavioural process modelled by the PCR system is depicted in figure 1.

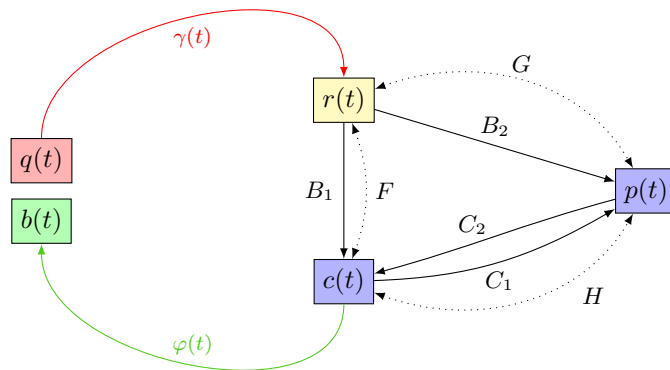


Figure 1: Behavioural evolutions and contagion process within a population affected by a catastrophic event, modelled by the PCR system: r , c , p , q , b correspond to the numbers of individuals in *reflex*, *control*, *panic*, *daily* and *back to daily* behaviours respectively.

The PCR system has been considered for simulations of concrete scenarios of catastrophic events, like an earthquake in Japan [10] or a tsunami on the Mediterranean coast [4]. The parameter r_m models the maximum capacity of individuals which can be in reflex behaviour. Without loss of generality, we will set $r_m = 1$ in the rest of the paper. The parameter b_m models the maximum capacity of individuals which can return to the daily behaviour. We assume that this maximum capacity coincides with the total population

$$\Pi = r + c + p + q + b$$

involved in the catastrophic event, thus we can reduce system (1) to a 4 equations system

$$\dot{x} = \psi(t, x), \tag{2}$$

where $x = (r, c, p, q)^T$ and

$$\psi(t, x) = \begin{pmatrix} \gamma(t)q(1-r) - Br + F(r, c)rc + G(r, p)rp \\ B_1r - C_2c + C_1p - F(r, c)rc + H(c, p)cp - \varphi(t)c(r+c+p+q) \\ B_2r + C_2c - C_1p - G(r, p)rp - H(c, p)cp \\ -\gamma(t)q(1-r) \end{pmatrix}.$$

The mathematical analysis of the PCR system (2) is presented in [3]. The following Theorem summaries its dynamics, and highlights the decisive role of the parameter C_1 which models the evolution from panic to control behaviour. The proof is detailed in [3].

Theorem 1. *For any initial condition $x_0 \in (\mathbb{R}^+)^4$, the Cauchy problem*

$$\begin{cases} \dot{x} = \psi(t, x), & t > 0, \\ x(0) = x_0, \end{cases}$$

admits a unique global solution whose components are non-negative and bounded.

If $C_1 > 0$, then the trivial equilibrium $0 \in \mathbb{R}^4$ is the only equilibrium, and it is globally asymptotically stable. If $C_1 = 0$, then the solution of system (2) stemming from any initial condition $x_0 = (r_0, c_0, p_0, q_0)$ such that

$$r_0 + c_0 + p_0 + q_0 > 0$$

presents a persistence of panic, that is

$$\lim_{t \rightarrow +\infty} p(t) = \bar{p} > 0.$$

The latter theorem shows that a bifurcation occurs in the PCR system (2) when the evolution parameter C_1 approaches 0 (see figure 2). This bifurcation is analysed in [3], where it is proved that it corresponds to a degenerate case of a saddle-node bifurcation at infinity.

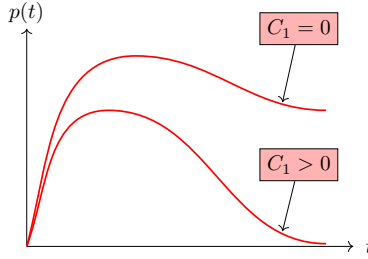


Figure 2: Panic level $p(t)$ for the PCR system (2). If $C_1 > 0$, then the panic level decreases to 0, which corresponds to a favourable return of all individuals to the daily behaviour. If $C_1 = 0$, then the panic persists.

2.2. PCR network

When considering the geographical relief of the zone impacted by the catastrophic event, it is natural to improve the previous modelling by a spatial modelling. One way is to construct a complex network whose nodes are coupled with multiple instances of the PCR system. Let us consider a simple graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ made with a finite set $\mathcal{V} = \{1, \dots, n\}$ of n vertices, where n is a positive integer, and a finite set $\mathcal{E} = \{e_1, \dots, e_k\}$ of k weighted edges [4, 2], with non-negative weights $\varepsilon_1, \dots, \varepsilon_k$. For each integer $l \in \{1, \dots, k\}$, there exists a unique pair of vertices (i, j) such that e_l connects vertex i towards vertex j .

We set $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k) \in (\mathbb{R}^+)^k$, and introduce the matrix of connectivity $L(\varepsilon)$ of order n , whose off-diagonal coefficients are given by

$$L_{ji}(\varepsilon) = \begin{cases} \varepsilon_l & \text{if } e_l = (i, j) \in \mathcal{E}, \\ 0 & \text{else,} \end{cases}$$

and whose diagonal coefficients satisfy

$$L_{ii}(\varepsilon) = - \sum_{\substack{j=1 \\ j \neq i}}^n L_{ji}(\varepsilon).$$

Next we couple each node in the graph with an instance of the PCR system (2). Thus we set

$$\begin{aligned} x_i &= (r_i, c_i, p_i, q_i)^T, & X &= (x_1, \dots, x_n)^T, \\ H &= \text{diag}\{1, 1, 1, 0\}, & HX &= (Hx_1, \dots, Hx_n)^T. \end{aligned} \tag{3}$$

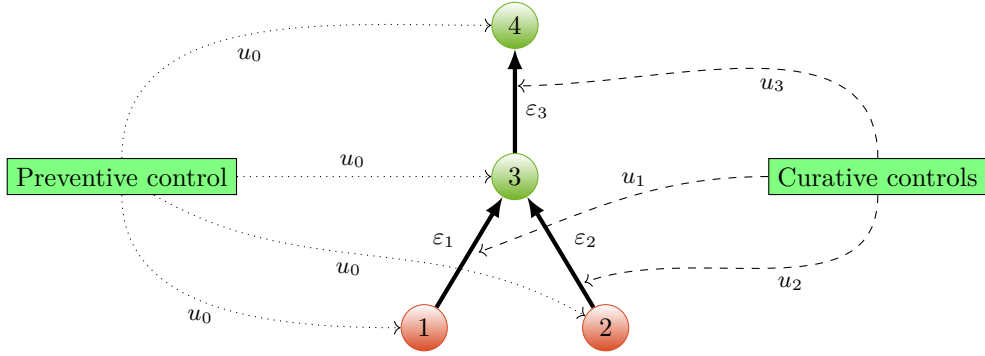


Figure 3: PCR network with 4 nodes and 3 weighted edges. Nodes 1 and 2 are of type (1), that is, are coupled with an instance of the PCR system such that $C_1 = 0$, whereas nodes 3 and 4 are of type (2), that is, are coupled with an instance of the PCR system such that $C_1 > 0$. Preventive control u_0 is exerted on each node, and curative controls (u_1, u_2, u_3) are exerted along each edge.

The definition of the matrix H means that individuals in daily behaviour q are not concerned with migrations in the network.

We allow the different instances of system (2) to admit different values of parameters, and we will especially focus on the effect of coupling PCR systems with different values of the parameter C_1 , identified previously as a bifurcation parameter.

Definition 1. We will call node of type (1) a node coupled with an instance of the PCR system such that $C_1 = 0$, and node of type (2) a node coupled with an instance of the PCR system such that $C_1 > 0$.

