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Abstract

We study a local multiplicity problem related to so-called generalized Shalika models. By establishing a local trace formula for these kinds of models, we are able to prove a multiplicity formula for discrete series. As a result, we can show that these multiplicities are, for discrete series, invariant under the local Jacquet-Langlands correspondence and are related to local exterior square L-functions.

Contents

1 Introduction

Let G be a p-adic reductive group, H a closed subgroup of G and χ a character of H (potentially the trivial one). To every smooth irreducible representation π of G, we associate a multiplicity

$$m(\pi, \chi) := \dim \mathrm{Hom}_H(\pi, \chi)$$

If the subgroup H is *spherical* (that is it admits an open orbit on the flag variety of G) then we expect these multiplicities to always be finite (This is already known in a certain number of cases see [10, Theorem 4.5] and [34, Theorem 5.1.5]) and to roughly detect certain kind of functorial lifts. For a good references on this circle of ideas, that has come to be called the *relative local Langlands program*, we refer the reader to [33] and to the monograph [34] that set forth a general formalism 'à la Langlands' for these kinds of problems.

In the foundational papers [38], [39], Waldspurger has discovered a new way to attack these questions by proving a certain integral formula computing the multiplicity $m(\pi, \chi)$ in the case of the so-called orthogonal Gross-Prasad models which, together with some twisted version of it related to epsilon factors of pair, has found a remarkable application to the local Gross-Prasad conjecture for orthogonal groups (see [40], [30]). This line of attack has then been adapted by the first author of this paper [4], [5] to deal with the local Gross-Prasad conjecture for unitary groups and by the second author [41], [42] in the setting of the so-called Ginzburg-Rallis models. Subsequently, in [6] the first author has also find another application of this method to a conjecture of Prasad concerning Galois pairs. In all these cases the basic tool to prove the aforementioned multiplicity formulas has been some new

kind of local (simple) trace formulas in the spirit of Arthur [2]. However, the proofs of these trace formulas, and particularly of their geometric sides, have each time been done in some ad hoc way pertaining to the particular features of the case at hand. It makes now little doubt that such trace formulas should exist in some generality. However, we provide here another example for the 'generalized Shalika models' in the hope that it can shed some light on the general features of a potential generalization.

1.1 Main results

Let F be a p-adic field and $\mathcal{A}$ be a central simple algebra over F of rank n (i.e. $\mathcal{A} = \mathrm{Mat}_m(\mathcal{D})$ where $\mathcal{D}/F$ is a division algebra of degree r and $n = mr$). We will denote by $\mathrm{Tr}_{\mathcal{A}/F} : \mathcal{A} \to F$ and $N_{\mathcal{A}/F} : \mathcal{A} \to F^\times$ the reduced trace and norm respectively. Set $G := \mathrm{GL}_2(\mathcal{A})$ and define the following subgroups of G:

- $H_0 := \{\begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} \mid \lambda \in \mathcal{A}^\times\}$;
- $N := \{\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix} \mid X \in \mathcal{A}\}$;
- $H := H_0 \ltimes N$.

Fix a continuous character $\omega : F^\times \to \mathbf{C}^\times$ that we identify with a character of H_0 through composition with the morphism $N_{\mathcal{A}/F} : H_0 \to F^\times$ defined by $N_{\mathcal{A}/F}(\begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}) = N_{\mathcal{A}/F}(\lambda)$. Let $\psi : F \to \mathbb{C}^\times$ be a nontrivial character and define $\xi : N \to \mathbf{C}^\times$ by

$$\xi \begin{pmatrix} 1 & X \\ & 1 \end{pmatrix} := \psi(\mathrm{Tr}_{\mathcal{A}/F} X), \quad X \in \mathcal{A}.$$

Then ξ is invariant under the H_0-conjugation and thus extends to a character, again denoted ξ, of H that is trivial on H_0. Similarly, we consider ω as a character on H by composition with the projection $H \twoheadrightarrow H_0$ and we denote by $\omega \otimes \xi$ the product of these two characters of H. We refer to the triple $(G, H, \omega \otimes \xi)$ as a *generalized Shalika triple*. In particular, if $\mathcal{A} = \mathrm{Mat}_n(F)$, we recover the usual Shalika model for GL_{2n}. For every irreducible admissible representation π of G, we define the *multiplicity* $m(\pi, \omega)$ by

$$m(\pi, \omega) := \dim \mathrm{Hom}_H(\pi, \omega \otimes \xi).$$

According to [10, Theorem 4.5], this multiplicity is always finite. The goal of this paper is to study the behavior of the multiplicity $m(\pi, \omega)$ under the local Jacquet-Langlands correspondence.

Remark 1.1. *In fact, by [23], [32] and [8], we even know the multiplicity $m(\pi, \omega)$ is less or equal to 1 (i.e. the generalized Shalika models are Gelfand pairs). But we don't need this result in the proof of the main theorem.*

Let $\mathcal{A}'$ be another degree n central simple algebra over F. Set $G' := \mathrm{GL}_2(\mathcal{A}')$ and define subgroups H'_0, N', $H' := H'_0 \ltimes N'$ analogous to the subgroups H_0, N and H of G. We also define similarly characters characters ξ', $\omega \otimes \xi'$ of N', H' respectively and for all irreducible admissible representation π' of G', we set

$$m(\pi', \omega) := \dim \mathrm{Hom}_{H'}(\pi', \omega \otimes \xi').$$

The main result of this paper is the following theorem that says that these multiplicities are invariant under the local Jacquet-Langlands correspondence for discrete series.

Theorem 1.2. *Let π (resp. π') be a discrete series of G (resp. G'). Assume that π and π' correspond to each other under the local Jacquet-Langlands correspondence (see [9]). Then*

$$m(\pi, \omega) = m(\pi', \omega).$$

Assume one moment that $\omega = \mathbf{1}$ (the trivial character) and set for simplicity $m(\pi) := m(\pi, \mathbf{1})$. Then, by work of Kewat [24], Kewat-Ragunathan [25], Jiang-Nien-Qin [21] and the multiplicity one theorem of Jacquet-Rallis [23], in the particular case where $\mathcal{A} = M_n(F)$ we know that for all discrete series π we have $m(\pi) = 1$ if and only if $L(s, \pi, \wedge^2)$ (the Artin exterior square L-function) has a pole at $s = 0$ (i.e. the Langlands parameter of π is symplectic) and $m(\pi) = 0$ otherwise. Actually, to our knowledge, a full proof of this result has not appeared in the literature and thus for completeness we provide the necessary complementary arguments in Section 6.1. Together with Theorem 1.2 this immediately implies

Theorem 1.3. *For all discrete series representation π of G, we have $m(\pi) = 1$ if and only if the local exterior square L-function $L(s, \pi, \wedge^2)$ has a pole at $s = 0$; and $m(\pi) = 0$ otherwise.*

We will prove Theorem 1.2 in Section 6.1. The key ingredient of our proof is a certain integral formula computing the multiplicity $m(\pi, \omega)$ that we now state. Recall that following Harish-Chandra, any irreducible representation π has a well-defined character Θ_π, which is a locally integrable function on G and locally constant on the regular semi-simple locus. Moreover, Harish-Chandra has completely described the possible singularities of Θ_π near singular semi-simple elements that lead to certain local expansions of the character near such point. Using these, we can define a certain regularization $x \mapsto c_\pi(x)$ of Θ_π at all semi-simple points by taking the average of the 'leading coefficients' of these local expansions. (See Section 2.2 for details, actually for the groups considered in this paper there is always at most one such leading coefficient.). Given this, our multiplicity formula can be stated as follows (see Proposition 3.3)

Theorem 1.4. *For all essentially square-integrable representation π of G with central character $\chi = \omega^n$ (seen as a character of $A_G = F^\times$), we have*

$$m(\pi, \omega) = \sum_{T \in \mathcal{T}_{\mathrm{ell}}(H_0)} |W(H_0, T)|^{-1} \int_{A_G \backslash T} D^H(t) c_\pi(t) \omega(t)^{-1} dt$$

where $\mathcal{T}_{\mathrm{ell}}(H_0)$ stands for a set of representatives of elliptic maximal tori in H_0, $W(H_0,T) = \mathrm{Norm}_{H_0}(T)/T$ is the corresponding Weyl group, $D^H(t)$ is the usual Weyl discriminant, the measure on the tori $A_G\backslash T$ are chosen to be of total mass one and the expression on the right hand side is absolutely convergent.

Theorem 1.2 is then an easy consequence of Theorem 1.4 and the characters relations that characterize the local Jacquet-Langlands correspondence (see §6.1 for details).

Remark 1.5. *In Appendix A, we will prove a slight generalization of a result of Mœglin and Waldspurger, a consequence of which is that the multiplicity formula above holds more generally for all irreducible admissible representations of G when $\mathcal{A} = \mathcal{D}$ is a division algebra. On the other hand, if $\mathcal{A}$ is not a division algebra, the multiplicity formula will only hold for discrete series (see Remark 3.4 for more details).*

Theorem 1.4 is a consequence of a certain local simple trace formula for the generalized Shalika models of the same kind as the local trace formulas developed in [5], [6] and [41]. To be specific, let $f \in {}^\circ\mathcal{C}(G)$ be an Harish-Chandra cusp form (see Section 2.5 for the definition of these) and for all $x, y \in G$, set

$$K_f(x,y) := \int_H f(x^{-1}hy)(\omega \otimes \xi)(h)^{-1}dh.$$

We define a distribution J on the space of cusp forms by

$$J(f) := \int_{H\backslash G} K_f(x,x)dx.$$

In later sections, we will show that both integrals above are absolutely convergent.

The aforementioned trace formula gives two expansions of $J(f)$: one geometric and one spectral. The geometric side is given by

$$J_{geom}(f) = \sum_{T \in \mathcal{T}_{\mathrm{ell}}(H_0)} |W(H_0,T)|^{-1} \int_T D^H(t) c_f(t) \omega(t)^{-1} dt$$

where $\mathcal{T}_{\mathrm{ell}}(H_0)$ denotes a set of representatives of conjugacy classes of maximal elliptic tori in H_0, $W(H_0,T)$ stands for the corresponding Weyl group, D^H is the usual Weyl discriminant and $c_f(t)$ is a certain weighted orbital integral of f in the sense of Arthur (see §2.5 for a precise definition). The spectral side, on the other hand, is given by the following expression

$$J_{spec}(f) = \sum_{\pi \in \Pi_2(G,\chi)} m(\pi,\omega) \operatorname{Tr} \pi^\vee(f)$$

where $\Pi_2(G,\chi)$ denotes the set of (isomorphism classes of) discrete series of G with central character $\chi = \omega^n$, seen as a character of $A_G = F^\times$, and $\pi^\vee$ stands for the contragredient of π. Then the trace formula we proved in this paper is just (see Theorem 3.1)

Theorem 1.6. *For all $f \in {}^{\circ}\mathcal{C}(G)$, we have*

$$J_{spec}(f) = J(f) = J_{geom}(f). \tag{1.1.1}$$

More precisely, the spectral side of the trace formula will be proved in Section 4 and the geometric side will be proved in Section 5. Moreover, Theorem 1.4 is, by standard means, an easy consequence of this trace formula (see §3.3).

In Section 6.2, we will discuss another application of the multiplicity formula. By applying Theorem 1.4 together with another multiplicity formula for the so-called Ginzburg-Rallis model proved in the previous papers [41] and [42] of the second author, we are able to establish some relationship between the two kinds of multiplicities (cf. Theorem 6.6 and Theorem 6.7). This will also allow us to prove the epsilon dichotomy conjecture for the Ginzburg-Rallis model in some cases. We refer the readers to Section 6.2 for details.

Finally, in Section 7, guided by the idea of beyond endoscopy, together with Theorem 1.3 that relates the multiplicities for generalized Shalika models to poles of local exterior square L-function, we restate our trace formula in the form of a (local) 'r-trace formula' for $r = \wedge^2$ the exterior square representation of the L-group ${}^{L}G = \mathrm{GL}_{2n}(\mathbb{C})$.

1.2 Organization of the paper and remarks on the proofs

In Section 2, we introduce basic notation and conventions of this paper. This includes some extended discussions of (θ-)weighted orbital integrals, germ expansions and the Harish-Chandra-Schwartz space. In Section 3, we state our (simple) local trace formula (Theorem 1.6) and prove that the multiplicity formula (Theorem 1.4) is a consequence of it.

Sections 4 and 5 are devoted to the proof of the trace formula. More precisely, in Section 4 we prove the spectral side of the trace formula. It is the easy part and moreover the arguments are very similar to [6, §4]. Section 5 contains the proof of the geometric side which is more involved. The general idea is inspired by the work of Waldspurger ([38], [39]) and the first author ([5], [6]) on the Gan-Gross-Prasad and Galois models. However, due to significant differences between generalized Shalika models and the previous cases, our proof of the geometric side is quite different. Indeed, as in the Gan-Gross-Prasad cases, singular orbits are contributing to the geometric side and these contributions are reflected in singularities of the original expression. Due to the fact that the generalized Shalika models are usually not *strongly tempered* in the sense of [34] (recall that a model (G, H) is said to be strongly tempered if all the tempered matrix coefficients of G are integrable on $H/H \cap Z_G$), we were unable to linearize the problem in order to perform a Fourier transform as in [38],[5] where it had the effect of killing the problematic singularities. As a result, we have to deal with them directly and for that we have in particular computed explicitly certain singular weighted (or rather *θ-weighted*) orbital integrals (see §5.5).

Sections 6 and 7 contain applications of the trace formula and multiplicity formula. In Section 6.1, we prove the two main theorems (Theorem 1.2 and Theorem 1.3) of this paper and in Section 6.2, we study the relations between the multiplicities for the generalized Shalika model and the Ginzburg-Rallis model. Using this, we will prove new cases of the

epsilon dichotomy conjecture for the Ginzburg-Rallis model. Finally, in Section 7, we rewrite our local trace formula as some kind of 'local r-trace formula'.

Finally, Appendix A contains a slight generalization of a result of Mœglin and Waldspurger concerning (generalized) Whittaker models.

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2 Preliminaries

2.1 Groups, measures, notations

Throughout this paper F will denote a p-adic field (i.e. a finite extension of $\mathbf{Q}_p$ for a certain prime number p) with ring of integers $\mathcal{O}_F$ and normalized absolute value $|.|_F$. We will denote by v_F the normalized valuation on F, by q the cardinal of the residue field of F and by log the logarithm in base q (so that $v_F(\lambda) = -\log|\lambda|_F$ for all $\lambda \in F^\times$). Moreover, for all finite extensions E of F we will set $v_E(\lambda) := \frac{1}{[E:F]} v_F(N_{E/F}(\lambda))$ for all $\lambda \in E$ where $N_{E/F} : E \to F$ stands for the norm (notice that v_E is NOT the normalized valuation on E unless the extension E/F is totally ramified). We fix throughout a nontrivial additive character $\psi : F \to \mathbf{C}^\times$. We will slightly abuse notation and denote algebraic groups and Lie algebras defined over F and their sets of F-points by the same letters.

Let G be a connected reductive group over F. We denote by $X^*(G)$ its group of algebraic characters and by A_G its maximal central split torus. We set

$$\mathcal{A}_G := \mathrm{Hom}(X^*(G), \mathbf{R}) = \mathrm{Hom}(X^*(A_G), \mathbf{R})$$

whose dual naturally identifies to

$$\mathcal{A}_G^* := X^*(G) \otimes \mathbf{R} = X^*(A_G) \otimes \mathbf{R}.$$

There is a natural morphism $H_G : G \to \mathcal{A}_G$ characterized by

$$\langle \chi, H_G(g) \rangle = \log(|\chi(g)|)$$

for all $\chi \in X^*(A_G)$ where we recall that log denotes the logarithm in base q. We set $\mathcal{A}_{G,F} := H_G(A_G)$. It is a lattice in $\mathcal{A}_G$. The same notations will be used for the Levi

subgroups of G (i.e. the Levi components of parabolic subgroups of G): if M is a Levi subgroup of G, we define similarly A_M, $\mathcal{A}_M$, H_M and $\mathcal{A}_{M,F}$. For such a Levi M, we will set

$$\mathcal{A}_M^G := \mathcal{A}_M/\mathcal{A}_G, \quad \mathcal{A}_{M,F}^G := \mathcal{A}_{M,F}/\mathcal{A}_{G,F}.$$

We will also use Arthur's notations [2, Sect. 1]: $\mathcal{P}(M)$, $\mathcal{F}(M)$ and $\mathcal{L}(M)$ will stand for the sets of parabolic subgroups with Levi component M, parabolic subgroups containing M and Levi subgroups containing M respectively. Let K be a special maximal compact subgroup of G. Then, for all parabolic subgroup P with Levi decomposition $P = MU$, the Iwasawa decomposition $G = MUK$ allows us to extend H_M to a map $H_P : G \to \mathcal{A}_M$ defined by $H_P(muk) := H_M(m)$ for all $m \in M$, $u \in U$ and $k \in K$. The Lie algebra of G will be denoted by $\mathfrak{g}$ and more generally for any algebraic group we will denote its Lie algebra by the corresponding Gothic letter. We will write Ad for the adjoint action of G on $\mathfrak{g}$. We denote by exp the exponential map, which is an F-analytic map from an open neighborhood of 0 in $\mathfrak{g}$ to G. We define G_{reg} as the open subset of regular semisimple elements of G. The notation $\mathcal{T}(G)$ (resp. $\mathcal{T}_{\mathrm{ell}}(G)$) will be used to denote a set of representatives for the G-conjugacy classes of maximal tori (resp. elliptic maximal tori) in G.

Let H be an algebraic group over F. For any subset $S \subset H$, we write $\mathrm{Cent}_H(S)$ (resp. $\mathrm{Norm}_H(S)$) for the centralizer of S in H (resp. the normalizer of S in H). If $S = \{x\}$ we will write H_x for the neutral connected component of $\mathrm{Cent}_H(x) := \mathrm{Cent}_H(\{x\})$. The Weyl discriminant D^H is defined by

$$D^H(x) := \left|\det(1 - \mathrm{Ad}(x)_{|\mathfrak{h}/\mathfrak{h}_x})\right|$$

for all semisimple element $x \in H$. For every subtorus T of H, we will denote by

$$W(H,T) := \mathrm{Norm}_H(T)/\mathrm{Cent}_H(T)$$

the corresponding Weyl group. If $A \subset H$ is a split subtorus that normalizes a unipotent subgroup $U \subset H$ we will write $R(A,U)$ for the set of roots of A in $\mathfrak{u}$.

If T is a torus over F, we will denote by T^c its maximal compact subgroup.

In this paper, we will assume that all the groups that we encounter have been equipped with Haar measures (left and right invariants as we will only consider measures on unimodular groups). In the particular case of tori T we normalize these Haar measures as follows: we fix on A_T the unique Haar measure giving A_T^c volume 1 and we choose on T the unique Haar measure such that $\mathrm{vol}(T/A_T) = 1$. For any connected reductive group G, we equip $\mathcal{A}_G$ with the unique Haar measure such that $\mathrm{vol}(\mathcal{A}_G/\mathcal{A}_{G,F}) = 1$. Thus this requirement also fixes Haar measures on $\mathcal{A}_M$ for all Levi subgroup M of G. If $M \subset L$ are two Levi subgroups then we give $\mathcal{A}_M^L \simeq \mathcal{A}_M/\mathcal{A}_L$ the quotient measure.

We will adopt the following slightly imprecise but convenient notation. If f and g are positive functions on a set X, we will write

$$f(x) \ll g(x) \text{ for all } x \in X,$$

and we will say that f is *essentially bounded* by g, if there exists a $c > 0$ such that

$$f(x) \leqslant cg(x) \text{ for all } x \in X.$$

We will also say that f and g are equivalent and we will write

$$f(x) \sim g(x) \text{ for all } x \in X$$

if both f is essentially bounded by g and g is essentially bounded by f.

In this paper we will freely use the notion of log-norms on varieties over F. The concept of norm on varieties over local fields was introduced by Kottwitz in [26, §18]. A log-norm is essentially just the log of a Kottwitz's norm and we refer the readers to [5, §1.2] for the definition and the basic properties of these log-norms. We will assume that all the algebraic varieties X over F that we encounter have been equipped with log norms σ_X. And for all $C > 0$, we will denote by $\mathbf{1}_{X,\leqslant C}$ (resp. $\mathbf{1}_{X,>C}$) the characteristic function of $\{x \in X; \sigma_X(x) \leqslant C\}$ (resp. $\{x \in X; \sigma_X(x) > C\}$).

For any connected reductive group G over F, we will denote by Ξ^G the Xi function of Harish-Chandra on G (see [5, §1.5] for the definition and basic properties of this function) and we will denote by $\mathcal{C}(G)$ the Harish-Chandra Schwartz space of G. This space consists of functions $f : G \to \mathbf{C}$ which are bi-invariant by a certain compact-open subgroup $J \subset G$ and such that for all $d > 0$, we have an inequality

$$|f(g)| \ll \Xi^G(g)\sigma_G(g)^{-d}$$

for all $g \in G$. Let χ be a unitary character of A_G. Then we will denote by $\mathcal{C}(G, \chi)$ the space of functions $f : G \to \mathbf{C}$ which are bi-invariant by a certain compact-open subgroup $J \subset G$ such that $f(ag) = \chi(a)f(g)$ for all $a \in A_G$ and $g \in G$, and such that for all $d > 0$, we have an inequality

$$|f(g)| \ll \Xi^G(g)\sigma_{A_G\backslash G}(g)^{-d}$$

for all $g \in G$. There is a natural surjective map

$$\mathcal{C}(G) \to \mathcal{C}(G, \chi) : \ f \mapsto f_\chi$$

given by

$$f_\chi(g) := \int_{A_G} f(ag)\chi(a)^{-1}da, \quad f \in \mathcal{C}(G), g \in G.$$

For any set S we will denote by $\mathbf{1}_S$ its characteristic function.

2.2 Representations

Let G be a connected reductive group over F. We will write $\mathrm{Irr}(G)$ for the set of isomorphism classes of (complex-valued) irreducible smooth representations of G. We will identify any element of $\mathrm{Irr}(G)$ with one of its representative. For $\pi \in \mathrm{Irr}(G)$, we will also write π for the space on which π acts. We will denote by $\Pi_2(G) \subset \mathrm{Irr}(G)$ the subset of essentially square-integrable representations. And if χ is a character of A_G, we will denote by $\Pi_2(G,\chi) \subset \Pi_2(G)$ the subset of representations with central character χ. When χ is unitary, the matrix coefficients of any representation $\pi \in \Pi_2(G,\chi)$ lie in $\mathcal{C}(G,\chi)$. For $\pi \in \mathrm{Irr}(G)$, we will denote by $\pi^\vee$ its smooth contragredient; and for $\pi \in \Pi_2(G)$, we will denote by $d(\pi)$ the formal degree of π. It is the unique positive real number (depending on the Haar measure on G) such that

$$\int_{A_G\backslash G} \langle \pi(g)v_1, v_1^\vee\rangle \langle v_2, \pi^\vee(g)v_2^\vee\rangle dg = \frac{1}{d(\pi)} \langle v_1, v_2^\vee\rangle \langle v_2, v_1^\vee\rangle$$

for all $v_1, v_2 \in \pi$ and $v_1^\vee, v_2^\vee \in \pi^\vee$. For any $f \in C_c^\infty(G)$ and $\pi \in \mathrm{Irr}(G)$, we write

$$\pi(f) := \int_G f(g)\pi(g)dg.$$

When $\pi \in \Pi_2(G,\chi)$ where the character χ is unitary, the map $f \mapsto \pi(f)$ extends by continuity to $\mathcal{C}(G)$ and $\mathcal{C}(G,\chi^{-1})$. In all cases, the operator $\pi(f)$ has finite rank. If f is a matrix coefficient of $\pi \in \Pi_2(G,\chi)$ (with χ unitary), we have

$$\mathrm{Tr}\, \pi^\vee(f) = d(\pi)^{-1} f(1). \tag{2.2.1}$$

Moreover, for any $\pi \in \mathrm{Irr}(G)$, Harish-Chandra has shown ([15, Theorem 16.3]) the existence of a locally integrable function Θ_π on G which is locally constant on G_{reg} and such that

$$\mathrm{Tr}\, \pi(f) = \int_G f(g)\Theta_\pi(g)dg$$

for all $f \in C_c^\infty(G)$. We shall refer to Θ_π as the *Harish-Chandra character* of π. Fixing a G-invariant symmetric bilinear pairing $\langle .,. \rangle : \mathfrak{g} \times \mathfrak{g} \to F$. Near every semi-simple element $x \in G$, there is a local expansion (see [15, Theorem 16.2])

$$\Theta_\pi(x\exp(X)) = \sum_{\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_x)} c_{\pi,\mathcal{O}}(x)\widehat{j}(\mathcal{O}, X)$$

for $X \in \mathfrak{g}_{x,\mathrm{reg}}$ sufficiently close to 0 and where

- $\mathrm{Nil}(\mathfrak{g}_x)$ stands for the set of nilpotent G_x-orbits in $\mathfrak{g}_x$ (for the adjoint action);
- $c_{\pi,\mathcal{O}}(x)$ are complex numbers;

- For all $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_x)$, $\widehat{j}(\mathcal{O},.)$ is the unique locally integrable function on $\mathfrak{g}_x$ (whose existence is guaranteed by [15, Theorem 7.7. and Lemma 7.9]) which is locally constant on $\mathfrak{g}_{x,\mathrm{reg}}$, and such that

$$\int_{\mathfrak{g}_x} \varphi(X)\widehat{j}(\mathcal{O},X)dX = \int_{\mathcal{O}} \widehat{\varphi}(Z)dZ, \quad \text{for all } \varphi \in C_c^\infty(\mathfrak{g}_x)$$

where dX is any Haar measure on $\mathfrak{g}_x$, $\varphi \in C_c^\infty(\mathfrak{g}_x) \mapsto \widehat{\varphi}$ is the Fourier transform given by $\widehat{\varphi}(Z) := \int_{\mathfrak{g}_x} \varphi(X)\psi(\langle Z,X\rangle)dX$ and dZ is the G_x-invariant measure on $\mathcal{O}$ associated to the self-dual Haar measure on F corresponding to ψ and the volume form on $\mathcal{O}$ derived from the symplectic form descended from $\langle .,.\rangle$ (see [29, §I.8] for more details on this).

For every semisimple element $x \in G$, we set

$$c_\pi(x) := \begin{cases} \frac{1}{|\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)|}\sum_{\mathcal{O}\in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)} c_{\pi,\mathcal{O}}(x), & \text{if } G_x \text{ is quasi-split;} \\ 0, & \text{otherwise} \end{cases}$$

where $\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)$ denotes the subset of regular nilpotent orbits in $\mathfrak{g}_x$ (this set is empty if G_x is not quasi-split). This value does not depend on the choices of $\langle .,.\rangle$ and ψ. If G_x is quasi-split and we fix a Borel subgroup $B_x \subset G_x$ and a maximal torus $T_{x,\mathrm{qd}} \subset B_x$, then by [5, Proposition 4.5.1 1.(ii)], we have

$$D^G(x)^{1/2}c_\pi(x) = |W(G_x,T_{x,\mathrm{qd}})|^{-1} \lim_{x'\in T_{x,\mathrm{qd}}\to x} D^G(x')^{1/2}\Theta_\pi(x'). \tag{2.2.2}$$

2.3 (G,M)- and (G,M,θ)-orthogonal sets

Let G be a connected reductive group over F and M be a Levi subgroup of G. For all $Q \in \mathcal{F}(M)$, we will denote by U_Q the unipotent radical of Q, L_Q the unique Levi component of Q such that $M \subset L_Q$ and $\overline{Q} = L_Q U_{\overline{Q}}$ the parabolic subgroup opposite to Q (with respect to L_Q). Let A_M be the split center of M. For all $P \in \mathcal{P}(M)$, denote by Δ_P (resp. Σ_P^+) the set of simple roots (resp. of all roots) of A_M in P. For all $\alpha \in \Sigma_P^+$, we shall denote by $\alpha^\vee \in \mathcal{A}_M$ the corresponding coroot and we set $\Delta_P^\vee := \{\alpha^\vee; \alpha \in \Delta_P\}$. We recall the notion of (G,M)-orthogonal set due to Arthur: a family $\mathcal{Y} = (\mathcal{Y}_P)_{P\in\mathcal{P}(M)}$ is a (G,M)-*orthogonal set* if for all $P,P' \in \mathcal{P}(M)$, we have

$$\mathcal{Y}_P - \mathcal{Y}_{P'} \in \sum_{\alpha\in\Sigma_P^+\cap -\Sigma_{P'}^+} \mathbf{R}\alpha^\vee.$$

Moreover, if the stronger relation

$$\mathcal{Y}_P - \mathcal{Y}_{P'} \in \sum_{\alpha\in\Sigma_P^+\cap -\Sigma_{P'}^+} \mathbf{R}_+\alpha^\vee$$

is satisfied for all $P, P' \in \mathcal{P}(M)$, then we say that the (G,M)-orthogonal set $\mathcal{Y}$ is *positive*. To a (G,M)-orthogonal set $\mathcal{Y}$ we can associate a smooth function $\gamma_M(.,\mathcal{Y})$ on $\mathcal{A}_M^*$ defined by (see [27, Lemme 1.9.3])

$$\gamma_M(\lambda,\mathcal{Y}) := \sum_{P\in\mathcal{P}(M)} \mathrm{vol}(\mathcal{A}_M^G/\mathbf{Z}[\Delta_P^\vee]) \prod_{\alpha\in\Delta_P} \langle \lambda, \alpha^\vee\rangle^{-1} e^{\langle\lambda,\mathcal{Y}_P\rangle}, \quad \lambda\in\mathcal{A}_M^*$$

where $\mathbf{Z}[\Delta_P^\vee] \subset \mathcal{A}_M^G$ denotes the lattice generated by $\Delta_P^\vee$, and we set

$$v_M(\mathcal{Y}) := \gamma_M(0,\mathcal{Y}).$$

More generally we can associate to a (G,M)-orthogonal set $\mathcal{Y}$ smooth functions $\gamma_M^Q(.,\mathcal{Y})$ on $\mathcal{A}_M^{L_Q}$ and complex numbers $v_M^Q(\mathcal{Y}) := \gamma_M^Q(0,\mathcal{Y})$ for all $Q\in\mathcal{F}(M)$ with $\gamma_M^G(.,\mathcal{Y}) = \gamma_M(.,\mathcal{Y})$ (see [27, §1.9]). If the (G,M)-family $\mathcal{Y}$ is positive, then $v_M^Q(\mathcal{Y})$ is just the volume of the convex hull of the projection of $(Y_P)_{P\in\mathcal{P}(M),P\subset Q}$ onto $\mathcal{A}_M^{L_Q}$. Also to a (G,M)-orthogonal set $\mathcal{Y}$ we can associate a certain function $\Gamma_M^G(.,\mathcal{Y})$ on $\mathcal{A}_M$ (see [27, §1.8]) that is just the characteristic function of the sum of the convex hull of $\mathcal{Y}$ with $\mathcal{A}_G$ if $\mathcal{Y}$ is positive.

An easy way to construct (G,M)-orthogonal sets is as follows. Let $M_0\subset M$ be a minimal Levi subgroup with split center A_0 and pick $P_0\in\mathcal{P}(M_0)$. Fix $Y\in\mathcal{A}_{M_0}$. For all $P\in\mathcal{P}(M)$, define Y_P to be the projection of wY onto $\mathcal{A}_M$ where $w\in W(G,A_0)$ is any element such that $wP_0\subset P$. Then $(Y_P)_{P\in\mathcal{P}(M)}$ is a (G,M)-orthogonal set.

Another way to construct (G,M)-orthogonal sets is as follows. Choose a maximal special compact subgroup K of G and use it to define maps $H_P : G\to\mathcal{A}_M$ as in §2.1. Then for all $g\in G$, the family $\mathcal{Y}_M(g) := (H_{\overline{P}}(g))_{P\in\mathcal{P}(M)}$ is a positive (G,M)-orthogonal set. In this situation, we define

$$v_M(g) := v_M(\mathcal{Y}_M(g)), \quad g\in G$$

and more generally

$$v_M^Q(g) := v_M^Q(\mathcal{Y}_M(g)), \quad g\in G$$

for all $Q\in\mathcal{F}(M)$.

