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CHOQUET-MONGE-AMPÈRE CLASSES

VINCENT GUEDJ*, SIBEL SAHIN** AND AHMED ZERIAHI*

ABSTRACT. We introduce and study Choquet-Monge-Ampère classes on compact Kähler manifolds. They consist of quasi-plurisubharmonic functions whose sublevel sets have small enough asymptotic Monge-Ampère capacity. We compare them with finite energy classes, which have recently played an important role in Kähler Geometry.

INTRODUCTION

Let (X, ω) be a compact Kähler manifold of complex dimension $n \geq 1$. Recall that a quasi-plurisubharmonic function (qpsh for short) on X is an upper semi-continuous function $\varphi : X \rightarrow \mathbb{R} \cup \{-\infty\}$ which is locally the sum of a plurisubharmonic and a smooth function. We write $\varphi \in PSH(X, \omega)$ if φ is qpsh and $\omega_\varphi := \omega + dd^c\varphi$ is a positive current. Here $d = \partial + \bar{\partial}$ and $d^c = \frac{i}{2\pi}(\bar{\partial} - \partial)$ are both real operators, so that $dd^c = \frac{i}{\pi}\partial\bar{\partial}$.

There are various ways to measure the singularities of such functions. We can measure the asymptotic size of the sublevel sets $(\varphi < -t)$ as $t \rightarrow +\infty$ through the Monge-Ampère capacity, which is defined by

$$C_\omega(K) := \sup \left\{ \int_K MA(u); u \in PSH(X, \omega), -1 \leq u \leq 0 \right\}.$$

where $MA(u) = \omega_u^n / \int_X \omega^n$ is a well-defined probability measure [BT82].

We can also consider the non-polar part of the Monge-Ampère measure

$$MA(\varphi) := \lim_{j \rightarrow +\infty} \mathbf{1}_{\{\varphi > -j\}} MA(\max(\varphi, -j)).$$

Following [GZ07] we say that $\varphi \in \mathcal{E}(X, \omega)$ if $MA(\varphi)$ is a probability measure and set

$$\mathcal{E}^p(X, \omega) := \{\varphi \in \mathcal{E}(X, \omega) \mid \varphi \in L^p(MA(\varphi))\}.$$

The finite energy classes $\mathcal{E}^p(X, \omega)$ have played an important role in recent applications of pluripotential theory to Kähler geometry (see for ex. [EGZ08, EGZ09, BBGZ13, BEG13]). While their local analogues can be well understood by measuring the size of the sublevel sets, this is not the case in the compact setting (see [BGZ08, BGZ09]).

In this article we introduce and initiate the study of the Choquet-Monge-Ampère classes

$$Ch^p(X, \omega) := \left\{ \varphi \in PSH(X, \omega) \mid \int_0^{+\infty} t^{p+n-1} C_\omega(\{\varphi \leq -t\}) dt < +\infty \right\}.$$

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We show in Theorem 2.7 that an ω -psh function φ belongs to $\mathcal{C}h^p(X, \omega)$ if and only if it has finite Choquet energy

$$\text{Ch}_p(\varphi) := \int_X (-\varphi)^p [(-\varphi)\omega + \omega_\varphi]^n < +\infty.$$

We establish in Corollary 2.8 that Choquet classes compare to finite energy classes as follows,

$$\mathcal{E}^{p+n-1}(X, \omega) \subset \mathcal{C}h^p(X, \omega) \subset \mathcal{E}^p(X, \omega).$$

These classes therefore coincide in dimension $n = 1$, but the inclusions are strict in general when $n \geq 2$: the first inclusion is sharp for functions with divisorial singularities (Proposition 3.11), while the second inclusion is sharp for functions with compact singularities (Proposition 3.8).

We briefly describe the range of the complex Monge-Ampère operator acting on Choquet classes in Proposition 3.3 and Proposition 3.6. This description is not as complete as the corresponding one for finite energy classes in [GZ07]; Choquet classes are rather meant to become a useful intermediate tool in the analysis of the complex Monge-Ampère operator.

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1. CHOQUET CLASSES

1.1. Choquet capacity.

1.1.1. *Generalized capacities.* Let Ω be a Hausdorff locally compact topological space which we assume is σ -compact. We denote by 2^Ω the set of all subsets of Ω . A set function $c : 2^\Omega \rightarrow \bar{\mathbb{R}}^+ := [0, +\infty]$ is called a capacity on Ω if it satisfies the following four properties:

- (i) $c(\emptyset) = 0$;
- (ii) c is monotone, i.e. $A \subset B \subset \Omega \implies 0 \leq c(A) \leq c(B)$;
- (iii) if $(A_n)_{n \in \mathbb{N}}$ is a non-decreasing sequence of subsets of Ω , then

$$c(\cup_n A_n) = \lim_{n \rightarrow +\infty} c(A_n) = \sup_n c(A_n);$$

- (iv) if (K_n) is a non-increasing sequence of compact subsets of Ω ,

$$c(\cap_n K_n) = \lim_n c(K_n) = \inf_n c(K_n).$$

A capacity c is said to be a Choquet capacity¹ if it is subadditive, i.e. if it satisfies the following extra condition

- (v) if $(A_n)_{n \in \mathbb{N}}$ is any sequence of subsets of Ω , then

$$c(\cup_n A_n) \leq \sum_n c(A_n),$$

Capacities are usually first defined for Borel subsets and then extended to all sets by building the appropriate outer set function.

¹We impose the subadditivity in our definition. Although not standard this property is satisfied for most capacities of interest for us.

Example 1.1. Let \mathcal{M} be a family of Borel measures on Ω . The set function defined on any Borel subsets $A \subset \Omega$ by the formula

$$c_{\mathcal{M}}(A) := \sup\{\mu(A); \mu \in \mathcal{M}\}$$

is a precapacity on Ω . It is called the upper envelope of \mathcal{M} . Observe that this precapacity need not be additive. The precapacity $c_{\mathcal{M}}$ need not be outer regular either unless \mathcal{M} is a finite set. However if \mathcal{M} is a compact set for the weak*-topology then $c_{\mathcal{M}}^*$ is a Choquet capacity.

If c is a Choquet capacity on Ω , every Borel subset $B \subset \Omega$ satisfies

$$c(B) = \sup\{c(K); K \text{ compact } K \subset B\}.$$

This is a special case of Choquet's capacitability theorem.

1.1.2. *Monge-Ampère capacities.* Let (X, ω) be a compact Kähler manifold of dimension n . We let $L^p(X) = L^p(X, \mathbb{R}, dV)$ denote the Lebesgue space of real valued measurable functions which are L^p -integrable with respect to a fixed volume form dV .