A PCR network is given by

$$\dot{X} = \Psi(t, X) + L(\varepsilon)HX, \quad (4)$$

where $\Psi(t, X) = (\psi^{(1)}(t, x_1), \dots, \psi^{(n)}(t, x_n))^T$. The above index in $\psi^{(i)}(t, x_i)$, $1 \leq i \leq n$, indicates that the values of parameter C_1 can differ from one node in the network to another. The next theorem, presented in [2], establishes a necessary and sufficient condition for the solution of the PCR network (4) to converge to the trivial equilibrium, which correspond to a global return of all individuals to the daily behaviour. It is also a condition for *synchronisation* in the network, since every node exhibits the same asymptotic dynamics under the considered assumptions.

Theorem 2. For any initial condition $X_0 \in (\mathbb{R}^+)^{4n}$, the Cauchy problem

$$\begin{cases} \dot{X} = \Psi(t, X) + L(\varepsilon)HX, & t > 0, \\ X(0) = X_0, \end{cases}$$

admits a unique solution whose components are non negative. The trivial equilibrium $0 \in \mathbb{R}^{4n}$ is the only equilibrium if and only if every node of type (1) is connected to at least one node of type (2) by an oriented chain. In that case, the trivial equilibrium is globally asymptotically stable.

Example 1. Consider the graph depicted in figure 3. The corresponding matrix of connectivity is given by

$$L(\varepsilon) = \begin{pmatrix} -\varepsilon_1 & 0 & 0 & 0 \\ 0 & -\varepsilon_2 & 0 & 0 \\ +\varepsilon_1 & +\varepsilon_2 & -\varepsilon_3 & 0 \\ 0 & 0 & +\varepsilon_3 & 0 \end{pmatrix}.$$

Furthermore, we have

$$L(\varepsilon) Hx = \begin{pmatrix} -\varepsilon_1 Hx_1 \\ -\varepsilon_2 Hx_2 \\ +\varepsilon_1 Hx_1 + \varepsilon_2 Hx_2 - \varepsilon_3 Hx_3 \\ +\varepsilon_3 Hx_3 \end{pmatrix}.$$

We assume that nodes 1 and 2 (depicted in red) are of type (1), that is, are coupled with an instance of the PCR system such that $C_1 = 0$, whereas nodes 3 and 4 (depicted in green) are of type (2), that is, are coupled with an instance of the PCR system such that $C_1 > 0$. According to Theorem 2, if $\varepsilon_1 = 0$ or $\varepsilon_2 = 0$, then the PCR network will exhibit a persistence of panic. At the opposite, if $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, then the solution of the PCR network will converge towards the trivial equilibrium $0 \in \mathbb{R}^{16}$, and the 4 nodes will synchronize.

As mentioned in our introduction, the evacuation of nodes of type (1) towards nodes of type (2) can be awkward in some particular situations. Furthermore, the catastrophic event can itself break some connexions in the network. For instance, an earthquake could damage a bridge connecting both sides of a river, or could hinder traffic flow on some avenue. In that case, the network would exhibit a persistence of panic. Now the question is to identify possible external actions in order to avoid such a situation.

2.3. Control problem

In order to avoid the possible persistence of panic pointed above, we consider a multiple control $u = (u_0, u_1, \dots, u_k)$ for the network problem (4), in which:

- u_0 models a preventive control introduced to facilitate the evolution from p to c , which appears in the system through the term $(C_1 + u_0)p$,
- (u_1, \dots, u_k) corresponds to a curative control introduced in order to increase the coupling strength along each edge in the network, with the terms $(\varepsilon_i + u_i)r_i$, $(\varepsilon_i + u_i)c_i$, $(\varepsilon_i + u_i)p_i$, $1 \leq i \leq k$.

Many concrete actions can be envisaged for those two types of controls. On the one hand, for the preventive control u_0 , awareness campaigns can be organised in the areas for which a high potential of catastrophic event is identified; those campaigns can be integrated to the educational programs to better prepare individuals to the known risks. On the other hand, curative controls u_1, \dots, u_k can be made through rescue services, in order to clear some hindered avenue, or to repair any urban installation. We indicate that it is a work in progress, in collaboration with geographers, to establish an exhaustive list of possible actions in concordance with the mathematical control functions (u_0, u_1, \dots, u_k) .

Next we consider the following general control problem

$$\dot{X} = \tilde{\Psi}(t, X, u) + \tilde{L}(\varepsilon, u) HX, \quad (5)$$

where $\tilde{\Psi}(t, X, u) = (\tilde{\psi}(t, x_1, u), \dots, \tilde{\psi}(t, x_n, u))^T$,

$$\tilde{\psi}(t, x, u) = \begin{pmatrix} -Br + \gamma(t)q(1-r) + F(r, c)rc + G(r, p)rp \\ B_1r - C_2c + (C_1 + u_0)p - F(r, c)rc + H(c, p)cp - \varphi(t)c(r+c+p+q) \\ B_2r + C_2c - (C_1 + u_0)p - G(r, p)rp - H(c, p)cp \\ -\gamma(t)q(1-r) \end{pmatrix},$$

and the matrix $\tilde{L}(\varepsilon, u)$ is defined by

$$\tilde{L}(\varepsilon, u) = L(\varepsilon_1 + u_1, \dots, \varepsilon_k + u_k). \quad (6)$$

The matrix $\tilde{L}(\varepsilon, u)$ is simply obtained by replacing ε_l by $\varepsilon_l + u_l$, $1 \leq l \leq k$ in $L(\varepsilon)$, that is, $\tilde{L}(\varepsilon, u) = L(\varepsilon + u)$. For brevity, we will denote this matrix \tilde{L} .

The next theorem shows that the solutions of the general control problem (5) corresponding to non-negative controls is global and admits non-negative components. In other words, the general control problem (5) is well-posed.

Theorem 3. *Let $u = (u_0, u_1, \dots, u_k)$ denote a multiple control with non-negative values. We assume that u is continuous in t and bounded. Then for any $X_0 \in (\mathbb{R}^+)^{4n}$, the Cauchy problem*

$$\begin{cases} \dot{X} = \tilde{\Psi}(t, X, u) + \tilde{L}(\varepsilon, u)HX, & t > 0, \\ X(0) = X_0, \end{cases}$$

admits a unique global solution whose components are non-negative and bounded.

Proof. Given any initial condition $X_0 \in (\mathbb{R}^+)^{4n}$, existence and uniqueness of a local in time solution $X(t, X_0)$ defined on some interval $[0, \tau]$ with $\tau > 0$, are a straightforward consequence of Cauchy-Lipschitz Theorem [9].

Non-negativity. In order to prove the non-negativity property, we introduce a modified problem as follows. For $\hat{x} = (\hat{r}, \hat{c}, \hat{p}, \hat{q}) \in \mathbb{R}^4$, define

$$\hat{\psi}(\hat{x}, u) = \left(\hat{\psi}_1(\hat{x}, u), \hat{\psi}_2(\hat{x}, u), \hat{\psi}_3(\hat{x}, u), \hat{\psi}_4(\hat{x}, u) \right)^T$$

by

$$\begin{aligned} \hat{\psi}_1(\hat{x}, u) &= +\gamma\hat{q}(1 - \hat{r}) - B\hat{r} + F(\hat{r}, \hat{c})\hat{r}\hat{c} + G(\hat{r}, \hat{p})\hat{r}\hat{p} \\ \hat{\psi}_2(\hat{x}, u) &= B_1\hat{r} - C_2\hat{c} + (C_1 + u_0)|\hat{p}| - F(\hat{r}, \hat{c})\hat{r}\hat{c} + H(\hat{c}, \hat{p})\hat{c}\hat{p} - \varphi\hat{c}(\hat{r} + \hat{c} + \hat{p} + \hat{q}) \\ \hat{\psi}_3(\hat{x}, u) &= B_2\hat{r} + C_2\hat{c} - (C_1 + u_0)\hat{p} - G(\hat{r}, \hat{p})\hat{r}\hat{p} - H(\hat{c}, \hat{p})\hat{c}\hat{p} \\ \hat{\psi}_4(\hat{x}, u) &= -\gamma\hat{q}(1 - \hat{r}), \end{aligned}$$

where we omit the dependence in t in order to lighten our notations.

