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A mathematics educator and a mathematician co-teaching mathematics – Affordances for teacher education

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Prospective secondary school teachers are required to take undergraduate courses in mathematics, which may be of limited relevance for their teaching. In this study, we investigate affordances of co-teaching for achieving such relevance. This is a qualitative study of an undergraduate course on Mathematical Proof and Proving, co-taught by a professor of mathematics education and a professor of mathematics. Analyzing an episode critiquing three different proofs, we show that the mathematician was concerned mainly with the written proof and its “correctness”, whereas the mathematics educator showed a sensitivity to the person behind the proof, and to pedagogical aspects of proof and proving. We propose that such a course may help students reconcile conflicts between how mathematics is taught and practiced in university and in high school, and suggest such co-teaching as a model for achieving relevance for teaching in mathematics courses.

Keywords: Mathematical proof and proving, teacher education, undergraduate mathematics.

Background and setting

The study reported herein was carried out in the context of an undergraduate course on mathematical proof and proving (MPP) taught at a major university in the USA¹. Typically, such courses are the responsibility of mathematics departments, yet this course was conceived and designed through collaboration between the university’s department of teaching and learning and its mathematics department, and was co-taught by a professor of mathematics education – the second author of this paper, herein MEI (Math Educator Instructor) – and a professor of mathematics, herein MI (Mathematician Instructor). The goal of this collaboration was to capitalize on the two departments’ complementary fields of expertise – mathematics education and mathematics. This setting provides a unique opportunity to investigate differences and interactions between these two different types of mathematical expertise. Some preliminary findings were reported by Sabouri, Thoms, & Zaslavsky (2013) and by Zaslavsky & Cooper (2017), where some aspects of the co-teaching were discussed.

The course was open to a wide range of students, but the majority of participants were enrolled in teacher preparation programs, for which it was a required course. Accordingly, we situate our study in the context of the mathematical preparation of pre-service secondary-school teachers. These students have two different needs for MPP. In the short term, they must be proficient in university routines of proving in order to succeed in their mathematics courses. The notion of formal proof is new for many university students, and the transition to the kind of mathematical proving that is required in undergraduate courses is known to be difficult (Harel & Sowder, 2007). In the longer term, the majority of these students will become secondary school mathematics teachers. As such, they will be expected to teach MPP in their classrooms and to assess students’ proficiency in and understanding of mathematical proving. Thus they are faced with the challenge of two transitions,

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first from high school to university MPP, and then back to high school. One way of addressing this challenge is to offer two separate courses, taught in mathematics departments and in schools of education, leaving it up to students to reconcile differences between MPP in the two contexts. Our study suggests an alternative; perhaps a course that is co-taught by professors of mathematics and of mathematics education will help reconcile the different, sometimes conflicting notions of MPP at high school and undergraduate levels. This is the overarching question that guides our study.

Literature review and research questions

Secondary school teachers are usually required to take some university level mathematics (ULM) courses in their pre-service training; in some contexts, they are even required to hold an undergraduate degree in mathematics or in a related field. Yet many ULM courses, taught in mathematics departments, deal with advanced mathematical content whose relevance for teaching is not immediately obvious. A number of researchers have investigated affordances of ULM courses for teaching, using a methodology of teachers' self-reporting (e.g., Adler, et al., 2014; Even, 2011; Zazkis & Leikin, 2010). Such methodologies have been found to be of limited value; teachers tended to report on general affordances of learning mathematics from mathematicians, however, Zazkis & Leikin (2010) found that teachers were generally unable to specify in what ways they made use of ULM in their teaching. In our study we utilize a different methodology for revealing affordances of the MPP course for teaching, in analysing the teaching in this unusual setting.

According to Harel & Sowder (2009), mathematicians teaching undergraduate courses are not fully aware of student difficulties in learning MPP. The conception of co-teaching in our context was similar in some aspects to the way it is utilized in special education (Friend et al., 2010), letting the mathematician and the mathematics educator share the responsibility for the content, while leaving it up to the educator to attend to students' "special needs". Thus, we hypothesize that the mathematician will take responsibility for epistemic aspects of the course, while the educator will take responsibility not only for addressing the students' mathematical difficulties, but also for the course's relevance for teaching. This hypothesis is consistent with Cooper's findings (2016) in his study of a professional development course for primary school teachers taught by research mathematicians, where the mathematicians took responsibility for the mathematical content, while the participating teachers themselves took responsibility for achieving relevance for teaching. Accordingly, our research questions are:

1. What are the instructors' views on MPP? In what ways are their views similar or different?
2. What are the affordances of different views on MPP for the students as future teachers?