Assume now given an algebraic involution θ of G. Then we recall that a parabolic subgroup P of G is said to be θ-split if $\theta(P)$ is opposite to P; and a Levi subgroup M of G is said to be θ-split if there exists a θ-split parabolic subgroup P such that $M = P\cap\theta(P)$. Also, a torus $T\subset G$ is said to be θ-split if $\theta(t) = t^{-1}$ for all $t\in T$ (here we remark that, unfortunately, the notions of θ-split torus and θ-split Levi do not necessarily coincide when a torus is also a minimal Levi, we hope that this will not create any confusion to the reader). We refer the reader to [6, §2.8.1] for a recapitulation of the basic structure of these θ-split subgroups. For M a θ-split Levi subgroup, we shall denote by $\mathcal{P}^\theta(M)$, resp. $\mathcal{F}^\theta(M)$, resp. $\mathcal{L}^\theta(M)$, the sets of θ-split parabolic subgroups with Levi component M, resp. θ-split parabolic subgroups containing M, resp. θ-split Levi subgroups containing M. For all $Q\in\mathcal{F}^\theta(M)$, we define

$$\mathcal{P}^{Q,\theta}(M) := \{P \in \mathcal{P}^\theta(M);\ P \subset Q\}.$$

This set is in bijection with the set of θ-split parabolic subgroups of L_Q with Levi component M by the map $P \mapsto P \cap L_Q$. We will denote by $A_{M,\theta}$ the maximal split and θ-split central subtorus of M and set

$$\mathcal{A}_{M,\theta} := X_*(A_{M,\theta}) \otimes \mathbf{R},$$

$$\mathcal{A}^L_{M,\theta} := \mathcal{A}_{M,\theta}/\mathcal{A}_{L,\theta}, \quad L \in \mathcal{L}^\theta(M).$$

Then for all θ-split Levi subgroup M, there is a natural decomposition

$$\mathcal{A}_M = \mathcal{A}_{M,\theta} \oplus \mathcal{A}^\theta_M$$

where $\mathcal{A}^\theta_M$ denotes the subspace of θ-invariant vectors and we define an homomorphism $H_{M,\theta} : M \to \mathcal{A}_{M,\theta}$ as the composition of H_M with the projection $\mathcal{A}_M \twoheadrightarrow \mathcal{A}_{M,\theta}$. There is also a natural decomposition

$$\mathcal{A}_{M,\theta} = \mathcal{A}^G_{M,\theta} \oplus \mathcal{A}_{G,\theta}$$

and more generally a natural decomposition

$$\mathcal{A}_{M,\theta} = \mathcal{A}^L_{M,\theta} \oplus \mathcal{A}_{L,\theta}$$

for all $L \in \mathcal{L}^\theta(M)$. Set $\mathcal{A}_{M,\theta,F} := H_{M,\theta}(A_M)$. We equip $\mathcal{A}_{M,\theta}$ with the unique Haar measure such that $\mathrm{vol}(\mathcal{A}_{M,\theta}/\mathcal{A}_{M,\theta,F}) = 1$ and $\mathcal{A}^L_{M,\theta}$ for $L \in \mathcal{L}^\theta(M)$ with the quotient Haar measure (where the Haar measure on $\mathcal{A}_{L,\theta}$ is defined similarly).

To every $P \in \mathcal{P}^\theta(M)$ is associated a cone $\mathcal{A}^+_{P,\theta} \subset \mathcal{A}_{M,\theta}$ defined by

$$\mathcal{A}^+_{P,\theta} := \{\Lambda \in \mathcal{A}_{M,\theta};\ \ \langle \alpha, \Lambda\rangle > 0\ \forall \alpha \in R(A_{M,\theta}, U_P)\}.$$

We shall denote by $\overline{\mathcal{A}^+_{P,\theta}}$ the closure of $\mathcal{A}^+_{P,\theta}$ and by $\tau^G_{P,\theta}$ the characteristic function of $\mathcal{A}^+_{P,\theta}$. For all $Q \in \mathcal{F}^\theta(M)$, we will also consider the function $\tau^G_{Q,\theta}$ as a function on $\mathcal{A}_{M,\theta}$ via the projection $\mathcal{A}_{M,\theta} \to \mathcal{A}_{L_Q,\theta}$.

In [6, §2.8.2], the first author has defined a notion of (G, M, θ)-orthogonal set which generalizes Arthur's classical notion of (G, M)-orthogonal set and we refer the reader to *loc. cit.* for basic definitions and properties of these. A (G, M, θ)-orthogonal set is a family $\mathcal{Y} = (\mathcal{Y}_P)_{P \in \mathcal{P}^\theta(M)}$ of points of $\mathcal{A}_{M,\theta}$ satisfying certain compatibility conditions. There is also a notion of positive (G, M, θ)-orthogonal set. If $\mathcal{Y}_P \in \mathcal{A}^+_{P,\theta}$ for all $P \in \mathcal{P}^\theta(M)$, then the (G, M, θ)-orthogonal set $\mathcal{Y}$ is positive. To any (G, M, θ)-orthogonal set $\mathcal{Y}$ is associated functions

$$\Gamma^Q_{L,\theta}(., \mathcal{Y}), \quad L \in \mathcal{L}^\theta(M), Q \in \mathcal{F}^\theta(L)$$

on $\mathcal{A}_{L,\theta}^{L_Q}$ and complex numbers

$$v_{L,\theta}^{Q}(\mathcal{Y}), \quad L \in \mathcal{L}^{\theta}(M), Q \in \mathcal{F}^{\theta}(L)$$

which are related by

$$v_{L,\theta}^{Q}(\mathcal{Y}) = \int_{\mathcal{A}_{L,\theta}^{L_Q}} \Gamma_{L,\theta}^{Q}(\Lambda, \mathcal{Y}) d\Lambda.$$

For simplicity, when $Q = G$, we will write $v_{L,\theta}(\mathcal{Y}) := v_{L,\theta}^{G}(\mathcal{Y})$. When $\mathcal{Y}$ is positive, $\Gamma_{M,\theta}^{G}(., \mathcal{Y})$ is the characteristic function of the sum of $\mathcal{A}_G$ with the convex hull of $\mathcal{Y}$; and more generally, for $Q \in \mathcal{F}^{\theta}(M)$, $\Gamma_{M,\theta}^{Q}(., \mathcal{Y})$ is the characteristic function of the sum of $\mathcal{A}_{L_Q,\theta}$ with the convex hull of $(\mathcal{Y}_P)_{P \in \mathcal{P}^{Q,\theta}(M)}$. We have the basic relation (see [27, Lemme 1.8.4 (3)] for the case of (G, M)-orthogonal sets, the proof being completely similar for (G, M, θ)-orthogonal sets)

$$\sum_{Q \in \mathcal{F}^{\theta}(M)} \Gamma_{M,\theta}^{Q}(\Lambda, \mathcal{Y}) \tau_{Q,\theta}^{G}(\Lambda - \mathcal{Y}_Q) = 1, \quad \Lambda \in \mathcal{A}_{M,\theta} \tag{2.3.1}$$

where for all $Q \in \mathcal{F}^{\theta}(M)$, we have denoted by $\mathcal{Y}_Q$ the projection of $\mathcal{Y}_P$ onto $\mathcal{A}_{L_Q,\theta}$ for any $P \in \mathcal{P}^{Q,\theta}(M)$ (the result does not depend on the choice of P). One basic property of the function $\Gamma_{M,\theta}^{G}(., \mathcal{Y})$ that we shall use repeatedly is the following (see [27, Corollaire 1.8.5], again for the case of (G, M)-orthogonal sets):

(2.3.2) Let $|.|$ be a norm on $\mathcal{A}_{M,\theta}$. Then, there exists a constant $c > 0$ independent of the (G, M, θ)-family $\mathcal{Y}$ such that for all $\Lambda \in \mathcal{A}_{M,\theta}$ in the support of $\Gamma_{M,\theta}^{G}(., \mathcal{Y})$, we have $|\Lambda^G| \leqslant c \sup_{P \in \mathcal{P}^{\theta}(M)} |\mathcal{Y}_P|$ where Λ^G is the projection of Λ onto $\mathcal{A}_{M,\theta}^{G}$.

In particular, this implies

(2.3.3) There exists $k > 0$ (for example $k = \dim(\mathcal{A}_{M,\theta}^{G})$ would work) such that

$$|v_{M,\theta}^{G}(\mathcal{Y})| \ll \left(\sup_{P \in \mathcal{P}^{\theta}(M)} |\mathcal{Y}_P| \right)^k$$

for all (G, M, θ)-orthogonal set $\mathcal{Y}$.

We will also need the following property:

(2.3.4) Let $Q \in \mathcal{F}^{\theta}(M)$ and $P \in \mathcal{P}^{\theta}(M)$. Then, for all (G, M, θ)-orthogonal set $\mathcal{Y}$ such that $\mathcal{Y}_{P'} \in \mathcal{A}_{P',\theta}^{+}$ for all $P' \in \mathcal{P}^{\theta}(M)$, the restriction of the function

$$\Lambda \mapsto \Gamma_{M,\theta}^{Q}(\Lambda, \mathcal{Y}) \tau_{Q,\theta}^{G}(\Lambda - \mathcal{Y}_Q)$$

to $\overline{\mathcal{A}_{P,\theta}^{+}}$ only depends on $\mathcal{Y}_P$.

Proof. Since $\mathcal{Y}$ is positive, the function $\Gamma^Q_{M,\theta}(.,\mathcal{Y})\tau^G_{Q,\theta}(.-\mathcal{Y}_Q)$ is the characteristic function of the sum of $\mathcal{A}^+_{Q,\theta}$ with the convex hull of $(\mathcal{Y}_{P'})_{P'\in\mathcal{P}^{Q,\theta}(M)}$. In particular, for all Λ in the support of this function, we have $\langle\alpha,\Lambda\rangle \geqslant \inf_{P'\in\mathcal{P}^{\theta,Q}(M)}\langle\alpha,\mathcal{Y}_{P'}\rangle > 0$ for all $\alpha\in R(A_{M,\theta},U_Q)$. This implies that if P is not included in Q, the restriction of $\Gamma^Q_{M,\theta}(.,\mathcal{Y})\tau^G_{Q,\theta}(.-\mathcal{Y}_Q)$ to $\overline{\mathcal{A}^+_{P,\theta}}$ is just identically zero. Assume now that $P\subset Q$. Then, by adapting [2, Lemma 3.1] to the case of (G,M,θ)-orthogonal sets, we see that the restriction of $\Gamma^Q_{M,\theta}(.,\mathcal{Y})$ to $\overline{\mathcal{A}^+_{P,\theta}}$ only depends on $\mathcal{Y}_P$. On the other hand, $\tau^G_{Q,\theta}(.-\mathcal{Y}_Q)$ only depends on $\mathcal{Y}_Q$ the projection of $\mathcal{Y}_P$ onto $\mathcal{A}_{L_Q,\theta}$. The claim follows. □

Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be two (G,M,θ)-orthogonal sets. Then, we have the following splitting formula (see [1, Corollary 7.4] for the case of (G,M)-orthogonal sets, the proof being again similar for (G,M,θ)-orthogonal sets)

$$v_{M,\theta}(\mathcal{Y}_1+\mathcal{Y}_2)=\sum_{L_1,L_2\in\mathcal{L}^\theta(M)} d^G_{M,\theta}(L_1,L_2)v^{Q_1}_{M,\theta}(\mathcal{Y}_1)v^{Q_2}_{M,\theta}(\mathcal{Y}_2) \tag{2.3.5}$$

where for all $L_1,L_2\in\mathcal{L}^\theta(M)$, Q_1 (resp. Q_2) is an element of $\mathcal{P}^\theta(L_1)$ (resp. $\mathcal{P}^\theta(L_2)$) which depends on the auxiliary choice of a generic point $\xi\in\mathcal{A}_{M,\theta}$, and $d^G_{M,\theta}(L_1,L_2)$ is a nonnegative real number. Moreover, $d^G_{M,\theta}(L_1,L_2)$ is nonzero if and only if $\mathcal{A}^G_{M,\theta}=\mathcal{A}^{L_1}_{M,\theta}\oplus\mathcal{A}^{L_2}_{M,\theta}$ and we have $d^G_{M,\theta}(G,M)=1$.

As for (G,M)-orthogonal sets, there is the following easy way to produce (G,M,θ)-orthogonal sets. Let $M_0\subset M$ be a minimal θ-split Levi subgroup and pick $P_0\in\mathcal{P}^\theta(M_0)$. Let A_0 be the maximal split and θ-split central subtorus of M_0 and set

$$W_0:=\mathrm{Norm}_G(A_0)/M_0$$

for the *little Weyl group* of M_0. Then, the natural action of W_0 on $\mathcal{P}^\theta(M_0)$ is simply transitive (see [18, Proposition 5.9]). To every point $Y\in\mathcal{A}_{M_0,\theta}$, we can now associate a (G,M,θ)-orthogonal set $(Y_P)_{P\in\mathcal{P}^\theta(M)}$ as follows: for each $P\in\mathcal{P}^\theta(M)$, set Y_P to be the projection of wY to $\mathcal{A}_{M,\theta}$ where $w\in W_0$ is any element such that $wP_0\subset P$.

Let K be a maximal special compact subgroup of G. By using the Iwasawa decomposition $G=PK$, we can define maps $H_{P,\theta}:G\to\mathcal{A}_{M,\theta}$ for all $P\in\mathcal{P}^\theta(M)$ by setting $H_{P,\theta}(muk):=H_{M,\theta}(m)$ for all $m\in M$, $u\in U_P$ and $k\in K$. Then for all $g\in G$, the family $\mathcal{Y}_{M,\theta}(g):=(H_{\overline{P},\theta}(g))_{P\in\mathcal{P}^\theta(M)}$ is a positive (G,M,θ)-orthogonal set. We set

$$v_{M,\theta}(g):=v_{M,\theta}(\mathcal{Y}_{M,\theta}(g)),\quad g\in G$$

and more generally

$$v^Q_{M,\theta}(g):=v^Q_{M,\theta}(\mathcal{Y}_{M,\theta}(g)),\quad g\in G$$

for all $Q\in\mathcal{F}^\theta(M)$. We have the following descent formula which is a special case of a general result of Arthur (see [1, Proposition 7.1] and [6, (2.8.4)])

$$v_{M,\theta}(g) = \sum_{L \in \mathcal{L}(M)} d^G_{M,\theta}(L) v^Q_M(g), \quad g \in G \tag{2.3.6}$$

where for all $L \in \mathcal{L}(M)$, Q is a certain parabolic subgroup with Levi component L which depends on the choice of a generic point $\xi \in \mathcal{A}_M$ and $d^G_{M,\theta}(L)$ is a real number. Moreover, $d^G_{M,\theta}(L)$ is nonzero only if $\mathcal{A}^G_M = \mathcal{A}^{G,\theta}_M \oplus \mathcal{A}^L_M$ and whenever $\mathcal{A}^{G,\theta}_M = 0$ we have $d^G_{M,\theta}(G) = 1$.

2.4 Weighted and θ-weighted orbital integrals

Let G be a connected reductive group over F, M be a Levi subgroup of G and $f \in \mathcal{C}(G)$. Fix a special maximal compact subgroup K of G that we use to define weights $g \mapsto v^Q_M(g)$, for $Q \in \mathcal{F}(M)$, as in the previous section. Then, for all $x \in M \cap G_{\mathrm{reg}}$ and $Q \in \mathcal{F}(M)$ we define, following Arthur, a *weighted orbital integral* by

$$\Phi^Q_M(x, f) := \int_{G_x \backslash G} f(g^{-1} x g) v^Q_M(g) dg.$$

By of [6, Lemma 2.5.1], the integral above is absolutely convergent. In the particular case where $Q = G$, we simply set $\Phi_M(x, f) := \Phi^G_M(x, f)$.

Let θ be an algebraic involution of G and assume that M is a θ-split Levi subgroup. Using the same special maximal compact subgroup K we associate, as in the previous paragraph, to any $Q \in \mathcal{F}^\theta(M)$, a weight $g \in G \mapsto v^Q_{M,\theta}(g)$. Then for all $x \in M \cap G_{\mathrm{reg}}$ and $Q \in \mathcal{F}^\theta(M)$, we define a *$\theta$-weighted orbital integral* by

$$\Phi^Q_{M,\theta}(x, f) := \int_{G_x \backslash G} f(g^{-1} x g) v^Q_{M,\theta}(g) dg.$$

In the particular case where $Q = G$, we simply set $\Phi_{M,\theta}(x, f) := \Phi^G_{M,\theta}(x, f)$.

2.5 Cusp forms and θ-strongly cuspidal functions

Let G be a connected reductive group over F. Following [38], we say that a function $f \in \mathcal{C}(G)$ is *strongly cuspidal* if for all proper parabolic subgroup $P = MU$ of G, we have

$$\int_U f(mu) du = 0$$

for all $m \in M$. By [5, Lemma 5.2.1 (i)], if $f \in \mathcal{C}(G)$ is strongly cuspidal, M is a Levi subgroup of G and $Q \in \mathcal{F}(M)$ is different from G, then we have

$$\Phi^Q_M(x, f) = 0$$

for all $x \in M \cap G_{\mathrm{reg}}$ where the weighted orbital integral $\Phi^Q_M(x, f)$ is defined by using any special maximal compact subgroup K of G.

Let θ an algebraic involution of G. We say that a function $f \in \mathcal{C}(G)$ is θ-*strongly cuspidal* if for all proper θ-split parabolic subgroup $P = MU \subset G$, we have

$$\int_U f(g^{-1}mug)du = 0$$

for all $m \in M$ and $g \in G$. By a standard change of variable, $f \in \mathcal{C}(G)$ is θ-strongly cuspidal if and only if for all proper θ-split parabolic subgroup $P = MU$, all $m \in M \cap G_{\mathrm{reg}}$ and all $g \in G$, we have

$$\int_U f(g^{-1}u^{-1}mug)du = 0.$$

By a proof similar to [5, Lemma 5.2.1 (i)], if $f \in \mathcal{C}(G)$ is θ-strongly cuspidal, M is a θ-split Levi subgroup of G and $Q \in \mathcal{F}^{\theta}(M)$ is different from G, then we have

$$\Phi^Q_{M,\theta}(x, f) = 0$$

for all $x \in M \cap G_{\mathrm{reg}}$ where the θ-weighted orbital integral $\Phi^Q_{M,\theta}(x, f)$ is defined by using any special maximal compact subgroup K of G.

To a strongly cuspidal function $f \in \mathcal{C}(G)$ we associate a function Θ_f on G_{reg} defined by

$$\Theta_f(x) := (-1)^{a_{G_x} - a_G} \Phi^G_{M(x)}(x, f)$$

where $M(x) := \mathrm{Cent}_G(A_{G_x})$ (i.e. the minimal Levi subgroup containing x), $a_{G_x} := \dim(A_{G_x})$, $a_G := \dim(A_G)$ and the weighted orbital integral $\Phi^G_{M(x)}(x, f)$ is defined by using any special maximal compact subgroup K of G (the result is independent of this choice, see [38, Lemme 5.2]). Then, by [38, Corollaire 5.9], Θ_f is a quasi-character in the sense of *loc. cit.*. This means that for all semi-simple elements $x \in G$, we have a local expansion

$$\Theta_f(x \exp(X)) = \sum_{\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_x)} c_{f,\mathcal{O}}(x) \widehat{j}(\mathcal{O}, X)$$

for all $X \in \mathfrak{g}_{x,\mathrm{reg}}$ sufficiently near 0, where $c_{f,\mathcal{O}}(x)$, $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_x)$, are complex numbers and the other notations have been defined in §2.2. For all semi-simple elements $x \in G$, we set

$$c_f(x) := \begin{cases} \frac{1}{|\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)|} \sum_{\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)} c_{f,\mathcal{O}}(x), & \text{if } G_x \text{ is quasi-split;} \\ 0, & \text{otherwise} \end{cases}$$

where we recall that $\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)$ denotes the subset of regular nilpotent orbits in $\mathfrak{g}_x$. This value does not depend on the choices of $\langle ., . \rangle$ and ψ. If G_x is quasi-split and we fix a Borel subgroup $B_x \subset G_x$ and a maximal torus $T_{x,\mathrm{qd}} \subset B_x$, then by [5, Proposition 4.5.1 1.(ii)], we have

$$D^G(x)^{1/2} c_f(x) = |W(G_x, T_{x,\mathrm{qd}})|^{-1} \lim_{x' \in T_{x,\mathrm{qd}} \to x} D^G(x')^{1/2} \Theta_f(x'). \tag{2.5.1}$$

Moreover, by [5, Proposition 4.5.1 1.(iii)], the function $(D^G)^{1/2}c_f$ is locally bounded on G.

Let χ be a unitary character of A_G. We say, following Harish-Chandra, that a function $f \in \mathcal{C}(G)$ or $f \in \mathcal{C}(G,\chi)$ is a *cusp form* if for all proper parabolic subgroup $P = MU$ of G, we have

$$\int_U f(xu)du = 0$$

for all $x \in G$. Of course, for functions in $\mathcal{C}(G)$ being a cusp form implies being strongly cuspidal. We shall denote by ${}^0\mathcal{C}(G)$ and ${}^0\mathcal{C}(G,\chi)$ the spaces of cusp forms in $\mathcal{C}(G)$ and $\mathcal{C}(G,\chi)$ respectively. For each $\pi \in \Pi_2(G,\chi)$, the matrix coefficients of π belong to ${}^0\mathcal{C}(G,\chi)$ ([15, Theorem 29]). And if f is such a matrix coefficient, we have (see [6, Proposition 2.6.1])

$$\Theta_f = d(\pi)^{-1} f(1)\Theta_\pi. \tag{2.5.2}$$

Moreover, any element in the space ${}^0\mathcal{C}(G,\chi)$ can be written as a finite linear combination of matrix coefficients of representations inside $\Pi_2(G,\chi)$. As a special case of Harish-Chandra-Plancherel formula ([37, Théorème VIII.4.2]), for all $f \in {}^0\mathcal{C}(G,\chi)$, we have an equality

$$f = \sum_{\pi \in \Pi_2(G,\chi)} d(\pi) f_\pi \tag{2.5.3}$$

where we have set $f_\pi(g) := \mathrm{Tr}(\pi^\vee(g^{-1})\pi^\vee(f))$ for all $\pi \in \Pi_2(G,\chi)$.

3 A multiplicity formula for generalized Shalika models

3.1 Generalized Shalika triples

From now on and until the end of the paper we fix a central simple algebra $\mathcal{A}$ over F of rank n (i.e. $\mathcal{A} = \mathrm{Mat}_{m\times m}(\mathcal{D})$ where $\mathcal{D}/F$ is a division algebra of degree r and $n = mr$). $\mathrm{Tr}_{\mathcal{A}/F} : \mathcal{A} \to F$ and $N_{\mathcal{A}/F} : \mathcal{A} \to F$ will stand for the reduced trace and norm respectively, and we will set $\nu(.) := |N_{\mathcal{A}/F}(.)|_F$. We also fix a maximal order $\mathcal{O}_\mathcal{A}$ of $\mathcal{A}$. Set $G := \mathrm{GL}_2(\mathcal{A})$ and define the following subgroups of G:

- $K := \mathrm{GL}_2(\mathcal{O}_\mathcal{A})$ (a maximal compact subgroup of G);
- $H_0 := \{\begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} \mid \lambda \in \mathcal{A}^\times\}$;
- $N := \{\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix} \mid X \in \mathcal{A}\}$;

- $H := H_0 \ltimes N$;
- $L := \{\begin{pmatrix} \lambda & \\ & \mu \end{pmatrix} \mid \lambda, \mu \in \mathcal{A}^\times\}$ (a Levi subgroup of G);
- $Q := LN = \{\begin{pmatrix} \lambda & X \\ & \mu \end{pmatrix} \mid \lambda, \mu \in \mathcal{A}^\times, X \in \mathcal{A}\}$ (a parabolic subgroup of G).

By [37, Lemme II.1.5 and Proposition II.4.5], we see that the subgroup H has the following property (we shall say that the pair (G, H) is *tempered* following [6, §2.7] and unlike [14] where these kind of pairs were called *strongly discrete*):

(3.1.1) There exists $d > 0$ such that the integral $\displaystyle\int_H \Xi^G(h)\sigma_G(h)^{-d}dh$ converges.

We fix a continuous character $\omega : F^\times \to \mathbf{C}^\times$ that we identify to a character of H_0 through composition with $N_{\mathcal{A}/F} : H_0 \to F^\times$. We then define a character $\xi : N \to \mathbf{C}^\times$ by

$$\xi \begin{pmatrix} 1 & X \\ & 1 \end{pmatrix} := \psi(\mathrm{Tr}_{\mathcal{A}/F}\, X), \quad X \in \mathcal{A}.$$

Then ξ is invariant under the H_0-conjugation and thus extends to a character ξ of H that is trivial on H_0. We can also consider ω as a character on H by composition with the projection $H \twoheadrightarrow H_0$ and we will denote by $\omega \otimes \xi$ the product of these two characters of H. We refer to the triple $(G, H, \omega \otimes \xi)$ as a *generalized Shalika triple*. In particular, if $\mathcal{A} = \mathrm{Mat}_n(F)$, this is the usual Shalika model. For all $\pi \in \mathrm{Irr}(G)$ we define the *multiplicity* $m(\pi, \omega)$ to be

$$m(\pi, \omega) := \dim \mathrm{Hom}_H(\pi, \omega \otimes \xi).$$

By [10, Theorem 4.5], this multiplicity is always finite.

3.2 A simple local trace formula for the generalized Shalika models

Here we assume that the character ω is unitary. Let $f \in \mathcal{C}(G)$. For all $x, y \in G$ we set

$$K_f(x, y) := \int_H f(x^{-1}hy)(\omega \otimes \xi)(h)^{-1}dh.$$

This integral is absolutely convergent by 3.1.1. Moreover, whenever convergent, we define the following expression

$$J(f) := \int_{H\backslash G} K_f(x, x)dx.$$

One of the main results of this paper is the following theorem which might be seen as some sort of simple local trace formula in the setting of the generalized Shalika models.

Theorem 3.1. *Assume that $f \in {}^0\mathcal{C}(G)$ and ω is unitary. Then, the expression defining $J(f)$ is absolutely convergent and we have the following two expansions of it:*

$$\sum_{T\in\mathcal{T}_{\mathrm{ell}}(H_0)} |W(H_0,T)|^{-1}\int_T D^H(t)c_f(t)\omega(t)^{-1}dt = J(f) = \sum_{\pi\in\Pi_2(G,\chi)} m(\pi,\omega)\operatorname{Tr}\pi^\vee(f)$$

where $\mathcal{T}_{\mathrm{ell}}(H_0)$ is a set of representatives of conjugacy classes of maximal elliptic tori in H_0, $c_f(t)$ is defined in Section 2.5, and $\chi = \omega^n$ be seen as a character of $A_G = F^\times$.

Note that the summation on the right hand side of the equality above is a finite sum (by [37, Théorème VIII.1.2]), hence it is convergent. The integrals on the left hand side are absolutely convergent by the following lemma.

Lemma 3.2. *With the same assumptions as in Theorem 3.1, the integral*

$$\int_T D^H(t)c_f(t)\omega(t)^{-1}dt$$

is absolutely convergent for all $T \in \mathcal{T}_{\mathrm{ell}}(H_0)$.

Proof. We can rewrite the integral as

$$\int_{A_G\backslash T} D^H(t)c_{f_\chi}(t)\omega(t)^{-1}dt$$

where we recall that $f_\chi(g) := \int_{A_G} f(ag)\chi(a)^{-1}da$. Since T is elliptic, $A_G\backslash T$ is compact. Together with the assumption that ω is unitary, it is enough to show that the function

$$t \in T_{\mathrm{reg}} \to D^H(t)c_{f_\chi}(t)$$

is locally bounded on T. This just follows from the fact that the function $(D^G)^{1/2}c_f$ is locally bounded on G ([5, Proposition 4.5.1 1.(iii)]), and $D^H(t) = D^G(t)^{1/2}$ for all $t \in T_{\mathrm{reg}}$. □

The proof of this Theorem will occupy Sections 4 and 5 entirely. In Section 4, we will prove the absolute convergence of $J(f)$ when $f \in {}^0\mathcal{C}(G)$ together with the spectral expansion (that is the second equality of the Theorem). It is the easy part and moreover the arguments are very similar to [6, §4]. Section 5 on the other hand contains the proof of the geometric side (i.e. the first equality of the Theorem) which is more involved than that of the spectral side.

3.3 The multiplicity formula

Let $\chi = \omega^n$ be seen as a character of $A_G = F^\times$. For all $\pi \in \mathrm{Irr}(G,\chi)$, define

$$m_{geom}(\pi,\omega) = \sum_{T\in\mathcal{T}_{\mathrm{ell}}(H_0)} |W(H_0,T)|^{-1}\int_{A_G\backslash T} D^H(t)c_\pi(t)\omega(t)^{-1}dt \tag{3.3.1}$$

where we recall that the Haar measures on the tori $A_G\backslash T$, $T \in \mathcal{T}_{\mathrm{ell}}(H_0)$, are chosen so that $\mathrm{vol}(A_G\backslash T) = 1$ (see §2.1). By a similar argument as in Lemma 3.2, we know that the integrals on the right hand side are absolutely convergent. The main interest of Theorem 3.1 is the following consequence of it.

Proposition 3.3. *For all $\pi \in \Pi_2(G,\chi)$, we have the multiplicity formula*

$$m(\pi,\omega) = m_{geom}(\pi,\omega) = \sum_{T\in\mathcal{T}_{\mathrm{ell}}(H_0)} |W(H_0,T)|^{-1} \int_{A_G\backslash T} D^H(t) c_\pi(t)\omega(t)^{-1} dt. \tag{3.3.2}$$

Moreover, if $\mathcal{A} = \mathcal{D}$ is a division algebra, then the same formula holds for all $\pi \in \mathrm{Irr}(G,\chi)$.

Proof. The case when $\mathcal{A}$ is a division algebra directly follows from Corollary A.2. Assume now that $\pi \in \Pi_2(G,\chi)$. The absolute value $|\chi|$ of χ extends uniquely to a positive valued character on G that we shall denote the same way. Then, up to multiplying ω by $|\chi|^{-1}$ and replacing π by $\pi \otimes |\chi|^{-\frac{1}{2n}}$ we may assume that χ is unitary and then so is ω. We can then use the equality of Theorem 3.1 that may be rewritten as

$$\sum_{T\in\mathcal{T}_{\mathrm{ell}}(H_0)} |W(H_0,T)|^{-1} \int_{A_G\backslash T} D^H(t) c_{f_\chi}(t)\omega(t)^{-1} dt = \sum_{\pi'\in\Pi_2(G,\chi)} m(\pi',\omega)\,\mathrm{Tr}(\pi')^\vee(f_\chi)$$

for all $f \in {}^0\mathcal{C}(G)$. Let $\pi \in \Pi_2(G,\chi)$. We choose $f \in {}^0\mathcal{C}(G)$ so that f_χ is a matrix coefficient of π with $f_\chi(1) \neq 0$. Then by Schur's orthogonality relations, the spectral side reduces to

$$m(\pi,\omega)\,\mathrm{Tr}\,\pi^\vee(f_\chi) = d(\pi)^{-1} m(\pi,\omega) f_\chi(1).$$

On the other hand, by 2.5.2, the geometric side equals

$$d(\pi)^{-1} f_\chi(1) \sum_{T\in\mathcal{T}_{\mathrm{ell}}(H_0)} |W(H_0,T)|^{-1} \int_{A_G\backslash T} D^H(t) c_\pi(t)\omega(t)^{-1} dt.$$

This proves the proposition. □

Remark 3.4. *In general, if $\mathcal{A}$ is not a division algebra, then the multiplicity formula* (3.3.2) *will not hold for all tempered (or generic) representations. For example, let $\mathcal{A} = \mathrm{Mat}_{m\times m}(D)$ with $m > 1$, and let π be a tempered representation of $G = \mathrm{GL}_{2m}(D)$ with central character χ. Assume that π is the parabolic induction of some discrete series $\tau = \tau_1 \otimes \cdots \otimes \tau_{2m}$ of the minimal parabolic subgroup $P_0 = M_0N_0$ of G (here $M_0 = (\mathrm{GL}_1(D))^{2m}$). Using [39, Lemme 2.3], we readily check that $c_\pi(t) = 0$ for all $T \in \mathcal{T}_{\mathrm{ell}}(H_0)$ and $t \in T_{\mathrm{reg}}$ (actually using van Dijk formula [11], we also have that the character Θ_π vanish on a neighborhood of any $t \in T_{\mathrm{reg}}$). This implies that $m_{geom}(\pi,\omega)$ is always equal to zero in this case. On the other hand, we can choose some nice τ such that $m(\pi,\omega) \neq 0$ (e.g. when the character ω is trivial, we just need to let $\tau = \tau_1 \otimes \cdots \otimes \tau_{2m}$ with $\tau_{2i-1} \simeq \tau_{2i}^\vee$ for $1 \leq i \leq m$).*

4 The spectral side

In this section we prove, following [6, §4], the absolute convergence as well as the spectral side of Theorem 3.1.

