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► **To cite this version:**

Thomas Lecorre. Q^2 game used in a task design of the double quantification. CERME 10, Feb 2017, Dublin, Ireland. hal-01941334

HAL Id: hal-01941334

<https://hal.science/hal-01941334>

Submitted on 30 Nov 2018

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Q² game used in a task design of the double quantification

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This article deals with student's interpretation of multiply-quantified statements. The efficient interpretation of such statements is useful, for example, for those who try to understand and use the formalism of the definition of limit of a function. Literature points out some students' difficulties in the interpretation of double quantified statements. I am using the scientific debate methodology to design and implement tasks by means of two fundamental steps. In the first step, I draw on the results of a questionnaire to identify students' difficulties related to the order of quantifiers in the interpretation of such statements. The aim of the second step is to design the targeted tasks by considering the results of the first step; these tasks are based on a game called Q². The implementation of the design shows that students understand that there are at least two kinds of interpretations, and that conventions of interpretations are needed.

Keywords: formalism, quantification, limit, scientific debate in class, game Q².

Quantified statements problem

Mathematics formalism uses massively quantification and specifically multiply-quantification. Research shows that students have difficulties with the interpretation of multiply-quantified statements (Dubinsky & Yiparaki, 2000; Chellougui, 2009; Piatek-Jimenez, 2010). EA statements corresponding to “Exists...for all...” sentences must be distinguished from AE ones corresponding to “For all...there exists...”. Dubinsky and Yiparaki (2000) study the impact of two main variables on the interpretation of double quantified statements. The first variable is the order of quantifiers (AE or EA) and the second one is the kind of statement, mathematical or non-mathematical. They show that the interpretation of non-mathematical statements is essentially correct but the interpretation of mathematical ones is difficult for students. For AE mathematical statements, the students' interpretation is very efficient, whereas their interpretation of EA ones seems to be done through an inversion: the EA statement is very often interpreted as the AE corresponding statement (the variables remain with their own quantifiers but the order of quantifiers are changed). Moreover, it is noticed that “the students did not appear to care of the syntax of a statement to analyze it [...] the student did not appear to be aware of having engaged in interpreting the questionnaire statements.” (Dubinsky & Yiparaki, 2000, p. 53). The inversion of interpretation of an EA statement is also noticed by Chellougui (2009) when she asks students to define the upper bound M of a set A and when almost all of them answer by what she calls “a strange definition”: « $\forall x \in A, \exists M \in \mathbb{R}, x \leq M$ ». Piatek-Jimenez (2010) confirms the asymmetric perception and interpretation of those two kinds of double quantification in the mathematical field. There seem to be two different problems: the use of “strange” conventions and the unawareness to interpret statements.

To overcome these difficulties Dubinsky and Yiparaki (2000) presented a game based on the dialogical logic (explained below) and used it with students to make them aware that two kinds of interpretation can be used and that this depends on rules of interpretation linked to the places of quantifiers. Results have shown that the game seems to help sometimes students to understand such statements but in some cases, it does not, and in a few cases this has created more confusion.

I want to pursue this research by designing tasks to overcome the difficulties of the interpretation of double quantified statements using the methodology of scientific debate. This implies two related goals. The first one deals with the identification of students' difficulties related to the order of quantifiers in a double quantified statement. I specifically explore the role of some other potential variables: the set of quantification (familiar or not, finite or infinite), the semiotic representation of the quantified variables (formal or usual language) and the kind of relation involved in the predicate (familiar or not). The second goal is to design and to implement tasks by considering the results of the study related to the first goal. A new game called Q^2 based on the interpretation of double quantified statements to determine a winning strategy will play a fundamental role in this design.

The aim of this paper is to study the following question: is there a domain of values of variables where students take into account the order of quantifiers and if yes how to take advantage of this domain to enlarge it? Specifically, how to make students aware that they are interpreting these statements and that they need rules for that?

The methodology of scientific debate: A tool for designing tasks

The scientific debate (Legrand 2001) is a socio-constructivist approach to learning and teaching mathematics based on two main principles: 1) the need of new knowledge can be obtained by making one realize that his/her previous conceptions may lead to contradiction; 2) the organization of appropriate debates among students permits firstly to express and share previous conceptions about the targeted subject, secondly to encounter the limits of these conceptions, and thirdly to be able to understand the institutionalisation related to this knowledge that is made by the teacher. Three fundamental steps shape the design of tasks by means of the scientific debate methodology:

1) The first step consists on epistemological and cognitive studies of the targeted mathematical concept: the role of quantifiers in formal statements and the interpretation of double quantified statements that stands on the use of a convention of interpretation. For this interpretation, I have chosen the dialogical logic (Lorezen, 1967). In this paper, we mainly explore the cognitive aspects and we use a questionnaire for that purpose.