Next, for $\hat{X} = (\hat{x}_1, \dots, \hat{x}_n)$, consider the modified network problem

$$\dot{\hat{x}}_i = \hat{\psi}(\hat{x}_i, u) + \sum_{\substack{k=1 \\ k \neq i}} \tilde{L}_{ik}H|\hat{x}_k| - \sum_{\substack{k=1 \\ k \neq i}} \tilde{L}_{ik}H\hat{x}_i, \quad 1 \leq i \leq n,$$

with the notation $|\hat{x}| = (|\hat{r}|, |\hat{c}|, |\hat{p}|, |\hat{q}|)$, and \tilde{L}_{ik} denoting the coefficient of indices (i, k) in the matrix $\tilde{L} = \tilde{L}(\varepsilon, u)$. For the same initial condition X_0 as in the non-modified problem, existence and uniqueness of a local in time solution $\hat{X}(t, X_0)$ defined on some interval $[0, \hat{\tau}]$ with $\hat{\tau} > 0$, are also obtained by Cauchy-Lipschitz Theorem.

Now, we recall that $H = \text{diag}\{1, 1, 1, 0\}$ (see Equation (3)), which means that the \hat{q}_i components, $1 \leq i \leq n$, are not coupled. It follows that

$$\hat{q}_i(t) = \hat{q}_i(0)e^{-\int_0^t \gamma(s)(1 - \hat{r}_i(s))ds}, \quad t \in [0, \hat{\tau}], \quad 1 \leq i \leq n, \quad (7)$$

which implies $\hat{q}_i(t) \geq 0$ for all $t \in [0, \hat{\tau}]$ and $1 \leq i \leq n$, since $\hat{q}_i(0) \geq 0$.

Let us next examine the non-negativity of other components \hat{r}_i, \hat{c}_i and \hat{p}_i , $1 \leq i \leq n$. We employ a truncation method presented in [15]. Define the real-valued function χ on \mathbb{R} by

$$\chi(s) = \begin{cases} 0 & \text{if } s > 0, \\ \frac{1}{2}s^2 & \text{if } s \leq 0. \end{cases}$$

The function χ is of class \mathcal{C}^1 on \mathbb{R} , with $\chi'(s) = 0$ if $s > 0$, $\chi'(s) = s$ if $s \leq 0$. Furthermore, it enjoys the properties

$$\chi(s) \geq 0, \quad \chi'(s) \leq 0, \quad 0 \leq \chi'(s)s = 2\chi(s), \quad \forall s \in \mathbb{R}. \quad (8)$$

Next we introduce for each i such that $1 \leq i \leq n$ the function ρ defined by

$$\rho(t) = \chi(\hat{r}_i(t)), \quad t \in [0, \hat{\tau}].$$

We have $\rho(0) = 0$, since $\hat{r}_i(0) \geq 0$. Moreover, ρ is continuously differentiable on $[0, \hat{\tau}]$, and we have

$$\rho'(t) = \dot{\hat{r}}_i(t)\chi'(\hat{r}_i(t)).$$

In the mean time, we have

$$\dot{\hat{r}}_i = \gamma\hat{q}_i(1 - \hat{r}_i) - B\hat{r}_i + F(\hat{r}_i, \hat{c}_i)\hat{r}_i\hat{c}_i + G(\hat{r}_i, \hat{p}_i)\hat{r}_i\hat{p}_i + \sum_{\substack{k=1 \\ k \neq i}} \tilde{L}_{ik}|\hat{r}_k| - \sum_{\substack{k=1 \\ k \neq i}} \tilde{L}_{ki}\hat{r}_i,$$

which leads to

$$\begin{aligned} \rho'(t) &= \gamma\hat{q}_i\chi'(\hat{r}_i) - \gamma\hat{q}_i\hat{r}_i\chi'(\hat{r}_i) - B\hat{r}_i\chi'(\hat{r}_i) + F(\hat{r}_i, \hat{c}_i)\hat{c}_i\hat{r}_i\chi'(\hat{r}_i) \\ &\quad + G(\hat{r}_i, \hat{p}_i)\hat{p}_i\hat{r}_i\chi'(\hat{r}_i) + \sum_{\substack{k=1 \\ k \neq i}} \tilde{L}_{ik}|\hat{r}_k|\chi'(\hat{r}_i) - \sum_{\substack{k=1 \\ k \neq i}} \tilde{L}_{ki}\hat{r}_i\chi'(\hat{r}_i). \end{aligned}$$

Since the control functions u_l , $0 \leq l \leq k$, are assumed to be non-negative, we have $\tilde{L}_{ij} \geq 0$ for all i, j such that $1 \leq i \leq n$ and $1 \leq j \leq n$. Moreover, the non-negativity of \hat{q}_i , combined with the properties (8) imply

$$\rho'(t) \leq 2M\rho(t),$$

where M is defined by

$$M = \sup_{t \in [0, \hat{\tau}]} [F(\hat{r}_i(t), \hat{c}_i(t))\hat{c}_i(t) + G(\hat{r}_i(t), \hat{p}_i(t))\hat{p}_i(t)] < +\infty.$$

Integrating the latter inequality leads to

$$\rho(t) \leq \rho(0)e^{2Mt}, \quad t \in [0, \hat{\tau}],$$

which implies $\rho(t) \leq 0$. Finally, we obtain $\rho(t) = 0$ for all $t \in [0, \hat{\tau}]$, thus $\hat{r}_i(t) \geq 0$ for all $t \in [0, \hat{\tau}]$. Applying the same method leads to $\hat{c}_i(t) \geq 0$ and $\hat{p}_i(t) \geq 0$ for all $t \in [0, \hat{\tau}]$.

The components of the solution $\hat{X}(t, X_0)$ of the modified problem are non-negative, so $\hat{X}(t, X_0)$ is also a solution of the initial non-modified problem on $[0, \hat{\tau}]$. By uniqueness, we have $\hat{X}(t, X_0) = X(t, X_0)$ on $[0, \tau] \cap [0, \hat{\tau}]$. Finally, it is easily seen that $\tau = \hat{\tau}$, thus we have proved the non-negativity of the components of $X(t, X_0)$ on $[0, \tau]$.

Boundedness. Introduce the function θ defined by

$$\theta(t) = \sum_{i=1}^n [r_i(t) + c_i(t) + p_i(t) + q_i(t)], \quad t \in [0, \tau].$$

It is differentiable on $[0, \tau]$. Since $\tilde{L}(\varepsilon, u)$ is a matrix whose sum of coefficients of each column, we have

$$\dot{\theta}(t) = -\varphi(t) \sum_{i=1}^n c_i(t) [r_i(t) + c_i(t) + p_i(t) + q_i(t)],$$

which implies that $\dot{\theta}(t) \leq 0$ for all $t \in [0, \tau]$. Consequently, we have $\theta(t) \leq \theta(0)$ for all $t > 0$, which proves the boundedness of the solution. The proof is complete. \square

The aim of introducing the multiple control $u = (u_0, u_1, \dots, u_k)$ is to limit panic at a reasonable level in complex networks for which we can predict, using Theorem 2, a persistence of panic on nodes of type (1). However, the latter proof shows that the solution of the control problem satisfies

$$q_i(t) > 0, \quad t > 0, \quad 1 \leq i \leq n,$$

for each initial condition such that $q_i(0) > 0$, $1 \leq i \leq n$ (see Equation (7)). This demonstrates that the trivial equilibrium $0 \in \mathbb{R}^{4n}$ cannot be reached in a finite time, when starting from such initial conditions. A more pragmatic goal would be to reach a neighbourhood \mathcal{N} of the trivial equilibrium. For instance, we can look for a multiple control so that the panic level is limited under a given proportion of the total population in a finite time.

