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On distinguished square-integrable representations for Galois pairs and a conjecture of Prasad

Raphaël Beuzart-Plessis *

November 30, 2018

Abstract

We prove an integral formula computing multiplicities of square-integrable representations relative to Galois pairs over p -adic fields and we apply this formula to verify two consequences of a conjecture of Dipendra Prasad. One concerns the exact computation of the multiplicity of the Steinberg representation and the other the invariance of multiplicities by transfer among inner forms.

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Introduction

Let F be a p -adic field (that is a finite extension of \mathbf{Q}_p for a certain prime number p) and H be a connected reductive group over F . Let E/F be a quadratic extension and set

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$G := R_{E/F}H_E$ where $R_{E/F}$ denotes Weil's restriction of scalars (so that $G(F) = H(E)$). To every complex smooth irreducible representation π of $G(F)$ and every continuous character χ of $H(F)$ we associate a multiplicity $m(\pi, \chi)$ (which is always finite by [18] Theorem 4.5) defined by

$$m(\pi, \chi) := \dim \text{Hom}_H(\pi, \chi)$$

where $\text{Hom}_H(\pi, \chi)$ stands for the space of $(H(F), \chi)$ -equivariant linear forms on (the space of) π . Recently, Dipendra Prasad [43] has proposed very general conjectures describing this multiplicity in terms of the Langlands parameterization of π , at least for representations belonging to the so-called 'generic' L -packets. These predictions, which generalize earlier conjectures of Jacquet ([27], [28]), are part of a larger stream that has come to be called the 'local relative Langlands program' and whose main aim is roughly to describe the 'spectrum' of general homogeneous *spherical* varieties $X = H \backslash G$ over local fields in terms of Langlands dual picture and correspondence. In the paper [44], and under the assumption that G is *split*, Sakellaridis and Venkatesh set up a very general framework to deal with these questions by introducing a certain complex reductive group \check{G}_X associated to the variety X , which generalizes Langlands construction of a dual group, together with a morphism $\check{G}_X \rightarrow \check{G}$ (actually, in the most general case, this should also include an extra SL_2 factor) which, according to them, should govern a great part of the spectral decomposition of $L^2(H \backslash G)$ (see [44] Conjecture 16.2.2). In a similar way, in the case where $G = R_{E/F}H_E$ as above (note that such a group is never split) Prasad introduces a certain L -group ${}^L H^{\text{op}}$ (further explanations on this notation are given below) and a morphism ${}^L H^{\text{op}} \rightarrow {}^L G$ which should govern, on the dual side, the behavior of the multiplicities $m(\pi, \chi)$ for a very particular quadratic character χ (denoted by $\omega_{H,E}$ below) that has also been defined by Prasad. The main goal of this paper is to present some coarse results supporting Prasad's very precise conjectures in the particular case of stable (essentially) square-integrable representations. In the rest of this introduction we will recall the part of Prasad's conjecture that we are interested in as well as the two consequences of it that we have been able to verify. We will also say some words on the proofs which are based on a certain simple local trace formula adapted to the situation and which takes its roots in Arthur's local trace formula ([4]) as well as in Waldspurger's work on the Gross-Prasad conjecture for orthogonal groups ([46], [47]).

Prasad associates a number of invariants to the situation at hand. First, he constructs a certain quadratic character $\omega_{H,E} : H(F) \rightarrow \{\pm 1\}$ as well as a certain quasi-split group H^{op} over F which is an E/F form of the quasi-split inner form of H . We refer the reader to [43]§7-8 for precise constructions of those and content ourself to give three examples here:

- If $H = GL_n$, then $H^{\text{op}} = U(n)_{qs}$ (quasi-split form) and $\omega_{H,E} = (\eta_{E/F} \circ \det)^{n+1}$ where $\eta_{E/F}$ is the quadratic character associated to E/F ;
- If $H = U(n)$ (a unitary group of rank n), then $H^{\text{op}} = GL_n$ and $\omega_{H,E} = 1$;

- If $H = SO(2n + 1)$ (any odd special orthogonal group), then $H^{\text{op}} = SO(2n + 1)_{qs}$ (the quasi-split inner form) and $\omega_{H,E} = \eta_{E/F} \circ N_{\text{spin}}$ where $N_{\text{spin}} : SO(2n + 1)(F) \rightarrow F^\times / (F^\times)^2$ denotes the spin norm.

To continue we need to restrict slightly the generality by only considering characters χ that are of ‘Galois type’ i.e. which are in the image of a map constructed by Langlands

$$H^1(W_F, Z(\check{H})) \rightarrow \text{Hom}_{\text{cont}}(H(F), \mathbf{C}^\times)$$

This map is always injective (because F is p -adic) but not always surjective (although it is most of the time, e.g. if H is quasi-split). We refer the reader to [37] for further discussion on these matters. The character $\omega_{H,E}$ is always of Galois type and, to every character χ of Galois type of $H(F)$, Prasad associates a certain ‘Langlands dual group’ $\mathcal{H}_\chi^{\text{op}}$ which sits in a short exact sequence

$$1 \rightarrow \check{H}^{\text{op}} \rightarrow \mathcal{H}_\chi^{\text{op}} \rightarrow W_F \rightarrow 1$$

together with a group embedding $\iota : \mathcal{H}_\chi^{\text{op}} \hookrightarrow {}^L G$ (where ${}^L G$ denotes the L -group of G) compatible with the projections to W_F and algebraic when restricted to \check{H}^{op} . In the particular case where $\chi = \omega_{H,E}$, we have $\mathcal{H}_\chi^{\text{op}} = {}^L H^{\text{op}}$ and ι is the homomorphism of quadratic base-change.

Remark 1 *Although the short exact sequence above always splits, there does not necessarily exist a splitting preserving a pinning of \check{H}^{op} and hence $\mathcal{H}_\chi^{\text{op}}$ is not always an L -group in the usual sense.*

Let $WD_F := W_F \times SL_2(\mathbf{C})$ be the Weil-Deligne group of F . An ‘ L -parameter’ taking values in $\mathcal{H}_\chi^{\text{op}}$ is defined as usual: a continuous Frobenius semi-simple morphism $WD_F \rightarrow \mathcal{H}_\chi^{\text{op}}$ which commutes with the projections to W_F and is algebraic when restricted to $SL_2(\mathbf{C})$. We are now ready to state (a slight generalization of) the stable version of Prasad’s conjecture for square-integrable representations:

Conjecture 1 *Let $\phi : WD_F \rightarrow {}^L G$ be a discrete L -parameter, $\Pi^G(\phi) \subseteq \text{Irr}(G)$ the corresponding L -packet and $\Pi_\phi = \sum_{\pi \in \Pi^G(\phi)} d(\pi)\pi$ the stable representation associated to ϕ . Then,*

we have

$$m(\Pi_\phi, \chi) = |\ker^1(F; H, G)|^{-1} \sum_{\psi} \frac{|Z(\phi)|}{|Z(\psi)|}$$

where

- *The sum is over the set of ‘ L -parameters’ $\psi : WD_F \rightarrow \mathcal{H}_\chi^{\text{op}}$ (taken up to \check{H}^{op} -conj) making the following diagram commute up to \check{G} -conj, i.e. there exists $g \in \check{G}$ such that $\iota \circ \psi = g\phi g^{-1}$,*

$$\begin{array}{ccc}
& & \mathcal{H}^{\text{op}} \\
& \nearrow \psi & \downarrow \iota \\
WD_F & \xrightarrow{\phi} & LG
\end{array}$$

- $\ker^1(F; H, G) := \text{Ker}(H^1(F, H) \rightarrow H^1(F, G))$ (corresponds to certain twists of the parameter ψ that become trivial in ${}^L G$);
- $Z(\phi) := \text{Cent}_{\check{G}}(\phi)/Z(\check{G})^{W_F}$ and $Z(\psi) := \text{Cent}_{\check{H}^{\text{op}}}(\psi)/Z(\check{H}^{\text{op}})^{W_F}$

As we said, this is only part of Prasad’s general conjectures which aim to describe (almost) all the multiplicities $m(\pi, \chi)$ explicitly. This version of the conjecture (and far more) is known in few particular cases: for $H = GL(n)$ by Kable and Anandavardhanan-Rajan ([29], [1]), for $H = U(n)$ by Feigon-Lapid-Offen ([21]) and for $H = GSp(4)$ by Hengfei Lu ([39]). The following theorems are both formal consequences of Conjecture 1 and are the main results of this paper (see Theorem 5.3.1 and Theorem 5.7.1):

Theorem 1 *Let H, H' be inner forms over F , $G := R_{E/F}H$, $G' := R_{E/F}H'$ and χ, χ' characters of Galois type of $H(F)$ and $H'(F)$ corresponding to each other (i.e. coming from the same element in $H^1(W_F, Z(\check{H})) = H^1(W_F, Z(\check{H}'))$). Let Π, Π' be (essentially) square-integrable representations of $G(F)$ and $G'(F)$ respectively which are stable (but not necessarily irreducible) and transfer of each other (i.e. $\Theta_{\Pi}(x) = \Theta_{\Pi'}(y)$ for all stably conjugate regular elements $x \in G_{\text{reg}}(F)$ and $y \in G'_{\text{reg}}(F)$ where $\Theta_{\Pi}, \Theta_{\Pi'}$ denote the Harish-Chandra characters of Π and Π' respectively). Then, we have*

$$m(\Pi, \chi) = m(\Pi', \chi')$$

Theorem 2 *For $\pi = \text{St}(G)$ the generalized Steinberg representation of $G(F)$ and χ a character of Galois type we have*

$$m(\text{St}(G), \chi) = \begin{cases} 1 & \text{if } \chi = \omega_{H,E} \\ 0 & \text{otherwise} \end{cases}$$

Theorem 2 also confirms an older conjecture of Prasad ([42], Conjecture 3) which was already known for split groups and tamely ramified extensions by work of Broussous-Courtès and Courtès ([12], [15], [16]) and for inner forms of GL_n by work of Matringe ([40]). The proof of Broussous and Courtès is mainly based on a careful study of the geometry of the building whereas Matringe’s work uses some Mackey machinery. Our approach is completely orthogonal to theirs and is based on a certain integral formula computing the multiplicity $m(\pi, \chi)$ in terms of the Harish-Chandra character of π . This formula is reminiscent and inspired by a similar result of Waldspurger in the context of the so-called Gross-Prasad conjecture ([46], [47]). It can also be seen as a ‘twisted’ version (‘twisted’ with respect to the non-split extension E/F) of the orthogonality relations between characters of discrete series due Harish-Chandra ([14], Theorem 3). It can be stated as follows (see Theorem 5.1.1):

Theorem 3 *Let π be a square-integrable representation of $G(F)$ and χ be a continuous character of $H(F)$. Assume that χ and the central character of π coincide on $A_H(F)$ (the maximal split central torus in $H(F)$). Then, we have*

$$m(\pi, \chi) = \int_{\Gamma_{\text{ell}}(\overline{H})} D^H(x) \Theta_\pi(x) \chi(x)^{-1} dx$$

where Θ_π denotes the Harish-Chandra character of π (a locally constant function on $G_{\text{reg}}(F)$), D^H is the usual Weyl discriminant and $\Gamma_{\text{ell}}(\overline{H})$ stands for the set of regular elliptic conjugacy classes in $\overline{H}(F) := H(F)/A_H(F)$ equipped with a suitable measure dx .

Theorem 1 is an easy consequence of this formula and Theorem 2 also follows from it with some extra work. Let us give an outline of the proof of Theorem 2 assuming Theorem 3. For notational simplicity we will assume that H is semi-simple. We have the following explicit formula for the character of the Steinberg representation (see §5.5 for a reminder)

$$D^G(x)^{1/2} \Theta_{\text{St}(G)}(x) = \sum_{P_0 \subseteq P = MU} (-1)^{a_P - a_{P_0}} \sum_{\{y \in M(F); y \sim_{\text{conj}} x\}/M\text{-conj}} D^M(y)^{1/2} \delta_P(y)^{1/2}$$

where P_0 is a minimal parabolic subgroup of G and we refer the reader to the core of the paper for other unexplained notations which are however pretty standard. Plugging this explicit formula in Theorem 3 and rearranging somewhat the terms we get

$$m(\text{St}(G), \chi) = \sum_{(\mathcal{M}, P)/\text{conj}} (-1)^{a_P - a_{P_0}} \int_{\Gamma_{\text{ell}}(\mathcal{M})} D^{\mathcal{M}}(x) \chi(x)^{-1} dx$$

where the sum runs over the $H(F)$ -conjugacy classes of pairs (\mathcal{M}, P) with

- \mathcal{M} an *elliptic twisted Levi subgroup* of H by which we mean an algebraic subgroup of H with trivial split center such that $R_{E/F} \mathcal{M}_E$ is a Levi subgroup of G ;
- P a parabolic subgroup of G with Levi component $R_{E/F} \mathcal{M}_E$.

Using a particular case of Harish-Chandra orthogonality relations between characters of discrete series ([14] Theorem 3), we can show that (see §5.6)

$$\int_{\Gamma_{\text{ell}}(\mathcal{M})} D^{\mathcal{M}}(x) \chi(x)^{-1} dx = (\chi|_{\mathcal{M}}, \mathbf{1})$$

where $\chi|_{\mathcal{M}}$ denotes the restriction of χ to $\mathcal{M}(F)$, $\mathbf{1}$ the trivial character of $\mathcal{M}(F)$ and (\cdot, \cdot) denotes the natural scalar product on the space of virtual characters of $\mathcal{M}(F)$. Then, in the above expression for $m(\text{St}(G), \chi)$, we can group together pairs (\mathcal{M}, P) according to their *stable* conjugacy classes ending up with an equality

$$m(\text{St}(G), \chi) = \sum_{(\mathcal{M}, P)/\text{stab}} (-1)^{a_P - a_{P_0}} |\ker^1(F; \mathcal{M}, H)| (\chi|_{\mathcal{M}}, \mathbf{1})$$

where $\ker^1(F; \mathcal{M}, H) := \text{Ker}(H^1(F, \mathcal{M}) \rightarrow H^1(F, H))$, a set which naturally parametrizes conjugacy classes inside the stable conjugacy class of (\mathcal{M}, P) . Set H_{ab} for the quotient of $H(F)$ by the common kernel of all the characters of Galois type (in case H is quasi-split it is just the *abelianization* of $H(F)$) and let \mathcal{M}_{ab} denote, for all elliptic twisted Levi \mathcal{M} , the image of $\mathcal{M}(F)$ in H_{ab} . Then, using Frobenius reciprocity, the last identity above can be rewritten as the equality between $m(\text{St}(G), \chi)$ and

$$\left(\chi, \sum_{(\mathcal{M}, P)/\text{stab}} (-1)^{a_P - a} |\ker^1(F; \mathcal{M}, H)| \text{Ind}_{\mathcal{M}_{ab}}^{H_{ab}}(\mathbf{1}) \right)$$

and thus Theorem 2 is now equivalent to the following identity in the Grothendieck group of H_{ab} :

$$(0.0.1) \quad \sum_{(\mathcal{M}, P)/\text{stab}} (-1)^{a_P - a} |\ker^1(F; \mathcal{M}, H)| \text{Ind}_{\mathcal{M}_{ab}}^{H_{ab}}(\mathbf{1}) = \omega_{H, E}$$

The proof of this identity in general is rather long and technical (see Proposition 5.4.1), so we content ourself (again) with giving two examples here:

- If $H = GL_n$, we have $H_{ab} = F^\times$ and $\omega_{H, E} = \eta_{E/F}^{n+1}$. If n is odd, $\mathcal{M} = H$ is the only elliptic twisted Levi and then 0.0.1 reduces to $\mathbf{1} = \mathbf{1}$. On the other hand, if n is even there are two (stable) conjugacy classes of pairs (\mathcal{M}, P) :

$$\mathcal{M}_0 = P_0 = H \text{ and } \mathcal{M}_1 = GL_{n/2}(E) \subseteq P_1 = \begin{pmatrix} GL_{n/2}(E) & * \\ & GL_{n/2}(E) \end{pmatrix}$$

we have $\mathcal{M}_{0, ab} = H_{ab} = F^\times \supset \mathcal{M}_{1, ab} = N(E^\times)$ and 0.0.1 reduces to the identity

$$\text{Ind}_{N(E^\times)}^{F^\times} \mathbf{1} - \mathbf{1} = \eta_{E/F}$$

- For $H = U(n)$ (a unitary group of rank n) we have $H_{ab} = \text{Ker } N_{E/F}$ and $\omega_{H, E} = \mathbf{1}$. In this case, stable conjugacy classes of pairs (\mathcal{M}, P) are parametrized by (ordered) partitions (n_1, \dots, n_k) of n as follows:

$$(n_1, \dots, n_k) \mapsto \mathcal{M} = U(n_1) \times \dots \times U(n_k) \subseteq P = \begin{pmatrix} GL_{n_1}(E) & * & * \\ & \ddots & * \\ & & GL_{n_k}(E) \end{pmatrix}$$

Moreover, $|\ker^1(F; \mathcal{M}, H)| = 2^{k-1}$ and $\mathcal{M}_{ab} = H_{ab}$ for all \mathcal{M} as above. Thus, in this case 0.0.1 reduces to the following combinatorial identity

$$\sum_{\substack{(n_1, \dots, n_k) \\ n_1 + \dots + n_k = n}} (-1)^{n-k} 2^{k-1} = 1$$

As we said, for its part, Theorem 3 is a consequence of a certain simple local trace formula adapted to the situation and to the proof of which most of the paper is devoted. Let us state briefly the content of this formula by assuming again, for simplicity, that the group H is semi-simple. Starting with a function $f \in C_c^\infty(G(F))$, we consider the following expression in two variables

$$K_f^\chi(x, y) := \int_{H(F)} f(x^{-1}hy)\chi(h)^{-1}dh, \quad x, y \in G(F)$$

This function is precisely the kernel of the operator on $L^2(H(F)\backslash G(F), \chi)$ given by convolution by f . Formally, the trace of such an operator should be given by the integral of this kernel over the diagonal that is

$$J^\chi(f) := \int_{H(F)\backslash G(F)} K_f^\chi(x, x)dx$$

Unfortunately, in general the convolution operator given by f isn't of trace-class and the above expression diverges. Nevertheless, we can still restrict our attention to some 'good' space of test functions for which the above integral is absolutely convergent. Recall, following Waldspurger [46], that the function f is said to be *strongly cuspidal* if for all proper parabolic subgroups $P = MU \subsetneq G$ we have

$$\int_{U(F)} f(xu)du = 0$$

for all $x \in M(F)$. We also say, following Harish-Chandra [23], that f is a *cuspid form* if the above kind of integrals vanish for all $x \in G(F)$ (thus, and contrary to what we might guess, being a cuspid form is stronger than being strongly cuspidal). Actually, it will be more convenient for us to work with functions that are not necessarily compactly supported: we will take f in the so-called *Harish-Chandra-Schwartz space* (see §1.3 for a reminder) denoted by $\mathcal{C}(G(F))$. The notions of strong cuspidality and of cuspid forms extend verbatim to this bigger space. The following theorem, whose proof is scattered all over this paper (see Theorem 2.1.1, Theorem 3.1.1 and Theorem 4.1.1), is our main technical result:

Theorem 4 *Let $f \in \mathcal{C}(G(F))$ be a strongly cuspidal function. Then, the expression defining $J^\chi(f)$ is absolutely convergent (see Theorem 2.1.1) and we have:*

(i) (see Theorem 4.1.1) *A geometric expansion*

$$J^\chi(f) = \int_{\Gamma_{\text{ell}}(H)} D^H(x)\Theta_f(x)\chi(x)^{-1}dx$$

where the function Θ_f is defined using weighted orbital integrals of Arthur (see §1.6);

(ii) (see Theorem 3.1.1) If f is moreover a cusp form, a spectral expansion

$$J^\chi(f) = \sum_{\pi \in \text{Irr}_{\text{sqr}}(G)} m(\pi, \chi) \text{Trace } \pi^\vee(f)$$

where $\text{Irr}_{\text{sqr}}(G)$ denotes the set of (equivalence classes of) irreducible square-integrable representations of $G(F)$ and for all $\pi \in \text{Irr}_{\text{sqr}}(G)$, π^\vee is the smooth contragredient of π .

We prove this theorem by following closely the general method laid down by [46], [47] and [11]. In particular, a crucial point to get the spectral expansion in the above theorem is to show that for π square-integrable the abstract multiplicity $m(\pi, \chi)$ is also the multiplicity of π in the discrete spectrum of $L^2(H(F)\backslash G(F))$. This fact is established in the course of the proof of Proposition 3.2.1 using the simple adaptation of an idea that goes back to Sakellaridis-Venkatesh ([44] Theorem 6.4.1) and Waldspurger ([47] Proposition 5.6).

Here is an outline of the contents of the different parts of the paper. In the first part, we set up the main notations and conventions as well as collect different results that will be needed in the subsequent sections. It includes in particular a discussion of a natural generalization of Arthur's (G, M) -families to symmetric pairs that we call (G, M, θ) -families. The second part contains the proof of the absolute convergence of $J^\chi(f)$ for strongly cuspidal functions f and in the third part we establish a spectral expansion of this distribution when f is a cusp form. These two parts are actually written in the more general setting of *tempered* symmetric pairs (G, H) (which were called *strongly discrete* in [22]) to which the proofs extend verbatim. The fourth part deals with the geometric expansion of $J^\chi(f)$. There we really have to restrict ourself to the setting of Galois pairs (that is when $G = R_{E/F}H_E$) since a certain equality of Weyl discriminants (see 4.1.1), which is only true in this particular case, plays a crucial role in allowing to control the uniform convergence of certain integrals. Finally, in the last part of this paper we prove the formula for the multiplicity (Theorem 3) and give two applications of it towards Prasad's conjecture (Theorem 1 and Theorem 2).

1 Preliminaries

1.1 Groups, measures, notations

Throughout this paper we will let F be a p -adic field (i.e. a finite extension of \mathbf{Q}_p for a certain prime number p) for which we will fix an algebraic closure \overline{F} . We will denote by $|\cdot|$ the canonical absolute value on F as well as its unique extension to \overline{F} . Unless specified otherwise, all groups and varieties that we consider in this paper will be tacitly assumed to be defined over F and we will identify them with their points over \overline{F} . Moreover for every finite extension K of F and every algebraic variety X defined over K we will denote by $R_{K/F}X$ Weil's restriction of scalars (so that in particular we have a canonical identification $(R_{K/F}X)(F) = X(K)$).

Let G be a connected reductive group over F and A_G be its maximal central split torus. We set $\overline{G} := G/A_G$ and

$$\mathcal{A}_G := X_*(A_G) \otimes \mathbf{R}$$

where $X_*(A_G)$ denotes the abelian group of cocharacters of A_G . If V is a real vector space we will always denote by V^* its dual. The space \mathcal{A}_G^* can naturally be identified with $X^*(A_G) \otimes \mathbf{R} = X^*(G) \otimes \mathbf{R}$ where $X^*(A_G)$ and $X^*(G)$ stand for the abelian groups of algebraic characters of A_G and G respectively. More generally, for every extension K/F we will denote by $X_K^*(G)$ the group of characters of G defined over K . There is a natural morphism $H_G : G(F) \rightarrow \mathcal{A}_G$ characterized by

$$\langle \chi, H_G(g) \rangle = \log(|\chi(g)|)$$

for all $\chi \in X^*(G)$. We set $\mathcal{A}_{G,F} := H_G(A_G(F))$. It is a lattice in \mathcal{A}_G . The same notations will be used for the Levi subgroups of G (i.e. the Levi components of parabolic subgroups of G): if M is a Levi subgroup of G we define similarly A_M , \mathcal{A}_M , H_M and $\mathcal{A}_{M,F}$. We will also use Arthur's notations: $\mathcal{P}(M)$, $\mathcal{F}(M)$ and $\mathcal{L}(M)$ will stand for the sets of parabolic subgroups with Levi component M , parabolic subgroups containing M and Levi subgroups containing M respectively. Let K be a maximal special compact subgroup of $G(F)$. Then, for all parabolic subgroups P with Levi decomposition $P = MU$ the Iwasawa decomposition $G(F) = M(F)U(F)K$ allows to extend H_M to a map $H_P : G(F) \rightarrow \mathcal{A}_M$ defined by $H_P(muk) := H_M(m)$ for all $m \in M(F)$, $u \in U(F)$ and $k \in K$. For all Levi subgroups $M \subset L$ there is a natural decomposition

$$\mathcal{A}_M = \mathcal{A}_M^L \oplus \mathcal{A}_L$$

where \mathcal{A}_M^L is generated by $H_M(\text{Ker}(H_{L|M(F)}))$ and we will set $a_M^L := \dim(\mathcal{A}_M^L)$. The Lie algebra of G will be denoted by \mathfrak{g} and more generally for any algebraic group we will denote its Lie algebra by the corresponding Gothic letter. We will write Ad for the adjoint action of G on \mathfrak{g} . We denote by \exp the exponential map which is an F -analytic map from an open neighborhood of 0 in $\mathfrak{g}(F)$ to $G(F)$. For all subsets $S \subset G$, we write $\text{Cent}_G(S)$, $\text{Cent}_{G(F)}(S)$ and $\text{Norm}_{G(F)}(S)$ for the centralizer of S in G , resp. the centralizer of S in $G(F)$, resp. the normalizer of S in $G(F)$. If $S = \{x\}$ we will denote by G_x the neutral connected component of $\text{Cent}_G(x) := \text{Cent}_G(\{x\})$. We define G_{reg} as the open subset of regular semisimple elements of G and for all subgroups H of G we will write $H_{\text{reg}} := H \cap G_{\text{reg}}$. Recall that a regular element $x \in G_{\text{reg}}(F)$ is said to be *elliptic* if $A_{G_x} = A_G$. We will denote by $G(F)_{\text{ell}}$ the set of regular elliptic elements in $G(F)$. The Weyl discriminant D^G is defined by

$$D^G(x) := \left| \det(1 - \text{Ad}(x)|_{\mathfrak{g}/\mathfrak{g}_x}) \right|$$

For every subtorus T of G we will write

$$W(G, T) := \text{Norm}_{G(F)}(T) / \text{Cent}_{G(F)}(T)$$

for its Weyl group. If $A \subset G$ is a split subtorus we will denote by $R(A, G)$ the set of roots of A in G i.e. the set of nontrivial characters of A appearing in the action of A on \mathfrak{g} . More

generally, if H is a subgroup of G and $A \subset H$ is a split subtorus we will denote by $R(A, H)$ the set of roots of A in H .

In all this paper we will assume that all the groups that we encounter have been equipped with Haar measures (left and right invariant as we will only consider measures on unimodular groups). In the particular case of tori we normalize these Haar measure by requiring that they give mass 1 to their maximal compact subgroups. For any Levi subgroup M of G we equip \mathcal{A}_M with the unique Haar measure such that $\text{vol}(\mathcal{A}_M/\mathcal{A}_{M,F}) = 1$. If $M \subset L$ are two Levi subgroups then we give $\mathcal{A}_M^L \simeq \mathcal{A}_M/\mathcal{A}_L$ the quotient measure.

Finally, we will adopt the following slightly imprecise but convenient notations. If f and g are positive functions on a set X , we will write

$$f(x) \ll g(x) \text{ for all } x \in X$$

and we will say that f is essentially bounded by g , if there exists a $c > 0$ such that

$$f(x) \leq cg(x), \text{ for all } x \in X$$

We will also say that f and g are equivalent and we will write

$$f(x) \sim g(x) \text{ for all } x \in X$$

if both f is essentially bounded by g and g is essentially bounded by f .

1.2 log-norms

All along this paper, we will assume that $\mathfrak{g}(F)$ has been equipped with a (classical) norm $|\cdot|_{\mathfrak{g}}$, that is a map $|\cdot|_{\mathfrak{g}} : \mathfrak{g}(F) \rightarrow \mathbb{R}_+$ satisfying $|\lambda X|_{\mathfrak{g}} = |\lambda| \cdot |X|_{\mathfrak{g}}$, $|X + Y|_{\mathfrak{g}} \leq |X|_{\mathfrak{g}} + |Y|_{\mathfrak{g}}$ and $|X|_{\mathfrak{g}} = 0$ if and only if $X = 0$ for all $\lambda \in F$ and $X, Y \in \mathfrak{g}(F)$. For any $R > 0$, we will denote by $B(0, R)$ the closed ball of radius R centered at the origin in $\mathfrak{g}(F)$.

In this paper we will freely use the notion of log-norms on varieties over F . The concept of norm on varieties over local fields has been introduced by Kottwitz in [36] §18. A log-norm is essentially just the logarithm of a Kottwitz's norm and we refer to [11] §1.2 for the basic properties of these log-norms. For convenience, we collect here the definition and basic properties of these objects.

First, an abstract log-norm on a set X is just a real-valued function $x \mapsto \sigma(x)$ on X such that $\sigma(x) \geq 1$, for all $x \in X$. For two abstract log-norms σ_1 and σ_2 on X , we will say that σ_2 dominates σ_1 and we will write $\sigma_1 \ll \sigma_2$ if

$$\sigma_1(x) \ll \sigma_2(x)$$

for all $x \in X$. We will say that σ_1 and σ_2 are equivalent if each of them dominates the other.

For an affine algebraic variety X over \overline{F} , choosing a set of generators f_1, \dots, f_m of its \overline{F} -algebra of regular functions $\mathcal{O}(X)$, we can define an abstract log-norm σ_X on X by setting

$$\sigma_X(x) = 1 + \log(\max\{1, |f_1(x)|, \dots, |f_m(x)|\})$$

for all $x \in X$. The equivalence class of σ_X doesn't depend on the particular choice of f_1, \dots, f_m and by a log-norm on X we will mean any abstract log-norm in this equivalence class. Note that if U is the principal Zariski open subset of X defined by the non-vanishing of $Q \in \mathcal{O}(X)$, then we have

$$\sigma_U(x) \sim \sigma_X(x) + \log(2 + |Q(x)|^{-1})$$

for all $x \in U$. More generally, for X any algebraic variety over \overline{F} , choosing a finite covering $(U_i)_{i \in I}$ of X by open affine subsets and fixing log-norms σ_{U_i} on each U_i , we can define an abstract log-norm on X by setting

$$\sigma_X(x) = \inf\{\sigma_{U_i}(x); i \in I \text{ such that } x \in U_i\}$$

Once again, the equivalence class of σ_X doesn't depend on the various choices and by a log-norm on X we will mean any abstract log-norm in this equivalence class.

We will assume that all varieties that we consider in this paper are equipped with log-norms and we will set $\sigma := \sigma_G$ and $\overline{\sigma} := \sigma_{\overline{G}}$.

Let $p : X \rightarrow Y$ be a regular map between algebraic varieties then we have

$$\sigma_Y(p(x)) \ll \sigma_X(x)$$

for all $x \in X$. If p is a closed immersion or more generally if p is a finite morphism ([36] Proposition 18.1(1)) we have

$$\sigma_Y(p(x)) \sim \sigma_X(x)$$

for all $x \in X$. We say that p has the *norm descent property* (with respect to F) if, denoting by p_F the induced map on F -points, we have

$$\sigma_Y(y) \sim \inf_{x \in p_F^{-1}(y)} \sigma_X(x)$$

for all $y \in p_F(X(F))$. By Proposition 18.3 of [36], if T is a subtorus of G then the projection $G \twoheadrightarrow T \backslash G$ has the norm descent property i.e. we have

$$(1.2.1) \quad \sigma_{T \backslash G}(g) \sim \inf_{t \in T(F)} \sigma(tg)$$

for all $g \in G(F)$. In section 1.9 we will prove that for H an F -spherical subgroup of G (i.e. a subgroup such that there exists a minimal parabolic subgroup P_0 of G with HP_0 open) the projection $G \twoheadrightarrow H \backslash G$ also has the norm descent property.

