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► **To cite this version:**

Karim Kellay, Florian Le Manach, Mohamed Zarrabi. Cyclicity in Dirichlet type spaces. *Complex Analysis and Spectral Theory. A conference in celebration of Thomas Ransford's 60th birthday, May 2018, Québec, Canada.* hal-01941825v1

**HAL Id: hal-01941825**

**<https://hal.science/hal-01941825v1>**

Submitted on 2 Dec 2018 (v1), last revised 7 May 2019 (v2)

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# CYCLICITY IN DIRICHLET TYPE SPACES

K. KELLAY, F. LE MANACH & M. ZARRABI

**Dedicated to Thomas Ransford on occasion of his 60th birthday.**

ABSTRACT. We study the cyclicity in the Dirichlet type spaces for outer functions with zeros set is countable.

## 1. INTRODUCTION AND MAIN RESULT

Let  $X$  be a Banach space of functions holomorphic in the open unit disk  $\mathbb{D}$ , such that the shift operator  $S : f(z) \rightarrow zf(z)$  is a continuous map from  $X$  into itself. Given  $f \in X$ , we denote by  $[f]_X$  the smallest closed  $S$ -invariant subspace of  $X$  containing  $f$ , namely

$$[f]_X = \overline{\{pf : p \text{ is a polynomial}\}}.$$

We say that  $f$  is cyclic in  $X$  if  $[f]_X = X$ .

The problem of cyclic vectors in the Dirichlet spaces goes back to the work of Beurling and Carleson (see [1, 3]). The classical Dirichlet space  $\mathcal{D}$  consists of holomorphic functions on the unit disc whose derivatives are square integrable. While Beurling characterizes cyclic vectors in the Hardy space  $H^2$ , the problem of characterizing the cyclic vectors in the Dirichlet space  $\mathcal{D}$  is much more difficult. Beurling's theorem says that the cyclic vectors in  $H^2$  are the outer functions. On the other hand we know that there are outer functions in the Dirichlet space which are not cyclic in  $\mathcal{D}$ . In fact, the cyclicity of such a function depends on the distribution of the zeros of the radial limit  $f^*$  of  $f$  on the unit circle. The Brown–Shields conjecture [11] claims that  $f \in \mathcal{D}$  is cyclic iff  $f$  is an outer function and the set of all zeros of  $f^*$  is a set of logarithmic capacity zero. A partial (positive) answer to this conjecture was given in [8, 7]. We mention the results of Beurling [1] about the boundary behavior for the functions of the Dirichlet spaces: if  $f \in \mathcal{D}$  we write  $f^*(\zeta) = \lim_{r \rightarrow 1^-} f(r\zeta)$ , then the radial limit  $f^*$  exists -q.e on  $\mathbb{T}$ , that is  $f^*$  exists outside a set of capacity logarithmic zero. As a consequence of weak-type inequality the invariant subspace  $\mathcal{D}_E$  defined by

$$\mathcal{D}_E = \{f \in \mathcal{D}, f^*|_E = 0 \text{ q.e.}\}$$

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2000 *Mathematics Subject Classification.* Primary 46E22; Secondary 31A05, 31A15, 31A20, 47B32.

*Key words and phrases.* Dirichlet spaces, Bergman spaces, Smirnov space, cyclic vectors.

The first author was supported by the Joint French-Russian Research Project PRC CNRS/RFBR,2017-2019.

is closed in  $\mathcal{D}$ . Carleson in [3] proved that for every closed subset  $E$  of the unit circle which has zero logarithmic capacity, there exists a cyclic function in  $\mathcal{D}$  which vanishes on  $E$ .

When the set of zeros of  $f^*$  is countable, Brown and Shields showed that if  $f$  is regular and outer then  $f$  is cyclic. Hedenmalm and Shields showed in [11] that this remains true when  $f \in \mathcal{D} \cap A(\mathbb{D})$  where  $A(\mathbb{D})$  is the disc algebra. Finally Richter and Sundberg [13] showed that this is still true when  $f$  is only in Dirichlet space. When the set of zeros of  $f^*$  is not countable see [6, 7, 8] in the case of the classical Dirichlet space  $\mathcal{D}_0^2$  and [9] in the case of  $\mathcal{D}_\alpha^2$ ,  $0 < \alpha < 1$ .

In this paper we are interested in cyclicity, in more general Dirichlet spaces, of outer function such that the zero set countable. We now introduce some notations. The Dirichlet space  $\mathcal{D}_\alpha^p$  with  $p \geq 1$  and  $\alpha > -1$  is given by

$$\mathcal{D}_\alpha^p = \left\{ f \in \text{Hol}(\mathbb{D}) : \|f\|_{\mathcal{D}_\alpha^p}^p = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p dA_\alpha(z) < \infty \right\}.$$

where  $dA_\alpha$  denote the finite measure on the unit disc  $\mathbb{D}$  given by

$$dA_\alpha(z) := (1 + \alpha)(1 - |z|^2)^\alpha dA(z),$$

and  $dA(z) = dx dy / \pi$  stands for the normalized area measure on  $\mathbb{D}$ . If  $p = 2$  and  $\alpha = 1$ , then  $\mathcal{D}_1^2$  is the Hardy space  $H^2$  and the classical Dirichlet space corresponds to  $p = 2$  and  $\alpha = 0$ ,  $\mathcal{D}_0^2 = \mathcal{D}$ . The following theorem is the main result of this paper.

**Theorem.** *Let  $p > 1$  such that  $\alpha + 1 < p \leq \alpha + 2$  and let  $f \in \mathcal{D}_\alpha^p \cap A(\mathbb{D})$ . If  $f$  is outer and  $\mathcal{Z}(f)$  is countable then  $f$  is cyclic in  $\mathcal{D}_\alpha^p$ .*

Notice that when  $1 < p < \alpha + 1$ ,  $H^p(\mathbb{D})$  is continuously embedded in  $\mathcal{D}_\alpha^p$  and every outer function  $f \in H^p(\mathbb{D})$  is cyclic for  $\mathcal{D}_\alpha^p$  (Proposition 3.1). On the other hand when  $p > \alpha + 2$  then every function which vanishes at least at one point is not cyclic in  $\mathcal{D}_\alpha^p$ .

The method used for the proof of Theorem 3.10 is inspired by that of the Hedenmalm and Shields [11] in the case of the classical Dirichlet space and the paper [8].

Throughout the paper, we use the following notations:

- $A \lesssim B$  means that there is an absolute constant  $C$  such that  $A \leq CB$ .
- $A \asymp B$  if both  $A \lesssim B$  and  $B \lesssim A$  hold.