For $\pi \in \Pi_2(G, \chi)$, let

$$\mathcal{B}_\pi : \pi \times \pi^\vee \to \mathbf{C}$$

be the bilinear form defined by

$$\mathcal{B}_\pi(v, v^\vee) := \int_{A_G \backslash H} \langle \pi(h)v, v^\vee \rangle (\omega \otimes \xi)(h)^{-1} dh$$

for all $(v, v^\vee) \in \pi \times \pi^\vee$. Note that the integral above is always absolutely convergent by 3.1.1. Obviously $\mathcal{B}_\pi$ descends to a bilinear pairing

$$\mathcal{B}_\pi : \pi_{\omega\otimes\xi} \times \pi^\vee_{(\omega\otimes\xi)^{-1}} \to \mathbf{C}$$

where $\pi_{\omega\otimes\xi}$ and $\pi^\vee_{(\omega\otimes\xi)^{-1}}$ denote the $(H, \omega\otimes\xi)$- and $(H, (\omega\otimes\xi)^{-1})$-coinvariant spaces of π and $\pi^\vee$ respectively. As in [6, §4] the main ingredient of the proof is the following proposition, which is a variation of [34, Theorem 6.4.1] (a similar idea also appears in [39, Proposition 5.6]):

Proposition 4.1. *$\mathcal{B}_\pi$ induces a perfect pairing between $\pi_{\omega\otimes\xi}$ and $\pi^\vee_{(\omega\otimes\xi)^{-1}}$.*

This proposition can be proved exactly the same way as [6, Proposition 4.2.1] once we establish the next lemma.

Lemma 4.2. *For all $\ell \in Hom_H(\pi, \omega \otimes \xi)$ and all $v \in \pi$, we have*

$$\int_{H\backslash G} |\ell(\pi(x)v)|^2 dx < \infty.$$

Moreover, for all $f \in \mathcal{C}(G)$, the integral

$$\int_G f(g)\ell(\pi(g)v)dg$$

is absolutely convergent and equals $\ell(\pi(f)v)$.

Proof of Lemma 4.2: Let $\ell \in Hom_H(\pi, \omega \otimes \xi)$ and $v \in \pi$. Set $G_1 := \mathcal{A}^\times$ and embed G_1 in G via $g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}$. Let $P_1 = L_1U_1 \subset G_1$ be a minimal parabolic subgroup and $A_1 \subset L_1$ the maximal split torus. Let Δ_1 be the set of simple roots of A_1 in P_1 and set

$$A_1^+ := \{a \in A_1; |\alpha(a)| \leqslant 1 \ \forall \alpha \in \Delta_1\},$$

$$A_1^{++} := \left\{ a \in A_1^+;\ |\alpha(a)| \leqslant 1\ \forall \alpha \in R(A_1, N) \right\}.$$

Then, by the Iwasawa decomposition $G = QK$ and the Cartan decomposition for G_1, there exists a compact subset $C_0 \subset G$ such that

$$G = HA_1^+ C_0. \tag{4.0.1}$$

Moreover, there exists a compact subset $C_A \subset A_1$ such that for all $a \in A_1^+$ with $\ell(\pi(a)v) \neq 0$, we have

$$a \in A_1^{++} C_A.$$

Indeed, it suffices to show that for all $\alpha \in R(A_1, N)$, there exists $c_\alpha > 0$ such that for all $a \in A_1$ with $\ell(\pi(a)v) \neq 0$, we have $|\alpha(a)| \leqslant c_\alpha$. Let α be such a root and note that the character ξ has a nontrivial restriction to the corresponding root subspace N_α. Let $K_\alpha \subset N_\alpha$ be a compact-open subgroup which leaves v invariant. Then, there exists $c_\alpha > 0$ such that for all $a \in A_1$ with $|\alpha(a)| > c_\alpha$, the restriction of ξ to $aK_\alpha a^{-1}$ is nontrivial. This easily implies that $\ell(\pi(a)v) = 0$. This proves the claim.

Thus there exists a compact-open subset $C \subset G$ such that

$$\text{The function } g \in G \mapsto \ell(\pi(g)v) \text{ has support in } HA_1^{++}C. \tag{4.0.2}$$

Let $\overline{P}_1 = L_1 \overline{U}_1$ be the parabolic subgroup opposite to P_1 and introduce the following parabolic subgroup of G:

$$P := \left\{ \begin{pmatrix} p_1 & 0 \\ X & g_1 \end{pmatrix};\ p_1 \in \overline{P}_1, g_1 \in G_1, X \in \mathcal{A} \right\}.$$

It has a Levi decomposition $P = L_P U_P$ where

$$L_P := \left\{ \begin{pmatrix} l_1 & 0 \\ 0 & g_1 \end{pmatrix};\ l_1 \in L_1, g_1 \in G_1 \right\}$$

and

$$U_P := \left\{ \begin{pmatrix} u_1 & 0 \\ X & 1 \end{pmatrix};\ u_1 \in \overline{U}_1, X \in \mathcal{A} \right\}.$$

Note that A_1 is contained in the center of L_P and

$$A_1^{++} = \{ a \in A_1;\ |\alpha(a)| \geqslant 1\ \forall \alpha \in R(A_1, U_P) \}. \tag{4.0.3}$$

Moreover, we have

(4.0.4) HP is open in G.

Define a function $\Xi^{H\backslash G}$ on $H\backslash G$ by

$$\Xi^{H\backslash G}(x) := \mathrm{vol}_{H\backslash G}(xC)^{-1/2}, \quad x \in H\backslash G.$$

Then as in [5, Proposition 6.7.1], we can show in turn that

(4.0.5) $\Xi^{H\backslash G}(xk) \sim \Xi^{H\backslash G}(x)$ and $\sigma_{H\backslash G}(xk) \sim \sigma_{H\backslash G}(x)$ for all $x \in H\backslash G$ and all $k \in C$;

(4.0.6) There exists $d > 0$ such that $\Xi^G(a) \ll \Xi^{H\backslash G}(a)\sigma_G(a)^d$ for all $a \in A_1^{++}$ (this uses 4.0.3 and 4.0.4);

(4.0.7) $\sigma_{H\backslash G}(a) \sim \sigma_G(a)$ for all $a \in A_1$ (this is because the regular map $G_1 \to H\backslash G$ is a closed embedding);

(4.0.8) There exists $d > 0$ such that the integral

$$\int_{H\backslash G} \Xi^{H\backslash G}(x)^2 \sigma_{H\backslash G}(x)^{-d} dx$$

converges (this uses decomposition 4.0.1);

(4.0.9) For all $d > 0$, there exists $d' > 0$ such that

$$\int_H \Xi^G(hx)\sigma_G(hx)^{-d'} dx \ll \Xi^{H\backslash G}(x)\sigma_{H\backslash G}(x)^{-d}$$

for all $x \in H\backslash G$ (this uses 4.0.4 together with decomposition 4.0.1).

Finally, by the above points and 4.0.2, in order to prove the lemma, it remains to show that

(4.0.10) For all $d > 0$, we have $|\ell(\pi(a)v)| \ll \Xi^G(a)\sigma_G(a)^{-d}$ for all $a \in A_1^{++}$.

Let $K_v \subset G$ be an open subgroup which stabilizes v. Then, from 4.0.3 and 4.0.4 we deduce the existence of a compact-open subgroup J of G such that $J \subset HaK_va^{-1}$ for all $a \in A_1^{++}$. It follows that

$$\ell(\pi(a)v) = \langle \pi(a)v, e_J * \ell \rangle$$

for all $a \in A_1^{++}$ where $e_J * \ell$ denotes the element of $\pi^\vee$ defined by

$$\langle w, e_J * \ell \rangle := \mathrm{vol}(J)^{-1} \int_J \ell(\pi(k)w)dk, \quad w \in \pi.$$

4.0.10 now follows from standard estimates for coefficients of square-integrable representations. ■

We now prove the absolute convergence and the spectral side of Theorem 3.1. Let $f \in {}^0\mathcal{C}(G)$. Then, we have

$$K_f(x,y) = \int_{A_G\backslash H} f_\chi(x^{-1}hy)(\omega\otimes\xi)(h)^{-1}dh$$

for all $x, y \in G$ where we recall that χ denotes the restriction of ω to A_G and $f_\chi(g) := \int_{A_G} f(ag)\chi(a)^{-1}da$ for all $g \in G$. By 2.5.3, we may assume that there exists $\pi \in \Pi_2(G,\chi)$ such that f_χ is a matrix coefficient of π, i.e. there exist $(v, v^\vee) \in \pi \times \pi^\vee$ such that $f_\chi(g) = \langle \pi(g)v, v^\vee\rangle$ for all $g \in G$. In this case, we have

$$K_f(x,y) = \mathcal{B}_\pi(\pi(y)v, \pi(x)v^\vee)$$

for all $x, y \in G$. Let $N := m(\pi,\omega)$ and let $v_1, \dots, v_N$ be vectors in π whose images in $\pi_{\omega\otimes\xi}$ form a basis. Let $v_1^\vee, \dots, v_N^\vee$ be vectors in $\pi^\vee$ whose images in $\pi^\vee_{(\omega\otimes\xi)^{-1}}$ form the dual basis with respect to $\mathcal{B}_\pi$. Then, we have

$$K_f(x,y) = \sum_{i=1}^N \mathcal{B}_\pi(\pi(y)v, v_i^\vee)\mathcal{B}_\pi(v_i, \pi(x)v^\vee)$$

for all $x, y \in G$. From there and Lemma 4.2 we deduce the absolute convergence of $J(f)$. Now the end of the proof is the same as that of [6, Theorem 4.1.1]: using Schur's orthogonality relations we have

$$\begin{aligned} J(f) &= \sum_{i=1}^N \int_{H\backslash G} \mathcal{B}_\pi(\pi(x)v, v_i^\vee)\mathcal{B}_\pi(v_i, \pi^\vee(x)v^\vee)dx = \sum_{i=1}^N \int_{A_G\backslash G} \langle \pi(g)v, v_i^\vee\rangle \mathcal{B}_\pi(v_i, \pi^\vee(g)v^\vee)dg \\ &= \sum_{i=1}^N \frac{\langle v, v^\vee\rangle}{d(\pi)}\mathcal{B}_\pi(v_i, v_i^\vee) = m(\pi,\omega)\frac{\langle v, v^\vee\rangle}{d(\pi)} = m(\pi,\omega)\operatorname{Tr}\pi(f). \end{aligned}$$

■

5 The geometric side

The goal of this chapter is to prove the geometric side of Theorem 3.1. The proof will be given in Section 5.9. We will continue to use the notations introduced in Chapter 3 and we will assume as in Theorem 3.1 that ω is unitary. We will also need the following extra notations:

- $\theta := \mathrm{Ad}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ (an involution of G);

- $G_1 := \mathcal{A}^\times$, $K_1 := \mathcal{O}_\mathcal{A}^\times$ and $\mathfrak{g}_1 := \mathcal{A}$ (the Lie algebra of G_1), note that we have a natural identification $L \simeq G_1 \times G_1$;
- There is a natural open embedding $G_1 \hookrightarrow \mathfrak{g}_1$ and we will denote by $\mathfrak{g}_1^*$ its image. Similarly, for any maximal torus T of G_1 the previous embedding restricts to an open embedding $T \hookrightarrow \mathfrak{t}$, where $\mathfrak{t}$ denotes the Lie algebra of T, and we will denote by $\mathfrak{t}^*$ its image (i.e. the open subset of all $X \in \mathfrak{t}$ such that $\nu(X) \neq 0$);
- $\langle ., . \rangle : \mathfrak{g}_1 \times \mathfrak{g}_1 \to F$ the bilinear pairing given by $\langle X, Y \rangle := \mathrm{Tr}_{\mathcal{A}/F}(XY)$;

Remark 5.1. *The introduction of the involution θ might seem unmotivated at first sight. It will however naturally appear in the computations of the next sections. A heuristic reason for this could be that distinction for the generalized Shalika model (G, H, ξ) considered in this paper is closely related to distinction for the* generalized linear model *associated to the pair (G, L) where L is as in Section 3.1 i.e. the subgroup of θ-fixed points. Indeed, when $\mathcal{A}$ is split by [23, Sect. 6] if an irreducible representation π admits a Shalika model then it also admits a linear model. Moreover, using the technique of local unfolding of Sakellaridis-Venkatesh [34, §9.5] it is highly plausible that we can prove (although we will not do it in this paper) that a tempered representation of G (or more generally a generic representation) has a generalized Shalika model* if and only if *it has a generalized linear model. If this is the case, the spectral sides of the (simple) trace formulae (of the kind considered in this paper) for these two models would be exactly the same and therefore so would their geometric sides. However, working with (generalized) Shalika models seemed easier to us since the space of semi-simple conjugacy classes in H is of half the dimension of the space of semi-simple conjugacy classes in L (and this plays somehow a crucial role in our proof of the geometric side).*

Recall that in this paper we are assuming that every algebraic variety X over F that we encounter has been equipped with a log-norm σ_X. For simplicity we will assume, as we may, that σ_{G_1} and $\sigma_{\mathfrak{g}_1^*}$ are both left and right invariant by K_1. Let $T \subset G_1$ be a maximal torus. By [26, Proposition 18.3], we may, and will, assume that for every $g \in G_1$, we have

$$\sigma_{T\backslash G_1}(g) = \inf_{t \in T} \sigma_{G_1}(tg). \tag{5.0.1}$$

We also recall the following inequality from [6, (2.2.2)]:

$$\sigma_{T\backslash G_1}(g) \ll \sigma_{G_1}(g^{-1}tg) \log\left(2 + D^{G_1}(t)^{-1}\right) \tag{5.0.2}$$

for all $g \in G_1$ and all $t \in T \cap G_{1,\mathrm{reg}}$. We will also need the following easy-to-check lemma:

Lemma 5.2. *Let K be a finite extension of F and set $v_K := \frac{1}{[K:F]} v_F \circ N_{K/F}$. Then for all $k > 0$, the inequality*

$$\int_{y\in K;\, v_K(y)\geqslant C} \max(1, v_K(y))^k dy \ll C^k \times \mathrm{vol}\{y \in K;\ v_K(y) \geqslant C\}$$

holds for all $C > 1$.

We now describe roughly how we will prove the geometric side of Theorem 3.1. In Section 5.1 we will introduce a sequence of truncations $(J_N(f))_{N\geqslant 1}$ of $J(f)$ such that $\lim_{N\to\infty} J_N(f) = J(f)$ whenever $J(f)$ is absolutely convergent. Then, in Section 5.8 we will show that $J_N(f)$ admits a limit whenever f is *θ-strongly cuspidal* (see §2.5) and we compute this limit. In the particular case where f is strongly cuspidal (in particular if f is a cusp form), we prove in Section 5.9 that this limit is equal to the geometric side of Theorem 3.1. The bulk of the proof is contained in Section 5.6, where we show that we can replace certain weights appearing naturally from our truncations by other weights that are related to certain (singular) θ-weighted orbital integrals.

5.1 Definition of a truncation

Fix a maximal split torus A_1 of G_1 such that K_1 is the fixator of a special point in the apartment associated to A_1. Let $M_1 := \mathrm{Cent}_{G_1}(A_1)$ and P_1 be a minimal parabolic subgroup with Levi component M_1. Let Δ_1 be the set of simple roots of A_1 in P_1. We have a Cartan decomposition

$$G_1 = K_1 M_1^+ K_1$$

with $M_1^+ := \{m \in M_1; \langle \alpha, H_{M_1}(m)\rangle \geqslant 0\ \forall \alpha \in \Delta_1\}$. Let $N \geqslant 1$ be an integer and let $T_N \in \mathcal{A}^{G_1}_{M_1,F}$ be the point characterized by $\langle \alpha, T_N\rangle = N$ for all $\alpha \in \Delta_1$. In [2, §3], Arthur has defined a certain characteristic function $u(., T_N)$ associated to T_N. More precisely, if we denote for all $\alpha \in \Delta_1$ by $\varpi_\alpha \in \mathcal{A}^*_{M_1}$ the corresponding simple weight and set

$$M_1^+(N) := \{m \in M_1^+;\ \langle H_{M_1}(m) - T_N, \varpi_\alpha\rangle \leqslant 0\ \forall \alpha \in \Delta_1\}.$$

Then $u(., T_N)$ is the characteristic function of $K_1 M_1^+(N) K_1$ (see [2, Lemma 3.1]). Because of the center, the set $K_1 M_1^+(N) K_1$ is not compact and we define another truncation function $\kappa_N : G_1 \to \{0, 1\}$, this time of compact support, by setting

$$\kappa_N(g) = \mathbf{1}_{[q^{-N}, q^N]}(\nu(g)) u(g, T_N), \quad g \in G_1$$

where $\mathbf{1}_{[q^{-N}, q^N]}$ stands for the characteristic function of the segment $[q^{-N}, q^N]$. The next lemma summarizes some basic properties of the sequence $(\kappa_N)_{N\geqslant 1}$ that we will need.

Lemma 5.3. *(i) There exist* $c_1, c_2 > 0$ *such that for all* $g \in G_1$ *and all* $N \geqslant 1$, *if* $\sigma_G(g) \leqslant c_1 N$, *then* $\kappa_N(g) = 1$; *and if* $\kappa_N(g) = 1$, *then* $\sigma_G(g) \leqslant c_2 N$.

(ii) Let $T \subset G_1$ *be a maximal torus. Then, there exist* $c > 0$ *and* $N_0 \geqslant 1$ *such that for all* $N \geqslant N_0$ *and all* $g, h \in G_1$ *with* $\max\left(\sigma_{T\backslash G_1}(g), \sigma_{T\backslash G_1}(h)\right) \leqslant cN$, *the function*

$$a \in T \mapsto \kappa_N(h^{-1}ag)$$

is invariant by the maximal compact subgroup T^c of T.

Proof. (i) is obvious;

(ii) Up to conjugating T we may assume that $T^c \subset K_1$ and $A_T \subset A_1$. Since the property we want to prove is invariant by left translation of both g and h by T, by 5.0.1 we may replace the condition $\max\left(\sigma_{T\backslash G_1}(g), \sigma_{T\backslash G_1}(h)\right) \leqslant cN$ by the seemingly stronger one $\max\left(\sigma_{G_1}(g), \sigma_{G_1}(h)\right) \leqslant cN$. Also, there exists a finite family $(t_i)_{i=1,\dots,k}$ of elements of T such that $T = \bigsqcup_{i=1}^{k} A_T T^c t_i$. Hence, it suffices to show the existence of $c > 0$ and $N_0 \geqslant 1$ such that for all $N \geqslant N_0$ and all $g, h \in G_1$ with $\max\left(\sigma_{G_1}(g), \sigma_{G_1}(h)\right) \leqslant cN$, we have

$$\kappa_N(h^{-1}tag) = \kappa_N(h^{-1}ag)$$

for all $t \in T^c$ and all $a \in A_T$. This property is obviously satisfied by the function $g \mapsto \mathbf{1}_{[q^{-N}, q^N]}(\nu(g))$ and thus we may replace κ_N by $u(., T_N)$. Set $M := \mathrm{Cent}_{G_1}(A_T)$ and define the maps $H_P : G_1 \to \mathcal{A}_M$, $P \in \mathcal{P}(M)$, by using the maximal compact subgroup K_1. These maps are T^c-invariant on the left. As σ_{G_1} is also T^c-invariant on the left (since we are assuming $T^c \subset K_1$), it suffices to show that for some $c > 0$ and $N_0 \geqslant 1$, the following holds: for all $N \geqslant N_0$ and $g, h \in G_1$ with $\max\left(\sigma_{G_1}(g), \sigma_{G_1}(h)\right) \leqslant cN$, the function

$$a \in A_T \mapsto u(h^{-1}ag, T_N)$$

only depends on the families $(H_P(h))_{P \in \mathcal{P}(M)}$ and $(H_P(g))_{P \in \mathcal{P}(M)}$. Such a property is provided by the proof of [2, Lemma 4.4]. Indeed, by the equation on the last line of p.38 of *loc. cit.*, there exists $c > 0$ and $N_0 \geqslant 1$ such that for all N, g and h as before, we have

$$u(h^{-1}ag, T_N) = \Gamma_M^{G_1}(H_M(a), \mathcal{Y}_M(h, g, T_N))$$

for all $a \in A_T(F)$ where the (G_1, M)-orthogonal set $\mathcal{Y}_M(h, g, T_N)$ is defined by

$$\mathcal{Y}_M(h, g, T_N)_P := T_{N,P} + H_P(h) - H_{\overline{P}}(g),\ P \in \mathcal{P}(M).$$

Obviously this (G_1, M)-orthogonal set only depends on the sets $(H_P(h))_{P \in \mathcal{P}(M)}$ and $(H_P(g))_{P \in \mathcal{P}(M)}$ and this proves the claim.

□

Choose Haar measures on G_1, $\mathfrak{g}_1$ and K such that

$$\int_G \varphi(g)dg = \int_{\mathfrak{g}_1}\int_{G_1\times G_1}\int_K \varphi\left(\begin{pmatrix}1 & X\\ & 1\end{pmatrix}\begin{pmatrix}g & \\ & h\end{pmatrix}k\right)dk\frac{dg}{\nu(g)^n}\nu(h)^n dhdX$$

and

$$\int_H \varphi(h)dh = \int_{G_1}\int_{\mathfrak{g}_1}\varphi\left(\begin{pmatrix}h_0 & \\ & h_0\end{pmatrix}\begin{pmatrix}1 & X\\ & 1\end{pmatrix}\right)dXdh_0$$

for all $\varphi \in C_c(G)$ (resp. for all $\varphi \in C_c(H)$).

For every integer $N \geqslant 1$ and every function $f \in \mathcal{C}(G)$, we define the following expression:

$$J_N(f) := \int_{G_1}\int_{G_1}\int_{\mathfrak{g}_1} f^K\left(\begin{pmatrix}g^{-1} & \\ & 1\end{pmatrix}\begin{pmatrix}h & \\ & h\end{pmatrix}\begin{pmatrix}1 & X\\ & 1\end{pmatrix}\begin{pmatrix}g & \\ & 1\end{pmatrix}\right)\xi(X)^{-1}dX\omega(h)^{-1}dh\kappa_N(g)\frac{dg}{\nu(g)^n}$$

where we have set

$$f^K(g) := \int_K f(k^{-1}gk)dk$$

for all $g \in G$. Note that if the integral defining $J(f)$ (see §3.2) is absolutely convergent, then we have

$$J(f) = \lim_{N\to\infty} J_N(f).$$

By Weyl's integration formula, we have

$$J_N(f) = \sum_{T\in\mathcal{T}(G_1)} |W(G_1,T)|^{-1} J_{N,T}(f) \tag{5.1.1}$$

for all $N \geqslant 1$ and all $f \in \mathcal{C}(G)$ where $\mathcal{T}(G_1)$ is a set of representatives of the G_1-conjugacy classes of maximal tori in G_1, and for all $T \in \mathcal{T}(G_1)$ we have set

$$J_{N,T}(f) := \int_T D^{G_1}(t)\int_{T\times T\backslash G_1\times G_1}\int_{\mathfrak{g}_1} f^K\left(\begin{pmatrix}g^{-1} & \\ & h^{-1}\end{pmatrix}\begin{pmatrix}t & \\ & t\end{pmatrix}\begin{pmatrix}1 & X\\ & 1\end{pmatrix}\begin{pmatrix}g & \\ & h\end{pmatrix}\right)$$
$$\kappa_{N,T,\xi}(g,h,X)dX\frac{dg}{\nu(g)^n}\nu(h)^n dh\omega(t)^{-1}dt$$

with

$$\kappa_{N,T,\xi}(g,h,X) := \int_T \xi(aX)^{-1}\kappa_N(h^{-1}ag)da.$$

From now on and until the end of Section 5.8 we fix a torus $T \in \mathcal{T}(G_1)$. Without loss of generality (i.e. up to conjugating T by an element of G_1), we may assume that $A_T \subset A_1$ and $T \cap K_1 = T^c$. From Lemma 5.3(i) and 5.0.1 we easily infer that there exists $k > 0$ such that the following estimate

$$|\kappa_{N,T,\xi}(g,h,X)| \ll \sigma_{T\backslash G_1}(g)^k \sigma_{T\backslash G_1}(h)^k N^k \tag{5.1.2}$$

holds for all $N \geqslant 1$, all $g, h \in G_1$ and all $X \in \mathfrak{g}_1$. More precisely, there exists $k > 0$ such that

$$\int_T \kappa_N(h^{-1}ag)da \ll \sigma_{T\backslash G_1}(g)^k \sigma_{T\backslash G_1}(h)^k N^k \tag{5.1.3}$$

for all $N \geqslant 1$ and all $g, h \in G_1$.

5.2 Concrete description of T and of related objects

Set $M_\natural := \mathrm{Cent}_{G_1}(A_T)$ (a Levi subgroup of G_1), $M := M_\natural \times M_\natural$ (a Levi subgroup of G). Then, the Levi M is θ-split. Moreover, there exist field extensions $K_1, \ldots, K_d$ of F such that

$$T \simeq K_1^\times \times \ldots \times K_d^\times. \tag{5.2.1}$$

More precisely, there exist:

- A division algebra $\mathcal{D}$, central over F and of degree r dividing n;
- A right $\mathcal{D}$-module V free of rank n/r;
- An isomorphism of F-algebras

  $$\mathcal{A} \simeq End_{\mathcal{D}}(V)$$

  inducing an isomorphism

  $$G_1 \simeq \mathrm{GL}_{\mathcal{D}}(V).$$

- For all $1 \leqslant i \leqslant d$, a degree r subextension F_i of K_i together with an embedding $F_i \hookrightarrow \mathcal{D}$ and an isomorphism of (right) $\mathcal{D}$-modules

  $$V \simeq K_1 \otimes_{F_1} \mathcal{D} \oplus \ldots \oplus K_d \otimes_{F_d} \mathcal{D} \tag{5.2.2}$$

  through which the action of T is given by multiplication by $K_i^\times$ on the i-th factor.

We fix such data (and isomorphisms) once and for all and we fix a basis of V compatible with the decomposition 5.2.2. Doing so we will identify G_1 (resp. G) with $\mathrm{GL}_{n/r}(\mathcal{D})$ (resp. $\mathrm{GL}_{2n/r}(\mathcal{D})$). Besides 5.2.1, we also get identifications

$$\mathfrak{t} \simeq K_1 \oplus \ldots \oplus K_d \tag{5.2.3}$$

and

$$M \simeq \mathrm{GL}_{k_1}(\mathcal{D}) \times \ldots \times \mathrm{GL}_{k_d}(\mathcal{D}) \times \mathrm{GL}_{k_1}(\mathcal{D}) \times \ldots \times \mathrm{GL}_{k_d}(\mathcal{D}) \tag{5.2.4}$$

where we have set $k_i := [K_i : F_i]$ for all $1 \leqslant i \leqslant d$. We will denote by $\mathfrak{t}^\perp$ the orthogonal of $\mathfrak{t}$ in $\mathfrak{g}_1$ with respect to the symmetric bilinear form $\langle ., . \rangle$ and by $X \mapsto X_\mathfrak{t}$ (resp. $X \mapsto X_{\mathfrak{t}^\perp}$) the projection $\mathfrak{g}_1 \to \mathfrak{t}$ (resp. $\mathfrak{g}_1 \to \mathfrak{t}^\perp$) relative to the decomposition $\mathfrak{g}_1 = \mathfrak{t} \oplus \mathfrak{t}^\perp$.

5.3 An estimate

For all $f \in \mathcal{C}(G)$, all $t_1, t_2 \in T \cap G_{1,\mathrm{reg}}$ and all $k > 0$, we define the following expressions:

$$I_k(f, t_1, t_2) := \int_{T\times T\backslash G_1 \times G_1} \int_{\mathfrak{g}_1} f\left(\begin{pmatrix} g^{-1} & \\ & h^{-1} \end{pmatrix}\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}\begin{pmatrix} g & \\ & h \end{pmatrix}\right)$$
$$\sigma_{T\backslash G_1}(g)^k \sigma_{T\backslash G_1}(h)^k \sigma_{\mathfrak{g}_1^*}(g^{-1}X_\mathfrak{t} h)^k \sigma_{\mathfrak{g}_1}(g^{-1}X_{\mathfrak{t}^\perp}h)^k dX \frac{dg}{\nu(g)^n}\nu(h)^n dh$$

and

$$I_{T,k}(f) := \int_T D^{G_1}(t) I_k(f, t, t) dt$$

where we recall that for all $X \in \mathfrak{g}_1$, $X_\mathfrak{t}$ and $X_{\mathfrak{t}^\perp}$ stand for the projections of X onto $\mathfrak{t}$ and $\mathfrak{t}^\perp$ respectively with respect to the decomposition $\mathfrak{g}_1 = \mathfrak{t} \oplus \mathfrak{t}^\perp$. Similarly, for all $f \in \mathcal{C}(G)$, all $t_1, t_2 \in T \cap G_{1,\mathrm{reg}}$, all $k > 0$ and all $C > 0$, we define the following expressions:

$$I_{k,\leqslant C}(f, t_1, t_2) := \int_{T\times T\backslash G_1 \times G_1} \int_{\mathfrak{g}_1} f\left(\begin{pmatrix} g^{-1} & \\ & h^{-1} \end{pmatrix}\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}\begin{pmatrix} g & \\ & h \end{pmatrix}\right)$$
$$\sigma_{T\backslash G_1}(g)^k \sigma_{T\backslash G_1}(h)^k \sigma_{\mathfrak{g}_1^*}(g^{-1}X_\mathfrak{t} h)^k \sigma_{\mathfrak{g}_1}(g^{-1}X_{\mathfrak{t}^\perp}h)^k$$
$$\mathbf{1}_{T\backslash G_1,\leqslant C}(g)\mathbf{1}_{T\backslash G_1,\leqslant C}(h)\mathbf{1}_{\mathfrak{g}_1^*,\leqslant C}(g^{-1}X_\mathfrak{t} h)\mathbf{1}_{\mathfrak{g}_1,\leqslant C}(g^{-1}X_{\mathfrak{t}^\perp}h) dX \frac{dg}{\nu(g)^n}\nu(h)^n dh$$

and

$$I_{T,k,\leqslant C}(f) := \int_T D^{G_1}(t) I_{k,\leqslant C}(f, t, t) dt.$$

Proposition 5.4. *(i) For all $f \in \mathcal{C}(G)$, all $t_1, t_2 \in T \cap G_{1,\mathrm{reg}}$ and all $k > 0$, the expressions defining $I_k(f, t_1, t_2)$ and $I_{T,k}(f)$ are absolutely convergent.*

(ii) For all $f \in \mathcal{C}(G)$, all $t_1, t_2 \in T \cap G_{1,\mathrm{reg}}$, all $k > 0$ and all $r > 0$, we have inequalities

$$|I_k(f, t_1, t_2) - I_{k,\leqslant C}(f, t_1, t_2)| \ll C^{-r}$$

and

$$|I_{T,k}(f) - I_{T,k,\leqslant C}(f)| \ll C^{-r}$$

for all $C > 0$.