Let $T \subset G$ again be a maximal subtorus. As the regular map $T \backslash G \times T_{\text{reg}} \rightarrow G_{\text{reg}}$, $(g, t) \mapsto g^{-1}tg$ is finite we get

$$(1.2.2) \quad \sigma_{T \backslash G}(g) \ll \sigma(g^{-1}tg) \log(2 + D^G(t)^{-1})$$

for all $g \in G$ and all $t \in T_{\text{reg}}$.

For every variety X defined over F , equipped with a log-norm σ_X , and all $M > 0$ we will denote by $X[< M]$, resp. $X[\geq M]$, the set of all $x \in X(F)$ such that $\sigma_X(x) < M$, resp. $\sigma_X(x) \geq M$. With this notation, if T is a torus over F and $k := \dim(A_T)$ we have

$$(1.2.3) \quad \text{vol}(T[< M]) \ll M^k$$

for all $M > 0$.

1.3 Function spaces

Let ω be a continuous character of $A_G(F)$. We define $\mathcal{S}_\omega(G(F)) := C_c^\infty(A_G(F) \backslash G(F), \omega)$ as the space of functions $f : G(F) \rightarrow \mathbf{C}$ which are smooth (i.e. locally constant), satisfy $f(ag) = \omega(a)f(g)$ for all $(a, g) \in A_G(F) \times G(F)$ and are compactly supported modulo $A_G(F)$.

Assume moreover that ω is unitary and let Ξ^G be Harish-Chandra Xi function associated to a special maximal compact subgroup K of $G(F)$ (see [45] §II.1). Then, we define the Harish-Chandra-Schwartz space $\mathcal{C}_\omega(G(F))$ as the space of functions $f : G(F) \rightarrow \mathbf{C}$ which are biinvariant by an open subgroup of $G(F)$, satisfy $f(ag) = \omega(a)f(g)$ for all $(a, g) \in A_G(F) \times G(F)$ and such that for all $d > 0$ we have $|f(g)| \ll \Xi^G(g) \bar{\sigma}(g)^{-d}$ for all $g \in G(F)$.

1.4 Representations

In this paper all representations we will consider are smooth and we will always use the slight abuse of notation of identifying a representation π with the space on which it acts. We will denote by $\text{Irr}(G)$ the set of equivalence classes of smooth irreducible representations of $G(F)$ and by $\text{Irr}_{\text{cusp}}(G)$, $\text{Irr}_{\text{sqr}}(G)$ the subsets of supercuspidal and essentially square-integrable representations respectively. If ω is a continuous unitary character of $A_G(F)$ we will also write $\text{Irr}_\omega(G)$ (resp. $\text{Irr}_{\omega, \text{cusp}}(G)$, $\text{Irr}_{\omega, \text{sqr}}(G)$) for the sets of all $\pi \in \text{Irr}(G)$ (resp. $\pi \in \text{Irr}_{\text{cusp}}(G)$, $\pi \in \text{Irr}_{\text{sqr}}(G)$) whose central character restricted to $A_G(F)$ equals ω . For all $\pi \in \text{Irr}(G)$ we will denote by π^\vee its contragredient and by $\langle \cdot, \cdot \rangle : \pi \times \pi^\vee \rightarrow \mathbf{C}$ the canonical pairing. For all $\pi \in \text{Irr}_{\omega, \text{sqr}}(G)$, $d(\pi)$ will stand for the formal degree of π . Recall that it depends on the Haar measure on $G(F)$ and that it is uniquely characterized by the following identity (Schur orthogonality relations)

$$\int_{A_G(F) \backslash G(F)} \langle \pi(g)v_1, v_1^\vee \rangle \langle v_2, \pi^\vee(g)v_2^\vee \rangle dg = \frac{1}{d(\pi)} \langle v_1, v_2^\vee \rangle \langle v_2, v_1^\vee \rangle$$

for all $v_1, v_2 \in \pi$ and all $v_1^\vee, v_2^\vee \in \pi^\vee$. From this, we easily infer that for every coefficient f of π we have

$$(1.4.1) \quad \text{Trace}(\pi^\vee(f)) = \frac{1}{d(\pi)} f(1)$$

Let $\pi \in \text{Irr}(G)$ and let ω be the inverse of the restriction of the central character of π to $A_G(F)$. Then, for all $f \in \mathcal{S}_\omega(G(F))$ we can define an operator $\pi(f)$ on π by

$$\langle \pi(f)v, v^\vee \rangle := \int_{A_G(F) \backslash G(F)} f(g) \langle \pi(g)v, v^\vee \rangle dg$$

for all $(v, v^\vee) \in \pi \times \pi^\vee$. For all $f \in \mathcal{S}_\omega(G(F))$ this operator is of finite rank and a very deep theorem of Harish-Chandra ([25] Theorem 16.3) asserts that the distribution

$$f \in \mathcal{S}_\omega(G(F)) \mapsto \text{Trace}(\pi(f))$$

is representable by a locally integrable function which is locally constant on $G_{\text{reg}}(F)$. This function, the Harish-Chandra character of π , will be denoted Θ_π . It is characterized by

$$\text{Trace}(\pi(f)) = \int_{A_G(F) \backslash G(F)} \Theta_\pi(g) f(g) dg$$

for all $f \in \mathcal{S}_\omega(G(F))$. If moreover the representation π is square-integrable (or more generally *tempered*) with unitary central character, then the integral defining $\pi(f)$ still makes sense for all $f \in \mathcal{C}_\omega(G(F))$, the resulting operator is again of finite rank and the above equality continues to hold.

1.5 Weighted orbital integrals

Let M be a Levi subgroup and fix a maximal special compact subgroup K of $G(F)$. Using K we can define maps $H_P : G(F) \rightarrow \mathcal{A}_M$ for all $P \in \mathcal{P}(M)$ (cf §1.1). Let $g \in G(F)$. The family

$$\{-H_P(g); P \in \mathcal{P}(M)\}$$

is a positive (G, M) -orthogonal set in the sense of Arthur (see [2] §2). In particular, following *loc. cit.* using this family we can define a weight $v_M^Q(g)$ for all $Q \in \mathcal{F}(M)$. Concretely, $v_M^Q(g)$ is the volume of the convex hull of the set $\{H_P(g); P \in \mathcal{P}(M), P \subset Q\}$ (this convex hull belongs to a certain affine subspace of \mathcal{A}_M with direction \mathcal{A}_M^L where $Q = LU$ with $M \subset L$ and we define the volume with respect to the fixed Haar measure on \mathcal{A}_M^L). If $Q = G$ we set $v_M(g) := v_M^G(g)$ for simplicity. For every character ω of $A_G(F)$, every function $f \in \mathcal{C}_\omega(G(F))$ and all $x \in M(F) \cap G_{\text{reg}}(F)$, we define, again following Arthur, a weighted orbital integral by

$$\Phi_M^Q(x, f) := \int_{G_x(F) \backslash G(F)} f(g^{-1}xg) v_M^Q(g) dg$$

The integral is absolutely convergent by the following lemma which is an immediate consequence of 1.2.2 and Lemma 1.9.2 (which will be proved later).

Lemma 1.5.1 *Let $x \in M(F) \cap G_{\text{reg}}(F)$. Then, for all $d > 0$ there exists $d' > 0$ such that the integral*

$$\int_{G_x(F) \backslash G(F)} \Xi^G(g^{-1}xg) \overline{\sigma}(g^{-1}xg)^{-d'} \sigma_{G_x \backslash G}(x)^d dg$$

converges.

Once again if $Q = G$, we will set $\Phi_M(x, f) := \Phi_M^G(x, f)$ for simplicity. If $M = G$ (so that necessarily $Q = G$), $\Phi_G(x, f)$ reduces to the usual orbital integral.

1.6 Strongly cuspidal functions

Let ω be a continuous unitary character of $A_G(F)$. Following [46], we say that a function $f \in \mathcal{C}_\omega(G(F))$ is *strongly cuspidal* if for every proper parabolic subgroup $P = MU$ of G we have

$$\int_{U(F)} f(mu) du = 0, \quad \forall m \in M(F)$$

(the integral is absolutely convergent by [45] Proposition II.4.5). By a standard change of variable, f is strongly cuspidal if and only if for every proper parabolic subgroup $P = MU$ and for all $m \in M(F) \cap G_{\text{reg}}(F)$ we have

$$\int_{U(F)} f(u^{-1}mu) du = 0$$

We will denote by $\mathcal{C}_{\omega, \text{scusp}}(G(F))$ the subspace of strongly cuspidal functions in $\mathcal{C}_\omega(G(F))$ and we will set $\mathcal{S}_{\omega, \text{scusp}}(G(F)) := \mathcal{S}_\omega(G(F)) \cap \mathcal{C}_{\omega, \text{scusp}}(G(F))$. Let K be a maximal special compact subgroup of $G(F)$. For $x \in G_{\text{reg}}(F)$ set $M(x) := \text{Cent}_G(A_{G_x})$ (it is the smallest Levi subgroup containing x). Then, by [46] Lemme 5.2, for all $f \in \mathcal{C}_{\omega, \text{scusp}}(G(F))$, all Levi subgroups M , all $Q \in \mathcal{F}(M)$ and all $x \in M(F) \cap G_{\text{reg}}(F)$ we have $\Phi_M^Q(x, f) = 0$ unless $Q = G$ and $M = M(x)$. For all $x \in G_{\text{reg}}(F)$ we set

$$\Theta_f(x) := (-1)^{a_{M(x)}^G} \Phi_{M(x)}(x, f)$$

Then the function Θ_f is independent of the choice of K and invariant by conjugation ([46] Lemme 5.2 and Lemme 5.3). Also by [46] Corollaire 5.9, the function $(D^G)^{1/2} \Theta_f$ is locally bounded on $G(F)$.

We say that a function $f \in \mathcal{C}_\omega(G(F))$ is a *cusp form* if it satisfies one of the following equivalent conditions (see [45] Théorème VIII.4.2 and Lemme VIII.2.1 for the equivalence between these two conditions):

- For every proper parabolic subgroup $P = MU$ and all $x \in G(F)$ we have

$$\int_{U(F)} f(xu)du = 0;$$

- f is a sum of matrix coefficients of representations in $\text{Irr}_{\omega, \text{sqr}}(G)$.

We will denote by ${}^0\mathcal{C}_\omega(G(F))$ the space of cusp forms. Let $f \in {}^0\mathcal{C}_\omega(G(F))$ and set $f_\pi(g) := \text{Trace}(\pi^\vee(g^{-1})\pi^\vee(f))$ for all $\pi \in \text{Irr}_{\omega, \text{sqr}}(G(F))$ and all $g \in G(F)$. Then, f_π belongs to ${}^0\mathcal{C}_\omega(G(F))$ for all $\pi \in \text{Irr}_{\omega, \text{sqr}}(G(F))$ ([24] Theorem 29) and we have an equality

$$(1.6.1) \quad f = \sum_{\pi \in \text{Irr}_{\omega, \text{sqr}}(G)} d(\pi) f_\pi$$

(This is a special case of Harish-Chandra-Plancherel formula, see [45] Theorem VIII.4.2).

Let ${}^0\mathcal{S}_\omega(G(F)) := \mathcal{S}_\omega(G(F)) \cap {}^0\mathcal{C}_\omega(G(F))$ be the space of compactly supported cusp forms. Similar to the characterization of ${}^0\mathcal{C}_\omega(G(F))$, a function $f \in \mathcal{S}_\omega(G(F))$ belongs to ${}^0\mathcal{S}_\omega(G(F))$ if and only if it satisfies one the the following equivalent conditions:

- For every proper parabolic subgroup $P = MU$ and all $x \in G(F)$ we have

$$\int_{U(F)} f(xu)du = 0;$$

- f is a sum of matrix coefficients of representations in $\text{Irr}_{\omega, \text{cusp}}(G)$.

Moreover, for $f \in {}^0\mathcal{S}_\omega(G(F))$, we have $f_\pi \in {}^0\mathcal{S}_\omega(G(F))$ for all $\pi \in \text{Irr}_{\omega, \text{cusp}}(G)$ and a spectral decomposition

$$(1.6.2) \quad f = \sum_{\pi \in \text{Irr}_{\omega, \text{cusp}}(G)} d(\pi) f_\pi$$

Finally, we will need the following proposition.

Proposition 1.6.1 *Let $\pi \in \text{Irr}_{\text{sqr}}(G)$ and let f be a matrix coefficient of π . Then, we have*

$$\Theta_f(x) = \frac{1}{d(\pi)} f(1) \Theta_\pi(x)$$

for all $x \in G_{\text{reg}}(F)$.

Proof: Unfortunately, the author has been unable to find a suitable reference for this probably well-known statement (however see [14] Proposition 5 for the case where x is elliptic and [2] for the case where π is supercuspidal). Let us say that it follows from a combination of Arthur's noninvariant local trace formula ([5], Proposition 4.1) applied to the case where one of the test functions is our f and of Schur orthogonality relations. Note that Arthur's local trace formula was initially only proved for compactly supported test functions, but see [6] Corollary 5.3 for the extension to Harish-Chandra Schwartz functions. ■

1.7 Tempered pairs

Let H be a unimodular algebraic subgroup of G (e.g. a reductive subgroup). We say that the pair (G, H) is *tempered* if there exists $d > 0$ such that the integral

$$\int_{H(F)} \Xi^G(h) \sigma(h)^{-d} dh$$

is convergent. This notion already appeared in [22] under the name of *strongly discrete* pairs. Following the referee suggestion we have decided to call these pairs *tempered* instead so that it is more in accordance with the notion of *strongly tempered* pairs introduced by Sakellaridis-Venkatesh in [44] §6 (since the latter implies the former but not conversely). This terminology is also justified by the fact that (G, H) is tempered if and only if the Haar measure on $H(F)$ defines a tempered distribution on $G(F)$ i.e. it extends to a continuous linear form on $\mathcal{C}(G(F))$. Moreover, by a result of Benoist and Kobayashi [8], when H is reductive and in the case where $F = \mathbb{R}$ (which is not properly speaking included in this paper) a pair (G, H) is tempered if and only if $L^2(H(F)\backslash G(F))$ is tempered as a unitary representation of $G(F)$. Although the author has not checked all the details, the proof of Benoist and Kobayashi seems to extend without difficulties to the p -adic case. However, we propose here a quick proof of one of the implications (but we won't use it in this paper).

Proposition 1.7.1 *Assume that the pair (G, H) is tempered. Then, the unitary representation of $G(F)$ on $L^2(H(F)\backslash G(F))$ given by right translation is tempered i.e. the Plancherel measure of $L^2(H(F)\backslash G(F))$ is supported on tempered representations.*

Proof: We will use the following criterion for temperedness due to Cowling-Haagerup-Howe [17]:

(1.7.1) Let (Π, \mathcal{H}) be a unitary representation of $G(F)$. Then (Π, \mathcal{H}) is tempered if and only if there exists $d > 0$ and a dense subspace $V \subset \mathcal{H}$ such that for all $u, v \in V$ we have

$$|(\Pi(g)u, v)| \ll \Xi^G(g) \sigma(g)^d$$

for all $g \in G(F)$ where (\cdot, \cdot) denotes the scalar product on \mathcal{H} .

We will check that this criterion is satisfied for $V = C_c^\infty(H(F)\backslash G(F)) \subset \mathcal{H} = L^2(H(F)\backslash G(F))$. Let $\varphi_1, \varphi_2 \in C_c^\infty(H(F)\backslash G(F))$ and choose $f_1, f_2 \in C_c^\infty(G(F))$ such that

$$\varphi_i(x) = \int_{H(F)} f_i(hx) dh$$

for $i = 1, 2$ and all $x \in H(F)\backslash G(F)$. Then, denoting by $R(g)$ the operator of right translation

by g and by (\cdot, \cdot) the L^2 -inner product on $L^2(H(F)\backslash G(F))$, we have

$$\begin{aligned}
(R(g)\varphi_1, \varphi_2) &= \int_{H(F)\backslash G(F)} \varphi_1(xg) \overline{\varphi_2(x)} dx \\
&= \int_{H(F)\backslash G(F)} \int_{H(F)\times H(F)} f_1(h_1xg) \overline{f_2(h_2x)} dh_2 dh_1 dx \\
&= \int_{H(F)\backslash G(F)} \int_{H(F)\times H(F)} f_1(h_1xg) \overline{f_2(h_2h_1x)} dh_2 dh_1 dx \\
&= \int_{G(F)} \int_{H(F)} f_1(\gamma g) \overline{f_2(h\gamma)} dh d\gamma
\end{aligned}$$

for all $g \in G(F)$. Let $d > 0$ that we will assume sufficiently large in what follows. As f_1 and f_2 are compactly supported, there obviously exist $C_1 > 0$ and $C_2 > 0$ such that

$$|f_1(\gamma)| \leq C_1 \Xi^G(\gamma) \sigma(\gamma)^{-2d} \text{ and } |f_2(\gamma)| \leq C_2 \Xi^G(\gamma) \sigma(\gamma)^{-d}$$

for all $\gamma \in G(F)$. It follows that for all $g \in G(F)$ we have

$$|(R(g)\varphi_1, \varphi_2)| \leq C_1 C_2 \int_{G(F)} \int_{H(F)} \Xi^G(\gamma g) \Xi^G(h\gamma) \sigma(h\gamma)^{-d} dh \sigma(\gamma g)^{-2d} d\gamma$$

Since $\sigma(\gamma_1\gamma_2)^{-1} \ll \sigma(\gamma_1)^{-1}\sigma(\gamma_2)$ for all $\gamma_1, \gamma_2 \in G(F)$, this last expression is essentially bounded by

$$\sigma(g)^{2d} \int_{G(F)} \int_{H(F)} \Xi^G(\gamma g) \Xi^G(h\gamma) \sigma(h)^{-d} dh \sigma(\gamma)^{-d} d\gamma$$

for all $g \in G(F)$. Let K be the special maximal compact subgroup used to define Ξ^G . Then Ξ^G is invariant both on the left and on the right by K and since $\sigma(k_1\gamma k_2)^{-1} \ll \sigma(\gamma)^{-1}$ for all $\gamma \in G(F)$ and $k_1, k_2 \in K$, we see that the last integral above is essentially bounded by

$$\sigma(g)^{2d} \int_{G(F)} \int_{H(F)} \int_{K \times K} \Xi^G(\gamma k_1 g) \Xi^G(h k_2 \gamma) dk_1 dk_2 \sigma(h)^{-d} dh \sigma(\gamma)^{-d} d\gamma$$

for all $g \in G(F)$. By the ‘doubling principle’ ([45] Lemme II.1.3), it follows that

$$|(R(g)\varphi_1, \varphi_2)| \ll \Xi^G(g) \sigma(g)^{2d} \int_{G(F)} \Xi^G(\gamma)^2 \sigma(\gamma)^{-d} d\gamma \times \int_{H(F)} \Xi^G(h) \sigma(h)^{-d} dh$$

for all $g \in G(F)$. By [45] Lemme II.1.5 and the assumption that (G, H) is tempered, for d sufficiently large the two integrals above are absolutely convergent. Thus, the criterion of Cowling-Haagerup-Howe is indeed satisfied for $V = C_c^\infty(H(F)\backslash G(F))$ and consequently $L^2(H(F)\backslash G(F))$ is tempered. ■

Finally, we include the following easy lemma which gives an alternative characterization of tempered pairs because it is how the tempered condition will be used in this paper.

Lemma 1.7.1 Set $A_G^H = (A_G \cap H)^0$. The pair (G, H) is tempered if and only if there exists $d > 0$ such that the integral

$$\int_{A_G^H(F) \backslash H(F)} \Xi^G(h) \bar{\sigma}(h)^{-d} dh$$

converges.

Proof: As Ξ^G is $A_G(F)$ invariant, it clearly suffices to show:

(1.7.2) For $d > 0$ sufficiently large, we have

$$\bar{\sigma}(h)^{-3d} \ll \int_{A_G^H(F)} \sigma(ah)^{-3d} da \ll \bar{\sigma}(h)^{-d}$$

for all $h \in H(F)$.

For this, we need first to observe that

$$(1.7.3) \quad \bar{\sigma}(h) \sim \inf_{a \in A_G^H(F)} \sigma(ah)$$

for all $h \in H(F)$. Indeed, as $A_G^H \backslash H$ is a closed subgroup of $A_G \backslash G$, this is equivalent to the fact that the projection $H \rightarrow A_G^H \backslash H$ has the norm descent property and this can be easily deduce from the existence of an algebraic subgroup H' of H such that the multiplication morphism $A_G^H \times H' \rightarrow H$ is surjective and finite (so that in particular $H'(F)$ has a finite number of orbits in $A_G^H(F) \backslash H(F)$).

By the inequalities $\bar{\sigma}(h) \ll \sigma(ah)$ and $\sigma(a) \ll \sigma(ah)\sigma(h)$ for all $a \in A_G^H(F)$ and all $h \in H(F)$, for any $d > 0$ we have

$$\int_{A_G^H(F)} \sigma(ah)^{-3d} da \ll \bar{\sigma}(h)^{-2d} \int_{A_G^H(F)} \sigma(ah)^{-d} da \ll \sigma(h)^d \bar{\sigma}(h)^{-2d} \int_{A_G^H(F)} \sigma(a)^{-d} da$$

for all $h \in H(F)$. For d sufficiently large, the last integral above is absolutely convergent. Moreover, as the left hand side of the above inequality is clearly invariant by $h \mapsto ah$ for any $a \in A_G^H(F)$, by 1.7.3, for d sufficiently large we get

$$\int_{A_G^H(F)} \sigma(ah)^{-3d} da \ll \left(\inf_{a \in A_G^H(F)} \sigma(ah) \right)^d \bar{\sigma}(h)^{-2d} \ll \bar{\sigma}(h)^{-d}$$

for all $h \in H(F)$ and this shows one half of 1.7.2. On the other hand, by the inequality $\sigma(ah) \ll \sigma(a)\sigma(h)$ for all $a \in A_G^H(F)$ and $h \in H(F)$, for any $d > 0$ we have

$$\sigma(h)^{-3d} \int_{A_G^H(F)} \sigma(a)^{-3d} da \ll \int_{A_G^H(F)} \sigma(ah)^{-3d} da$$

for all $h \in H(F)$. Once again, for d sufficiently large the two integrals above are absolutely convergent and, as the right hand side of the inequality is invariant by $h \mapsto ah$ for any $a \in A_G^H(F)$, by 1.7.3 for d sufficiently large we get

$$\bar{\sigma}(h)^{-3d} \ll \left(\inf_{a \in A_G^H(F)} \sigma(ah) \right)^{-3d} \ll \int_{A_G^H(F)} \sigma(ah)^{-3d} da$$

for all $h \in H(F)$ and this proves the second half of 1.7.2. ■

1.8 Symmetric varieties

1.8.1 Basic definition, θ -split subgroups

Let H be an algebraic subgroup of G . Recall that H is said to be *symmetric* if there exists an involutive automorphism θ of G (defined over F) such that

$$(G^\theta)^0 \subset H \subset G^\theta$$

where G^θ denotes the subgroup of θ -fixed elements. If this is the case, we say that H and θ are associated. The involution θ is not, in general, determined by H but by [26] Proposition 1.2, its restriction to the derived subgroup of G is. From now on and until the end of §1.8.2 we fix a symmetric subgroup H of G and we will denote by θ an associated involutive automorphism.

Let $T \subset G$ be a subtorus. We say that T is θ -split if $\theta(t) = t^{-1}$ for all $t \in T$ and we say that it is (θ, F) -split if it is θ -split as well as split as a torus over F . For every F -split subtorus $A \subset G$ we will denote by A_θ the maximal (θ, F) -split subtorus of A . A parabolic subgroup $P \subset G$ is said to be θ -split if $\theta(P)$ is a parabolic subgroup opposite to P . If this is the case, HP is open, for the Zariski topology, in G (this is because $\mathfrak{h} + \mathfrak{p} = \mathfrak{g}$) and similarly $H(F)P(F)$ is open, for the analytic topology, in $G(F)$. If P is a θ -split parabolic subgroup, we will say that the Levi component $M := P \cap \theta(P)$ of P is a θ -split Levi subgroup. Note that this terminology can be slightly confusing since a torus can be a θ -split Levi without being θ -split as a torus (e.g. for $G = GL(2)$, T the standard maximal torus and θ given by $\theta(g) = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} g \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$). Nevertheless, the author believe that no confusion should arise in this paper as the context will clarify which notion is being used.

Actually, a Levi subgroup $M \subset G$ is θ -split (i.e. it is the θ -split Levi component of a θ -split parabolic) if and only if M is the centralizer of a (θ, F) -split subtorus if and only if M is the centralizer of $A_{M,\theta}$. We will adapt Arthur's notation to θ -split Levi and parabolic subgroups as follows: if M is a θ -split Levi subgroup we will denote by $\mathcal{P}^\theta(M)$, resp. $\mathcal{F}^\theta(M)$, resp. $\mathcal{L}^\theta(M)$ the set of all θ -split parabolic subgroups with Levi component M , resp. containing M , resp. the set of all θ -split Levi subgroups containing M .

Let $M \subset G$ be a θ -split Levi subgroup. We set

$$\mathcal{A}_{M,\theta} := X_*(A_{M,\theta}) \otimes \mathbf{R}$$

and $a_{M,\theta} := \dim \mathcal{A}_{M,\theta}$. Note that we have

$$\mathcal{A}_M = \mathcal{A}_{M,\theta} \oplus \mathcal{A}_M^\theta$$

where as before a θ superscript indicates the subset of θ -fixed points. This decomposition is compatible with the decompositions

$$\mathcal{A}_M = \mathcal{A}_M^L \oplus \mathcal{A}_L$$

for all $L \in \mathcal{L}^\theta(M)$. Hence, we also have

$$\mathcal{A}_{M,\theta} = \mathcal{A}_{M,\theta}^L \oplus \mathcal{A}_{L,\theta}$$

for all $L \in \mathcal{L}^\theta(M)$, where we have set $\mathcal{A}_{M,\theta}^L := \mathcal{A}_{M,\theta} \cap \mathcal{A}_M^L$. Also, we let $a_{M,\theta}^L := \dim \mathcal{A}_{M,\theta}^L = a_{M,\theta} - a_{L,\theta}$. We define an homomorphism $H_{M,\theta} : M(F) \rightarrow \mathcal{A}_{M,\theta}$ as the composition of the homomorphism H_M with the projection $\mathcal{A}_M \twoheadrightarrow \mathcal{A}_{M,\theta}$. For all $P \in \mathcal{P}^\theta(M)$, the roots $R(\mathcal{A}_{M,\theta}, U_P)$ of $\mathcal{A}_{M,\theta}$ in the unipotent radical U_P of P can be considered as elements of the dual space $\mathcal{A}_{M,\theta}^*$ of $\mathcal{A}_{M,\theta}$. There is a unique subset $\Delta_{P,\theta} \subset R(\mathcal{A}_{M,\theta}, U_P)$ such that every element of $(\mathcal{A}_{M,\theta}, U_P)$ is in a unique way a nonnegative integral linear combination of elements of $\Delta_{P,\theta}$. The set $\Delta_{P,\theta}$ is the image of Δ_P by the natural projection $\mathcal{A}_M^* \twoheadrightarrow \mathcal{A}_{M,\theta}^*$ and it forms a basis of $(\mathcal{A}_{M,\theta}^G)^*$. We call it the set of simple roots of $\mathcal{A}_{M,\theta}$ in P . Define

$$\mathcal{A}_{P,\theta}^+ := \{X \in \mathcal{A}_M; \langle \alpha, X \rangle > 0 \forall \alpha \in \Delta_{P,\theta}\}$$

Then, we have the decomposition

$$\mathcal{A}_{M,\theta} = \bigsqcup_{Q \in \mathcal{F}^\theta(M)} \mathcal{A}_{Q,\theta}^+$$

More precisely the set $R(\mathcal{A}_{M,\theta}, G)$ of roots of $\mathcal{A}_{M,\theta}$ in G divides $\mathcal{A}_{M,\theta}$ into certain facets which are exactly the cones $\mathcal{A}_{Q,\theta}^+$ where $Q \in \mathcal{F}^\theta(M)$. In a similar way, the subspaces supporting the facets of this decomposition are precisely the subspaces of the form $\mathcal{A}_{L,\theta}$, $L \in \mathcal{L}^\theta(M)$, whereas the chambers (i.e. the open facets) are precisely the cones $\mathcal{A}_{P,\theta}^+$ for $P \in \mathcal{P}^\theta(M)$. Fixing a maximal special compact subgroup K of $G(F)$, for all $P \in \mathcal{P}^\theta(M)$, we define a map

$$H_{P,\theta} : G(F) \rightarrow \mathcal{A}_{M,\theta}$$

as the composition of H_P with the projection $\mathcal{A}_M \twoheadrightarrow \mathcal{A}_{M,\theta}$ i.e. we have $H_{P,\theta}(muk) = H_{M,\theta}(m)$ for all $m \in M(F)$, $u \in U_P(F)$ and $k \in K$.

Let A_0 be a maximal (θ, F) -split subtorus and let M_0 be its centralizer in G . For simplicity we set $\mathcal{A}_0 := \mathcal{A}_{M_0,\theta}$. It is known that the set of roots $R(\mathcal{A}_0, G)$ of \mathcal{A}_0 in G forms a root system in the dual space $(\mathcal{A}_0^G)^*$ to \mathcal{A}_0^G (Proposition 5.9 of [26]). The Weyl group associated to this root system is naturally isomorphic to

$$W(G, A_0) := \text{Norm}_{G(F)}(A_0)/M_0(F)$$

and is called the little Weyl group (associated to A_0) (again Proposition 5.9 of [26]). Two maximal (θ, F) -split subtori are not necessarily $H(F)$ -conjugate (e.g. for $G = GL_n$ and $H = O(n)$) but they are always $G(F)$ -conjugate ¹.

Let M be a θ -split Levi subgroup and let $\alpha \in R(A_{M,\theta}, G)$. Then we define a ‘coroot’ $\alpha^\vee \in \mathcal{A}_{M,\theta}^G$ as follows. First assume that α is a reduced root (i.e. $\frac{\alpha}{2} \notin R(A_{M,\theta}, G)$). Let M_α be the unique θ -split Levi containing M such that $\mathcal{A}_{M_\alpha,\theta} = \text{Ker}(\alpha)$. Let Q_α be the unique θ -split parabolic subgroup of M_α with θ -split Levi M such that $\Delta_{Q_\alpha} = \{\alpha\}$. Let $P_0^{M_\alpha}$ be a minimal θ -split parabolic subgroup of M_α contained in Q_α and set $M_0 := P_0^{M_\alpha} \cap \theta(P_0^{M_\alpha})$, $A_0 := A_{M_0,\theta}$. Let $\Delta_0^{M_\alpha}$ be the set of simple roots of A_0 in $P_0^{M_\alpha}$. Then there is a unique simple root $\beta \in \Delta_0^{M_\alpha}$ whose projection to $\mathcal{A}_{M,\theta}^*$ equals α . Let $\beta^\vee \in \mathcal{A}_0$ be the corresponding coroot. Then we define α^\vee as the image of β^\vee by the projection $\mathcal{A}_0 \rightarrow \mathcal{A}_{M,\theta}$. We easily check that this construction does not depend on the choice of $P_0^{M_\alpha}$ since for another choice $P_0'^{M_\alpha}$ with $M_0' := P_0'^{M_\alpha} \cap \theta(P_0'^{M_\alpha})$ and $A_0' := A_{M_0',\theta}$ there exists $m \in M(F)$ with $mA_0'm^{-1} = A_0$ and $mP_0'^{M_\alpha}m^{-1} = P_0^{M_\alpha}$. If α is nonreduced, there exists $\alpha_0 \in R(M,\theta, G)$ such that $\alpha = 2\alpha_0$ and we simply set $\alpha^\vee = \frac{\alpha_0^\vee}{2}$.