We denote by $PSH(X, \omega)$ the set of ω -plurisubharmonic functions: these are functions $\varphi : X \rightarrow \mathbb{R} \cup \{-\infty\}$ which are locally the sum of a plurisubharmonic and a smooth function, and such that $\omega_{\varphi} := \omega + dd^c\varphi \geq 0$ in the sense of currents. Recall that for all $p \geq 1$,

$$PSH(X, \omega) \subset L^p(X).$$

The Monge-Ampère capacity C_{ω} is defined for Borel sets $K \subset X$ by

$$C_{\omega}(K) := \sup \left\{ \int_K MA(u); u \in PSH(X, \omega), -1 \leq u \leq 0 \right\}.$$

where $MA(u) = \omega_u^n / \int_X \omega^n$ is a well-defined probability measure [BT82].

It follows from the work of Bedford-Taylor [BT82] that C_{ω} is a Choquet capacity. We refer the reader to [GZ05] for basics on $PSH(X, \omega)$ and C_{ω} .

1.2. **The Choquet integral.** Let C be a Choquet capacity on X , a compact topological space.

Definition 1.2. The Choquet's integral of a non negative Borel function $f : X \rightarrow \mathbb{R}^+$ is

$$\int_X f dC := \int_0^{+\infty} C(\{f \geq t\}) dt.$$

A change of variables shows that for any exponent $p \geq 1$,

$$\int_X f^p dC := p \int_0^{+\infty} t^{p-1} C(\{f \geq t\}) dt.$$

Observe that if $K \subset X$ is a Borel set then

$$\int_X \mathbf{1}_K dC = C(K).$$

Definition 1.3. We set, for $p \geq 1$,

$$\mathcal{L}^p(X, C) := \{f \in \mathcal{B}(X, \mathbb{R}); \|f\|_{L^p(X, C)} < +\infty\},$$

where $\mathcal{B}(X, \mathbb{R})$ is the space of real-valued Borel functions in X and

$$\|f\|_{L^p(X, C)} := \left(\int_X |f|^p dC \right)^{1/p}.$$

Lemma 1.4. *Let $f, g \in \mathcal{L}^p(X, C)$ and $\lambda \in \mathbb{R}$. Then*

1. *If $0 \leq f \leq g$ then $\int_X f dC \leq \int_X g dC$.*
2. *$\|\lambda f\|_{L^p(X, C)} = |\lambda| \|f\|_{L^p(X, C)}$.*
3. *$\|f + g\|_{L^p(X, C)} \leq 2(\|f\|_{L^p(X, C)} + \|g\|_{L^p(X, C)})$,*

In particular $\mathcal{L}^p(X, C)$ is a vector space. The above quasi-triangle inequality defines a uniform structure, which is furthermore metrizable for general reasons [BOUR], i.e. we can equip $\mathcal{L}^p(X, C)$ with an invariant metric ρ such that a sequence (f_j) converges to f for the metric ρ if and only if $\lim_{j \rightarrow +\infty} \|f_j - f\|_{L^p(X, C)} = 0$.

Proof. The first two items are obvious. The third one follows from the subadditivity of the capacity and the inclusion

$$\{f + g \geq t\} \subset \{f \geq \frac{t}{2}\} \cup \{g \geq \frac{t}{2}\}.$$

□

Lemma 1.5. *Let (f_j) be a sequence of non-negative Borel functions on X .*

1. *If (f_j) is non-decreasing and $f := \sup_j f_j$, then*

$$\int_X f dC = \lim_{j \rightarrow +\infty} \int_X f_j dC = \sup_j \int_X f_j dC.$$

2. *If (f_j) is a decreasing sequence of positive upper semi-continuous functions and $f := \inf_j f_j$, then*

$$\int_X f dC = \lim_{j \rightarrow +\infty} \int_X f_j dC = \inf_j \int_X f_j dC.$$

This lemma follows from continuity properties of the Choquet capacity; the proof is left to the reader.

1.3. Choquet-Monge-Ampère classes. Let (X, ω) be a compact Kähler manifold of dimension n .

Definition 1.6. *The Choquet-Monge-Ampère class is*

$$Ch^p(X, \omega) := PSH(X, \omega) \cap \mathcal{L}^{p+n}(X, C_\omega).$$

Observe that when $\varphi \in Ch^p(X, \omega)$ and $\varphi \leq 0$ then

$$\int_X (-\varphi)^{p+n} dC_\omega = (p+n) \int_0^{+\infty} t^{p+n-1} C_\omega(\{\varphi \leq -t\}) dt,$$

and

$$C_\omega(\{\varphi \leq -t\}) \leq t^{-p-n} \int_X (-\varphi)^{p+n} dC_\omega.$$

Proposition 1.7. *The class $Ch^p(X, \omega)$ is convex.*

If $(\varphi_j) \in Ch^p(X, \omega)^{\mathbb{N}}$ converges in $L^1(X)$ to $\varphi \in PSH(X, \omega)$ and satisfies $\sup_j \int_X (-\varphi_j)^{p+n} dC_\omega < +\infty$, then $\varphi \in Ch^p(X, \omega)$ and

$$\int_X (-\varphi)^{p+n} dC_\omega \leq \liminf_{j \rightarrow +\infty} \int_X (-\varphi_j)^{p+n} dC_\omega.$$

Proof. Set

$$\tilde{\varphi}_j := \left(\sup_{\ell \geq j} \varphi_\ell \right)^*.$$

Then $(\tilde{\varphi}_j)$ is a non-increasing sequence of $PSH(X, \omega)$ which converges to φ pointwise. Since $\varphi_j \leq \tilde{\varphi}_j \leq 0$ for all j , we infer that $\tilde{\varphi}_j \in Ch^p(X, \omega)$ and

$$\int_X (-\tilde{\varphi}_j)^{p+n} dC_\omega \leq \int_X (-\varphi_j)^{p+n} dC_\omega \leq M := \sup_j \int_X (-\varphi_j)^{p+n} dC_\omega.$$

By Lemma 1.5 we conclude that

$$\begin{aligned} \int_X (-\varphi)^{p+n} dC_\omega &= \lim_j \int_X (-\tilde{\varphi}_j)^{p+n} dC_\omega \\ &\leq \liminf_j \int_X (-\varphi_j)^{p+n} dC_\omega \leq M. \end{aligned}$$

Hence $\varphi \in Ch^p(X, \omega)$ and the required inequality follows. \square

2. ENERGY ESTIMATES

We now compare the Choquet-Monge-Ampère classes with the finite energy classes $\mathcal{E}^q(X, \omega)$ introduced in [GZ07].

2.1. Finite energy classes. Given $\varphi \in PSH(X, \omega)$, we consider its canonical approximants

$$\varphi_j := \max(\varphi, -j) \in PSH(X, \omega) \cap L^\infty(X).$$

It follows from the Bedford-Taylor theory that the measures $MA(\varphi_j)$ are well defined probability measures. Since the φ_j 's are decreasing, it is natural to expect that these measures converge (in the weak sense). The following strong monotonicity property holds:

Lemma 2.1. *The sequence $\mu_j := \mathbf{1}_{\{\varphi > -j\}} MA(\varphi_j)$ is an increasing sequence of Borel measures.*

The proof is an elementary consequence of the maximum principle (see [GZ07, p.445]). Since the μ_j 's all have total mass bounded from above by 1 (the total mass of the measure $MA(\varphi_j)$), we can consider

$$\mu_\varphi := \lim_{j \rightarrow +\infty} \mu_j,$$

which is a positive Borel measure on X , with total mass ≤ 1 .

