Praxeological analysis: The case of ideals in Ring Theory

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The importance of studying structuralist praxeologies has been highlighted by Hausberger (2016). In this communication, we illustrate it on the case of ideals in Ring Theory. We provide a detailed study of a proof extracted from a textbook in Abstract Algebra showing that structuralist praxeologies involve interplay between intertwined algebraic, set-theoretic and logical praxeologies, revealing a hidden complexity.

Keywords: Mathematical structuralism, structuralist praxeologies, ideals in Ring Theory, logic.

Introduction

Hausberger (2016), with the introduction of the notion of *structuralist praxeology*, underlined the importance of *praxeological analysis* in the didactical study of phenomena related to the teaching and learning of Abstract Algebra at University level. His work is based on an epistemological investigation of algebraic structuralism that showed that mathematical practice in Abstract Algebra may be interpreted as an application of the *axiomatic method*, structures being used as tools by mathematicians in order to prove statements on objects. In the Anthropological Theory of the Didactics (Bosch &Gascon, 2014), a method is a *set of techniques*. In fact, ATD poses the general model that every human activity may be described by quadruples $[T,\tau,\theta,\Theta]$, called *praxeologies*, which correspond to the organisations it sets up: these combine a praxis (a type of tasks T and a set of techniques τ) with a logos that include two levels of description and justification of the praxis: the technology θ and the theory Θ. Hausberger (2016) made the assumption that clarifying the *structuralist techniques* may illuminate practices in Abstract Algebra, make their rationale more visible and ground them as a coherent whole. Hausberger (2016) described common tasks and techniques in the arithmetic of abstract rings and studied the structuralist praxeologies developed by students on a mathematical forum online. By contrast, the empirical data presented here is an extract of the solution of an exercise on Noetherian rings proposed by teachers in a textbook. The central mathematical notion at stake is the notion of *idea*l. By a detailed study of this example, we will develop the argument that structuralist praxeologies involve interplays between algebraic, set-theoretic and logical praxeologies, thus revealing a hidden complexity.

Structuralist praxeologies as intertwined algebraic, set-theoretic and logical ones

The notion of structuralist praxeology

Structuralist techniques are the by-products of the complete rewriting of classical algebra operated by Noether's school in the 1920s (Hausberger, 2013 & 2016). They are based on the now standard structuralist constructs: sub-structures, homomorphisms, isomorphism theorems, products or sums of structures, quotients, etc. Hausberger (2016) stressed that common tasks in Abstract Algebra may often be solved using elementary techniques. Whenever its logos block contains a theorem on structures, the praxeology may be called structuralist. Nevertheless, a gradation of its structuralist dimension (loc. cit.) may be observed. In fact, structuralist praxeologies reflect the concrete-abstract and particular-general dialectics that are at stake in Abstract Algebra: tasks involving concrete and particular objects are completed by using abstract and general considerations on structures. Examples will be given in the sequel. The particularity of structuralist praxeologies that will be investigated in this article is that they often involve sub-praxeologies of algebraic, set-theoretic or logical type.

Algebraic and set-theoretic praxeologies

Noether qualified her own work of "set-theoretic foundation for algebra" (Hausberger, 2013), following Dedekind. On an epistemological point of view, it is characterised by the transition from thinking about operations on elements to thinking in terms of selected subsets and homomorphisms. The distinguished subsets are the kernels of homomorphisms, hence the normal subgroups in Group Theory and the ideals in Ring Theory. Noether uncovered the importance of the chain condition on ideals that led to the definition of Noetherian rings (see below). In other words, set-theoretic operations on ideals are connected to algebraic properties on elements. We will present below this connection by means of a "dictionary". It explains the intertwining of algebraic praxeologies (on the level of elements) and set-theoretic praxeologies (on the level of structures), but it leads also to the use of logical praxeologies, notably for the descent from the ideals toward the elements at stake.