Let $\tilde{\alpha} \in R(A_M, G)$ be a root extending α and $\tilde{\alpha}^\vee \in \mathcal{A}_M$ the corresponding coroot. Then, in general the projection of $\tilde{\alpha}^\vee$ to $\mathcal{A}_{M,\theta}$ does not coincide with α^\vee as defined above but, however, the two are always positively proportional. Finally, we remark that when M is a minimal θ -split Levi subgroup, so that $R(A_{M,\theta}, G)$ is a root system, then for all $\alpha \in R(A_{M,\theta}, G)$, α^\vee coincides with the usual coroot defined using this root system.

Let P be a θ -split parabolic subgroup. We set $\mathcal{A}_{P,\theta} := \mathcal{A}_{M,\theta}$ and $a_{P,\theta} = a_{M,\theta}$ where $M := P \cap \theta(P)$ and we let $\Delta_{P,\theta}^\vee \subseteq \mathcal{A}_{M,\theta}^G$ be the set of simple coroots corresponding to $\Delta_{P,\theta} \subseteq (\mathcal{A}_{M,\theta}^G)^*$ and $\hat{\Delta}_{P,\theta} \subseteq (\mathcal{A}_{M,\theta}^G)^*$ be the basis dual to $\Delta_{P,\theta}^\vee$. More generally, let $Q \supset P$ be another θ -split Levi subgroup. We set $\mathcal{A}_{P,\theta}^Q := \mathcal{A}_{M,\theta}^L$ and $a_{P,\theta}^Q := a_{M,\theta}^L$ where $L := Q \cap \theta(Q)$ and we let $\Delta_{P,\theta}^Q \subseteq (\mathcal{A}_{P,\theta}^Q)^*$ be the set of simple roots of $A_{M,\theta}$ in $P \cap L$, $(\Delta_{P,\theta}^Q)^\vee \subseteq \mathcal{A}_{P,\theta}^Q$ be the corresponding set of simple coroots and $\hat{\Delta}_{P,\theta}^Q \subseteq (\mathcal{A}_{P,\theta}^Q)^*$ be the basis dual to $(\Delta_{P,\theta}^Q)^\vee$. We have decompositions

$$\mathcal{A}_{P,\theta} = \mathcal{A}_{P,\theta}^Q \oplus \mathcal{A}_{Q,\theta}, \quad \mathcal{A}_{P,\theta}^* = (\mathcal{A}_{P,\theta}^Q)^* \oplus \mathcal{A}_{Q,\theta}^*$$

for which $\Delta_{P,\theta}^Q \subseteq \Delta_{P,\theta}$, $(\Delta_{P,\theta}^Q)^\vee \subseteq \Delta_{P,\theta}^\vee$ and moreover $\Delta_{Q,\theta}$ (resp. $\Delta_{Q,\theta}^\vee$) is the image of $\Delta_{P,\theta} - \Delta_{P,\theta}^Q$ (resp. $\Delta_{P,\theta}^\vee - (\Delta_{P,\theta}^Q)^\vee$) by the projection $\mathcal{A}_{P,\theta}^* \rightarrow \mathcal{A}_{Q,\theta}^*$ (resp. $\mathcal{A}_{P,\theta} \rightarrow \mathcal{A}_{Q,\theta}$). We define the following functions:

- $\tau_{P,\theta}^Q$: characteristic function of the set of $X \in \mathcal{A}_{M,\theta}$ such that $\langle \alpha, X \rangle > 0$ for all $\alpha \in \Delta_{P,\theta}^Q$;
- $\hat{\tau}_{P,\theta}^Q$: characteristic function of the set of $X \in \mathcal{A}_{M,\theta}$ such that $\langle \varpi_\alpha, X \rangle > 0$ for all $\varpi_\alpha \in \hat{\Delta}_{P,\theta}^Q$;

¹Indeed if A_0 and A_0' are two maximal (θ, F) -split tori, $M_0 := \text{Cent}_G(A_0), M_0' := \text{Cent}_G(A_0')$, $P_0 \in \mathcal{P}^\theta(M_0)$ and $P_0' \in \mathcal{P}^\theta(M_0')$ then by [26] Proposition 4.9, P_0 and P_0' are conjugated by an element of $g \in G(F) \cap HP_0$ and it suffices to show that we can take g in $G(F) \cap HM_0$ but this follows from the fact that $H \cap P_0 = H \cap M_0$

- $\delta_{M,\theta}^Q$: characteristic function of the subset $\mathcal{A}_{L,\theta}$ of $\mathcal{A}_{M,\theta}$.

We also define a function $\Gamma_{P,\theta}^Q$ on $\mathcal{A}_{M,\theta} \times \mathcal{A}_{M,\theta}$, whose utility will be revealed in the next section, by

$$\Gamma_{P,\theta}^Q(H, X) := \sum_{R \in \mathcal{F}^\theta(M); P \subseteq R \subseteq Q} (-1)^{a_{R,\theta} - a_{Q,\theta}} \tau_{P,\theta}^R(H) \widehat{\tau}_{R,\theta}^Q(H - X)$$

Let M be a θ -split Levi subgroup. Then, for all $P \in \mathcal{P}^\theta(M)$ we set

$$\mathcal{A}_{P,\theta}^{+,*} := \{\lambda \in \mathcal{A}_{P,\theta}^*; \langle \alpha^\vee, \lambda \rangle > 0 \forall \alpha^\vee \in \Delta_{P,\theta}^\vee\}$$

As for $\mathcal{A}_{M,\theta}$, the set of coroots $R(A_{M,\theta}, G)^\vee$ divides $\mathcal{A}_{M,\theta}^*$ into facets which are exactly the cones $\mathcal{A}_{Q,\theta}^{+,*}$ for $Q \in \mathcal{F}^\theta(M)$ and the chambers for this decomposition are the $\mathcal{A}_{P,\theta}^{+,*}$, $P \in \mathcal{P}^\theta(M)$. As usual, we say that two parabolics $P, P' \in \mathcal{P}^\theta(M)$ are adjacent if the intersection of the closure of their corresponding chambers contains a facet of codimension one. If this is the case, the hyperplane generated by this intersection is called the wall separating the two chambers.

1.8.2 (G, M, θ) -families and orthogonal sets

As we have recalled in the previous section, the combinatorics of θ -split Levi and parabolic subgroups is entirely governed, as is the case for classical Levi and parabolic subgroups, by a root system. As a consequence, for M a θ -split Levi subgroup of G the classical theory of (G, M) -families due to Arthur extends without difficulty to a theory of (G, M, θ) -families indexed by θ -split parabolics that we now introduce. By definition, a (G, M, θ) -family is a family $(\varphi_{P,\theta})_{P \in \mathcal{P}^\theta(M)}$ of C^∞ functions on $i\mathcal{A}_{M,\theta}^*$ such that for any two adjacent parabolic subgroups $P, P' \in \mathcal{P}^\theta(M)$, the functions $\varphi_{P,\theta}$ and $\varphi_{P',\theta}$ coincide on the wall separating the chambers $i\mathcal{A}_{P,\theta}^{+,*}$ and $i\mathcal{A}_{P',\theta}^{+,*}$. To a (G, M, θ) -family $(\varphi_{P,\theta})_{P \in \mathcal{P}^\theta(M)}$ we can associate a scalar $\varphi_{M,\theta}$ as follows: the function

$$\varphi_{M,\theta}(\lambda) := \sum_{P \in \mathcal{P}^\theta(M)} \varphi_{P,\theta}(\lambda) \varepsilon_{P,\theta}(\lambda), \quad \lambda \in i\mathcal{A}_{M,\theta}^*$$

where we have set

$$\varepsilon_{P,\theta}(\lambda) := \text{meas}(\mathcal{A}_{M,\theta}^G / \mathbf{Z}[\Delta_{P,\theta}^\vee]) \prod_{\alpha^\vee \in \Delta_{P,\theta}^\vee} \langle \lambda, \alpha^\vee \rangle^{-1}$$

is C^∞ and we define $\varphi_{M,\theta} := \varphi_{M,\theta}(0)$. Note that we need a Haar measure on $\mathcal{A}_{M,\theta}^G$ for the definition of the functions $\varepsilon_{P,\theta}$ ($P \in \mathcal{P}^\theta(M)$) to make sense. We fix one as follows. Let $\overline{A}_{M,\theta}$ be the image of $A_{M,\theta}$ in $\overline{G} := G/A_G$ and let $A'_{M,\theta}$ be the inverse image of $\overline{A}_{M,\theta}$ in G . Let $\mathcal{A}_{M,\theta,F}^G \subseteq \mathcal{A}_{M,\theta}^G$ denote the image of $A'_{M,\theta}(F)$ by $\dot{H}_{M,\theta}^G$. It is a lattice of $\mathcal{A}_{M,\theta}^G$ and we choose our measure so that the quotient $\mathcal{A}_{M,\theta}^G / \mathcal{A}_{M,\theta,F}^G$ is of measure one.

We will actually only need (G, M, θ) -families of a very particular shape obtained as follows. We say that a family of points $\mathcal{Y}_{M,\theta} = (Y_{P,\theta})_{P \in \mathcal{P}^\theta(M)}$ in $\mathcal{A}_{M,\theta}$ is a (G, M, θ) -orthogonal set if for all adjacent $P, P' \in \mathcal{P}^\theta(M)$ we have

$$Y_{P,\theta} - Y_{P',\theta} = r_{P,P'} \alpha^\vee$$

where $r_{P,P'} \in \mathbf{R}$ and $\{\alpha^\vee\} = \Delta_{P,\theta}^\vee \cap -\Delta_{P',\theta}^\vee$. We say that the family is a *positive* (G, M, θ) -orthogonal set if it is a (G, M, θ) -orthogonal set and moreover $r_{P,P'} \geq 0$ for all adjacent $P, P' \in \mathcal{P}^\theta(M)$. To a (G, M, θ) -orthogonal set $\mathcal{Y}_{M,\theta} = (Y_{P,\theta})_{P \in \mathcal{P}^\theta(M)}$ we associate the (G, M, θ) -family $(\varphi_{P,\theta}(\cdot, \mathcal{Y}_{M,\theta}))_{P \in \mathcal{P}^\theta(M)}$ defined by

$$\varphi_{P,\theta}(\lambda, \mathcal{Y}_{M,\theta}) := e^{\langle \lambda, Y_{P,\theta} \rangle}, \quad \lambda \in i\mathcal{A}_{M,\theta}^*$$

and we let $v_{M,\theta}(\mathcal{Y}_{M,\theta}) := \varphi_{M,\theta}(0, \mathcal{Y}_{M,\theta})$ be the scalar associated to this (G, M, θ) -family. If $\mathcal{Y}_{M,\theta}$ is a positive (G, M, θ) -orthogonal set then $v_{M,\theta}(\mathcal{Y}_{M,\theta})$ is just the volume of the convex hull of the elements in the family $\mathcal{Y}_{M,\theta}$ (with respect to the fixed Haar measure on $\mathcal{A}_{M,\theta}^G$). For any (G, M, θ) -orthogonal set $\mathcal{Y}_{M,\theta} = (Y_{P,\theta})_{P \in \mathcal{P}^\theta(M)}$ and all $Q \in \mathcal{F}^\theta(M)$ we define $Y_{Q,\theta}$ to be the projection of $Y_{P,\theta}$ to $\mathcal{A}_{Q,\theta}$ for any $P \in \mathcal{P}^\theta(M)$ with $P \subset Q$ (the result is independent of the choice of P) and more generally for any $Q, R \in \mathcal{F}^\theta(M)$ with $Q \subset R$ we let $Y_{Q,\theta}^R$ be the projection of $Y_{P,\theta}$ to $\mathcal{A}_{Q,\theta}^R$ for any $P \in \mathcal{P}^\theta(M)$ with $P \subset Q$. Then, for any $L \in \mathcal{L}^\theta(M)$ the family $\mathcal{Y}_{L,\theta} := (Y_{Q,\theta})_{Q \in \mathcal{P}^\theta(L)}$ forms a (G, L, θ) -orthogonal set.

Fixing a maximal special compact subgroup K of $G(F)$ to define maps $H_{P,\theta}$ ($P \in \mathcal{P}^\theta(M)$), for all $g \in G(F)$ the family $\mathcal{Y}_{M,\theta}(g) := (-H_{P,\theta}(g))_{P \in \mathcal{P}^\theta(M)}$ is a positive (G, M, θ) -orthogonal set. Indeed, the family $(-H_P(g))_{P \in \mathcal{P}(M)}$ is a positive (G, M) -orthogonal set in the classical sense of Arthur (see [2] §2) and thus for all $P = MU_P, P' = MU_{P'} \in \mathcal{P}^\theta(M)$ we have

$$-H_P(g) + H_{P'}(g) \in \sum_{\alpha \in R(A_M, U_P) \cap -R(A_M, U_{P'})} \mathbf{R}_+ \alpha^\vee.$$

As the projection of $R(A_M, U_P)$ (resp. $R(A_M, U_{P'})$) to $\mathcal{A}_{M,\theta}^*$ is $R(A_{M,\theta}, U_P)$ (resp. $R(A_{M,\theta}, U_{P'})$) and for all $\alpha \in R(A_M, G)$ the projection of the coroot α^\vee to $\mathcal{A}_{M,\theta}^*$ is positively proportional to $\bar{\alpha}^\vee$, where $\bar{\alpha} \in R(A_{M,\theta}, G)$ denotes the projection of α to $\mathcal{A}_{M,\theta}^*$, it follows that

$$-H_{P,\theta}(g) + H_{P',\theta}(g) \in \sum_{\alpha \in R(A_{M,\theta}, U_P) \cap -R(A_{M,\theta}, U_{P'})} \mathbf{R}_+ \alpha^\vee$$

for all $P, P' \in \mathcal{P}^\theta(M)$ i.e. $\mathcal{Y}_{M,\theta}(g)$ is a positive (G, M, θ) -orthogonal set.

We define

$$v_{M,\theta}(g) := v_{M,\theta}(\mathcal{Y}_{M,\theta}(g))$$

There is another easier way to obtain (G, M, θ) -orthogonal sets. It is as follows. Let $M_0 \subset M$ be a minimal θ -split Levi subgroup with little Weyl group W_0 . Fix $P_0 \in \mathcal{P}^\theta(M_0)$. Then, for all $X \in \mathcal{A}_0 := \mathcal{A}_{M_0,\theta}$ we define a (G, M_0, θ) -orthogonal set $\mathcal{Y}[X]_0 := (Y[X]_{P'_0,\theta})_{P'_0 \in \mathcal{P}^\theta(M_0)}$ by setting $Y[X]_{P'_0,\theta} := w_{P'_0} X$ for all $P'_0 \in \mathcal{P}^\theta(M_0)$ where $w_{P'_0} \in W_0$ is the unique element such

that $w_{P'_0}P_0 = P'_0$. By the general construction explained above this also yields a (G, M, θ) -orthogonal set $\mathcal{Y}[X]_{M,\theta} = (Y[X]_{P,\theta})_{P \in \mathcal{P}^\theta(M)}$.

Let $\mathcal{Y}_{M,\theta} = (Y_{P,\theta})_{P \in \mathcal{P}^\theta(M)}$ be a (G, M, θ) -orthogonal set. For $Q \in \mathcal{F}^\theta(M)$ we define a function $\Gamma_{M,\theta}^Q(\cdot, \mathcal{Y}_{M,\theta})$ on $\mathcal{A}_{M,\theta}$ by

$$\Gamma_{M,\theta}^Q(H, \mathcal{Y}_{M,\theta}) := \sum_{R \in \mathcal{F}^\theta(M); R \subset Q} \delta_{M,\theta}^R(H) \Gamma_{R,\theta}^Q(H, Y_{R,\theta})$$

where the functions $\delta_{M,\theta}^R$ and $\Gamma_{R,\theta}^Q$ have been defined in the previous section. Let $L = Q \cap \theta(Q)$. Fixing a norm $|\cdot|$ on $\mathcal{A}_{M,\theta}^L$, we have the following basic property concerning the support of this function (see [38] Corollaire 1.8.5):

(1.8.1) There exists $c > 0$ independent of $\mathcal{Y}_{M,\theta}$ such that for all $H \in \mathcal{A}_{M,\theta}$ with $\Gamma_{M,\theta}^Q(H, \mathcal{Y}_{M,\theta}) \neq 0$ we have $|H^Q| \leq c \sup_{P \in \mathcal{P}^\theta(M); P \subset Q} |Y_{P,\theta}^Q|$ where H^Q denotes the projection of H to $\mathcal{A}_{M,\theta}^L$.

Moreover, if the (G, M, θ) -orthogonal set $\mathcal{Y}_{M,\theta}$ is positive then $\Gamma_{M,\theta}^Q(\cdot, \mathcal{Y}_{M,\theta})$ is just the characteristic function of the set of $H \in \mathcal{A}_{M,\theta}$ such that H^Q belongs to the convex hull of $(Y_{P,\theta}^Q)_{P \in \mathcal{P}^\theta(M); P \subset Q}$ ([38] Proposition 1.8.7). Without assuming the positivity of our (G, M, θ) -orthogonal set, we have the identity ([38] Lemme 1.8.4(3))

$$(1.8.2) \quad \sum_{Q \in \mathcal{F}^\theta(M)} \Gamma_{M,\theta}^Q(H, \mathcal{Y}_{M,\theta}) \tau_{Q,\theta}^G(H - Y_{Q,\theta}) = 1$$

for all $H \in \mathcal{A}_{M,\theta}$.

Let \mathcal{R} be a free \mathbf{Z} -module of finite type. Recall that a *exponential-polynomial* on \mathcal{R} is a function on \mathcal{R} of the following form

$$f(Y) = \sum_{\chi \in \widehat{\mathcal{R}}} \chi(Y) p_\chi(Y)$$

where $\widehat{\mathcal{R}}$ denotes the group of complex (not necessarily unitary) characters of \mathcal{R} and for all $\chi \in \widehat{\mathcal{R}}$, p_χ is a ‘complex polynomial’ function on \mathcal{R} , i.e. an element of $Sym((\mathbf{C} \otimes \mathcal{R})^*)$, which is zero for all but finitely many $\chi \in \widehat{\mathcal{R}}$. If f is an exponential-polynomial on \mathcal{R} then a decomposition as above is unique, the set of characters $\chi \in \widehat{\mathcal{R}}$ such that $p_\chi \neq 0$ is called the *set of exponents* of f and p_1 (corresponding to $\chi = 1$ the trivial character) is called the *purely polynomial part* of f . Finally, we define the *degree* of f as the maximum, over all $\chi \in \widehat{\mathcal{R}}$, of the degree of the polynomials p_χ . We record the following lemma whose proof is elementary:

Lemma 1.8.1 *Let \mathcal{R} be a free \mathbf{Z} -module of finite type, let f be a exponential-polynomial on \mathcal{R} and let $C \subset \mathbf{R} \otimes \mathcal{R}$ be an open cone (with a vertex possibly different from the origin). Then, if the limit*

$$\lim_{\substack{Y \in \mathcal{R} \cap C \\ |Y| \rightarrow \infty}} f(Y)$$

exists it equals the constant term of the purely polynomial part of f .

Let $M_0 \subset M$ be a minimal θ -split Levi subgroup, $\mathcal{A}_0 := \mathcal{A}_{M_0, \theta}$ and fix $P_0 \in \mathcal{P}^\theta(M_0)$. For all $X \in \mathcal{A}_0$ we dispose of the (G, M, θ) -orthogonal set $\mathcal{Y}[X]_{M, \theta} = (Y[X]_{P, \theta})_{P \in \mathcal{P}^\theta(M)}$ defined above. We let $\mathcal{Y}_{M, \theta} + \mathcal{Y}[X]_{M, \theta} := (Y_{P, \theta} + Y[X]_{M, \theta})_{P \in \mathcal{P}^\theta(M)}$ be the sum of the two (G, M, θ) -orthogonal sets $\mathcal{Y}_{M, \theta}$ and $\mathcal{Y}[X]_{M, \theta}$. Obviously, it is also a (G, M, θ) -orthogonal set. We let

$$\tilde{v}_{M, \theta}(\mathcal{Y}_{M, \theta} + \mathcal{Y}[X]_{M, \theta}) := \int_{A_G(F) \backslash A'_{M, \theta}(F)} \Gamma_{M, \theta}^G(H_{M, \theta}(a), \mathcal{Y}_{M, \theta} + \mathcal{Y}[X]_{M, \theta}) da$$

where we recall that $A'_{M, \theta}$ is the subtorus generated by A_G and $A_{M, \theta}$. Let $\mathcal{A}_{0, F}$ denote the image of $A_{M_0, \theta}(F)$ by $H_{M_0, \theta}$. It is a lattice in \mathcal{A}_0 and we have the following lemma (combine [41] Lemme 1.7(ii) with equalities 1.5(2) and 1.3(7) of *loc.cit.*):

Lemma 1.8.2 *For every lattice $\mathcal{R} \subset \mathcal{A}_{0, \theta, F} \otimes \mathbf{Q}$ the function $X \in \mathcal{R} \mapsto \tilde{v}_{M, \theta}(\mathcal{Y}_{M, \theta} + \mathcal{Y}[X]_{M, \theta})$ is an exponential-polynomial whose degree and exponents belong to finite sets which are independent of $\mathcal{Y}_{M, \theta}$. Moreover, if we denote by $\tilde{v}_{M, \theta, 0}(\mathcal{Y}_{M, \theta}, \mathcal{R})$ the constant coefficient of the purely polynomial part of this exponential-polynomial there exists $c > 0$ depending only on \mathcal{R} such that for all $k \geq 1$ we have*

$$\left| \tilde{v}_{M, \theta, 0}(\mathcal{Y}_{M, \theta}, \frac{1}{k}\mathcal{R}) - v_{M, \theta}(\mathcal{Y}_{M, \theta}) \right| \leq ck^{-1} \left(\sup_{P \in \mathcal{P}^\theta(M)} |Y_{P, \theta}| \right)^{a_{M, \theta}^G}$$

Let now $\mathcal{Y}_M = (Y_P)_{P \in \mathcal{P}(M)}$ be a usual (G, M) -orthogonal set. This induces a (G, M, θ) -orthogonal set $\mathcal{Y}_{M, \theta} := (Y_{P, \theta})_{P \in \mathcal{P}^\theta(M)}$ where, for all $P \in \mathcal{P}^\theta(M)$, we denote by $Y_{P, \theta}$ the projection of Y_P to $\mathcal{A}_{M, \theta}$. The subspace $\mathcal{A}_{M, \theta} + \mathcal{A}_G$ of \mathcal{A}_M being special in the sense of [3]§7², we have a descent formula (Proposition 7.1 of *loc.cit.*)

$$(1.8.3) \quad v_{M, \theta}(\mathcal{Y}_{M, \theta}) = \sum_{L \in \mathcal{L}(M)} d_{M, \theta}^G(L) v_M^Q(\mathcal{Y}_M)$$

where for all $L \in \mathcal{L}(M)$, Q is a parabolic with Levi component L which depends on the choice of a generic point $\xi \in \mathcal{A}_M$ and $d_{M, \theta}^G(L)$ is a coefficient which is nonzero only if $\mathcal{A}_M^G = \mathcal{A}_M^{G, \theta} \oplus \mathcal{A}_M^L$. Moreover if $\mathcal{A}_M^{G, \theta} = 0$ then we have $d_{M, \theta}^G(G) = 1$. Let K be a special maximal compact subgroup of $G(F)$ that we use to define the maps H_P for $P \in \mathcal{P}(M)$

²Indeed with the notations of *loc.cit.* we need to check that for every root $\beta \in R(A_{M, \theta}, G)$ the sum $\sum_{\alpha \in \Sigma(\beta)} m_\alpha \alpha$ is trivial on \mathcal{A}_M^Q but this is trivial since for all $\alpha \in \Sigma(\beta)$ we have $\iota(\alpha) := -\theta(\alpha) \in \Sigma(\beta)$ and $m_{\iota(\alpha)} = m_\alpha$

and $H_{P,\theta}$ for $P \in \mathcal{P}^\theta(M)$. Then, the formula 1.8.3 applied to the particular case where $Y_P = H_P(g)$ for all $P \in \mathcal{P}(M)$ and for some $g \in G(F)$ yields

$$(1.8.4) \quad v_{M,\theta}(g) = \sum_{L \in \mathcal{L}(M)} d_{M,\theta}^G(L) v_M^Q(g)$$

1.9 Estimates

In this section we collect some estimates that we will need in the core of the paper. We start with four lemmas concerning maximal tori and integrals over regular orbits in G .

Lemma 1.9.1 *Let $T \subset G$ be a maximal torus. Then, we have*

$$\sigma(t) \ll \sigma(g^{-1}tg)$$

for all $t \in T$ and all $g \in G$.

Proof: Let $W := W(G_{\overline{F}}, T_{\overline{F}})$ be the absolute Weyl group of T and set $\mathcal{B} := G//G - \text{Ad}$ (i.e. the GIT quotient of G acting on itself by the adjoint action). Let $p : G \rightarrow \mathcal{B}$ be the natural projection. By Chevalley theorem, the inclusion $T \hookrightarrow G$ induces an isomorphism $T//W \simeq \mathcal{B}$ and thus the restriction of p to T is a finite morphism. Hence, we have

$$\sigma(t) \sim \sigma_{\mathcal{B}}(p(t))$$

for all $t \in T$ and it follows that

$$\sigma(t) \sim \sigma_{\mathcal{B}}(p(t)) = \sigma_{\mathcal{B}}(p(g^{-1}tg)) \ll \sigma(g^{-1}tg)$$

for all $t \in T$ and all $g \in G$. ■

Lemma 1.9.2 (Harish-Chandra, Clozel) *Let $T \subset G$ be a maximal torus. Then, for all $d > 0$ there exists $d' > 0$ such that*

$$D^G(t)^{1/2} \int_{T(F) \backslash G(F)} \Xi^G(g^{-1}tg) \sigma(g^{-1}tg)^{-d'} dg \ll \sigma(t)^{-d}$$

for all $t \in T_{\text{reg}}(F)$.

Proof: By Corollary 2 of [14] there exists $d_0 > 0$ such that

$$\sup_{t \in T_{\text{reg}}(F)} D^G(t)^{1/2} \int_{T(F) \backslash G(F)} \Xi^G(g^{-1}tg) \sigma(g^{-1}tg)^{-d_0} dg < \infty$$

Thus by Lemma 1.9.1, for all $d > 0$ we have

$$D^G(t)^{1/2} \int_{T(F) \backslash G(F)} \Xi^G(g^{-1}tg) \sigma(g^{-1}tg)^{-d_0-d} dg \ll \sigma(t)^{-d}$$

for all $t \in T_{\text{reg}}(F)$. ■

Lemma 1.9.3 *Let $T \subset G$ be a subtorus such that $T_{\text{reg}} := T \cap G_{\text{reg}}$ is nonempty (i.e. T contains nonsingular elements). Then, for all $k > 0$, there exists $d > 0$ such that the integral*

$$\int_{T(F)} \log(2 + D^G(t)^{-1})^k \sigma(t)^{-d} dt$$

converges.

Proof: We denote by $X_{\overline{F}}^*(T)$ the group of regular characters of T defined over \overline{F} . There exists a multiset Σ of nontrivial elements in $X_{\overline{F}}^*(T)$ such that

$$D^G(t) = \prod_{\alpha \in \Sigma} |\alpha(t) - 1|$$

for all $t \in T_{\text{reg}}(F)$ where we have denoted by $|\cdot|$ the unique extension of the absolute value over F to \overline{F} . We have

$$\log(2 + D^G(t)^{-1}) \ll \prod_{\alpha \in \Sigma} \log(2 + |\alpha(t) - 1|^{-1})$$

Thus, by Cauchy-Schwartz, it suffices to prove the following claim:

(1.9.1) For all $\alpha \in X_{\overline{F}}^*(T) - \{1\}$ and all $k > 0$ there exists $d > 0$ such that the integral

$$\int_{T(F)} \log(2 + |\alpha(t) - 1|^{-1})^k \sigma(t)^{-d} dt$$

converges.

Let $\alpha \in X_{\overline{F}}^*(T) - \{1\}$ and let $\Gamma_\alpha \subset \Gamma_F$ be the stabilizer of α for the natural Galois action. Write $\Gamma_\alpha = \text{Gal}(\overline{F}/F_\alpha)$ where F_α/F is a finite extension. By the universal property of restriction of scalars, α induces a morphism $\tilde{\alpha} : T \rightarrow R_{F_\alpha/F} \mathbf{G}_m$. Denoting by $\text{Ker}(\tilde{\alpha})$ the kernel of $\tilde{\alpha}$, for all $k > 0$ and all $d > 0$ we have

$$\int_{T(F)} \log(2 + |\alpha(t) - 1|^{-1})^k \sigma(t)^{-k} dt = \int_{T(F)/\text{Ker}(\tilde{\alpha})(F)} \log(2 + |\alpha(t) - 1|^{-1})^k \int_{\text{Ker}(\tilde{\alpha})(F)} \sigma(tt')^{-d} dt' dt$$

As there exists a subtorus $T' \subset T$ such that the multiplication map $T' \times \text{Ker}(\tilde{\alpha}) \rightarrow T$ is an isogeny (so that $\sqrt{\sigma(t)\sigma(t')} \ll \sigma(tt')$ and $\sigma(t) \sim \sigma_{T/\text{Ker}(\tilde{\alpha})}(t)$ for all $(t, t') \in \text{Ker}(\tilde{\alpha}) \times T'$), we see that for all d sufficiently large (i.e. so that the integral below converges) we have

$$\int_{\text{Ker}(\tilde{\alpha})(F)} \sigma(tt')^{-d} dt' \ll \sigma_{T/\text{Ker}(\tilde{\alpha})}(t)^{-d/2}$$

for all $t \in T(F)/\text{Ker}(\tilde{\alpha})(F)$. Since $T(F)/\text{Ker}(\tilde{\alpha})(F)$ is an open subset of $(T/\text{Ker}(\tilde{\alpha}))(F)$ we are thus reduced to the case where $\tilde{\alpha}$ is an embedding.

Define $N(\alpha) \in X^*(T)$ by

$$N(\alpha) := \prod_{\sigma \in \Gamma_F / \Gamma_\alpha} \sigma(\alpha)$$

We distinguish two cases. First, if $N(\alpha) \neq 1$ then we have an inequality

$$\log(2 + |\alpha(t) - 1|^{-1}) \ll \log(2 + |N(\alpha)(t) - 1|^{-1})$$

for all $t \in T(F)$ with $N(\alpha)(t) \neq 1$. Hence, up to replacing α by $N(\alpha)$ we may assume that $\alpha \in X^*(T)$ in which case by the previous reduction we are left to prove 1.9.1 in the particular case where $T = \mathbf{G}_m$ and $\alpha = Id$ in which case it is easy to check. Assume now that $N(\alpha) = 1$. Since $\tilde{\alpha}$ is an embedding this implies that T is anisotropic and we just need to prove that for all $k > 0$ the function

$$t \in T(F) \mapsto \log(2 + |\alpha(t) - 1|^{-1})^k$$

is locally integrable. Using the exponential map we are reduced to proving a similar statement for vector spaces where we replace $\alpha(t) - 1$ by a linear form which is easy to check directly. ■

Combining 1.2.2 with Lemma 1.9.2 and Lemma 1.9.3 we get the following:

Lemma 1.9.4 *Let $T \subset G$ be a subtorus such that $T_{\text{reg}} := T \cap G_{\text{reg}}$ is nonempty and let T^G be the centralizer of T in G (a maximal torus). Then, for all $k > 0$ there exists $d > 0$ such that the integral*

$$\int_{T(F)} D^G(t)^{1/2} \int_{T^G(F) \backslash G(F)} \Xi^G(g^{-1}tg) \sigma(g^{-1}tg)^{-d} \sigma_{T^G \backslash G}(g)^k dg dt$$

converges.