Theoretical framework and epistemic analysis

We join researchers such as Nardi et al. (2014) in taking a Commognitive approach (Sfard, 2008). Mathematical "knowledge" is conceived as a particular community's established modes of communication, called discourses. These are constituted in commonly used keywords (e.g., "proof", "given"), in narratives that are endorsed or rejected by the community (e.g., proofs), in visual mediation that is considered useful (e.g., two-column format of a proof), and in repetitive routines (e.g., proving). The very different mathematical discourses of two communities - research mathematicians and researchers in mathematics education - differ in their use of common keywords, in the types of routines they engage in, and in the rules and norms that determine which narratives

will be endorsed and which will be rejected, and these differences are grounded in the communities' activities – mathematical research on the one hand and mathematics education on the other.

Proof is a *genre* in mathematical discourse, a type of narrative that is expected to adhere to a community's conventions. It usually includes text, and may employ a variety of means to visually mediate its mathematical ideas. A proof may be endorsed or rejected based on the metarules of a community's mathematical discourse. Proving is a routine in mathematical discourse, with the goal of producing a proof. Ideally, the prover should take responsibility for producing a “valid” (i.e. endorsable) proof, which may subsequently be endorsed or rejected by a teacher or by peers (classmates or fellow researchers). Routines of proving, and the proofs that are produced, are governed by very different metarules in school and in university.

Method and data

The study reported herein is part of a larger study, for which extensive data were collected, including video recording and field notes of each of the 13 lessons taught in 2013, audio recording and field notes of weekly meetings held following each lesson, email exchanges among members of the teaching team, and students' written homework with TAs' comments and grading. The rich data offers insight into the instructors' intentions, however, in the current analysis we are concerned with affordances of the co-teaching as it played out, and thus focus our attention on the video recordings of the lessons, and, in particular, on a single episode from lesson 8, selected for being rich in interactions between the MI and MEI. We assumed that the two instructors referring to each other's ideas, possibly disagreeing with each other, would reveal their tacit norms and ideas about MPP, and highlight differences between them.

Analysis

In the following episode, MI and MEI discuss a homework assignment, where an exemplary proof had been provided for the claim $|x \cdot y| = |x| \cdot |y|$. Three other responses (also ostensibly proofs) were included in the assignment, and students were asked to critique them. These “proofs”, labeled 2.1, 2.2, 2.3, were copied onto the board, and were discussed.

Format: we present one or more utterances followed by a short annotation raising points that we later elaborate in the discussion. For brevity, utterances not relevant for our analysis were omitted.

Sample proof 2.1

MI One thing I told you in the very first lecture is to do what? Write the “given” and write RTP [remains to prove] ... This is a mistake that is commonly made because people confuse what is given and what you need to prove.

MI considered proof 2.1 unacceptable, since it begins with what needs to be proven, and manipulates it to obtain the incontrovertible $a \cdot b = a \cdot b$. He believes that following a simple rule of writing the given and writing RTP at the outset can help students avoid this common mistake.

2.1
 CASE 1: LET $x = -a$, $y = -b$
 $|-a \cdot -b| = |-a| \cdot |-b|$
 $a \cdot b = a \cdot b$
 CASE 2: LET $x = -a$, $y = b$
 $|-a \cdot b| = |-a| \cdot |b| \Rightarrow a \cdot b = a \cdot b$
 CASE 3: LET $x = a$, $y = b$
 $|a \cdot b| = |a| \cdot |b| \Rightarrow a \cdot b = a \cdot b$

Figure 1: Proof 2.1

MEI [Where] we provided proof [in the HW], we actually wrote the given and what you have to find, and if the people who wrote this proof [2.1] started by saying: “given”, then there would be no confusion of what should the last line be. The last line being verifying this.

MEI elaborates MI’s point, but soon after gives a different perspective on this rule of thumb:

MEI It's more than writing what's given and what you have to prove. It's also accounting in each line ... what is the status of what you wrote. Is it given, is it a known fact that you bring from some other place, which is fine. You have to annotate and write where it comes from, how you got to there... If we can infer the following [line], we have to say it... We need these words to make sense of what the status of each line is. Because you'll suddenly ask what is this? How do I know it? It's a mean of communication, but it's also a means of sort of control of what you're doing... ..We need to know [where it came from], not just us to follow you, but mainly for you to produce a correct proof. If you skip and don't account for each line, you're more likely to make a mistake.