Definition 2.2. *We set*

$$\mathcal{E}(X, \omega) := \{\varphi \in PSH(X, \omega) \mid \mu_\varphi(X) = 1\}.$$

For $\varphi \in \mathcal{E}(X, \omega)$, we set $MA(\varphi) := \mu_\varphi$.

The notation is justified by the following important fact: the complex Monge-Ampère operator $\varphi \mapsto MA(\varphi)$ is well defined on the class $\mathcal{E}(X, \omega)$, i.e. for every decreasing sequence of bounded (in particular smooth) ω -psh functions φ_j , the probability measures $MA(\varphi_j)$ weakly converge towards μ_φ , if $\varphi \in \mathcal{E}(X, \omega)$.

Every bounded ω -psh function clearly belongs to $\mathcal{E}(X, \omega)$ since in this case $\{\varphi > -j\} = X$ for j large enough, hence

$$\mu_\varphi \equiv \mu_j = MA(\varphi_j) = MA(\varphi).$$

The class $\mathcal{E}(X, \omega)$ also contains many ω -psh functions which are unbounded. When X is a compact Riemann surface ($n = \dim_{\mathbb{C}} X = 1$), the set $\mathcal{E}(X, \omega)$ is the set of ω -sh functions whose Laplacian does not charge polar sets.

Remark 2.3. *If $\varphi \in PSH(X, \omega)$ is normalized so that $\varphi \leq -1$, then $(-\varphi)^\varepsilon$ belongs to $\mathcal{E}(X, \omega)$ whenever $0 \leq \varepsilon < 1$. The functions which belong to the class $\mathcal{E}(X, \omega)$, although usually unbounded, have relatively mild singularities. In particular they have zero Lelong numbers.*

It is shown in [GZ07] that the comparison principle holds in $\mathcal{E}(X, \omega)$:

Proposition 2.4. *Fix $u, v \in \mathcal{E}(X, \omega)$. Then*

$$\int_{\{v < u\}} MA(u) \leq \int_{\{v < u\}} MA(v).$$

The class $\mathcal{E}(X, \omega)$ is the largest class for which the complex Monge-Ampère is well defined and the maximum principle holds.

Definition 2.5. *We let $\mathcal{E}^p(X, \omega)$ denote the set of ω -psh functions with finite p -energy, i.e.*

$$\mathcal{E}^p(X, \omega) := \{\varphi \in \mathcal{E}(X, \omega) / (|\varphi|)^p \in L^1(MA(\varphi))\}.$$

Here follow a few important properties of these classes (see [GZ07]):

- when $p \geq 1$, any $\varphi \in \mathcal{E}^p(X, \omega)$ is such that $\nabla_\omega \varphi \in L^2(\omega^n)$;
- $\varphi \in \mathcal{E}^p(X, \omega)$ if and only if for any (resp. one) sequence of bounded ω -functions decreasing to φ , $\sup_j \int_X (-\varphi_j)^p MA(\varphi_j) < +\infty$;
- the class $\mathcal{E}^p(X, \omega)$ is convex.

2.2. Choquet energy. For $\varphi \in PSH^-(X, \omega)$ and $p \geq 1$, we set

$$\text{Ch}_p(\varphi) := \sum_{j=0}^n C_n^j \int_X (-\varphi)^{p+j} \omega_\varphi^{n-j} \wedge \omega^j = \int_X (-\varphi)^p [(-\varphi)\omega + \omega_\varphi]^n.$$

Here and in the sequel we use the french notation $C_n^j := \binom{n}{j}$.

We recall the following useful result:

Lemma 2.6. *Fix $\varphi, \psi \in \mathcal{E}(X, \omega)$. Then for all $t < 0$ and $0 \leq \delta \leq 1$,*

$$\delta^n C_\omega(\{\varphi - \psi < -t - \delta\}) \leq \int_{\{\varphi - \psi < -t - \delta\psi\}} MA(\varphi).$$

In particular

$$\delta^n C_\omega(\{\varphi < -t - \delta\}) \leq \int_{\{\varphi < -t\}} MA(\varphi).$$

Proof. If u is a ω -psh function such that $0 \leq u \leq 1$, then

$$\{\varphi < -t - \delta\} \subset \{\varphi < \delta u - t - \delta\} \subset \{\varphi < -t\}.$$

Since $\delta^n \text{MA}(u) \leq \text{MA}(\delta u)$ and $\varphi \in \mathcal{E}(X, \omega)$ it follows from the comparison principle that

$$\begin{aligned} \delta^n \int_{\{\varphi < -t - \delta\}} \text{MA}(u) &\leq \int_{\{\varphi < \delta u - t - \delta\}} \text{MA}(\delta u) \\ &\leq \int_{\{\varphi < \delta u - t - \delta\}} \text{MA}(\varphi) \leq \int_{\{\varphi < -t\}} \text{MA}(\varphi). \end{aligned}$$

This proves the last inequality. The first one is a refinement of the second, we refer the reader to [EGZ09] for a proof. \square

Theorem 2.7. *For all $p \geq 1$ and $0 \geq \varphi \in \text{PSH}(X, \omega) \cap L^\infty(X)$,*

$$(2.1) \quad \int_X (-\varphi)^{n+p} dC_\omega \leq 2^{n+p} \text{Ch}_p(\varphi),$$

and

$$(2.2) \quad \text{Ch}_p(\varphi) \leq V_\omega(X) + (n+1)2^n \int_X (-\varphi)^{n+p} dC_\omega.$$

In particular

$$\text{Ch}^p(X, \omega) = \{\varphi \in \text{PSH}(X, \omega); \text{Ch}_p(\varphi) < +\infty\},$$

and the inequalities (2.1) and (2.2) hold for all $\varphi \in \text{Ch}^p(X, \omega)$.