The following lemma will be needed in the proof of the next proposition. As it might be of independent interest we present it separately.

Lemma 1.9.5 *Let G be an anisotropic group over F and Y an affine G -variety. Set $Y' = Y/G$ for the GIT quotient (it is an affine algebraic variety over F) and denote by $p : Y \rightarrow Y'$ the natural projection. Then we have*

$$\sigma_Y(y) \sim \sigma_{Y'}(p(y))$$

for all $y \in Y(F)$.

Proof: First, we have $\sigma_{Y'}(p(y)) \ll \sigma_Y(y)$ for all $y \in Y$ since p is a morphism of algebraic varieties. Let $f \in F[Y]$, we need to show that

$$\log(2 + |f(y)|) \ll \sigma_{Y'}(p(y))$$

for all $y \in Y(F)$. Let W be the G -submodule of $F[Y]$ generated by f and V be its dual. There is a natural morphism $\varphi : Y \rightarrow V$ and we have a commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & Y' = Y/G \\ \downarrow \varphi & & \downarrow \varphi' \\ V & \longrightarrow & V' := V/G \end{array}$$

By definition there is a function $f_V \in F[V]$ such that $f = f_V \circ \varphi$ and moreover $\sigma_{V'}(y') \ll \sigma_Y(y')$ for all $y' \in Y'$. Hence, we are reduced to the case where $Y = V$ and we may assume that f is homogeneous. By Kempf's extension of the stability criterion of Mumford over any perfect field ([30], Corollary 5.1) and since G is anisotropic, for every $v \in V(F)$ the G -orbit $G.v \subset V$ is closed. It follows that there exist homogeneous polynomials $P_1, \dots, P_N \in F[V'] = F[V]^G$ whose only common zero in $V(F)$ is 0. We only need to show that for some $R, C > 0$ we have

$$(1.9.2) \quad \max(1, |f(v)|) \leq C \max(1, |P_1(v)|, \dots, |P_N(v)|)^R$$

for all $v \in V(F)$. Up to replacing f, P_1, \dots, P_N by some powers, we may assume that they are all of the same degree. Then, for every $1 \leq i \leq N$, f/P_i is a rational function on the projective space $\mathbf{P}(V)$ and the map

$$[v] \in \mathbf{P}(V)(F) \mapsto \min(|(f/P_1)([v])|, \dots, |(f/P_N)([v])|) \in \mathbf{R}_+$$

is continuous for the analytic topology hence bounded (as $\mathbf{P}(V)(F)$ is compact) and this proves that inequality 1.9.2 is true for $R = 1$ and some constant C . ■

Following [31] Definition 4.9, we say that a subgroup $H \subset G$ is *F-spherical* if there exists a minimal parabolic subgroup P_0 of G such that HP_0 is open, in the Zariski topology, in G . For example symmetric subgroups (see §1.8.1) are *F-spherical*. Recall that in §1.2 we have defined a ‘norm descent property’ for regular maps between *F*-varieties.

Proposition 1.9.1 *Let $H \subset G$ be an *F-spherical* subgroup. Then, the natural projection $p : G \rightarrow H \backslash G$ has the norm descent property.*

Proof: Set $X := H \backslash G$. By [36] Proposition 18.2 (1), it suffices to show that X can be covered by Zariski open subsets over which the projection p has the norm descent property. Since G acts transitively on X it even suffices to construct only one such open subset (because its G -translates will have the same property). By the local structure theorem ([31] Corollary 4.12), there exists a parabolic subgroup $Q = LU$ of G such that

- $\mathcal{U} = HQ$ is open in G ;
- $H \cap Q = H \cap L$ and this subgroup contains the non-anisotropic factors of the derived subgroup of L .

Obviously, to show that the restriction of p to \mathcal{U} has the norm descent property it is sufficient to establish that $L \rightarrow H \cap L \backslash L$ has the norm descent property. We are thus reduced to the case where H contains all the non-anisotropic factors of the derived subgroup of G . Let G_{der} denote the derived subgroup of G , $G_{der,nc}$ the product of the non-anisotropic factors of G_{der} , $G_{der,c}$ the product of the anisotropic factors of G_{der} and set $G' = G_{der,c}Z(G)^0$, $H' = H \cap G'$. Then, we have $H = G_{der,nc}H'$ and the multiplication map $G_{der,nc} \times G' \rightarrow G$ is an isogeny. It follows that there exists a finite set $\{\gamma_i; i \in I\}$ of elements of $G(F)$ such that

$$G(F) = \bigsqcup_{i \in I} G_{der,nc}(F)G'(F)\gamma_i$$

and

$$H(F) \backslash G(F) = \bigcup_{i \in I} H'(F) \backslash G'(F)\gamma_i$$

From these decompositions, we infer that we only need to prove the norm descent property for $G' \rightarrow H' \backslash G'$ i.e. we may assume that $G_{der,nc} = 1$. Let G_c be the product of $G_{der,c}$ with the maximal anisotropic subtorus of $Z(G)^0$ and consider the projection

$$p' : X := H \backslash G \rightarrow X' := HG_c \backslash G$$

We claim that

$$(1.9.3) \quad \sigma_X(x) \sim \sigma_{X'}(p'(x))$$

for all $x \in X(F)$. As H is reductive (since $G(F)$ contains no unipotent element), X is affine and the claim follows from the Lemma 1.9.5.

Now because of 1.9.3, we may replace X by X' i.e. we may assume that H contains G_c . As the multiplication map $G_c \times A_G \rightarrow G$ is an isogeny, by a similar argument as before we are reduced to the case where G is a split torus for which the proposition is easy to establish directly. ■

2 Definition of a distribution for all symmetric pairs

2.1 The statement

Let G be a connected reductive group over F , H be a symmetric subgroup of G and θ be the involution of G associated to H (see §1.8.1). Set $A_G^H = (A_G \cap H)^0$, $\overline{G} := G/A_G$, $\overline{H} := H/A_H$, $X := A_G(F)H(F) \backslash G(F)$, $\mathbf{X} := HA_G \backslash G$, $\sigma_X := \sigma_{\mathbf{X}}$ and $\overline{\sigma} := \sigma_{\overline{G}}$. Note that X is an open subset of $\mathbf{X}(F)$. Let χ and ω be continuous unitary characters of $H(F)$ and $A_G(F)$ respectively such that $\chi|_{A_G^H(F)} = \omega|_{A_G^H(F)}$. Then, for all $f \in \mathcal{S}_\omega(G(F))$ we define a function K_f^χ on X by

$$K_f^\chi(x) := \int_{A_G^H(F) \backslash H(F)} f(x^{-1}hx) \chi(h)^{-1} dh$$

If the pair (G, H) is tempered then the expression defining K_f^χ makes sense for all $f \in \mathcal{C}_\omega(G(F))$ (by Lemma 1.7.1). The goal of this chapter is to show that if f is strongly cuspidal then the expression

$$J^\chi(f) := \int_X K_f^\chi(x) dx$$

is convergent. More precisely we will prove the following

Theorem 2.1.1 *For all $f \in \mathcal{S}_{\omega, \text{scusp}}(G(F))$, the expression defining $J^\chi(f)$ is absolutely convergent. Moreover, if the pair (G, H) is tempered then the expression defining $J^\chi(f)$ is also absolutely convergent for all $f \in \mathcal{C}_{\omega, \text{scusp}}(G(F))$.*

2.2 Some estimates

Let $A \subset G$ be a (θ, F) -split subtorus, set $M := \text{Cent}_G(A)$ and let $Q = LU_Q \in \mathcal{F}^\theta(M)$ where $L := Q \cap \theta(Q)$. Let $\overline{Q} = \theta(Q) = LU_{\overline{Q}}$ be the opposite parabolic subgroup and set

$$A_Q^+ := \{a \in A; |\alpha(a)| \geq 1 \forall \alpha \in R(A, U_{\overline{Q}})\}$$

and

$$A_Q^+(\delta) := \{a \in A; |\alpha(a)| \geq e^{\delta \overline{\sigma}(a)} \forall \alpha \in R(A, U_{\overline{Q}})\}$$

for all $\delta > 0$. Recall that if Y is an algebraic variety over F and $M > 0$ we denote by $Y[< M]$ the subset of $y \in Y(F)$ with $\sigma_Y(y) < M$. We also recall that we have fixed a (classical) norm $|\cdot|_{\mathfrak{g}}$ and that for any $R > 0$, $B(0, R)$ denotes the closed ball of radius R centered at the origin for this norm (see §1.2).

Lemma 2.2.1 (i) *Let $\epsilon > 0$ and $\delta > 0$. Then, we have*

$$\overline{\sigma}(a) \ll \sup(\overline{\sigma}(g), \overline{\sigma}(a^{-1}ga))$$

for all $a \in A_Q^+(\delta)$ and all $g \in G(F) \setminus (Q(F)aU_Q[< \epsilon \overline{\sigma}(a)]a^{-1})$;

(ii) *Let $0 < \delta' < \delta$ and $c_0 > 0$. Then, if $\epsilon > 0$ is sufficiently small we have*

$$aU_Q[< \epsilon \overline{\sigma}(a)]a^{-1} \subseteq \exp\left(B(0, c_0 e^{-\delta' \overline{\sigma}(a)}) \cap \mathfrak{u}_Q(F)\right)$$

for all $a \in A_Q^+(\delta)$.

(iii) We have

$$\bar{\sigma}(h) \ll \bar{\sigma}(a^{-1}ha)$$

and

$$\bar{\sigma}(h) + \bar{\sigma}(a) \ll \bar{\sigma}(ha)$$

for all $a \in A$ and all $h \in H$.

(iv) Set $L := Q \cap \theta(Q)$, $H_L := H \cap L$ and $H^Q := H_L \times U_Q$ where U_Q denotes the unipotent radical of Q . Then, we have

$$\bar{\sigma}(h^Q) \ll \bar{\sigma}(a^{-1}h^Qa)$$

for all $a \in A_Q^+$ and all $h^Q \in H^Q$.

Proof: (i) and (ii) are essentially [11] Lemma 1.3.1 (i) and (ii) applied to the group $\bar{G} := A_G \backslash G$. To prove (iii), we first observe that

$$\theta(a^{-1}ha) = aha^{-1} \text{ and } \theta(ha)^{-1}ha = a^2$$

for all $a \in A$, $h \in H$. Hence, we have

$$\sup(\bar{\sigma}(aha^{-1}), \bar{\sigma}(a^{-1}ha)) \ll \bar{\sigma}(a^{-1}ha) \text{ and } \bar{\sigma}(a^2) \ll \bar{\sigma}(ha)$$

for all $a \in A$, $h \in H$. Since $\bar{\sigma}(a) \sim \bar{\sigma}(a^2)$ for all $a \in A$ and $\bar{\sigma}(h) \ll \bar{\sigma}(ha) + \bar{\sigma}(a)$, this already suffices to establish the second inequality. To prove the first one, it only remains to show the following

(2.2.1) We have

$$\bar{\sigma}(g) \ll \sup(\bar{\sigma}(aga^{-1}), \bar{\sigma}(a^{-1}ga))$$

for all $a \in A$ and all $g \in G$.

Fix an embedding $\iota : \bar{G} \hookrightarrow SL_n$ for some $n \geq 1$ which sends the torus A into the standard maximal torus A_n of SL_n . Then, we are reduced to proving 2.2.1 in the particular case where $G = SL_n$ and $A = A_n$. For every matrix $g \in SL_n$, denote by $g_{i,j}$ ($1 \leq i, j \leq n$) the (i, j) th-entry of g and for $a \in A_n$, set $a_i = a_{i,i}$. Then, we have

$$\bar{\sigma}(g) \sim \sup_{i,j} \log(2 + |g_{i,j}|)$$

Hence, it suffices to show that for all $1 \leq i, j \leq n$ we have

$$\log(2 + |g_{i,j}|) \leq \sup(\log(2 + |(aga^{-1})_{i,j}|), \log(2 + |(a^{-1}ga)_{i,j}|))$$

for all $g \in SL_n$ and $a \in A_n$. However, $(aga^{-1})_{i,j} = a_i a_j^{-1} g_{i,j}$, $(a^{-1}ga)_{i,j} = a_j a_i^{-1} g_{i,j}$ and at least one of the quotients $a_i a_j^{-1}$, $a_j a_i^{-1}$ is of absolute value greater than 1. The result follows.

We now prove (iv). Every $h^Q \in H^Q(F)$ can be written $h^Q = h_L u_Q$ where $h_L \in H_L \subset L$ and $u_Q \in U_Q$. Moreover, we have $\bar{\sigma}(lu_Q) \sim \bar{\sigma}(l) + \bar{\sigma}(u_Q)$ and $\bar{\sigma}(u_Q) \ll \bar{\sigma}(a^{-1}u_Q a)$ for all $l \in L$, $u_Q \in U_Q$ and $a \in A_Q^+$. Besides, as $H_L \subset H$, by (iii) we have $\bar{\sigma}(h_L) \ll \bar{\sigma}(a^{-1}h_L a)$ for all $h_L \in H_L$ and $a \in A$. It follows that

$$\bar{\sigma}(h^Q) \sim \bar{\sigma}(h_L) + \bar{\sigma}(u_Q) \ll \bar{\sigma}(a^{-1}h_L a) + \bar{\sigma}(a^{-1}u_Q a) \sim \bar{\sigma}(a^{-1}h^Q a)$$

for all $h^Q = h_L u_Q \in H^Q = H_L \times U_Q$ and all $a \in A_Q^+$. This proves (iv). ■

2.3 Weak Cartan decompositions and Harish-Chandra-Schwartz space of X

Let $A_{0,j}$, $j \in J$, be representatives of the $H(F)$ -conjugacy classes of maximal (θ, F) -split tori of G . There are a finite number of them and by a result of Benoist-Oh and Delorme-Sécherre ([9] and [19]), there exists a compact subset $\mathcal{K}_G \subset G(F)$ such that

$$(2.3.1) \quad G(F) = \bigcup_{j \in J} H(F)A_{0,j}(F)\mathcal{K}_G$$

This decomposition is called a weak Cartan decomposition.

Let $C \subset G(F)$ be a compact subset with nonempty interior and set

$$\Xi_C^X(x) = \text{vol}_X(xC)^{-1/2}$$

for all $x \in X$ and where vol_X refers to a $G(F)$ -invariant measure on X (which exists as H is reductive hence unimodular). If $C' \subset G(F)$ is another compact subset with nonempty interior, the functions Ξ_C^X and $\Xi_{C'}^X$ are equivalent and we will denote by Ξ^X any such function (for some choice of C).

Proposition 2.3.1 (i) *For every compact subset $\mathcal{K} \subseteq G(F)$, we have the following equivalences*

$$(2.3.2) \quad \Xi^X(xk) \sim \Xi^X(x)$$

$$(2.3.3) \quad \sigma_X(xk) \sim \sigma_X(x)$$

for all $x \in X$ and all $k \in \mathcal{K}$.

(ii) Let A_0 be a (θ, F) -split subtorus of G . Then, there exists $d > 0$ such that

$$(2.3.4) \quad \Xi^G(a)\bar{\sigma}(a)^{-d} \ll \Xi^X(a) \ll \Xi^G(a)$$

$$(2.3.5) \quad \sigma_X(a) \sim \bar{\sigma}(a)$$

for all $a \in A_0(F)$.

(iii) There exists $d > 0$ such that the integral

$$\int_X \Xi^X(x)^2 \sigma_X(x)^{-d} dx$$

is absolutely convergent.

Assume moreover that the pair (G, H) is tempered, then we have

(iv) For all $d > 0$ there exists $d' > 0$ such that

$$\int_{A_G^H(F) \backslash H(F)} \Xi^G(hx)\bar{\sigma}(hx)^{-d'} dh \ll \Xi^X(x)\sigma_X(x)^{-d}$$

for all $x \in X$.

(v) There exist $d > 0$ and $d' > 0$ such that

$$\int_{A_G^H(F) \backslash H(F)} \Xi^G(x^{-1}hx)\bar{\sigma}(x^{-1}hx)^{-d} dh \ll \Xi^X(x)^2 \sigma_X(x)^{d'}$$

for all $x \in X$.

(vi) More generally, let Q be a θ -split parabolic subgroup of G and set $L := Q \cap \theta(Q)$, $H_L := H \cap L$ and $H^Q := H_L \rtimes U_Q$ where U_Q denotes the unipotent radical of Q . Let $A_0 \subset L$ be a maximal (θ, F) -split subtorus and set

$$A_Q^+ := \{a \in A_0(F); |\alpha(a)| \geq 1 \forall \alpha \in R(A_0, U_Q)\}$$

where U_Q denotes the unipotent radical of \bar{Q} . Then, H^Q is a unimodular algebraic group, (G, H^Q) is a tempered pair and, fixing a Haar measure dh^Q on $H^Q(F)$, there exists $d > 0$ and $d' > 0$ such that

$$\int_{A_G^H(F) \backslash H^Q(F)} \Xi^G(a^{-1}h^Q a)\bar{\sigma}(a^{-1}h^Q a)^{-d} dh^Q \ll \Xi^X(a)^2 \sigma_X(a)^{d'}$$

for all $a \in A_Q^+$.

Proof:

- (i) is easy and left to the reader.
(ii) Let M_0 be the centralizer of A_0 in G . For all $P_0 \in \mathcal{P}^\theta(M_0)$ set

$$A_{P_0}^+ := \{a \in A_0(F); |\alpha(a)| \geq 1 \forall \alpha \in R(A_0, P_0)\}$$

Then

$$(2.3.6) \quad A_0(F) = \bigcup_{P_0 \in \mathcal{P}^\theta(M_0)} A_{P_0}^+$$

Thus, we may fix $P_0 \in \mathcal{P}^\theta(M)$, set $A^+ := A_{P_0}^+$ and prove 2.3.4 and 2.3.5 for those a belonging to A^+ . Since P_0 is θ -split, HP_0 is open in G . The proof of (ii) is now the same as Proposition 6.7.1(ii) of [11] after replacing Proposition 6.4.1(iii) of *loc. cit.* by Lemma 2.2.1 (iii).

- (iii) The proof is exactly the same as for Proposition 6.7.1 (iii) of [11]: we use the weak Cartan decomposition 2.3.1 to show that X has polynomial growth in the sense of [10] and then we conclude as in *loc. cit.*
(iv) Let A_0 be a maximal (θ, F) -split subtorus of G , $M_0 := \text{Cent}_G(A_0)$ and $P_0 \in \mathcal{P}^\theta(M_0)$. By the decompositions 2.3.1 and 2.3.6, points (i) and (ii) and Lemma 2.2.1(iii) it suffices to show the existence of $d > 0$ such that

$$\int_{A_G^H(F) \backslash H(F)} \Xi^G(ha) \bar{\sigma}(h)^{-d} dh \ll \Xi^G(a)$$

for all $a \in A_{P_0}^+$. Since P_0 is θ -split, $H(F)P_0(F)$ is open in $G(F)$. It follows that if K is a maximal compact subgroup of $G(F)$ by which Ξ^G is right invariant there exists an open-compact subgroups $J \subset G(F)$ and $J_H \subset H(F)$ such that $J \subset J_H a K a^{-1}$ for all $a \in A_{P_0}^+$. Hence, for all $d > 0$, all $k \in J$ and all $a \in A_{P_0}^+$, writing $k = k_H a k_G a^{-1}$ with $k_H \in J_H$ and $k_G \in K$, we have

$$\int_{A_G^H(F) \backslash H(F)} \Xi^G(hka) \bar{\sigma}(h)^{-d} dh = \int_{A_G^H(F) \backslash H(F)} \Xi^G(ha) \bar{\sigma}(hk_H^{-1})^{-d} dh$$

It follows that for all $d > 0$ we have

$$\int_{A_G^H(F) \backslash H(F)} \Xi^G(ha) \bar{\sigma}(h)^{-d} dh \ll \int_{A_G^H(F) \backslash H(F)} \int_K \Xi^G(hka) dk \bar{\sigma}(h)^{-d} dh$$

for all $a \in A_{P_0}^+$ and we conclude by the ‘doubling principle’ (see [11] Proposition 1.5.1) and the fact that the pair (G, H) is tempered.

- (v) Once again, the proof is very similar to the proof of Proposition 6.7.1 (v) of [11] so that we shall only sketch the argument. Let A_0 be a maximal (θ, F) -split torus of G and $P_0 \in \mathcal{P}^\theta(M_0)$ where $M_0 := \text{Cent}_G(A_0)$. By the weak Cartan decomposition, (i), (ii), the inequality $\bar{\sigma}(h) \ll \bar{\sigma}(a^{-1}ha)\bar{\sigma}(a)$ and the decomposition 2.3.6, we are reduced to proving the existence of $d, d' > 0$ such that

$$\int_{A_G^H(F) \backslash H(F)} \Xi^G(a^{-1}ha)\bar{\sigma}(h)^{-d}dh \ll \Xi^G(a)^2\bar{\sigma}(a)^{d'}$$

for all $a \in A_{P_0}^+$. Using the fact that $H(F)P_0(F)$ is open in $G(F)$ we show as in the proof of (iv) that if K is a maximal compact subgroup of $G(F)$ we have

$$\int_{A_G^H(F) \backslash H(F)} \Xi^G(a^{-1}ha)\bar{\sigma}(h)^{-d}dh \ll \int_{A_G^H(F) \backslash H(F)} \int_{K \times K} \Xi^G(a^{-1}k_1hk_2a)\bar{\sigma}(h)^{-d}dk_1dk_2dh$$

for all $a \in A_{P_0}^+$ and then we conclude again by the ‘doubling principle’ (see [11] Proposition 1.5.1 (vi)) and the fact that the pair (G, H) is tempered.

- (vi) The proof that H^Q is unimodular is similar to the proof of Proposition 6.8.1 (ii) of [11] noticing that \bar{Q} is a good parabolic subgroup with respect to H (that is $\bar{Q}H$ is open in G) and that $H_L = H \cap \bar{Q}$. Moreover, the fact that the pair (G, H^Q) is tempered and the estimate can be proved in much the same way as Proposition 6.8.1 (iv)-(vi) of [11]. Indeed, if we denote by M_0 the centralizer of A_0 in G , we have

$$A_{\bar{Q}}^+ = \bigcup_{\substack{P_0 \in \mathcal{P}^\theta(M_0) \\ P_0 \subset \bar{Q}}} A_{P_0}^+$$

and thus, fixing $P_0 \in \mathcal{P}^\theta(M_0)$ with $P_0 \subset \bar{Q}$, it suffices to show the existence of $d, d' > 0$ such that

$$(2.3.7) \quad \text{The integral } \int_{A_G^H(F) \backslash H^Q(F)} \Xi^G(h^Q)\bar{\sigma}(h^Q)^{-d}dh^Q \text{ converges;}$$

and

$$(2.3.8) \quad \int_{A_G^H(F) \backslash H^Q(F)} \Xi^G(a^{-1}h^Qa)\bar{\sigma}(a^{-1}h^Qa)^{-d}dh^Q \ll \Xi^X(a)^2\sigma_X(a)^{d'} \text{ for all } a \in A_{P_0}^+.$$

Then, 2.3.7 can be proved exactly as Proposition 6.8.1 (iv) of [11] where the only inputs used are the facts that $\bar{Q}H$ is open in G and that the pair (G, H) is tempered. Also, 2.3.8 can be proved exactly as Proposition 6.8.1 (vi) of *loc. cit.* where this time the only inputs are the estimates of (ii) and of Lemma 2.2.1 (iv), the convergence of 2.3.7 and the fact that \bar{P}_0H is open in G . ■

2.4 Proof of theorem 2.1.1

By Proposition 2.3.1 (iii), it suffices to establish the two following claims

(2.4.1) For all $f \in \mathcal{S}_{\omega, \text{scusp}}(G(F))$ the function $x \mapsto K_f^\chi(x)$ is compactly supported.

(2.4.2) Assume that the pair (G, H) is tempered. Then, for all $d > 0$ and all $f \in \mathcal{C}_{\omega, \text{scusp}}(G(F))$, we have

$$|K_f^\chi(x)| \ll \Xi^X(x)^2 \sigma_X(x)^{-d}$$

for all $x \in X$.

We will only show 2.4.2, the proof of 2.4.1 being similar and actually slightly easier. Moreover, the proof of 2.4.2 is also very similar to the proof of Theorem 8.1.1 (ii) of [11]. We will thus content ourself with outlining the main steps. Let A_0 be a maximal (θ, F) -split subtorus of G , $M_0 := \text{Cent}_G(A_0)$ and $P_0 = M_0 U_0 \in \mathcal{P}^\theta(M_0)$. Let $\overline{P}_0 = \theta(P_0)$ be the opposite parabolic subgroup and set

$$A_{\overline{P}_0}^+ := \{a \in A_0(F); |\alpha(a)| \geq 1 \forall \alpha \in R(A_0, \overline{P}_0)\}$$

By the weak Cartan decomposition 2.3.1 as well as Proposition 2.3.1(i), it suffices to show 2.4.2 only for $x = a \in A_{\overline{P}_0}^+$. For all $Q \in \mathcal{F}^\theta(M_0)$ and $\delta > 0$ set

$$A_{\overline{Q}}^+(\delta) := \{a \in A_0(F); |\alpha(a)| \geq e^{\delta \overline{\sigma}(a)} \forall \alpha \in R(A_0, U_{\overline{Q}})\}$$

where $\overline{Q} := \theta(Q)$ and $U_{\overline{Q}}$ denotes the unipotent radical of \overline{Q} . Then, if δ is sufficiently small we have

$$A_{\overline{P}_0}^+ = \bigcup_{Q \in \mathcal{F}^\theta(M_0) - \{G\}, P_0 \subset Q} A_{\overline{Q}}^+(\delta) \cap A_{\overline{P}_0}^+$$

Thus, fixing $Q \in \mathcal{F}^\theta(M_0) - \{G\}$ with $P_0 \subset Q$ and $\delta > 0$, it suffices to prove the estimate 2.4.2 only for $x = a \in A_{\overline{Q}}^+(\delta) \cap A_{\overline{P}_0}^+$. We fix such a Q and such a δ henceforth. Let U_Q be the unipotent radical of Q and set $L := Q \cap \overline{Q}$, $H_L := H \cap L$ and $H^Q := H_L \times U_Q$. We define a unitary character χ^Q of $H^Q(F)$ by setting $\chi^Q(h_L u_Q) = \chi(h_L)$ for all $h_L \in H_L(F)$ and $u_Q \in U_Q(F)$. Then by Proposition 2.3.1(vi), H^Q is a unimodular algebraic group and the pair (G, H^Q) is tempered. Thus, fixing a Haar measure dh^Q on $H^Q(F)$, we can define

$$K_f^{\chi^Q}(x) := \int_{A_G^H(F) \backslash H^Q(F)} f(x^{-1} h^Q x) \chi^Q(h^Q)^{-1} dh^Q$$

for all $f \in \mathcal{C}_\omega(G(F))$ and all $x \in G(F)$. As $U_Q \subset H^Q \subset Q$, for any strongly cuspidal function $f \in \mathcal{C}_{\omega, \text{scusp}}(G(F))$, the function $K_f^{\chi^Q}$ vanishes identically. Therefore, it is sufficient to show the existence of $c > 0$ such that for every $f \in \mathcal{C}_\omega(G(F))$ and $d > 0$ we have

$$(2.4.3) \quad \left| K_f^\chi(a) - cK_f^{\chi, Q}(a) \right| \ll \Xi^X(a)^2 \sigma_X(a)^{-d}$$

for all $a \in A_{\overline{Q}}^+(\delta) \cap A_{P_0}^+$. We prove this following closely the proof of Proposition 8.1.4 of [11]. We henceforth fix $f \in \mathcal{C}_\omega(G(F))$ and $d > 0$.

The F -analytic map

$$\begin{aligned} H(F) \cap \overline{P}_0(F)U_0(F) &\rightarrow U_0(F) \\ h = \overline{p}u &\mapsto u \end{aligned}$$

is submersive at the origin (this follows from the fact that $\overline{P}_0(F)H(F)$ is open in $G(F)$). Therefore, we may find a compact-open neighborhood \mathcal{U}_0 of 1 in $U_0(F)$ together with an F -analytic map

$$h : \mathcal{U}_0 \rightarrow H(F)$$

such that $h(u) \in \overline{P}_0(F)u$ for all $u \in \mathcal{U}_0$ and $h(1) = 1$. Set $\mathcal{U}_Q := \mathcal{U}_0 \cap U_Q(F)$, $\mathcal{H} := H_L(F)h(\mathcal{U}_Q)$ and fix Haar measures dh_L and du_Q on $H_L(F)$ and $U_Q(F)$ whose product is the fixed Haar measure on $H^Q(F)$. The following fact is easy and can be proved exactly the same way as point (8.1.7) of [11] (note that here $H \cap \overline{Q} = H \cap L$):

(2.4.4) The map $H_L(F) \times \mathcal{U}_Q \rightarrow H(F)$, $(h_L, u_Q) \mapsto h_L h(u_Q)$, is an F -analytic open embedding with image \mathcal{H} and there exists a smooth function $j \in C^\infty(\mathcal{U}_Q)$ such that

$$\int_{\mathcal{H}} \varphi(h) dh = \int_{H_L(F)} \int_{\mathcal{U}_Q} \varphi(h_L h(u_Q)) j(u_Q) du_Q dh_L$$

for all $\varphi \in L^1(\mathcal{H})$.

Fix $\epsilon > 0$ that we will assume sufficiently small in what follows. By Lemma 2.2.1(ii), for ϵ small enough we have

$$aU_Q [< \epsilon \overline{\sigma}(a)] a^{-1} \subseteq \mathcal{U}_Q$$

for all $a \in A_{\overline{Q}}^+(\delta)$. This allows us to define

$$\begin{aligned} H^{<\epsilon, a} &:= H_L(F)h(aU_Q [< \epsilon \overline{\sigma}(a)] a^{-1}), \\ H^{Q, <\epsilon, a} &:= H_L(F)aU_Q [< \epsilon \sigma(a)] a^{-1} \end{aligned}$$

and the following expressions

$$\begin{aligned} K_f^{\chi, <\epsilon}(a) &:= \int_{A_G^H(F) \setminus H^{<\epsilon, a}} f(a^{-1}ha) \chi(h)^{-1} dh \\ K_f^{\chi, Q, <\epsilon}(a) &:= \int_{A_G^H(F) \setminus H^{Q, <\epsilon, a}} f(a^{-1}h^Q a) \chi^Q(h^Q)^{-1} dh^Q \end{aligned}$$

for all $a \in A_{\overline{Q}}^+(\delta)$. Set $c = j(1)$ (where the function $j(\cdot)$ is the one appearing in 2.4.4). Obviously 2.4.3 will follow if we can show that for ϵ sufficiently small we have:

$$(2.4.5) \quad |K_f^{\chi}(a) - K_f^{\chi, < \epsilon}(a)| \ll \Xi^X(a)^2 \sigma_X(a)^{-d}$$

$$(2.4.6) \quad |K_f^Q(a) - K_f^{\chi, Q, < \epsilon}(a)| \ll \Xi^X(a)^2 \sigma_X(a)^{-d}$$

$$(2.4.7) \quad K_f^{\chi, < \epsilon}(a) = c K_f^{\chi, Q, < \epsilon}(a)$$

for all $a \in A_{\overline{Q}}^+(\delta) \cap A_{P_0}^+$.