## 2. DIRICHLET SPACE AND DUALITY

The Bergman spaces  $\mathcal{A}_\alpha^p$  with  $p \geq 1$ ,  $\alpha > -1$  is given by

$$\mathcal{A}_\alpha^p(\mathbb{D}) = \left\{ f \in \text{Hol}(\mathbb{D}), \|f\|_{\mathcal{A}_\alpha^p}^p = \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) < \infty \right\}.$$

We define the Bergman spaces  $\mathcal{A}_\alpha^p(\mathbb{D}_e)$  on the exterior disk  $\mathbb{D}_e = (\mathbb{C} \cup \{\infty\}) \setminus \overline{\mathbb{D}}$  with  $p \geq 1$  and  $\alpha > -1$  by

$$\mathcal{A}_\alpha^p(\mathbb{D}_e) = \left\{ g \in \text{Hol}(\mathbb{D}_e), g(\infty) = 0 \text{ and } \|g\|_{\mathcal{A}_\alpha^p}^p = \int_{\mathbb{D}_e} |g(z)|^p \frac{(|z|^2 - 1)^\alpha}{|z|^{4-p+2\alpha}} dA(z) < \infty \right\}.$$

Note that  $\mathcal{A}_\alpha^p(\mathbb{D})$  and  $\mathcal{A}_\alpha^p(\mathbb{D}_e)$  are isometrically isomorphic via the isometry  $R : f \mapsto Rf$  defined on  $\mathcal{A}_\alpha^p(\mathbb{D})$  by

$$(2.1) \quad Rf(z) = \frac{1}{z} \overline{f\left(\frac{1}{\bar{z}}\right)}, \quad z \in \mathbb{D}_e.$$

Indeed, by the variable change  $z \mapsto 1/\bar{z}$ ,

$$\int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) = \int_{\mathbb{D}_e} |\overline{f(1/\bar{z})}/z|^p \frac{(|z|^2 - 1)^\alpha}{|z|^{4-p+2\alpha}} dA(z)$$

Futhermore if  $f = \sum_{n \geq 0} a_n z^n \in \mathcal{A}_\alpha^p(\mathbb{D})$  then by (2.1)

$$(2.2) \quad Rf(z) = \sum_{n=0}^{\infty} \frac{\overline{a_n}}{z^{n+1}}, \quad z \in \mathbb{D}_e.$$

Denote by  $S$  the shift operator on  $\mathcal{A}_\alpha^p(\mathbb{D})$  for  $p \geq 1$  and  $\alpha > -1$ , that is the multiplication by  $z$  on  $\mathcal{A}_\alpha^p(\mathbb{D})$ . Let  $S^*$  denote the backward shift, that is

$$S^*f(z) = \frac{f(z) - f(0)}{z}.$$

Notice that  $S^*$  is continuous on  $\mathcal{A}_\alpha^p(\mathbb{D})$  for  $p \geq 1$  and  $\alpha > -1$ . Indeed, for  $f \in \mathcal{A}_\alpha^p(\mathbb{D})$  we get by subharmonicity ([12, proposition 1.1]) that

$$\left| \frac{f(z) - f(0)}{z} \right| \leq \sup_{|w| \leq 1/2} |f'(w)| \lesssim \|f\|_{\mathcal{A}_\alpha^p(\mathbb{D})}, \quad |z| < 1/2.$$

Since  $f \mapsto f(0)$  is continuous on  $\mathcal{A}_\alpha^p(\mathbb{D})$  ([12, proposition 1.1]), we have

$$\begin{aligned} \|S^*f\|_{\mathcal{A}_\alpha^p}^p &\leq \int_{|z| \leq 1/2} \|f\|_{\mathcal{A}_\alpha^p}^p dA_\alpha(z) + 2^p \int_{1/2 < |z| < 1} |f(z) - f(0)|^p dA_\alpha(z) \\ &\lesssim \|f\|_{\mathcal{A}_\alpha^p(\mathbb{D})}^p + \|f - f(0)\|_{\mathcal{A}_\alpha^p(\mathbb{D})}^p \\ &\lesssim \|f\|_{\mathcal{A}_\alpha^p(\mathbb{D})}^p. \end{aligned}$$

From now we suppose that  $p > 1$  and we denote by  $q = \frac{p}{p-1}$ .

**Lemma 2.1.** *Suppose that  $-1 < \alpha < p - 1$ . Then  $\langle \cdot, \cdot \rangle$  defined on  $\mathcal{D}_\alpha^p \times \mathcal{A}_{-\alpha q/p}^q(\mathbb{D})$  by*

$$(2.3) \quad \langle f, g \rangle = \int_{\mathbb{D}} f'(z) \overline{S^*g(z)} dA(z) + f(0) \overline{g(0)}, \quad f \in \mathcal{D}_\alpha^p, g \in \mathcal{A}_{-\alpha q/p}^q(\mathbb{D}),$$

is linear on the left, antilinear on the right and

$$|\langle f, g \rangle| \lesssim \|f\|_{\mathcal{D}_\alpha^p} \|g\|_{\mathcal{A}_{-\alpha q/p}^q(\mathbb{D})}.$$

*Proof.* Since  $-\alpha q/p > -1$ ,  $(f, g) \rightarrow \langle f, g \rangle$  is well defined. Clearly this map is linear on the left and antilinear on the right. It is therefore sufficient to show that

$$|\langle f, g \rangle| \lesssim \|f\|_{\mathcal{D}_\alpha^p} \|g\|_{\mathcal{A}_{-\alpha q/p}^q(\mathbb{D})}.$$

Using Hölder inequality and the fact that the maps  $S^*$  and  $f \mapsto f(0)$  are continuous on the space  $\mathcal{A}_{-\alpha q/p}^q(\mathbb{D})$ , we get

$$\begin{aligned} |\langle f, g \rangle| &\leq \int_{\mathbb{D}} |f'(z)| \frac{(1-|z|^2)^{\alpha/p}}{(1-|z|^2)^{\alpha/p}} |S^*g(z)| dA(z) + |f(0)\overline{g(0)}| \\ &\leq \left( \int_{\mathbb{D}} |f'(z)|^p dA_\alpha(z) \right)^{1/p} \left( \int_{\mathbb{D}} |S^*g(z)|^q dA_{-\alpha q/p} \right)^{1/q} + |f(0)||g(0)| \\ &\leq \|f\|_{\mathcal{D}_\alpha^p} \|S^*g\|_{\mathcal{A}_{-\alpha q/p}^q(\mathbb{D})} + |f(0)||g(0)| \\ &\lesssim \|f\|_{\mathcal{D}_\alpha^p} \|g\|_{\mathcal{A}_{-\alpha q/p}^q(\mathbb{D})}. \end{aligned}$$

□

The previous lemma shows that  $\langle \cdot, \cdot \rangle$  defines a duality between  $\mathcal{D}_\alpha^p$  and  $\mathcal{A}_{-\alpha q/p}^q(\mathbb{D})$ . The following result shows that  $\mathcal{A}_{-\alpha q/p}^q(\mathbb{D})$  can be identified as the dual of  $\mathcal{D}_\alpha^p$ .