Proof. (i) Fix $f \in \mathcal{C}(G)$ and $k > 0$. Up to the replacement of f by its absolute value, we may assume that f only takes nonnegative values. By [6, Lemma 1.9.2 and Lemma 1.9.3], it suffices to prove that for every $r > 0$, we have an inequality

$$\begin{aligned}
&\left(\frac{\nu(h)}{\nu(g)}\right)^n \int_{\mathfrak{g}_1} f\left(\begin{pmatrix} g^{-1} & \\ & h^{-1}\end{pmatrix}\begin{pmatrix} t_1 & \\ & t_2\end{pmatrix}\begin{pmatrix} 1 & X \\ & 1\end{pmatrix}\begin{pmatrix} g & \\ & h\end{pmatrix}\right)\sigma_{\mathfrak{g}_1^*}(g^{-1}X_{\mathfrak{t}}h)^k\sigma_{\mathfrak{g}_1}(g^{-1}X_{\mathfrak{t}^\perp}h)^k dX \\
&\ll \left(\frac{\nu(t_2)}{\nu(t_1)}\right)^{n/2} \Xi^{G_1}(g^{-1}t_1 g)\Xi^{G_1}(h^{-1}t_2 h)\sigma_{G_1}(g^{-1}t_1 g)^{-r}\sigma_{G_1}(h^{-1}t_2 h)^{-r} \\
&\times \log(2 + D^{G_1}(t_1)^{-1})^{d+5k}\log(2 + D^{G_1}(t_2)^{-1})^{d+5k}\sigma_{T\backslash G_1}(g)^{-k}\sigma_{T\backslash G_1}(h)^{-k}
\end{aligned}$$

for all $g, h \in G_1$ and $t_1, t_2 \in T \cap G_{1,\mathrm{reg}}$, where we recall that d denotes the rank of A_T (see §5.2). By 5.0.2, we have

$$\sigma_{T\backslash G_1}(g)\sigma_{T\backslash G_1}(h) \ll \sigma_{G_1}(g^{-1}t_1 g)\sigma_{G_1}(h^{-1}t_2 h)\log(2 + D^{G_1}(t_1)^{-1})\log(2 + D^{G_1}(t_2)^{-1})$$

for all $g, h \in G_1$ and $t_1, t_2 \in T \cap G_{1,\mathrm{reg}}$. Thus we are reduced to show that for every $r > 0$ we have

$$\begin{aligned}
&\left(\frac{\nu(h)}{\nu(g)}\right)^n \int_{\mathfrak{g}_1} f\left(\begin{pmatrix} g^{-1} & \\ & h^{-1}\end{pmatrix}\begin{pmatrix} t_1 & \\ & t_2\end{pmatrix}\begin{pmatrix} 1 & X \\ & 1\end{pmatrix}\begin{pmatrix} g & \\ & h\end{pmatrix}\right)\sigma_{\mathfrak{g}_1^*}(g^{-1}X_{\mathfrak{t}}h)^k\sigma_{\mathfrak{g}_1}(g^{-1}X_{\mathfrak{t}^\perp}h)^k dX \\
&\ll \left(\frac{\nu(t_2)}{\nu(t_1)}\right)^{n/2} \Xi^{G_1}(g^{-1}t_1 g)\Xi^{G_1}(h^{-1}t_2 h)\sigma_{G_1}(g^{-1}t_1 g)^{-r}\sigma_{G_1}(h^{-1}t_2 h)^{-r}\sigma_{T\backslash G_1}(g)^{(d+4)k}\sigma_{T\backslash G_1}(h)^{(d+4)k}
\end{aligned}$$

for all $g, h \in G_1$ and $t_1, t_2 \in T$. As the left hand side of the above inequality is invariant by left translations of both g and h by T, by 5.0.1 we see that we may replace $\sigma_{T\backslash G_1}(g)$ and $\sigma_{T\backslash G_1}(h)$ by $\sigma_{G_1}(g)$ and $\sigma_{G_1}(h)$ respectively in the right hand side of the inequality. Moreover, we have

$$\begin{aligned}\sigma_{\mathfrak{g}_1^*}(g^{-1}X_{\mathfrak{t}}h)\sigma_{\mathfrak{g}_1}(g^{-1}X_{\mathfrak{t}^\perp}h) &\ll \log\left(2+\nu(X_{\mathfrak{t}}^{-1})\right)\sigma_{\mathfrak{g}_1}(X)^2\sigma_{G_1}(g)^2\sigma_{G_1}(h)^2\\ &\ll \log\left(2+\nu(X_{\mathfrak{t}}^{-1})\right)\sigma_{\mathfrak{g}_1}(g^{-1}Xh)^2\sigma_{G_1}(g)^4\sigma_{G_1}(h)^4\\ &\ll \log\left(2+\nu(X_{\mathfrak{t}}^{-1})\right)\sigma_{G_1}(g)^4\sigma_{G_1}(h)^4\\ &\times \sigma_G\left(\begin{pmatrix}g^{-1} & \\ & h^{-1}\end{pmatrix}\begin{pmatrix}t_1 & \\ & t_2\end{pmatrix}\begin{pmatrix}1 & X\\ & 1\end{pmatrix}\begin{pmatrix}g & \\ & h\end{pmatrix}\right)^2\end{aligned}$$

for all $g,h\in G_1$ and $X\in\mathfrak{g}_1$. Thus, as the function $\gamma\in G\mapsto f(\gamma)\sigma_G(\gamma)^{2k}$ is again Harish-Chandra Schwartz, up to the replacement of f by this function, we just need to verify that for every $r>0$ we have

$$\begin{aligned}&\left(\frac{\nu(h)}{\nu(g)}\right)^n\int_{\mathfrak{g}_1} f\left(\begin{pmatrix}g^{-1} & \\ & h^{-1}\end{pmatrix}\begin{pmatrix}t_1 & \\ & t_2\end{pmatrix}\begin{pmatrix}1 & X\\ & 1\end{pmatrix}\begin{pmatrix}g & \\ & h\end{pmatrix}\right)\log\left(2+\nu(X_{\mathfrak{t}}^{-1})\right)^k dX\\ &\ll\left(\frac{\nu(t_2)}{\nu(t_1)}\right)^{n/2}\Xi^{G_1}(g^{-1}t_1g)\Xi^{G_1}(h^{-1}t_2h)\sigma_{G_1}(g^{-1}t_1g)^{-r}\sigma_{G_1}(h^{-1}t_2h)^{-r}\sigma_{G_1}(g)^{dk}\sigma_{G_1}(h)^{dk}\end{aligned}$$

for all $g,h\in G_1$ and $t_1,t_2\in T$. Fix henceforth $r>0$ and for all $Y\in\mathfrak{t}$, let $Y=Y_1+\ldots+Y_d$ be the decomposition of Y according to the identification 5.2.3 (so that $Y_i\in K_i$ for all $1\leqslant i\leqslant d$). Then,

$$\log\left(2+\nu(Y^{-1})\right)\ll\prod_{i=1}^d\max(1,v_{K_i}(Y_i))$$

for all $Y\in\mathfrak{t}^*$ and thus, by Hölder inequality,

$$\begin{aligned}&\int_{\mathfrak{g}_1} f\left(\begin{pmatrix}g^{-1} & \\ & h^{-1}\end{pmatrix}\begin{pmatrix}t_1 & \\ & t_2\end{pmatrix}\begin{pmatrix}1 & X\\ & 1\end{pmatrix}\begin{pmatrix}g & \\ & h\end{pmatrix}\right)\log\left(2+\nu(X_{\mathfrak{t}}^{-1})\right)^k dX\\ &\ll\left(\prod_{i=1}^d\int_{\mathfrak{g}_1} f\left(\begin{pmatrix}g^{-1} & \\ & h^{-1}\end{pmatrix}\begin{pmatrix}t_1 & \\ & t_2\end{pmatrix}\begin{pmatrix}1 & X\\ & 1\end{pmatrix}\begin{pmatrix}g & \\ & h\end{pmatrix}\right)\max(1,v_{K_i}(X_{\mathfrak{t},i}))^{dk}dX\right)^{1/d}.\end{aligned}$$

It follows that we just need to establish the following estimates for every $1\leqslant i\leqslant d$:

(5.3.1)

$$\begin{aligned}&\left(\frac{\nu(h)}{\nu(g)}\right)^n\int_{\mathfrak{g}_1} f\left(\begin{pmatrix}g^{-1} & \\ & h^{-1}\end{pmatrix}\begin{pmatrix}t_1 & \\ & t_2\end{pmatrix}\begin{pmatrix}1 & X\\ & 1\end{pmatrix}\begin{pmatrix}g & \\ & h\end{pmatrix}\right)\max(1,v_{K_i}(X_{\mathfrak{t},i}))^{dk}dX\\ &\ll\left(\frac{\nu(t_2)}{\nu(t_1)}\right)^{n/2}\Xi^{G_1}(g^{-1}t_1g)\Xi^{G_1}(h^{-1}t_2h)\sigma_{G_1}(g^{-1}t_1g)^{-r}\sigma_{G_1}(h^{-1}t_2h)^{-r}\sigma_{G_1}(g)^{dk}\sigma_{G_1}(h)^{dk}\end{aligned}$$

for all $g, h \in G_1$ and all $t_1, t_2 \in T$. Fix $1 \leqslant i \leqslant d$ and let $\mathfrak{t}_i$ be the subspace of $\mathfrak{t}$ corresponding to K_i via the identification 5.2.3. Denote by $\mathfrak{t}_i^\perp$ the orthogonal complement of $\mathfrak{t}_i$ in $\mathfrak{g}_1$ with respect to the bilinear pairing $\langle ., . \rangle$. We can write

$$\begin{aligned}
&\int_{\mathfrak{g}_1} f\left(\begin{pmatrix} g^{-1} & \\ & h^{-1} \end{pmatrix}\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}\begin{pmatrix} g & \\ & h \end{pmatrix}\right) \max(1, v_{K_i}(X_{\mathfrak{t},i}))^{dk} dX \\
&= \int_{\mathfrak{t}_i^\perp}\int_{\mathfrak{t}_i} f\left(\begin{pmatrix} g^{-1} & \\ & h^{-1} \end{pmatrix}\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}\begin{pmatrix} 1 & X+Y \\ & 1 \end{pmatrix}\begin{pmatrix} g & \\ & h \end{pmatrix}\right) \max(1, v_{K_i}(Y))^{dk} dY dX
\end{aligned}$$

for all $g, h \in G_1$ and all $t_1, t_2 \in T$. Let $J \subset G$ be an open subgroup by which f is biinvariant. Then, there exists $A > 1$ such that $\begin{pmatrix} 1 & g^{-1}Yh \\ & 1 \end{pmatrix} \in J$ for all $g, h \in G_1$ and all $Y \in \mathfrak{t}_i$ with $v_{K_i}(Y) \geqslant A\sigma_{G_1}(g)\sigma_{G_1}(h)$. For all $g, h \in G_1$, set

$$\mathfrak{t}_i[<, g, h] := \{Y \in \mathfrak{t}_i \mid v_{K_i}(Y) < A\sigma_{G_1}(g)\sigma_{G_1}(h)\},$$

$$\mathfrak{t}_i[\geqslant, g, h] := \{Y \in \mathfrak{t}_i \mid v_{K_i}(Y) \geqslant A\sigma_{G_1}(g)\sigma_{G_1}(h)\}.$$

Then by further decomposing the above integral, we have

$$\begin{aligned}
&\int_{\mathfrak{g}_1} f\left(\begin{pmatrix} g^{-1} & \\ & h^{-1} \end{pmatrix}\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}\begin{pmatrix} g & \\ & h \end{pmatrix}\right) \max(1, v_{K_i}(X_{\mathfrak{t},i}))^{dk} dX \\
&= \int_{\mathfrak{t}_i^\perp}\int_{\mathfrak{t}_i[<,g,h]} f\left(\begin{pmatrix} g^{-1} & \\ & h^{-1} \end{pmatrix}\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}\begin{pmatrix} 1 & X+Y \\ & 1 \end{pmatrix}\begin{pmatrix} g & \\ & h \end{pmatrix}\right) \max(1, v_{K_i}(Y))^{dk} dY dX \\
&+ \int_{\mathfrak{t}_i^\perp}\int_{\mathfrak{t}_i[\geqslant,g,h]} f\left(\begin{pmatrix} g^{-1} & \\ & h^{-1} \end{pmatrix}\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}\begin{pmatrix} g & \\ & h \end{pmatrix}\right) \max(1, v_{K_i}(Y))^{dk} dY dX
\end{aligned}$$

for all $g, h \in G_1$ and all $t_1, t_2 \in T$. By applying Lemma 5.2, we see that this last expression is essentially bounded by the product of $\sigma_{G_1}(g)^{dk}\sigma_{G_1}(h)^{dk}$ times

$$\begin{aligned}
&\int_{\mathfrak{t}_i^\perp}\int_{\mathfrak{t}_i[<,g,h]} f\left(\begin{pmatrix} g^{-1} & \\ & h^{-1} \end{pmatrix}\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}\begin{pmatrix} 1 & X+Y \\ & 1 \end{pmatrix}\begin{pmatrix} g & \\ & h \end{pmatrix}\right) dY dX \\
&+ \int_{\mathfrak{t}_i^\perp}\int_{\mathfrak{t}_i[\geqslant,g,h]} f\left(\begin{pmatrix} g^{-1} & \\ & h^{-1} \end{pmatrix}\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}\begin{pmatrix} g & \\ & h \end{pmatrix}\right) dY dX \\
&= \int_{\mathfrak{t}} f\left(\begin{pmatrix} g^{-1} & \\ & h^{-1} \end{pmatrix}\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}\begin{pmatrix} g & \\ & h \end{pmatrix}\right) dX
\end{aligned}$$

for all $g, h \in G_1$ and all $t_1, t_2 \in T$. Finally, it follows from [37, Proposition II.4.5] that

$$\left(\frac{\nu(h)}{\nu(g)}\right)^n \int_{\mathfrak{t}} f\left(\begin{pmatrix} g^{-1} & \\ & h^{-1}\end{pmatrix}\begin{pmatrix} t_1 & \\ & t_2\end{pmatrix}\begin{pmatrix} 1 & X \\ & 1\end{pmatrix}\begin{pmatrix} g & \\ & h\end{pmatrix}\right) dX$$
$$= \int_{\mathfrak{t}} f\left(\begin{pmatrix} g^{-1}t_1 g & \\ & h^{-1}t_2 h\end{pmatrix}\begin{pmatrix} 1 & X \\ & 1\end{pmatrix}\right) dX$$
$$\ll \left(\frac{\nu(t_2)}{\nu(t_1)}\right)^{n/2} \Xi^{G_1}(g^{-1}t_1 g)\Xi^{G_1}(h^{-1}t_2 h)\sigma_{G_1}(g^{-1}t_1 g)^{-r}\sigma_{G_1}(h^{-1}t_2 h)^{-r}$$

for all $g, h \in G_1$ and all $t_1, t_2 \in T$. This proves inequality 5.3.1 and ends the proof of (i).

(ii) We prove the first inequality, the proof of the second one being similar. Fix $f \in \mathcal{C}(G)$, $t_1, t_2 \in T \cap G_{1,\mathrm{reg}}$, $k > 0$ and $r > 0$. For all $C > 0$ we have

$$|I_k(f, t_1, t_2) - I_{k,\leqslant C}(f, t_1, t_2)| \leqslant I^1_{k,>C}(|f|, t_1, t_2) + I^2_{k,>C}(|f|, t_1, t_2) + I^3_{k,>C}(|f|, t_1, t_2) + I^4_{k,>C}(|f|, t_1, t_2)$$

where

$$I^1_{k,>C}(|f|, t_1, t_2) := \int_{T\times T\backslash G_1\times G_1}\int_{\mathfrak{g}_1} \left| f\left(\begin{pmatrix} g^{-1} & \\ & h^{-1}\end{pmatrix}\begin{pmatrix} t_1 & \\ & t_2\end{pmatrix}\begin{pmatrix} 1 & X \\ & 1\end{pmatrix}\begin{pmatrix} g & \\ & h\end{pmatrix}\right)\right|$$
$$\sigma_{T\backslash G_1}(g)^k \sigma_{T\backslash G_1}(h)^k \sigma_{\mathfrak{g}_1^*}(g^{-1}X_{\mathfrak{t}}h)^k \sigma_{\mathfrak{g}_1}(g^{-1}X_{\mathfrak{t}^\perp}h)^k$$
$$\mathbf{1}_{T\backslash G_1,>C}(g) dX \frac{dg}{\nu(g)^n}\nu(h)^n dh,$$

and $I^2_{k,>C}(|f|, t_1, t_2)$, $I^3_{k,>C}(|f|, t_1, t_2)$ and $I^4_{k,>C}(|f|, t_1, t_2)$ are defined similarly by replacing $\mathbf{1}_{T\backslash G_1,>C}(g)$ by $\mathbf{1}_{T\backslash G_1,>C}(h)$, $\mathbf{1}_{\mathfrak{g}_1^*,>C}(g^{-1}X_{\mathfrak{t}}h)$ and $\mathbf{1}_{\mathfrak{g}_1,>C}(g^{-1}X_{\mathfrak{t}^\perp}h)$ respectively. Then for all $1 \leqslant j \leqslant 4$ and all $C > 0$, we have

$$I^j_{k,>C}(|f|, t_1, t_2) \leqslant C^{-r} I_{k+r}(|f|, t_1, t_2).$$

By (i) $I_{k+r}(|f|, t_1, t_2)$ is absolutely convergent. This proves the claimed inequality and finishes the proof of the Proposition.

□

5.4 Computation of certain (G, M, θ)-orthogonal sets

Recall that we have fixed a basis of V compatible with the decomposition 5.2.2. In this basis, the maximal (θ, F)-split central torus $A_{M,\theta}$ of M is the subgroup of matrices of the form

$$\begin{pmatrix} \lambda_1 I_{k_1} & & & & & \\ & \ddots & & & & \\ & & \lambda_d I_{k_d} & & & \\ & & & \lambda_1^{-1} I_{k_1} & & \\ & & & & \ddots & \\ & & & & & \lambda_d^{-1} I_{k_d} \end{pmatrix}$$

where $\lambda_i \in F^\times$ for all $1 \leqslant i \leqslant d$ and we recall that $k_i := [K_i : F_i]$. We identify $\mathcal{A}_{M,\theta}^*$ with $\mathbf{R}^d$ through the choice of the basis $(-\chi_1, \ldots, -\chi_d)$ where for every $1 \leqslant i \leqslant d$, χ_i denotes the character of $A_{M,\theta}$ sending an element as written above to λ_i. From this, we get an identification of $\mathcal{A}_{M,\theta}$ with $\mathbf{R}^d$ such that for every $m = (m_1, \ldots, m_{2d}) \in M$ (decomposition according to the identification 5.2.4), we have

$$H_{M,\theta}(m) = \left(\frac{v_F(N_{M_{k_1}(\mathcal{D})/F} m_1) - v_F(N_{M_{k_1}(\mathcal{D})/F} m_{d+1})}{2rk_1}, \ldots, \frac{v_F(N_{M_{k_d}(\mathcal{D})/F} m_d) - v_F(N_{M_{k_d}(\mathcal{D})/F} m_{2d})}{2rk_d} \right)$$

where for every $1 \leqslant i \leqslant d$, $N_{M_{k_i}(\mathcal{D})/F} : M_i(\mathcal{D}) \to F$ denotes the reduced norm and r stands as before for the degree of $\mathcal{D}$ over F (Recall that the homomorphism H_M was defined, see Section 2.1, using the logarithm in base q explaining why there is no $\log(q)$ in the above formula). For all $X = (X_1, \ldots, X_d) \in \mathfrak{t}^*$, we will also set

$$V(X) := (v_{K_1}(X_1), \ldots, v_{K_d}(X_d))$$

where we recall that $v_{K_i} = \frac{1}{[K_i:F]} v_F \circ N_{K_i/F}$ for every $1 \leqslant i \leqslant d$. Note that we have

$$H_{M,\theta} \begin{pmatrix} X & 0 \\ 0 & X^{-1} \end{pmatrix} = V(X) \tag{5.4.1}$$

for all $X \in \mathfrak{t}^*$. We extend the map $H_{M,\theta} : M \to \mathcal{A}_{M,\theta}$ to the maps $H_{P,\theta} : G \to \mathcal{A}_{M,\theta}$ for all $P \in \mathcal{P}^\theta(M)$ by using the maximal compact subgroup K.

Set $W := (\mathbf{Z}/2\mathbf{Z})^d$. We identify W with a subgroup of G by sending the i-th element e_i of the canonical basis of W to the element of G which switch the two copies of $K_i \otimes_{F_i} \mathcal{D}$ in $V \oplus V$. By the assumption that $T^c \subset K_1$, we have $W \subset K$. Conjugation by W preserves M and commutes with θ. Hence it induces an action on the set $\mathcal{F}^\theta(M)$ of θ-split parabolic subgroup containing M and also on $\mathcal{A}_{M,\theta}$ that we shall simply denote by $(w, P) \mapsto wP$ and $(w, X) \mapsto wX$ respectively. We have

$$\mathcal{P}^\theta(M) = \bigsqcup_{w \in W} w\mathcal{P}^{\theta,Q}(M) \tag{5.4.2}$$

where we recall that $\mathcal{P}^\theta(M)$ is the set of θ-split parabolic subgroups with Levi component M and $\mathcal{P}^{\theta,Q}(M)$ is the subset of $P \in \mathcal{P}^\theta(M)$ such that $P \subset Q$.

Proof. It suffices to show that

$$\bigcup_{P\in\mathcal{P}^{\theta,Q}(M)} \overline{\mathcal{A}^+_{P,\theta}}$$

is a fundamental domain for the action of W on $\mathcal{A}_{M,\theta}$. With the identification above, we have

$$\bigcup_{P\in\mathcal{P}^{\theta,Q}(M)} \overline{\mathcal{A}^+_{P,\theta}} = \{X \in \mathcal{A}_{M,\theta};\ \langle \alpha, X\rangle \geqslant 0\ \forall \alpha \in R(A_{M,\theta}, U_Q)\} = \mathbf{R}^d_-.$$

Now the claim follows easily from the fact that the action of W on $\mathcal{A}_{M,\theta}$ is given by sign changes of the coordinates. □

Lemma 5.5. *There exist $c > 0$ and $N_0 \geqslant 1$ such that for all $X \in \mathfrak{t}^*$, all $Y \in \mathfrak{t}^\perp$, all $g, h \in G_1$ and all $\lambda \in F^\times$ satisfying*

- $v_F(\lambda) \geqslant N_0$;
- $\sigma_{G_1}(g) \leqslant cv_F(\lambda)$, $\sigma_{G_1}(h) \leqslant cv_F(\lambda)$, $\sigma_{\mathfrak{t}^*}(X) \leqslant cv_F(\lambda)$ *and* $\sigma_{\mathfrak{g}_1}(Y) \leqslant cv_F(\lambda)$,

we have

$$H_{wP,\theta}\left(\begin{pmatrix} 1 & \lambda^{-1}X+Y \\ & 1\end{pmatrix}\begin{pmatrix} g & \\ & h\end{pmatrix}\right) - \frac{1}{2}V(\lambda^{-1}X) = w(H_{P,\theta}\begin{pmatrix} g & \\ & h\end{pmatrix} - \frac{1}{2}V(\lambda^{-1}X))$$

for all $P \in \mathcal{P}^{\theta,Q}(M)$ and all $w \in W$.

Proof. Let $P \in \mathcal{P}^{\theta,Q}(M)$, $w \in W$ and fix $N_0 \geqslant 1$ and $c > 0$. Let X, Y, g, h, λ satisfy the conditions of the lemma. We will show that the claimed equality is true provided N_0 is sufficiently large and c sufficiently small. For simplicity, we will set $z = \lambda^{-1}$.

Since $W \subset K$, we have

$$H_{wP,\theta}\left(\begin{pmatrix} 1 & zX+Y \\ & 1\end{pmatrix}\begin{pmatrix} g & \\ & h\end{pmatrix}\right) = wH_{P,\theta}\left(w\begin{pmatrix} 1 & zX+Y \\ & 1\end{pmatrix}\begin{pmatrix} g & \\ & h\end{pmatrix}\right).$$

Up to reordering the K_i's, we may assume that $w = e_1 + \ldots + e_t$ for some $1 \leqslant t \leqslant d$. In this case we have

$$w = \begin{pmatrix} 0_k & 0 & I_k & 0 \\ 0 & I_{m-k} & 0 & 0_{m-k} \\ I_k & 0 & 0_k & 0 \\ 0 & 0_{m-k} & 0 & I_{m-k}\end{pmatrix}$$

where $k := k_1 + \ldots + k_t$ and $m = n/r$. Write $X = \begin{pmatrix} X_k & \\ & X_{m-k} \end{pmatrix}$ and $Y = \begin{pmatrix} Y_k & Z \\ W & Y_{m-k} \end{pmatrix}$ where $(X_k, Y_k) \in M_k(\mathcal{D})^2$, $(X_{m-k}, Y_{m-k}) \in M_{m-k}(\mathcal{D})^2$, $W \in M_{m-k,k}(\mathcal{D})$ and $Z \in M_{k,m-k}(\mathcal{D})$. Note that if N_0 is sufficiently large and c sufficiently small, the matrix $zX_k + Y_k = zX_k(I_k + \lambda X_k^{-1} Y_k)$ is invertible. Direct computation then gives

$$
\begin{aligned}
& w \begin{pmatrix} 1 & zX + Y \\ & 1 \end{pmatrix} = \\
& \begin{pmatrix} -(zX_k + Y_k)^{-1} & & & \\ & I_{m-k} & & \\ & & zX_k + Y_k & \\ & & & I_{m-k} \end{pmatrix} \begin{pmatrix} I_k & & -(zX_k + Y_k) & Z \\ & I_{m-k} & W & zX_{m-k} + Y_{m-k} \\ & & I_k & \\ & & & I_{m-k} \end{pmatrix} \\
& \times \begin{pmatrix} I_k & 0 & 0_k & 0 \\ -W(zX_k + Y_k)^{-1} & I_{m-k} & 0 & -W(zX_k + Y_k)^{-1} Z \\ (zX_k + Y_k)^{-1} & 0 & I_k & (zX_k + Y_k)^{-1} Z \\ 0 & 0_{m-k} & 0 & I_{m-k} \end{pmatrix}.
\end{aligned}
$$

For N_0 sufficiently large and c sufficiently small, the matrix

$$
\begin{pmatrix} 0_k & 0 & 0_k & 0 \\ -W(zX_k + Y_k)^{-1} & 0_{m-k} & 0 & -W(zX_k + Y_k)^{-1} Z \\ (zX_k + Y_k)^{-1} & 0 & 0_k & (zX_k + Y_k)^{-1} Z \\ 0 & 0_{m-k} & 0 & 0_{m-k} \end{pmatrix}
$$

is so small compared to g and h that

$$
\begin{pmatrix} g^{-1} & \\ & h^{-1} \end{pmatrix} \begin{pmatrix} I_k & 0 & 0_k & 0 \\ -W(zX_k + Y_k)^{-1} & I_{m-k} & 0 & -W(zX_k + Y_k)^{-1} Z \\ (zX_k + Y_k)^{-1} & 0 & I_k & (zX_k + Y_k)^{-1} Z \\ 0 & 0_{m-k} & 0 & I_{m-k} \end{pmatrix} \begin{pmatrix} g & \\ & h \end{pmatrix}
$$

belongs to K. Hence, for N_0 sufficiently large and c sufficiently small, we have

$$
H_{P,\theta}\left(w \begin{pmatrix} 1 & zX + Y \\ & 1 \end{pmatrix} \begin{pmatrix} g & \\ & h \end{pmatrix} \right) = H_{P,\theta} \begin{pmatrix} \begin{pmatrix} -(zX_k + Y_k)^{-1} & \\ & I_{m-k} \end{pmatrix} g & \\ & \begin{pmatrix} zX_k + Y_k & \\ & I_{m-k} \end{pmatrix} h \end{pmatrix}.
$$

Once again, for N_0 sufficiently large and c sufficiently small, the matrix $z^{-1} X_k^{-1} Y_k$ is so small compared to g and h that we have

$$
g^{-1} \begin{pmatrix} (I_k + z^{-1} X_k^{-1} Y_k)^{-1} & \\ & I_{m-k} \end{pmatrix} g \in K_1, \quad h^{-1} \begin{pmatrix} I_k + z^{-1} X_k^{-1} Y_k & \\ & I_{m-k} \end{pmatrix} h \in K_1.
$$

So finally we get that

$$H_{wP,\theta}\left(\begin{pmatrix}1 & zX+Y\\ & 1\end{pmatrix}\begin{pmatrix}g & \\ & h\end{pmatrix}\right)=wH_{M,\theta}\begin{pmatrix}-z^{-1}X_k^{-1} & & & \\ & I_{m-k} & & \\ & & zX_k & \\ & & & I_{m-k}\end{pmatrix}+wH_{P,\theta}\begin{pmatrix}g & \\ & h\end{pmatrix}$$

for N_0 sufficiently large and c sufficiently small. Now, from 5.4.1 we easily check that

$$wH_{M,\theta}\begin{pmatrix}-z^{-1}X_k^{-1} & & & \\ & I_{m-k} & & \\ & & zX_k & \\ & & & I_{m-k}\end{pmatrix}=\frac{V(zX)-wV(zX)}{2}$$

and this ends the proof of the lemma. □

5.5 Computation of certain singular θ-weighted orbital integrals

To all $g,h\in G_1$ and $X\in\mathfrak{t}^*\oplus\mathfrak{t}^\perp$ we associate a (G,M,θ)-orthogonal set $\mathcal{Z}(g,h,X)=(\mathcal{Z}(g,h,X)_P)_{P\in\mathcal{P}^\theta(M)}$ defined by

$$\mathcal{Z}(g,h,X)_{w\overline{P}}:=wH_{P,\theta}\begin{pmatrix}g & \\ & h\end{pmatrix}-\frac{1}{2}(V(X_\mathfrak{t})-wV(X_\mathfrak{t}))$$

for all $P\in\mathcal{P}^{\theta,Q}(M)$ and $w\in W$ where we recall that $X_\mathfrak{t}$ denotes the projection of X onto $\mathfrak{t}$. The fact that $\mathcal{Z}(g,h,X)$ is indeed a (G,M,θ)-orthogonal set easily follows from Lemma 5.5. We define

$$w_{M,\theta}(g,h,X):=v_{M,\theta}(\mathcal{Z}(g,h,X))$$

for all $g,h\in G_1$ and $X\in\mathfrak{t}^*\oplus\mathfrak{t}^\perp$. Note that $w_{M,\theta}(g,h,X)$ is NOT always the volume of the convex hull of the set $\mathcal{Z}(g,h,X)$ (this is because this (G,M,θ)-orthogonal set is not always positive). However, there exists $k>0$ such that

$$|w_{M,\theta}(g,h,X)|\ll\sigma_{T\backslash G_1}(g)^k\sigma_{T\backslash G_1}(h)^k\sigma_{\mathfrak{g}_1^*}(g^{-1}X_\mathfrak{t}h)^k \tag{5.5.1}$$

for all $g,h\in G_1$ and $X\in\mathfrak{t}^*\oplus\mathfrak{t}^\perp$. Indeed, by 2.3.3, there exists $k>0$ such that

$$|w_{M,\theta}(g,h,X)|\ll\sigma_{G_1}(g)^k\sigma_{G_1}(h)^k\sigma_{\mathfrak{t}^*}(X_\mathfrak{t})^k\ll\sigma_{G_1}(g)^k\sigma_{G_1}(h)^k\sigma_{\mathfrak{g}_1^*}(g^{-1}X_\mathfrak{t}h)^k$$

for all $(g,h,X)\in G_1^2\times(\mathfrak{t}^*\oplus\mathfrak{t}^\perp)$. We easily check that $w_{M,\theta}(t_1g,t_2h,t_1Xt_2^{-1})=w_{M,\theta}(g,h,X)$ for all $t_1,t_2\in T$ and thus we get

$$|w_{M,\theta}(g,h,X)|\ll\sigma_{G_1}(t_1g)^k\sigma_{G_1}(t_2h)^k\sigma_{\mathfrak{g}_1^*}(g^{-1}X_\mathfrak{t}h)^k$$

for all $g, h \in G_1$, $X \in \mathfrak{t}^* \oplus \mathfrak{t}^\perp$ and $t_1, t_2 \in T$. Taking the infimum over t_1 and t_2 of the right hand side yields the desired inequality by 5.0.1.

Recall that in §2.4, we have defined a θ-weighted orbital integral $f \in \mathcal{C}(G) \mapsto \Phi_{M,\theta}(x, f)$ for all $x \in M \cap G_{\mathrm{reg}}$ (where we again use the maximal compact subgroup K to define the maps $H_{P,\theta} : G \to \mathcal{A}_{M,\theta}$ for $P \in \mathcal{P}^\theta(M)$). For all $t \in T \cap G_{1,\mathrm{reg}}$, we define a θ-weighted orbital integral at the non-regular point $\begin{pmatrix} t & \\ & t \end{pmatrix}$ by

$$\Phi_{M,\theta}\left(\begin{pmatrix} t & \\ & t \end{pmatrix}, f\right) := D^{G_1}(t)^{-1} \int_{T\times T\backslash G_1 \times G_1} \int_{\mathfrak{g}_1} f^K\left(\begin{pmatrix} g^{-1} & \\ & h^{-1} \end{pmatrix}\begin{pmatrix} t & \\ & t \end{pmatrix}\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}\begin{pmatrix} g & \\ & h \end{pmatrix}\right) w_{M,\theta}(g, h, X) dX \frac{dg}{\nu(g)^n} \nu(h)^n dh$$

for all $f \in \mathcal{C}(G)$. Note that this integral is absolutely convergent by 5.5.1 and Lemma 5.4(i).

Proposition 5.6. *For all θ-strongly cuspidal function $f \in \mathcal{C}(G)$ and all $t \in T \cap G_{1,\mathrm{reg}}$, we have*

$$\lim_{\lambda \to 1} D^G\begin{pmatrix} t & \\ & \lambda t \end{pmatrix}^{1/2} \Phi_{M,\theta}\left(\begin{pmatrix} t & \\ & \lambda t \end{pmatrix}, f\right) = D^{G_1}(t)^2 \Phi_{M,\theta}\left(\begin{pmatrix} t & \\ & t \end{pmatrix}, f\right).$$

(In particular the limit exists).