First we prove 2.4.5 and 2.4.6. By Lemma 2.2.1 (i)-(iii)-(iv) and noticing that

$$H^{< \epsilon, a} = H(F) \cap \overline{Q}(F) a U_Q [< \epsilon \overline{\sigma}(a)] a^{-1} \text{ and } H^{Q, < \epsilon, a} = H^Q(F) \cap \overline{Q}(F) a U_Q [< \epsilon \overline{\sigma}(a)] a^{-1},$$

we see that $\overline{\sigma}(a) \ll \overline{\sigma}(a^{-1} h a)$ and $\overline{\sigma}(a) \ll \overline{\sigma}(a^{-1} h^Q a)$ for all $a \in A_{\overline{Q}}^+(\delta)$, $h \in H(F) \setminus H^{< \epsilon, a}$ and $h^Q \in H^Q(F) \setminus H^{Q, < \epsilon, a}$. Thus, by definition of $\mathcal{C}_\omega(G(F))$, for every $d_1 > 0$ and $d_2 > 0$, the left hand sides of 2.4.5 and 2.4.6 are essentially bounded by

$$\overline{\sigma}(a)^{-d_1} \int_{A_G^H(F) \setminus H(F)} \Xi^G(a^{-1} h a) \overline{\sigma}(a^{-1} h a)^{-d_2} dh$$

and

$$\overline{\sigma}(a)^{-d_1} \int_{A_G^H(F) \setminus H^Q(F)} \Xi^G(a^{-1} h^Q a) \overline{\sigma}(a^{-1} h^Q a)^{-d_2} dh^Q$$

for all $a \in A_{\overline{Q}}^+(\delta)$ respectively. By Proposition 2.3.1 (v)-(vi), there exists $d_3 > 0$ such that for d_2 sufficiently large the last two expressions above are essentially bounded by $\Xi^X(a)^2 \sigma_X(a)^{d_3} \overline{\sigma}(a)^{-d_1}$ for all $a \in A_{\overline{Q}}^+$. Finally, by Proposition 2.3.1 (ii), if d_1 is sufficiently large we have $\sigma_X(a)^{d_3} \overline{\sigma}(a)^{-d_1} \ll \sigma_X(a)^{-d}$ for all $a \in A_0(F)$. This proves 2.4.5 and 2.4.6.

It only remains to prove 2.4.7. By 2.4.4 and the choice of Haar measures, we have

$$(2.4.8) \quad K_f^{\chi, < \epsilon}(a) = \int_{A_G^H(F) \setminus H_L(F)} \int_{a U_Q [< \epsilon \overline{\sigma}(a)] a^{-1}} f(a^{-1} h_L h(u_Q) a) \chi(h_L h(u_Q))^{-1} j(u_Q) du_Q dh_L$$

and

$$(2.4.9) \quad K_f^{\chi, Q, < \epsilon}(a) = \int_{A_G^H(F) \setminus H_L(F)} \int_{a U_Q [< \epsilon \overline{\sigma}(a)] a^{-1}} f(a^{-1} h_L u_Q a) \chi(h_L)^{-1} du_Q dh_L$$

for all $a \in A_Q^\pm(\delta)$. Since the function $u_Q \mapsto \chi(h(u_Q))^{-1}j(u_Q)$ is smooth, by Lemma 2.2.1(ii) for ϵ sufficiently small we have

$$(2.4.10) \quad \chi(h(u_Q))^{-1}j(u_Q) = j(1) = c$$

for all $a \in A_Q^\pm(\delta)$ and all $u_Q \in aU_Q[< \epsilon\bar{\sigma}(a)]a^{-1}$. Let $J \subset G(F)$ be a compact-open subgroup by which f is right invariant. By definition, the map $u_Q \mapsto h(u_Q)u_Q^{-1}$ is F -analytic, sends 1 to 1 and takes values in $\bar{P}_0(F)$. By Lemma 2.2.1(ii) again, it follows that for all $0 < \delta' < \delta$ and all $c_0 > 0$ if ϵ is sufficiently small we have

$$a^{-1}h(u_Q)u_Q^{-1}a \in \exp\left(B(0, c_0e^{-\delta'\bar{\sigma}(a)}) \cap \bar{\mathfrak{p}}_0(F)\right)$$

for all $a \in A_Q^\pm(\delta) \cap A_{\bar{P}_0}^\pm$ and all $u_Q \in aU_Q[< \epsilon\bar{\sigma}(a)]a^{-1}$. Moreover, there exists $\alpha > 0$ such that $|\text{Ad}(g^{-1})X|_{\mathfrak{g}} \leq e^{\alpha\bar{\sigma}(g)}|X|_{\mathfrak{g}}$ for all $g \in G(F)$ and all $X \in \mathfrak{g}(F)$. Hence, if ϵ is sufficiently small we have

$$(2.4.11) \quad a^{-1}u_Q^{-1}h(u_Q)a = (a^{-1}u_Qa)^{-1}(a^{-1}h(u_Q)u_Q^{-1}a)(a^{-1}u_Qa) \in J$$

for all $a \in A_Q^\pm(\delta) \cap A_{\bar{P}_0}^\pm$ and all $u_Q \in aU_Q[< \epsilon\bar{\sigma}(a)]a^{-1}$. It is clear that 2.4.7 follows from 2.4.8, 2.4.9, 2.4.10 and 2.4.11 and this ends the proof of Theorem 2.1.1. ■

3 The spectral side

3.1 The statement

In this chapter G is a connected reductive group over F , $\bar{G} := G/A_G$ and H is a symmetric subgroup of G . As in the previous chapter, we let A_G^H as the connected component of $H \cap A_G$. Let ω and χ be continuous unitary characters of $A_G(F)$ and $H(F)$ respectively such that $\omega|_{A_G^H(F)} = \chi|_{A_G^H(F)}$. Set

$$\nu(H) := [H(F) \cap A_G(F) : A_H(F)]$$

In §2.1, we have defined a linear form

$$f \in \mathcal{S}_{\omega, \text{scusp}}(G(F)) \mapsto J^\chi(f)$$

which, if (G, H) is a tempered pair, extends to a continuous linear form

$$f \in \mathcal{C}_{\omega, \text{scusp}}(G(F)) \mapsto J^\chi(f)$$

For all $\pi \in \text{Irr}(G)$ we define a multiplicity $m(\pi, \chi)$ by

$$m(\pi, \chi) := \dim \operatorname{Hom}_H(\pi, \chi)$$

where $\operatorname{Hom}_H(\pi, \chi)$ denotes the space of linear forms $\ell : \pi \rightarrow \mathbf{C}$ such that $\ell \circ \pi(h) = \chi(h)\ell$ for all $h \in H(F)$. By Theorem 4.5 of [18], we know that this space is always finite dimensional so that the multiplicity $m(\pi, \chi)$ is well-defined.

Recall that $\operatorname{Irr}_{\omega, \text{cusp}}(G)$, resp. $\operatorname{Irr}_{\omega, \text{sqr}}(G)$, denote the sets of equivalence classes of irreducible supercuspidal, resp. square-integrable, representations of $G(F)$ with central character ω . Define the following linear forms

$$f \in \mathcal{S}_{\omega, \text{scusp}}(G(F)) \mapsto J_{\text{spec, cusp}}^{\chi}(f) := \nu(H) \sum_{\pi \in \operatorname{Irr}_{\omega, \text{cusp}}(G)} m(\pi, \chi) \operatorname{Trace}(\pi^{\vee}(f))$$

and

$$f \in \mathcal{C}_{\omega, \text{scusp}}(G(F)) \mapsto J_{\text{spec, disc}}^{\chi}(f) := \nu(H) \sum_{\pi \in \operatorname{Irr}_{\omega, \text{sqr}}(G)} m(\pi, \chi) \operatorname{Trace}(\pi^{\vee}(f))$$

Notice that the sums defining these linear forms are always finite by the result of Harish-Chandra that for every compact-open subgroup $J \subset G(F)$ the set of $\pi \in \operatorname{Irr}_{\omega, \text{sqr}}(G)$ with $\pi^J \neq 0$ is finite ([45] Théorème VIII.1.2). Recall that in §1.6 we have introduced certain spaces ${}^0\mathcal{S}_{\omega}(G(F))$, ${}^0\mathcal{C}_{\omega}(G(F))$ of cusp forms. The goal of this chapter is to prove the following

Theorem 3.1.1 *For all $f \in {}^0\mathcal{S}_{\omega}(G(F))$ we have*

$$J^{\chi}(f) = J_{\text{spec, cusp}}^{\chi}(f)$$

Moreover, if (G, H) is a tempered pair, for all $f \in {}^0\mathcal{C}_{\omega}(G(F))$ we have

$$J^{\chi}(f) = J_{\text{spec, disc}}^{\chi}(f)$$

3.2 Explicit description of the intertwinings

For all $\pi \in \operatorname{Irr}_{\omega, \text{cusp}}(G(F))$ we define a bilinear form

$$\mathcal{B}_{\pi} : \pi \times \pi^{\vee} \rightarrow \mathbf{C}$$

by

$$\mathcal{B}_{\pi}(v, v^{\vee}) := \int_{A_G^H(F) \backslash H(F)} \langle \pi(h)v, v^{\vee} \rangle \chi(h)^{-1} dh$$

for all $(v, v^{\vee}) \in \pi \times \pi^{\vee}$. If (G, H) is a tempered pair and $\pi \in \operatorname{Irr}_{\omega, \text{sqr}}(G(F))$ then the above integral is also absolutely convergent and thus also defines a bilinear form $\mathcal{B}_{\pi} : \pi \times \pi^{\vee} \rightarrow \mathbf{C}$. In all cases, we have

$$\mathcal{B}_{\pi}(\pi(h_1)v, \pi^{\vee}(h_2)v^{\vee}) = \chi(h_1)\chi^{-1}(h_2)\mathcal{B}_{\pi}(v, v^{\vee})$$

for all $(v, v^\vee) \in \pi \times \pi^\vee$ and all $h_1, h_2 \in H(F)$. Thus \mathcal{B}_π factorizes through a bilinear form

$$\mathcal{B}_\pi : \pi_\chi \times \pi_{\chi^{-1}}^\vee \rightarrow \mathbf{C}$$

where π_χ and $\pi_{\chi^{-1}}^\vee$ denote the spaces of $(H(F), \chi)$ - and $(H(F), \chi^{-1})$ -coinvariants in π and π^\vee respectively i.e. the quotients of π and π^\vee by the subspaces generated by vectors of the form $\pi(h)v - \chi(h)v$ ($h \in H(F)$, $v \in \pi$) and $\pi^\vee(h)v^\vee - \chi(h)^{-1}v^\vee$ ($h \in H(F)$, $v^\vee \in \pi^\vee$) respectively. The following proposition has been proved in more generality in [44] Theorem 6.4.1 when the subgroup H is *strongly tempered* (in the sense of *loc. cit.*). The same kind of idea already appears in [47] Proposition 5.6.

Proposition 3.2.1 \mathcal{B}_π induces a nondegenerate pairing between π_χ and $\pi_{\chi^{-1}}$.

Proof: We will prove the proposition when (G, H) is a tempered pair and $\pi \in \text{Irr}_{\omega, \text{sqf}}(G)$, the case where $\pi \in \text{Irr}_{\omega, \text{cusp}}(G)$ being similar and easier. Fix a $G(F)$ -invariant scalar product (\cdot, \cdot) on π . We can define the following sesquilinear version of \mathcal{B}_π

$$\mathcal{L}_\pi : \pi \times \pi \rightarrow \mathbf{C}$$

$$(v, v') \mapsto \int_{A_G^H(F) \backslash H(F)} (\pi(h)v, v') \chi(h)^{-1} dh$$

which factorizes through a sesquilinear pairing $\mathcal{L}_\pi : \pi_\chi \times \pi_\chi \rightarrow \mathbf{C}$. Obviously, it suffices to show that this pairing is non degenerate. Since π_χ is finite dimensional, this is equivalent to saying that the map

$$v \in \pi \mapsto \mathcal{L}_\pi(\cdot, v) \in \text{Hom}_H(\pi, \chi)$$

is surjective. To continue we need the following lemma, a consequence of the weak Cartan decomposition (2.3.1):

Lemma 3.2.1 For all $\ell \in \text{Hom}_H(\pi, \chi)$ and all $v \in \pi$ we have

$$\int_X |\ell(\pi(x)v)|^2 dx < \infty$$

and moreover for all $f \in \mathcal{C}_{\omega^{-1}}(G(F))$ the integral

$$\int_{A_G(F) \backslash G(F)} f(g) \ell(\pi(g)v) dg$$

is absolutely convergent and equals $\ell(\pi(f)v)$.

Proof: For every compact-open subgroup $J \subset G(F)$ we will denote by $e_J * \ell$ the smooth linear form (i.e. an element of π^\vee) $v \in \pi \mapsto \frac{1}{\text{vol}(J)} \int_J \ell(\pi(k)v) dk$. Let A_0 be a maximal (θ, F) -split subtorus of G , $M_0 := \text{Cent}_G(A_0)$ (a minimal θ -split Levi subgroup) and $P_0 \in \mathcal{P}^\theta(M_0)$. Set

$$A_{P_0}^+ := \{a \in A_0(F); |\alpha(a)| \geq 1 \forall \alpha \in \Delta_{P_0}\}$$

Then, by the weak Cartan decomposition 2.3.1 and Proposition 2.3.1(i)-(ii)-(iii)-(iv), in order to prove that the two integrals of the proposition are convergent it suffices to show that for all $d > 0$ we have

$$(3.2.1) \quad |\ell(\pi(a)v)| \ll \Xi^G(a) \bar{\sigma}(a)^{-d}$$

for all $a \in A_{P_0}^+$. Let $J \subset G(F)$ be a compact-open subgroup such that $v \in \pi^J$. Since P_0 is θ -split, $H(F)P_0(F)$ is open in $G(F)$ and consequently there exists a compact-open subgroup $J' \subset G(F)$ such that

$$J' \subset H(F)a(J \cap P_0(F))a^{-1}$$

for all $a \in A_{P_0}^+$. Thus, for all $a \in A_{P_0}^+$ we have $\ell(\pi(a)v) = \langle e_{J'} * \ell, \pi(a)v \rangle$ and the inequality 3.2.1 now follows from the known asymptotics of smooth coefficients of square-integrable representations.

To prove the last part of the proposition, choose $J \subset G(F)$ a compact-open subgroup by which f is invariant on the left. Then, we have

$$\begin{aligned} \int_{A_G(F) \backslash G(F)} f(g) \ell(\pi(g)v) dg &= \frac{1}{\text{vol}(J)} \int_{A_G(F) \backslash G(F)} f(g) \int_J \ell(\pi(kg)v) dk dg \\ &= \int_{A_G(F) \backslash G(F)} f(g) \langle e_J * \ell, \pi(g)v \rangle dg \\ &= \langle e_J * \ell, \pi(f)v \rangle = \ell(\pi(f)v) \blacksquare \end{aligned}$$

By the lemma we can define a scalar product, also denoted (\cdot, \cdot) , on $\text{Hom}_H(\pi, \chi)$ characterized by

$$\int_X \ell(\pi(x)v) \overline{\ell'(\pi(x)v')} dx = (\ell, \ell')(v, v')$$

for all $\ell, \ell' \in \text{Hom}_H(\pi, \chi)$ and all $v, v' \in \pi$. Let $\ell \in \text{Hom}_H(\pi, \chi)$ which is orthogonal for this scalar product to all the forms $\mathcal{L}_\pi(\cdot, v)$ for $v \in \pi$. To conclude it suffices to show that this implies $\ell = 0$. Since for all $v, v' \in \pi$ we have $(v, \pi(\cdot)v') \in \mathcal{C}_{\omega^{-1}}(G(F))$, by the lemma we have

$$0 = \int_X \ell(\pi(x)v') \overline{\mathcal{L}_\pi(\pi(x)v', v)} dx = \nu(H) \int_{A_G(F) \backslash G(F)} \ell(\pi(g)v')(v, \pi(g)v') dg = \frac{\nu(H)}{d(\pi)} \ell(v)(v', v')$$

for all $v, v' \in \pi$ and where $d(\pi)$ stands for the formal degree of π . Hence, $\ell = 0$. \blacksquare

3.3 Proof of Theorem 3.1.1

Once again we will prove the theorem in the case where (G, H) is a tempered pair and $f \in {}^0\mathcal{C}_\omega(G(F))$. The case where $f \in {}^0\mathcal{S}_\omega(G(F))$ is completely similar since compactly supported cusp forms are linear combinations of matrix coefficients of supercuspidal representations whereas a general cusp form $f \in {}^0\mathcal{C}_\omega(G(F))$ is a linear combination of matrix coefficients of square-integrable representations.

More precisely, let $f \in {}^0\mathcal{C}_\omega(G(F))$ and for all $\pi \in \text{Irr}_{\omega, \text{sqr}}(G(F))$, set $f_\pi(g) := \text{Trace}(\pi^\vee(g^{-1})\pi^\vee(f))$ for all $g \in G(F)$. Then, we have $f_\pi \in {}^0\mathcal{C}_\omega(G(F))$ for all $\pi \in \text{Irr}_{\omega, \text{sqr}}(G(F))$ and by the Harish-Chandra-Plancherel formula for cusp forms 1.6.1 we have

$$J^X(f) = \sum_{\pi \in \text{Irr}_{\omega, \text{sqr}}(G(F))} d(\pi) J^X(f_\pi)$$

Thus, it suffices to show that

$$J^X(f_\pi) = \nu(H) d(\pi)^{-1} m(\pi, \chi) \text{Trace}(\pi^\vee(f))$$

for all $\pi \in \text{Irr}_{\omega, \text{sqr}}(G(F))$. Fix $\pi \in \text{Irr}_{\omega, \text{sqr}}(G(F))$. As f_π is a sum of coefficients of π , the equality above is equivalent to

$$J^X(f_{v, v^\vee}) = \nu(H) d(\pi)^{-1} m(\pi, \chi) f_{v, v^\vee}(1)$$

for all $(v, v^\vee) \in \pi \times \pi^\vee$ where $f_{v, v^\vee}(g) := \langle \pi(g)v, v^\vee \rangle$ for all $g \in G(F)$. Fix $(v, v^\vee) \in \pi \times \pi^\vee$. Then, we have

$$K_{f_{v, v^\vee}}^X(x) = \mathcal{B}_\pi(\pi(x)v, \pi^\vee(x)v^\vee)$$

for all $x \in X$. Choose a basis $(\bar{v}_1, \dots, \bar{v}_N)$ of π_χ (where $N = m(\pi, \chi)$) and let $(\bar{v}_1^\vee, \dots, \bar{v}_N^\vee)$ be the dual basis of $\pi_{\chi^{-1}}^\vee$ with respect to \mathcal{B}_π (such a dual basis exists thanks to Proposition 3.2.1). Let (v_1, \dots, v_N) and $(v_1^\vee, \dots, v_N^\vee)$ be any lifting of these basis to π and π^\vee respectively. Then we have

$$\mathcal{B}_\pi(\pi(x)v, \pi^\vee(x)v^\vee) = \sum_{i=1}^N \mathcal{B}_\pi(\pi(x)v, v_i^\vee) \mathcal{B}_\pi(v_i, \pi^\vee(x)v^\vee)$$

for all $x \in X$. Now by Lemma 3.2.1, we have

$$\begin{aligned}
J^X(f_{v,v^\vee}) &= \int_X \mathcal{B}_\pi(\pi(x)v, \pi^\vee(x)v^\vee) dx = \sum_{i=1}^N \int_X \mathcal{B}_\pi(\pi(x)v, v_i^\vee) \mathcal{B}_\pi(v_i, \pi^\vee(x)v^\vee) dx \\
&= \nu(H) \sum_{i=1}^N \int_{A_G(F) \backslash G(F)} \langle \pi(g)v, v_i^\vee \rangle \mathcal{B}_\pi(v_i, \pi^\vee(g)v^\vee) dg \\
&= \nu(H) \sum_{i=1}^N \frac{\langle v, v^\vee \rangle}{d(\pi)} \mathcal{B}_\pi(v_i, v_i^\vee) \\
&= \nu(H) N \frac{\langle v, v^\vee \rangle}{d(\pi)} = \nu(H) d(\pi)^{-1} m(\pi, \chi) f_{v,v^\vee}(1) \blacksquare
\end{aligned}$$

4 The geometric side

4.1 The statement

In this chapter E/F is a quadratic extension, H is a connected reductive group over F and $G := R_{E/F}H_E$. We have a natural inclusion $H \hookrightarrow G$ and we shall denote by θ involution of G induced by the nontrivial element of $\text{Gal}(E/F)$. Hence $H = G^\theta$. Note that in this case, with the notations of §2.1 we have $A_G^H = A_H$. Set

$$\nu(H) := [H(F) \cap A_G(F) : A_H(F)]$$

As in §2.1, we let $\overline{G} := G/A_G$, $\overline{H} := H/A_H$, $X := A_G(F)H(F) \backslash G(F)$, $\mathbf{X} := HA_G \backslash G$, $\sigma_X := \sigma_{\mathbf{X}}$ and $\overline{\sigma} := \sigma_{\overline{G}}$. Note that X is an open subset of $\mathbf{X}(F)$. We have the following identity between Weyl discriminants

$$(4.1.1) \quad D^H(h) = D^G(h)^{1/2}, \quad h \in H_{\text{reg}}(F)$$

which will be crucial in what follows.

Let ω and χ be continuous unitary characters of $A_G(F)$ and $H(F)$ respectively such that $\omega|_{A_H(F)} = \chi|_{A_H(F)}$. In §2.1, we have defined a continuous linear form $f \in \mathcal{C}_{\omega, \text{scusp}}(G(F)) \mapsto J^X(f)$. We define a second continuous linear form $f \in \mathcal{C}_{\omega, \text{scusp}}(G(F)) \mapsto J_{\text{geom}}^X(f)$ by setting

$$J_{\text{geom}}^X(f) := \nu(H) \sum_{T \in \mathcal{T}_{\text{ell}}(H)} |W(H, T)|^{-1} \int_{\overline{T}(F)} D^H(t) \Theta_f(t) \chi(t)^{-1} dt$$

for all $f \in \mathcal{C}_{\omega, \text{scusp}}(G(F))$, where $\mathcal{T}_{\text{ell}}(H)$ denotes a set of representatives of the $H(F)$ -conjugacy classes of maximal elliptic tori in H , we have set $\overline{T} := T/A_H$ for all $T \in \mathcal{T}_{\text{ell}}(H)$ and we recall that $\overline{T}(F)$ is equipped with the Haar measure of total mass 1. Since for all $f \in \mathcal{C}_{\omega, \text{scusp}}(G(F))$ the function $(D^G)^{1/2} \Theta_f$ is locally bounded, by 4.1.1 we see that the expression defining $J_{\text{geom}}^X(f)$ is absolutely convergent. The goal of this chapter is to show the following

Theorem 4.1.1 For all $f \in \mathcal{C}_{\omega, \text{scusp}}(G(F))$, we have

$$J^X(f) = J_{\text{geom}}^X(f)$$

We fix a function $f \in \mathcal{C}_{\omega, \text{scusp}}(G(F))$ until the end of this chapter.

4.2 Truncation and first decomposition

We fix a sequence $(\kappa_N)_{N \geq 1}$ of functions $\kappa_N : X(F) \rightarrow \{0, 1\}$ satisfying the two following conditions:

(4.2.1) There exist $C_1, C_2 > 0$ such that for all $x \in X(F)$ and all $N \geq 1$, we have:

$$\sigma_X(x) \leq C_1 N \Rightarrow \kappa_N(x) = 1$$

$$\kappa_N(x) \neq 0 \Rightarrow \sigma_X(x) \leq C_2 N$$

(4.2.2) There exists an open-compact subgroup $K' \subset G(F)$ such that the function κ_N is right-invariant by K' for all $N \geq 1$.

Such a sequence of truncation functions is easy to construct (see [11] §10.9). Set

$$J_N^X(f) = \int_X K^X(f, x) \kappa_N(x) dx$$

for all $N \geq 1$. Then we have

$$(4.2.3) \quad J^X(f) = \lim_{N \rightarrow \infty} J_N^X(f)$$

Let $\mathcal{T}(H)$ be a set of representatives of the $H(F)$ -conjugacy classes of maximal tori in H . By the Weyl integration formula for H , we have

$$K^X(f, x) = \sum_{T \in \mathcal{T}(H)} |W(H, T)|^{-1} \int_{\overline{T}(F)} D^H(t) \int_{T(F) \backslash H(F)} f(x^{-1} h^{-1} t h x) dh \chi(t)^{-1} dt$$

where we have set $\overline{T} := T/A_H$ for all $T \in \mathcal{T}(H)$. At least formally, it follows that for all $N \geq 1$

$$(4.2.4) \quad J_N^\chi(f) = \sum_{T \in \mathcal{T}(H)} |W(H, T)|^{-1} \int_{\overline{T}(F)} D^H(t) \int_{T^G(F) \backslash G(F)} f(g^{-1}tg) \kappa_{N,T}(g) dg \chi(t)^{-1} dt$$

where for all $T \in \mathcal{T}(H)$ we have denoted by T^G the centralizer of T in G (a maximal torus in G) and we have set

$$\kappa_{N,T}(g) := \int_{A_G(F)T(F) \backslash T^G(F)} \kappa_N(ag) da$$

for all $g \in G(F)$. Define

$$J_{N,T}^\chi(f) := \int_{\overline{T}(F)} D^H(t) \int_{T^G(F) \backslash G(F)} f(g^{-1}tg) \kappa_{N,T}(g) dg \chi(t)^{-1} dt$$

for all $N \geq 1$ and all $T \in \mathcal{T}(H)$. The equality 4.2.4 can thus be restated as

$$(4.2.5) \quad J_N^\chi(f) = \sum_{T \in \mathcal{T}(H)} |W(H, T)|^{-1} J_{N,T}^\chi(f)$$

The previous formal manipulations are justified a posteriori by the following lemma:

Lemma 4.2.1 (i) *There exists $k \geq 1$ such that*

$$\kappa_{N,T}(g) \ll N^k \sigma_{T^G \backslash G}(g)^k$$

for all $N \geq 1$ and all $g \in G(F)$.

(ii) *For all $T \in \mathcal{T}(H)$, the expression defining $J_{N,T}^\chi(f)$ is absolutely convergent and the identity 4.2.5 is valid.*

Proof:

(i) Since the natural inclusion $A_G T \backslash T^G \subset \mathbf{X}$ is a closed immersion (essentially because T^G is θ -stable), we have $\sigma_X(a) \sim \sigma_{A_G T \backslash T^G}(a)$ for all $a \in T^G(F)$. As $\sigma_X(x) \ll \sigma_X(xg)\sigma(g)$ for all $(x, g) \in X \times G(F)$, it follows from 4.2.1 that there exists $c_1 > 0$ such that for all $N \geq 1$, all $a \in T^G(F)$ and all $g \in G(F)$ we have

$$\kappa_N(ag) \neq 0 \Rightarrow \sigma_{A_G T \backslash T^G}(a) < c_1 N \sigma(g)$$

Hence, since the function κ_N is nonnegative and bounded by 1, by 1.2.3 there exists $k > 0$ such that

$$\kappa_{N,T}(g) \leq \text{meas}((A_G T \backslash T^G)[< c_1 N \sigma(g)]) \ll N^k \sigma(g)^k$$

for all $N \geq 1$ and all $g \in G(F)$. The function $g \mapsto \kappa_{N,T}(g)$ being left invariant by $T^G(F)$ we may replace $\sigma(g)$ in the inequality above by $\inf_{a \in T^G(F)} \sigma(ag)$ which by 1.2.1 is equivalent to $\sigma_{T^G \backslash G}(g)$. This proves (i).

(ii) Since f belongs to the Harish-Chandra-Schwartz space $\mathcal{C}_\omega(G(F))$, this follows from a combination of (i), 4.1.1 and Lemma 1.9.4. ■

From now on and until the end of §4.5, we fix a torus $T \in \mathcal{T}(H)$.

4.3 Change of truncation

Set $\overline{T}^G := T^G/A_G$ (a maximal torus of \overline{G}) and let \overline{A} be the maximal (θ, F) -split subtorus of \overline{T}^G . Let $A \subset T^G$ be the inverse image of \overline{A} and set

$$\kappa_{N,A}(g) := \int_{A_G(F) \backslash A(F)} \kappa_N(ag) da$$

for all $g \in G(F)$ and all $N \geq 1$. We define the following quantity

$$\nu(T) := [H(F) \cap A_G(F) : A_H(F)] \times [A(F) \cap A_T(F) : A_H(F)]^{-1}$$

Then, we have

$$(4.3.1) \quad J_{N,T}^\chi(f) = \nu(T) \int_{\overline{T}(F)} D^H(t) \int_{A_T(F)A(F) \backslash G(F)} f(g^{-1}tg) \kappa_{N,A}(g) dg \chi(t)^{-1} dt$$

Indeed, by our choices of Haar measures on tori (see §1.1) and noting that $T(F) \cap A_G(F) = H(F) \cap A_G(F)$, we have

$$\begin{aligned} \kappa_{N,T}(g) &= \int_{A_G(F)T(F) \backslash T^G(F)} \kappa_N(ag) da = [(T \cap A_G)(F) : (A_T \cap A_G)(F)] \int_{A_G(F)A_T(F) \backslash T^G(F)} \kappa_N(ag) da \\ &= [(H \cap A_G)(F) : (A_T \cap A_G)(F)] \int_{A_T(F)A(F) \backslash T^G(F)} \int_{A_G(F)(A \cap A_T)(F) \backslash A(F)} \kappa_N(at^G g) dadt^G \\ &= [(H \cap A_G)(F) : (A_G \cap A_T)(F)] [(A \cap A_T)(F) : (A_G \cap A_T)(F)]^{-1} \int_{A_T(F)A(F) \backslash T^G(F)} \kappa_{N,A}(t^G g) dt^G \\ &= \nu(T) \int_{A_T(F)A(F) \backslash T^G(F)} \kappa_{N,A}(t^G g) dt^G \end{aligned}$$

for all $g \in G(F)$ and all $N \geq 1$, hence the result.

Since $A(F)A_T(F) \backslash T^G(F)$ is compact by 1.2.1 we have

$$(4.3.2) \quad \sigma_{T^G \backslash G}(g) \sim \inf_{a \in A(F)A_T(F)} \sigma(ag)$$

for all $g \in G(F)$ and hence the same proof as that of Lemma 4.2.1(i) shows that there exists $k > 0$ such that

$$(4.3.3) \quad \kappa_{N,A}(g) \ll N^k \sigma_{TG \setminus G}(g)^k$$

for all $N \geq 1$ and all $g \in G(F)$.