Proof. By Lemma 1.5 and the continuity properties for the Monge-Ampère operators, it suffices to prove the estimates (2.1) and (2.2) when $0 \geq \varphi \in \text{PSH}(X, \omega) \cap L^\infty(X)$. Now

$$\int_X (-\varphi)^{n+p} dC_\omega = (n+p) \int_0^{+\infty} t^{n+p-1} C_\omega(\{\varphi \leq -t\}) dt.$$

Fix $t \geq 1$ and $u \in \text{PSH}(X, \omega)$ such that $-1 \leq u \leq 0$. Observe that $\varphi/t \in \text{PSH}^-(X, \omega) \cap L^\infty(X)$ and

$$\{\varphi < -2t\} \subset \{\varphi/t < u - 1\} \subset \{\varphi < -t\}.$$

Set $\psi_t := \varphi/t$. This is a bounded ω -psh function in X such that $\omega + dd^c \psi_t \leq t^{-1} \omega_\varphi + \omega$. The comparison principle (Proposition 2.4) yields

$$\int_{\{\varphi < -2t\}} \omega_u^n \leq \int_{\{\psi_t < u-1\}} \omega_u^n \leq \int_{\{\varphi < -t\}} (t^{-1} \omega_\varphi + \omega)^n.$$

Since $(t^{-1} \omega_\varphi + \omega)^n = \sum_{j=0}^n C_n^j t^{-n+j} \omega_\varphi^{n-j} \wedge \omega^j$, we infer, for all $t \geq 1$,

$$t^n C_\omega(\{\varphi < -2t\}) \leq \sum_{j=0}^n C_n^j t^j \int_{\{\varphi < -t\}} \omega_\varphi^{n-j} \wedge \omega^j.$$

It follows on the other hand from Lemma 2.6 that for $0 < t \leq 1$,

$$t^n C_\omega(\{\varphi < -2t\}) \leq \int_{\{\varphi < -t\}} \omega_\varphi^n.$$

Thus for all $t > 0$

$$t^{n+p-1}C_\omega(\{\varphi < -2t\}) \leq 2 \sum_{j=0}^n C_n^j(p+j)t^{p+j-1} \int_{\{\varphi < -t\}} \omega_\varphi^{n-j} \wedge \omega^j,$$

hence

$$\int_X (-\varphi)^{n+p} dC_\omega \leq (n+1)2^{n+p+1} \text{Ch}_p(\varphi).$$

Conversely fix $\varphi \in PSH(X, \omega) \cap L^\infty(X)$. Then for $j = 0, \dots, n$

$$\int_X (-\varphi)^{p+j} \omega_\varphi^{n-j} \wedge \omega^j = V_\omega(X) + (p+j) \int_1^{+\infty} t^{p+j-1} \omega_\varphi^{n-j} \wedge \omega^j(\{\varphi \leq -t\}).$$

Observe that if we set $\varphi_t := \sup\{\varphi, -t\}$, then

$$\int_{\{\varphi \leq -t\}} \omega_\varphi^{n-j} \wedge \omega^j = \int_{\{\varphi \leq -t\}} \omega_{\varphi_t}^{n-j} \wedge \omega^j.$$

Since for $t \geq 1$, $t^{-1}\omega_{\varphi_t} \leq \omega_{\psi_t}$, where $\psi_t := \sup\{\varphi/t, -1\}$, we infer

$$\int_{\{\varphi \leq -t\}} \omega_\varphi^{n-j} \wedge \omega^j \leq t^{n-j} \int_{\{\varphi \leq -t\}} \omega_{\psi_t}^{n-j} \wedge \omega^j.$$

Now

$$C_n^j \omega_{\psi_t}^{n-j} \wedge \omega^j \leq (\omega_{\psi_t} + \omega)^n = 2^n (\omega + dd^c(\psi_t/2))^n$$

and $-1 \leq \psi_t \leq 0$, therefore

$$\sum_{j=0}^n C_n^j \int_0^{+\infty} t^{p+j} \omega_\varphi^{n-j} \wedge \omega^j \leq 2^n V_\omega(X) + n2^n \int_X (-\varphi)^{n+p} dC_\omega,$$

hence

$$\sum_{j=0}^n C_n^j \int_X (-\varphi)^{p+j} \omega_\varphi^{n-j} \wedge \omega^j \leq 2^n V_\omega(X) + n2^n \int_X (-\varphi)^{n+p} dC_\omega.$$

□

Corollary 2.8.

$$\mathcal{E}^{p+n-1}(X, \omega) \subset \mathcal{Ch}^p(X, \omega) \subset \mathcal{E}^p(X, \omega).$$

Proof. The second inclusion follows from the fact that

$$\int_X (-\varphi)^p \omega_\varphi^n \leq \text{Ch}_p(\varphi).$$

To prove the first inclusion we can assume that $\varphi \leq -1$. Observe that when $\varphi \in \mathcal{E}^{p+n-1}(X, \omega)$ so does $\varphi/2$ and for $j = 1, \dots, n-1$

$$\int_X (-\varphi)^{p+j} \omega_\varphi^{n-j} \wedge \omega^j \leq 2^n \int_X (-\varphi)^{p+j} \omega_{\varphi/2}^n$$

and for $j = 0$, we always have $\int_X (-\varphi)^p \omega_\varphi^n < +\infty$. □

3. RANGE OF THE MONGE-AMPÈRE OPERATOR

In this section X is a compact Kähler manifold equipped with a semi-positive form ω such that $\int_X \omega^n = 1$, where $n = \dim_{\mathbb{C}} X$.

3.1. The Monge-Ampère operator on $Ch^p(X, \omega)$.

Lemma 3.1. *Fix $0 \geq \varphi, \psi \in Ch^p(X, \omega)$ and $0 \leq j \leq n$. Then*

$$\int_X (-\varphi)^{p+j} \omega_\psi^{n-j} \wedge \omega^j \leq 2^{p+j} \int_X (-\varphi)^{p+j} \omega_\varphi^{n-j} \wedge \omega^j + 2^{p+j} \int_X (-\psi)^{p+j} \omega_\psi^{n-j} \wedge \omega^j.$$

Proof. Set $\chi(t) = -(-t)^{p+j}$. The proof is slightly different if $j = 0$ and $0 < p < 1$ or if $p + j \geq 1$ (χ is convex or concave). We only treat the second case and leave the modifications to the reader. Observe that $0 \leq \chi'(2t) = M\chi'(t)$, with $M = 2^{p+j-1}$, hence

$$\begin{aligned} \int_X (-\chi) \circ \varphi \omega_\psi^{n-j} \wedge \omega^j &= \int_{-\infty}^0 \chi'(t) \omega_\psi^{n-j} \wedge \omega^j (\varphi < t) dt \\ &\leq 2M \int_{-\infty}^0 \chi'(t) \omega_\psi^{n-j} \wedge \omega^j (\varphi < 2t) dt. \end{aligned}$$

Now $(\varphi < 2t) \subset (\varphi < \psi + t) \cup (\psi < t)$, hence $0 \leq \chi'(2t) = M\chi'(t)$, with $M = 2^{p+j-1}$, hence

$$\begin{aligned} \int_X (-\chi) \circ \varphi \omega_\psi^{n-j} \wedge \omega^j &\leq 2M \int_{-\infty}^0 \chi'(t) \omega_\psi^{n-j} \wedge \omega^j (\varphi < \psi + t) dt \\ &\quad + 2M \int_X (-\psi)^{p+j} \omega_\psi^{n-j} \wedge \omega^j. \end{aligned}$$