Proof. Recall that by definition

$$\Phi_{M,\theta}\left(\begin{pmatrix} t & \\ & \lambda t \end{pmatrix}, f\right) = \int_{T\times T\backslash G} f\left(g^{-1}\begin{pmatrix} t & \\ & \lambda t \end{pmatrix} g\right) v_{M,\theta}(\mathcal{Y}(g)) dg$$

where for $g \in G$, $\mathcal{Y}(g)$ is the (G, M, θ)-orthogonal set defined by

$$\mathcal{Y}(g)_P = H_{\overline{P},\theta}(g), \quad P \in \mathcal{P}^\theta(M).$$

For all $z \in F^\times$, we define a (G, M, θ)-orthogonal set $\mathcal{X}(z) = (\mathcal{X}(z)_P)_{P\in\mathcal{P}^\theta(M)}$ by

$$\mathcal{X}(z)_{w\overline{P}} := \frac{1}{2}(V(z) - wV(z)), \text{ for all } P \in \mathcal{P}^{Q,\theta}(M) \text{ and all } w \in W.$$

Then for all $\lambda \in F^\times\backslash\{1\}$, we have

$$\Phi_{M,\theta}\left(\begin{pmatrix} t & \\ & \lambda t \end{pmatrix}, f\right) = \int_{T\times T\backslash G} f\left(g^{-1}\begin{pmatrix} t & \\ & \lambda t \end{pmatrix} g\right) v_{M,\theta}(\mathcal{Y}(g) + \mathcal{X}(1-\lambda)) dg.$$

Indeed, this follows from 2.3.5 and the equality

$$\Phi^Q_{M,\theta}\left(\begin{pmatrix} t & \\ & \lambda t \end{pmatrix}, f\right) = 0$$

for all $Q \in \mathcal{F}^\theta(M)$ with $Q \neq G$, which is a consequence of the fact that f is θ-strongly cuspidal.

Now by the Iwasawa decomposition $G = (G_1 \times G_1)NK$, we can write

$$\Phi_{M,\theta}\left(\begin{pmatrix} t & \\ & \lambda t \end{pmatrix}, f\right) = \int_{T\times T\backslash G_1\times G_1}\int_{\mathfrak{g}_1} f^K\left(\begin{pmatrix} g^{-1} & \\ & h^{-1}\end{pmatrix}\begin{pmatrix} 1 & -X \\ & 1\end{pmatrix}\begin{pmatrix} t & \\ & \lambda t\end{pmatrix}\begin{pmatrix} 1 & X \\ & 1\end{pmatrix}\begin{pmatrix} g & \\ & h\end{pmatrix}\right)$$
$$v_{M,\theta}\left(\mathcal{Y}\left(\begin{pmatrix} 1 & X \\ & 1\end{pmatrix}\begin{pmatrix} g & \\ & h\end{pmatrix}\right) + \mathcal{X}(1-\lambda)\right) dX \frac{dg}{\nu(g)^n}\nu(h)^n dh.$$

After the variable change $X \mapsto (1 - \lambda \mathrm{Ad}(t^{-1}))^{-1}X$, for λ sufficiently close to 1, we get

$$D^G\begin{pmatrix} t & \\ & \lambda t\end{pmatrix}^{1/2}\Phi_{M,\theta}\left(\begin{pmatrix} t & \\ & \lambda t\end{pmatrix}, f\right) =$$
$$D^{G_1}(t)\int_{T\times T\backslash G_1\times G_1}\int_{\mathfrak{g}_1} f^K\left(\begin{pmatrix} g^{-1} & \\ & h^{-1}\end{pmatrix}\begin{pmatrix} t & \\ & \lambda t\end{pmatrix}\begin{pmatrix} 1 & X \\ & 1\end{pmatrix}\begin{pmatrix} g & \\ & h\end{pmatrix}\right)\widetilde{w}(g,h,X,\lambda)dX\frac{dg}{\nu(g)^n}\nu(h)^n dh$$

where we have set

$$\widetilde{w}(g,h,X,\lambda) := v_{M,\theta}\left(\mathcal{Y}\left(\begin{pmatrix} 1 & (1-\lambda\mathrm{Ad}(t^{-1}))^{-1}X \\ & 1\end{pmatrix}\begin{pmatrix} g & \\ & h\end{pmatrix}\right) + \mathcal{X}(1-\lambda)\right).$$

By 2.3.3, there exists $k > 0$ such that

$$|\widetilde{w}(g,h,X,\lambda)| \ll \sigma_{T\backslash G_1}(g)^k\sigma_{T\backslash G_1}(h)^k\sigma_{\mathfrak{g}_1}(g^{-1}Xh)^k(v_F(1-\lambda))^k \tag{5.5.2}$$

for all $(g,h,X,\lambda) \in G_1^2 \times (\mathfrak{t}^* \oplus \mathfrak{t}^\perp) \times F^\times$ with λ sufficiently close to 1.

For all $C > 0$, set

$$\Phi_{M,\theta,\leqslant C}\left(\begin{pmatrix} t & \\ & \lambda t\end{pmatrix}, f\right) := D^G\begin{pmatrix} t & \\ & \lambda t\end{pmatrix}^{-1/2} D^{G_1}(t)\times$$
$$\int_{T\times T\backslash G_1\times G_1}\int_{\mathfrak{g}_1} f^K\left(\begin{pmatrix} g^{-1} & \\ & h^{-1}\end{pmatrix}\begin{pmatrix} t & \\ & \lambda t\end{pmatrix}\begin{pmatrix} 1 & X \\ & 1\end{pmatrix}\begin{pmatrix} g & \\ & h\end{pmatrix}\right)\widetilde{w}(g,h,X,\lambda)$$
$$\mathbf{1}_{T\backslash G_1,\leqslant C}(g)\mathbf{1}_{T\backslash G_1,\leqslant C}(h)\mathbf{1}_{\mathfrak{g}_1^*,\leqslant C}(g^{-1}X_{\mathfrak{t}}h)\mathbf{1}_{\mathfrak{g}_1,\leqslant C}(g^{-1}X_{\mathfrak{t}^\perp}h)dX\frac{dg}{\nu(g)^n}\nu(h)^n dh$$

and

$$\Phi_{M,\theta,\leqslant C}\left(\begin{pmatrix} t & \\ & t\end{pmatrix}, f\right) := D^{G_1}(t)^{-1}\int_{T\times T\backslash G_1\times G_1}\int_{\mathfrak{g}_1} f^K\left(\begin{pmatrix} g^{-1} & \\ & h^{-1}\end{pmatrix}\begin{pmatrix} t & \\ & t\end{pmatrix}\begin{pmatrix} 1 & X \\ & 1\end{pmatrix}\begin{pmatrix} g & \\ & h\end{pmatrix}\right)$$
$$w_{M,\theta}(g,h,X)\,\mathbf{1}_{T\backslash G_1,\leqslant C}(g)\mathbf{1}_{T\backslash G_1,\leqslant C}(h)\mathbf{1}_{\mathfrak{g}_1^*,\leqslant C}(g^{-1}X_{\mathfrak{t}}h)\mathbf{1}_{\mathfrak{g}_1,\leqslant C}(g^{-1}X_{\mathfrak{t}^\perp}h)dX\frac{dg}{\nu(g)^n}\nu(h)^n dh.$$

Fix $0 < \epsilon < 1$. By 5.5.1, 5.5.2 and Proposition 5.4(ii), we have (Note that for λ sufficiently close to 1, f^K is invariant by $\begin{pmatrix} 1 & \\ & \lambda \end{pmatrix}$ on the left):

$$\lim_{\lambda \to 1} D^G \begin{pmatrix} t & \\ & \lambda t \end{pmatrix}^{1/2} \left(\Phi_{M,\theta} \left(\begin{pmatrix} t & \\ & \lambda t \end{pmatrix}, f \right) - \Phi_{M,\theta,\leqslant v_F(\lambda-1)^\epsilon} \left(\begin{pmatrix} t & \\ & \lambda t \end{pmatrix}, f \right) \right) = 0, \tag{5.5.3}$$

$$\lim_{\lambda \to 1} \Phi_{M,\theta} \left(\begin{pmatrix} t & \\ & t \end{pmatrix}, f \right) - \Phi_{M,\theta,\leqslant v_F(\lambda-1)^\epsilon} \left(\begin{pmatrix} t & \\ & t \end{pmatrix}, f \right) = 0. \tag{5.5.4}$$

Moreover, it follows from Lemma 5.5 that for all λ sufficiently close to 1 and all $g, h \in G_1$, $X \in \mathfrak{g}_1$ with $\sigma_{T\backslash G_1}(g) \leqslant v_F(\lambda-1)^\epsilon$, $\sigma_{T\backslash G_1}(h) \leqslant v(\lambda-1)^\epsilon$, $\sigma_{\mathfrak{g}_1^*}(g^{-1}X_{\mathfrak{t}}h) \leqslant v_F(\lambda-1)^\epsilon$ and $\sigma_{\mathfrak{g}_1}(g^{-1}X_{\mathfrak{t}^\perp}h) \leqslant v_F(\lambda-1)^\epsilon$, we have

$$\mathcal{Y}\left(\begin{pmatrix} 1 & (1-\lambda \mathrm{Ad}(t^{-1}))^{-1}X \\ & 1 \end{pmatrix} \begin{pmatrix} g & \\ & h \end{pmatrix} \right) + \mathcal{X}(1-\lambda) = \mathcal{Z}(g,h,X),$$

and so

$$\widetilde{w}(g,h,X,\lambda) = w_{M,\theta}(g,h,X).$$

Since f^K is left invariant by $\begin{pmatrix} \lambda & \\ & 1 \end{pmatrix}$ for λ sufficiently close to 1, it follows that

$$D^G \begin{pmatrix} t & \\ & \lambda t \end{pmatrix}^{1/2} \Phi_{M,\theta,\leqslant v(\lambda-1)^\epsilon} \left(\begin{pmatrix} t & \\ & \lambda t \end{pmatrix}, f \right) = D^{G_1}(t)^2 \Phi_{M,\theta,\leqslant v(\lambda-1)^\epsilon} \left(\begin{pmatrix} t & \\ & t \end{pmatrix}, f \right)$$

for λ sufficiently close to 1 and hence, by 5.5.3 and 5.5.4,

$$\lim_{\lambda \to 1} D^G \begin{pmatrix} t & \\ & \lambda t \end{pmatrix}^{1/2} \Phi_{M,\theta} \left(\begin{pmatrix} t & \\ & \lambda t \end{pmatrix}, f \right) = D^{G_1}(t)^2 \Phi_{M,\theta} \left(\begin{pmatrix} t & \\ & t \end{pmatrix}, f \right).$$

□

5.6 Change of weight

Choose a minimal θ-split Levi subgroup M_0 contained in M and fix $P_0 \in \mathcal{P}^\theta(M_0)$. Using these data we can associate to any point $Y \in \mathcal{A}_{M_0,\theta}$ a (G,M,θ)-orthogonal set $(Y_P)_{P \in \mathcal{P}^\theta(M)}$ as in §2.3. For every $Y \in \mathcal{A}_{M_0,\theta}$, every $g, h \in G_1$ and every $X \in \mathfrak{t}^* \oplus \mathfrak{t}^\perp$, we define a (G,M,θ)-orthogonal set $\mathcal{Z}(g,h,X,Y) := (\mathcal{Z}(g,h,X,Y)_P)_{P \in \mathcal{P}^\theta(M)}$ by setting

$$\mathcal{Z}(g,h,X,Y)_P := Y_P - \mathcal{Z}(g,h,X)_P$$

for all $P \in \mathcal{P}^\theta(M)$ where $\mathcal{Z}(g,h,X)_P$ was defined in Section 5.5. To such a (G,M,θ)-orthogonal set is associated a function $\Gamma^G_{M,\theta}(.,\mathcal{Z}(g,h,X,Y))$ on $\mathcal{A}_{M,\theta}$ (see §2.3). If the

(G, M, θ)-orthogonal set $\mathcal{Z}(g, h, X, Y)$ is positive, this is just the characteristic function of its convex hull.

For all $Y \in \mathcal{A}_{M_0,\theta}$, we define a new weight $\widetilde{w}_{M,\theta}(.,.,.,Y)$ on $G_1 \times G_1 \times \mathfrak{g}_1$ by setting

$$\widetilde{w}_{M,\theta}(g, h, X, Y) := \int_T \Gamma^G_{M,\theta}\left(H_{M,\theta}(a), \mathcal{Z}(g, h, X, Y)\right) da$$

where for simplicity we have written $H_{M,\theta}(a)$ for $H_{M,\theta}\begin{pmatrix} a & \\ & 1 \end{pmatrix}$. A proof similar to that of 5.5.1 shows that for some $k > 0$, we have

$$|\widetilde{w}_{M,\theta}(g, h, X, Y)| \ll \sigma_{T\backslash G_1}(g)^k \sigma_{T\backslash G_1}(h)^k \sigma_{\mathfrak{g}_1^*}(g^{-1} X_{\mathfrak{t}} h)^k \left(1 + |Y|\right)^k \tag{5.6.1}$$

and even

$$\int_T |\Gamma^G_{M,\theta}\left(H_{M,\theta}(a), \mathcal{Z}(g, h, X, Y)\right)| da \ll \sigma_{T\backslash G_1}(g)^k \sigma_{T\backslash G_1}(h)^k \sigma_{\mathfrak{g}_1^*}(g^{-1} X_{\mathfrak{t}} h)^k \left(1 + |Y|\right)^k \tag{5.6.2}$$

for all $g, h \in G_1$, $X \in \mathfrak{t}^* \oplus \mathfrak{t}^\perp$ and all $Y \in \mathcal{A}_{M_0,\theta}$, where $|.|$ denotes a norm on $\mathcal{A}_{M_0,\theta}$. Using this weight we define a new expression

$$J_{Y,T}(f) := \int_T D^{G_1}(t) \int_{T\times T\backslash G_1 \times G_1} \int_{\mathfrak{g}_1} f^K\left(\begin{pmatrix} g^{-1} & \\ & h^{-1} \end{pmatrix}\begin{pmatrix} t & \\ & t \end{pmatrix}\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}\begin{pmatrix} g & \\ & h \end{pmatrix}\right)$$
$$\widetilde{w}_{M,\theta}(g, h, X, Y) dX \frac{dg}{\nu(g)^n} \nu(h)^n dh \omega(t)^{-1} dt$$

which is absolutely convergent by 5.6.1 and Proposition 5.4(ii).

Let A_0 be the maximal central (θ, F)-split subtorus of M_0 and let Δ_0 be the set of simple roots of A_0 in P_0. The goal of this section is to show the following:

Proposition 5.7. *Let $0 < \epsilon_1 < \epsilon_2 < 1$ and assume that $f \in \mathcal{C}(G)$ is θ-strongly cuspidal. Then for all $r > 0$, we have*

$$\left| J_{N,T}(f) - \frac{1}{2^d} J_{Y,T}(f) \right| \ll N^{-r}$$

for all $N \geqslant 1$ and all $Y \in \mathcal{A}^+_{P_0,\theta}$ satisfying the following two inequalities

$$N^{\epsilon_1} \leqslant \inf_{\alpha \in \Delta_0} \alpha(Y), \tag{5.6.3}$$

$$\sup_{\alpha \in \Delta_0} \alpha(Y) \leqslant N^{\epsilon_2}. \tag{5.6.4}$$

Proof. For all $C > 0$, all $N \geqslant 1$ and all $Y \in \mathcal{A}^+_{P_0,\theta}$, we set

$$J_{N,T,\leqslant C}(f) := \int_T D^{G_1}(t) \int_{T\times T\backslash G_1\times G_1} \int_{\mathfrak{g}_1} f^K\left(\begin{pmatrix} g^{-1} & \\ & h^{-1}\end{pmatrix}\begin{pmatrix} t & \\ & t\end{pmatrix}\begin{pmatrix} 1 & X \\ & 1\end{pmatrix}\begin{pmatrix} g & \\ & h\end{pmatrix}\right)$$
$$\kappa_{N,T,\xi}(g,h,X)\mathbf{1}_{T\backslash G_1,\leqslant C}(g)\mathbf{1}_{T\backslash G_1,\leqslant C}(h)\mathbf{1}_{\mathfrak{g}_1^*,\leqslant C}(g^{-1}X_{\mathfrak{t}}h)dX\frac{dg}{\nu(g)^n}\nu(h)^n dh\omega(t)^{-1}dt$$

and

$$J_{Y,T,\leqslant C}(f) := \int_T D^{G_1}(t) \int_{T\times T\backslash G_1\times G_1} \int_{\mathfrak{g}_1} f^K\left(\begin{pmatrix} g^{-1} & \\ & h^{-1}\end{pmatrix}\begin{pmatrix} t & \\ & t\end{pmatrix}\begin{pmatrix} 1 & X \\ & 1\end{pmatrix}\begin{pmatrix} g & \\ & h\end{pmatrix}\right)$$
$$\widetilde{w}_{M,\theta}(g,h,X,Y)\mathbf{1}_{T\backslash G_1,\leqslant C}(g)\mathbf{1}_{T\backslash G_1,\leqslant C}(h)\mathbf{1}_{\mathfrak{g}_1^*,\leqslant C}(g^{-1}X_{\mathfrak{t}}h)dX\frac{dg}{\nu(g)^n}\nu(h)^n dh\omega(t)^{-1}dt.$$

We fix henceforth an $\epsilon > 0$ satisfying $\epsilon < \epsilon_1$. It follows from 5.6.1, 5.1.2 and Proposition 5.4(ii) that for all $r > 0$, we have

$$|J_{N,T}(f) - J_{N,T,\leqslant N^\epsilon}(f)| \ll N^{-r}$$

and

$$|J_{Y,T}(f) - J_{Y,T,\leqslant N^\epsilon}(f)| \ll N^{-r}$$

for all $N \geqslant 1$ and all $Y \in \mathcal{A}^+_{P_0,\theta}$ satisfying inequality 5.6.4. Thus it suffices to establish that for all $r > 0$, we have

$$\left|J_{N,T,\leqslant N^\epsilon}(f) - \frac{1}{2^d}J_{Y,T,\leqslant N^\epsilon}(f)\right| \ll N^{-r} \tag{5.6.5}$$

for all $N \geqslant 1$ and all $Y \in \mathcal{A}^+_{P_0,\theta}$ satisfying inequalities 5.6.3 and 5.6.4.

For all $g, h \in G_1$, $X \in \mathfrak{t}^* \oplus \mathfrak{t}^\perp$ and $Y \in \mathcal{A}_{M_0,\theta}$, we have the identity (see 2.3.1)

$$\sum_{R\in\mathcal{F}^\theta(M)} \Gamma^R_{M,\theta}(\Lambda, \mathcal{Z}(g,h,X,Y))\tau^G_{R,\theta}(\Lambda - \mathcal{Z}(g,h,X,Y)_R) = 1 \tag{5.6.6}$$

for all $\Lambda \in \mathcal{A}_{M,\theta}$. We are going to modify this decomposition slightly by taking its convolution with a function $\varphi_N \in C_c^\infty(T)$ of small support (relative to N^{ϵ_1}) and satisfying the condition $\int_T \varphi_N = 1$. The function we need is provided by the next lemma. Before we state the lemma, we need to introduce some more notations. Let $\varphi \mapsto \widehat{\varphi}$ be the usual Fourier transform on $C_c^\infty(\mathfrak{t})$ defined by

$$\widehat{\varphi}(X) = \int_{\mathfrak{t}} \varphi(Y)\xi(XY)^{-1}dY, \quad X \in \mathfrak{t}.$$

For $\varphi \in C_c^\infty(T)$, we also set

$$\varphi^\sharp(X) := \int_T \varphi(a)\xi(aX)^{-1}da, \quad X \in \mathfrak{t}.$$

Note that $\varphi^\sharp = c_0(\nu^{-1}\varphi)\widehat{\ }$ where c_0 is the constant such that $da = c_0\frac{dY}{\nu(Y)}$.
We define a function $L \in C_c^\infty(\mathfrak{t})$ by

$$L(X) := L_{K_1}(X_1)\dots L_{K_d}(X_d), \quad X \in \mathfrak{t}$$

where $X = X_1 + \ldots + X_d$ is the decomposition of X according to the identification 5.2.3, and for all $1 \leqslant i \leqslant d$, the function $L_{K_i}(\cdot)$ is defined to be

$$L_{K_i}(X_i) := \begin{cases} 1, & \text{if } v_{K_i}(X_i) > 0; \\ \frac{1}{2}, & \text{if } v_{K_i}(X_i) = 0; \\ 0, & \text{otherwise.} \end{cases}$$

This function factorizes through the map $X \in \mathfrak{t} \mapsto V(X) \in \mathcal{A}_{M,\theta}$ introduced in §5.4 and we set $L(V(X)) := L(X)$ for all $X \in \mathfrak{t}$. A crucial property of the function L is that it satisfies the identity

$$\sum_{w \in W} L(wX) = 1 \tag{5.6.7}$$

for all $X \in \mathfrak{t}$, where W is the subgroup of the normalizer of M that was introduced in §5.4.

Lemma 5.8. *There exists a sequence of nonnegative functions $(\varphi_N)_{N \geqslant N_0}$ in $C_c^\infty(T/T^c)$ with the following properties*

- *$Supp(\varphi_N) \subseteq \{a \in T;\ \sigma_T(a) \leqslant N^\epsilon\}$ for all $N \geqslant N_0$;*
- *$\int_T \varphi_N = 1$ for all $N \geqslant N_0$;*
- *There exists $c > 0$ such that $\left|\varphi_N^\sharp(X) - L(X)\right| \leqslant e^{-cN^\epsilon}$ for all $X \in \mathfrak{t}$ and all $N \geqslant N_0$.*

Proof. Setting $\varphi'_N = c_0\nu^{-1}\varphi_N$, we are looking for a sequence of nonnegative functions $(\varphi'_N)_{N \geqslant N_0}$ in $C_c^\infty(T/T^c)$ such that

- $Supp(\varphi'_N) \subseteq \{a \in T;\ \sigma_T(a) \leqslant N^\epsilon\}$ for all $N \geqslant N_0$;
- $\widehat{\varphi'_N}(0) = L(0) = 1$ for all $N \geqslant N_0$;
- There exists $c > 0$ such that $\left|\widehat{\varphi'_N}(X) - L(X)\right| \leqslant e^{-cN^\epsilon}$ for all $X \in \mathfrak{t}$ and all $N \geqslant N_0$.

Let $\varphi \in C_c^\infty(\mathfrak{t}) \mapsto \check{\varphi} \in C_c^\infty(\mathfrak{t})$ be the inverse Fourier transform and for $C > 0$ denote by $\mathbf{1}_{\leqslant C}$ the characteristic function of $\mathfrak{t}[\leqslant C] := \{a \in T;\ \sigma_T(a) \leqslant C\}$. Notice that $\check{L} \geqslant 0$ as L is a positive linear combination of characteristic functions of lattices and set $\varphi_N'' = \mathbf{1}_{\leqslant N^\epsilon}\check{L}$ for every $N \geqslant 1$. Then,

$$\sup_{X\in\mathfrak{t}} \left|\widehat{\varphi_N''}(X) - L(X)\right| \leqslant \int_{\mathfrak{t}} (1 - \mathbf{1}_{\leqslant N^\epsilon}(X))\check{L}(X)dX \leqslant \mathrm{vol}\left(\mathfrak{t}[> N^\epsilon] \cap \mathrm{supp}(\check{L})\right) \sup_{X\in\mathfrak{t}} \check{L}(X)$$

for every $N \geqslant 1$ where we have set $\mathfrak{t}[> C] = \mathfrak{t} \setminus \mathfrak{t}[\leqslant C]$ for $C > 0$. Moreover, there exists $c' > 0$ such that $\mathrm{vol}(\mathfrak{t}[> C] \cap \mathrm{supp}(\check{L})) \ll e^{-c'C}$ for all $C > 0$. Then, we easily check that the sequence $\varphi_N' = \widehat{\varphi_N''}(0)^{-1}\varphi_N''$ (which is well-defined for $N \gg 1$) satisfies all the required conditions for any $c < c'$ provided N_0 is large enough. □

We choose a sequence of functions $(\varphi_N)_{N\geqslant N_0}$ as in the above lemma. For all $g, h \in G_1$, all $X \in \mathfrak{t}^* \oplus \mathfrak{t}^\perp$, all $Y \in \mathcal{A}_{M_0,\theta}$ and all $R \in \mathcal{F}^\theta(M)$, we set

$$\Delta_R(\Lambda, g, h, X, Y) := \Gamma^R_{M,\theta}(\Lambda, \mathcal{Z}(g, h, X, Y))\tau^G_{R,\theta}(\Lambda - \mathcal{Z}(g, h, X, Y)_R),\ \Lambda \in \mathcal{A}_{M,\theta}.$$

And for all $\varphi \in C_c^\infty(T)$, we define the convolution $\varphi * \Delta_R(., g, h, X, Y)$ as usual by

$$\varphi * \Delta_R(\Lambda, g, h, X, Y) := \int_T \varphi(a)\Delta_R(\Lambda - H_{M,\theta}(a), g, h, X, Y)da,\ \Lambda \in \mathcal{A}_{M,\theta}.$$

By 5.6.6 and the second property satisfied by the sequence $(\varphi_N)_{N\geqslant N_0}$, we have

$$\sum_{R\in\mathcal{F}^\theta(M)} \varphi_N * \Delta_R(\Lambda, g, h, X, Y) = 1 \tag{5.6.8}$$

for all $N \geqslant N_0$, all $\Lambda \in \mathcal{A}_{M,\theta}$ and all $(g, h, X, Y) \in G_1^2 \times (\mathfrak{t}^* \oplus \mathfrak{t}^\perp) \times \mathcal{A}^+_{P_0,\theta}$.

Set

$$J^{R,Y}_{N,T,\leqslant C}(f) := \int_T D^{G_1}(t) \int_{T\times T\backslash G_1\times G_1} \int_{\mathfrak{g}_1} f^K\left(\begin{pmatrix} g^{-1} & \\ & h^{-1}\end{pmatrix}\begin{pmatrix} t & \\ & t\end{pmatrix}\begin{pmatrix} 1 & X \\ & 1\end{pmatrix}\begin{pmatrix} g & \\ & h\end{pmatrix}\right)$$
$$\kappa^{R,Y}_{N,T,\xi}(g, h, X)\mathbf{1}_{T\backslash G_1,\leqslant C}(g)\mathbf{1}_{T\backslash G_1,\leqslant C}(h)\mathbf{1}_{\mathfrak{g}_1^*,\leqslant C}(g^{-1}X_{\mathfrak{t}}h)dX\frac{dg}{\nu(g)^n}\nu(h)^n dh\omega(t)^{-1}dt$$

for all $N \geqslant N_0$, all $Y \in \mathcal{A}_{M_0,\theta}$, all $R \in \mathcal{F}^\theta(M)$ and all $C > 0$, where

$$\kappa^{R,Y}_{N,T,\xi}(g, h, X) := \int_T \xi(aX)^{-1}\kappa_N(h^{-1}ag)\varphi_N * \Delta_R(H_{M,\theta}(a), g, h, X, Y)da$$

for all $g, h \in G_1$ and $X \in \mathfrak{t}^* \oplus \mathfrak{t}^\perp$. Then, by 5.6.8 we have

$$J_{N,T,\leqslant C}(f) = \sum_{R\in\mathcal{F}^\theta(M)} J^{R,Y}_{N,T,\leqslant C}(f)$$

for all $C > 0$, all $N \geqslant N_0$ and all $Y \in \mathcal{A}_{M_0,\theta}$. Thus in order to prove 5.6.5, it suffices to establish the following two facts

(5.6.9) There exists $c > 0$ such that

$$\left| J^{G,Y}_{N,T,\leqslant N^\epsilon}(f) - \frac{1}{2^d} J_{Y,T,\leqslant N^\epsilon}(f) \right| \ll e^{-cN^\epsilon}$$

for all $N \geqslant N_0$ and all $Y \in \mathcal{A}^+_{P_0,\theta}$ satisfying inequality 5.6.4.

(5.6.10) For all $R \in \mathcal{F}^\theta(M)$ with $R \neq G$ and all $r > 0$, we have

$$\left| J^{R,Y}_{N,T,\leqslant N^\epsilon}(f) \right| \ll N^{-r}$$

for all $N \geqslant N_0$ and all $Y \in \mathcal{A}^+_{P_0,\theta}$ satisfying inequality 5.6.3.

From now on and until the end of the proof, when $N \geqslant 1$ is fixed, we will say that $(g, h, X) \in G_1^2 \times \mathfrak{g}_1$ is *in the good range* if $X \in \mathfrak{t}^* \oplus \mathfrak{t}^\perp$ and we have the inequalities $\sigma_{T\backslash G_1}(g) \leqslant N^\epsilon$, $\sigma_{T\backslash G_1}(h) \leqslant N^\epsilon$ and $\sigma_{\mathfrak{g}_1^*}(g^{-1}X_{\mathfrak{t}}h) \leqslant N^\epsilon$.

Proof of 5.6.9 : By Proposition 5.4(i), it suffices to show the existence of $c > 0$ and $k > 0$ such that

(5.6.11)
$$\left| \kappa^{G,Y}_{N,T,\xi}(g, h, X) - \frac{1}{2^d} \widetilde{w}_{M,\theta}(g, h, X, Y) \right| \ll e^{-cN^\epsilon} \sigma_{T\backslash G_1}(g)^k \sigma_{T\backslash G_1}(h)^k \sigma_{\mathfrak{g}_1^*}(g^{-1}X_{\mathfrak{t}}h)^k (1 + |Y|)^k$$

for all $N \geqslant N_0$, all $Y \in \mathcal{A}^+_{P_0,\theta}$ satisfying inequality 5.6.4 and all $(g, h, X) \in G_1 \times G_1 \times \mathfrak{g}_1$ in the good range.

By 2.3.2 and the first property satisfied by the sequence $(\varphi_N)_{N\geqslant N_0}$, there exists $C > 0$ such that for all $(g, h, X, Y) \in G_1 \times G_1 \times (\mathfrak{t}^* \oplus \mathfrak{t}^\perp) \times \mathcal{A}^+_{P_0,\theta}$ and all $N \geqslant 1$, we have

$$\varphi_N * \Gamma^G_{M,\theta}(H_{M,\theta}(a), \mathcal{Z}(g, h, X, Y)) \neq 0 \Rightarrow \sigma_{G_1}(a) \leqslant C \left(\sigma_{G_1}(g) + \sigma_{G_1}(h) + \sigma_{\mathfrak{g}_1^*}(g^{-1}X_{\mathfrak{t}}h) + |Y| + N^\epsilon \right)$$

for all $a \in T$. Hence by Lemma 5.3(i), together with the inequality $\epsilon < \epsilon_2 < 1$, for N sufficiently large, we have

$$\varphi_N * \Gamma^G_{M,\theta}(H_{M,\theta}(a), \mathcal{Z}(g, h, X, Y)) \neq 0 \Rightarrow \kappa_N(h^{-1}ag) = 1$$

for all $Y \in \mathcal{A}^+_{P_0,\theta}$ satisfying inequality 5.6.4 and all $(g,h,X) \in G_1 \times G_1 \times \mathfrak{g}_1$ in the good range. This implies, again for N sufficiently large, that

$$\begin{aligned}\kappa^{G,Y}_{N,T,\xi}(g,h,X) &= \int_T \xi(aX)^{-1}\varphi_N * \Gamma^G_{M,\theta}(H_{M,\theta}(a), \mathcal{Z}(g,h,X,Y))da \\ &= \int_T \varphi^\sharp_N(aX_{\mathfrak{t}})\Gamma^G_{M,\theta}(H_{M,\theta}(a), \mathcal{Z}(g,h,X,Y))da\end{aligned}$$

for all $Y \in \mathcal{A}^+_{P_0,\theta}$ satisfying inequality 5.6.4 and all $(g,h,X) \in G_1 \times G_1 \times \mathfrak{g}_1$ in the good range. By 5.6.2 and the third property satisfied by the sequence $(\varphi_N)_{N\geqslant N_0}$, there exist $c>0$ and $k>0$ such that

$$\begin{aligned}&\left|\int_T \varphi^\sharp_N(aX_{\mathfrak{t}})\Gamma^G_{M,\theta}(H_{M,\theta}(a), \mathcal{Z}(g,h,X,Y))da - \int_T L(aX_{\mathfrak{t}})\Gamma^G_{M,\theta}(H_{M,\theta}(a), \mathcal{Z}(g,h,X,Y))da\right| \\ &\leqslant e^{-cN^\epsilon}\int_T |\Gamma^G_{M,\theta}(H_{M,\theta}(a), \mathcal{Z}(g,h,X,Y))|da \ll e^{-cN^\epsilon}\sigma_{T\backslash G_1}(g)^k\sigma_{T\backslash G_1}(h)^k\sigma_{\mathfrak{g}_1^*}(g^{-1}X_{\mathfrak{t}}h)^k(1+|Y|)^k\end{aligned}$$

for all $N \geqslant N_0$ and all $(g,h,X,Y) \in G_1 \times G_1 \times (\mathfrak{t}^* \oplus \mathfrak{t}^\perp) \times \mathcal{A}_{M_0,\theta}$. Thus in order to show 5.6.11 , it only remains to establish the following identity

$$\int_T L(aX_{\mathfrak{t}})\Gamma^G_{M,\theta}(H_{M,\theta}(a), \mathcal{Z}(g,h,X,Y))da = \frac{1}{2^d}\widetilde{w}_{M,\theta}(g,h,X,Y) \tag{5.6.12}$$

for all $(g,h,X,Y) \in G_1 \times G_1 \times (\mathfrak{t}^* \oplus \mathfrak{t}^\perp) \times \mathcal{A}_{M_0,\theta}$. Fix such (g,h,X,Y). After the variable change $a \mapsto aX_{\mathfrak{t}}^{-1}$, we get

$$\int_T L(aX_{\mathfrak{t}})\Gamma^G_{M,\theta}(H_{M,\theta}(a), \mathcal{Z}(g,h,X,Y))da = \int_T L(a)\Gamma^G_{M,\theta}(H_{M,\theta}(a), \mathcal{Z}(g,h,X,Y)+\frac{1}{2}V(X_{\mathfrak{t}}))da$$

(Note that $H_{M,\theta}(X_{\mathfrak{t}}) = \frac{1}{2}V(X_{\mathfrak{t}})$). Moreover, we easily check from the definitions that the (G,M,θ)-orthogonal set $\mathcal{X}(g,h,X,Y) := \mathcal{Z}(g,h,X,Y) + \frac{1}{2}V(X_{\mathfrak{t}})$ has the property that $\mathcal{X}(g,h,X,Y)_{wP} = w\mathcal{X}(g,h,X,Y)_P$ for all $w \in W$ and all $P \in \mathcal{P}^\theta(M)$. This implies that the function $\Gamma^G_{M,\theta}(.,\mathcal{X}(g,h,X,Y))$ is W-invariant and by 5.6.7 it follows that

$$\begin{aligned}\int_T L(a)\Gamma^G_{M,\theta}(H_{M,\theta}(a), \mathcal{X}(g,h,X,Y))da &= \frac{1}{|W|}\sum_{w\in W}\int_T L(wa)\Gamma^G_{M,\theta}(H_{M,\theta}(a), \mathcal{X}(g,h,X,Y))da \\ &= \frac{1}{2^d}\int_T \Gamma^G_{M,\theta}(H_{M,\theta}(a), \mathcal{X}(g,h,X,Y))da \\ &= \frac{1}{2^d}\int_T \Gamma^G_{M,\theta}(H_{M,\theta}(a), \mathcal{Z}(g,h,X,Y))da\end{aligned}$$

where to get the last equality we have performed the variable change $a \mapsto aX_{\mathfrak{t}}$. This proves 5.6.12 and ends the proof of 5.6.9 .