Let M be the centralizer in G of A . It is a θ -split Levi subgroup with $A_{M,\theta} = A_\theta$. Indeed, the inclusion $A_\theta \subset A_{M,\theta}$ is obvious and T^G is a maximal torus of M hence $A_{M,\theta}$ is included in $A_{T^G,\theta} = A_\theta$. Let \overline{A}_0 be a maximal (θ, F) -split subtorus of \overline{G} containing \overline{A} and denote by A_0 its inverse image in G . Let M_0 be the centralizer in G of A_0 . It is a minimal θ -split Levi subgroup, we again have $A_{0,\theta} = A_{M_0,\theta}$ and we set $\mathcal{A}_{0,\theta} := \mathcal{A}_{M_0,\theta}$. Let K be a special maximal compact subgroup of $G(F)$. We use K to define the functions $H_{Q,\theta}$ for all $Q \in \mathcal{F}^\theta(M_0)$ (see §1.8.1). Fix $P_0 \in \mathcal{P}^\theta(M_0)$ and let Δ_0 be the set of simple roots of A_0 in P_0 . To every $Y \in \mathcal{A}_{P_0,\theta}^+$ we associate a positive (G, M_0, θ) -orthogonal set $(Y_{P'_0})_{P'_0 \in \mathcal{P}^\theta(M_0)}$ by setting $Y_{P'_0} = wY$ where w is the unique element in the little Weyl group $W(G, A_0)$ such that $wP_0 = P'_0$. By the general constructions of §1.8.2, this also induces a positive (G, M, θ) -orthogonal set $(Y_P)_{P \in \mathcal{P}^\theta(M)}$. For all $g \in G(F)$, we define another (G, M, θ) -orthogonal set $\mathcal{Y}(g) = (\mathcal{Y}(g)_P)_{P \in \mathcal{P}^\theta(M)}$ by setting

$$\mathcal{Y}(g)_P := Y_P - H_{\overline{P},\theta}(g)$$

for all $P \in \mathcal{P}^\theta(M)$ where $\overline{P} := \theta(P)$. Recall that this (G, M, θ) -orthogonal set induces a function $\Gamma_{M,\theta}^G(\cdot, \mathcal{Y}(g))$ on $\mathcal{A}_{M,\theta}$ (see §1.8.2). If $\mathcal{Y}(g)$ is a positive (G, M, θ) -orthogonal set then this is just the characteristic function of the convex hull of $\{\mathcal{Y}(g)_P; P \in \mathcal{P}^\theta(M)\}$. Define

$$\tilde{v}_{M,\theta}(Y, g) := \int_{A_G(F) \setminus A(F)} \Gamma_{M,\theta}^G(H_{M,\theta}(a), \mathcal{Y}(g)) da$$

for all $Y \in \mathcal{A}_{P_0,\theta}^+$ and all $g \in G(F)$. Fixing a norm $|\cdot|$ on $\mathcal{A}_{0,\theta}$, by 1.2.3 there exists $k > 0$ such that we have an inequality

$$(4.3.4) \quad |\tilde{v}_{M,\theta}(Y, g)| \ll (1 + |Y|)^k \sigma_{TG \setminus G}(g)^k$$

for all $Y \in \mathcal{A}_{P_0,\theta}^+$ and all $g \in G(F)$. Define the following expression

$$J_{Y,T}^X(f) := \nu(T) \int_{\overline{T}(F)} D^H(t) \int_{A_T(F)A(F) \setminus G(F)} f(g^{-1}tg) \tilde{v}_{M,\theta}(Y, g) dg \chi(t)^{-1} dt$$

for all $Y \in \mathcal{A}_{P_0,\theta}^+$. Using 4.3.4 and reasoning as in the proof of Lemma 4.2.1(ii), we can show that this expression is absolutely convergent.

Proposition 4.3.1 *Let $0 < \epsilon_1 < \epsilon_2 < 1$. Then, for all $k > 0$ we have*

$$|J_{N,T}^X(f) - J_{Y,T}^X(f)| \ll N^{-k}$$

for all $N \geq 1$ and all $Y \in \mathcal{A}_{P_0, \theta}^+$ satisfying the two inequalities

$$(4.3.5) \quad N^{\epsilon_1} \leq \inf_{\alpha \in \Delta_0} \alpha(Y)$$

$$(4.3.6) \quad \sup_{\alpha \in \Delta_0} \alpha(Y) \leq N^{\epsilon_2}$$

Proof: Let $0 < \epsilon_1 < \epsilon_2 < 1$. For $M > 0$ we will denote by $\mathbf{1}_{<M}$ and $\mathbf{1}_{\geq M}$ the characteristic functions of the sets of $g \in G(F)$ satisfying $\inf_{a \in A(F)A_T(F)} \sigma(ag) < M$ and $\inf_{a \in A(F)A_T(F)} \sigma(ag) \geq M$ respectively. For all $M > 0$, we can write

$$J_{N,T}^X(f) = J_{N,T,<M}^X(f) + J_{N,T,\geq M}^X(f)$$

$$J_{Y,T}^X(f) = J_{Y,T,<M}^X(f) + J_{Y,T,\geq M}^X(f)$$

for all $N \geq 1$ and all $Y \in \mathcal{A}_{P_0, \theta}^+$, where

$$J_{N,T,<M}^X(f) := \nu(T) \int_{\overline{T(F)}} D^H(t) \int_{A_T(F)A(F) \setminus G(F)} \mathbf{1}_{<M}(g) f(g^{-1}tg) \kappa_{N,A}(g) dg \chi(t)^{-1} dt$$

$$J_{N,T,\geq M}^X(f) := \nu(T) \int_{\overline{T(F)}} D^H(t) \int_{A_T(F)A(F) \setminus G(F)} \mathbf{1}_{\geq M}(g) f(g^{-1}tg) \kappa_{N,A}(g) dg \chi(t)^{-1} dt$$

and $J_{Y,T,<M}^X(f)$, $J_{Y,T,\geq M}^X(f)$ are defined by similar expressions. First we show

(4.3.7) For all $\epsilon > 0$ and all $k > 0$ we have

$$|J_{N,T,\geq N^\epsilon}^X(f)| \ll N^{-k}$$

and

$$|J_{Y,T,\geq N^\epsilon}^X(f)| \ll N^{-k}$$

for all $N \geq 1$ and all $Y \in \mathcal{A}_{P_0, \theta}^+$ satisfying inequality 4.3.6.

By 4.3.3 and 4.3.4 and the fact that f belongs to the Harish-Chandra-Schwartz space $\mathcal{C}_\omega(G(F))$, we only need to show that for all $k, k' > 0$ and $\epsilon > 0$ there exists $d > 0$ such that

$$\int_{\overline{T(F)}} D^H(t) \int_{A(F)A_T(F) \setminus G(F)} \mathbf{1}_{\geq N^\epsilon}(g) \Xi^G(g^{-1}tg) \overline{\sigma}(g^{-1}tg)^{-d} \sigma_{T^G \setminus G}(g)^k dg dt \ll N^{-k'}$$

for all $N \geq 1$. By 4.3.2, for all $r > 0$ this integral is essentially bounded by

$$N^{-\epsilon r} \int_{\overline{T}(F)} D^H(t) \int_{T^G(F) \setminus G(F)} \Xi^G(g^{-1}tg) \overline{\sigma}(g^{-1}tg)^{-d} \sigma_{T^G \setminus G}(g)^{k+r} dg dt$$

for all $N \geq 1$. By 4.1.1 and Lemma 1.9.4 for all $k, r > 0$ there exists $d > 0$ making the last integral above convergent. The claim follows.

Choose $\epsilon > 0$ such that $\epsilon < \epsilon_1$. By 4.3.7, it suffices to show that for all $k > 0$ we have

$$(4.3.8) \quad |J_{N,T,<N^\epsilon}^X(f) - J_{Y,T,<N^\epsilon}^X(f)| \ll N^{-k}$$

for all $N \geq 1$ and all $Y \in \mathcal{A}_{P_0,\theta}^+$ satisfying inequalities 4.3.5 and 4.3.6. For $Q \in \mathcal{F}^\theta(M)$, $Y \in \mathcal{A}_{P_0,\theta}^+$ and $N \geq 1$, we set

$$\kappa_{N,A}^{Y,Q}(g) := \int_{A_G(F) \setminus A(F)} \Gamma_{M,\theta}^Q(H_{M,\theta}(a), \mathcal{Y}(g)) \tau_{Q,\theta}^G(H_{M,\theta}(a) - \mathcal{Y}(g)_Q) \kappa_N(ag) da$$

for all $g \in G(F)$ where the functions $\Gamma_{M,\theta}^Q(\cdot, \mathcal{Y}(g))$ and $\tau_{Q,\theta}^G$ have been defined in §1.8.2. Note that

$$\Gamma_{M,\theta}^Q(\cdot, \mathcal{Y}(ag)) = \Gamma_{M,\theta}^Q(\cdot + H_{M,\theta}(a), \mathcal{Y}(g))$$

for all $a \in A(F)$. Hence the functions $\kappa_{N,A}^{Y,Q}$ are $A(F)$ -invariant on the left and this allows us to define the following expressions

$$J_{N,T,<N^\epsilon}^{X,Y,Q}(f) := \int_{\overline{T}(F)} D^H(t) \int_{A(F)A_T(F) \setminus G(F)} \mathbf{1}_{<N^\epsilon}(g) f(g^{-1}tg) \kappa_{N,A}^{Y,Q}(g) dg \chi(t)^{-1} dt$$

for all $N \geq 1$, all $Y \in \mathcal{A}_{P_0,\theta}^+$ and all $Q \in \mathcal{F}^\theta(M)$. By 1.8.2, we have

$$J_{N,T,<N^\epsilon}^X(f) = \sum_{Q \in \mathcal{F}^\theta(M)} J_{N,T,<N^\epsilon}^{X,Y,Q}(f)$$

for all $N \geq 1$ and all $Y \in \mathcal{A}_{P_0,\theta}^+$. Thus, to show 4.3.8 it suffices to establish the two following facts

(4.3.9) There exists $N_0 \geq 1$ such that

$$J_{N,T,<N^\epsilon}^{X,Y,G}(f) = J_{Y,T,<N^\epsilon}^X(f)$$

for all $N \geq N_0$ and all $Y \in \mathcal{A}_{P_0,\theta}^+$ satisfying inequality 4.3.6.

(4.3.10) For all $Q \in \mathcal{F}^\theta(M)$, $Q \neq G$, and all $k > 0$ we have

$$\left| J_{N,T,<N^\epsilon}^{\chi,Y,Q}(f) \right| \ll N^{-k}$$

for all $N \geq 1$ and all $Y \in \mathcal{A}_{P_0,\theta}^+$ satisfying inequality 4.3.5.

First we prove 4.3.9. By definition of $J_{N,T,<N^\epsilon}^{\chi,Y,G}(f)$ and $J_{Y,T,<N^\epsilon}^\chi(f)$ it suffices to show that there exists $N_0 \geq 1$ such that

$$\kappa_{N,A}^{Y,G}(g) = \tilde{v}_{M,\theta}(Y, g)$$

for all $N \geq N_0$, all $Y \in \mathcal{A}_{P_0,\theta}^+$ satisfying inequality 4.3.6 and all $g \in G(F)$ with $\sigma(g) < N^\epsilon$. Unraveling the definitions of $\kappa_{N,A}^{Y,Q}(g)$ and $\tilde{v}_{M,\theta}(Y, g)$, we see that it would follow if we can show the implication

$$(4.3.11) \quad \Gamma_{M,\theta}^G(H_{M,\theta}(a), \mathcal{Y}(g)) \neq 0 \implies \kappa_N(ag) = 1$$

for all $N \gg 1$, all $Y \in \mathcal{A}_{P_0,\theta}^+$ satisfying inequality 4.3.6, all $g \in G(F)$ with $\sigma(g) < N^\epsilon$ and all $a \in A(F)$. By 1.8.1, there exists $C > 0$ such that for all $Y \in \mathcal{A}_{P_0,\theta}^+$, all $g \in G(F)$ and all $a \in A(F)$ we have

$$\Gamma_{M,\theta}^G(H_{M,\theta}(a), \mathcal{Y}(g)) \neq 0 \implies \sigma_{A_G \setminus A}(a) \leq C \left(\sup_{\alpha \in \Delta_0} \alpha(Y) + \sigma(g) \right)$$

As $\sigma_{A_G \setminus A}(a) \sim \sigma_X(a)$ (this is a consequence of the facts that $A^\theta A_G \setminus A$ is closed in \mathbf{X} and $A_G \setminus A^\theta A_G$ is finite) and $\sigma_X(ag) \ll \sigma_X(a) + \sigma(g)$ for all $a \in A(F)$ and all $g \in G(F)$, it follows that there exists $C' > 0$ such that for all $N \geq 1$, all $Y \in \mathcal{A}_{P_0,\theta}^+$ satisfying inequality 4.3.6, all $g \in G(F)$ with $\sigma(g) < N^\epsilon$ and all $a \in A(F)$ we have

$$\Gamma_{M,\theta}^G(H_{M,\theta}(a), \mathcal{Y}(g)) \neq 0 \implies \sigma_X(ag) \leq C' (N^{\epsilon_1} + N^\epsilon)$$

Since $\epsilon, \epsilon_1 < 1$, by property 4.2.1 of our sequence of truncation functions the last inequality above implies $\kappa_N(ag) = 1$ whenever $N \gg 1$. This shows 4.3.11 and ends the proof of 4.3.9.

It only remains to prove claim 4.3.10. Fix $Q \in \mathcal{F}^\theta(M)$, $Q \neq G$, with Levi decomposition $Q = LU_Q$ where $L := Q \cap \theta(Q)$. Let $\overline{Q} = \theta(Q) = L\overline{U}_Q$ be the opposite parabolic subgroup. We have the Iwasawa decomposition $G(F) = L(F)\overline{U}_Q(F)K$ and accordingly we can decompose the integral

$$\begin{aligned} \int_{A(F)A_T(F) \setminus G(F)} \mathbf{1}_{<N^\epsilon}(g) f(g^{-1}tg) \kappa_{N,A}^{Y,Q}(g) dg = \\ \int_{A(F)A_T(F) \setminus L(F) \times \overline{U}_Q(F) \times K} \mathbf{1}_{<N^\epsilon}(luk) f(k^{-1}u^{-1}l^{-1}tluk) \kappa_{N,A}^{Y,Q}(luk) dl dudk \end{aligned}$$

for all $N \geq 1$, all $Y \in \mathcal{A}_{P_0, \theta}^+$ and all $t \in \overline{T}_{\text{reg}}(F)$. To continue we need the following fact which we will establish after we finish the proof of 4.3.10:

(4.3.12) There exists $N_0 \geq 1$ such that for all $N \geq N_0$, all $Y \in \mathcal{A}_{P_0, \theta}^+$ satisfying inequality 4.3.5 and all $l \in L(F)$, $u \in \overline{U}_Q(F)$, $k \in K$ with $\sigma(luk) < N^\epsilon$ we have

$$\kappa_{N, A}^{Y, Q}(luk) = \kappa_{N, A}^{Y, Q}(lk)$$

Taking 4.3.12 for granted we get

$$\int_{A(F)A_T(F) \backslash G(F)} \mathbf{1}_{<N^\epsilon}(g) f(g^{-1}tg) \kappa_{N, A}^{Y, Q}(g) dg = \int_{A(F)A_T(F) \backslash L(F) \times \overline{U}_Q(F) \times K} \mathbf{1}_{<N^\epsilon}(luk) f(k^{-1}u^{-1}l^{-1}tluk) \kappa_{N, A}^{Y, Q}(lk) dl dudk$$

for all $N \gg 1$, all $t \in \overline{T}_{\text{reg}}(F)$ and all $Y \in \mathcal{A}_{P_0, \theta}^+$ satisfying inequality 4.3.5. As f is strongly cuspidal if we forget the term $\mathbf{1}_{<N^\epsilon}(luk)$ in the integrand of the last integral above we get zero. Thus, we have

$$\int_{A(F)A_T(F) \backslash G(F)} \mathbf{1}_{<N^\epsilon}(g) f(g^{-1}tg) \kappa_{N, A}^{Y, Q}(g) dg = - \int_{A(F)A_T(F) \backslash L(F) \times \overline{U}_Q(F) \times K} \mathbf{1}_{\geq N^\epsilon}(luk) f(k^{-1}u^{-1}l^{-1}tluk) \kappa_{N, A}^{Y, Q}(lk) dl dudk$$

for all $N \gg 1$, all $t \in \overline{T}_{\text{reg}}(F)$ and all $Y \in \mathcal{A}_{P_0, \theta}^+$ satisfying inequality 4.3.5. Hence, to get 4.3.10 it only remains to show that for all $k > 0$ we have

$$(4.3.13) \quad \int_{\overline{T}(F)} D^H(t) \int_{A(F)A_T(F) \backslash L(F) \times \overline{U}_Q(F) \times K} \mathbf{1}_{\geq N^\epsilon}(luk) |f(k^{-1}u^{-1}l^{-1}tluk)| \left| \kappa_{N, A}^{Y, Q}(lk) \right| dl dudk dt \ll N^{-k}$$

for all $N \geq 1$ and all $Y \in \mathcal{A}_{\theta, P_0}^+$. Since for all (G, M, θ) -orthogonal set $\mathcal{Z} = (\mathcal{Z}_P)_{P \in \mathcal{P}^\theta(M)}$ the function $X \in \mathcal{A}_\theta \mapsto \Gamma_{M, \theta}^Q(X, \mathcal{Z})$ is uniformly bounded independently of \mathcal{Z} (this follows from the definition of this function) and $\tau_{Q, \theta}^G$ is a characteristic function, we have

$$\left| \kappa_{N, A}^{Y, Q}(g) \right| \ll \kappa_{N, A}(g)$$

for all $N \geq 1$, all $Y \in \mathcal{A}_{P_0, \theta}^+$ and all $g \in G(F)$. Hence, by 4.3.3 and the fact that f belongs to the Harish-Chandra-Schwartz space $\mathcal{C}_\omega(G(F))$ there exists $k > 0$ such that for all $d > 0$ the left hand side of 4.3.13 is essentially bounded by the product of N^k with

$$\int_{\overline{T}(F)} D^H(t) \int_{A(F)A_T(F)\backslash L(F)\times \overline{U}_Q(F)\times K} \mathbf{1}_{\geq N^\epsilon}(luk)\Xi^G(k^{-1}u^{-1}l^{-1}tluk)\overline{\sigma}(k^{-1}u^{-1}l^{-1}tluk)^{-d} \sigma_{T^G\backslash G}(lk)^k dldudkdt$$

for all $N \geq 1$ and all $Y \in \mathcal{A}_{P_0, \theta}^+$. Since $T^G \subset L$ we have $\sigma_{T^G\backslash G}(lk) \ll \sigma_{T^G\backslash G}(luk)$ for all $l \in L(F)$, $u \in \overline{U}_Q(F)$ and all $k \in K$. Thus, the last expression above is essentially bounded by

$$\int_{\overline{T}(F)} D^H(t) \int_{A(F)A_T(F)\backslash G(F)} \mathbf{1}_{\geq N^\epsilon}(g)\Xi^G(g^{-1}tg)\overline{\sigma}(g^{-1}tg)^{-d} \sigma_{T^G\backslash G}(g)^k dg$$

for all $N \geq 1$. We already saw that for all $k' > 0$ we can find $d > 0$ such that this last integral is essentially bounded by $N^{-k'}$ for all $N \geq 1$. This shows 4.3.13 and ends the proof of 4.3.10 granting 4.3.12.

We now prove 4.3.12. For $g \in G(F)$, the function $\Gamma_{M, \theta}^Q(\cdot, \mathcal{Y}(g))\tau_{Q, \theta}^G(\cdot - \mathcal{Y}(g)_Q)$ depends only on the points $\mathcal{Y}(g)_P$ for all $P \in \mathcal{P}^\theta(M)$ with $P \subset Q$ and those points remain invariant by left translation of g by $\overline{U}_Q(F)$. Hence, it suffices to show the following:

(4.3.14) There exists $N_0 \geq 1$ such that for all $N \geq N_0$, all $Y \in \mathcal{A}_{P_0, \theta}^+$ satisfying inequality 4.3.5, all $l \in L(F)$, $u \in \overline{U}_Q(F)$, $k \in K$ with $\sigma(luk) < N^\epsilon$ and all $a \in A(F)$ we have

$$\Gamma_{M, \theta}^Q(H_{M, \theta}(a), \mathcal{Y}(l))\tau_{Q, \theta}^G(H_{M, \theta}(a) - \mathcal{Y}(l)_Q) \neq 0 \Rightarrow \kappa_N(aluk) = \kappa_N(alk)$$

Let $N \geq 1$ and Y, l, u, k be as above (in particular Y satisfies condition 4.3.5 and $\sigma(luk) < N^\epsilon$). We will show that the conclusion of 4.3.14 holds provided N is sufficiently large. Let $a \in A(F)$ be such that

$$\Gamma_{M, \theta}^Q(H_{M, \theta}(a), \mathcal{Y}(l))\tau_{Q, \theta}^G(H_{M, \theta}(a) - \mathcal{Y}(l)_Q) \neq 0$$

We need to show that $\kappa_N(aluk) = \kappa_N(alk)$.

There exists $C > 0$ such that for all $g \in G(F)$ and all $Y \in \mathcal{A}_{P_0, \theta}^+$ if

$$\sigma(g) \leq C \inf_{\alpha \in \Delta_0} \alpha(Y)$$

then $\mathcal{Y}(g)_P \in \mathcal{A}_{P, \theta}^+$ for all $P \in \mathcal{P}^\theta(M)$ and thus $\mathcal{Y}(g)$ is a positive (G, M, θ) -orthogonal set. As $\epsilon < \epsilon_1$ it follows that for N sufficiently large the (G, M, θ) -orthogonal set $\mathcal{Y}(l)$ is positive. In particular, again for N sufficiently large, the function

$$X \in \mathcal{A}_{M, \theta} \mapsto \Gamma_{M, \theta}^Q(X, \mathcal{Y}(l))\tau_{Q, \theta}^G(X - \mathcal{Y}(l)_Q)$$

is the characteristic function of the sum of $\mathcal{A}_{Q,\theta}^+$ with the convex hull of the family $(\mathcal{Y}(l)_P)_{P \subset Q}$. As $\epsilon < \epsilon_1$ and $\sigma(l) \ll N^\epsilon$, it follows that

$$(4.3.15) \quad \log|\beta(a)| \geq \inf_{P \subset Q} \beta(Y_P - H_{\overline{P},\theta}(l)) \gg \inf_{\alpha \in \Delta_0} \alpha(Y) - \sigma(l) \gg N^{\epsilon_1}$$

for all $\beta \in R(A, U_Q)$. Fix a norm $|\cdot|$ on $\mathfrak{g}(F)$ and let us denote by $B(0, r)$ the open ball of radius r centered at the origin for all $r > 0$. Since $\sigma(lul^{-1}) \ll N^\epsilon$ and $\epsilon < \epsilon_1$, we deduce from 4.3.15 that there exists a constant $c_1 > 0$ such that for N big enough we have

$$alul^{-1}a^{-1} \in \exp(B(0, e^{-c_1 N^{\epsilon_1}}))$$

Let $P_a \in \mathcal{P}^\theta(M)$ be such that $H_{M,\theta}(a) \in \overline{\mathcal{A}_{P_a,\theta}^+}$ (the closure of the positive chamber associated to P_a). Since P_a is θ -split, the multiplication map $H(F) \times P_a(F) \rightarrow G(F)$ is submersive at the origin and hence this map admits an F -analytic section defined on a neighborhood of 1 in $G(F)$. It follows that there exists $c_2 > 0$ (independent of a since there is only a finite number of possibilities for P_a) so that for N large enough

$$alul^{-1}a^{-1} \in H(F) \exp(B(0, e^{-c_2 N^{\epsilon_1}}) \cap \mathfrak{p}_a(F))$$

Choose $X \in B(0, e^{-c_2 N^{\epsilon_1}}) \cap \mathfrak{p}_a(F)$ with $alul^{-1}a^{-1} \in H(F) \exp(X)$. Since κ_N is left invariant by $H(F)$ we have

$$\kappa_N(aluk) = \kappa_N(\exp(X)alk)$$

As $a \in \overline{\mathcal{A}_{P_a,\theta}^+}$, $\epsilon < \epsilon_1$ and $\sigma(l) \ll N^\epsilon$, there exists a constant $c_3 > 0$ such that

$$k^{-1}l^{-1}a^{-1}Xalk \in B(0, e^{-c_3 N^{\epsilon_1}})$$

By property 4.2.2 of our sequence of truncation functions, we deduce that for N sufficiently large κ_N is right invariant by $\exp(k^{-1}l^{-1}a^{-1}Xalk)$. Hence,

$$\kappa_N(\exp(X)alk) = \kappa_N(alk)$$

This proves claim 4.3.14 and ends the proof of the proposition. ■

4.4 First computation of the limit

Recall the function $g \in G(F) \mapsto v_{M,\theta}(g)$ introduced in §1.8.2. Its value at $g \in G(F)$ is given by the volume of the convex hull of the set $\{H_{P,\theta}(g); P \in \mathcal{P}^\theta(M)\}$.

Proposition 4.4.1 *We have*

$$\lim_{N \rightarrow \infty} J_{N,T}^X(f) = (-1)^{a_{M,\theta}^G(T)} \int_{\overline{T}(F)} D^H(t) \int_{A_T(F)A(F) \setminus G(F)} f(g^{-1}tg) v_{M,\theta}(g) dg \chi(t)^{-1} dt$$

Proof: Let $0 < \epsilon_1 < \epsilon_2 < 1$, $0 < \delta < 1$ and set

$$\mathcal{A}_{P_0, \theta}^+(\delta) := \{Y \in \mathcal{A}_{P_0, \theta}^+; \inf_{\alpha \in \Delta_0} \alpha(Y) \geq \delta \sup_{\alpha \in \Delta_0} \alpha(Y)\}$$

Then $\mathcal{A}_{P_0, \theta}^+(\delta)$ is a cone in $\mathcal{A}_{0, \theta}$ with nonempty interior. By Proposition 4.3.1 for all $k > 0$ we have

$$|J_{N, T}^X(f) - J_{Y, T}^X(f)| \ll N^{-k}$$

for all $N \geq 1$ and all $Y \in \mathcal{A}_{P_0, \theta}^+(\delta)$ with $N^{\epsilon_1} \leq \inf_{\alpha \in \Delta_0} \alpha(Y) \leq \delta^{-1} N^{\epsilon_2}$. As for N sufficiently large the two sets

$$\left\{ Y \in \mathcal{A}_{P_0, \theta}^+(\delta); N^{\epsilon_1} \leq \inf_{\alpha \in \Delta_0} \alpha(Y) \leq \delta^{-1} N^{\epsilon_2} \right\}$$

and

$$\left\{ Y \in \mathcal{A}_{P_0, \theta}^+(\delta); (N+1)^{\epsilon_1} \leq \inf_{\alpha \in \Delta_0} \alpha(Y) \leq \delta^{-1} (N+1)^{\epsilon_2} \right\}$$

intersect, it follows that the two limits

$$\lim_{N \rightarrow \infty} J_{N, T}^X(f), \quad \lim_{Y \in \mathcal{A}_{P_0, \theta}^+(\delta) \rightarrow \infty} J_{Y, T}^X(f)$$

exist and are equal. We will denote by $J_{\infty, T}^X(f)$ this common limit.

Let $\mathcal{A}_{0, \theta, F}$ denote the image of $A_{0, \theta}(F)$ by $H_{M_0, \theta}$. Then by Lemma 1.8.2, we know that for every lattice $\mathcal{R} \subset \mathcal{A}_{0, \theta, F} \otimes \mathbf{Q}$ and all $g \in G(F)$ the function $Y \in \mathcal{R} \cap \mathcal{A}_{P_0, \theta}^+(\delta) \mapsto \tilde{v}_{M, \theta}(Y, g)$ coincides with the restriction of an exponential-polynomial of bounded degree and with exponents in a fixed finite set (both independent of g). Let us denote by $\tilde{v}_{M, \theta, 0}(\mathcal{R}, g)$ the constant term of the purely polynomial part of this exponential-polynomial. Then by Lemma 1.8.1, we have

$$(4.4.1) \quad J_{\infty, T}^X(f) = \nu(T) \int_{\overline{T}(F)} D^H(t) \int_{A_T(F)A(F) \backslash G(F)} f(g^{-1}tg) \tilde{v}_{M, \theta, 0}(\mathcal{R}, g) dg \chi(t)^{-1} dt$$

for every lattice $\mathcal{R} \subset \mathcal{A}_{0, \theta, F} \otimes \mathbf{Q}$. Fix such a lattice \mathcal{R} . By Lemma 1.8.2, there exists $r > 0$ such that

$$\left| \tilde{v}_{M, \theta, 0}\left(\frac{1}{k}\mathcal{R}, g\right) - (-1)^{a_{M, \theta}^G} v_{M, \theta}(g) \right| \ll \sigma(g)^r k^{-1}$$

for all $k \geq 1$ and all $g \in G(F)$. Since the left hand side is invariant by left translation of g by $T^G(F)$, we also have

$$\left| \tilde{v}_{M,\theta,0}\left(\frac{1}{k}\mathcal{R}, g\right) - (-1)^{a_{M,\theta}^G} v_{M,\theta}(g) \right| \ll \sigma_{TG \setminus G}(g)^r k^{-1}$$

for all $k \geq 1$ and all $g \in G(F)$. By Lemma 1.9.4, this implies that

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{\overline{T}(F)} D^H(t) \int_{A_T(F)A(F) \setminus G(F)} f(g^{-1}tg) \tilde{v}_{M,\theta,0}\left(\frac{1}{k}\mathcal{R}, g\right) dg \chi(t)^{-1} dt = \\ (-1)^{a_{M,\theta}^G} \int_{\overline{T}(F)} D^H(t) \int_{A_T(F)A(F) \setminus G(F)} f(g^{-1}tg) v_{M,\theta}(g) dg \chi(t)^{-1} dt \end{aligned}$$

From this and 4.4.1 (which is of course also true if we replace \mathcal{R} by $\frac{1}{k}\mathcal{R}$, $k \geq 1$) we deduce the proposition. ■

4.5 End of the proof

By the descent formula 1.8.4 and Proposition 4.4.1 we have

$$\lim_{N \rightarrow \infty} J_{N,T}^X(f) = (-1)^{a_{M,\theta}^G} \nu(T) [A_{TG}(F) : A_T(F)A(F)] \sum_{L \in \mathcal{L}(M)} d_{M,\theta}^G(L) \int_{\overline{T}(F)} \Phi_M^Q(t, f) \chi(t)^{-1} dt$$

Since f is strongly cuspidal only the term corresponding to $L = G$ can contribute to the sum above so that

$$(4.5.1) \quad \lim_{N \rightarrow \infty} J_{N,T}^X(f) = (-1)^{a_{M,\theta}^G} \nu(T) [A_{TG}(F) : A_T(F)A(F)] d_{M,\theta}^G(G) \int_{\overline{T}(F)} \Phi_M(t, f) \chi(t)^{-1} dt$$

Assume first that T is not elliptic in H . We distinguish two cases:

- If $M \neq \text{Cent}_G(A_{TG})$ we have $\Phi_M(t, f) = 0$ for all $t \in \overline{T}_{\text{reg}}(F)$ as $M \neq M(t)$ so that the limit 4.5.1 vanishes.
- If $M = \text{Cent}_G(A_{TG})$, we have $\mathcal{A}_M^{G,\theta} = \mathcal{A}_T^H \neq 0$ (as T is not elliptic in H), thus $d_{M,\theta}^G(G) = 0$ and the limit 4.5.1 also vanishes in this case.