Logical praxeologies

Many tasks in Abstract Algebra involve proof and proving, thus logical praxeologies. Durand-Guerrier (2008) has enlightened that the natural deduction developed by Copi (1954) provides a powerful tool to analyse and check mathematical proofs. In particular, it allows identifying those steps where mathematical arguments are silenced, supporting the claim that mathematics and logic are closely intertwined in proof. We will rely on Copi's natural deduction to describe logical praxeologies likely to appear in proof and proving: elimination and introduction of implication, universal quantifiers and existential quantifiers, restriction of the domain of quantification. The theory is the First order logic (Predicate calculus) and the technologies are logical theorems (i.e. statements true for every interpretation in any non-empty domain). In Copi's natural deduction, one deals with a generic non-empty universe, and some aspects need pragmatic control in order to ensure validity, as we will see below. The following table details common logical praxeologies that can be involved in a proof and hence in the study of structuralist praxeologies.

We provide triplets (type of tasks, technique, technology):

index	Type of tasks	Technique	Technology	Example of use
L1	Elimination of an implication	Asserting the antecedent – asserting the consequent	$[(P \Rightarrow Q) \wedge P] \Rightarrow Q$	Deduction based on a conditional theorem
L2	Introduction of implication	Recognizing that Q has been proved under the hypothesis P, and assert "$P \Rightarrow Q$"	$\neg(P \wedge \neg Q) \Leftrightarrow (P \Rightarrow Q)$	Conclusion of the proof of a conditional statement
L3	Elimination of a universal quantifier	Deleting the quantifier, introducing of a generic element of the universe, assigning this element to every occurrence of the variable in the open statement.	$[\forall x\, (F(x)] \Rightarrow F(y)$	Using a universal statement in a proof by generic element.
L4	Introduction of a universal	Given a true statement involving a generic element of a domain U,	No logical theorem. Need to control that the	Conclusion of proof by generic element.

	quantifier	assert the corresponding universal statement	element is actually a generic element of U (no other assumption on this element has been done)	
L5	Introduction of an existential statement	Given an element of the universe U satisfying an open sentence, assert that the corresponding existential statement is true.	$F(y) \Rightarrow \exists x\, F(x)$	Conclusion of the proof of an existential statement.
L6	Elimination of an existential statement	Given a true existential statement, introduce an element satisfying the corresponding open sentence.	No logical theorem. Need to control that the name of the element has not been used prior in the proof	Using an AE statement ("For all, Exists") in a proof.
L7	Restriction of the domain of quantification	Given a universal statement true in a domain A, assert it on a subdomain B of A.	$[(\forall x\, (A(x) \Rightarrow F(x)) \wedge (\forall x\, (B(x) \Rightarrow A(x))] \Rightarrow [(\forall x\, (B(x) \Rightarrow F(x))$	Fitting the statement with the antecedent of a conditional statement
L8	Transformatio n of a statement preserving its truth value	Substitute an equivalent statement to a given statement	In the case of implication: $[(\forall x\ (P(x) \Rightarrow R(x)) \wedge (\forall x\ (Q(x) \Leftrightarrow R(x))] \Rightarrow [(\forall x\ (P(x) \Rightarrow Q(x))$	Using the dictionary of properties elements/structures (*cf.* table 2)

Figure 1: List of a priori logical praxeologies according to Copi

The case of ideals in Ring Theory

The notion of ideal and its ecology

An *ideal* I of a ring $(A, +, \cdot)$ is, by definition, a subset of A with these properties: (i) I is a subgroup of the additive group $(A, +)$; (ii) if $a \in A$ and $x \in I$, then $a \cdot x \in I$. As part of her Master's degree dissertation, the first author conducted an epistemological and didactic study of the concept of ideal in order to explore the ecology, including the *habitats* and *niche* (Artaud, 1997) of this concept in French university education. This epistemological study started with the creation of ideal numbers by Kumer in 1847 and it enhanced the rise towards abstraction leading in the 1920s through the work of Noether to the concept we use today (Jovignot, to appear). As far as the ecology of the concept of ideal is concerned, the epistemological study allowed the identification of the following *a priori* main *habitats:* general Ring Theory (quotient rings and isomorphism theorems), arithmetic of abstract rings and elimination theory. Bearing on those results, Jovignot developed an analytical framework to identify *habitats* and *niches* of the notion of ideal in algebra textbooks addressed to undergraduates and Master's students. A first study of 3 textbooks has led to improve this grid, that was then applied to a sample of 7 French textbooks that were considered as representative of the ecology of the concept of ideal and of its use in the different post-secondary institutions in which this concept is taught in France. This study confirmed general Ring Theory and arithmetic of abstract rings as major *habitats* of the concept of ideal, but it also allowed the exhibition of *habitats* that had not been previously identified, such as the theory of modules and algebraic geometry, which suggests the importance of leading a complementary study in contemporary epistemology. Finally, elimination theory appeared, in our sample, only in the specialized computer algebra manual.