MEI stresses the importance of accounting for everything that is written. The most obvious reason, stressed by MI earlier in the lesson, is that these are the norms of the genre; this is the way you communicate with others within the mathematical community. However, MEI’s words suggest additional considerations. First, in her words “we need to know... not just to follow you” she appears to be conscious of the pedagogical setting, where instructors need to follow the student’s thinking in order to assess their work. Additionally, she sees the written proof not only as a means of communicating with others, but also for communicating with oneself (i.e., thinking). Her word “control” alludes to metacognitive aspects of proving, and MEI seems to be suggesting that following norms of writing a proof may contribute to the process of producing a valid proof.

Sample proof 2.2

MI The proof started by saying... this $[(xy)^2 = x^2 \cdot y^2]$ is correct, right? Does [it] imply $[\sqrt{(xy)^2} = \sqrt{x^2 \cdot y^2}]$?

MEI The thinking of this person was that they're taking what we need to prove and squaring it and getting there

For MI, it is the proof that is “saying” something, whereas MEI draws attention to the thinking of the person behind the proof.

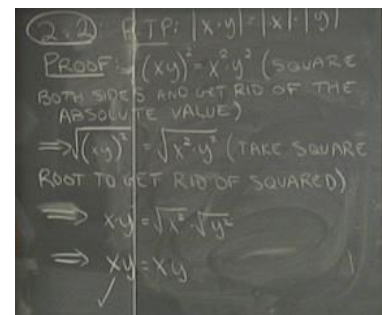


Figure 2: Proof 2.2

MI I'm asking you simply does anyone in this room disagree up to this step?

The implication of MI’s question (“does anyone disagree”) is that the proof may be correct (at least “up to this step”) regardless of the thinking of the person who wrote the proof (e.g., squaring the RTP instead of beginning with the given).

MEI This would be ok without this [striking through lines in the parentheses “square both sides and get rid of the absolute value”]. Because what it says here, that you took this as given in a way, and squared it and got this.

In spite of MEI’s previous attention to the prover, she is now showing how the written proof can be fixed by striking out the parts that were a consequence of the prover’s misguided thinking.

MI Sorry, I did not read that. I was just looking at the equations. Absolutely right, this is making the same mistake [starting with RTP instead of with Given].

MEI I do want to say very nicely that at least the logic of the thinking was clear here, because whoever did it provided the explanation, and this is easier to follow, and important.

MI, in assessing the correctness of the proof, had not read the text in parentheses. This further supports the suggestion that he is concerned with the proof as a mathematical product (i.e., the equations), and not with the prover's thinking, which is represented in the explanations. MEI, in contrast, values not only the mathematical correctness of the proof, but also the clarity of thinking that is revealed. Here again she is showing concern for pedagogical aspects of MPP.

MI Whoever wrote the proof had a very good idea at the beginning, and choked at this point [$xy = xy$]

Here MI refers for the first time to the person who wrote the proof. However, his claim that this person "had a very good idea" is not justified. This claim was based on the fact that the first line on the board - $\sqrt{(xy)^2} = \sqrt{x^2 \cdot y^2}$ - can serve as the beginning of a valid proof. In this he is ignoring the proving process and the thoughts of the prover, as reflected in the words that MEI crossed out.

Sample proof 2.3

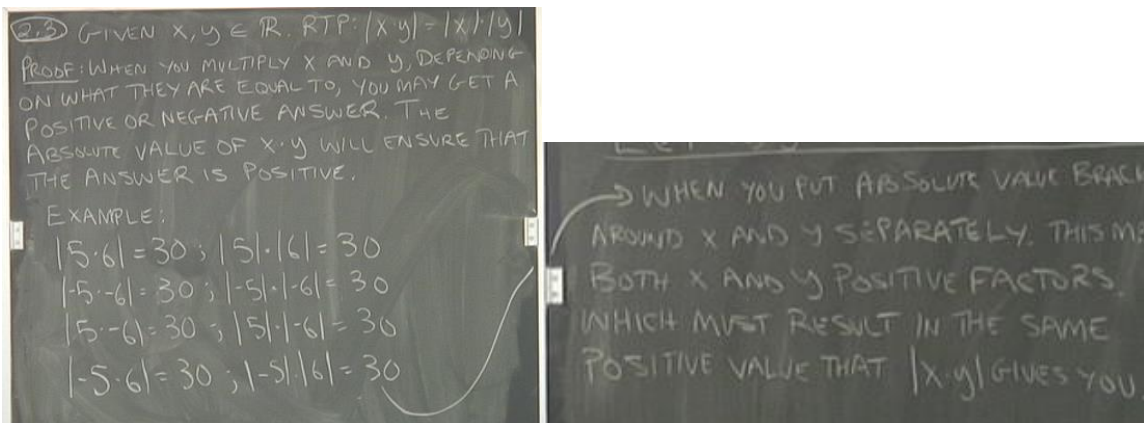


Figure 3: Proof 2.3

MI How many of you think this is a proof? How many of you don't think this is a proof?