2) The second step concerns the design of the tasks, which is done by using mainly two kinds of questions to initiate the debate. The first kind of question concerns the truth of a conjecture: is it true or false? This conjecture can be given by the teacher or can come from the students after a call for conjecture. The second kind of question concerns the nature of an object for a conjecture or a property: is it an example, a counter example, or off topic (neither example nor counter-example) for this conjecture, and is it an example for a property? The example can come from the teacher or from a call for an example. A vote is made: do you think it is true, false or *something else* (an example, a counter example, off topic or *something else*)? The possibility of voting for *something else* is given to preserve the authenticity of the other votes (voting True or False must be a choice founded on convictions). Then a debate is organized by the teacher among those who have different view-points. The teacher never gives any opinion about what is debated but tries to maintain a level of interaction by emphasizing the contradictions between students.

3) The third step concerns the level of experimentation and its analysis: what actually happens is confronted to what was expected to happen. Specifically, this analysis leads to discuss the efficiency of the choices made in the two first steps.

The whole study is conducted according to the aforementioned three steps. In this paper, I show my findings from the study related to the first step and I give more details about the elaboration and the results of the two final steps. For the first step, I have chosen to use a questionnaire on a sample of 181 students in their last year of secondary school to identify students' difficulties with the interpretation of double quantified statements in formal and non-formal context. For the second step, the results of this questionnaire are used in a way that is aiming to make students aware that the lack of the rules of interpretation is not a problem in a certain domain but leads them to conflicts out of this domain. This design is based on a game, called Q^2 , in which the interpretation of double quantified statements is crucial: the question that will initiate the debate concerns the way to win this game. For the third step, I experiment this design for students in their last year of secondary school in a scientific class composed of 29 students (experimentations at university are planned).

Task design

Background: The result of the questionnaire

I have mainly studied five variables in the interpretation of double quantified statements: place of quantifiers, kind of statements (mathematical or not), the set of quantification (familiar or not, finite or infinite), the semiotic representation of the variables (formal or informal language) and the kind of relation involved (familiar or not). In the questionnaire, words (for all, exists) are used instead of symbols because symbols are introduced after the game Q^2 as a solution to the conflicts that appear about quantifiers and associated conventions. A questionnaire has been given to 182 students. We will only give the findings here (for more details see Lecorre, 2016a). The first finding is that the rule of “correct interpretation for AE statements and inversed one for EA statements” seems not so obvious: some EA statements are perfectly interpreted whereas some AE statements are interpreted through an inversion of the quantifiers and the variables. The second finding is that some other variables are correlated with difficulties in the interpretation (e.g. width of the quantified set, formalization of variables...). The third finding is that there exists a domain of correct interpretation for both EA and AE. This domain is made of non-mathematical statements quantified on a “small” finite set (less than ten values) without any formal variable and with a familiar relation. These findings are then used to design the tasks aiming at discussing the interpretation of double quantified statements.

The game Q^2

The game presented by Dubinsky and al. (2000) is based on the dialogical logic of Lorenzen (1967) which gives a way to decide on the truth of quantified statements using a codified dialog between a proponent and an opponent. In this game, for example, if the sentence is “for All x there Exists y such that $R(x;y)$ ”, the A-player chooses x and the E-player has to find a y such that $R(x;y)$ is verified. If he/she fails to find such y , the A-player wins, otherwise the A-player can give another x (same kind of rule for EA). I call this game a direct game: given a statement and sets of quantification, you have to decide the truth. The Q^2 game is an inverted game: given a statement and the truth, the players construct the set of quantification to make a statement true or false.

For the Q^2 game I choose values of the variables that make it an easy game to play: non-mathematical field, small set of quantification, familiar relation. This choice is made to permit students to get into the game and into the interpretation of associated statements.

This is a two players' game. This game is given by four elements: a starting rule, a winning rule, a statement and a grid. For example, red player has to start (starting rule) and plays for the falsification (winning rule) of the statement S: "For all red letters, there exists the same black letter" and the given grid:

H	U	H
L	U	L
U	H	T

Figure 1: A grid for the game Q^2

So red has, first, to circle one letter with his red pencil. Then the black player circles one letter and so on until all the letters have been circled. With the given elements of this example (starting and winning rule, statement, and grid) and the coloration of letters, if the statement S is false then the red player wins, but if it's true the black one wins. This game has, of course, many variants, beginning with the filling of the grid and the starting and winning rules. This game is the heart of a design which aims precisely to enlarge the domain of good interpretation. I am going to show that a smart use of Q^2 in the design has the potential to reach such a goal by emphasizing the lack of the convention of interpretation.