Hence, in both cases the limit 4.5.1 equals zero for T nonelliptic in H . Now, if T is elliptic in H we have $A_T = A_H$, $A_{TG} = A$ and $\mathcal{A}_M^{G,\theta} = 0$ so that $d_{M,\theta}^G(G) = 1$, $[A_{TG}(F) : A_T(F)A(F)] = 1$ and $\nu(T) = \nu(H)$ and we get

$$\lim_{N \rightarrow \infty} J_{N,T}^X(f) = (-1)^{a_M^G} \nu(H) \int_{\overline{T}(F)} \Phi_M(t, f) \chi(t)^{-1} dt = \nu(H) \int_{\overline{T}(F)} D^H(t) \Theta_f(t) \chi(t)^{-1} dt$$

Theorem 4.1.1 now follows from the above equality, 4.2.3 and 4.2.5. ■

5 Applications to a conjecture of Prasad

In this chapter E/F is a quadratic extension, H is a connected reductive group over F and $G := R_{E/F}H_E$. We will denote by θ the involution of G induced by the nontrivial element of $\text{Gal}(E/F)$. Hence $H = G^\theta$. As before we set $\overline{G} := G/A_G$ and $\overline{H} := H/A_H$. If Q is an algebraic subgroup of H then $R_{E/F}Q_E$ is an algebraic subgroup of G . Also, note that if P (resp. M) is a parabolic (resp. Levi) subgroup of G which is θ -stable (i.e. $\theta(P) = P$, resp. $\theta(M) = M$) then there exists a parabolic (resp. Levi) subgroup \mathcal{P} (resp. \mathcal{M}) of H such that $P = R_{E/F}\mathcal{P}_E$ (resp. $M = R_{E/F}\mathcal{M}_E$). We will denote by $\mathcal{R}(G)$ the space of *virtual* representations of $G(F)$ that is the complex vector spaces with basis $\text{Irr}(G)$. Similarly, if A is an abelian group we will denote by $\mathcal{R}(A)$ the space of virtual characters of A . We will write $H^i(F, \cdot)$ for the functors of Galois cohomology and if \mathcal{H}, \mathcal{G} are algebraic groups over F with \mathcal{H} a subgroup of \mathcal{G} we will set

$$\ker^1(F; \mathcal{H}, \mathcal{G}) := \text{Ker} (H^1(F, \mathcal{H}) \rightarrow H^1(F, \mathcal{G}))$$

By [35] Theorem 1.2, for every connected reductive group over F there exists a natural structure of abelian group on $H^1(F, \mathcal{G})$ which is uniquely characterized by the fact that for every elliptic maximal torus $T \subset \mathcal{G}$ the natural map $H^1(F, T) \rightarrow H^1(F, \mathcal{G})$ is a group morphism. Moreover, for every connected reductive groups \mathcal{H}, \mathcal{G} and every morphism $\mathcal{H} \rightarrow \mathcal{G}$ the induced map $H^1(F, \mathcal{H}) \rightarrow H^1(F, \mathcal{G})$ is a group morphism. Indeed, if \mathcal{T} is an elliptic maximal torus in \mathcal{H} (whose existence is guaranteed by [33] p.271) and \mathcal{T}' is a maximal torus of \mathcal{G} containing the image of \mathcal{T} , then we have a commuting square

$$\begin{array}{ccc} H^1(F, \mathcal{T}) & \longrightarrow & H^1(F, \mathcal{T}') \\ \downarrow & & \downarrow \\ H^1(F, \mathcal{H}) & \longrightarrow & H^1(F, \mathcal{G}) \end{array}$$

where the upper, left and right arrows are morphisms of groups and moreover the map $H^1(F, \mathcal{T}) \rightarrow H^1(F, \mathcal{H})$ is surjective ([35] Lemma 10.2). From these, it easily follows that $H^1(F, \mathcal{H}) \rightarrow H^1(F, \mathcal{G})$ is a morphism of abelian groups.

Finally, for every $\pi \in \text{Irr}(G)$ and every continuous character χ of $H(F)$ we recall that in 3.1 we have defined a multiplicity

$$m(\pi, \chi) := \dim \text{Hom}_H(\pi, \chi)$$

which is always finite by [18], Theorem 4.5. The function $\pi \in \text{Irr}(G) \mapsto m(\pi, \chi)$ extends by linearity to $\mathcal{R}(G)$.

5.1 A formula for the multiplicity

We will denote by $\Gamma_{\text{ell}}(\overline{H})$ the set of regular elliptic conjugacy classes in $\overline{H}(F)$ and we equip this set with a topology and a measure characterized by the fact that for all $x \in \overline{H}_{\text{reg}}(F)$

the map $t \in G_x(F) \mapsto tx \in \Gamma_{\text{ell}}(\overline{H})$, which is well-defined in a neighborhood of the identity, is a local isomorphism preserving measures near 1 (recall that in §1.1 we have fixed Haar measures on the F -points of any torus and in particular on $G_x(F)$). More concretely, if we fix a set $\mathcal{T}_{\text{ell}}(H)$ of representatives of the $H(F)$ -conjugacy classes of elliptic maximal tori in H , then for every integrable function φ on $\Gamma_{\text{ell}}(\overline{H})$ we have the following integration formula

$$\int_{\Gamma_{\text{ell}}(\overline{H})} \varphi(x) dx = \sum_{T \in \mathcal{T}_{\text{ell}}(H)} |W(H, T)|^{-1} \int_{\overline{T}(F)} \varphi(t) dt$$

where we have set $\overline{T} := T/A_H$ for all $T \in \mathcal{T}_{\text{ell}}(H)$ and we recall since \overline{T} is anisotropic, the Haar measure on $\overline{T}(F)$ is of total mass 1. For all $\pi \in \text{Irr}(G)$ and every continuous character χ of $H(F)$ with $\omega_{\pi|_{A_H(F)}} = \chi|_{A_H(F)}$, set

$$m_{\text{geom}}(\pi, \chi) := \int_{\Gamma_{\text{ell}}(\overline{H})} D^H(x) \Theta_{\pi}(x) \chi(x)^{-1} dx$$

This expression makes sense since semisimple regular elements of H are also semisimple and regular in G and the function $x \in H_{\text{reg}}(F) \mapsto D^H(x)^{1/2} \Theta_{\pi}(x)$ is locally bounded on $H(F)$ by [25] Theorem 16.3 and the identity $D^H(x) = D^G(x)^{1/2}$.

Recall that we are denoting by $\text{Irr}_{\text{sqr}}(G)$ the set of (equivalence classes of) irreducible essentially square-integrable representations of $G(F)$. The main theorem of this section is the following.

Theorem 5.1.1 *For all $\pi \in \text{Irr}_{\text{sqr}}(G)$ and every continuous character χ of $H(F)$ with $\omega_{\pi|_{A_H(F)}} = \chi|_{A_H(F)}$ we have*

$$m(\pi, \chi) = m_{\text{geom}}(\pi, \chi)$$

Proof: Up to twisting π and χ by real unramified characters, we may assume that ω_{π} and χ are unitary. Set $\omega := \omega_{\pi|_{A_G(F)}}$. By Theorem 3.1.1 and Theorem 4.1.1, for all $f \in {}^0\mathcal{C}_{\omega}(G)$ we have

$$\sum_{\sigma \in \text{Irr}_{\omega, \text{sqr}}(G)} m(\sigma, \chi) \text{Trace}(\sigma^{\vee}(f)) = \int_{\Gamma_{\text{ell}}(\overline{H})} D^H(x) \Theta_f(x) \chi(x)^{-1} dx$$

By 1.4.1 and Proposition 1.6.1, when we apply this equality to a coefficient of π we get the identity of the theorem. ■

5.2 Galoisian characters and Prasad's character $\omega_{H,E}$

Let \check{H} denote the complex dual group of H , $Z(\check{H})$ be its center and W_F be the Weil group of F . Denoting by $\text{Hom}_{\text{cont}}(H(F), \mathbf{C}^{\times})$ the group of continuous characters of $H(F)$, Langlands has defined an homomorphism

$$\alpha_H : H^1(W_F, Z(\check{H})) \rightarrow \text{Hom}_{\text{cont}}(H(F), \mathbf{C}^{\times})$$

which is injective since F is p -adic but is not always surjective although it is most of the time (e.g. if H is quasi-split). We refer the reader to [37] for discussion of these matters. We will call the image of α_H the set of *Galoisian characters* (of $H(F)$). Assume that H is semi-simple. Let H_{sc} be the simply connected cover of H and $\pi_1(H)$ be the kernel of the projection $H_{sc} \rightarrow H$. By Tate-Nakayama duality, we have an isomorphism $H^1(W_F, Z(\check{H})) \simeq H^1(F, \pi_1(H))^D$, where $(\cdot)^D$ denotes duality for finite abelian groups, and the morphism α_H is the composition of this isomorphism with the (dual of the) connecting map $H(F) \rightarrow H^1(F, \pi_1(H))$. In particular, in this case, a character of $H(F)$ is Galoisian if and only if it factorizes through $H^1(F, \pi_1(H))$.

In [43], Prasad has defined a quadratic character $\omega_{H,E} : H(F) \rightarrow \{\pm 1\}$ which depends not only on H but also on the quadratic extension E/F . It is a Galoisian character whose simplest definition is as the image by α_H of the cocycle c defined by $c(w) = 1$ if $w \in W_E$ (the Weil group of E) and $c(w) = z$ if $w \in W_F \setminus W_E$, where z denotes the image of the central element $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \in SL_2(\mathbf{C})$ by any principal SL_2 -morphism $SL_2(\mathbf{C}) \rightarrow \check{H}$. In what follows we shall need another description of Prasad's character (see [43] §8 for the equivalence between the two definitions). First of all, $\omega_{H,E}$ is the pullback by $H(F) \rightarrow H_{ad}(F)$ of $\omega_{H_{ad},E}$, where H_{ad} denotes the adjoint group of H . Thus, to describe $\omega_{H,E}$ we may assume that H is semisimple. We introduce notations as before: H_{sc} is the simply connected cover of H and $\pi_1(H)$ stands for the kernel of the projection $H_{sc} \rightarrow H$. Let $B \subset H_{sc, \overline{F}}$ and $T \subset B$ be a Borel subgroup and a maximal torus thereof (both a priori only defined over \overline{F}). Let $\rho \in X_{\overline{F}}^*(T)$ be the half sum of the positive roots of T with respect to B (this belongs to the character lattice of T since H_{sc} is simply-connected). Then, it can be easily shown that the restriction of ρ to $\pi_1(H)$ induces a morphism $\pi_1(H) \rightarrow \mu_2$ defined over F . Pushing this through the inclusion $\mu_2 \hookrightarrow \text{Ker } N_{E/F}$ we get a morphism $\pi_1(H) \rightarrow \text{Ker } N_{E/F}$ and $\omega_{H,E}$ is simply the composition of the connecting map $H_{ad}(F) \rightarrow H^1(F, \pi_1(H))$ with the corresponding homomorphism between Galoisian H^1 's:

$$H^1(F, \pi_1(H)) \rightarrow H^1(F, \text{Ker } N_{E/F}) \simeq \{\pm 1\}$$

5.3 First application: comparison between inner forms

Let H' be another connected reductive group over F and let $\psi^H : H_{\overline{F}} \simeq H'_{\overline{F}}$ be an inner twisting. Set $G' := R_{E/F}H'$. Then ψ^H induces an inner twisting $\psi^G : G_{\overline{F}} \simeq G'_{\overline{F}}$ (actually, there are natural isomorphisms $G_{\overline{F}} \simeq H_{\overline{F}} \times H'_{\overline{F}}$, $G'_{\overline{F}} \simeq H_{\overline{F}} \times H'_{\overline{F}}$ and using these as identifications we just have $\psi^G = \psi^H \times \psi^H$).

Recall that two regular elements $x \in G_{\text{reg}}(F)$ and $x' \in G'_{\text{reg}}(F)$ are *stably conjugate* if there exists $g \in G(\overline{F})$ such that $x' = \psi^G(gxg^{-1})$ and the isomorphism $\psi^G \circ \text{Ad}(g) : G_{x, \overline{F}} \simeq G'_{x', \overline{F}}$ is defined over F . Similarly, two regular elements of $G(F)$ are *stably conjugate* if they are conjugate by an element of $G(\overline{F})$ which induces an isomorphism defined over F between their connected centralizers.

We say that a virtual representation $\Pi \in \mathcal{R}(G)$ (or $\Pi' \in \mathcal{R}(G')$) is *stable* if its character

Θ_Π (or $\Theta_{\Pi'}$) is constant on regular stable conjugacy classes in $G(F)$ (resp. in $G'(F)$). Two stable virtual representations $\Pi \in \mathcal{R}(G)$ and $\Pi' \in \mathcal{R}(G')$ are said to be *transfer of each other* if for all pairs $(x, x') \in G_{\text{reg}}(F) \times G'_{\text{reg}}(F)$ of stably conjugate regular elements we have $\Theta_\Pi(x) = \Theta_{\Pi'}(x')$. By the main results of [7], every stable virtual representation $\Pi \in \mathcal{R}(G)$ is the transfer of a stable virtual representation $\Pi' \in \mathcal{R}(G')$ and conversely.

We define similarly the notion of stable conjugacy for regular elements in $H(F)$ and $H'(F)$ and of transfer between (virtual) representations of $H(F)$ and $H'(F)$. The inner twist ψ^H allows to identify the L -groups of H and H' and thus to get an identification $H^1(W_F, Z(\check{H})) = H^1(W_F, Z(\check{H}'))$. We say that two Galoisian characters χ, χ' of $H(F)$ and $H'(F)$ *correspond to each other* if they originate from the same element of $H^1(W_F, Z(\check{H}))$. Galoisian characters are always stable and if χ, χ' are Galoisian characters of $H(F)$, $H'(F)$ respectively that correspond to each other then they are also transfer of each other.

Theorem 5.3.1 *Let Π and Π' be stable virtual essentially square-integrable representations of $G(F)$ and $G'(F)$ respectively. Let χ and χ' be Galoisian characters of $H(F)$ and $H'(F)$ respectively. Then, if Π, Π' are transfer of each other and χ, χ' correspond to each other, we have*

$$m(\Pi, \chi) = m(\Pi', \chi')$$

Proof: Let $\Gamma_{\text{ell}}(\overline{H})/\text{stab}$ be the set of stable conjugacy classes in $\Gamma_{\text{ell}}(\overline{H})$. It is easy to see that we can equip $\Gamma_{\text{ell}}(\overline{H})/\text{stab}$ with a unique topology and a unique measure such that the natural projection $p : \Gamma_{\text{ell}}(\overline{H}) \rightarrow \Gamma_{\text{ell}}(\overline{H})/\text{stab}$ is a local isomorphism preserving measures locally. We define $\Gamma_{\text{ell}}(\overline{H}')/\text{stab}$ and equip it with a topology and a measure in a similar way. Let $p' : \Gamma_{\text{ell}}(\overline{H}') \rightarrow \Gamma_{\text{ell}}(\overline{H}')/\text{stab}$ be the natural projection. Since Π and Π' are stable and essentially square-integrable, by Theorem 5.1.1 we have

$$m(\Pi, \chi) = \int_{\Gamma_{\text{ell}}(\overline{H})/\text{stab}} |p^{-1}(x)| D^H(x) \Theta_\Pi(x) \chi(x)^{-1} dx$$

and

$$m(\Pi', \chi') = \int_{\Gamma_{\text{ell}}(\overline{H}')/\text{stab}} |p'^{-1}(y)| D^{H'}(y) \Theta_{\Pi'}(y) \chi'(y)^{-1} dy$$

Since we can always transfer elliptic regular elements to all inner forms (see [35] §10), there is a bijection

$$\Gamma_{\text{ell}}(\overline{H})/\text{stab} \simeq \Gamma_{\text{ell}}(\overline{H}')/\text{stab}$$

characterized by: $x \mapsto y$ if and only if x and y are stably conjugate. It is not hard to see that this bijection preserves measures locally and hence globally. Let $x \in H_{\text{reg}}(F)$ and $y \in H'_{\text{reg}}(F)$ be two stably conjugate elements. As $\Theta_\Pi, \Theta_{\Pi'}$ on the one hand and χ, χ' on the other hand are transfer of each other, we have $\Theta_\Pi(x) = \Theta_{\Pi'}(y)$ and $\chi(x) = \chi'(y)$. Moreover, we also have $D^H(x) = D^{H'}(y)$. Therefore, to get the theorem it only remains to

show that $|p^{-1}(x)| = |p'^{-1}(y)|$. By standard cohomological arguments we have $|p^{-1}(x)| = |\ker^1(F; T, G)|$ and $|p'^{-1}(y)| = |\ker^1(F; T', G')|$ where $T := G_x$ and $T' := G'_y$. Since F is p -adic, by [35] Theorem 1.2 there exist structures of abelian groups on $H^1(F, G)$ and $H^1(F, G')$ such that the natural maps $H^1(F, T) \rightarrow H^1(F, G)$ and $H^1(F, T') \rightarrow H^1(F, G')$ are morphisms of groups. Moreover, by [35] Lemma 10.2 these are surjective. It follows that $|\ker^1(F; T, G)| = |H^1(F, T)||H^1(F, G)|^{-1}$ and $|\ker^1(F; T', G')| = |H^1(F, T')||H^1(F, G')|^{-1}$. As T and T' are F -isomorphic we have $H^1(F, T) \simeq H^1(F, T')$ and by [35] Theorem 1.2 again we have $H^1(F, G) \simeq H^1(F, G')$ (since G and G' have isomorphic L -groups). This suffices to conclude that $|\ker^1(F; T, G)| = |\ker^1(F; T', G')|$ and therefore that $|p^{-1}(x)| = |p'^{-1}(y)|$. ■

5.4 Elliptic twisted Levi subgroups

In this section we assume for simplicity that H is semi-simple and quasi-split. (All the results presented in this section are still true, with obvious modifications, in general. However, the assumption that H is semi-simple and quasi-split simplifies a lot the proofs and, in any case, we will only need to apply them for such groups.)

We say that an algebraic subgroup \mathcal{M} of H is a *twisted Levi subgroup* if $R_{E/F}\mathcal{M}_E$ is a Levi subgroup of G . If \mathcal{M} is a twisted Levi subgroup of H , we say that it is *elliptic* if $A_{\mathcal{M}} = \{1\}$.

Lemma 5.4.1 *Let M be a Levi subgroup of G and set $\mathcal{M} := M \cap H$. Then, the following assertions are equivalent:*

- (i) \mathcal{M} is an elliptic twisted Levi subgroup of H ;
- (ii) \mathcal{M} contains an elliptic maximal torus of H ;
- (iii) A_M is θ -split;
- (iv) M is θ -split and $\mathcal{P}(M) = \mathcal{P}^\theta(M)$.

Proof:(i) \Rightarrow (ii): By [33] p.271, \mathcal{M} contains a maximal torus T such that $A_T = A_{\mathcal{M}} = \{1\}$. Thus, T is elliptic and since \mathcal{M} is of the same (absolute) rank as H , it is also maximal in H . This proves the first implication.

(ii) \Rightarrow (iii): Assume that \mathcal{M} contains an elliptic maximal torus \mathcal{T} of H and set $T := R_{E/F}\mathcal{T}_E$. Then we have $A_M \subset A_T$ and A_T is θ -split (as $(A_T^\theta)^0$ is a split torus contained in \mathcal{T} and so is trivial) from which it follows that A_M is also θ -split.

(iii) \Rightarrow (iv): Assume that A_M is θ -split. Then, M is θ -split since it is the centralizer of A_M . Moreover θ acts on \mathcal{A}_M as $-Id$ thus sending any positive chamber \mathcal{A}_P^+ corresponding to $P \in \mathcal{P}(M)$ to its opposite. This shows that $\mathcal{P}(M) = \mathcal{P}^\theta(M)$.

(iv) \Rightarrow (i): Assume that M is θ -split and $\mathcal{P}(M) = \mathcal{P}^\theta(M)$. In particular M is θ -stable and since M comes by restriction of scalars from a subgroup of H_E (as is any evi subgroup of G), it follows that $M = R_{E/F}\mathcal{M}_E$ i.e. \mathcal{M} is a twisted Levi. It only remains to show that \mathcal{M} is elliptic. Assume, by way of contradiction, that it is not the case i.e. $A_{\mathcal{M}} \neq \{1\}$. Then

there exists a parabolic $P \in \mathcal{P}(M)$ such that $H_M(A_{\mathcal{M}}(F))$ contains a nonzero element of the closure of the positive chamber associated to P . Since this element is θ -fixed and the intersection of the closures of the positive chambers associated to P and \overline{P} (the parabolic subgroup opposite to P) is reduced to $\{0\}$ we cannot have $\theta(P) = \overline{P}$ thus contradicting the fact that $\mathcal{P}(M) = \mathcal{P}^\theta(M)$. ■

Let P_0 be a minimal θ -split parabolic subgroup of G . We claim that P_0 is also a minimal parabolic subgroup of G (hence a Borel subgroup since G is quasi-split).

Proof: Let B be a Borel subgroup of G . Since two minimal θ -split parabolic subgroups are always $G(F)$ -conjugate ([26] Proposition 4.9), it suffices to show the existence of $g \in G(F)$ such that gBg^{-1} is θ -split. Over the algebraic closure we have $G_{\overline{F}} \simeq H_{\overline{F}} \times H_{\overline{F}}$ with θ exchanging the two copies. Since B is in the same class as its opposite Borel this shows the existence of $g_1 \in G(\overline{F})$ such that $g_1 B_{\overline{F}} g_1^{-1}$ is θ -split. Then, the set $\mathcal{U} := H_{\overline{F}} g_1 B_{\overline{F}}$ is a Zariski open subset of $G_{\overline{F}}$ with the property that for all $g \in \mathcal{U}$ the Borel $g B_{\overline{F}} g^{-1}$ is θ -split. Since $G(F)$ is dense in G for the Zariski topology ([20], Exp XIV, 6.5, 6.7) we can find $g \in G(F) \cap \mathcal{U}$ and it has the desired property. ■

Set $T_0 := P_0 \cap \theta(P_0)$, $A_{min} := A_{T_0}$, $A_0 := A_{min, \theta}$ and denote by Δ_{min} and Δ_0 the sets of simple roots of A_{min} and A_0 in P_0 . All the parabolic subgroups that we will consider in this section will be standard with respect to P_0 (i.e. contain P_0) and when we write $P = MU$ for such a parabolic subgroup we always mean that U is the unipotent radical and M the unique Levi component containing T_0 . We have a natural projection $\Delta_{min} \twoheadrightarrow \Delta_0$ and θ naturally acts on $R(A_{min}, G)$ sending Δ_{min} to $-\Delta_{min}$. Let Δ_- be the set of simple roots $\alpha \in \Delta_{min}$ such that $\theta(\alpha) = -\alpha$. It can be identified with a subset of Δ_0 through the projection $\Delta_{min} \twoheadrightarrow \Delta_0$ (i.e. the restriction of this projection to Δ_- is injective). Let $I \subset \Delta_-$. We will denote by $P_I = M_I U_I$ the unique parabolic subgroup containing P_0 such that the set of simple roots of A_{min} in $M_I \cap P_0$ is precisely $\Delta_{min} - I$. Then P_I is θ -split and we have $M_I = P_I \cap \theta(P_I)$. Moreover, as $\mathcal{A}_{M_I}^*$ is generated by the restrictions of the roots in I , θ acts as $-Id$ on this space showing that A_{M_I} is θ -split and thus by point (ii) of the previous lemma that $\mathcal{M}_I := M_I \cap H$ is an elliptic twisted Levi subgroup of H . Let H_{ab} be the quotient of $H(F)$ by the common kernel of all the Galoisian characters χ of $H(F)$. It is an abelian group and for all $I \subset \Delta_-$ we will denote by $\mathcal{M}_{I, ab}$ the image of $\mathcal{M}_I(F)$ in H_{ab} .

Define $\underline{\mathcal{C}}$ as the set of pairs (\mathcal{M}, P) with \mathcal{M} an elliptic twisted Levi subgroup of H and $P \in \mathcal{P}(M)$ where $M := R_{E/F} \mathcal{M}_E$ and $\underline{\mathcal{D}}$ as the set of triples (T, \mathcal{M}, P) with $(\mathcal{M}, P) \in \underline{\mathcal{C}}$ and $T \subset \mathcal{M}$ an elliptic maximal torus. We let \mathcal{C} and \mathcal{D} denote the $H(F)$ -conjugacy classes in $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ respectively. We say that two pairs $(\mathcal{M}, P), (\mathcal{M}', P') \in \mathcal{C}$ are *stably conjugate*, and we will write $(\mathcal{M}, P) \sim_{stab} (\mathcal{M}', P')$, if there exists $h \in H(\overline{F})$ such that $h \mathcal{M}_{\overline{F}} h^{-1} = \mathcal{M}'_{\overline{F}}$ and $h P_{\overline{F}} h^{-1} = P'_{\overline{F}}$. Write $\mathcal{C}/stab$ for the set of stable conjugacy classes in \mathcal{C} . We let $\mathcal{T}_{ell}(H)$ be a set of representatives of the $H(F)$ -conjugacy classes of elliptic maximal tori in H . Finally for all $P = MU \supset P_0$ and all $T \in \mathcal{T}_{ell}(H)$, we define

$$\Gamma_M(T) := \{\gamma \in G(F); \gamma^{-1} T \gamma \subset M\} / M(F)$$

Proposition 5.4.1 (i) For all $T \in \mathcal{T}_{\text{ell}}(H)$, all $P = MU \supseteq P_0$ and all $\gamma \in \Gamma_M(T)$ we have $(T, \gamma M \gamma^{-1} \cap H, \gamma P \gamma^{-1}) \in \mathcal{D}$;

(ii) The map

$$\bigsqcup_{T \in \mathcal{T}_{\text{ell}}(H)} \bigsqcup_{P_0 \subseteq P = MU} \Gamma_M(T) \rightarrow \mathcal{D}$$

$$\gamma \in \Gamma_M(T) \mapsto (T, \gamma M \gamma^{-1} \cap H, \gamma P \gamma^{-1})$$

(which is well-defined by (i)) is surjective and the fiber over $(T, \mathcal{M}, P) \in \mathcal{D}$ is of cardinality $|W(H, T)| |W(\mathcal{M}, T)|^{-1}$.

(iii) For all $(\mathcal{M}, P) \in \mathcal{C}$ the fiber of the map $\mathcal{C} \rightarrow \mathcal{C}/\text{stab}$ containing (\mathcal{M}, P) is of cardinality $|\ker^1(F; \mathcal{M}, H)|$.

(iv) For all $(\mathcal{M}, P), (\mathcal{M}', P') \in \mathcal{C}$ we have $(\mathcal{M}, P) \sim_{\text{stab}} (\mathcal{M}', P')$ if and only if P and P' are in the same class and the map $I \subseteq \Delta_- \mapsto (\mathcal{M}_I, P_I) \in \mathcal{C}/\text{stab}$ is a bijection.

(v) Let $(\mathcal{M}, P), (\mathcal{M}', P') \in \mathcal{C}$ be such that $(\mathcal{M}, P) \sim_{\text{stab}} (\mathcal{M}', P')$. Then, for every Galoisian character χ of $H(F)$ we have $\chi|_{\mathcal{M}} = 1$ if and only if $\chi|_{\mathcal{M}'} = 1$, where we have denoted by $\chi|_{\mathcal{M}}$ and $\chi|_{\mathcal{M}'}$ the restrictions of χ to $\mathcal{M}(F)$ and $\mathcal{M}'(F)$ respectively.

(vi) We have the following identity in $\mathcal{R}(H_{ab})$:

$$\sum_{I \subseteq \Delta_-} (-1)^{|\Delta_- - I|} |\ker^1(F; \mathcal{M}_I, H)| \text{Ind}_{\mathcal{M}_I, ab}^{H_{ab}}(\mathbf{1}) = \omega_{H, E}$$

where $\omega_{H, E} : H_{ab} \rightarrow \{\pm 1\}$ is Prasad's character (see §5.2).

Proof:

(i) For all $T \in \mathcal{T}_{\text{ell}}(H)$, all $P = MU \supseteq P_0$ and all $\gamma \in \Gamma_M(T)$, the subgroup $\gamma M \gamma^{-1} \cap H$ contains an elliptic maximal torus of H (namely T) and thus by Lemma 5.4.1 is an elliptic twisted Levi subgroup. This shows that $(T, \gamma M \gamma^{-1} \cap H, \gamma P \gamma^{-1}) \in \mathcal{D}$.

(ii) Let $(T, \mathcal{M}, P) \in \mathcal{D}$. Up to conjugation we may assume that $T \in \mathcal{T}_{\text{ell}}(H)$. As P_0 is a minimal parabolic subgroup of G , there exist $\gamma \in G(F)$ such that $\gamma^{-1} P \gamma \supseteq P_0$ and $\gamma^{-1} R_{E/F} \mathcal{M}_E \gamma \supseteq T_0$. This shows the surjectivity. The claim about the cardinality of the fibers is a consequence of the two following facts:

(5.4.1) Let $T, T' \in \mathcal{T}_{\text{ell}}(H)$, $P = MU \supseteq P_0$, $P' = M'U' \supseteq P_0$, $\gamma \in \Gamma_M(T)$ and $\gamma' \in \Gamma_{M'}(T')$. Then the two triples $(T, \gamma M \gamma^{-1} \cap H, \gamma P \gamma^{-1})$ and $(T', \gamma' M' \gamma'^{-1} \cap H, \gamma' P' \gamma'^{-1})$ are $H(F)$ -conjugate if and only if $T = T'$, $M = M'$, $P = P'$ and $\gamma' \in \text{Norm}_{H(F)}(T) \gamma M(F)$.

(5.4.2) Let $T \in \mathcal{T}_{\text{ell}}(H)$, $P = MU \supset P_0$ and $\gamma \in \Gamma_M(T)$. Then, the image of the map

$$\begin{aligned} \text{Norm}_{H(F)}(T) &\rightarrow \Gamma_M(T) \\ h &\mapsto h\gamma \end{aligned}$$

is of cardinality $|W(H, T)||W(\mathcal{M}, T)|^{-1}$ where $\mathcal{M} := \gamma M \gamma^{-1} \cap H$.

Proof of 5.4.1: Assume that $(T, \gamma M \gamma^{-1} \cap H, \gamma P \gamma^{-1})$ and $(T', \gamma M' \gamma^{-1} \cap H, \gamma P' \gamma^{-1})$ are $H(F)$ -conjugate. By definition of $\mathcal{T}_{\text{ell}}(H)$ we have $T = T'$ and since P and P' are both standard with respect to P_0 we also have $P = P'$ and $M = M'$. Thus there exists $h \in H(F)$ such that

$$h(T, \mathcal{M}, \gamma P \gamma^{-1})h^{-1} = (T, \mathcal{M}', \gamma P \gamma'^{-1})$$

where $\mathcal{M} := \gamma M \gamma^{-1} \cap H$ and $\mathcal{M}' := \gamma' M \gamma'^{-1} \cap H$. This equality immediately implies that $h \in \text{Norm}_{H(F)}(T)$. Moreover, since $\gamma M \gamma^{-1} = R_{E/F} \mathcal{M}_E$ and $\gamma' M \gamma'^{-1} = R_{E/F} \mathcal{M}'_E$ we also have $h \gamma M \gamma^{-1} h^{-1} = \gamma' M \gamma'^{-1}$ and $h \gamma P \gamma^{-1} h^{-1} = \gamma' P \gamma'^{-1}$ proving that $\gamma'^{-1} h \gamma$ normalizes both M and P i.e. $\gamma' \in h \gamma M(F)$ and hence $\gamma' \in \text{Norm}_{H(F)}(T) \gamma M(F)$. This proves one direction of the claim the other being obvious.

Proof of 5.4.2: As $|W(H, T)||W(\mathcal{M}, T)|^{-1}$ is the cardinality of the quotient

$$\text{Norm}_{H(F)}(T) / \text{Norm}_{\mathcal{M}(F)}(T)$$

it suffices to show that for all $h, h' \in \text{Norm}_{H(F)}(T)$ we have $h\gamma = h'\gamma$ in $\Gamma_M(T)$ if and only if $h'^{-1}h \in \mathcal{M}(F)$. But as $\mathcal{M} = \gamma M \gamma^{-1} \cap H$ this immediately follows from the definition of $\Gamma_M(T)$.