Proof of 5.6.10 : Let $R \in \mathcal{F}^\theta(M)$ with $R \neq G$. We will need to rewrite slightly the weight $\kappa^{R,Y}_{N,T,\xi}(g,h,X)$. By Lemma 5.3(ii) and since the functions $(\varphi_N)_{N \geqslant N_0}$ are T^c-invariant, for N big enough, all $Y \in \mathcal{A}_{M_0,\theta}$ and all $(g,h,X) \in G_1 \times G_1 \times \mathfrak{g}_1$ in the good range, we have

$$\kappa^{R,Y}_{N,T,\xi}(g,h,X) = \mathrm{vol}(T^c)^{-1} \int_T \int_{T^c} \xi(a_0 a X)^{-1} da_0 \kappa_N(h^{-1}ag)\varphi_N * \Delta_R(H_{M,\theta}(a), g, h, X, Y) da$$

or equivalently

$$\kappa^{R,Y}_{N,T,\xi}(g,h,X) = \int_T \varphi_c^\sharp(aX_{\mathfrak{t}})\kappa_N(h^{-1}ag)\varphi_N * \Delta_R(H_{M,\theta}(a), g, h, X, Y) da$$

where $\varphi_c := \mathrm{vol}(T^c)^{-1}\mathbf{1}_{T^c}$.

In what follows, for every $a \in T$, we will write $a = (a_1, \ldots, a_d)$ for the decomposition of a according to the identification 5.2.1 (so that $a_i \in K_i^\times$ for all $1 \leqslant i \leqslant d$). Recall that we have identified $\mathcal{A}_{M,\theta}$ with $\mathbf{R}^d$ (see §5.4), the identification being so that

$$H_{M,\theta}(a) = \frac{1}{2}(v_{K_1}(a_1), \ldots, v_{K_d}(a_d))$$

for all $a \in T$ where $v_{K_i} := \frac{1}{[K_i:F]} v_F \circ N_{K_i/F}$ for all $1 \leqslant i \leqslant d$. Moreover the set of roots of $A_{M,\theta}$ in U_Q, identified to a subset of $\mathcal{A}^*_{M,\theta}$, can be explicitly described as

$$R(A_{M,\theta}, U_Q) = \{\chi_i + \chi_j \mid 1 \leqslant i, j \leqslant d\}$$

where for all $1 \leqslant i \leqslant d$, we have denoted by χ_i the functional $\mathcal{A}_{M,\theta} = \mathbf{R}^d \to \mathbf{R}$ given by $\chi_i(x_1, \ldots, x_d) = -x_i$.
Write $R = L_R U_R$ for the unique Levi decomposition of R with $M \subset L_R$ and let $\overline{R} = L_R U_{\overline{R}}$ be the parabolic subgroup opposite to R (with respect to L_R). We now distinguish two cases:

- First assume that $U_{\overline{R}}$ is not included in the parabolic subgroup Q. Then, we will actually prove that for N big enough, we have

  $$\kappa^{R,Y}_{N,T,\xi}(g,h,X) = 0$$

  for all $Y \in \mathcal{A}^+_{P_0,\theta}$ satisfying inequality 5.6.3 and all $(g,h,X) \in G_1 \times G_1 \times \mathfrak{g}_1$ in the good range. This would imply that

  $$J^{R,Y}_{N,T,\leqslant N^\epsilon}(f) = 0$$

  for N big enough and all $Y \in \mathcal{A}^+_{P_0,\theta}$ satisfying inequality 5.6.3 .

  By our assumption, we have $U_R \cap U_Q \neq \{1\}$ which implies $R(A_{M,\theta}, U_R) \cap R(A_{M,\theta}, U_Q) \neq \emptyset$. Let $\alpha \in R(A_{M,\theta}, U_R) \cap R(A_{M,\theta}, U_Q)$. Then, by the previous concrete description of $R(A_{M,\theta}, U_Q)$, there exist $1 \leqslant i, j \leqslant d$ such that

$$\langle \alpha, H_{M,\theta}(a)\rangle = -\frac{v_{K_i}(a_i) + v_{K_j}(a_j)}{2} \tag{5.6.13}$$

for all $a \in T$. As $\epsilon < \epsilon_1$, there exists $c_1 > 0$ such that for all N sufficiently large, all $Y \in \mathcal{A}^+_{P_0,\theta}$ satisfying inequality 5.6.3 and all $(g,h,X) \in G_1 \times G_1 \times \mathfrak{g}_1$ in the good range, we have $\langle \beta, \mathcal{Z}(g,h,X,Y)_P\rangle \geqslant c_1 N^{\epsilon_1}$ for all $P \in \mathcal{P}^\theta(M)$ and all $\beta \in R(A_{M,\theta}, U_P)$. In particular, $\mathcal{Z}(g,h,X,Y)_P \in \mathcal{A}^+_{P,\theta}$ for all $P \in \mathcal{P}^\theta(M)$ and the (G,M,θ)-orthogonal set $\mathcal{Z}(g,h,X,Y)$ is positive. This implies that $\Delta_R(.,\mathcal{Z}(g,h,X,Y))$ is the characteristic function of the sum of $\mathcal{A}^+_{R,\theta}$ with the convex hull of the family $(\mathcal{Z}(g,h,X,Y)_P)_{P\subset R}$ and hence that

$$\Delta_R(\Lambda, \mathcal{Z}(g,h,X,Y)) \neq 0 \Rightarrow \langle \alpha, \Lambda\rangle \geqslant c_1 N^{\epsilon_1}$$

for all $\Lambda \in \mathcal{A}_{M,\theta}$. Again since $\epsilon < \epsilon_1$ and by the first property satisfied by the sequence $(\varphi_N)_{N\geqslant N_0}$, it follows that there exists $c_2 > 0$ such that

$$\varphi_N * \Delta_R(\Lambda, \mathcal{Z}(g,h,X,Y)) \neq 0 \Rightarrow \langle \alpha, \Lambda\rangle \geqslant c_2 N^{\epsilon_1}$$

for all $\Lambda \in \mathcal{A}_{M,\theta}$, all N sufficiently large, all $Y \in \mathcal{A}^+_{P_0,\theta}$ satisfying inequality 5.6.3 and all $(g,h,X) \in G_1 \times G_1 \times \mathfrak{g}_1$ in the good range. On the other hand, since the function $\varphi_c^\sharp$ is compactly supported on $\mathfrak{t}$, by 5.6.13 there exists $c_3 > 0$ such that

$$\varphi_c^\sharp(aX_\mathfrak{t}) \neq 0 \Rightarrow \langle \alpha, H_{M,\theta}(a)\rangle \leqslant c_3 N^{\epsilon}$$

for all $N \geqslant 1$, all $a \in T$ and all $(g,h,X) \in G_1 \times G_1 \times \mathfrak{g}_1$ in the good range. As $\epsilon < \epsilon_1$, it follows that for N sufficiently large, the supports of the functions

$$a \in T \mapsto \varphi_N * \Delta_R(H_{M,\theta}(a), \mathcal{Z}(g,h,X,Y))$$

and

$$a \in T \mapsto \varphi_c^\sharp(aX_\mathfrak{t})$$

are disjoint and hence

$$\kappa^{R,Y}_{N,T,\xi}(g,h,X) = 0$$

for all $Y \in \mathcal{A}^+_{P_0,\theta}$ satisfying inequality 5.6.3 and all $(g,h,X) \in G_1 \times G_1 \times \mathfrak{g}_1$ in the good range. This proves the claim and ends the proof of 5.6.10 in this case.

- Now assume that $U_{\overline{R}} \subset Q$ or equivalently $U_Q \subset \overline{R}$. Let $V_{\overline{R}}$ and $V_{\overline{R}}^{\perp}$ be the subspaces of $\mathfrak{g}_1$ such that

$$U_{\overline{R}} \cap U_Q = \left\{ \begin{pmatrix} 1 & X \\ & 1 \end{pmatrix} \mid X \in V_{\overline{R}} \right\},$$

$$L_R \cap U_Q = \left\{ \begin{pmatrix} 1 & X \\ & 1 \end{pmatrix} \mid X \in V_{\overline{R}}^{\perp} \right\}$$

(we can show that $V_{\overline{R}}^{\perp}$ is the orthogonal of $V_{\overline{R}}$ with respect to $\langle ., . \rangle$ but we won't need this fact). Let U_1, U_2 be the unipotent subgroups of G_1 such that $U_{\overline{R}} \cap L = U_1 \times U_2$ and let L_1 be the Levi subgroup of G_1 such that $L_R \cap L = L_1 \times L_1$ (that $L_R \cap L$ is of this form follows from the fact that R is θ-split). Then, we have $\mathfrak{g}_1 = V_{\overline{R}} \oplus V_{\overline{R}}^{\perp}$ (this follows from the fact that $U_Q = (L_R \cap U_Q)(U_{\overline{R}} \cap U_Q)$) and $u_1 X u_2^{-1} - X \in V_{\overline{R}}^{\perp}$ for all $(u_1, u_2) \in U_1 \times U_2$ and all $X \in V_{\overline{R}}^{\perp}$ (this follows from the fact that $l^{-1} u l u^{-1} \in U_{\overline{R}}$ for all $(l, u) \in L_R \times U_{\overline{R}}$). We will denote by $X \mapsto X_{\overline{R}}^{\perp}$ the projection $\mathfrak{g}_1 \to V_{\overline{R}}^{\perp}$ relative to the decomposition $\mathfrak{g}_1 = V_{\overline{R}} \oplus V_{\overline{R}}^{\perp}$.

For all $(g, h, Y) \in G_1 \times G_1 \times \mathcal{A}_{P_0,\theta}^{+}$, we introduce a new (G, M, θ)-orthogonal set $\mathcal{Z}(g, h, Y)$ defined by

$$\mathcal{Z}(g, h, Y)_{w\overline{P}} := Y_{w\overline{P}} - w H_{P,\theta} \begin{pmatrix} g & \\ & h \end{pmatrix}$$

for all $P \in \mathcal{P}^{\theta,Q}(M)$ and all $w \in W$ (that it indeed defines a (G, M, θ)-orthogonal set follows from Lemma 5.5). As before, we set

$$\Delta_R(\Lambda, \mathcal{Z}(g, h, Y)) := \Gamma_{M,\theta}^{R}(\Lambda, \mathcal{Z}(g, h, Y)) \tau_{R,\theta}^{G}(\Lambda - \mathcal{Z}(g, h, Y)), \quad \Lambda \in \mathcal{A}_{M,\theta}.$$

For all $N \geqslant N_0$ and all $Y \in \mathcal{A}_{P_0,\theta}^{+}$, we define a new weight by

$$\kappa_{N,T,\xi}^{R,Y,\star}(g, h, X) := \int_T \xi(a X_{\mathfrak{t}})^{-1} \kappa_N(h^{-1} a g) \varphi_N * \Delta_R(H_{M,\theta}(a), \mathcal{Z}(g, h, Y)) da$$

for all $(g, h, X) \in G_1 \times G_1 \times \mathfrak{g}_1$. We will now need the following lemma whose proof is postponed to the next section:

Lemma 5.9. *For N big enough, we have*

$$\kappa_{N,T,\xi}^{R,Y}(u_1 l_1 k_1, u_2 l_2 k_2, X) = \kappa_{N,T,\xi}^{R,Y,\star}(l_1, l_2, X_{\overline{R}}^{\perp})$$

for all $(u_1, u_2) \in U_1 \times U_2$, $(l_1, l_2) \in L_1 \times L_1$, $(k_1, k_2) \in K_1 \times K_1$, $X \in \mathfrak{g}_1$ and $Y \in \mathcal{A}_{P_0,\theta}^{+}$ such that Y satisfies inequality 5.6.3 and $(u_1 l_1 k_1, u_2 l_2 k_2, X)$ is in the good range.

Assuming the above lemma, by the Iwasawa decompositions $G_1 = U_1 L_1 K_1 = U_2 L_1 K_1$ and since $K_1 \times K_1 \subset K$, for N sufficiently large, we have

$$\begin{aligned} J^{R,Y}_{N,T,\leqslant N^\epsilon}(f) = \int_T D^{G_1}(t) \int_{T\times T\backslash L_1\times L_1} \int_{U_1\times U_2} \int_{\mathfrak{g}_1} \\ f^K\left(\begin{pmatrix} l_1^{-1}u_1^{-1} & \\ & l_2^{-1}u_2^{-1}\end{pmatrix}\begin{pmatrix} t & \\ & t\end{pmatrix}\begin{pmatrix} 1 & X \\ & 1\end{pmatrix}\begin{pmatrix} u_1 l_1 & \\ & u_2 l_2\end{pmatrix}\right) \\ \kappa^{R,Y,\star}_{N,T,\xi}(l_1,l_2,X^\perp_{\overline{R}})\mathbf{1}_{T\backslash G_1,\leqslant N^\epsilon}(u_1 l_1)\mathbf{1}_{T\backslash G_1,\leqslant N^\epsilon}(u_2 l_2)\mathbf{1}_{\mathfrak{g}_1^*,\leqslant N^\epsilon}(l_1^{-1}u_1^{-1}X_{\mathfrak{t}}u_2 l_2) \\ dX du_1 du_2 \delta_{P_1}(l_1)^{-1}\frac{dl_1}{\nu(l_1)^n}\delta_{P_2}(l_2)^{-1}\nu(l_2)^n dl_2 \omega(t)^{-1}dt \end{aligned}$$

for all $Y \in \mathcal{A}^+_{P_0,\theta}$ satisfying inequality 5.6.3 where we have denoted by δ_{P_1} and δ_{P_2} the modular characters of the parabolic subgroups $P_1 = L_1U_1$ and $P_2 = L_1U_2$ of G_1 respectively.

Introduce the following expression:

$$\begin{aligned} J^{R,Y,\star}_{N,T}(f) := \int_T D^{G_1}(t) \int_{T\times T\backslash L_1\times L_1} \int_{U_1\times U_2} \int_{\mathfrak{g}_1} \\ f^K\left(\begin{pmatrix} l_1^{-1}u_1^{-1} & \\ & l_2^{-1}u_2^{-1}\end{pmatrix}\begin{pmatrix} t & \\ & t\end{pmatrix}\begin{pmatrix} 1 & X \\ & 1\end{pmatrix}\begin{pmatrix} u_1 l_1 & \\ & u_2 l_2\end{pmatrix}\right) \\ \kappa^{R,Y,\star}_{N,T,\xi}(l_1,l_2,X^\perp_{\overline{R}})dX du_1 du_2 \delta_{P_1}(l_1)^{-1}\frac{dl_1}{\nu(l_1)^n}\delta_{P_2}(l_2)^{-1}\nu(l_2)^n dl_2 \omega(t)^{-1}dt \end{aligned}$$

for all $N \geqslant N_0$ and all $Y \in \mathcal{A}^+_{P_0,\theta}$. By 5.1.3, for some $k > 0$, we have

$$\left|\kappa^{R,Y,\star}_{N,T,\xi}(l_1,l_2,X^\perp_{\overline{R}})\right| \ll \sigma_{T\backslash G_1}(u_1 l_1 k_1)^k \sigma_{T\backslash G_1}(u_2 l_2 k_2)^k$$

for all $(l_1,l_2) \in L_1\times L_1$, $(u_1,u_2) \in U_1\times U_2$ and $(k_1,k_2) \in K_1\times K_1$. Hence, by Proposition 5.4 (and using the Iwasawa decompositions $G_1 = U_1L_1K_1 = U_2L_1K_1$ backwards), the expression defining $J^{R,Y,\star}_{N,T}(f)$ is absolutely convergent. And for all $r > 0$, we have

$$\left|J^{R,Y}_{N,T,\leqslant N^\epsilon}(f) - J^{R,Y,\star}_{N,T}(f)\right| \ll N^{-r}$$

for all $N \geqslant N_0$ and all $Y \in \mathcal{A}^+_{P_0,\theta}$ satisfying inequality 5.6.3 . Thus to prove 5.6.10 in this case, it suffices to establish that

$$J^{R,Y,\star}_{N,T}(f) = 0$$

for all $N \geqslant N_0$ and all $Y \in \mathcal{A}^+_{P_0,\theta}$. This vanishing follows from the fact that f is θ-strongly cuspidal by the following sequence of equalities:

$$\begin{aligned}
&\int_{U_1\times U_2}\int_{\mathfrak{g}_1} f^K\left(\begin{pmatrix} l_1^{-1}u_1^{-1} & \\ & l_2^{-1}u_2^{-1}\end{pmatrix}\begin{pmatrix} t & \\ & t\end{pmatrix}\begin{pmatrix} 1 & X \\ & 1\end{pmatrix}\begin{pmatrix} u_1l_1 & \\ & u_2l_2\end{pmatrix}\right)\kappa^{R,Y,\star}_{N,T,\xi}(l_1,l_2,X^{\perp}_{\overline{R}})dXdu_1du_2\\
&=\int_{U_1\times U_2}\int_{\mathfrak{g}_1} f^K\left(\begin{pmatrix} l_1^{-1} & \\ & l_2^{-1}\end{pmatrix}\begin{pmatrix} u_1^{-1}tu_1 & \\ & u_2^{-1}tu_2\end{pmatrix}\begin{pmatrix} 1 & X \\ & 1\end{pmatrix}\begin{pmatrix} l_1 & \\ & l_2\end{pmatrix}\right)\kappa^{R,Y,\star}_{N,T,\xi}(l_1,l_2,X^{\perp}_{\overline{R}})dXdu_1du_2\\
&=\Delta(t)\int_{U_1\times U_2}\int_{\mathfrak{g}_1} f^K\left(\begin{pmatrix} l_1^{-1} & \\ & l_2^{-1}\end{pmatrix}\begin{pmatrix} tu_1 & \\ & tu_2\end{pmatrix}\begin{pmatrix} 1 & X \\ & 1\end{pmatrix}\begin{pmatrix} l_1 & \\ & l_2\end{pmatrix}\right)\kappa^{R,Y,\star}_{N,T,\xi}(l_1,l_2,X^{\perp}_{\overline{R}})dXdu_1du_2\\
&=\Delta(t)\int_{V^{\perp}_{\overline{R}}}\int_{U_{\overline{R}}} f^K\left(\begin{pmatrix} l_1^{-1} & \\ & l_2^{-1}\end{pmatrix}\begin{pmatrix} t & \\ & t\end{pmatrix}U\begin{pmatrix} 1 & X^{\perp}_{\overline{R}} \\ & 1\end{pmatrix}\begin{pmatrix} l_1 & \\ & l_2\end{pmatrix}\right)\kappa^{R,Y,\star}_{N,T,\xi}(l_1,l_2,X^{\perp}_{\overline{R}})dUdX^{\perp}_{\overline{R}}\\
&=0
\end{aligned}$$

for all $N \geqslant N_0$, all $t \in T \cap G_{1,\mathrm{reg}}$, all $l_1, l_2 \in L_1$ and all $Y \in \mathcal{A}^+_{P_0,\theta}$, where $\Delta(t) := D^{G_1}(t)^{-1}D^{L_1}(t)$ and where the first equality follows from the variable change $X \mapsto u_1Xu_2^{-1}$ (recall that $(u_1Xu_2^{-1})^{\perp}_{\overline{R}} = X^{\perp}_{\overline{R}}$), the second equality is a consequence of the classical change of variable inverse to $(u_1,u_2) \mapsto (t^{-1}u_1^{-1}tu_1, t^{-1}u_2^{-1}tu_2)$ (whose jacobian equals $D^{G_1/L_1}(t)^{-1}$ because $\delta_{P_1}(t) = \delta_{P_2}(t)^{-1}$), in the third equality we have merged an integral over $U_1 \times U_2$ and an integral over $V_{\overline{R}}$ into an integral over $U_{\overline{R}}$, and finally the last equality follows from the fact that f is θ-strongly cuspidal (note that $\begin{pmatrix} 1 & X^{\perp}_{\overline{R}} \\ & 1\end{pmatrix}$ belongs to L_R) so that the inner integral already vanish identically. This finishes the proof of 5.6.10 in this case and thus also of the proposition.

□

5.7 Proof of Lemma 5.9

For convenience we recall the statement here.

Lemma 5.10. *Let $0 < \epsilon < \epsilon_1 < 1$ and let $R \in \mathcal{F}^{\theta}(M)$ with $U_{\overline{R}} \subset Q$. Then, for N big enough, we have*

$$\kappa^{R,Y}_{N,T,\xi}(u_1l_1k_1, u_2l_2k_2, X) = \kappa^{R,Y,\star}_{N,T,\xi}(l_1,l_2,X^{\perp}_{\overline{R}})$$

for all $(u_1,u_2) \in U_1 \times U_2$, $(l_1,l_2) \in L_1 \times L_1$, $(k_1,k_2) \in K_1 \times K_1$, $X \in \mathfrak{g}_1$ and $Y \in \mathcal{A}^+_{P_0,\theta}$ satisfying the inequalities

- $N^{\epsilon_1} \leqslant \inf_{\alpha\in\Delta_0}\alpha(Y)$;
- $\sigma_{T\backslash G_1}(u_1l_1) \leqslant N^{\epsilon}$, $\sigma_{T\backslash G_1}(u_2l_2) \leqslant N^{\epsilon}$;

- $\sigma_{\mathfrak{g}_1^*}(l_1^{-1}u_1^{-1}X_{\mathfrak{t}}u_2l_2) \leqslant N^{\epsilon}$.

Proof. We will use the notation introduced in the proof of Proposition 5.7. In particular, we have a decomposition $\mathfrak{g}_1 = V_{\overline{R}} \oplus V_{\overline{R}}^{\perp}$. Recall that the linear map $X \mapsto X_{\overline{R}}^{\perp}$ is just the projection onto $V_{\overline{R}}^{\perp}$ relative to this decomposition. We will set $V_{\overline{R},\mathfrak{t}} := V_{\overline{R}} \cap \mathfrak{t}$ and $V_{\overline{R},\mathfrak{t}}^{\perp} := V_{\overline{R}}^{\perp} \cap \mathfrak{t}$. Then, we have $\mathfrak{t} = V_{\overline{R},\mathfrak{t}} \oplus V_{\overline{R},\mathfrak{t}}^{\perp}$ (this follows from the fact that the adjoint action of T preserves the decomposition $\mathfrak{g}_1 = V_{\overline{R}} \oplus V_{\overline{R}}^{\perp}$). For all $X \in \mathfrak{g}_1$, we will set $X_{\overline{R},\mathfrak{t}}^{\perp} := (X_{\overline{R}}^{\perp})_{\mathfrak{t}} = (X_{\mathfrak{t}})_{\overline{R}}^{\perp}$. Note that we have $X_{\mathfrak{t}} - X_{\overline{R},\mathfrak{t}}^{\perp} \in V_{\overline{R},\mathfrak{t}}$ for all $X \in \mathfrak{g}_1$.

By the definition of $\kappa_{N,T,\xi}^{R,Y}$, it is invariant by right translation by K_1 in both the first and the second variables. Thus, we just need to prove the lemma in the case where $k_1 = k_2 = 1$. Moreover, since $\kappa_{N,T,\xi}^{R,Y}(g,h,X) = \kappa_{N,T,\xi}^{R,Y}(t_1g, t_2h, t_1Xt_2^{-1})$ for all $(g,h,X,t_1,t_2) \in G_1^2 \times \mathfrak{g}_1 \times T^2$, by 5.0.1 we may replace the conditions that $\sigma_{T\backslash G_1}(u_1l_1) \leqslant N^{\epsilon}$ and $\sigma_{T\backslash G_1}(u_2l_2) \leqslant N^{\epsilon}$ by the conditions that $\sigma_{G_1}(u_1l_1) \leqslant N^{\epsilon}$ and $\sigma_{G_1}(u_2l_2) \leqslant N^{\epsilon}$. Note that there exists $c > 0$ such that for all $(l_1, l_2, u_1, u_2, X) \in L_1^2 \times U_1 \times U_2 \times \mathfrak{g}_1$, the inequalities $\sigma_{G_1}(u_1l_1) \leqslant N^{\epsilon}$, $\sigma_{G_1}(u_2l_2) \leqslant N^{\epsilon}$ and $\sigma_{\mathfrak{g}_1^*}(l_1^{-1}u_1^{-1}X_{\mathfrak{t}}u_2l_2) \leqslant N^{\epsilon}$ imply that

$$\max\left(\sigma_{G_1}(l_1), \sigma_{G_1}(l_2), \sigma_{G_1}(u_1), \sigma_{G_1}(u_2), \sigma_{\mathfrak{t}^*}(X_{\mathfrak{t}})\right) \leqslant cN^{\epsilon}.$$

Fix such a c. From now on and until the end of the proof, for a fixed N, we will say that $(l_1, l_2, u_1, u_2, X, Y) \in L_1^2 \times U_1 \times U_2 \times \mathfrak{g}_1 \times \mathcal{A}_{P_0,\theta}^{+}$ is in the *good range* if it satisfies the inequality above and moreover $N^{\epsilon_1} \leqslant \inf_{\alpha \in \Delta_0} \alpha(Y)$. Then in order to prove the lemma, we just need to establish the following fact:

(5.7.1) For N big enough and all $(l_1, l_2, u_1, u_2, X, Y)$ in the good range, we have

$$\kappa_{N,T,\xi}^{R,Y}(u_1l_1, u_2l_2, X) = \kappa_{N,T,\xi}^{R,Y,\star}(l_1, l_2, X_{\overline{R}}^{\perp}).$$

First of all, by Lemma 5.3(ii) and since the functions $(\varphi_N)_{N \geqslant N_0}$ are T^c-invariant, for N sufficiently large and all $(l_1, l_2, u_1, u_2, X, Y)$ in the good range, we have

$$\kappa_{N,T,\xi}^{R,Y}(u_1l_1, u_2l_2, X) = \int_T \varphi_c^{\sharp}(aX_{\mathfrak{t}})\kappa_N(l_2^{-1}u_2^{-1}au_1l_1)\varphi_N * \Delta_R(H_{M,\theta}(a), \mathcal{Z}(u_1l_1, u_2l_2, X, Y))da$$

and

$$\kappa_{N,T,\xi}^{R,Y,\star}(l_1, l_2, X_{\overline{R}}^{\perp}) = \int_T \varphi_c^{\sharp}(aX_{\overline{R},\mathfrak{t}}^{\perp})\kappa_N(l_2^{-1}al_1)\varphi_N * \Delta_R(H_{M,\theta}(a), \mathcal{Z}(l_1, l_2, Y))da$$

where $\varphi_c := \mathrm{vol}(T^c)^{-1}\mathbf{1}_{T^c}$. Therefore, 5.7.1 is a consequence of the two following facts:

(5.7.2) For N sufficiently large and all $(l_1, l_2, u_1, u_2, X, Y)$ in the good range, we have

$$\kappa_N(l_2^{-1}u_2^{-1}au_1l_1) = \kappa_N(l_2^{-1}al_1)$$

and

$$\varphi_c^\sharp(aX_{\mathfrak{t}}) = \varphi_c^\sharp(aX_{\overline{R},\mathfrak{t}}^\perp)$$

for all $a \in T$ with $\varphi_N * \Delta_R(H_{M,\theta}(a), \mathcal{Z}(l_1, l_2, X, Y)) \neq 0$.

(5.7.3) For N sufficiently large and all $(l_1, l_2, u_1, u_2, X, Y)$ in the good range, we have

$$\varphi_N * \Delta_R(H_{M,\theta}(a), \mathcal{Z}(u_1 l_1, u_2 l_2, X, Y)) = \varphi_N * \Delta_R(H_{M,\theta}(a), \mathcal{Z}(l_1, l_2, Y))$$

for all $a \in T$ such that $\varphi_c^\sharp(aX_{\mathfrak{t}}) \neq 0$.

Proof of 5.7.2: As $\overline{R}$ is θ-split, the parabolic subgroups $P_1 = L_1U_1$ and $P_2 = L_2U_2$ of G_1 are opposite to each other. Therefore, $R(A_T, U_2) = -R(A_T, U_1)$. Let us denote by $R(A_T, V_{\overline{R},\mathfrak{t}})$ the set of weights of A_T in $V_{\overline{R},\mathfrak{t}}$ for the linear action $(a, X) \mapsto aX$. Then, we will need the following

(5.7.4) For every $\alpha \in R(A_T, U_1)$ (resp. $\alpha \in R(A_T, V_{\overline{R},\mathfrak{t}})$), there exists a root $\beta \in R(A_{M,\theta}, U_{\overline{R}})$ such that

$$\langle \alpha, H_T(a) \rangle = 2\langle \beta, H_{M,\theta}(a) \rangle \text{ (resp. } \langle \alpha, H_T(a) \rangle = \langle \beta, H_{M,\theta}(a) \rangle)$$

for all $a \in T$.

Let $\mathfrak{u}_{1,\alpha}$ (resp. $V_{\overline{R},\mathfrak{t},\alpha}$) be the root (resp. weight) subspace corresponding to α. Then, we easily check that $A_M = A_T \times A_T$ acts on this subspace by a character γ where in the case $\alpha \in R(A_T, V_{\overline{R},\mathfrak{t}})$, we consider the action of A_M on $V_{\overline{R},\mathfrak{t}}$ given by $(a_1, a_2) \cdot X = a_1 X a_2^{-1} = a_1 a_2^{-1} X$. Denoting by β the restriction of γ to $A_{M,\theta}$ (so that in particular $\beta \in R(A_{M,\theta}, U_{\overline{R}})$), we have

$$\langle \beta, H_{M,\theta}(a) \rangle = \frac{1}{2}\langle \beta, H_{M,\theta}\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} \theta \begin{pmatrix} a & \\ & 1 \end{pmatrix}^{-1}\right) \rangle = \frac{1}{2}\langle \beta, H_{M,\theta}\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \rangle = \frac{1}{2}\log\left|\beta\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}\right|$$

for all $a \in T$. On the other hand, the action of $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$ on $\mathfrak{u}_{1,\alpha}$ (resp. $V_{\overline{R},\mathfrak{t},\alpha}$) coincides with the action of a (resp. a^2). Hence, $\beta\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} = \alpha(a)$ (resp. $\beta\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} = \alpha(a)^2$) and this shows the claim.