The comparison principle yields $\omega_\psi^{n-j} \wedge \omega^j (\varphi < \psi + t) \leq \omega_\varphi^{n-j} \wedge \omega^j (\varphi < \psi + t)$. The desired inequality follows by observing that $(\varphi < \psi + t) \subset (\varphi < t)$. \square

Lemma 3.2. *Let μ be a probability measure. Then $Ch^p(X, \omega) \subset L^q(\mu)$ if and only if there exists $C_\mu > 0$ such that $\forall \varphi \in Ch^p(X, \omega)$ with $\sup_X \varphi = -1$,*

$$\int_X (-\varphi)^q d\mu \leq C_\mu [\text{Ch}_p(\varphi)]^{\frac{q}{p+n}}.$$

Proof. One implication is obvious. Assume that $Ch^p(X, \omega) \subset L^q(\mu)$, we want to establish the quantitative integrability property. Assume on the contrary that there exists a sequence $\varphi_j \in Ch^p(X, \omega)$ with $\sup_X \varphi_j = -1$ and

$$\int_X (-\varphi_j)^q d\mu \geq 4^{jq} \text{Ch}_p(\varphi_j)^{\frac{q}{p+n}}.$$

Assume first that $M_j := \text{Ch}_p(\varphi_j)$ is uniformly bounded. Note that $M_j \geq 1$ since $\varphi_j \leq -1$. It follows from Proposition 1.7 that $\varphi = \sum_{j \geq 1} 2^{-j} \varphi_j$ belongs to $Ch^p(X, \omega)$. Now for all $k \geq 1$,

$$\int_X (-\varphi)^q d\mu \geq 2^{-kq} \int_X (-\varphi_k)^q d\mu \geq 2^{kq},$$

hence $\int_X (-\varphi)^q d\mu = +\infty$, a contradiction.

Extracting and relabelling we can thus assume $M_j := \text{Ch}_p(\varphi_j) \rightarrow +\infty$.

Set $\psi_j = \varepsilon_j \varphi_j$ with $\varepsilon_j = M_j^{-\frac{1}{n+p}}$ and $\psi = \sum_{j \geq 1} 2^{-j} \psi_j$. We note again that for all $k \geq 1$,

$$\int_X (-\psi)^q d\mu \geq 2^{-kq} \int_X (-\psi_k)^q d\mu \geq 2^{kq} \varepsilon_k^q M_k^{\frac{q}{p+n}} = 2^{kq},$$

hence $\psi \notin L^q(\mu)$. We now show that $\psi \in \mathcal{Ch}^p(X, \omega)$ to get a contradiction. It suffices to show that $\text{Ch}_p(\psi_j)$ is uniformly bounded from above. Observe that $\omega_{\psi_j} = \varepsilon_j \omega_{\varphi_j} + (1 - \varepsilon_j) \omega \leq \varepsilon_j \omega_{\varphi_j} + \omega$. We need to control each term

$$\varepsilon_j^{p+n-k} \int_X (-\varphi_j)^{p+\ell} \omega_{\varphi_j}^{n-\ell-k} \wedge \omega^{\ell+k},$$

where $0 \leq \ell \leq n$ and $0 \leq k \leq n - \ell$. Hölder inequality yields

$$\int_X (-\varphi_j)^{p+\ell} \omega_{\varphi_j}^{n-\ell-k} \wedge \omega^{\ell+k} \leq \left(\int_X (-\varphi_j)^{p+\ell+k} \omega_{\varphi_j}^{n-\ell-k} \wedge \omega^{\ell+k} \right)^{\frac{p+\ell}{p+\ell+k}},$$

therefore

$$\text{Ch}_p(\psi_j) \leq C \max_{\ell, k} \left(\varepsilon_j^{p+n-k} M_j^{\frac{p+\ell}{p+\ell+k}} \right) = C \max_{\ell, k} \left(M_j^{-\frac{k(n-\ell-k)}{(p+\ell+k)(n+p)}} \right) \leq C',$$

since $\varepsilon_j = M_j^{-\frac{1}{n+p}}$. \square

The range of the complex Monge-Ampère operator acting on finite energy classes has been characterized in [GZ07]. The situation is more subtle for Choquet-Monge-Ampère classes.

We now connect the way a non pluripolar measure is dominated by the Monge-Ampère capacity to integrability properties with respect to Choquet-Monge-Ampère classes:

Proposition 3.3. *Let μ be a probability measure on X . If $\mu \leq AC_\omega^\alpha$ with $0 < A$ and $q/(p+n) < \alpha < 1$, then*

$$\mathcal{Ch}^p(X, \omega) \subset L^q(\mu).$$

Proof. Let $\varphi \in \mathcal{Ch}^p(X, \omega)$ with $\sup_X \varphi = -1$. It follows from Hölder inequality that

$$\begin{aligned} 0 &\leq \int_X (-\varphi)^q d\mu = 1 + q \int_1^{+\infty} t^{q-1} \mu(\varphi < -t) dt \\ &\leq 1 + qA \int_1^{+\infty} t^{q-1} [C_\omega(\varphi < -t)]^\alpha dt \\ &\leq 1 + qA \left[\int_1^{+\infty} t^{\frac{q-\alpha(p+n)}{1-\alpha}-1} dt \right]^{1-\alpha} \cdot \left[\int_1^{+\infty} t^{n+p-1} C_\omega(\varphi < -t) dt \right]^\alpha. \end{aligned}$$

The first integral in the last line converges when $q/(p+n) < \alpha$ since $q-\alpha(p+n) < 0$. The last one is bounded from above by definition. Therefore $\mathcal{Ch}^p(X, \omega) \subset L^q(\mu)$. \square

We now investigate conditions under which the converse of this result holds. We start by considering the problem for the finite energy classes $\mathcal{E}^p(X, \omega)$:

Proposition 3.4. *If $\mathcal{E}^p(X, \omega) \subset L^p(\mu)$ for $p > 1$, then there exists an $A > 0$ such that $\mu \leq AC_\omega^\alpha$ where $\alpha = (1 - 1/p)^n$.*

Proof. Suppose that $\mathcal{E}^p(X, \omega) \subset L^p(\mu)$ then by [GZ07] $\mu = \omega_\psi^n$ for some $\psi \in \mathcal{E}^p(X, \omega)$ such that $\sup_X \psi = -1$. Let $\varphi \in PSH(X, \omega)$ with $-1 \leq \varphi \leq 0$ then

$$\begin{aligned} \int_X (-\varphi)^p \omega_\psi^n &= \int_X (-\varphi)^p \omega_\psi \wedge \omega_\psi^{n-1} \\ &= \int_X (-\psi)(-dd^c(-\varphi)^p) \wedge \omega_\psi^{n-1} + \int_X (-\varphi)^p \omega \wedge \omega_\psi^{n-1}. \end{aligned}$$