- (iii) This follows from a standard cohomological argument by noticing that \mathcal{M} is the normalizer of the pair (\mathcal{M}, P) in H .
- (iv) If $(\mathcal{M}, P) \sim_{\text{stab}} (\mathcal{M}', P')$ then in particular P and P' are conjugate in $G(\overline{F})$ and thus are in the same class (i.e. are $G(F)$ -conjugate). Conversely, assume that P and P' are in the same class and set $M := R_{E/F} \mathcal{M}_E$, $M' := R_{E/F} \mathcal{M}'_E$. Then, there exists $g \in G(F)$ such that $gPg^{-1} = P'$ and $gMg^{-1} = M'$. Let \overline{P} and \overline{P}' be the parabolic subgroups opposite to P and P' with respect to M and M' respectively. Then $g\overline{P}g^{-1} = \overline{P}'$ and since $\theta(P) = \overline{P}$, $\theta(P') = \overline{P}'$, $\theta(M) = M$ and $\theta(M') = M'$ we also have

$$\theta(g)\overline{P}\theta(g)^{-1} = \theta(gPg^{-1}) = \overline{P}'$$

and

$$\theta(g)M\theta(g)^{-1} = \theta(gMg^{-1}) = M'$$

Therefore $g^{-1}\theta(g)$ normalizes both M and \overline{P} and thus $g^{-1}\theta(g) \in M(F)$. Since the map

$$M(\overline{F}) \rightarrow \{m \in M(\overline{F}); \theta(m) = m^{-1}\}$$

$$m \mapsto m^{-1}\theta(m)$$

is surjective, there exist $m \in M(\overline{F})$ and $h \in H(\overline{F})$ such that $g = hm$. We have $hP_{\overline{F}}h^{-1} = P'_{\overline{F}}$ and $hM_{\overline{F}}h^{-1} = M'_{\overline{F}}$ showing that (\mathcal{M}, P) and (\mathcal{M}', P') are stably conjugate. This shows the first part of (iv). Notice that for all $P \in \mathcal{P}^\theta(M_0)$ with $P \supset P_0$ the torus A_M , where $M := P \cap \theta(P)$, is θ -split if and only if $P = P_I$ for some $I \subset \Delta_-$. Hence, the second part of (iv) follows from the first and Lemma 5.4.1 since every class of parabolic subgroups contains a unique element which is standard with respect to P_0 .

- (v) We need to show that $\mathcal{M}_{ab} = \mathcal{M}'_{ab}$. Let H_{sc} be the simply connected cover of H and $\pi_1(H)$ be the kernel of the projection $H_{sc} \rightarrow H$. Then we have $H_{ab} = H^1(F, \pi_1(H))$. Let \mathcal{M}_{sc} and \mathcal{M}'_{sc} denote the inverse images of \mathcal{M} and \mathcal{M}' in H_{sc} . From the short exact sequences $1 \rightarrow \pi_1(H) \rightarrow \mathcal{M}_{sc} \rightarrow \mathcal{M} \rightarrow 1$ and $1 \rightarrow \pi_1(H) \rightarrow \mathcal{M}'_{sc} \rightarrow \mathcal{M}' \rightarrow 1$, we get exact sequences $1 \rightarrow \mathcal{M}_{ab} \rightarrow H_{ab} \rightarrow H^1(F, \mathcal{M}_{sc})$ and $1 \rightarrow \mathcal{M}'_{ab} \rightarrow H_{ab} \rightarrow H^1(F, \mathcal{M}'_{sc})$. Thus, we need to show that $\text{Ker}(H_{ab} \rightarrow H^1(F, \mathcal{M}_{sc})) = \text{Ker}(H_{ab} \rightarrow H^1(F, \mathcal{M}'_{sc}))$. By hypothesis, there exists $h \in H(\overline{F})$ such that $hM_{\overline{F}}h^{-1} = M'_{\overline{F}}$ and $hP_{\overline{F}}h^{-1} = P'_{\overline{F}}$. Set $M := R_{E/F}M_E$ and $M' := R_{E/F}M'_E$. Then, we also have $hM_{\overline{F}}h^{-1} = M'_{\overline{F}}$ and it follows that for all $\sigma \in \Gamma_F$ we have $h^\sigma h^{-1} \in H(\overline{F}) \cap M'(\overline{F}) = \mathcal{M}'(\overline{F})$. Choose $h_{sc} \in H_{sc}(\overline{F})$ which lifts h . Then, there is a bijection $\iota : H^1(F, \mathcal{M}_{sc}) \simeq H^1(F, \mathcal{M}'_{sc})$ given at the level of cocycles by

$$c \mapsto (\sigma \in \Gamma_F \mapsto h_{sc}c(\sigma)^\sigma h_{sc}^{-1})$$

Let c_0 be the 1-cocycle $\sigma \in \Gamma_F \mapsto h_{sc}^\sigma h_{sc}^{-1} \in \mathcal{M}'_{sc}(\overline{F})$ and denote by $[c_0]$ its class in $H^1(F, \mathcal{M}'_{sc})$. Then $\iota - [c_0] : H^1(F, \mathcal{M}_{sc}) \simeq H^1(F, \mathcal{M}'_{sc})$ is a bijection of pointed sets (and even an isomorphism of abelian groups) making the following square commute

$$\begin{array}{ccc} H_{ab} & \longrightarrow & H^1(F, \mathcal{M}_{sc}) \\ \parallel & & \downarrow \wr \\ H_{ab} & \longrightarrow & H^1(F, \mathcal{M}'_{sc}) \end{array}$$

This immediately implies that the kernels of the upper and bottom arrows are identical.

- (vi) Let $I \subset \Delta_-$ and denote by $\mathcal{M}_{I,sc}$ the inverse image of \mathcal{M}_I in H_{sc} . Then, from the short exact sequence $1 \rightarrow \pi_1(H) \rightarrow \mathcal{M}_{I,sc} \rightarrow \mathcal{M}_I \rightarrow 1$ we get an exact sequence

$$1 \rightarrow \mathcal{M}_{I,ab} \rightarrow H_{ab} \rightarrow H^1(F, \mathcal{M}_{I,sc}) \rightarrow H^1(F, \mathcal{M}_I) \rightarrow H^2(F, \pi_1(H))$$

By [32] the natural connecting map $H^1(F, H) \rightarrow H^2(F, \pi_1(H))$ is an isomorphism and it follows that the previous exact sequence can be rewritten as

$$1 \rightarrow \mathcal{M}_{I,ab} \rightarrow H_{ab} \rightarrow H^1(F, \mathcal{M}_{I,sc}) \rightarrow \ker^1(F; \mathcal{M}_I, H) \rightarrow 1$$

From this exact sequence, we deduce the following equality in $\mathcal{R}(H_{ab})$:

$$(5.4.3) \quad |\ker^1(F; \mathcal{M}_I, H)| \text{Ind}_{\mathcal{M}_{I,ab}}^{H_{ab}}(\mathbf{1}) = \text{Res}_I^{H_{ab}} \text{Ind}_1^I(\mathbf{1})$$

where $\text{Res}_I^{H_{ab}}$ denotes the restriction functor with respect to the morphism $H_{ab} \rightarrow H^1(F, \mathcal{M}_{I,sc})$ and Ind_1^I denotes the induction functor with respect to the morphism $1 \rightarrow H^1(F, \mathcal{M}_{I,sc})$. Moreover the morphism $H^1(F, \mathcal{M}_{\Delta_-,sc}) \rightarrow H^1(F, \mathcal{M}_{I,sc})$ induced by the inclusion $\mathcal{M}_{\Delta_-} \subset \mathcal{M}_I$, makes the following square commute

$$\begin{array}{ccc} H_{ab} & \longrightarrow & H^1(F, \mathcal{M}_{\Delta_-,sc}) \\ \parallel & & \downarrow \\ H_{ab} & \longrightarrow & H^1(F, \mathcal{M}_{I,sc}) \end{array}$$

Hence, we have a factorization $\text{Res}_I^{H_{ab}} = \text{Res}_{\Delta_-}^{H_{ab}} \circ \text{Res}_I^{\Delta_-}$ where $\text{Res}_I^{\Delta_-}$ denotes the restriction functor with respect to the morphism $H^1(F, \mathcal{M}_{\Delta_-,sc}) \rightarrow H^1(F, \mathcal{M}_{I,sc})$. Combining this with 5.4.3, we get the identity

$$(5.4.4) \quad \sum_{I \subset \Delta_-} (-1)^{|\Delta_- - I|} |\ker^1(F; \mathcal{M}_I, H)| \text{Ind}_{\mathcal{M}_{I,ab}}^{H_{ab}}(\mathbf{1}) = \text{Res}_{\Delta_-}^{H_{ab}} \left(\sum_{I \subset \Delta_-} (-1)^{|\Delta_- - I|} \text{Res}_I^{\Delta_-} \text{Ind}_1^I(\mathbf{1}) \right)$$

To continue, we need to compute the groups $H^1(F, \mathcal{M}_{I,sc})$ and the morphisms

$$H^1(F, \mathcal{M}_{\Delta_-,sc}) \rightarrow H^1(F, \mathcal{M}_{I,sc})$$

explicitly for all $I \subset \Delta_-$. Let $T_{0,sc}$ and $P_{0,sc}$ denote the inverse image of T_0 and P_0 in $G_{sc} := R_{E/F}H_{sc,E}$. Since $T_{0,sc}$ is θ -stable, there exists a maximal torus $\mathcal{T}_{0,sc}$ of H such that $T_{0,sc} = R_{E/F}\mathcal{T}_{0,sc,E}$. Moreover, there exists a Borel subgroup $\mathcal{P}_{0,sc}$ of $H_{sc,E}$ such that $P_{0,sc} = R_{E/F}\mathcal{P}_{0,sc}$. In what follows, we fix an algebraic closure \bar{F} of F containing E and we set $\Gamma_E := \text{Gal}(\bar{F}/E)$. Let $\Delta_{min,\bar{F}}$ be the set of simple roots of $\mathcal{T}_{0,sc,\bar{F}}$ in

$\mathcal{P}_{0,sc,\overline{F}}$. It is a subset of $X_{\overline{F}}^*(\mathcal{T}_{0,sc})$ which is Γ_E -stable (as $\mathcal{P}_{0,sc}$ is defined over E) and we have a natural surjection $\Delta_{min,\overline{F}} \twoheadrightarrow \Delta_{min}$ (obtained by restriction to the maximal split subtorus of $\mathcal{T}_{0,sc,E}$) whose fibers are precisely the Γ_E -orbits in $\Delta_{min,\overline{F}}$. For all $\beta \in \Delta_{min,\overline{F}}$, we will denote by $\varpi_\beta \in X_{\overline{F}}^*(\mathcal{T}_{0,sc})$ the corresponding weight and for all $\alpha \in \Delta_{min}$ we define

$$\varpi_\alpha := \sum_{\beta \in \Delta_{min,\overline{F}}; \beta \mapsto \alpha} \varpi_\beta$$

where the sum is over the set of simple roots $\beta \in \Delta_{min,\overline{F}}$ mapping to α through the projection $\Delta_{min,\overline{F}} \twoheadrightarrow \Delta_{min}$. We always have $\varpi_\alpha \in X_E^*(\mathcal{T}_{0,sc})$ but we warn the reader that in general ϖ_α is NOT the weight associated to the simple root α in the usual sense (although it is proportional to it). Let $I \subset \Delta_-$. For all $\alpha \in I$ the character ϖ_α extends to $\mathcal{M}_{I,sc,\overline{F}}$ and is defined over E and thus gives rise to a character $R_{E/F}\mathcal{M}_{I,sc,E} \rightarrow R_{E/F}\mathbf{G}_{m,E}$. Since $\theta(\varpi_\alpha) = -\varpi_\alpha$, this last character induces a morphism $\mathcal{M}_{I,sc} \rightarrow \text{Ker } N_{E/F}$ that we will also denote by ϖ_α . Consider the torus $T_I := (\text{Ker } N_{E/F})^I$ and the morphism

$$\kappa_I := (\varpi_\alpha)_{\alpha \in I} : \mathcal{M}_{I,sc} \rightarrow T_I$$

Then we claim that

(5.4.5) The induced map $H^1(\kappa_I) : H^1(F, \mathcal{M}_{I,sc}) \rightarrow H^1(F, T_I)$ is an isomorphism.

Let $\mathcal{M}_{I,sc,der}$ be the derived subgroup of $\mathcal{M}_{I,sc}$ and set

$$T'_I := \mathcal{M}_{I,sc} / \mathcal{M}_{I,sc,der}$$

Then, we have an exact sequence

$$H^1(F, \mathcal{M}_{I,sc,der}) \rightarrow H^1(F, \mathcal{M}_{I,sc}) \rightarrow H^1(F, T'_I)$$

and $H^1(F, \mathcal{M}_{I,sc,der})$ is trivial by [32] since $\mathcal{M}_{I,sc,der}$ is simply connected. Moreover the morphism $H^1(F, \mathcal{M}_{I,sc}) \rightarrow H^1(F, T'_I)$ is surjective. Indeed, if $T \subset \mathcal{M}_{I,sc}$ is a maximal elliptic torus (which exists by [33] p.271) then the kernel of the projection $T \rightarrow T'_I$ is a maximal anisotropic torus of $\mathcal{M}_{I,sc,der}$ and thus by Tate-Nakayama duality its H^2 vanishes and the morphism $H^1(F, T) \rightarrow H^1(F, T'_I)$, which obviously factorizes through $H^1(F, \mathcal{M}_{I,sc})$, is surjective. Therefore, the morphism $H^1(F, \mathcal{M}_{I,sc}) \rightarrow H^1(F, T'_I)$ is an isomorphism and it only remains to show that the natural map $H^1(F, T'_I) \rightarrow H^1(F, T_I)$ is also an isomorphism. Let T''_I be the kernel of the projection $T'_I \rightarrow T_I$. Then T''_I is connected since it is a quotient of the common kernel of all the ϖ_α 's, $\alpha \in I$, which is

a connected group (this follows from the fact that $\{\varpi_\alpha; \alpha \in I\}$ generates $X_E^*(\mathcal{M}_{I,sc})$). Therefore, T_I'' is an anisotropic torus (since T_I' is) and by Tate-Nakayama duality again we just need to prove the injectivity of $H^1(F, T_I'') \rightarrow H^1(F, T_I')$ or, equivalently, that the map $H^1(F, T_I'') \rightarrow H^1(F, T_I')$ has trivial image. We have norm maps $N : R_{E/F}T_{I,E}' \rightarrow T_I'$ and $N : R_{E/F}T_{I,E}'' \rightarrow T_I''$ giving rise to a commuting square

$$\begin{array}{ccc} H^1(F, T_I'') & \longrightarrow & H^1(F, T_I') \\ N \uparrow & & N \uparrow \\ H^1(F, R_{E/F}T_{I,E}'') & \longrightarrow & H^1(F, R_{E/F}T_{I,E}') \end{array}$$

Since $T_{I,E} \simeq \mathbf{G}_{m,E}^I$ is the maximal split torus quotient of $\mathcal{M}_{I,sc,E}$, the torus $R_{E/F}T_{I,E}''$ is anisotropic and therefore so is the kernel of the norm map $R_{E/F}T_{I,E}'' \rightarrow T_I''$ (which is automatically connected). Hence, by Tate-Nakayama again, the map $H^1(F, R_{E/F}T_{I,E}'') \rightarrow H^1(F, T_I'')$ is surjective and it follows, by the above commuting square, that to conclude we only need to show that $H^1(F, R_{E/F}T_{I,E}') is trivial. By an argument similar to what we have done before, the map $H^1(F, M_{I,sc}) \rightarrow H^1(F, R_{E/F}T_{I,E}')$, where $M_{I,sc} := R_{E/F}\mathcal{M}_{I,sc,E}$, is surjective. Since $M_{I,sc}$ is a Levi subgroup of G_{sc} , the map $H^1(F, M_{I,sc}) \rightarrow H^1(F, G_{sc})$ has trivial kernel and by [32] it follows that $H^1(F, M_{I,sc}) = 1$. Hence, $H^1(F, R_{E/F}T_{I,E}') = 1$ also and this ends the proof of 5.4.5.$

By 5.4.5, we have isomorphisms

$$(5.4.6) \quad H^1(F, \mathcal{M}_{I,sc}) \simeq H^1(F, T_I) \simeq (\mathbf{Z}/2\mathbf{Z})^I$$

for all $I \subset \Delta_-$ such that the maps $H^1(F, \mathcal{M}_{\Delta_-,sc}) \rightarrow H^1(F, \mathcal{M}_{I,sc})$ correspond to the natural projections

$$\begin{aligned} (\mathbf{Z}/2\mathbf{Z})^{\Delta_-} &\rightarrow (\mathbf{Z}/2\mathbf{Z})^I \\ (e_\alpha)_{\alpha \in \Delta_-} &\mapsto (e_\alpha)_{\alpha \in I} \end{aligned}$$

We can now compute the right hand side of 5.4.4. For all $I \subset \Delta_-$, let A_I denote the kernel of the projection $H^1(F, \mathcal{M}_{\Delta_-,sc}) \rightarrow H^1(F, \mathcal{M}_{I,sc})$. In particular we have $A_\emptyset = H^1(F, \mathcal{M}_{\Delta_-,sc}) \simeq (\mathbf{Z}/2\mathbf{Z})^{\Delta_-}$ and A_I is the subgroup $(\mathbf{Z}/2\mathbf{Z})^{\Delta_- \setminus I}$. For all $I \subset \Delta_-$ we have

$$\text{Res}_I^{\Delta_-} \text{Ind}_1^I(\mathbf{1}) = \sum_{\chi \in \text{Irr}(A_\emptyset); \chi|_{A_I} = 1} \chi$$

where $\text{Irr}(A_\emptyset)$ denotes the set of characters of A_\emptyset . Thus the sum

$$\sum_{I \subset \Delta_-} (-1)^{|\Delta_- - I|} \text{Res}_I^{\Delta_-} \text{Ind}_1^I(\mathbf{1})$$

is equal to

$$\sum_{I \subset \Delta_-} (-1)^{|\Delta_- - I|} \sum_{\chi \in \text{Irr}(A_\emptyset); \chi|_{A_I} = 1} \chi = \sum_{\chi \in \text{Irr}(A_\emptyset)} \left(\sum_{I \subset \Delta_-; \chi|_{A_I} = 1} (-1)^{|\Delta_- - I|} \right) \chi$$

Let $\chi \in \text{Irr}(A_\emptyset)$. By the above description of the subgroups $A_I \subset A_\emptyset$, we see that $A_I + A_J = A_{I \cap J}$ for all $I, J \subset \Delta_-$. Hence there exists a smallest subset $I_\chi \subset \Delta_-$ such that $\chi|_{A_{I_\chi}} = 1$ and we have

$$\sum_{I \subset \Delta_-; \chi|_{A_I} = 1} (-1)^{|\Delta_- - I|} = \sum_{I_\chi \subset I \subset \Delta_-} (-1)^{|\Delta_- - I|}$$

This sum is zero unless $I_\chi = \Delta_-$ in which case it is equal to 1. Using again the explicit determination of the subgroups $A_I \subset A_\emptyset$, it is easy to see that there is only one character $\chi \in \text{Irr}(A_\emptyset)$ with $I_\chi = \Delta_-$, namely the character ω defined (via the isomorphism 5.4.6) by

$$\begin{aligned} (\mathbf{Z}/2\mathbf{Z})^{\Delta_-} &\rightarrow \{\pm 1\} \\ (e_\alpha)_{\alpha \in \Delta_-} &\mapsto (-1)^{\sum_{\alpha \in \Delta_-} e_\alpha} \end{aligned}$$

Therefore by 5.4.4, we get

$$\sum_{I \subset \Delta_-} (-1)^{|\Delta_- - I|} |\ker^1(F; \mathcal{M}_I, H)| \text{Ind}_{\mathcal{M}_{I,ab}}^{H_{ab}}(\mathbf{1}) = \text{Res}_{\Delta_-}^{H_{ab}}(\omega)$$

and it only remains to show that $\text{Res}_{\Delta_-}^{H_{ab}}(\omega) = \omega_{H,E}$. Set

$$\rho := \sum_{\alpha \in \Delta_{min}} \varpi_\alpha = \sum_{\beta \in \Delta_{min}, \bar{F}} \varpi_\beta, \quad \rho_1 := \sum_{\alpha \in \Delta_-} \varpi_\alpha \quad \text{and} \quad \rho_2 := \sum_{\alpha \in \Delta_{min} \setminus \Delta_-} \varpi_\alpha$$

Then, by restriction ρ , ρ_1 and ρ_2 define three morphisms $\pi_1(H) \rightarrow \text{Ker } N_{E/F}$. By definition, $\omega_{H,E}$ and $\text{Res}_{\Delta_-}^{H_{ab}}(\omega)$ are the morphisms

$$H_{ab} = H^1(F, \pi_1(H)) \rightarrow H^1(F, \text{Ker } N_{E/F}) \simeq \{\pm 1\}$$

induced by ρ and ρ_1 respectively. Thus it suffices to show that the morphism

$$H^1(\rho_2) : H^1(F, \pi_1(H)) \rightarrow H^1(F, \text{Ker } N_{E/F})$$

induced by ρ_2 is trivial. By definition of Δ_- we can find a subset $S \subset \Delta_{min} \setminus \Delta_-$ such that $\Delta_{min} \setminus \Delta_- = S \sqcup -\theta(S)$ (disjoint union). For all $\alpha \in S$, the character ϖ_α induces a morphism $\pi_1(H) \rightarrow R_{E/F} \mathbf{G}_{m,E} \simeq R_{E/F}(\text{Ker } N_{E/F})_E$. By the decomposition $\Delta_{min} \setminus \Delta_- = S \sqcup -\theta(S)$, we see that $H^1(\rho_2)$ is the composition of

$$\sum_{\alpha \in S} H^1(\varpi_\alpha) : H^1(F, \pi_1(H)) \rightarrow H^1(F, R_{E/F}(\text{Ker } N_{E/F})_E)$$

with the norm map $H^1(F, R_{E/F}(\text{Ker } N_{E/F})_E) \rightarrow H^1(F, \text{Ker } N_{E/F})$. By Hilbert 90, we have

$$H^1(F, R_{E/F}(\text{Ker } N_{E/F})_E) = 1$$

and thus $H^1(\rho_2) = 0$. This ends the proof of the proposition. ■

5.5 Reminder on the Steinberg representation

Fix a minimal parabolic subgroup P_0 of G with Levi decomposition $P_0 = M_0 U_0$. Then, the Steinberg representation of $G(F)$ is by definition the following virtual representation

$$\text{St}(G) := \sum_{P_0 \subset P = MU} (-1)^{a_M - a_{M_0}} i_P^G(\delta_P^{1/2})$$

where i_P^G denotes the functor of normalized parabolic induction. It follows from [13] that $\text{St}(G)$ is in fact a true representation of $G(F)$ which is moreover irreducible and square-integrable. Obviously, the Steinberg representation has trivial central character. Moreover, if G_{ad} denotes the adjoint group of G then $\text{St}(G)$ is the pullback of $\text{St}(G_{ad})$ by the projection $G(F) \rightarrow G_{ad}(F)$. For all $x \in G_{\text{reg}}(F)$ and all $P = MU \supset P_0$, let us set

$$\Gamma_M(x) := \{\gamma \in G(F); \gamma^{-1}x\gamma \in M(F)\}/M(F)$$

Then, the character $\Theta_{\text{St}(G)}$ of $\text{St}(G)$ is given by the following formula ([24] Theorem 30)

$$(5.5.1) \quad D^G(x)^{1/2} \Theta_{\text{St}(G)}(x) = \sum_{P_0 \subset P = MU} (-1)^{a_M - a_{M_0}} \sum_{\gamma \in \Gamma_M(x)} D^M(\gamma^{-1}x\gamma)^{1/2} \delta_P(\gamma^{-1}x\gamma)^{1/2}$$

for all $x \in G_{\text{reg}}(F)$. In particular, we have

$$(5.5.2) \quad \Theta_{\text{St}(G)}(x) = (-1)^{a_G - a_{M_0}}$$

for all $x \in G(F)_{\text{ell}}$

The representation $\text{St}(G)$ is stable: this follows from the fact that parabolic induction sends stable distributions to stable distributions. Let H' be another connected reductive group over F and $\psi^H : H_{\overline{F}} \simeq H'_{\overline{F}}$ be an inner twisting. Following notations of §5.3 we define $G' := R_{E/F}H'_E$ and the inner twisting $\psi^G : G_{\overline{F}} \simeq G'_{\overline{F}}$. Let $P'_0 = M'_0N'_0$ be a minimal parabolic subgroup of G' . Since transfer is compatible with parabolic induction we have that $(-1)^{a_{M_0}} \text{St}(G)$ and $(-1)^{a_{M'_0}} \text{St}(G')$ are transfer of each other. Moreover, in our situation we have $(-1)^{a_{M_0} - a_{M'_0}} = 1$. Indeed, by the main result of [34] we have $(-1)^{a_{M_0} - a_{M'_0}} = e(G)e(G')$ where $e(G)$ and $e(G')$ are the so-called *Kottwitz signs* of G and G' respectively and it follows from points (4) and (5) of the Corollary of *loc.cit.* that $e(G) = e(H)^2 = 1$ and $e(G') = e(H')^2 = 1$. Thus, we have that

(5.5.3) $\text{St}(G)$ and $\text{St}(G')$ are transfer of each other.

5.6 Harish-Chandra's orthogonality relations for discrete series

Let π and σ be essentially square-integrable representations of $H(F)$ with central characters coinciding on $A_H(F)$. Then, we have the following orthogonality relation between the characters of π and σ^\vee (the smooth contragredient of σ) which is due to Harish-Chandra (see [14] Theorem 3):

$$\int_{\Gamma_{\text{ell}}(\overline{H})} D^H(x) \Theta_\pi(x) \Theta_{\sigma^\vee}(x) dx = \begin{cases} 1 & \text{if } \pi \simeq \sigma \\ 0 & \text{otherwise} \end{cases}$$

where the measure on $\Gamma_{\text{ell}}(\overline{H})$ is the one introduced in §5.1. These relations can be seen as an analog of Theorem 5.1.1 in the case where $E = F \times F$. Let χ be a continuous character of $H(F)$. In the particular case where $\pi = \text{St}(H)$ and $\sigma = \text{St}(H) \otimes \chi$, by §5.5.2 we get the relation

$$(5.6.1) \quad \int_{\Gamma_{\text{ell}}(\overline{H})} D^H(x) \chi(x) dx = \begin{cases} 1 & \text{if } \chi = \mathbf{1} \\ 0 & \text{otherwise} \end{cases}$$

Indeed if χ is nontrivial then $\text{St}(H) \otimes \chi \not\simeq \text{St}(H)$ since $\text{St}(H)$ and $\text{St}(H) \otimes \chi$ have different cuspidal supports.

5.7 Second application: multiplicity of the Steinberg representation

Theorem 5.7.1 *For every Galoisian character χ of $H(F)$ we have*

$$m(\text{St}(G), \chi) = \begin{cases} 1 & \text{if } \chi = \omega_{H,E} \\ 0 & \text{otherwise} \end{cases}$$

Proof: Let χ be a Galoisian character of $H(F)$. If the restriction of χ to the center of $H(F)$ is nontrivial then obviously $m(\text{St}(G), \chi) = 0$ since $\text{St}(G)$ has trivial central character. We

assume now that χ restricted to the center of $H(F)$ is trivial. Let H_{ad} be the adjoint group of H and set $G_{ad} := R_{E/F}H_{ad,E}$. Let $H(F)_{ad}$ denote the image of $H(F)$ by the projection $H(F) \rightarrow H_{ad}(F)$ (i.e. the quotient $H(F)/Z_H(F)$). Then, since $\text{St}(G)$ is the pullback of $\text{St}(G_{ad})$ to $G(F)$, by Frobenius reciprocity we have

$$m(\text{St}(G), \chi) = \dim \text{Hom}_{H(F)_{ad}}(\text{St}(G), \chi) = \dim \text{Hom}_{H_{ad}(F)}(\text{St}(G_{ad}), \text{Ind}_{H(F)_{ad}}^{H_{ad}(F)} \chi)$$

The representation $\text{Ind}_{H(F)_{ad}}^{H_{ad}(F)} \chi$ is a multiplicity-free sum of Galoisian characters containing $\omega_{H_{ad,E}}$ if and only if $\chi = \omega_{H,E}$. This shows that the statement of the theorem for H_{ad} implies the statement of the theorem for H . Thus, we may assume that H is adjoint. Moreover, by §5.5.3 and Theorem 5.3.1, up to replacing H by its quasi-split inner form, we may also assume that H is quasi-split. We will now use freely the notations introduced in §5.4. By Theorem 5.1.1 and §5.5.1, we have

$$m(\text{St}(G), \chi) = \sum_{T \in \mathcal{T}_{\text{ell}}(H)} |W(H, T)|^{-1} \sum_{P_0 \subseteq P = MU} (-1)^{a_M - a_{M_0}} \sum_{\gamma \in \Gamma_M(T)} \int_{T(F)} D^M(\gamma^{-1}t\gamma)^{1/2} \chi(t)^{-1} dt$$

(Note that we have $\delta_P(\gamma^{-1}t\gamma) = 1$ for all γ and t as in the expression above since $\gamma^{-1}t\gamma$ is a compact element). By Proposition 5.4.1(ii), it follows that

$$m(\text{St}(G), \chi) = \sum_{(\mathcal{M}, P) \in \mathcal{C}} (-1)^{a_{\mathcal{M}} - a_{M_0}} \int_{\Gamma_{\text{ell}}(\mathcal{M})} D^{\mathcal{M}}(x) \chi(x)^{-1} dx$$

where for all $(\mathcal{M}, P) \in \mathcal{C}$ we have set $a_{\mathcal{M}} := a_M$ with $M := R_{E/F}\mathcal{M}_E$. Thus, by 5.6.1 we also have

$$m(\text{St}(G), \chi) = \sum_{(\mathcal{M}, P) \in \mathcal{C}} (-1)^{a_{\mathcal{M}} - a_{M_0}} (\chi|_{\mathcal{M}}, \mathbf{1})$$

where $\chi|_{\mathcal{M}}$ denotes the restriction of χ to $\mathcal{M}(F)$ and (\cdot, \cdot) denotes the natural scalar product on the space of virtual characters of \mathcal{M}_{ab} . By Proposition 5.4.1(iii), (iv) and (v), this can be rewritten as

$$m(\text{St}(G), \chi) = \sum_{I \subset \Delta_-} (-1)^{a_{\mathcal{M}_I} - a_{M_0}} |\ker^1(F; \mathcal{M}_I, H)| (\chi|_{\mathcal{M}_I}, \mathbf{1})$$

By Frobenius reciprocity, it follows that

$$m(\text{St}(G), \chi) = \sum_{I \subset \Delta_-} (-1)^{a_{\mathcal{M}_I} - a_{M_0}} |\ker^1(F; \mathcal{M}_I, H)| (\chi, \text{Ind}_{\mathcal{M}_I, ab}^{H_{ab}} \mathbf{1})$$

It is easy to see that $(-1)^{a_{\mathcal{M}_I} - a_{M_0}} = (-1)^{|\Delta_- - I|}$ for all $I \subset \Delta_-$ and therefore by Proposition 5.4.1(vi) the last expression above is equal to 1 if $\chi = \omega_{H,E}$ and 0 otherwise. ■

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