2.4. Optimality criterion

In what follows, we denote by \mathcal{U} the set of admissible control functions, composed with Lebesgue-integrable functions $u = (u_0, u_1, \dots, u_k)$ for which there exists $T > 0$ such that u is defined on $[0, T]$ with values in $\tilde{K} = [0, 1]^{k+1}$. Note that T may depend on u .

Let $u = (u_0, u_1, \dots, u_k) \in \mathcal{U}$ denote an admissible control for the general control problem (5). Applying this multiple control, we aim to reach a neighbourhood \mathcal{N} of the trivial equilibrium $0 \in \mathbb{R}^{4n}$. Additionally, we would like to minimize on \mathcal{U} the performance index on Bolza type [5]:

$$\tilde{J}(X_0, u, T) = \alpha_0 T + \int_0^T \left[\alpha_1 \sum_{i=1}^n p_i^2(t) + \alpha_2 u_0^2(t) + \alpha_3 \sum_{l=1}^k u_l^2(t) \right] dt, \quad (9)$$

where the parameters α_i , $0 \leq i \leq 3$, are non-negative real coefficients such that

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 > 0,$$

introduced in order to weight the impact of each component of the multiple control. For instance, setting $\alpha_2 = 0$ means that the control function u_0 is cost-free. The expression of $\tilde{J}(X_0, u, T)$ corresponds to the wish to limit the level of panic during the control process, while mobilising the less rescue services to operate during the catastrophic event. If the coefficient α_0 is set to 1, whereas α_1 , α_2 and α_3 are set to 0, we simply obtain a time optimal problem.

Finally, we can state the optimal control problem for the complex network of non-identical PCR systems. The problem is to find a pair (X, u) defined on some interval $[0, T]$, such that

$$\begin{cases} \dot{X} = \tilde{\Psi}(t, X, u) + \tilde{L}(\varepsilon, u) HX, & t > 0, \\ X(0) = X_0, \\ X(T) \in \mathcal{N}, \\ \min_{u \in \mathcal{U}} \tilde{J}(X_0, u, T), \end{cases} \quad (10)$$

where X_0 is a given initial datum in $(\mathbb{R}^+)^{4n}$, and \mathcal{N} denotes a neighbourhood of the trivial equilibrium $0 \in \mathbb{R}^{4n}$. In the next section, we shall see that problem (10) can also be written as an optimal control problem for a complex network of controlled PCR systems.

§ 3. Existence of an optimal control for complex networks

This section is devoted to establishing an existence theorem for optimal control problems in complex networks. In order to generalize our study, so that it can be used for application to other problems, we consider abstract complex networks built with non-identical instances of a given controlled dynamical system, and we deduce from the general case the existence of a solution to the optimal control problem (10) corresponding to complex networks of PCR systems.

3.1. Preliminary results

Let us consider an optimal control problem of the general form

$$\begin{cases} \dot{x} = f(t, x(t), u(t)), & t_0 \leq t \leq T, \\ (t_0, T, x(t_0), x(T)) \in S, \\ \min_{u \in \mathcal{U}} J(x_0, u, T), \end{cases} \quad (11)$$

where f is a function defined on $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^M$ with values in \mathbb{R}^N and positive integers N, M ; S is a subset of \mathbb{R}^{2+2N} , and \mathcal{U} is the set of admissible control functions, composed with Lebesgue-integrable functions $u = (u_1, \dots, u_M)$ for which there exists $T > 0$ such that u is defined on $[t_0, T]$ with values in a subset K of \mathbb{R}^M . We assume that the performance criterion $J(x_0, u, T)$ can be written

$$J(x_0, u, T) = \phi(t_0, T, x_0, x(T)) + \int_{t_0}^T \delta(t, x(t), u(t)) dt, \quad (12)$$

where ϕ is a function defined on S and δ a function defined on $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^M$.

Introduce the class \mathcal{F} of feasible pairs (x_0, u) such that there exist $T > 0$ and an integrable function x defined on $[t_0, T]$, such that

$$x(t) = x_0 + \int_{t_0}^t f(t, x(t), u(t)) dt, \quad t_0 \leq t \leq T,$$

with $u \in \mathcal{U}$ and $(t_0, T, x(t_0), x(T)) \in S$. From [5] (Chapter III, Theorem 4.1), we have the following theorem.

Theorem 4. *Suppose that K is a compact subset of \mathbb{R}^M , and f is a continuous function such that there exist positive constants k_1, k_2 with*

$$\begin{aligned} |f(t, x, u)| &\leq k_1(1 + |x|), \\ |f(t, x, u) - f(t, y, u)| &\leq k_2|x - y|, \end{aligned} \quad (13)$$

for all $t \in \mathbb{R}$, $x, y \in \mathbb{R}^N$ and $u \in \mathcal{U}$. Assume furthermore that δ is continuous and

- (i) \mathcal{F} is not empty,
- (ii) S is compact and ϕ is continuous on S ,
- (iii) K is convex, $f(t, x, u) = \alpha(t, x) + \beta(t, x)u$, where α, β are defined on $\mathbb{R} \times \mathbb{R}^N$, and $\delta(t, x, \cdot)$ is convex on K .

Then there exists a solution (x_0, u) to the optimal control problem (11).

For all $x \in \mathbb{R}^N$, $|x|$ denotes the usual quadratic norm of x on \mathbb{R}^N , defined by

$$|x|^2 = \sum_{i=1}^N x_i^2.$$

The product $\beta(t, x)u$ has to be understood as the product of a matrix $\beta(t, x)$ of M columns and N lines with a vector $u \in \mathbb{R}^M$. It is worth noting that x and y in property (13) can be replaced by the solutions $x(t)$ and $y(t)$ starting from any initial conditions x_0, y_0 such that (x_0, u) and (y_0, u) belong to \mathcal{F} (see [5], Lemmas 5.2 and 5.4 in Chapter III). Finally, we mention that the compacity assumption on S can be weakened (see [5]).

3.2. Existence theorem for abstract complex networks

In this section, we aim to prove an existence theorem for an optimal control problem in complex networks of dynamical systems, built with multiple instances of problem (11).