Now

$-dd^c(-\varphi)^p = -p(p-1)(-\varphi)^{p-2}d\varphi \wedge d^c\varphi + p(-\varphi)^{p-1}dd^c\varphi \leq p(-\varphi)^{p-1}dd^c\varphi$
and $(-\varphi)^p \leq (-\varphi)^{p-1}$ since $0 \leq -\varphi \leq 1$, hence

$$\begin{aligned} \int_X (-\varphi)^p \omega_\psi^n &\leq p \int_X (-\psi)(-\varphi)^{p-1}dd^c\varphi \wedge \omega_\psi^{n-1} + \int_X (-\varphi)^{p-1} \omega \wedge \omega_\psi^{n-1} \\ &\leq p \int_X (-\psi)(-\varphi)^{p-1}dd^c\varphi \wedge \omega_\psi^{n-1} + \int_X (-\psi)(-\varphi)^{p-1} \omega \wedge \omega_\psi^{n-1} \\ &= p \int_X (-\psi)(-\varphi)^{p-1} \omega_\varphi \wedge \omega_\psi^{n-1}. \end{aligned}$$

Hölder inequality thus yields

$$\begin{aligned} \int_X (-\varphi)^p \omega_\psi^n &\leq p \left(\int_X (-\psi)^p \omega_\varphi \wedge \omega_\psi^{n-1} \right)^{\frac{1}{p}} \left(\int_X (-\varphi)^p \omega_\varphi \wedge \omega_\psi^{n-1} \right)^{1-\frac{1}{p}} \\ &\leq p \left(\int_X (-\psi)^p \omega_\psi^n \right)^{\frac{1}{p}} \left(\int_X (-\varphi)^p \omega_\varphi \wedge \omega_\psi^{n-1} \right)^{1-\frac{1}{p}}. \end{aligned}$$

Repeating the same argument n times we end up with

$$\int_X (-\varphi)^p \omega_\psi^n \leq A \left(\int_X (-\varphi)^p \omega_\varphi^n \right)^{(1-1/p)^n}$$

Fix $E \subset X$ a compact set. The conclusion follows by applying this inequality to the extremal function $\varphi = h_{\omega, E}^*$, observing that

$$\mu(E) \leq \int_X (-h_{\omega, E}^*)^p \omega_\psi^n,$$

while $C_\omega(E) = \int_X (-h_{\omega, E}^*)^p \omega_{h_{\omega, E}^*}^n$, as shown in [GZ05]. \square

Lemma 3.5. *Let μ be a probability measure. Then $\mathcal{E}^p(X, \omega) \subset L^q(\mu)$ if and only if there exists a constant $C > 0$ such that for all $\psi \in PSH(X, \omega) \cap L^\infty(X)$ with $\sup_X \psi = -1$*

$$(3.1) \quad 0 \leq \int_X (-\psi)^q d\mu \leq C \left(\int_X (-\psi)^p \omega_\psi^n \right)^{\frac{q}{p+1}}$$

Proof. One implication is clear so suppose that $\mathcal{E}^p(X, \omega) \subset L^q(\mu)$ and assume for a contradiction that there exists $\psi_j \in PSH(X, \omega) \cap L^\infty(X)$ with $\sup_X \psi_j = -1$ such that

$$\int_X (-\psi_j)^q d\mu \geq 4^{jq} M_j^{\frac{q}{p+1}}$$

where $M_j = \int_X (-\psi_j)^p \omega_{\psi_j}^n$.

If M_j is uniformly bounded then $\psi = \sum_{j \geq 1} 2^{-j} \psi_j$ belongs to $\mathcal{E}^p(X, \omega)$.
Now

$$\int_X (-\psi)^q d\mu \geq \int_X \frac{(-\psi_j)^q}{2^{jq}} d\mu \geq 2^{jq} M_j^{\frac{q}{p+1}} \geq 2^{jq}$$

since $\psi_j \leq -1$, $M_j \geq 1$. So $\int_X (-\psi)^q d\mu \rightarrow \infty$, a contradiction.

We obtain the same contradiction if $\{M_j\}$ admits a bounded subsequence so we can assume $M_j \rightarrow \infty$ and $M_j \geq 1$. Set $\varphi_j = \varepsilon_j \psi_j$ where $\varepsilon_j = M_j^{-\frac{1}{1+p}}$ and $\psi = \sum_{j \geq 1} 2^{-j} \varphi_j$ then,

$$\int_X (-\psi)^q d\mu \geq \int_X \frac{(-\varphi_j)^q}{2^{jq}} d\mu = 2^{-jq} \varepsilon_j^q \int_X (-\psi_j)^q d\mu \geq 2^{jq} \rightarrow \infty$$

so $\psi \notin L^q(\mu)$.

We now check that $\varphi_j \in \mathcal{E}^p(X, \omega)$ to derive a contradiction. Since $\omega_{\varphi_j} \leq \varepsilon_j \omega_{\psi_j} + \omega$, we get

$$\begin{aligned} \int_X (-\varphi_j)^p \omega_{\varphi_j}^n &= \varepsilon_j^p \int_X (-\psi_j)^p \omega_{\varphi_j}^n \\ &\leq \varepsilon_j^p \left(\int_X (-\psi_j)^p \omega^n + 2^n \varepsilon_j \int_X (-\psi_j)^p \omega_{\psi_j}^n \right) = O(1), \end{aligned}$$

because

$$\begin{aligned} \int_X (-\psi_j)^p \omega_{\psi_j}^n &= \int_X (-\psi_j)^p \omega \wedge \omega_{\psi_j}^{n-1} + \int_X p(-\psi_j)^{p-1} d\psi_j \wedge d^c \psi_j \wedge \omega_{\psi_j}^{n-1} \\ &\geq \int_X (-\psi_j)^p \omega \wedge \omega_{\psi_j}^{n-1} \geq \dots \geq \int_X (-\psi_j)^p \omega^k \wedge \omega_{\psi_j}^{n-k} \end{aligned}$$

for all $1 \leq k \leq n-1$ and $\int_X (-\psi_j)^p \omega^n$ is bounded since $\psi_j \in PSH(X, \omega) \cap L^\infty(X)$ and $\sup_X \psi_j = -1$. \square

We are now ready to give necessary conditions for a non-pluripolar measure to be dominated by the Monge-Ampère capacity, in terms of its integrability condition properties with respect to Choquet-Monge-Ampère classes:

Proposition 3.6. *If $\psi \in Ch^p(X, \omega)$ and $\mu = MA(\psi)$ then $\mu \leq (C_\omega)^{\frac{p}{p+n}}$.*