**Proposition 2.2.** *Let  $p > 1$  and  $-1 < \alpha < p - 1$ . The dual of  $\mathcal{D}_\alpha^p$ , noted by  $\mathcal{D}_\alpha^{p'}$ , is isomorphic to  $\mathcal{A}_{-\alpha q/p}^q(\mathbb{D})$ .*

*Proof.* We will show that the application  $g \mapsto \langle \cdot, g \rangle$  is an isomorphism of  $\mathcal{A}_{-\alpha q/p}^q(\mathbb{D})$  in  $\mathcal{D}_\alpha^{p'}$  the dual of  $\mathcal{D}_\alpha^p$ . This application is well defined, antilinear, continuous and injective. Let's show that it's surjective. Take  $L$  in  $\mathcal{D}_\alpha^{p'}$ . For all  $f \in \mathcal{A}_\alpha^p(\mathbb{D})$ , we consider  $F$  the primitive of  $f$  on  $\mathbb{D}$  such that  $F(0) = 0$ . It's easy to see that  $F \in \mathcal{D}_\alpha^p$ . We define the application  $L_0$  on  $\mathcal{A}_\alpha^p(\mathbb{D})$  by  $L_0(f) = L(F)$ . Thus  $L_0$  belong to the dual of  $\mathcal{A}_\alpha^p(\mathbb{D})$  since

$$|L_0(f)| = |L(F)| \leq \|L\| \|F\|_{\mathcal{D}_\alpha^p} = \|L\| \|f\|_{\mathcal{A}_\alpha^p}.$$

By the Hahn-Banach theorem,  $L_0$  extends to  $L_\alpha^p(\mathbb{D}) = L^p(\mathbb{D}, dA_\alpha)$  in a continuous linear form  $\widetilde{L}_0$ . By Riesz representation theorem, there exists  $\psi_0 \in L_{-\alpha q/p}^p(\mathbb{D}) = L_\alpha^p(\mathbb{D})'$  such that for any  $g \in L_\alpha^p(\mathbb{D})$ ,

$$\widetilde{L}_0(g) = \int_{\mathbb{D}} g(z) \overline{\psi_0(z)} dA(z).$$

Let  $P$  be the linear map defined by

$$P : f \mapsto \left( z \mapsto \int_{\mathbb{D}} \frac{f(w)}{(1-z\bar{w})^2} dA(w) \right).$$

According to [12, Théorème 1.10],  $P$  is a bounded projection from  $L_\gamma^s(\mathbb{D})$  onto  $\mathcal{A}_\gamma^s(\mathbb{D})$  for  $\gamma < s - 1$  which is the case when  $(s, \gamma) = (p, \alpha)$  and  $(s, \gamma) = (q, -\alpha q/p)$ . Set  $\psi = P(\psi_0) \in \mathcal{A}_{-\alpha q/p}^q(\mathbb{D})$ . So for  $f \in \mathcal{A}_\alpha^p(\mathbb{D})$ , we get

$$\begin{aligned} L_0(f) = \widetilde{L}_0(f) &= \int_{\mathbb{D}} f(z) \overline{\psi_0(z)} dA(z) \\ &= \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{f(w)}{(1 - z\bar{w})^2} \overline{\psi_0(z)} dA(w) dA(z) \\ &= \int_{\mathbb{D}} f(w) \int_{\mathbb{D}} \frac{\overline{\psi_0(z)}}{(1 - w\bar{z})^2} dA(z) dA(w) \\ &= \int_{\mathbb{D}} f(w) \overline{\psi(w)} dA(w). \end{aligned}$$

Thus we showed that there is  $\psi \in \mathcal{A}_{-\alpha q/p}^q(\mathbb{D})$  such that for any  $F \in \mathcal{D}_\alpha^p$  with  $F(0) = 0$ , we have

$$L(F) = \int_{\mathbb{D}} F'(z) \overline{\psi(z)} dA(z).$$

Set  $\varphi(z) = z\psi(z) + \overline{L(1)} \in \mathcal{A}_{-\alpha q/p}^q(\mathbb{D})$ . We have  $S^*\varphi = \psi$ . Hence for  $h \in \mathcal{D}_\alpha^p$

$$\begin{aligned} L(h) &= L(h - h(0)) + L(h(0)) \\ &= \int_{\mathbb{D}} h'(z) \overline{\psi(z)} dA(z) + h(0)L(1) \\ &= \int_{\mathbb{D}} h'(z) \overline{S^*\varphi(z)} dA(z) + h(0)\overline{\varphi(0)} = \langle h, \varphi \rangle. \end{aligned}$$

This shows that the application  $g \mapsto \langle \cdot, g \rangle$  is surjective and defines an isomorphism from  $\mathcal{A}_{-\alpha q/p}^q(\mathbb{D})$  onto  $\mathcal{D}_\alpha^{p'}$ .  $\square$

**Remarks.** If  $p > 1$  and  $\alpha < p - 1$ , the dual of  $\mathcal{D}_\alpha^p$  is identified as  $\mathcal{A}_{-\alpha q/p}^q(\mathbb{D})$ . Also the spaces  $\mathcal{A}_{-\alpha q/p}^q(\mathbb{D})$  and  $\mathcal{A}_{-\alpha q/p}^q(\mathbb{D}_e)$  are isomorphic, so we can identify the dual of  $\mathcal{D}_\alpha^p$  with  $\mathcal{A}_{-\alpha q/p}^q(\mathbb{D}_e)$  by the duality

$$\langle f, g \rangle_e = \langle f, R^{-1}g \rangle, \quad f \in \mathcal{D}_\alpha^p, \quad g \in \mathcal{A}_{-\alpha q/p}^q(\mathbb{D}_e).$$

In the following we will introduce the tools to use the Hedenmalm and Shields Theorem [11, Theorem 1]. For all  $\varphi \in \mathcal{D}_\alpha^{p'}$ , we set

$$\widetilde{\varphi}(\lambda) = \langle f_\lambda, \varphi \rangle, \quad \lambda \in \mathbb{D}_e$$

where  $f_\lambda$  is given by

$$f_\lambda(z) = (\lambda - z)^{-1}, \quad z \in \mathbb{D}.$$

We define then as in [11]

$$\mathcal{D}_\alpha^{p*} = \{ \widetilde{\varphi}, \varphi \in \mathcal{D}_\alpha^{p'} \}.$$