Let us consider again a simple graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ made with a finite set $\mathcal{V} = \{1, \dots, n\}$ of n vertices, where n is a positive integer, and a finite set $\mathcal{E} = \{e_1, \dots, e_k\}$ of k weighted edges, with non-negative weights $\varepsilon_1, \dots, \varepsilon_k$. We suppose that each node is coupled with an instance of problem (11), and we allow additional controls along each edge of the network. Thus we consider the optimal control problem for the complex network of the following form:

$$\begin{cases} \dot{X} = g(t, X(t), U(t)) + \tilde{L}(\varepsilon, U)HX, & t_0 \leq t \leq T, \\ (t_0, T, X(t_0), X(T)) \in \tilde{S}, \\ \min_{U \in \tilde{\mathcal{U}}} \tilde{J}(X_0, U, T). \end{cases} \quad (14)$$

The latter optimal control problem (14) is a generalisation of problem (10). Here, we have introduced $X = (x_1, \dots, x_n)^T \in \mathbb{R}^{nN}$, where $x_i \in \mathbb{R}^N$ denotes the state of node i , $1 \leq i \leq n$; $U = (u_1, \dots, u_n, u_{n+1}, \dots, u_{n+k})^T \in \mathbb{R}^{nM+k}$, where $u_i \in \mathbb{R}^M$ denotes the controls exerted on node i , $1 \leq i \leq n$, and $u_{n+l} \in \mathbb{R}$, $1 \leq l \leq k$, denotes the control exerted along each edge $e_l \in \mathcal{E}$. Note that for $1 \leq i \leq n$, we have $u_i \in \mathbb{R}^M$, whereas $u_{n+l} \in \mathbb{R}$ for $1 \leq l \leq k$. The function g is defined by

$$g(t, X, U) = (f^{(1)}(t, x_1, u_1), \dots, f^{(n)}(t, x_n, u_n))^T,$$

where the above index in $f^{(i)}(t, x_i, u_i)$ indicates that the complex network is composed with non-identical instances of problem (11), that is, instances for which parameters in the function f may differ from one node to another. The matrix of connectivity $\tilde{L}(\varepsilon, U)$ is defined as in equation (6), and the matrix H is a diagonal matrix of order N , whose diagonal coefficient H_{ii} is equal to 0 or 1, whether the corresponding component x_i is coupled; as in equation (3), we have

$$HX = (Hx_1, \dots, Hx_n)^T.$$

Next, \tilde{S} is a subset of \mathbb{R}^{2+2nN} , and \tilde{J} is a performance index defined by

$$\tilde{J}(X_0, U, T) = \sum_{i=1}^n J^{(i)}(x_i(0), u_i, T) + \int_{t_0}^T \tilde{\delta}(t, X, u_{n+1}, \dots, u_{n+k}) dt, \quad (15)$$

where $\tilde{\delta}$ is defined on $\mathbb{R} \times \mathbb{R}^{nN} \times \mathbb{R}^k$, which means that \tilde{J} is obtained by superposition of the performance indices $J^{(i)}$ of each node i , and a second term which takes into account the controls exerted along each edge in the network. Finally, $\tilde{\mathcal{U}}$ denotes the set of Lebesgue-integrable functions

$$U = (u_1, \dots, u_n, u_{n+1}, \dots, u_{n+k})$$

for which there exists $T > 0$ such that U is defined on $[0, T]$ with values in a subset \tilde{K} of \mathbb{R}^{nM+k} .

Theorem 5. *Suppose that the hypotheses of Theorem 4 hold for each instance of problem (11). Furthermore, assume that \tilde{S} is compact, \tilde{K} is compact and convex, and $\tilde{\delta}(t, X, \cdot)$ is continuous and convex on \tilde{K} . Then the optimal control problem for the complex network (14) admits a solution (X_0, U) .*

Proof. The optimal control problem (14) for the complex network can be written

$$\begin{cases} \dot{X} = \tilde{f}(t, X(t), U(t)), & t_0 \leq t \leq T, \\ (t_0, T, X(t_0), X(T)) \in \tilde{S}, \\ \min_{u \in \tilde{\mathcal{U}}} \tilde{J}(X_0, U, T), \end{cases} \quad (16)$$

with the following notations:

$$\begin{aligned}\tilde{f}(t, X(t), U(t)) &= g(t, X(t), U(t)) + \tilde{L}(\varepsilon, U)HX, \\ \tilde{J}(X_0, U, T) &= \tilde{\phi}(t_0, T, X_0, X(T)) + \int_{t_0}^T \Delta(t, X_0, U)dt,\end{aligned}$$

where

$$\begin{aligned}\tilde{\phi}(t_0, T, X_0, X(T)) &= \sum_{i=1}^n \phi^{(i)}(t_0, T, x_i(0), x_i(T)), \\ \Delta(t, X_0, U) &= \tilde{\delta}(t, X_0, u_{n+1}, \dots, u_{n+k}) + \sum_{i=1}^n \delta^{(i)}(t, x_i, u_i).\end{aligned}$$

Let us check that the hypotheses of Theorem 4 are satisfied for the rewritten complex network problem (16). First, the set \tilde{K} is assumed to be compact. Next, the continuity of $f^{(i)}$ for $1 \leq i \leq n$ implies the continuity of \tilde{f} , as well as the properties (13). The continuity of $\tilde{\delta}$ and $\delta^{(i)}$, $1 \leq i \leq n$, implies that of Δ .

The existence of a feasible pair for each instance of problem (11) guaranty the existence of a feasible pair for problem (16), so hypothesis (i) in Theorem 4 is satisfied. The continuity of $\tilde{\phi}$ follows from that of $\phi^{(i)}$, $1 \leq i \leq n$, thus hypothesis (ii) in Theorem 4 is satisfied as well.

It remains to check hypothesis (iii) for the rewritten network problem (16). Each instance of problem (11) satisfies hypothesis (iii), thus we have

$$f^{(i)}(t, x_i, u_i) = \alpha^{(i)}(t, x_i) + \beta^{(i)}(t, x_i)u_i, \quad 1 \leq i \leq n.$$

Let us introduce

$$A(t, X) = (\alpha^{(i)}(t, x_i))_{1 \leq i \leq n}, \quad B(t, X) = (\beta^{(i)}(t, x_i))_{1 \leq i \leq n}.$$

It follows that we can write

$$\begin{aligned}\tilde{f}(t, X, U) &= \tilde{A}(t, X) + \tilde{B}(t, X), \\ \tilde{A}(t, X, U) &= A(t, X) + L(\varepsilon)HX, \\ \tilde{B}(t, X, U)U &= B(t, X)U + L(U)HX.\end{aligned}$$

Finally, the convexity of $\Delta(t, X, \cdot)$ follows from that of $\tilde{\delta}(t, X, \cdot)$ and $\delta(t, x_i, \cdot)$, $1 \leq i \leq n$. The proof is complete. \square

In the next section, we apply Theorem 5 to the optimal problem (10) for complex networks of PCR systems stated above. Obviously, Theorem 5 can be applied to many other problems, e.g. air traffic networks [11].

3.3. Existence of an optimal control for complex networks of PCR systems

Now we are ready to prove the existence of an optimal control for problem (10). Let X_0 denote any initial datum in $(\mathbb{R}^+)^{4n}$, and \mathcal{N} a compact neighbourhood of the trivial equilibrium $0 \in \mathbb{R}^{4n}$. We introduce $\tilde{S} = \{0\} \times [0, \theta] \times \{X_0\} \times \mathcal{N}$, with $\theta > 0$ sufficiently large, $K = [0, 1]$ and $\tilde{K} = K^{k+1}$. We obtain the following existence theorem.