Proof. We assume without loss of generality that $\sup_X \psi = -1$. For $\varphi \in PSH(X, \omega)$ with $-1 \leq \varphi \leq 0$ Hölder inequality and integration by parts yields

$$\begin{aligned} \int_X (-\varphi)^{p+n} \omega_\psi^n &\leq (p+n) \int_X (-\varphi)^{p+n-1} (-\psi) \omega_\varphi \wedge \omega_\psi^{n-1} \\ &\leq (p+n) \left(\int_X (-\psi)^{p+1} \omega_\varphi \wedge \omega_\psi^{n-1} \right)^{\frac{1}{p+1}} \left(\int_X (-\varphi)^{\frac{(p+n)(p+n-1)}{p}} \omega_\varphi \wedge \omega_\psi^{n-1} \right)^{\frac{p}{p+1}} \end{aligned}$$

To handle the second term observe that

$$\begin{aligned} &\int_X (-\varphi)^{\frac{(p+n)(p+n-1)}{p}} \omega_\varphi \wedge \omega_\psi^{n-1} \\ &\leq c_{p,n} \left(\int_X (-\psi)^{p+2} \omega_\varphi^2 \wedge \omega_\psi^{n-2} \right)^{\frac{1}{p+2}} \left(\int_X (-\varphi)^{\frac{(p+2)(p^2+(p+1)(n-1))}{p(p+1)}} \omega_\varphi^2 \wedge \omega_\psi^{n-2} \right)^{\frac{p+1}{p+2}} \end{aligned}$$

As it can be observed, at each step the power of $(-\varphi)$ is obtained by reducing the previous power by 1 first and then multiplying by $\frac{p+m}{p+m-1}$ where m is the

number of the corresponding step. Hence the power of $(-\varphi)$ at the m 'th step, σ_m , is given by induction by

$$\sigma_{m+1} = \frac{p+m}{p+m-1}(\sigma_m - 1)$$

Therefore we have to justify that σ_m is bigger than $p+n-m$ and we can continue the procedure n -times. We will show this by induction:

For $m=1$, $\sigma_1 = \frac{p+1}{p}(p+n-1) > p+n-1$ since $\frac{p+1}{p} > 1$ and assume that $\sigma_m > p+n-m$ then

$$\sigma_{m+1} = \frac{p+m}{p+m-1}(\sigma_m - 1) > \frac{p+m}{p+m-1}(p+n-m-1) > p+n-(m+1)$$

since $\frac{p+m}{p+m-1} > 1$.

Now at the n 'th step we have,

$$\int_X (-\varphi)^{p+n} \omega_\psi^n \leq c_{p,n} \left(\prod_{i=1}^n \left(\int_X (-\psi)^{p+i} \omega_\varphi^i \wedge \omega_\psi^{n-i} \right)^{\frac{1}{p+i}} \right) \left(\int_X (-\varphi)^{\sigma_n} \omega_\varphi^n \right)^{\frac{p}{p+n}}$$

and since $0 \leq (-\varphi) \leq 1$ and $\sigma_n > p$ we have

$$\leq c_{p,n} \left(\prod_{i=1}^n \left(\int_X (-\psi)^{p+i} \omega_\varphi^i \wedge \omega_\psi^{n-i} \right)^{\frac{1}{p+i}} \right) \left(\int_X (-\varphi)^p \omega_\varphi^n \right)^{\frac{p}{p+n}}$$

Each term in the product is bounded since $\psi \in Ch^p(X, \omega)$ so we have

$$\int_X (-\varphi)^{p+n} \omega_\psi^n \leq A \left(\int_X (-\varphi)^p \omega_\varphi^n \right)^{\frac{p}{p+n}}$$

and the conclusion follows by applying this inequality to the extremal function $\varphi = h_{\omega, E}^*$, where $E \subset X$ is an arbitrary compact set. \square

Remark 3.7. *In the case where $Ch^p(X, \omega) \subset L^q(\mu)$ and $q \geq p+n-1$, $p > 1$ as an immediate consequence of Corollary 2.8 and Proposition 3.4 we obtain that there exists $A > 0$ such that $\mu \leq AC_\omega^\alpha$ where $\alpha = (1 - \frac{1}{p})^n$.*

3.2. Examples. It follows from Corollary 2.8 that the classes $Ch^p(X, \omega)$ and $\mathcal{E}^p(X, \omega)$ coincide when $n=1$. We describe in this section the finite Choquet energy classes in special cases.

3.2.1. Compact singularities. The class $Ch^p(X, \omega)$ is similar to $\mathcal{E}^p(X, \omega)$ for functions with "compact singularities":

Proposition 3.8. *let D be an ample \mathbb{Q} -divisor. Let φ be a ω -psh function which is bounded in a neighborhood of D . Then*

$$\varphi \in Ch^p(X, \omega) \iff \varphi \in \mathcal{E}^p(X, \omega).$$

The inclusions $\mathcal{E}^{p+n-1}(X, \omega) \subset Ch^p(X, \omega) \subset \mathcal{E}^p(X, \omega)$ are strict in general when $n \geq 2$, as we show in Example 3.9 below .

Proof. Let V be a neighborhood of D where φ is bounded. For simplicity we assume that $c_1(D) = \{\omega\}$. Let ω' be a smooth semi-positive closed form cohomologous to ω , such that $\omega' \equiv 0$ outside V . Let ρ be a smooth ω -psh

function such that $\omega' = \omega + dd^c \rho$. Shifting by a constant, we can assume that $0 \leq \rho \leq M$. Observe that

$$\begin{aligned} -dd^c(-\varphi)^{p+j} &= -(p+j)(p+j-1)(-\varphi)^{p+j-2}d\varphi \wedge d^c\varphi + (p+j)(-\varphi)^{p+j-1}\omega_\varphi \\ &\leq (p+j)(-\varphi)^{p+j-1}\omega_\varphi. \end{aligned}$$

Therefore

$$\begin{aligned} \int (-\varphi)^{p+j}\omega_\varphi^{n-j} \wedge \omega^j &= \int (-\varphi)^{p+j}\omega_\varphi^{n-j} \wedge \omega^{j-1} \wedge \omega' \\ &+ \int (-\varphi)^{p+j}\omega_\varphi^{n-j} \wedge \omega^{j-1} \wedge dd^c\rho \\ &= O(1) + \int \rho dd^c[-(-\varphi)^{p+j}] \wedge \omega_\varphi^{n-j} \wedge \omega^{j-1} \\ &\leq O(1) + (p+j)M \int (-\varphi)^{p+j-1}\omega_\varphi^{n-j+1} \wedge \omega^{j-1}. \end{aligned}$$

We denote here by $O(1)$ the first term $\int (-\varphi)^{p+j}\omega_\varphi^{n-j} \wedge \omega^{j-1} \wedge \omega'$ which is bounded, since φ is bounded on the support of ω' .