Let  $\varphi \in \mathcal{D}_\alpha^{p'}$ , we have

$$\tilde{\varphi}(\lambda) = \left\langle \sum_{n=0}^{\infty} \frac{z^n}{\lambda^{n+1}}, \varphi \right\rangle = \sum_{n=0}^{\infty} \frac{\langle z^n, \varphi \rangle}{\lambda^{n+1}}.$$

We identify  $\varphi$  as an element of  $\mathcal{A}_{-\alpha q/p}^q(\mathbb{D})$  that we write

$$\varphi(z) = \sum_{n \geq 0} a_n z^n, \quad z \in \mathbb{D}.$$

So if  $n = 0$ ,  $\langle z^n, \varphi \rangle = \overline{\varphi(0)} = \overline{a_0}$  and if  $n \geq 1$ ,

$$\begin{aligned} \langle z^n, \varphi \rangle &= \int_{\mathbb{D}} n z^{n-1} \overline{S^* \varphi(z)} dA(z) \\ &= \int_{\mathbb{D}} n z^{n-1} \sum_{m=1}^{\infty} \overline{a_m} \bar{z}^{m-1} dA(z) \\ &= \sum_{m=1}^{\infty} n \overline{a_m} \int_0^1 \int_0^{2\pi} r^{n+m-2} e^{i\theta(n-m)} d\theta / \pi r dr \\ (2.4) \quad &= \overline{a_n} \int_0^1 2nr^{2n-1} dr = \overline{a_n}. \end{aligned}$$

Thus for  $\lambda \in \mathbb{D}_e$ ,

$$\tilde{\varphi}(\lambda) = \sum_{n=0}^{\infty} \frac{\overline{a_n}}{\lambda^{n+1}}.$$

Moreover, according to (2.2), we also have

$$R\varphi(\lambda) = \sum_{n=0}^{\infty} \frac{\overline{a_n}}{\lambda^{n+1}}, \quad \lambda \in \mathbb{D}_e.$$

So

$$\mathcal{D}_\alpha^{p*} = \mathcal{A}_{-\alpha q/p}^q(\mathbb{D}_e).$$

The following lemma will be useful for expressing duality (voir [11, Lemma 3]).

**Lemma 2.3.** *Let  $p > 1$  and  $-1 < \alpha < p - 1$ . Let  $f \in \mathcal{D}_\alpha^p$  and  $g \in \mathcal{A}_{-\alpha q/p}^q(\mathbb{D}_e)$ . For  $0 \leq r < 1$ , we set*

$$f_r(z) = f(rz), \quad z \in \mathbb{D} \quad \text{and} \quad g_{1/r}(z) = g(z/r), \quad z \in \mathbb{D}_e.$$

Then

$$\langle f, g \rangle_e = \lim_{r \rightarrow 1^-} \langle f_r, g_{1/r} \rangle_e = \lim_{r \rightarrow 1^-} \sum_{n=0}^{\infty} a_n b_n r^n = \lim_{r \rightarrow 1^-} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) g(e^{i\theta}/r) e^{i\theta} d\theta.$$

where  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  and  $g(1/z) = \sum_{n=0}^{\infty} b_n z^{n+1}$ ,  $z \in \mathbb{D}$ .

3. CYCLICITY IN  $\mathcal{D}_\alpha^p$ 

We start this part by comparing the spaces  $\mathcal{D}_\alpha^p$  and the Hardy spaces  $H^p(\mathbb{D})$ . We suppose  $p \geq 1$  and  $\alpha > -1$ . Let  $H^\infty(\mathbb{D})$  be the algebra of bounded analytic functions on the open unit disc  $\mathbb{D}$  and let  $H^p(\mathbb{D})$  be the Hardy space of analytic functions  $f$  on  $\mathbb{D}$  such that

$$\|f\|_{H^p} = \sup_{r < 1} M_p(f, r) < \infty,$$

where

$$M_p(f, r) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}.$$

Let  $\mathcal{N}$  be the Nevanlinna class of analytic functions  $f$  on  $\mathbb{D}$  for which

$$\sup_{r < 1} \int_{\mathbb{T}} \ln^+ |f(r\zeta)| |d\zeta| < \infty.$$

By Fatou's Theorem, the radial limit  $f^*(\zeta) = \lim_{r \rightarrow 1^-} f(r\zeta)$  exists a.e on  $\mathbb{T}$  and  $\ln |f^*| \in L^1(\mathbb{T})$ . Recall that  $f \in \mathcal{N}$  if and only if  $f = \varphi/\psi$ , where  $\varphi, \psi \in H^\infty(\mathbb{D})$ .

Let  $\mathcal{N}^+$  be the Smirnov class of analytic functions  $f \in \mathcal{N}$  such that

$$\sup_{r < 1} \int_{\mathbb{T}} \ln^+ |f(r\zeta)| |d\zeta| = \int_{\mathbb{T}} \ln^+ |f^*(\zeta)| |d\zeta|.$$

The function  $f \in \mathcal{N}^+$  if and only if  $f = \varphi/\psi$  where  $\varphi, \psi \in H^\infty(\mathbb{D})$  and  $\psi$  is an outer function that is  $\psi$  has the form

$$\psi(z) = \exp \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \psi^*(\zeta) \frac{|d\zeta|}{2\pi}, \quad z \in \mathbb{D}.$$

A function  $f \in H^p(\mathbb{D})$  is cyclic for  $H^p(\mathbb{D})$  if and only if  $f$  is outer [?, 4.8.4]. We then study the different possible inclusions between the spaces  $\mathcal{D}_\alpha^p$  and  $H^p(\mathbb{D})$  to obtain first conditions on the cyclicity in the Dirichlet spaces.