Corollary 1. *The optimal control problem (10) for the complex network of PCR systems admits a solution (X_0, u^*) , with u^* minimizing (9).*

Proof. First, we show that the optimal control problem (4) can be rewritten as an optimal control for multiple instances of controlled PCR systems. Indeed, it suffices to introduce

$$\psi(t, x, u_0) = \begin{pmatrix} \gamma(t)q(1-r) - Br + F(r, c)rc + G(r, p)rp \\ B_1r - C_2c + (C_1 + u_0)p - F(r, c)rc + H(c, p)cp - \varphi(t)c(r + c + p + q) \\ B_2r + C_2c - (C_1 + u_0)p - G(r, p)rp - H(c, p)cp \\ -\gamma(t)q(1-r) \end{pmatrix},$$

and

$$J(x_0, u_0, T) = \frac{\alpha_0}{n}T + \int_0^T \delta(t, x(t), u_0(t))dt, \quad \delta(t, x(t), u_0(t)) = \alpha_1 p^2(t) + \alpha_2 u_0^2(t).$$

Now we prove that the hypotheses of Theorem 4 hold for each instance of the previous system. By virtue of Theorem 3, it is seen that the solution $X(t)$ of problem (4) starting from any initial condition $X_0 \in (\mathbb{R}^+)^{4n}$, associated with any control $u \in \mathcal{U}$, evolves in a compact set $M \subset (\mathbb{R}^+)^{4n}$. This implies the boundedness of the solution $x(t)$ of the Cauchy problem

$$\begin{cases} \dot{x} = \psi(t, x, u_0), & t > 0, \\ x(0) = x_0, \end{cases}$$

corresponding to any control u_0 with values in K . Since K is compact as well, we obtain property (13), by applying the mean value theorem, ψ being of class \mathcal{C}^1 . Next, we easily prove that \mathcal{F} is non-empty, since (x_0, η) with $\eta \in]0, 1]$, is a feasible pair. Assumption (ii) in Theorem 4 hold as well, since $\tilde{S} = \{0\} \times [0, \theta] \times \{X_0\} \times \mathcal{N}$ is compact and

$$\phi(t_0, T, x_0, x_T) = \frac{\alpha_0}{n}T.$$

Finally, assumption (iii) is directly verified, since $K = [0, 1]$ is convex, and $\delta(t, x(t), \cdot)$ is convex.

It remains to introduce

$$\tilde{\delta}(t, X_0, u_1, \dots, u_k) = \alpha_3 \sum_{l=1}^k u_l^2(t).$$

The convexity of $\tilde{K} = [0, 1]^{k+1}$ and $\tilde{\delta}(t, X_0, \cdot)$ are clear, and this achieves the proof. \square

3.4. Synchronisation under control in PCR networks

In this section, we briefly show how control can be used to achieve synchronisation in the PCR network (4). Let us consider a PCR network built with non-identical instances of PCR systems (2). Assume that there exists at least one node (x) of type (1), that is, with $C_1 = 0$ (see Definition 1), which is not evacuated towards any node of type (2). By virtue of Theorem 2, the corresponding PCR network will exhibit a persistence of panic on node (x), whereas nodes of type (2) will converge to the trivial equilibrium corresponding to a return to daily behaviour. In other words, one can find at least two nodes in the network which do not present the same dynamics.

Now, assume that a multiple control (u_0, u_1, \dots, u_k) is applied on the network, and consider the controlled PCR network (10). By virtue of Corollary 1, for any compact neighbourhood \mathcal{N} of the trivial equilibrium, and any given initial datum $X_0 \in (\mathbb{R}^+)^{4n}$, there exists such a multiple control (u_0, u_1, \dots, u_k) satisfying $X(T) \in \mathcal{N}$, with $T > 0$ (depending on \mathcal{N}), and minimizing the performance index given by (9). Generalising this situation leads to the following definition, which extends the classical definition of synchronisation in complex networks [1, 8].

Definition 2. Consider an optimal control problem (14) for an abstract complex network made of n nodes (x_i) , $1 \leq i \leq n$.

If for every initial condition X_0 , there exists a control U defined on $[0, T]$, such that

$$x_i(T) = x_j(T),$$

for any pair of nodes (x_i, x_j) , we say that the network synchronises under control.

If for every $\varepsilon > 0$ and every initial condition X_0 , there exists a control U defined on $[0, T]$, such that

$$|x_i(T) - x_j(T)| < \varepsilon,$$

for any pair of nodes (x_i, x_j) , we say that the network asymptotically synchronises under control.

By virtue of Corollary 1, we obtain the following proposition.

Proposition 1. The controlled PCR network (5) asymptotically synchronises under control.

As mentioned previously, the trivial equilibrium $0 \in \mathbb{R}^{4n}$ cannot be reached in finite time for some initial condition X_0 such that $\sum_{i=1}^n q_i(0) > 0$. This means that the controlled PCR network (5) can only synchronise asymptotically under control.

More generally, it is a work in progress to identify minimal hypotheses on the dynamics of a given complex network, to guaranty that it can synchronise or asymptotically synchronise under control.

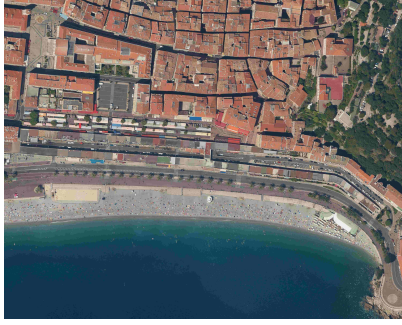
§ 4. Numerical simulations

In this final section, we present a selection of numerical simulations for the optimal control problem of PCR networks (10). By virtue of Corollary 1, the existence of a solution to that problem is guaranteed. However, we aim to measure the effect of each control on the panic level. Is the preventive control u_0 more efficient than the curative control (u_1, \dots, u_n) ? Does time optimal control imply a high mobilisation of control along the edges of the network? Is it possible to control the panic level without controlling a part of the network, or is it necessary to act on each node and each edge? These are some of the questions we investigate in what follows.

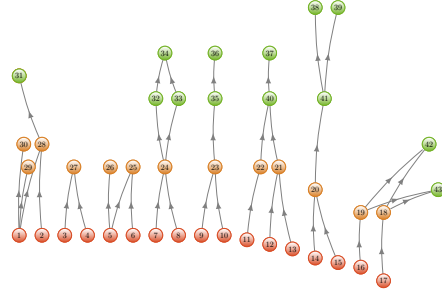
4.1. Risk of tsunami on the Mediterranean coast

It is known [6] that a submarine fracture, at about 80 km away from the french Mediterranean littoral, potentially generates seismic events of variable intensity, which in turn can provoke tsunamis of low intensity, which roll over the coast after a very short time. The french city of Nice is particularly exposed to that risk. Nevertheless, the size of the wave is expected to remain under a high-risk level, which mean that the panic provoked by the arrival of the tsunami would represent the major danger.

Figure 4(a) shows an aerial picture of the old city-centre of Nice, bordering the littoral, divided into many small beaches, which are separated from the urban installations by the famous avenue *Promenade des Anglais*. The graph which is depicted in figure 4(b) is the result of a collaboration with geographers [10], [4], and represents the network of high-density places and their possible connexions. We remark that this graph can be split into a finite number of small connected graphs, which have a structure similar to that of the graph presented in figure 3. For that reason, we propose to test our model on such a 4 nodes network. The red nodes (1) and (2) are coupled with an instance of the PCR system such that $C_1 = 0$, since they correspond to beach places where the risk of panic is expected to be high, while the green nodes (3) and (4) are coupled with an instance of the PCR system with $C_1 > 0$, since they correspond to protected places.



(a)



(b)

Figure 4: (a) Aerial picture of the city of Nice, France. The city-centre borders the littoral, divided into many small beaches, which are separated from the urban installations by the *Promenade des Anglais*. (b) Complex network of high-density places and their possible connexions.

Table 1: Parameters of the 4 nodes PCR network depicted in figure 3 for the numerical simulations.