By induction we obtain that each term $\int (-\varphi)^{p+j}\omega_\varphi^{n-j} \wedge \omega^j$ is controlled by $\int (-\varphi)^p\omega_\varphi^n$. Thus $\text{Ch}_p(\varphi)$ is finite if and only if so is $\int (-\varphi)^p\omega_\varphi^n$. \square

This proposition allows us to cook up examples of ω -psh functions φ such that $\varphi \in \text{Ch}^p(X, \omega)$ but $\varphi \notin \mathcal{E}^{p+n-1}(X, \omega)$. The next example shows how to cook up examples such that $\varphi \in \mathcal{E}^p(X, \omega)$ but $\varphi \notin \text{Ch}^p(X, \omega)$:

Example 3.9. Assume $X = \mathbb{C}\mathbb{P}^{n-1} \times \mathbb{C}\mathbb{P}^1$ and $\omega(x, y) := \alpha(x) + \beta(y)$, where α is the Fubini-Study form on $\mathbb{C}\mathbb{P}^{n-1}$ and β is the Fubini-Study form on $\mathbb{C}\mathbb{P}^1$. Fix $u \in \text{PSH}(\mathbb{C}\mathbb{P}^{n-1}, \alpha) \cap \mathcal{C}^\infty(\mathbb{C}\mathbb{P}^{n-1})$ and $v \in \mathcal{E}(\mathbb{C}\mathbb{P}^1, \beta)$.

The function φ defined by $\varphi(x, y) := u(x) + v(y)$ for $(x, y) \in X$ belongs to $\mathcal{E}(X, \omega)$. Moreover $\omega_\varphi = \alpha_u + \beta_v$ and for any $1 \leq \ell \leq n$, we have

$$\omega_\varphi^{n-j} = \alpha_u^{n-j} + (n-j)\alpha_u^{n-j-1} \wedge \beta_v$$

and

$$\omega_\varphi^{n-j} \wedge \omega^j = \alpha_u^{n-j} \wedge \alpha^j + j\alpha^{j-1} \wedge \alpha_u^{n-j} \wedge \beta + (n-j)\alpha^j \wedge \alpha_u^{n-j-1} \wedge \beta_v.$$

Thus for $j \leq n-1$,

$$\varphi \in L^{p+j}(\omega_\varphi^{n-j} \wedge \omega^j) \iff v \in L^{p+j}(\beta_v)$$

hence

$$\varphi \in \text{Ch}^p(X, \omega) \iff v \in \mathcal{E}^{p+n-1}(\mathbb{C}\mathbb{P}^1, \beta)$$

while

$$\varphi \in \mathcal{E}^p(X, \omega) \iff v \in \mathcal{E}^p(\mathbb{C}\mathbb{P}^1, \beta).$$

Choosing $v \in L^p(\beta_v) \setminus L^{p+n-1}(\beta_v)$, we obtain an example of a ω -psh function φ such that $\varphi \in \mathcal{E}^p(X, \omega)$ but $\varphi \notin \text{Ch}^p(X, \omega)$.

Remark 3.10. In the above examples, we can choose u and v toric, hence both inclusions in Corollary 2.8 are sharp in the toric setting as well. For details on toric singularities, we refer the reader to [G14, DN16].

3.2.2. *Divisorial singularities.* Let D be an ample \mathbb{Q} -divisor, s a holomorphic defining section of L_D and h a smooth positive metric of L . We assume for simplicity that the curvature of h is ω , so that the Poincaré-Lelong formula can be written

$$dd^c \log |s|_h = [D] - \omega,$$

where $[D]$ denotes the current of integration along D .

Let χ be a smooth convex increasing function and set $\varphi = \chi \circ \log |s|_h$. We normalize h so that $\chi' \circ \log |s|_h \leq 1/2$. It follows that φ is strictly ω -psh, since

$$dd^c \varphi = \chi'' \circ L dL \wedge d^c L + \chi' \circ L dd^c L \geq -\chi' \circ L \omega \geq -\omega/2,$$

where $L := \log |s|_h$.

Proposition 3.11. *Set $\varphi = \chi \circ \log |s|_h \in PSH(X, \omega)$. Then*

$$\varphi \in Ch^p(X, \omega) \iff \varphi \in \mathcal{E}^{p+n-1}(X, \omega).$$

Proof. Set $L = \log |s|_h$. Observe that

$$\omega + dd^c \varphi = \chi'' \circ L dL \wedge d^c L + \chi' \circ L [D] + (1 - \chi' \circ L)\omega.$$

A necessary condition for φ to belong to a finite energy class is that ω_φ does not charge pluripolar sets, hence $\chi'(-\infty) = 0$ and

$$\omega + dd^c \varphi = \chi'' \circ L dL \wedge d^c L + (1 - \chi' \circ L)\omega.$$

Since $\frac{1}{2} \leq 1 - \chi' \circ L \leq 1$, we infer

$$\omega_\varphi^{n-j} \wedge \omega^j \sim \chi'' \circ L dL \wedge d^c L \wedge \omega^{n-1} + \omega^n,$$

for $0 \leq j \leq n-1$. We write here $\mu \sim \mu'$ if the positive Radon measures μ, μ' are uniformly comparable, i.e. $C^{-1}\mu \leq \mu' \leq C\mu$ for some constant $C > 0$. Thus

$$\begin{aligned} \varphi \in Ch^p(X, \omega) &\iff \varphi \in L^{p+n-1}(\chi'' \circ L dL \wedge d^c L \wedge \omega^{n-1}) \\ &\iff \varphi \in L^{p+n-1}(\omega_\varphi^n) \iff \varphi \in \mathcal{E}^{p+n-1}(X, \omega). \end{aligned}$$

□

Example 3.12. *For $\chi(t) = -(-t)^\alpha$, $0 < \alpha < 1$, we obtain*

$$\varphi = -(-\log |s|_h)^\alpha \in \mathcal{E}^p(X, \omega) \text{ iff } \alpha < \frac{1}{p+1}$$

and

$$\varphi = -(-\log |s|_h)^\alpha \in Ch^p(X, \omega) \text{ iff } \alpha < \frac{1}{p+n}$$

We refer the reader to [DN16] for more information on Monge-Ampère measures with divisorial singularities.

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INSTITUT UNIVERSITAIRE DE FRANCE & INSTITUT MATHÉMATIQUES DE TOULOUSE,,
 UNIVERSITÉ PAUL SABATIER, 31062 TOULOUSE CEDEX 09, FRANCE
E-mail address: `vincent.guedj@math.univ-toulouse.fr`

ÖZYEĞİN UNIVERSITY (ISTANBUL) & INSTITUT DE MATHÉMATIQUES DE TOULOUSE,,
 UNIVERSITÉ PAUL SABATIER, 118 ROUTE DE NARBONNE, F-31062 TOULOUSE CEDEX 09,
E-mail address: `sibel.sahin@ozyegin.edu.tr`

INSTITUT DE MATHÉMATIQUES DE TOULOUSE,, UNIVERSITÉ PAUL SABATIER, 118
 ROUTE DE NARBONNE, F-31062 TOULOUSE CEDEX 09,
E-mail address: `ahmed.zeriahi@math.univ-toulouse.fr`