**Proposition 3.1.** *Let  $p \geq 1$  and  $\alpha > -1$ . If  $p < \alpha + 1$  then  $H^p(\mathbb{D})$  is continuously embedded in  $\mathcal{D}_\alpha^p$ . Consequently, if  $f \in H^p(\mathbb{D})$  is outer then  $f$  is cyclic for  $\mathcal{D}_\alpha^p$ .*

*Proof.* Let  $f \in H^p(\mathbb{D})$ ,  $z = re^{it} \in \mathbb{D}$  et  $r < \rho < 1$ . By Cauchy formula,

$$f'(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\rho e^{i(\theta+t)})}{(\rho e^{i\theta} - r)^2} \rho e^{i(\theta-t)} d\theta.$$



Now, by Minkowski inequality,

$$\begin{aligned}
M_p(f', r) &= \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\rho e^{i(\theta+t)})}{(\rho e^{i\theta} - r)^2} \rho e^{i(\theta-t)} d\theta \right|^p dt \right)^{1/p} \\
&\leq \frac{\rho}{2\pi} \int_0^{2\pi} \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(\rho e^{i(\theta+t)})|^p}{|\rho e^{i\theta} - r|^{2p}} dt \right)^{1/p} d\theta \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho}{|\rho e^{i\theta} - r|^2} d\theta M_p(f, \rho) \\
&= \frac{\rho}{\rho^2 - r^2} M_p(f, \rho) \leq \frac{1}{\rho - r} M_p(f, \rho).
\end{aligned}$$

Now letting  $\rho \rightarrow 1$ , we get

$$\frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^p d\theta \leq \frac{1}{(1-r)^p} \|f\|_{H^p}^p,$$

Since  $p < \alpha + 1$ ,

$$\begin{aligned}
\int_{\mathbb{D}} |f'(z)|^p dA_\alpha(z) &= \int_0^1 \int_0^{2\pi} |f'(re^{i\theta})|^p d\theta (1-r^2)^\alpha r dr / \pi \\
&\leq 2^{\alpha+1} \int_0^1 \frac{(1-r)^\alpha}{(1-r)^p} dr \|f\|_{H^p}^p \\
&= \frac{2^{\alpha+1}}{\alpha+1-p} \|f\|_{H^p}^p.
\end{aligned}$$

So  $H^p(\mathbb{D})$  is continuously embedded in  $\mathcal{D}_\alpha^p$ . Now the result follows from the fact that an outer function is cyclic in  $H^p(\mathbb{D})$ .  $\square$

**Remark.** If  $p < \alpha + 1$ , the Dirichlet space  $\mathcal{D}_\alpha^p = \mathcal{A}_{\alpha-p}^p(\mathbb{D})$ , see [16]. Therefore, in this case, there exists an inner function which is cyclic in  $\mathcal{D}_\alpha^p$ , see [14]. If  $p > \alpha + 1$  we have the following result.

**Proposition 3.2.** *Let  $p > 1$  and  $p > \alpha + 1$ . The Dirichlet space  $\mathcal{D}_\alpha^p$  is continuously embedded in  $H^p(\mathbb{D})$ . Therefore if  $f \in \mathcal{D}_\alpha^p$  is cyclic in  $\mathcal{D}_\alpha^p$  then  $f$  is an outer function.*

*Proof.* Let  $f \in \mathcal{D}_\alpha^p$  and  $r \in [1/2, 1[$ . We have

$$f(re^{i\theta}) = \int_0^r f'(se^{i\theta}) e^{i\theta} ds + f(0).$$

Note that  $|f(0)| \leq \|f\|_{\mathcal{D}_\alpha^p}$  and by subharmonicity, there exists  $C > 0$  such that  $|f'(se^{i\theta})| \leq C\|f\|_{\mathcal{D}_\alpha^p}$ ,  $0 \leq s \leq 1/2$ . So

$$|f(re^{i\theta})| \leq \int_{1/2}^r |f'(se^{i\theta})| ds + (C/2 + 1)\|f\|_{\mathcal{D}_\alpha^p}.$$

By Hölder inequality, and since  $\alpha q/p = \alpha/(p-1) < 1$ ,

$$\begin{aligned}
\left( \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} &\lesssim \left( \int_0^{2\pi} \left( \int_{1/2}^r |f'(se^{i\theta})| ds \right)^p d\theta \right)^{1/p} + \|f\|_{\mathcal{D}_\alpha^p} \\
&\lesssim \left( \int_0^{2\pi} \left( \int_{1/2}^r |f'(se^{i\theta})|^p (1-s^2)^\alpha ds \right) \left( \int_{1/2}^r (1-s^2)^{-\alpha q/p} ds \right)^{p/q} d\theta \right)^{1/p} + \|f\|_{\mathcal{D}_\alpha^p} \\
&\lesssim \left( \int_0^{2\pi} \int_{1/2}^1 |f'(se^{i\theta})|^p (1-s^2)^\alpha 2s ds d\theta \right)^{1/p} \left( \int_{1/2}^1 (1-s^2)^{-\alpha q/p} ds \right)^{1/q} + \|f\|_{\mathcal{D}_\alpha^p} \\
&\lesssim \left( 1 - \frac{\alpha q}{p} \right) \|f\|_{\mathcal{D}_\alpha^p} + \|f\|_{\mathcal{D}_\alpha^p}
\end{aligned}$$

So  $\|f\|_{H^p} \lesssim \|f\|_{\mathcal{D}_\alpha^p}$ . Hence if  $f$  is cyclic for  $\mathcal{D}_\alpha^p$  then  $f$  is also cyclic for  $H^p(\mathbb{D})$  and  $f$  is then an outer function.  $\square$

**Remark.** We have  $\mathcal{D}_1^2(\mathbb{D}) = H^2(\mathbb{D})$  and  $\mathcal{D}_0^2(\mathbb{D}) = \mathcal{D}$ . So if  $1 \leq p \leq 2$  and  $p = \alpha + 1$ , we obtain by interpolation theorem [16, (3.8)], that  $\mathcal{D}_\alpha^p$  is continuously embedded in  $H^p(\mathbb{D})$ . Furthermore if  $p > \alpha + 2$ , then  $\mathcal{D}_\alpha^p$  is continuously embedded in  $H^\infty(\mathbb{D})$  (see the prove of [16, Theorem 4.2]).

We can summarize here all the inclusions obtained:

$$\begin{aligned}
p < \alpha + 1 &\implies H^p(\mathbb{D}) \subset \mathcal{D}_\alpha^p = \mathcal{A}_{\alpha-p}^p(\mathbb{D}) \\
1 \leq p \leq 2 \text{ et } p = \alpha + 1 &\implies \mathcal{D}_\alpha^p \subset H^p(\mathbb{D}) \\
p > \alpha + 1 &\implies \mathcal{D}_\alpha^p \subset H^p(\mathbb{D}) \\
p > \alpha + 2 &\implies \mathcal{D}_\alpha^p \subset H^\infty(\mathbb{D}).
\end{aligned}$$

We assume in the following that  $p > \alpha + 1$ . We will prove that any outer function of  $A(\mathbb{D}) \cap \mathcal{D}_\alpha^p$  whose set of zeros is reduced to a single point is cyclic in  $\mathcal{D}_\alpha^p$ . For that we will use a Hedenmalm-Shields Theorem [11, Theorem 1]. We first need to define the following notions. Let's say  $X \subset \text{Hol}(\mathbb{D})$  a Banach space. The multiplier of  $X$ , noted  $M(X)$ , is defined by

$$M(X) = \{\varphi \in \text{Hol}(\mathbb{D}), \varphi f \in X, \forall f \in X\}.$$

If  $X \subset \text{Hol}(\mathbb{D}_e)$  we define in a similar way  $M(X)$ .