Parameter	Value
B_1	0.3
B_2	0.3
C_1 on nodes 3, 4	0.1
C_1 on nodes 1, 2	0.0
C_2	0.2
ε_1	0.1
ε_2	0
ε_3	0.1

We emphasize that the null value of ε_2 implies that node (2) is not evacuated. By virtue of Theorem 2, we can deduce that in absence of control, the panic level on node (2) will persist. On the other hand, by virtue of Proposition 1, we know that the network asymptotically synchronises under control. The values of other parameters are given in table 1. Those values have been chosen with a qualitative approach, since the fine calibration of the model is a work in progress [13]. The imitation functions are given by Holling type [7] non linear terms:

$$\begin{aligned}
 F(r, c)rc &= \frac{-\beta rc}{1+r} + \frac{\beta rc}{1+c} \\
 G(r, p)rp &= \frac{-\beta rp}{1+r} + \frac{\beta rp}{1+p} \\
 H(c, p)cp &= \frac{-\beta cp}{1+c} + \frac{\beta cp}{1+p}.
 \end{aligned}$$

Here, we assume that the imitation process can be neglected with respect to the evolution process, thus we set $\beta = 10^{-3}$. Finally, the initial condition is chosen to model the fact that all individuals are in the daily behaviour when the catastrophic event occurs. Thus we set

$$r_i(0) = 0, \quad c_i(0) = 0, \quad p_i(0) = 0, \quad q_i(0) = 1, \quad 1 \leq i \leq n.$$

Numerical simulations have been performed with the free and open-source software BOCOP developed by the INRIA [12], in a Debian/GNU-Linux environment. In the next sections, we present the results of 7 different choices of parameters α_i , $0 \leq i \leq 3$, involved in the performance criterion \tilde{J} given by (9), and p_f , which represents the panic level expected under the effect of the control (u_0, u_1, u_2, u_3) (see table 2).

Table 2: Values of parameters α_i , $0 \leq i \leq 3$, involved in the performance criterion \tilde{J} given by (9), and of the final panic level p_f , for the 7 numerical simulations; numerical results for the final time T_f ; numerical results for the minimum \tilde{J}_{min} of the performance criterion.

Scenario	1	2	3	4	5	6	7
α_0	0	0	0	0	0	1	1
α_1	1	1	1	1	1	1	1
α_2	1	0	1	$u_0 = 0$	$u_0 = 0$	1	0
α_3	1	1	0	1	1	1	0
p_f	10%	10%	10%	10%	5%	5%	5%
T_f	18.513	16.529	18.769	64.11	94.17	17.94	14.39
\tilde{J}_{min}	2.918	0.285	2.910	16.152	16.376	23.383	14.769

4.2. Comparison of preventive control and curative controls

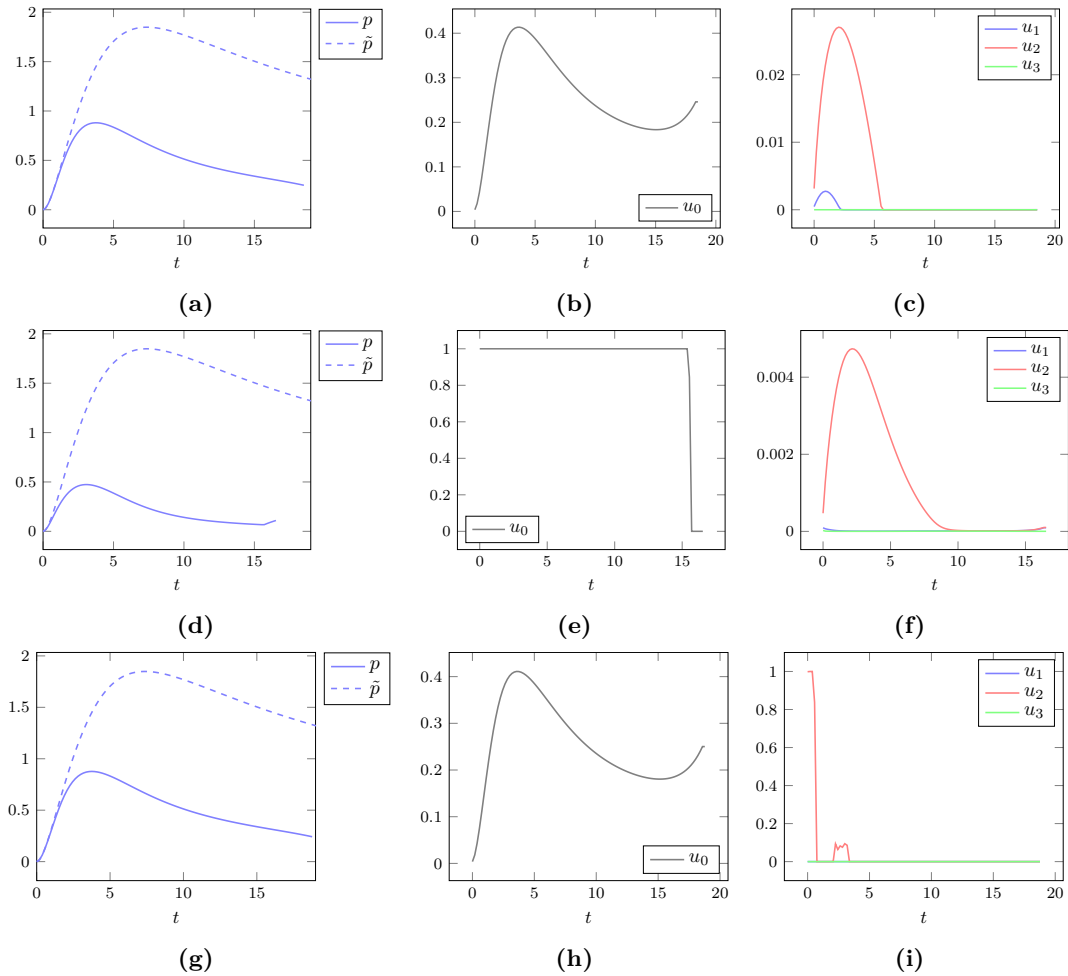


Figure 5: Control of panic in a PCR network. The panic level \tilde{p} corresponds to the situation without control (dashed lines in a, d, g), whereas the panic level p corresponds to the situation with control (continuous lines in a, d, g). The first line (a, b, c) shows the numerical results for scenario 1, the second line (d, e, f) for scenario 2 and the third line for scenario 3.

We begin by assuming that parameter α_0 is null in the performance criterion given by (9), which means that time is not taken into account in the minimizing process. We would like to compare the efficiency of the preventive control u_0 which is exerted on each node, and the curative control (u_1, u_2, u_3) with is exerted along the edges of the network.

The first scenario corresponds to the situation with $\alpha_1 = \alpha_2 = \alpha_3$, which takes into account the total number of individuals in panic during the event, and does not favour one of both controls u_0 and (u_1, u_2, u_3) . We set the final panic level at 10%. Numerical results for this first scenario are shown in figure 5 (a, b, c). We remark that the control u_0 seems to be efficient, whereas the control u_3 is weak. The minimum of the performance criterion is $\tilde{J}_{min} \simeq 2.918$ (see table 2, last line).

In the second scenario, we suppose that $\alpha_1 = \alpha_3 = 1$, whereas $\alpha_2 = 0$, which means that the preventive control u_0 is cost-free. The numerical results are shown in figure 5 (d, e, f), and the minimum of the performance criterion is $\tilde{J}_{min} \simeq 0.285$. Not surprisingly, we remark that u_0 is greater than in scenario 1.

In the third scenario, we investigate the effect of setting $\alpha_3 = 0$ and $\alpha_1 = \alpha_2 = 1$, which means that the curative control (u_1, u_2, u_3) is cost-free. The numerical results are shown in figure 5 (g, h, i), and the minimum of the performance criterion is $\tilde{J}_{min} \simeq 2.910$. We observe that u_0 has a similar shape as in scenario 1, whereas the curative control u_2 exerted along the edge (2, 3), which is the only possible evacuation path to escape from node (2), reveals a *bang-bang* shape with greater values than in scenarios 1 and 2.

For those three first scenarios, we remark that the final times are approximately equal to each other, since $T_f \in [16, 19]$ (see table 2). It seems reasonable to conclude that the preventive control u_0 is more efficient than the curative control (u_1, u_2, u_3) .