Now, since $\epsilon < \epsilon_1$, there exists $c_1 > 0$ such that for N sufficiently large and all $(l_1, l_2, u_1, u_2, X, Y)$ in the good range, the (G, M, θ)-orthogonal set $\mathcal{Z}(l_1, l_2, X, Y)$ is positive and $\langle \beta, \mathcal{Z}(l_1, l_2, X, Y)_P \rangle \leqslant -c_1 N^{\epsilon_1}$ for all $P \in \mathcal{P}^\theta(M)$ with $P \subset R$ and all $\beta \in R(A_{M,\theta}, U_{\overline{R}})$. Since the (G, M, θ)-orthogonal set $\mathcal{Z}(l_1, l_2, X, Y)$ is positive, the function $\Delta_R(., \mathcal{Z}(l_1, l_2, X, Y))$ is the characteristic function of the sum of $\mathcal{A}_{R,\theta}^+$ with the convex hull of $(\mathcal{Z}(l_1, l_2, X, Y)_P)_{P \subset R}$ and it follows

that $\langle \beta, \Lambda \rangle \leqslant -c_1 N^{\epsilon_1}$ for all $\Lambda \in \mathcal{A}_{M,\theta}$ in the support of this function and all $\beta \in R(A_{M,\theta}, U_{\overline{R}})$. Again by using the fact that $\epsilon < \epsilon_1$, we deduce from this and the first property satisfied by the sequence $(\varphi_N)_{N \geqslant N_0}$ (see Lemma 5.8), that there exists $c_2 > 0$ such that for N sufficiently large and all $(l_1, l_2, u_1, u_2, X, Y)$ in the good range, we have $\langle \beta, \Lambda \rangle \leqslant -c_2 N^{\epsilon_1}$ for all $\Lambda \in \mathcal{A}_{M,\theta}$ in the support of the function $\varphi_N * \Delta_R(., \mathcal{Z}(l_1, l_2, X, Y))$ and all $\beta \in R(A_{M,\theta}, U_{\overline{R}})$. By 5.7.4, it follows that we have, again for N sufficiently large and all $(l_1, l_2, u_1, u_2, X, Y)$ in the good range,

$$\langle \alpha, H_T(a) \rangle \leqslant -c_2 N^{\epsilon_1}$$

for all $\alpha \in R(A_T, U_1) \cup R(A_T, V_{\overline{R},\mathfrak{t}})$ and all $a \in T$ such that

$$\varphi_N * \Delta_R(H_{M,\theta}(a), \mathcal{Z}(l_1, l_2, X, Y)) \neq 0$$

Still assuming that $(l_1, l_2, u_1, u_2, X, Y)$ is in the good range, this implies (recall that $R(A_T, U_2) = -R(A_T, U_1)$ and $X_{\mathfrak{t}} - X^{\perp}_{\overline{R},\mathfrak{t}} \in V_{\overline{R},\mathfrak{t}}$)

$$l_2^{-1} u_2^{-1} a u_1 a^{-1} u_2 l_2 \in K_1, \;\; l_1^{-1} a^{-1} u_2^{-1} a l_1 \in K_1 \text{ and } a(X_{\mathfrak{t}} - X^{\perp}_{\overline{R},\mathfrak{t}}) \in \mathcal{L}$$

for N sufficiently large, where $\mathcal{L}$ is a lattice in $\mathfrak{t}$ by which $\varphi_c^{\sharp}$ is invariant. Hence, we get

$$\kappa_N(l_2^{-1} u_2^{-1} a u_1 l_1) = \kappa_N(l_2^{-1} u_2^{-1} a l_1) = \kappa_N(l_2^{-1} a l_1)$$

and

$$\varphi_c^{\sharp}(aX_{\mathfrak{t}}) = \varphi_c^{\sharp}(aX^{\perp}_{\overline{R},\mathfrak{t}})$$

for all N sufficiently large, all $(l_1, l_2, u_1, u_2, X, Y)$ in the good range and all $a \in T$ such that $\varphi_N * \Delta_R(H_{M,\theta}(a), \mathcal{Z}(l_1, l_2, X, Y)) \neq 0$.

Proof of 5.7.3: Since the function $\varphi_c^{\sharp}$ is compactly supported on $\mathfrak{t}$, there exists $c_3 > 0$ such that for all N, all $(l_1, l_2, u_1, u_2, X, Y)$ in the good range and all $a \in T$, if $\varphi_c^{\sharp}(aX_{\mathfrak{t}}) \neq 0$, then $v_{K_i}(a_i) \geqslant -c_3 N^{\epsilon}$ for all $1 \leqslant i \leqslant d$ (recall that we are denoting by $(a_1, \ldots, a_d)$ the decomposition of a according to the identification 5.2.1). These last inequalities mean that $H_{M,\theta}(a) \in \mathbf{R}_+^d + c_3 Z_N$ where $Z_N := (-N^{\epsilon}, \ldots, -N^{\epsilon})$ (recall that we are identifying $\mathcal{A}_{M,\theta}$ with $\mathbf{R}^d$ as in §5.4). Moreover, by the first property satisfied by the sequence $(\varphi_N)_{N \geqslant N_0}$ (see Lemma 5.8), there exists $c_4 > 0$ such that, for all $N \geqslant N_0$ and all function Δ on $\mathcal{A}_{M,\theta}$, the restriction of the function $\varphi_N * \Delta$ to $c_3 Z_N + \mathbf{R}_+^d$ only depends on the restriction of Δ to $c_4 Z_N + \mathbf{R}_+^d$. Note that

$$\mathbf{R}_+^d = \bigcup_{P \in \mathcal{P}^{Q,\theta}(M)} \overline{\mathcal{A}^+_{P,\theta}}$$

(this follows from the fact that $\mathbf{R}_+^d = \{\Lambda \in \mathcal{A}_{M,\theta} \mid \langle \alpha, \Lambda \rangle \leqslant 0 \; \forall \alpha \in R(A_{M,\theta}, U_Q)\}$). Hence, it suffices to show that for all $P \in \mathcal{P}^{Q,\theta}(M)$, all N sufficiently large and all $(l_1, l_2, u_1, u_2, X, Y)$ in the good range, the two functions

$$\Delta_R(.,\mathcal{Z}(u_1l_1,u_2l_2,X,Y))$$

and

$$\Delta_R(.,\mathcal{Z}(l_1,l_2,Y))$$

coincide on $c_4Z_N+\overline{\mathcal{A}^+_{\overline{P},\theta}}$. Or equivalently that the two functions

$$\Delta_R(.,\mathcal{Z}(u_1l_1,u_2l_2,X,Y)-c_4Z_N)$$

and

$$\Delta_R(.,\mathcal{Z}(l_1,l_2,Y)-c_4Z_N)$$

coincide on $\overline{\mathcal{A}^+_{\overline{P},\theta}}$. Fix $P\in\mathcal{P}^{Q,\theta}(M)$. As $\epsilon<\epsilon_1$, for N sufficiently large and all (l_1,l_2,u_1,u_2,X,Y) in the good range, we have

$$\mathcal{Z}(u_1l_1,u_2l_2,X,Y)_{P'}-c_4Z_N\in\mathcal{A}^+_{P',\theta}\text{ and }\mathcal{Z}(l_1,l_2,Y)_{P'}-c_4Z_N\in\mathcal{A}^+_{P',\theta}$$

for all $P'\in\mathcal{P}^\theta(M)$. By 2.3.4, it follows that the restrictions of the functions

$$\Delta_R(.,\mathcal{Z}(u_1l_1,u_2l_2,X,Y)-c_4Z_N)\text{ and }\Delta_R(.,\mathcal{Z}(l_1,l_2,Y)-c_4Z_N)$$

to $\overline{\mathcal{A}^+_{\overline{P},\theta}}$ only depend on $\mathcal{Z}(u_1l_1,u_2l_2,X,Y)_{\overline{P}}$ and $\mathcal{Z}(l_1,l_2,Y)_{\overline{P}}$ respectively. Returning to the definitions, we see that

$$\mathcal{Z}(u_1l_1,u_2l_2,X,Y)_{\overline{P}}=Y_{\overline{P}}-H_{P,\theta}\begin{pmatrix}u_1l_1 & \\ & u_2l_2\end{pmatrix}=Y_{\overline{P}}-H_{P,\theta}\begin{pmatrix}l_1 & \\ & l_2\end{pmatrix}=\mathcal{Z}(l_1,l_2,Y)_{\overline{P}}$$

for all $(l_1,l_2,u_1,u_2,X,Y)\in L_1^2\times U_1\times U_2\times\mathfrak{g}_1\times\mathcal{A}^+_{P_0,\theta}$. Hence, for N sufficiently large and all (l_1,l_2,u_1,u_2,X,Y) in the good range, the functions $\Delta_R(.,\mathcal{Z}(u_1l_1,u_2l_2,X,Y)-c_4Z_N)$ and $\Delta_R(.,\mathcal{Z}(l_1,l_2,Y)-c_4Z_N)$ indeed coincide on $\overline{\mathcal{A}^+_{\overline{P},\theta}}$. This finishes the proof of 5.7.3 and also of the lemma. □

5.8 Computation of $\lim_{N\to\infty}J_{N,T}(f)$ for θ-strongly cuspidal functions

Proposition 5.11. *Let $f\in\mathcal{C}(G)$ be a θ-strongly cuspidal function. Then, we have*

$$\lim_{N\to\infty}J_{N,T}(f)=\frac{(-1)^d}{2^d}\int_T D^{G_1}(t)^2\Phi_{M,\theta}\left(\begin{pmatrix}t & \\ & t\end{pmatrix},f\right)\omega(t)^{-1}dt$$

where $\Phi_{M,\theta}\left(\begin{pmatrix}t & \\ & t\end{pmatrix},f\right)$ is the 'singular θ-weighted orbital integral' introduced in §5.5.

Proof. By the same argument as in the proof of [6, Proposition 5.4.1], except that we replace Lemma 2.9.4 and Proposition 5.3.1 of *loc. cit.* by Proposition 5.4 and Proposition 5.7 of this paper, we have

$$\begin{aligned}\lim_{N\to\infty} J_{N,T}(f) &= \frac{1}{2^d}\int_T D^{G_1}(t)\int_{T\times T\backslash G_1\times G_1}\int_{\mathfrak{g}_1} f^K\left(\begin{pmatrix} g^{-1} & \\ & h^{-1}\end{pmatrix}\begin{pmatrix} t & \\ & t\end{pmatrix}\begin{pmatrix} 1 & X \\ & 1\end{pmatrix}\begin{pmatrix} g & \\ & h\end{pmatrix}\right)\\ &\quad v_{M,\theta}(-\mathcal{Z}(g,h,X))dX\frac{dg}{\nu(g)^n}\nu(h)^n dh\omega(t)^{-1}dt\\ &= \frac{(-1)^d}{2^d}\int_T D^{G_1}(t)\int_{T\times T\backslash G_1\times G_1}\int_{\mathfrak{g}_1} f^K\left(\begin{pmatrix} g^{-1} & \\ & h^{-1}\end{pmatrix}\begin{pmatrix} t & \\ & t\end{pmatrix}\begin{pmatrix} 1 & X \\ & 1\end{pmatrix}\begin{pmatrix} g & \\ & h\end{pmatrix}\right)\\ &\quad w_{M,\theta}(g,h,X)dX\frac{dg}{\nu(g)^n}\nu(h)^n dh\omega(t)^{-1}dt\\ &= \frac{(-1)^d}{2^d}\int_T D^{G_1}(t)^2\Phi_{M,\theta}\left(\begin{pmatrix} t & \\ & t\end{pmatrix}, f\right)\omega(t)^{-1}dt.\end{aligned}$$

□

5.9 Proof of the geometric side

The geometric side of Theorem 3.1 now follows immediately from 5.1.1, Proposition 5.11 and the following proposition:

Proposition 5.12. *Assume that the function $f\in\mathcal{C}(G)$ is strongly cuspidal. Then,*

(i) If T is not elliptic in G_1 (i.e. $d>1$), then

$$\Phi_{M,\theta}\left(\begin{pmatrix} t & \\ & t\end{pmatrix}, f\right) = 0$$

for all $t\in T\cap G_{1,\mathrm{reg}}$.

(ii) If T is elliptic in G_1 (i.e. $d=1$), then

$$\Phi_{M,\theta}\left(\begin{pmatrix} t & \\ & t\end{pmatrix}, f\right) = -2c_f\begin{pmatrix} t & \\ & t\end{pmatrix}$$

for all $t\in T\cap G_{1,\mathrm{reg}}$.

Proof. (i) By Proposition 5.6, it suffices to show that

$$\Phi_{M,\theta}\left(\begin{pmatrix} t_1 & \\ & t_2\end{pmatrix}, f\right) = 0$$

for all $\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \in (T \times T) \cap G_{\mathrm{reg}}$. This follows from the descent formula 2.3.6 which gives

$$\Phi_{M,\theta}\left(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}, f\right) = \sum_{L \in \mathcal{L}(M)} d^G_{M,\theta}(L) \Phi^Q_M \left(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}, f\right);$$

and the fact that for $Q \neq G$, since f is strongly cuspidal, we have (see §2.5)

$$\Phi^Q_M \left(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}, f\right) = 0$$

for all $\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \in (T \times T) \cap G_{\mathrm{reg}}$ (Note that here $d^G_{M,\theta}(G) = 0$ since $\mathcal{A}^{G,\theta}_M \neq 0$).

(ii) By Proposition 5.6 and 2.5.1, it suffices to notice that

$$\Phi_{M,\theta}\left(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}, f\right) = \Phi_M\left(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}, f\right), D^G\begin{pmatrix} t & \\ & t \end{pmatrix}^{1/2} = D^{G_1}(t)^2 \text{ and } |W(G_x, T_{x,\mathrm{qd}})| = 2$$

for all $\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \in (T \times T) \cap G_{\mathrm{reg}}$ and all $t \in T \cap G_{1,\mathrm{reg}}$ where we have set $x = \begin{pmatrix} t & \\ & t \end{pmatrix}$. □

6 Applications

In this section, we give applications of the multiplicity formula we proved in Proposition 3.3. In Section 6.1, we study the behavior of the multiplicities under the local Jacquet-Langlands correspondence. We also study the relations between the multiplicities and the local exterior square L-function. Then in Section 6.2, we study the relations between generalized Shalika models and Ginzburg-Rallis models. We keep the notation introduced in Section 3.1.

6.1 The multiplicities over L-packets

In this subsection, we are going to prove the main theorems (Theorem 1.2 and Theorem 1.3) of this paper. Let $\mathcal{A}'$ be another degree n central simple algebra over F. Set $G' := \mathrm{GL}_2(\mathcal{A}')$ and define subgroups H'_0, N', $H' := H'_0 \rtimes N'$ analogous to the subgroups H_0, N and H of G. We define a character $\xi' : N' \to \mathbf{C}^\times$ in the same way as the character ξ of N and we extend it to a character of H' trivial on H'_0. We also identify the character $\omega : F^\times \to \mathbf{C}^\times$ with a character of H' by composition with projection $H' \to H'_0$ and the reduced norm. This gives us a character $\omega \otimes \xi'$ of H'. For all $\pi' \in \mathrm{Irr}(G')$, we set

$$m(\pi', \omega) := \dim \mathrm{Hom}_{H'}(\pi', \omega \otimes \xi').$$

We want to prove the following theorem.

Theorem 6.1. *Let $\pi \in \Pi_2(G)$ and $\pi' \in \Pi_2(G')$ which correspond to each other under the local Jacquet-Langlands correspondence (see [9]). Then, we have*

$$m(\pi, \omega) = m(\pi', \omega).$$

Proof. If the central characters of π and π' does not coincide with ω^n, then by looking at the action of the center, we see that $m(\pi, \omega) = m(\pi', \omega) = 0$. Assume now that the central characters of π and π' are equal to ω^n. Set $G_1 := \mathcal{A}^\times$ and $G_1' := (\mathcal{A}')^\times$. Then, by the multiplicity formula of Proposition 3.3, we have

$$m(\pi, \omega) = \sum_{T \in \mathcal{T}_{\mathrm{ell}}(G_1)} |W(G_1, T)|^{-1} \int_{A_{G_1} \backslash T} D^{G_1}(t)^2 c_\pi \begin{pmatrix} t & \\ & t \end{pmatrix} \omega(t)^{-1} dt,$$

$$m(\pi', \omega) = \sum_{T' \in \mathcal{T}_{\mathrm{ell}}(G_1')} |W(G_1', T')|^{-1} \int_{A_{G_1'} \backslash T'} D^{G_1'}(t')^2 c_{\pi'} \begin{pmatrix} t' & \\ & t' \end{pmatrix} \omega(t')^{-1} dt'.$$

There is a natural bijection $\mathcal{T}_{\mathrm{ell}}(G_1) \simeq \mathcal{T}_{\mathrm{ell}}(G_1')$ which sends $T \in \mathcal{T}_{\mathrm{ell}}(G_1)$ to the unique torus $T' \in \mathcal{T}_{\mathrm{ell}}(G_1')$ such that $T \simeq T'$. Moreover, such an isomorphism can be obtained as conjugation by an element in $G_1(\overline{F}) \simeq G_1'(\overline{F})$ where $G_1(\overline{F})$ and $G_1'(\overline{F})$ denote the groups of points of G_1, G_1' over a fixed algebraic closure $\overline{F}$ of F and the isomorphism $G_1(\overline{F}) \simeq G_1'(\overline{F})$ is induced by an $\overline{F}$-isomorphism $\mathcal{A} \otimes_F \overline{F} \simeq \mathcal{A}' \otimes_F \overline{F}$. Let $T \in \mathcal{T}_{\mathrm{ell}}(G_1)$ and $T' \in \mathcal{T}_{\mathrm{ell}}(G_1')$ corresponding to each other and fix an isomorphism $T \simeq T'$ as before (i.e. induced from conjugation by an element over the algebraic closure). Then, conjugation by the same element induces an isomorphism $W(G_1, T) \simeq W(G_1', T')$. Let $t \in T$ and denote by t' its image in T'. We easily check that $D^{G_1}(t)^2 = D^{G_1'}(t')^2$ and $\omega(t) = \omega(t')$. Moreover, if $t \in T \cap G_{1,\mathrm{reg}}$, then $t' \in T' \cap G_{1,\mathrm{reg}}'$; and by 2.2.2, we have

$$c_\pi \begin{pmatrix} t & \\ & t \end{pmatrix} = \frac{1}{2} D^G \begin{pmatrix} t & \\ & t \end{pmatrix}^{-1/2} \lim_{\lambda \in F^\times \to 1} D^G \begin{pmatrix} t & \\ & \lambda t \end{pmatrix}^{1/2} \Theta_\pi \begin{pmatrix} t & \\ & \lambda t \end{pmatrix},$$

$$c_{\pi'} \begin{pmatrix} t' & \\ & t' \end{pmatrix} = \frac{1}{2} D^{G'} \begin{pmatrix} t' & \\ & t' \end{pmatrix}^{-1/2} \lim_{\lambda \in F^\times \to 1} D^{G'} \begin{pmatrix} t' & \\ & \lambda t' \end{pmatrix}^{1/2} \Theta_{\pi'} \begin{pmatrix} t' & \\ & \lambda t' \end{pmatrix}.$$

Once again we can easily check that

$$D^G \begin{pmatrix} t & \\ & t \end{pmatrix} = D^{G'} \begin{pmatrix} t' & \\ & t' \end{pmatrix}, \quad D^G \begin{pmatrix} t & \\ & \lambda t \end{pmatrix} = D^{G'} \begin{pmatrix} t' & \\ & \lambda t' \end{pmatrix}$$

for all $\lambda \in F^\times$. Furthermore, by the relations of the distribution characters under the local Jacquet-Langlands correspondence (see [9]), we also have

$$\Theta_\pi\begin{pmatrix} t & \\ & \lambda t\end{pmatrix} = \Theta_{\pi'}\begin{pmatrix} t' & \\ & \lambda t'\end{pmatrix}$$

for all $\lambda \in F^\times$. This shows that

$$c_\pi\begin{pmatrix} t & \\ & t\end{pmatrix} = c_\pi\begin{pmatrix} t' & \\ & t'\end{pmatrix}.$$

All in all, we get the equality

$$\begin{aligned}&|W(G_1,T)|^{-1}\int_{A_{G_1}\backslash T} D^{G_1}(t)^2 c_\pi\begin{pmatrix} t & \\ & t\end{pmatrix}\omega(t)^{-1}dt\\ =&|W(G_1',T')|^{-1}\int_{A_{G_1'}\backslash T'} D^{G_1'}(t')^2 c_{\pi'}\begin{pmatrix} t' & \\ & t'\end{pmatrix}\omega(t')^{-1}dt'.\end{aligned}$$

Summing this over all $T \in \mathcal{T}_{\mathrm{ell}}(G_1)$, we get the desired equality. □

Then we study the relations between the multiplicity and the local exterior square L-function. Assume that $\mathcal{A}' = M_n(F)$, $\pi' \in \Pi_2(G')$ and $\omega = \mathbf{1}$ is the trivial character of H_0' in which case we will simply set

$$m(\pi') := m(\pi', \mathbf{1})$$

The following theorem seems to be well-known but apparently only half of its proof has appeared in the literature (by combining results of different sources). For sake of completeness, we provide a proof of the other half. The exterior square L-function appearing in the statement is the Artin local L-function defined through the local Langlands correspondence of Harris-Taylor [16], Henniart [19] and Scholze [35].

Theorem 6.2. *With the notations above, the local exterior square L-function $L(s, \pi', \wedge^2)$ has a pole at $s = 0$ if and only if $m(\pi') = 1$.*

Proof. By the multiplicity one result of [23] (see [8] for a more general result), in order to prove the theorem, it is enough to show that the local exterior square L-function $L(s, \pi', \wedge^2)$ has a pole at $s = 0$ if and only if $m(\pi') \neq 0$. This has been proved in [21] when π' is supercuspidal. When π' is a discrete series, by [24, Theorem 4.3, Corollary 4.4] and [25, Corollary 1.4], if the local exterior square L-function $L(s, \pi', \wedge^2)$ has a pole at $s = 0$, then $m(\pi') \neq 0$. Hence it remains to prove the other direction. We will follow the method introduced in [21].

Assume that $m(\pi') \neq 0$. Then, of course, the central character of π' is trivial. Let $\mathrm{SO}_{4n}(F)$ be the F-split special orthogonal group of rank $2n$, and let $P = MN$ be a Siegel parabolic subgroup of $\mathrm{SO}_{4n}(F)$ with Levi component M isomorphic to $\mathrm{GL}_{2n}(F)$. For $s \in \mathbb{C}$, let $\Pi(s) = I_P^{\mathrm{SO}_{4n}(F)}(\pi' \otimes |\det|^s)$ be the normalized parabolic induction of $\pi' \otimes |\det|^s$ to $\mathrm{SO}_{4n}(F)$. By [22, Theorem 3.1] and [21, Proposition 2.3], together with the assumption that $m(\pi') \neq 0$,

we know that the representation $\Pi(\frac{1}{2})$ of $\mathrm{SO}_{4n}(F)$ is reducible. Hence it remains to prove the following lemma.

Lemma 6.3. *With the notations above, the representation $\Pi(\frac{1}{2})$ is reducible if and only if then the local exterior square L-function $L(s,\pi',\wedge^2)$ has a pole at $s=0$.*

Proof. Let $N_M \subset M$ be the unipotent radical of a Borel subgroup of M and set $N_0 := N_M \ltimes N$. Then, N_0 is the unipotent radical of a Borel subgroup of $\mathrm{SO}_{4n}(F)$. Let ξ_0 be a generic character on N_0 and set $\xi_M := \xi_0 \mid_{N_M}$ for the restriction of ξ_0 to N_M. As π' is square-integrable it is in particular generic with respect to (N_M, ξ_M). By [12, Lemma B.2] and the Standard Module Conjecture [17], we have that $\Pi(\frac{1}{2})$ is reducible if and only if $\gamma^{Sh}(2s,\pi',\wedge^2,\psi) = \epsilon(2s,\pi',\wedge^2,\psi)\frac{L(1-2s,\pi',\wedge^2)}{L(2s,\pi',\wedge^2)}$ has a pole at $s=1/2$ where $\gamma^{Sh}(s,\pi',\wedge^2,\psi)$ denotes the gamma factor defined by Shahidi (with respect to any non-trivial additive character ψ since it doesn't change the existence or not of a pole at $s=1/2$). Since π' is square-integrable, $L(s,\pi',\wedge^2)$ has no pole with positive real part and in particular at $s=1$. Hence, by the main result of [20], $\Pi(\frac{1}{2})$ is reducible if and only if $L(s,\pi',\wedge^2)$ has a pole at $s=0$. □

□

As the Jacquet-Langlands correspondence preserves L-function, we get a Corollary from the two theorems above which proves Theorem 1.3.

Corollary 6.4. *For all $\pi \in \Pi_2(G)$, the local exterior square L-function $L(s,\pi,\wedge^2)$ has a pole at $s=0$ if and only if $m(\pi)=1$.*

Remark 6.5. *If ω is the square of a character of $F^\times$, then by twisting π by a suitable character of G together with the above corollary, we see that for $\pi \in \Pi_2(G)$, the local twisted exterior square L-function $L(s,\pi,\wedge^2\otimes\omega^{-1})$ has a pole at $s=0$ if and only if $m(\pi,\omega)=1$. The same should be true in general (i.e. when ω is not a square of a character of $F^\times$) although it would necessitate to redo the works of [21], [22], [24] and [25] by introducing this twist. However, note that in the case where $n=2$, this more general result has been proved in [13, Theorem 1.5].*

6.2 The relations between the generalized Shalika model and the Ginzburg-Rallis model

In this subsection, we will consider the particular cases where $\mathcal{A} = M_2(F)$ or a non-split quaternion algebra D over F. More precisely, we will give some relations between the multiplicities for the Ginzburg-Rallis model of $\mathrm{GL}_3(D)$ (resp. $\mathrm{GL}_6(F)$) and for the generalized Shalika model for $\mathrm{GL}_2(D)$ (resp. $\mathrm{GL}_4(F)$).

Assume ω unitary. Let π_D be a tempered representation of $\mathrm{GL}_2(D)$ with central character ω^2 and let Π_D be the normalized parabolic induction of the representation $\pi_D \boxtimes \omega^{-1}$ of the Levi subgroup $\mathrm{GL}_2(D)\times\mathrm{GL}_1(D)$ to $\mathrm{GL}_3(D)$ (where again we have identified ω^{-1} with a

character of $\mathrm{GL}_1(D)$ by composition with the reduced norm). Then Π_D is a tempered representation of $\mathrm{GL}_3(D)$ with trivial central character. Let $m_{GR}(\Pi_D)$ be the multiplicity of the Ginzburg-Rallis model (with trivial character). We refer the readers to the previous papers [41] and [42] of the second author for the definition of the Ginzburg-Rallis model. The following theorem tells us a relation between the Ginzburg-Rallis model for $\mathrm{GL}_3(D)$ and the generalized Shalika model for $\mathrm{GL}_2(D)$.

Theorem 6.6. *With the notations above, we have*

$$m(\pi_D, \omega) = m_{GR}(\Pi_D).$$

Proof. In [42, Section 8.1], we proved a multiplicity formula $m_{GR}(\Pi_D) = m_{geom}(\Pi_D)$ for the Ginzburg-Rallis model. By the definition of $m_{geom}(\Pi_D)$ in [42], together with the definition of Π_D above and [39, Lemme 2.3], we can easily show that

$$m_{geom}(\Pi_D) = \sum_{T \in \mathcal{T}_{\mathrm{ell}}(\mathrm{GL}_1(D))} |W(\mathrm{GL}_1(D), T)|^{-1} \int_{T/A_G} D^{\mathrm{GL}_1(D)}(t)^2 c_{\pi_D} \begin{pmatrix} t & \\ & t \end{pmatrix} \omega(t)^{-1} dt.$$

Hence by Proposition 3.3, we get $m(\pi_D) = m(\Pi_D)$, and this proves the theorem. □

Next, we will prove a relation between the Ginzburg-Rallis model for $\mathrm{GL}_6(F)$ and the Shalika model for $\mathrm{GL}_4(F)$. Let π be a discrete series of $\mathrm{GL}_4(F)$ with central character ω^2, and let π_D be the Jacquet-Langlands lift of π to $\mathrm{GL}_2(D)$. Let $St(\omega^{-1})$ be the Steinberg representation of $\mathrm{GL}_2(F)$ twisted by the character ω^{-1} and let Π be the normalized parabolic induction of the representation $\pi \boxtimes St(\omega^{-1})$ of the Levi subgroup $\mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$ to $\mathrm{GL}_6(F)$. Then Π is a tempered representation of $\mathrm{GL}_6(F)$ with trivial central character. Let $m_{GR}(\Pi)$ denote the multiplicity of Π relative to the Ginzburg-Rallis model (with trivial character) of $\mathrm{GL}_6(F)$.

Theorem 6.7. *With the notations above, we have*

$$m(\pi, \omega) + m_{GR}(\Pi) = 1.$$

Proof. Let Π_D be the Jacquet-Langlands lift of Π to $\mathrm{GL}_3(D)$. Then Π_D is the parabolic induction of the representation $\pi_D \otimes \omega^{-1}$ of the Levi subgroup $\mathrm{GL}_2(D) \times \mathrm{GL}_1(D)$. By Theorem 6.6 and Theorem 6.1, we have

$$m_{GR}(\Pi_D) = m(\pi_D, \omega) = m(\pi, \omega).$$

On the other hand, in [41] and [42], we have proved that $m_{GR}(\Pi) + m_{GR}(\Pi_D) = 1$. The theorem follows. □

Remark 6.8. *It is worth to mention that Theorem 6.6 can also be directly proved by Mackey theory. On the other hand, it is not clear to us how to prove Theorem 6.7 by Mackey theory (we can however show that the sum of the multiplicities it at least one in this way).*

For the rest part of this section, we are going to prove some partial results for the epsilon dichotomy conjecture of the Ginzburg-Rallis model. Let Π be an irreducible generic representation of $\mathrm{GL}_6(F)$ with central character β^2 where β is some character on $F^\times$, and let $m_{GR}(\Pi)$ be the multiplicity relative to the Ginzburg-Rallis model with character β. We first recall the epsilon dichotomy conjecture from [42].

Conjecture 6.9. *With the notations above, we have*

$$\begin{aligned} m_{GR}(\Pi) = 1 \quad &\Longleftrightarrow \quad \epsilon(1/2, (\wedge^3\Pi) \otimes \beta^{-1}) = 1, \\ m_{GR}(\Pi) = 0 \quad &\Longleftrightarrow \quad \epsilon(1/2, (\wedge^3\Pi) \otimes \beta^{-1}) = -1. \end{aligned}$$

In [42], we have proved the conjecture when Π is not a discrete series, or the parabolic induction of a discrete series of $\mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$. Now we are going to prove the conjecture when Π is the parabolic induction of a discrete series $\pi \boxtimes \pi_0$ of $\mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$ with π_0 being a twist of the Steinberg representation of $\mathrm{GL}_2(F)$.

Let $\pi_0 = St(\alpha)$ for some character $\alpha : F^\times \to \mathbb{C}^\times$. Then the central character of π_0 is α^2, which implies that the central character of π is $\alpha^{-2}\beta^2$. Let ϕ_π (resp. ϕ_{π_0}) be the Langlands parameter of π (resp. π_0). Then the Langlands parameter ϕ_Π of Π is just $\phi_\pi \oplus \phi_{\pi_0}$.

Proposition 6.10. *With the notations above, we have*

$$\epsilon(1/2, (\wedge^3\Pi) \otimes \beta^{-1}) = \epsilon(1/2, \wedge^2(\phi_\pi) \otimes \phi_{\pi_0} \otimes \beta^{-1}). \tag{6.2.1}$$

Proof. Since $\phi_\Pi = \phi_\pi \oplus \phi_{\pi_0}$, we have

$$\wedge^3(\phi_\Pi) = (\wedge^2(\phi_\pi) \otimes \phi_{\pi_0}) \oplus (\phi_\pi \otimes \det(\phi_{\pi_0})) \oplus (\wedge^3(\phi_\pi)).$$

Since the central character of π is $\alpha^{-2}\beta^2$ and the central character of π_0 is α^2, we have $(\wedge^3(\phi_\pi) \otimes \beta^{-1})^\vee = \phi_\pi \otimes \alpha^2 \otimes \beta^{-1} = \phi_\pi \otimes \det(\phi_{\pi_0}) \otimes \beta^{-1}$. This implies that

$$\begin{aligned} \epsilon(1/2, (\wedge^3\Pi) \otimes \beta^{-1}) &= \epsilon(1/2, \wedge^2(\phi_\pi) \otimes \phi_{\pi_0} \otimes \beta^{-1}) \epsilon(1/2, \phi_\pi \otimes \det(\phi_{\pi_0}) \otimes \beta^{-1}) \\ &\quad \epsilon(1/2, \wedge^3(\phi_\pi) \otimes \beta^{-1}) \\ &= \det(\wedge^3(\phi_\pi) \otimes \beta^{-1})(-1) \times \epsilon(1/2, \wedge^2(\phi_\pi) \otimes \phi_{\pi_0} \otimes \beta^{-1}) \\ &= (\det(\phi_\pi))^3(-1)\beta^{-4}(-1) \times \epsilon(1/2, \wedge^2(\phi_\pi) \otimes \phi_{\pi_0} \otimes \beta^{-1}) \\ &= \epsilon(1/2, \wedge^2(\phi_\pi) \otimes \phi_{\pi_0} \otimes \beta^{-1}). \end{aligned}$$

This proves the proposition. □

Now up to twist π_0 by the character β^{-1}, we may assume that $\beta = 1$. This is allowable since twist by characters will not change the multiplicity and the epsilon factor. In fact, by the proposition above, we know that the epsilon factor will not be changed if we twist π_0 by characters. As for the multiplicity, by [42, Corollary 5.15], the multiplicity $m_{GR}(\Pi)$ for the Ginzburg-Rallis model is equal to the multiplicity $m_{MM}(\pi \otimes \pi_0)$ of the middle model (we refer the readers to [42, Appendix A] for the definition of the middle model). But it is easy

to see from the definition that the multiplicity of the middle model will not be changed if we twist π_0 by characters.