MEI What do you think J's opinion is? Does he think it's acceptable or not?

In her question, MEI is allowing for the possibility that the students' opinion will be different from MI's, but she is suggesting that they should be coming around to seeing things Jim's way.

MI It is a proof. It's a badly written proof, but it is a proof. Whoever wrote this made a lousy job of writing [it]. That's the only mistake that person made... because they don't know how to write a proof. How to present it. So the mistake here is not in the content but in the presentation. The person ended up writing two paragraphs for two lines.

These words highlight three aspects of a proof in MI's discourse. There is the end product, what he calls the "presentation", which in this case he considers "lousy" (Jim later rewrites the proof in two condensed lines of mathematical expressions). There is the "content" of the proof, the mathematical

ideas that underlie the presentation and are revealed in it, which in this case are valid. Finally, there is the thinking of the prover, which MEI is keenly aware of, but Jim appears to be ignoring.

Discussion

In this section, we discuss findings from the analyzed episode, along with some additional findings from other episodes whose analysis we omitted for brevity. Regarding our first research question, we found, as hypothesized, that MEI and MI stressed different aspects of MPP. We present similarities and differences between their discourse, as it pertains to proof and proving.

The human element in a mathematical proof

MI and MEI both held the view that a text purporting to be a mathematical proof must adhere to specific norms of communication. For Jim, the question of validity was central: does the proof begin with the given, end with what needs to be proven, and is each line in the proof mathematically justified. This is a consequence of the communicational role of proof in his discourse – to convince members of the community that a claim is valid. MEI, too, was concerned with the mathematical validity of arguments, yet she was also sensitive to pedagogical aspects of proving, and considered the prover's communication with a teacher, who is not only assessing the validity of a mathematical text, but also the nature of the mathematical thinking that produced it. There were also differences in the instructors' attitude to the prover's communication with herself. MI saw two distinct phases in proving, a draft phase (which he called a "scratch") where the prover does her thinking and is not accountable for what is written, and the final product which will be scrutinized by others. Thus, for MEI a student's proof should reveal the prover's underlying thinking, whereas for MI the final product should conceal thinking. MEI held a more integrated view regarding the phases of producing a proof, where the "accounting" in the final product could serve a metacognitive role in the process of proving, by "controlling" the flow and minimizing mistakes. These differences in the instructors' attitudes to proof and proving are evident in their use of language. MI spoke of what the proof "is saying", whereas for MEI it is a human agent who is "saying" something. Jim frequently asked if the students "agree up to this step", where he is referring to a step in the written proof. In sample proof 2.2 MI did not pay attention to the student's thinking, as reflected in the text in parentheses. It was MEI who suggested striking out these lines, but though this would "fix" the proof, she realized that it would not fix the thinking of the person who produced it. MI, on the other hand, when speaking of the process of producing proof 2.2, attributed a "good idea" to the prover, based on the fact that a valid proof could have begun with the first line. He felt that after this promising start the person had "choked", and this just a few seconds after he conceded that the prover had in fact made "the same mistake" as the prover in sample 2.1 - beginning with what needs to be proven.

In spite of MI's attention in sample proofs 2.1 and 2.2 to proof as a text, in proof 2.3 he suggested that this text may be a "representation" (i.e., a visual mediation) for something else. Though the proof was badly written, he felt that there was no mistake in the "content" of the proof. The nature of this underlying content and its relationship with its representation as a text is not clear. MI does not appear to be alluding to a human agent's thinking, but this point requires further analysis of his discourse, drawing on additional data.

The role of proof in mathematical discourse

Later in the lesson MEI pointed out how the pedagogical context of the course may give a distorted view of the role of proving in mathematics: “Mathematicians do proof in order to establish theories... But what happens in this course, because the focus is on how to really construct proofs, sometimes we're doing it about facts that may be trivial to you... we may be giving you a wrong message”. This may explain the importance MEI attributed to a comment from MI in lesson 6, regarding a proof of the claim: if $n > 10$, then $n^5 - 6n^4 + 27 \geq 0$. At MEI's prompting, MI showed that a careful analysis of the proof reveals that the expression is not only greater than 0, it is in fact greater than 40,000! Why should this be important? Jim explained that “you might need [this “stronger” fact] later on in the proof”. MEI re-voiced this idea, but we suspect that for her it had a second role, in addressing “the wrong message” about the nature of proving. The task, given by some anonymous agent, was to prove “greater than 0”. In showing more than was required, MI and MEI were modeling proving as an investigation, where the prover has some agency in deciding what to prove.