As in [11] we identify the dual  $X'$  of  $X$  with a space  $X^*$  of holomorphic functions on  $\mathbb{D}_e$ . Finally for  $E \subset \mathbb{T}$  a closed set of zero Lebesgue measure, we set

$$\mathcal{H}_E(\mathcal{N}^+, X^*) = \{\varphi \in \text{Hol}(\mathbb{C} \cup \{\infty\} \setminus E), \varphi|_{\mathbb{D}} \in \mathcal{N}^+(\mathbb{D}), \varphi|_{\mathbb{D}_e} \in X^*\}.$$

We note  $\text{Hol}(\overline{\mathbb{D}})$ , respectively  $\text{Hol}(\overline{\mathbb{D}_e})$ , the space of all holomorphic functions in a neighbourhood of  $\overline{\mathbb{D}}$ , respectively  $\overline{\mathbb{D}_e}$ .

**Theorem 3.3** (Hedenmalm-Shields [11]). *Let  $X \subset \text{Hol}(\overline{\mathbb{D}})$  be a Banach space. Assume that*

- (1) *The embedding map of  $X$  into  $\text{Hol}(\overline{\mathbb{D}})$  is continuous and  $X$  contains  $\text{Hol}(\overline{\mathbb{D}})$  as a dense subspace*
- (2)  *$X \cap A(\mathbb{D})$  is a Banach algebra, containing  $\text{Hol}(\overline{\mathbb{D}})$  as a dense algebra.*
- (3)  *$\text{Hol}(\overline{\mathbb{D}}) \subset M(X)$ .*
- (4)  *$\text{Hol}(\overline{\mathbb{D}_e}) \subset M(X^*) = H^\infty(\mathbb{D}_e)$ .*

*If  $f \in X \cap A(\mathbb{D})$  is an outer function and if*

$$\mathcal{H}_{\mathcal{Z}(f)}(\mathcal{N}^+, X^*) = \{0\}$$

*then  $f$  is cyclic in  $X$ .*

Hedenmalm and Shields show that if  $f \in A(\mathbb{D}) \cap \mathcal{D}_0^2(\mathbb{D})$  is an outer function and  $\mathcal{Z}(f) = \{1\}$  then  $\mathcal{H}_{\mathcal{Z}(f)}(\mathcal{N}^+, \mathcal{D}_0^2(\mathbb{D})^*) = \{0\}$  and so  $f$  is cyclic (see also [8, 6]). We will prove a similar result for  $\mathcal{D}_\alpha^p$  where  $\alpha + 1 < p \leq \alpha + 2$ .

**Theorem 3.4.** *Let  $p > 1$  and  $p > \alpha + 1$ . If  $f \in A(\mathbb{D}) \cap \mathcal{D}_\alpha^p$  is outer and if*

$$\mathcal{H}_{\mathcal{Z}(f)}(\mathcal{N}^+, \mathcal{D}_\alpha^{p*}) = \{0\}$$

*then  $f$  is cyclic on  $\mathcal{D}_\alpha^p$ .*

To prove this result, we will use Theorem 3.3. For that we need only to show the following lemma (see the proof of [5, lemma 11]).

**Lemma 3.5.** *Let  $p > 1$  and  $\alpha > -1$ . Then  $M(\mathcal{A}_\alpha^p(\mathbb{D}_e)) = H^\infty(\mathbb{D}_e)$ .*

*Proof.* Let  $f \in \mathcal{A}_\alpha^p(\mathbb{D}_e)$  and  $g \in H^\infty(\mathbb{D}_e)$ . We have

$$\int_{\mathbb{D}_e} |f(z)g(z)|^p \frac{(|z|^2 - 1)^\alpha}{|z|^{4-p+2\alpha}} dA(z) \leq \|g\|_\infty^p \|f\|_{\mathcal{A}_\alpha^p}^p$$

So  $fg \in \mathcal{A}_\alpha^p(\mathbb{D}_e)$  and  $H^\infty(\mathbb{D}_e) \subset M(\mathcal{A}_\alpha^p(\mathbb{D}_e))$ .

Now let  $g \in M(\mathcal{A}_\alpha^p(\mathbb{D}_e))$  and let  $M_g : \mathcal{A}_\alpha^p(\mathbb{D}_e) \rightarrow \mathcal{A}_\alpha^p(\mathbb{D}_e)$  be the operator given by  $M_g(f) = fg$ . By the closed graph theorem,  $M_g$  is bounded. For  $z \in \mathbb{D}_e$ , the linear functional  $\Lambda_z : \mathcal{A}_\alpha^p(\mathbb{D}_e) \rightarrow \mathbb{C}$  defined by  $\Lambda_z(f) = f(z)$ , is continuous ([12, proposition 1.1]). So for  $f \in \mathcal{A}_\alpha^p(\mathbb{D}_e)$  and  $z \in \mathbb{D}_e$ ,

$$|f(z)g(z)| = |\Lambda_z(M_g f)| \leq \|\Lambda_z\| \|M_g\| \|f\|_{\mathcal{A}_\alpha^p}.$$

Hence

$$\|\Lambda_z\| |g(z)| \leq \|\Lambda_z\| \|M_g\|$$

and  $g \in H^\infty(\mathbb{D}_e)$ . So  $M(\mathcal{A}_\alpha^p(\mathbb{D}_e)) \subset H^\infty(\mathbb{D}_e)$ . On the other hand the inclusion  $H^\infty(\mathbb{D}_e) \subset M(\mathcal{A}_\alpha^p(\mathbb{D}_e))$  is obvious.  $\square$

By identifying the dual of  $\mathcal{D}_\alpha^p$  with  $\mathcal{A}_{-\alpha q/p}^q(\mathbb{D}_e)$ , we have for  $f \in \mathcal{D}_\alpha^p$  and  $\varphi \in \mathcal{A}_{-\alpha q/p}^q(\mathbb{D}_e)$ ,

$$\varphi \in \left( [f]_{\mathbb{N}}^{\mathcal{D}_\alpha^p} \right)^\perp \iff \langle z^n f, \varphi \rangle_e = 0, \quad \forall n \in \mathbb{N}.$$

**Lemma 3.6.** *Let  $p > 1$  and  $p > \alpha + 1$ . Let  $E \subset \mathbb{T}$  a closed set of Lebesgue measure,  $\varphi \in \mathcal{H}_E(\mathcal{N}^+, \mathcal{D}_\alpha^{p*})$  and  $f \in \mathcal{D}_\alpha^p$ . If the family of functions*

$$z \in \mathbb{T} \mapsto f(rz)\varphi(z/r), \quad 1/2 < r < 1,$$

*is uniformly integrable on  $\mathbb{T}$ . Then  $\varphi \in \left( [f]_{\mathbb{N}}^{\mathcal{D}_\alpha^p} \right)^\perp$ .*