Assuming that $\beta = 1$, by Theorem 6.7, we have $m_{GR}(\Pi) + m(\pi, \alpha^{-1}) = 1$. Let π_D be the Jacquet-Langlands lift of π to $\mathrm{GL}_2(D)$. By Theorem 6.1, we have $m(\pi, \alpha^{-1}) = m(\pi_D, \alpha^{-1})$. Combining with the proposition above, we see that in order to prove the epsilon dichotomy conjecture for Π, it is enough to prove that

$$\begin{aligned} m(\pi_D, \alpha^{-1}) = 0 \quad &\Longleftrightarrow \quad \epsilon(1/2, \wedge^2(\phi_\pi) \otimes \phi_{\pi_0}) = 1, \\ m(\pi_D, \alpha^{-1}) = 1 \quad &\Longleftrightarrow \quad \epsilon(1/2, \wedge^2(\phi_\pi) \otimes \phi_{\pi_0}) = -1. \end{aligned}$$

But the above relations have already been proved in [13, Theorem 1.5(2)]. This finishes the proof of the epsilon dichotomy conjecture for Π.

Remark 6.11. *It is worth to mention that the cases so far proven of the epsilon dichotomy conjecture of the Ginzburg-Rallis model all reduce to the epsilon dichotomy conjecture of the Gross-Prasad model.*

7 A connection to the local r-trace formula

In this section, we will rewrite the local trace formula for the Shalika model in terms of the local r-trace formula. In Langlands' proposal [28] for beyond endoscopy, one of the most important ingredients is a global r-trace formula (the name "r-trace formula" was first introduced by Arthur in his note [3] for beyond endoscopy). To be more specific, let G be a connected reductive group defined over a number field k, and let r be a finite dimensional algebraic representation of the L-group of G. For any cuspidal automorphic representation π of Ramanujan type of $G(\mathbb{A}_k)$, we let $L(s, \pi, r)$ be the associated global automorphic L-function. For a given test function f on $G(\mathbb{A}_k)$, we let $S^1_{cusp}(f)$ be the cuspidal part of the stable Arthur-Selberg trace formula (we refer the reader to [3] for more details on this; let us just say here that $S^1_{cusp}(f)$ ought to be a sum over every *cuspidal global L-packet* of $G(\mathbb{A}_k)$ of a certain *stable* linear combination of the characters associated to this global L-packet). Then we define as in [3], $S^r_{cusp}(f)$ as the generalization of $S^1_{cusp}(f)$ in which the stable multiplicities of representations π that occur in $S^1_{cusp}(f)$ are weighted by the order of the pole

$$m_r(\pi) := -ord_{s=1}(L(s, \pi, r))$$

at $s = 1$ of the automorphic L-function $L(s, \pi, r)$. The goal of the r-trace formula is to find a decomposition of $S^r_{cusp}(f)$ in terms of stable distributions on some smaller groups G'. Here G' should run over elliptic "beyond endoscopic data". Like the theory of endoscopy, the most important step is to find a geometric expansion for the distribution $S^r_{cusp}(f)$.

Guided by the same philosophy, we can also consider some kind of *local r-trace formula*. Let G be a connected reductive group defined over a local field F and r, as before, be a finite dimensional algebraic representation of the L-group of G. For π an irreducible admissible representation of G, we denote as usual by $L(s, \pi, r)$ the associated local L-function (which

is well-defined as soon as the local Langlands correspondence is known for G). For a test function f on G, we can define the local analog of the distribution $S^r_{cusp}(f)$ to be

$$I^r_{disc}(f) := \sum_{\pi \in \Pi_2(G)} m_r(\pi) \operatorname{Tr}(\pi(f)) \tag{7.0.1}$$

where $m_r(\pi) := -ord_{s=0}(L(s, \pi, r))$ is the order of the poles at $s = 0$ of the L-function $L(s, \pi, r)$. When G has non-trivial split center, we should actually fix a central character χ of A_G and only consider in the sum above the square-integrable representations with central character χ i.e. sum over $\Pi_2(G, \chi)$ instead. As in the global case, the main problem is to find a geometric expansion for $I^r_{disc}(f)$.

For the rest part of this section, we will discuss a special case of such local r-trace formula. We consider the case where F is a p-adic field, $G = \mathrm{GL}_{2n}(F)$, $r = \wedge^2$ is the exterior square representation of the L-group ${}^LG = \mathrm{GL}_{2n}(\mathbb{C})$ and we fix the central character of our representations to be trivial. For $f \in {}^\circ\mathcal{C}(G, 1)$, by applying the local trace formula of Theorem 3.1 in the case where ω is trivial, we will give a geometric expansion for $I^r_{disc}(f)$. Let us denote by $I_{\mathrm{geom}}(f)$ and $I_{\mathrm{spec}}(f)$ the geometric and spectral sides of this trace formula respectively (i.e. the left and right hand sides of the identity of Theorem 3.1 respectively). Then the spectral side reads

$$I_{\mathrm{spec}}(f) = \sum_{\pi \in \Pi_2(G,1)} \operatorname{Tr}(\pi(f)) m(\pi^\vee).$$

where, for simplicity, we have set $m(\pi^\vee) := m(\pi^\vee, \mathbf{1})$. As every $\pi \in \Pi_2(G, 1)$ is unitary, it is easy to see that $m(\pi^\vee) = m(\pi)$. Moreover, by Theorem 6.2, the multiplicity $m(\pi)$ is nonzero if and only if the exterior square L-function $L(s, \pi, \wedge^2)$ has a pole at $s = 0$. Since π is a discrete series, the order of the pole of $L(s, \pi, \wedge^2)$ at $s = 0$ is either 0 or 1. On the other hand, we know that $m(\pi) \leq 1$ (see [23]). Therefore, for all discrete series $\pi \in \Pi_2(G, 1)$, we have

$$m(\pi^\vee) = -ord_{s=0}(L(s, \pi, r)) = m_r(\pi).$$

This implies that

$$I_{\mathrm{spec}}(f) = \sum_{\pi \in \Pi_2(G,1)} \operatorname{Tr}(\pi(f)) m_r(\pi) = I^r_{\mathrm{disc}}(f).$$

Therefore, by Theorem 3.1 we have

$$I^r_{\mathrm{disc}}(f) = I_{\mathrm{geom}}(f).$$

We summarize the above discussion in the following proposition.

Proposition 7.1. *We have the identity*

$$I_{geom}(f) = I^r_{disc}(f)$$

for every $f \in {}^\circ\mathcal{C}(G, 1)$ where $G = \mathrm{GL}_{2n}(F)$, $I_{\mathrm{geom}}(f)$ denotes the left-hand side of Theorem 3.1 and $I^r_{disc}(f)$ stands for the local r-trace formula as defined above for the exterior square representation $r = \wedge^2$ of ${}^LG = \mathrm{GL}_{2n}(\mathbb{C})$.

Remark 7.2. *In general, the local multiplicity problem for many spherical pairs are closely related to the Langlands functoriality and the poles of some L-functions $L(s,\pi,r)$. Hence if we can prove the local trace formula for these models, we can find the geometric expansion of the corresponding local r-trace formulas.*

A Slight generalization of a result of Mœglin and Waldspurger

Let F be a nonarchimedean field of characteristic zero with ring of integer $\mathcal{O}_F$ and normalized absolute value $|.|$. We fix henceforth an uniformizer $\varpi_F \in \mathcal{O}_F$. As in the core of this paper, we will abuse notations by denoting algebraic groups over F and the corresponding group of F-points by the same letter. We will do the same for Lie algebras.

Let G be a connected reductive group over F with Lie algebra $\mathfrak{g}$. Fix a G-invariant non-degenerate symmetric bilinear pairing $\langle .,. \rangle : \mathfrak{g} \times \mathfrak{g} \to F$ and a nontrivial additive character $\psi : F \to \mathbb{C}^\times$. Let $P_0 = M_0 N_0$ be a minimal parabolic subgroup of G with unipotent radical N_0 and a fixed Levi component M_0. Let $\overline{P}_0 = M_0 \overline{N}_0$ be the opposite parabolic subgroup and $A_0 \subset M_0$ the maximal central split torus. Let $\mathfrak{n}_0$, $\overline{\mathfrak{n}}_0$ and $\overline{\mathfrak{p}}_0$ denote the Lie algebra of N_0, $\overline{N}_0$ and $\overline{P}_0$ respectively. Let Δ_0 be the set of simple roots of A_0 in $\mathfrak{n}_0$ and for all $\alpha \in \Delta_0$ fix a nonzero vector $Y_{-\alpha}$ in the root subspace $\mathfrak{g}_{-\alpha}$ corresponding to $-\alpha$. Set $Y := \sum_{\alpha \in \Delta_0} Y_{-\alpha} \in \overline{\mathfrak{n}}_0$. It is well-known that

$$[Y, \overline{\mathfrak{p}}_0] = \overline{\mathfrak{n}}_0. \tag{A.0.1}$$

The exponential map induces a regular isomorphism $\exp : \mathfrak{n}_0 \simeq N_0$. Let $\log : N_0 \to \mathfrak{n}_0$ be its inverse and set

$$\xi(n) := \psi(\langle Y, \log n \rangle), \quad n \in N_0.$$

Then ξ is a character of N_0 which is generic (i.e. with a stabilizer in M_0 of minimal dimension). We will denote by $M_{0,\xi}$ the stabilizer of ξ in M_0 (i.e. the centralizer of Y in M_0).

Let π be a smooth irreducible representation of G and Θ_π its Harish-Chandra character. Recall that for all semi-simple element $x \in G$ there is a local expansion (see [15, Theorem 16.2])

$$\Theta_\pi(x \exp(X)) = \sum_{\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_x)} c_{\pi,\mathcal{O}}(x) \widehat{j}(\mathcal{O}, X)$$

for $X \in \mathfrak{g}_{x,\mathrm{reg}}$ sufficiently close to 0.

For all $x \in M_{0,\xi}$, denote by $\mathcal{O}_x$ the G_x-adjoint orbit of Y in $\mathfrak{g}_x$. Then $\mathcal{O}_x$ is an element of $\mathrm{Nil}(\mathfrak{g}_x)$ which is maximal for the order defined by $\mathcal{O}' \leqslant \mathcal{O} \Leftrightarrow \mathcal{O}' \subset \overline{\mathcal{O}}$ where $\overline{\mathcal{O}}$ denotes closure of $\mathcal{O}$ for the F-analytic topology.

Let W be the space of π and let $W(N_0, \xi)$ be the subspace generated by all vectors of the form $\pi(n)v - \xi(n)v$ for $v \in V$ and $n \in N_0$. The coinvariant space $W/W(N_0, \xi)$ carries a natural representation of $M_{0,\xi}$ that we shall denote by $\pi_{N_0,\xi}$. In [29] it was shown that (see [36] for the case of residual characteristic 2)

$$\dim \pi_{N_0,\xi} = c_{\pi,\mathcal{O}_1}(1).$$

The goal of this appendix is to extend slightly this result by using the same techniques. More precisely we will prove:

Proposition A.1. *For all $x \in M_{0,\xi}$ we have*

$$\operatorname{Tr} \pi_{N_0,\xi}(x) = D^{G/M_0}(x)^{1/2} c_{\pi,\mathcal{O}_x}(x)$$

where we have set

$$D^{G/M_0}(x) := D^G(x) D^{M_0}(x)^{-1}.$$

Proof. In order to include the case of residual characteristic 2, we will use [36] as our main reference (which however follows very closely [29]). Let $x \in M_{0,\xi}$. Note that $\mathrm{Ad}(x)$ is semi-simple and compact (indeed this follows from the facts that M_0 is anisotropic modulo A_0 and $M_{0,\xi} \cap A_0$ is contained in the center of G). To agree with the conventions of [36], we will assume that ψ is unramified (i.e. its conductor is $\mathcal{O}_F$). This is no loss in generality since we can always scale ψ so that it become unramified and up to scaling $\langle ., . \rangle$ by the corresponding inverse factor the character ξ doesn't change and the same holds for the coefficient $c_{\pi,\mathcal{O}_x}(x)$ as we easily check from our normalizations. Let φ be the sum of the positive coroots of A_0 with respect to P_0. Then we have $\varphi(s) Y \varphi(s)^{-1} = s^{-2} Y$ for all $s \in F^\times$. The adjoint action of $\varphi(F^\times)$ on $\mathfrak{g}$ induces a decomposition

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$$

where $\mathfrak{g}_i := \{X \in \mathfrak{g};\ \varphi(s) X \varphi(s)^{-1} = s^i X \ \forall s \in F^\times\}$. Set $\mathfrak{g}^- := \bigoplus_{i \leqslant 0} \mathfrak{g}_i$ and $\mathfrak{g}^2 := \bigoplus_{i \geqslant 2} \mathfrak{g}_i$. Then we have $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{g}^2$ and $\mathfrak{n}_0 = \mathfrak{g}^2$(this is because $\mathfrak{g}_1 = 0$). Let $Y^\sharp$ be the centralizer of Y in $\mathfrak{g}$ and V be a complement subspace to $Y^\sharp$ which is invariant by $\varphi(F^\times) M_{0,\xi}$ (in particular it is invariant by $\varphi(F^\times)$). By $\varphi(F^\times)$-invariance, we have a decomposition $V = V^- \oplus V^2$ where $V^- := V \cap \mathfrak{g}^-$ and $V^2 := V \cap \mathfrak{g}^2$. Note that

$$V^2 = \mathfrak{g}^2 = \mathfrak{n}_0. \tag{A.0.2}$$

Indeed, this follows from A.0.1 and the fact that $\langle ., . \rangle$ induces a perfect pairing between $\overline{\mathfrak{n}}_0$ and $\mathfrak{n}_0$.

Since $\mathrm{Ad}(x)$ is semi-simple, we have a decomposition $\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{g}^x$ where $\mathfrak{g}_x$ and $\mathfrak{g}^x$ stand respectively for the kernel and the image of $\mathrm{Ad}(x) - 1$ in $\mathfrak{g}$. As V is $\mathrm{Ad}(x)$-stable, we have a similar decomposition $V = V_x \oplus V^x$. The bilinear form

$$B_Y : (Z, X) \in \mathfrak{g} \times \mathfrak{g} \mapsto B_Y(Z, X) := \langle Y, [Z, X] \rangle$$

is alternating $M_{0,\xi}$-invariant and nondegenerate when restricted to V. Moreover, the decomposition $V = V_x \oplus V^x$ is orthogonal for this alternating form. Set $V_x^- := V_x \cap \mathfrak{g}^-$, $V_x^2 := V_x \cap \mathfrak{g}^2$, $V^{x,-} := V^x \cap \mathfrak{g}^-$ and $V^{x,2} := V^x \cap \mathfrak{g}^2$. Then we have $V_x = V_x^- \oplus V_x^2$ and $V^x = V^{x,-} \oplus V^{x,2}$ (indeed this follows from the facts that V is $\varphi(F^\times)$-stable and x centralizes $\varphi(F^\times)$). Moreover, the form B_Y induces a perfect pairing between V_x^- and V_x^2 on the one hand and between $V^{x,-}$ and $V^{x,2}$ on the other hand. Since $\mathrm{Ad}(x)$ is compact and commutes with $\mathrm{Ad}(\varphi(F^\times))$, we can find a lattice of $L^{x,-}$ of $V^{x,-}$ which is $\mathrm{Ad}(x)$-stable and such that

$$L^{x,-} = \bigoplus_{i \leqslant 0} L^{x,-} \cap \mathfrak{g}_i.$$

Let $L^{x,2}$ be the dual lattice of $V^{x,2}$ with respect to B_Y, that is

$$L^{x,2} := \{X \in V^{x,2};\ B_Y(Z, X) \in \mathcal{O}_F\ \forall Z \in L^{x,-}\}.$$

Similarly, we fix a lattice $L_x^- \subset V_x^-$ with the property that

$$L_x^- = \bigoplus_{i \leqslant 0} L_x^- \cap \mathfrak{g}_i$$

and denote by $L_x^2 \subset V_x^2$ the dual lattice. Finally, we also fix a lattice $L_Y \subset Y^\sharp$ which is $\mathrm{Ad}(x)$-invariant and such that $L_Y = \bigoplus_{i \in \mathbf{Z}} L_Y \cap \mathfrak{g}_i$, $L_Y = L_Y \cap \mathfrak{g}_x \oplus L_Y \cap \mathfrak{g}^x$. We set

$$L := L_Y \oplus L_x^- \oplus L_x^2 \oplus L^{x,-} \oplus L^{x,2}$$

Then, L is a lattice of $\mathfrak{g}$ which is $\mathrm{Ad}(x)$-invariant and satisfies properties (i) and (ii) of [36, §3]. Moreover, by construction we have

$$L = L_x \oplus L^x$$

where $L_x := L \cap \mathfrak{g}_x$ and $L^x := L \cap \mathfrak{g}^x$.

Set $t := \varphi(\varpi_F)$. For all integer n sufficiently large we set

$$G_n := \exp(\varpi_F^n L), \quad G'_n := t^{-n} G_n t^n.$$

When n is large enough, these are compact-open subgroups of G. Again for n sufficiently large, we define two characters $\xi_n : G_n \to \mathbf{C}^\times$ and $\xi'_n : G'_n \to \mathbf{C}^\times$ by

$$\xi_n(\exp(X)) := \psi(\varpi_F^{-2n} \langle Y, X + \frac{1}{2}[X_+, X_-] \rangle), \quad X \in \varpi_F^n L;$$

$$\xi'_n(\gamma) := \xi_n(t^n \gamma t^{-n}), \quad \gamma \in G'_n$$

where $X \mapsto X_-$ and $X \mapsto X_+$ denote respectively the projections onto $\mathfrak{g}^-$ and $\mathfrak{g}^2$ relative to the decomposition $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{g}^2$. These characters are $\mathrm{Ad}(x)$-invariant (since x is in the

centralizer of both Y and t). Moreover, we can easily check, using the Campbell-Hausdorff formula, that for n large enough, the character ξ_n coincide with the character χ_n constructed in [36, Lemma 6]. For all n for which G'_n and ξ'_n are defined, we set

$$W'_n := \{v \in W \mid \ \pi(\gamma)v = \xi'_n(\gamma)v \ \forall \gamma \in G'_n\}.$$

These subspaces are invariant by $\pi(x)$ and by [36, Lemma 8 and Lemma 9.(b)], when n is large enough, the natural projection $W \twoheadrightarrow W/W(N_0, \xi)$ restricts to an isomorphism $W'_n \simeq W/W(N_0, \xi)$. From there it easily follows that

$$\mathrm{Tr}\, \pi_{N_0,\xi}(x) = \mathrm{Tr}\, \pi(x)_{|W'_n} \tag{A.0.3}$$

for all n large enough. Fix a Haar measure dg on G and for n sufficiently large, let $\varphi_n, \varphi'_n \in C_c^\infty(G)$ be the functions defined by

$$\varphi_n(\gamma) = \begin{cases} \frac{1}{\mathrm{vol}(G_n)}\xi_n(\gamma^{-1}), & \text{if } \gamma \in G_n; \\ 0, & \text{otherwise.} \end{cases}$$

$$\varphi'_n(\gamma) := \varphi_n(t^n \gamma t^{-n}).$$

Then $\pi(\varphi'_n)$ is a projection onto W'_n and thus

$$\mathrm{Tr}\, \pi(x)_{|W'_n} = \mathrm{Tr}\, \pi(x)\pi(\varphi'_n) = \mathrm{Tr}\, \pi(x)\pi(t^{-n})\pi(\varphi_n)\pi(t^n) = \mathrm{Tr}\, \pi(L(x)\varphi_n) \tag{A.0.4}$$

where $(L(x)\varphi_n)(\gamma) := \varphi_n(x^{-1}\gamma)$ for all $\gamma \in G$.

Fix open neighborhoods $\omega \subset \mathfrak{g}$ and $\Omega \subset G$ of 0 and 1 respectively such that the exponential map induces an F-analytic isomorphism $\exp : \omega \simeq \Omega$. Let $\log : \Omega \to \omega$ denote the inverse of this map. We fix a Haar measure on $\mathfrak{g}$ such that the exponential map preserves measures locally in a neighborhood of 0 and Haar measures on $\mathfrak{g}^x$ and $\mathfrak{g}_x$ whose product is equal to the Haar measure on $\mathfrak{g}$. Choose $e \geqslant 1$ such that $(1 - \mathrm{Ad}(x^{-1}))L^x \supset \varpi_F^e L^x$ and $\langle Y, L\rangle \subset \varpi_F^{-e}\mathcal{O}_F$. We can find an integer $B \geqslant 0$ satisfying the following conditions:

(A.0.5) The map

$$(\varpi_F^B L^x) \times (\varpi_F^B L_x) \to G : \ \ (Z, X) \mapsto \exp(-x^{-1}Zx)\exp(X)\exp(Z)$$

is an F-analytic isomorphism onto an open neighborhood of 1 contained in Ω and the Jacobian of this map is a constant equal to $D^G(x)$.

(A.0.6) For all $m, m' \geqslant B$, all $Z \in \varpi_F^m L^x$ and all $X \in \varpi_F^{m'} L_x$ we have

$$\log\left(e^{-x^{-1}Zx}e^X e^Z\right) \in X + Z - x^{-1}Zx + \frac{1}{2}[X, Z + x^{-1}Zx] + \frac{1}{2}[Z, x^{-1}Zx] + \varpi_F^{m+m'+e}L.$$

(A.0.7) $[L, L] \subset 2\varpi_F^{2e+1-B}L$.

Assume n is large. By A.0.3 and A.0.4, we have

$$\operatorname{Tr} \pi_{N_0,\xi}(x) = \int_G \Theta_\pi(x\gamma)\varphi_n(\gamma)d\gamma.$$

And by A.0.5, this equals

$$D^G(x) \int_{\varpi_F^B L^x} \int_{\varpi_F^B L_x} \Theta_\pi(x\exp(X))\varphi_n(\exp(-x^{-1}Zx)\exp(X)\exp(Z))dXdZ.$$

By A.0.6 and A.0.7, we check that for $Z \in \varpi_F^B L^x$ and $X \in \varpi_F^B L_x$, we have

$$\exp(-x^{-1}Zx)\exp(X)\exp(Z) \in G_n$$

if and only if $X \in \varpi_F^n L_x$ and $Z - x^{-1}Zx \in \varpi_F^n L^x$ and that in that case, we have

$$\xi_n(\exp(-x^{-1}Zx)\exp(X)\exp(Z)) = \psi\Big(\varpi_F^{-2n}\Big\langle Y, X + Z - x^{-1}Zx + \frac{1}{2}\big([X, Z + x^{-1}Zx] + [Z, x^{-1}Zx] + [X_+ + Z_+ - x^{-1}Z_+x, X_- + Z_- - x^{-1}Z_-x]\big)\Big\rangle\Big).$$

Since x centralizes Y, we have $\langle Y, \mathfrak{g}^x\rangle = 0$ and thus Y is orthogonal to $Z - x^{-1}Zx$, $[X, Z + x^{-1}Zx]$, $[X_+, Z_- - x^{-1}Z_-x]$ and $[Z_+ - x^{-1}Z_+x, X_-]$. Similarly, $\langle Y, \mathfrak{g}_i\rangle = 0$ for $i \neq 2$ so that Y is orthogonal to $[Z_-, x^{-1}Z_-x]$ and $[Z_+, x^{-1}Z_+x]$. Finally, since $\langle ., .\rangle$ is $\mathrm{Ad}(x)$-invariant and x centralizes Y, we have $\langle Y, [x^{-1}Z_+, x^{-1}Z_-x]\rangle = \langle Y, [Z_+, Z_-]\rangle$. From all of these, we get by direct computation that the above expression equals

$$\xi_n(\exp(X))\psi\left(\varpi_F^{-2n} B_Y(Z_-, x^{-1}Z_+x - Z_+)\right).$$

Finally we end up with

$$\begin{aligned}\operatorname{Tr} \pi_{N_0,\xi}(x) =& D^G(x)\operatorname{vol}(L^x)^{-1} \int_{(1-\mathrm{Ad}(x^{-1}))^{-1}L^x} \psi\left(B_Y(Z_-, x^{-1}Z_+x - Z_+)\right) dZ \\ &\times \operatorname{vol}(\varpi_F^n L_x)^{-1} \int_{\varpi_F^n L_x} \xi_n(\exp(X))\Theta_\pi(x\exp(X))dX.\end{aligned}$$

Notice that the lattice L_x of $\mathfrak{g}_x$ satisfies the assumptions (i) and (ii) of [36, §3], that is

- $L_x = \bigoplus_{i\in\mathbf{Z}} L_x \cap \mathfrak{g}_{x,i}$ where $\mathfrak{g}_{x,i} := \mathfrak{g}_x \cap \mathfrak{g}_i$;
- The lattice $L_x/(L_x \cap Y^\sharp)$ is self-dual with respect to $\psi \circ B_Y$.

Hence, the same computation as that of the proof of [36, Lemma 7] shows that

$$\operatorname{vol}(\varpi_F^n L_x)^{-1} \int_{\varpi_F^n L_x} \xi_n(\exp(X))\Theta_\pi(x\exp(X))dX = c_{\pi,\mathcal{O}_x}(x)$$

when n is large. Thus, it only remains to show that

$$D^G(x)\,\mathrm{vol}(L^x)^{-1}\int_{(1-\mathrm{Ad}(x^{-1}))^{-1}L^x}\psi\left(B_Y(Z_-,x^{-1}Z_+x-Z_+)\right)dZ=D^{G/M_0}(x)^{1/2}. \tag{A.0.8}$$

By construction, we have a decomposition $L^x=L^x_Y\oplus L^{x,-}\oplus L^{x,2}$ where $L^x_Y:=L^x\cap Y^\sharp$. Fix Haar measures on $\mathfrak{g}^x_Y:=\mathfrak{g}^x\cap Y^\sharp$, $V^{x,-}$ and $V^{x,2}$ whose product gives the (already fixed) Haar measure on $\mathfrak{g}^x$ through the decomposition $\mathfrak{g}^x=\mathfrak{g}^x_Y\oplus V^{x,-}\oplus V^{x,2}$. Up to scaling all these Haar measures, we may assume that

$$\mathrm{vol}(L^x)=\mathrm{vol}(L^x_Y)=\mathrm{vol}(L^{x,-})=\mathrm{vol}(L^{x,2})=1.$$

Then, as the function $Z\in\mathfrak{g}\mapsto\psi\left(B_Y(Z_-,x^{-1}Z_+x-Z_+)\right)$ is invariant by translation by $Y^\sharp$, we have

$$\begin{aligned}
&\int_{(1-\mathrm{Ad}(x^{-1}))^{-1}L^x}\psi\left(B_Y(Z_-,x^{-1}Z_+x-Z_+)\right)dZ\\
&=\mathrm{vol}((1-\mathrm{Ad}(x^{-1}))^{-1}L^x_Y)\int_{(1-\mathrm{Ad}(x^{-1}))^{-1}L^{x,-}}\int_{(1-\mathrm{Ad}(x^{-1}))^{-1}L^{x,2}}\psi\left(B_Y(Z_-,(1-\mathrm{Ad}(x^{-1}))Z_+)\right)dZ_+dZ_-\\
&=\left|\det(1-\mathrm{Ad}(x))_{|\mathfrak{g}^x_Y\oplus V^{x,2}}\right|^{-1}\int_{(1-\mathrm{Ad}(x^{-1}))^{-1}L^{x,-}}\int_{L^{x,2}}\psi\left(B_Y(Z_-,Z_+)\right)dZ_+dZ_-.
\end{aligned}$$

In order to get the last line we have used the fact that if $\mathcal{V}$ is an F-vector space, $\mathcal{L}\subset\mathcal{V}$ a lattice and T an endomorphism without the eigenvalue 1 such that $T\mathcal{L}=\mathcal{L}$, then $\mathrm{vol}((1-T^{-1})^{-1}\mathcal{L})=|\det(1-T)|^{-1}\,\mathrm{vol}(\mathcal{L})$. Since $L^{x,2}$ is the lattice dual to $L^{x,-}$ with respect to B_Y and ψ unramified, we have

$$\int_{L^{x,2}}\psi\left(B_Y(Z_-,Z_+)\right)dZ_+=\begin{cases}1, & \text{if } Z_-\in L^{x,-};\\ 0, & \text{otherwise}\end{cases}$$

for all $Z_-\in V^{x,-}$. Thus, we have

$$\int_{(1-\mathrm{Ad}(x^{-1}))^{-1}L^{x,-}}\int_{L^{x,2}}\psi\left(B_Y(Z_-,Z_+)\right)dZ_+dZ_-=\mathrm{vol}(L^{x,-})=1$$

and consequently

$$\begin{aligned}
D^G(x)\int_{(1-\mathrm{Ad}(x^{-1}))^{-1}L^x}\psi\left(B_Y(Z_-,x^{-1}Z_+x-Z_+)\right)dZ&=D^G(x)\left|\det(1-\mathrm{Ad}(x))_{|\mathfrak{g}^x_Y\oplus V^{x,2}}\right|^{-1}\\
&=\left|\det(1-\mathrm{Ad}(x))_{|V^{x,-}}\right|
\end{aligned}$$

where for the last equality we have used the fact that

$$D^G(x)=\left|\det(1-\mathrm{Ad}(x))_{|\mathfrak{g}^x}\right|=\left|\det(1-\mathrm{Ad}(x))_{|\mathfrak{g}^x_Y\oplus V^{x,2}\oplus V^{x,-}}\right|.$$

Furthermore, since $V^{x,-}$ and $V^{x,2}$ are in duality under the form B_Y which is $\mathrm{Ad}(x)$-invariant, we have

$$\left|\det(1-\mathrm{Ad}(x))_{|V^{x,-}}\right| = \left|\det(1-\mathrm{Ad}(x^{-1}))_{|V^{x,2}}\right| = \left|\det(1-\mathrm{Ad}(x))_{|V^{x,2}}\right| = \left|\det(1-\mathrm{Ad}(x))_{|\mathfrak{g}^x\cap\mathfrak{n}_0}\right|$$

where in the second equality we have used the fact that $\mathrm{Ad}(x)$ is compact and in the last one we have used A.0.2. Similarly, since $\mathfrak{g}^x \cap \mathfrak{n}_0$ and $\mathfrak{g}^x \cap \overline{\mathfrak{n}}_0$ are in duality under the form $\langle ., . \rangle$ which is $\mathrm{Ad}(x)$-invariant, we have

$$\left|\det(1-\mathrm{Ad}(x))_{|\mathfrak{g}^x\cap\mathfrak{n}_0}\right| = \left|\det(1-\mathrm{Ad}(x))_{|\mathfrak{g}^x\cap(\mathfrak{n}_0\oplus\overline{\mathfrak{n}}_0)}\right|^{1/2} = D^{G/M_0}(x)^{1/2}.$$

This shows A.0.8 and ends the proof of the proposition. □

The following corollary is a direct consequence of the last proposition.

Corollary A.2. *Set $H := M_{0,\xi} \ltimes N_0$. Let τ be a smooth irreducible representation of $M_{0,\xi}$ (necessarily of finite dimension) and let χ_τ be its character. Assume that the central characters of π and χ_τ coincide on the split center A_G of G. Then, we have*

$$\dim \mathrm{Hom}_H(\pi, \tau \otimes \xi) = \int_{M_{0,\xi}/A_G} D^{G/M_0}(x)^{1/2} c_{\pi,\mathcal{O}_x}(x)\chi_\tau(x^{-1})dx$$

where the Haar measure on $M_{0,\xi}/A_G$ is chosen so that $\mathrm{vol}(M_{0,\xi}/A_G) = 1$.

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