Affordances of the co-taught course for future teachers: Explicit and implicit goals

We hypothesized that a course on MPP co-taught by a mathematician and a researcher in mathematics education would have special affordances for futures teachers, in presenting and reconciling different aspects of MPP that are crucial for teaching. Our analysis has demonstrated some affordances.

The explicit goal of the course was transitional – to help students learn the mathematics department's norms of MPP, and to develop proficiency in university level routines of proving. This goal was addressed by both instructors, with Jim taking a leading role. MEI accepted and encouraged Jim's role as the mathematical authority, in calling on him to take over when crucial mathematical issues were at stake. It was MI who modeled the kind of proving that will be expected in advanced university courses. Furthermore, if students internalize MI's discourse, they may eventually bring a commitment to mathematical precision and rigor to their own classrooms. MEI did not disagree with Jim. However, in stressing other aspects of MPP, we feel that she was modeling not only how to produce an acceptable proof, but also how to teach MPP. In her attention to the thinking behind students' proofs she was modeling how these future teachers should be concerned not only with what their students are writing, but also with the mathematical thinking that drives their work. She was further concerned with some meta-level issues, such as the investigative nature of mathematical activity, and the development of metacognitive skills of self-monitoring the process of proving.

Thus, students were exposed to two different perspectives on MPP, both of which they will need to internalize as teachers. However, MI and MEI by and large taught in the *one teach, one observe* approach (Friend et al., 2010), and their points of view remained disjoint. In fact, being aware of their differences and wishing to present a unified front, they often tried to resolve them at the planning stage of lessons (Sabouri, Thoms, & Zaslavsky, 2013). Had they taught in the *teaming* approach, described as “representing opposite views in a debate, illustrating two ways to solve a problem, and so on” (Friend et al., p. 12), there may have been opportunities to openly discuss differences, encouraging students to reconcile different aspects of MPP relevant for their future teaching.

References

- Adler, J., Hossain, S., Stevenson, M., Clarke, J., Archer, R., & Grantham, B. (2014). Mathematics for teaching and deep subject knowledge: Voices of mathematics enhancement course students in England. *Journal of Mathematics Teacher Education*, *17*, 129–148.
- Cooper, J. (2016). *Mathematicians and primary school teachers learning from each other*. Unpublished doctoral dissertation.
- Even, R. (2011). The relevance of advanced mathematics studies to expertise in secondary school mathematics teaching: practitioners' views. *ZDM*, *43*(6-7), 941–950.
- Friend, M., Cook, L., Hurley-Chamberlain, D., & Shamberger, C. (2010). Co-teaching: An illustration of the complexity of collaboration in special education. *Journal of Educational and Psychological Consultation*, *20*(1), 9–27.
- Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 805–842). Charlotte, NC: Information Age Publishing.
- Harel, G., & Sowder, L. (2009). College Instructors' views of students vis-à-vis proof. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and Learning Proof Across Grades* (pp. 275–289). New York: Routledge Taylor & Francis Group.
- Nardi, E., Ryve, A., Stadler, E., & Viirman, O. (2014). Commognitive analyses of the learning and teaching of mathematics at university level: the case of discursive shifts in the study of Calculus. *Research in Mathematics Education*, *16*(2), 182–198.
- Sabouri, P., Thoms, M., & Zaslavsky, O. (2013). The merits of collaboration between mathematicians and mathematics educators on the design and implementation of an undergraduate course on mathematical proof and proving. In S. Brown, G. Karakok, K. H. Roh, & M. Oehrman (Eds.), *Proceedings of the 16th Annual Conference on Research in Undergraduate Mathematics Education*, *2* (pp. 307–313). Denver, Colorado.
- Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge: Cambridge University Press.
- Zaslavsky, O., & Cooper, J. (2017). What constitutes a proof? Complementary voices of a mathematician and a mathematics educator in a co-taught undergraduate course on mathematical proof and proving. *Online Proceedings of the 20th conference on Research in Undergraduate Mathematics Education*. San Diego.
- Zazkis, R., & Leikin, R. (2010). Advanced mathematical knowledge in teaching practice: Perceptions of secondary mathematics teachers. *Mathematical Thinking and Learning*, *12*(4), 263–281.