*Proof.* This result holds by using the analogue arguments like those in [8, Lemma 3.4] for the classical Dirichlet space. For the sake of completeness, we put it here. Let  $f \in \mathcal{D}_\alpha^p$  and  $\varphi|_{\mathbb{D}_e} \in \mathcal{D}_\alpha^{p*} = \mathcal{A}_{-\alpha q/p}^q(\mathbb{D}_e)$ . By Proposition 2.3, we have

$$\langle f, \varphi \rangle = \lim_{r \rightarrow 1^-} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta})\varphi(e^{i\theta}/r)e^{i\theta} d\theta.$$

By Proposition 3.2,  $\mathcal{D}_\alpha^p \subset H^p(\mathbb{D})$  and so  $f^*$ , the radial limite of  $f$ , exists a.e on  $\mathbb{T}$ . Since  $\varphi \in \text{Hol}(\mathbb{C} \setminus E)$  and  $E$  is a closed set of Lebesgue measure zero,  $\varphi(z/r) \rightarrow \varphi(z)$  exists a.e on  $\mathbb{T}$  when  $r \rightarrow 1^-$ . So the family of the functions  $z \mapsto f(rz)\varphi(z/r)$  converges a.e to  $f^*\varphi$  when  $r \rightarrow 1^-$ . By the uniform integrability, this family of functions converges in  $L^1(\mathbb{T})$  norm. Then

$$\langle f, \varphi \rangle = \frac{1}{2\pi} \int_0^{2\pi} f^*(e^{i\theta})\varphi(e^{i\theta})e^{i\theta} d\theta.$$

Futhermore  $\varphi \in \mathcal{N}^+$  and  $f \in H^p(\mathbb{D}) \subset \mathcal{N}^+$ , then  $f\varphi \in \mathcal{N}^+$ . By Smirnov's generalized maximum principl  $\widehat{f^*\varphi}(n) = 0$

$$\widehat{f^*\varphi}(n) = \langle f, \varphi \rangle = \frac{1}{2\pi} \int_0^{2\pi} f^*(e^{i\theta})\varphi(e^{i\theta})e^{i\theta} d\theta = 0.$$

Repeating the same argument with  $f$  replaced by  $z^n f$ , we get  $\langle z^n f, \varphi \rangle = 0$  for all  $n \in \mathbb{N}$ .  $\square$

We have the following classical Lemma

**Lemma 3.7.** *Let  $p > 1$  and  $p > \alpha + 1$ . Let  $E \subset \mathbb{T}$  be a closed set of Lebesgue measure zero and  $\varphi \in \mathcal{H}_E(\mathcal{N}^+, \mathcal{D}_\alpha^{p*})$ . Then there exists a constant  $C > 0$  such that*

$$|\varphi(z)| \leq \frac{C}{\text{dist}(z, E)^4}, \quad 1 < |z| < 2.$$

*Proof.* Let  $\varphi \in \mathcal{H}_E(\mathcal{N}^+, \mathcal{D}_\alpha^{p*})$ . Since  $\varphi|_{\mathbb{D}} \in \mathcal{N}^+$ ,  $\varphi|_{\mathbb{D}} = \varphi_i \varphi_o$ , where  $\varphi_i$  is an inner function and  $\varphi_o$  is an outer function in  $\mathcal{N}$  (see [4, p. 25]). Furthermore, since  $E$  has Lebesgue measure zero,  $\varphi(z) = \varphi^*(z) = \lim_{r \rightarrow 1^-} \varphi(rz)$  exists a.e on  $\mathbb{T}$ . The function  $\log |\varphi|$  being in  $L^1(\mathbb{T})$ , we get

$$\begin{aligned} |\varphi(z)| &\leq |\varphi_o(z)| = \left| \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log |\varphi(e^{it})| dt \right) \right| \\ &\leq \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} \log |\varphi(e^{it})| dt \right) \\ &\leq \exp \left( \frac{1 - |z|^2}{(1 - |z|)^2} \int_0^{2\pi} |\log |\varphi(e^{it})|| dt \right) \\ &\leq \exp \left( \frac{2}{1 - |z|} \|\log |\varphi|\|_{L^1(\mathbb{T})} \right) \\ &\leq \exp \left( \frac{C_1}{1 - |z|} \right), \end{aligned}$$

for some constant  $C_1 > 0$ . Let  $z \in \mathbb{D}_e$  with  $|z| \leq 2$ . The disc of radius  $(|z| - 1)/2$  centered at  $z$ ,  $D(z, (|z| - 1)/2)$  is contained in  $\mathbb{D}_e$ . Since  $\varphi|_{\mathbb{D}_e} \in \mathcal{A}_{-\alpha q/p}^q(\mathbb{D}_e)$ , by subharmonicity of  $|\varphi|$  and for  $q = p/(p - 1) \geq 1$ , we obtain

$$\begin{aligned} \frac{(|z| - 1)^2}{4} |\varphi(z)|^q &\leq \frac{1}{\pi} \int_{D(z, (|z| - 1)/2)} |\varphi(w)|^q dA(w) \\ &\leq \frac{1}{\pi} \int_{D(z, (|z| - 1)/2)} |\varphi(w)|^q \frac{(|w|^2 - 1)^{-\alpha q/p}}{|w|^{4 - q - 2\alpha q/p}} \frac{|w|^{4 - q - 2\alpha q/p}}{(|w|^2 - 1)^{-\alpha q/p}} dA(w) \\ &\leq \max(2^{2\alpha q/p}, 2^{4 - q}) \int_{\mathbb{D}_e} |\varphi(w)|^q \frac{(|w|^2 - 1)^{-\alpha q/p}}{|w|^{4 - q - 2\alpha q/p}} dA(w) \\ &\leq \max(2^{2\alpha q/p}, 2^{4 - q}) \|\varphi|_{\mathbb{D}_e}\|_{\mathcal{B}_{-\alpha q/p}^q}^q. \end{aligned}$$

So

$$|\varphi(z)| \leq \frac{C_2}{(|z| - 1)^2}, \quad 1 < |z| \leq 2,$$

for some constant  $C_2 > 0$ . Since  $\log |\varphi|$  is subharmonic function, By Taylor-Williams estimates [15, lemma 5.8 and 5.9] and [6, Lemma 9.6.5], we get the lemma.  $\square$