4.3. Absence of preventive control

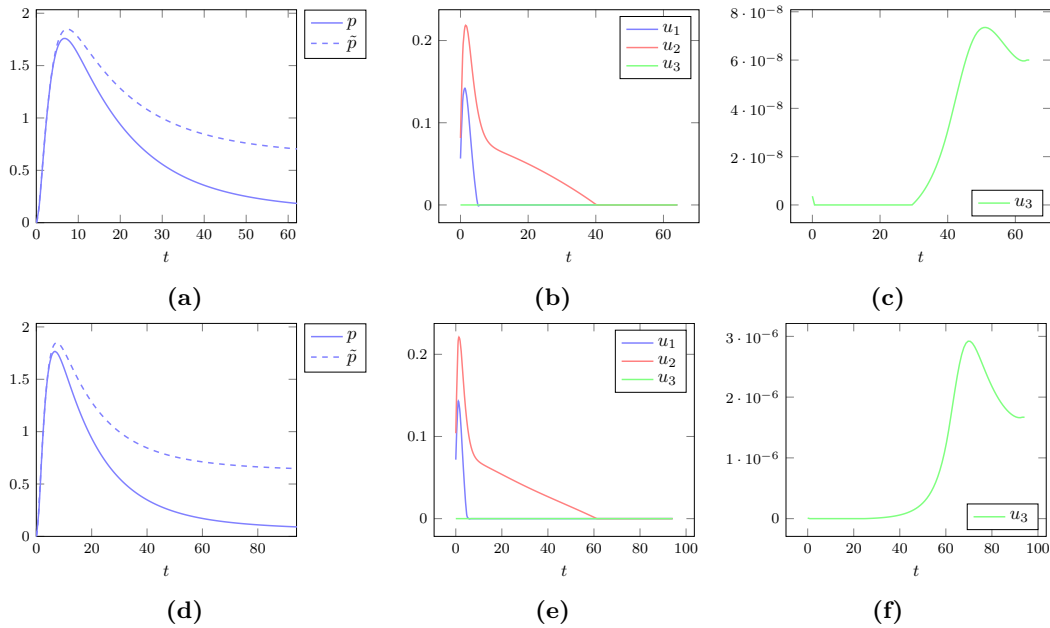


Figure 6: Control of panic in a PCR network without preventive control. The panic level \tilde{p} corresponds to the situation without control (dashed lines in a, d), whereas the panic level p corresponds to the situation with control (continuous lines in a, d). The first line (a, b, c) shows the numerical results for scenario 4, the second line (d, e, f) for scenario 5.

Next, in order to underpin the latter observation, we explore the effect of setting $u_0 = 0$ in the control problem (5). Corollary 1 and Proposition 1 still apply and guaranty the existence of a control (u_1, u_2, u_3) such that the PCR network asymptotically synchronises. Thus we test two novel scenarios with $\alpha_1 = \alpha_3 = 1$. We set the final panic level at 10% for scenario 4, and at 5% for scenario 5. Numerical results are shown in figure 6 (a, b, c) for scenario 4, and figure 6 (d, e, f) for scenario 5. We observe that the values of the final times are much greater than in scenarios 1, 2, 3. The control u_2 exerted along the edge (2, 3) is greater than the control u_1 exerted along the edge (1, 3), which is not surprising, since $\varepsilon_2 = 0 < \varepsilon_1$. In the mean time, the minimum \tilde{J}_{min} of the performance criterion, given in table 2 (last line), is much greater than in scenarios 1, 2, 3 as well, which seems to corroborate the affirmation that preventive control u_0 should be favoured. Roughly speaking, the desired final state can be reached in absence of preventive control, but it takes more time, with a greater cost.

4.4. Time optimal problem

We finish this section with two additional scenarios taking into account the minimization of time in the process of control. Thus we set $\alpha_0 = 1$. The corresponding numerical results are presented in figure 7. In scenario 6, we suppose $\alpha_1 = \alpha_2 = \alpha_3 = 1$, and $p_f = 5\%$. We remark that the final time T_f is lesser than in scenario 1, despite the final panic level p_f is more exigent. In counterpart, the minimum \tilde{J}_{min} of the performance criterion is much greater (see table 2).

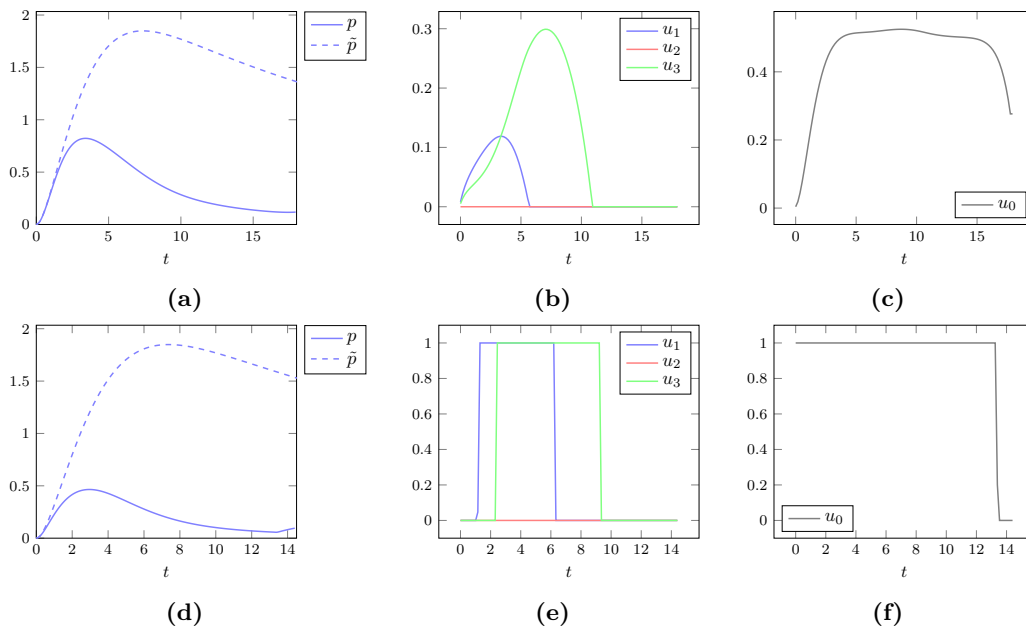


Figure 7: Time optimal control of panic in a PCR network. The panic level \tilde{p} corresponds to the situation without control (dashed lines in a, d), whereas the panic level p corresponds to the situation with control (continuous lines in a, d). The first line (a, b, c) shows the numerical results for scenario 6, the second line (d, e, f) for scenario 7.

In scenario 7, we set $\alpha_1 = 1$, and meanwhile $\alpha_2 = \alpha_3 = 0$, which means that the performance criterion takes into account the panic level during the whole event, but that both preventive control u_0 and curative controls (u_1, u_2, u_3) are cost-free. We observe that the optimal controls present a *bang-bang* structure (see figure 7, second line). It is remarkable that the control u_2 exerted along the edge (2, 3), expected to act as an evacuation path, admits low values. This

might be interpreted again in the following sense: it seems cost-less to apply a preventive control on high-risk areas, than mobilising curative controls in order to displace individuals.

§ 5. Conclusion and perspectives

In this paper, we have presented novel results for the problem of controlling panic in complex networks of dynamical systems for an original behavioural model. The existence of a minimizing control has been rigorously proved, in a sufficiently wide framework, which allows application to other problems. Numerical simulations tend to show that preventive control should be favoured with respect to curative control, which can represent a precious help for decision makers.

In a future work, we aim to generalize our study to the optimal control of complex networks of partial differential equations of parabolic type, in order to take into account in a hybrid model the superposition of local diffusion and long distance diffusion.

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