Ideals and structuralist praxeologies in the arithmetic of abstract rings

Arithmetic of abstract rings as a mathematical domain is characterized by a mathematical structure in “Russian dolls”: it involves Euclidean, principal ideal domains (PID) and unique factorization domains (UFD), which generalize properties of the ring of integers, and mathematical theorems that state inclusions from the former class to the latter. Common tasks consist in proving that a given ring, for instance Gauss’s ring of integers **Z**[i], belongs to a class or the other. More abstract tasks, such as the one that will be analyzed below, involve making new connections between such classes. The central notion is the notion of ideal. In fact, the class may be defined directly by a property on ideals (such as PIDs) or by properties on elements (such as UFDs) which may be related to properties of ideals by means of the following “dictionary” which was already mentioned above. This dictionary will be useful to understand used praxeologies in the task studied below.

index	Conditions of validity	Level of elements	Level of structures
D1		a divides b	(a) contains (b)
D2		a and b are associates	$(a) = (b)$
D3	$p \neq 0$	p is a prime element	(p) is a prime ideal
D4	A is a principal ideal domain	p is irreducible in A	(p) is a maximal ideal of A
D5	A is a unique factorisation domain	d is a *gcd* of a and b	$(d) = (a) + (b)$

Figure 2: dictionary of properties elements/structures

The task under study

In the next section, we will present the praxeological analysis of an exercise involving the concept of ideal which is extracted from a book addressed to Master’s degree students preparing the French *Agrégation*[1]: Francinou, S. & Gianella, H. (1994). This book is widely used by university students in France. The authors sampled classical exercises in Algebra and provided proofs. In the chosen exercise, students are requested to establish a connection between Noetherian integral domains endowed with an extra property and PIDs. We clarify that this praxeological analysis is not a tool for teaching but could help us later for the design of experimentation with students.

The exercise is the following (our translation):

Let A be a Noetherian integral domain. We assume that every maximal ideal of A is principal.

1) Show that A is a unique factorization domain.
2) Show that every non-zero prime ideal is maximal, principal and of the form (p) where p is irreducible.
3) Show that A is a principal ideal domain. (loc. cit. p.57)

We will restrict our study to question 1. The authors introduce the following classical criteria, in which E designates the property of existence of a factorization and U the property of unicity:

A is a UFD if and only if:

a) every increasing chain $(a_1) < (a_2) < (a_3) <\ldots$ of principal ideals is stationary (equivalent to E)
b) every irreducible element is prime (equivalent to U)

[1] Competitive exam for prospective teachers for secondary and tertiary education

The proof provided by the authors is the following (our translation from French):

Since *A* is Noetherian, *A* satisfies (E). To establish that *A* is a unique factorization domain, it suffices to prove that if *p* is irreducible, the ideal (*p*) is prime. Let us consider a maximal ideal *M* containing (*p*). By hypothesis *M* is principal generated by *a*. Thus *a* divides *p*. Since *a* is not a unit (because $M \neq A$), *p* and *a* are associates and (*p*) = *M* is maximal. In particular (*p*) is prime.