The following result allows us to reduce the study of cyclic vectors vanishing on a closed set  $E$  to the study of cyclicity of a particular specific functions. More precisely we have

**Theorem 3.8.** *Let  $p > 1$  and  $p > \alpha + 1$ . Let  $f \in \mathcal{D}_\alpha^p$  and  $E \subset \mathbb{T}$  be a closed set of Lebesgue measure zero. If there exists a constant  $C_1 > 0$  such that,*

$$|f(z)| \leq C_1 \text{dist}(z, E)^4, \quad z \in \mathbb{D},$$

then

$$\mathcal{H}_E(\mathcal{N}^+, \mathcal{D}_\alpha^{p*}) \subset \left( [f]_{\mathbb{N}}^{\mathcal{D}_\alpha^p} \right)^\perp.$$

This means that for all  $g \in \mathcal{H}_E(\mathcal{N}^+, \mathcal{D}_\alpha^{p*})$ ,  $g|_{\mathbb{D}_e} \in \left( [f]_{\mathbb{N}}^{\mathcal{D}_\alpha^p} \right)^\perp$  i.e.

$$\langle z^n f, g|_{\mathbb{D}_e} \rangle_e = 0, \quad \forall n \in \mathbb{N}.$$

*Proof.* Let  $\varphi \in \mathcal{H}_E(\mathcal{N}^+, \mathcal{D}_\alpha^{p*})$ . By lemma 3.7, there exists a constant  $C_2 > 0$  such that

$$|\varphi(z)| \leq \frac{C_2}{\text{dist}(z, E)^4}, \quad 1 < |z| < 2.$$

So for  $1/2 < r < 1$  and  $z \in \mathbb{T}$ , we have

$$|f(rz)\varphi(z/r)| \leq C_1 C_2 \frac{\text{dist}(rz, E)^4}{\text{dist}(z/r, E)^4} \leq C_1 C_2.$$

The family of the functions  $z \mapsto f(rz)\varphi(z/r)$  is uniformly integrable on  $\mathbb{T}$  pour  $1/2 < r < 1$ , thus by Lemma 3.6,  $\varphi \in \left( [f]_{\mathbb{N}}^{\mathcal{D}_\alpha^p} \right)^\perp$ , which finishes the proof.  $\square$

**Corollary 3.9.** *Let  $p > 1$  such that  $\alpha + 1 < p \leq \alpha + 2$ . We have*

$$\mathcal{H}_{\{1\}}(\mathcal{N}^+, \mathcal{D}_\alpha^{p*}) = \{0\}.$$

*Proof.* Let  $f(z) := (z - 1)^4$ . We have  $f \in \mathcal{D}_\alpha^p$  and  $|f(z)| \leq |z - 1|^4$ . By Theorem 3.8 ,

$$\mathcal{H}_{\{1\}}(\mathcal{N}^+, \mathcal{D}_\alpha^{p*}) \subset \left( [f]_{\mathbb{N}}^{\mathcal{D}_\alpha^p} \right)^\perp.$$

It suffices to prove that  $f$  is cyclic. Let  $\varphi \in \mathcal{A}_{-\alpha q/p}^q(\mathbb{D})$  such that

$$\langle z^n(z - 1), \varphi \rangle = 0, \quad \forall n \in \mathbb{N}.$$

Write  $\varphi(z) = \sum_{n \geq 0} a_n z^n$ , we get by (2.4),

$$\overline{a_n} = \langle z^n, \varphi \rangle = \langle z^{n+1}, \varphi \rangle = \overline{a_{n+1}}.$$

Then

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n = \frac{a_0}{1 - z}, \quad z \in \mathbb{D}.$$

Suppose that  $\varphi \neq 0$ . Since  $\varphi \in \mathcal{A}_{-\alpha q/p}^q(\mathbb{D})$ , we have

$$(3.1) \quad \int_{\mathbb{D}} \frac{(1 - |z|^2)^{-\alpha q/p}}{|1 - z|^q} dA(z) < \infty,$$

and so  $q + \alpha q/p < 2$  (see [12, Theorem 1.7]), which contradicts the assumptions on  $p$  and  $\alpha$ . So  $\varphi = 0$  and  $[z - 1]_{\mathbb{N}}^{\mathcal{D}_\alpha^p} = \mathcal{D}_\alpha^p$ . In particular  $z - 1 \in [(z - 1)^2]_{\mathbb{N}}^{\mathcal{D}_\alpha^p}$  and then

$$[(z - 1)^2]_{\mathbb{N}}^{\mathcal{D}_\alpha^p} = [z - 1]_{\mathbb{N}}^{\mathcal{D}_\alpha^p} = \mathcal{D}_\alpha^p.$$

With the same argument we obtain

$$[(z-1)^4]_{\mathbb{N}}^{\mathcal{D}_\alpha^p} = \mathcal{D}_\alpha^p,$$

and  $f(z) = (z-1)^4$  is cyclic in  $\mathcal{D}_\alpha^p$ . □

**Remark.** The proof of the previous result also gives us that for  $p > \alpha + 2$ , the function  $f(z) = z - 1$  is not cyclic in  $\mathcal{D}_\alpha^p$ . Indeed by (3.1),  $\varphi(z) = 1/(1-z) \in \mathcal{A}_{-\alpha q/p}^q(\mathbb{D})$  and  $\varphi \perp z^n f$ ,  $n \in \mathbb{N}$ . More generally if  $f \in A(\mathbb{D}) \cap \mathcal{D}_\alpha^p$  with  $f(1) = 0$ , then  $f$  is not cyclic in  $\mathcal{D}_\alpha^p$ . Indeed for  $p > \alpha + 2$ , we have  $\mathcal{D}_\alpha^p \subset H^\infty(\mathbb{D})$  with  $\|\cdot\|_{H^\infty} \lesssim \|\cdot\|_{\mathcal{D}_\alpha^p}$  which implies

$$[f]_{\mathbb{N}}^{\mathcal{D}_\alpha^p} \subset \{g \in A(\mathbb{D}), g(1) = 0\}.$$

**Theorem 3.10.** *Let  $p > 1$  such that  $\alpha + 1 < p \leq \alpha + 2$  and let  $f \in A(\mathbb{D}) \cap \mathcal{D}_\alpha^p$ . If  $f$  is an outer function and  $\mathcal{Z}(f)$  is countable then  $f$  is cyclic in  $\mathcal{D}_\alpha^p$ .*

*Proof.* Since  $\mathcal{Z}(f)$  is countable, by [2, Theorem 3] it suffices to prove the theorem when the zero set is reduced to a single point. The result now follows by Theorem 3.4 and Corollary 3.9. □

**Acknowledgement.** The authors thank Javad Mashreghi and Laval University for its support and hospitality.

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