Praxeological analysis of the task

Supplementing the proof

Reading the proof of the authors, it appears that a lot of steps remain implicit. In order to be able to study the full set of praxeologies involved in the proof, either explicit or implicit, we have supplemented it. We consider that the proof is complete when all the statements are obtained by natural deduction from previously established results or standard theorems in Abstract Algebra. We do not examine in detail in this paper the question of which of these supplements should be taught, but we will provide hypothesis that will be studied in further steps of this research. The steps of the proof presented in the textbook are numbered, our supplements appear in italic and are designated by letters whenever several steps are involved. The supplemented proof reads as follows:

1. Since A is noetherian, *A* satisfies (E).

a. Indeed, A is Noetherian so every increasing chain of ideals is stationary by definition.

b. In particular, every increasing chain of principal ideals is stationary.

c. So, thanks to the criteria, A satisfies(E).

2. To establish that *A* is a UFD it suffices to prove that if *p* is irreducible, the ideal (*p*) is prime.

a. Indeed, we need to show that every irreducible element is prime (criteria, b)

*b. And "p is prime" is equivalent to "(*p*) is prime"*

c. In fact, we will show that (p) is maximal. It is enough since every maximal ideal is prime in a ring.

3. *Let p be an irreducible element of A and* M a maximal ideal containing (*p*).

a. If there aren't any irreducible elements, we are done. In fact, irreducible elements exist since A is Noetherian, except if A is a field.

b. p is not an unit, so (p) is proper and M exists according to Krull's theorem.

4. By hypothesis *M* is principal. *Let a be a generator of M.*

5. Thus *a* divides *p*.

a. Indeed, (p) is included in M and M = (a), so (p) is included in (a).

b. And (p) is included in (a) if and only if a divides p.

6. Since *a* is not a unit (because $M \neq A$), *p* and *a* are associates - *indeed, a|p so there exists b in A such that p=ab; moreover, p is irreducible so, since a is not a unit, b must be a unit and p and a are associates -*

7. and (*p*) = *M* is maximal *since two principal ideals are equal if and only if their generators are associates.*

8. In particular (*p*) is prime.

Praxeological analysis

We present the praxeological analysis as a tabular; in the column labelled "steps", we are indicating in which steps of the proof the studied praxeology appears. Only tasks, techniques and technologies are mentioned; the theory in the sense of ATD is Ring Theory. We note S structuralist praxeologies and A algebraic ones.

steps	Type of task	Technique	Technology
1 – 8 (S1)	Show that a ring is UFD	Use of the criteria	Equivalence between the criteria and the definition of a UFD
1	L7		
2	L8		
2 b – 8 (S2)	Show that an element *p* is prime	Associate to *p* the ideal *(p)* and show that (*p*) is prime	The dictionary of properties elements/structures (D3)
2 c – 8 (S3)	Show that an ideal is prime	Try to show that the ideal is maximal	Every maximal ideal is prime
2 c	L1		
3 - 7 (S4)	Show that an ideal *I* is maximal	Take a maximal ideal *M* containing *I* and show that *M*=*I*	Krull's theorem
3 a	L6 (making explicit the two existential statements permitting the introduction of *p* and *M*)		
4 - 7 (S5)	Show that two principal ideals are equal	Show that two generators of those ideals are associates	The dictionary of properties elements/structures (D2)
5 (S6)	Show that *a* divides *b*	Show that (*a*) contains (*b*)	The dictionary of properties elements/structures (D1)
6 (A1)	Show that two elements *a* and *b* are associates	Show that *a* divides *b*; it is enough to conclude whenever *b* is irreducible and *a* is not a unit	Definition of units, irreducible elements and associates; *a* and *b* are associates if and only if $a \mid b$ and $b \mid a$

Figure 3: praxeological analysis of the task

Conclusions of our analysis

This praxeological study allows us to highlight significant characteristics of the praxeologies used by the authors that we summarize below.

The dictionary elements/structure is used along the proof; indeed the proof involves relationships between properties of elements of the ring (being an irreducible or a prime element) and properties of subsets (being a principal, maximal or prime ideal). Moreover, the algebraic notion of generator and the dictionary of properties elements/structures allow the replacement of common set-theoretic praxeologies (such as proving an equality of two sets by double inclusion) by more powerful algebraic praxeologies (involving A1). This cultural shift that is characteristic of structuralist algebra may be pointed out as a potential obstacle (previous praxeologies hindering the use of the new praxeologies to be acquired).

The structuralist praxeology S1 decomposes into several sub-praxeologies S2-S6, A1, L1, L7, L8. In the authors' proof, only structuralist steps of the proof are given; the steps involving algebraic and logical praxeologies are nearly systematically hidden. We may hypothesize that these authors see structuralist steps as the architecture of the proof and expect students to be able to reconstruct the missing elements by themselves. On the contrary, we will argue in favour of setting out the non-structuralist praxeologies and elaborate on their role in connection with structuralist praxeologies.

The nearly systematic omission of logical praxeologies raises the following issues and comments. First of all, the proof deals with generic objects, which is due to the level of generality of the statement of the exercise. It is already explicit in the statement itself, therefore both rules of elimination (L3) and introduction (L4) of a universal quantifier on the ring are not needed. In the sequel, the ideal M is introduced (step 3) without justification of its existence (Krull's theorem). The introduction of the generator of M is allusive and could be misinterpreted, letting think that this element has already been introduced. In both cases, the elimination of the existential quantifier (L6) remains implicit, letting thus implicit the statements themselves. The generic element p that plays a central role in the proof is not introduced, while it is a delicate step. Indeed, a classical way to prove a conditional statement by generic element is to introduce an element satisfying the antecedent, under the implicit assumption that such element exist; indeed, if not, there is nothing to prove (step 3.a). In addition, letting silent the restriction of the quantification domain (L7, step 1) hides the fact that this rule does not apply for existential statements, which might not be clear for some students. Finally, the substitution rule (L8) is a key for using the dictionary elements/structure by substituting a property of elements for a property of structure and vice-versa.

Giving such a proof requires the availability of the praxeologies cited above and a suitable understanding of their interrelations, or enough experience on the structuralist methodology in order to apply these praxeologies *en acte*. We hypothesise that the textbook's proof does not permit the appropriation of the structuralist praxeologies at stake. A didactical strategy to reach this goal may include, for instance, a "meta-discourse" on the crucial role of the dictionary elements/structure, together with making explicit the logical praxeologies whose role has been underlined above.

The particular construction of the proof (related to the decomposition of S1 into S2-S6-A1) can be understood by analysing the interplay between the blocks of the praxis and that of the logos of the different praxeologies engaged in the proof. However, the technological elements are seldom present in the proof written by the authors. For example, the properties of the dictionary are used but barely cited. Even if the students own in their praxeological equipment those technologies, the proof doesn't offer them the opportunity to identify those technologies in the context of the proof and thus build the associated structuralist praxeologies in order to be able to use them by themselves in another proof situation.

General conclusion and perspectives

Our praxeological analysis has highlighted the complexity of the chosen exercise. This complexity comes, in particular, from the decomposition of structuralist praxeologies into several structuralist sub-praxeologies and their interrelation with logical and algebraic praxeologies. These are fundamental in order to make the structuralist technologies practically operative. A sketchy proof which restricts to the structuralist steps, although it is seen as a clear and synthetic account by mathematicians, may therefore appear quite inadequate for self-learning by students who are not familiar with the structuralist methodology. In other words, our study contributes to break the "illusion of transparency" behind proofs that may be found in Abstract Algebra textbooks.

We aim to record and analyse the work of students who attempt to reconstruct the proof as we did, or to write a proof from scratch. In this forthcoming empirical study, our praxeological analysis will serve as an *a priori* analysis. It may also be used as a starting point in order to prepare clues for the students and other types of didactical intervention, as well as to lead semi-structured interviews. Moreover, we intend to interview the authors of the book in order to get insights in their goals and motivations for the choices they made when writing down the proof. More generally, it is expected from these praxeological analyses, conducted on a larger scale, a deeper understanding of structuralist praxeologies with a view to setting up didactic engineerings dedicated to the teaching of structuralist concepts and in particular the ideal concept.

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