"The Q^2 situation"

The principle of the Q^2 situation is to provoke conflicts of interpretations that will lead students to the need of the convention of interpretation. The situation Q^2 is divided into four periods:

- The first period aims an appropriation of the game Q^2 by playing.
- The second period deals with the concept of winning grid for the game Q^2 .
- The third period targets a conflict of interpretation, in a way that students feel the need of conventions of interpretation. At that stage, the conventions are given.
- The fourth period is just an application of these conventions on the unsuccessful domain where the values of the variables lead students to difficulties of interpretation.

In the first period, a paper is given that contains eight games of Q^2 to play (each game is defined by a statement, a starting rule and a grid). These games are designed for the two players to have opportunities to win and to start to have ideas on how to play to have good chances of winning. In fact, with such a game, with "good players", the winner depends only on the statement (EA or AE), the starting rule, and the grid given.

The second period aims at the definition of winning grid. The students are asked to give the winning grids for a given rule, then a debate is organized about these propositions: are they winning grids, or not? The contradictory opinions about the propositions should lead students to identify the lack of a definition of winning grid. A winning grid is, in fact, quite difficult to define in a mathematical way for pre-university students (double recursive definition). Here, a definition such as "a grid is a winning grid for red if when red plays "cleverly", he is sure to win, even if black also plays cleverly" is largely sufficient for this design. When the students show a need for a definition, the above definition is given.

The third period aims to highlight the lack of convention of interpretation. Once again, for a given rule, the students are asked to give winning grids. There should be no more conflicts about what is a winning grid in general, but new conflicts should appear about the propositions: is this grid really a winning grid with this rule? This should happen because the winning grids depend on the interpretation of the rule which is a double quantified statement. And the need for a convention should appear with the impossibility to find a common agreement (is it a winning grid or not?). The didactical principle which is used here is the following: it seems very difficult to organize a direct confrontation of the different rules used by students to interpret a double quantified statement, because this problem, taken as a general one, is too theoretical and depends on too many variables (findings of the questionnaire). On the contrary, it is much easier to create a conflict on concrete consequences of the interpretation of such statements. Here, the conflict holds about the question "is the grid a winning one or not?" Then, trying to understand each other, and trying to convince their peers, students are going to explain their own interpretation. And then it will appear that the implicit conventions used by students are contradictory. Students can realize that without common conventions, no agreement is possible. Deciding if a grid is a winning grid or not is possibly more complicated than the logical principles of deciding who won, but it leads students to materialize their own conventions through these grids and permit to confront these conventions.

Then these conventions are given in the manner of the dialogic logic (Lorezen, 1967), which is described above with the game of Dubinsky and Yiparaki (2000). At this stage, the game Q^2 plays as a preparation to such rules by simulating a game between a proponent and an opponent. The design here aims much more the awareness of the need of convention than the "right rules". The aim is to make students aware of the necessity to check the validity of their interpretation relatively to the adopted conventions.

The fourth period consists in verifying, still using debates, that the given rules can lead to agreements, and even in the domain where students used to fail: the rules are helpful, efficient.

Results

The first period of the situation Q^2 (playing the game) shows a good appropriation of the game: the winning grids were almost always won by the one for who the grid was a winning one, which means that the students were playing "cleverly". Some strategies seemed to begin to be used. And, above all, almost all the decisions about who is the winner were correct. All this is coherent with the results from the questionnaire in terms of domain of interpretation.

The second period led to the question of the definition of a winning grid. The definition is given.

The third period begun with a question of the interpretation raised by students in a debate. I present an extract of the script of this debate to explain this unexpected acceleration. Student are asked to give winning grids for red for the game "There exists a red square such that all black squares have the same symbol" where red starts and plays for true. The given statement does not specify if black squares shall have the same symbol as the red one or not. So, after six winning grids had been given by the students in the second period, this lack of information was intensely discussed.

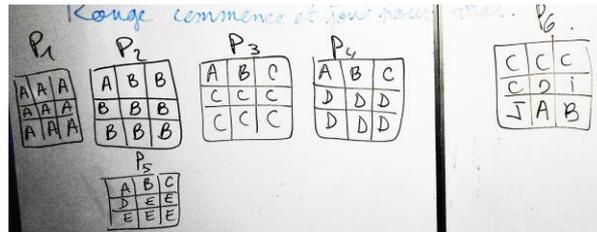


Figure 2: The winning grids for red given by the students

The grid P1 was put into debate and everyone agreed that it was winning grid for red. Then P2 was put into debate (28 votes for a winning grid for red and 1 for “something else”). Loïc who had voted “something else” changed his mind and explained why, for him, it was a winning grid for red:

Loïc: Because red starts and as he plays to win he takes the square A and...

Teacher: You’re saying that “red plays A first” yes and what?

Loïc: Then black takes only B squares.

Teacher: Black only takes B squares. Why, in the end, red wins?

Loïc: Because....

Hadrien: Because Black has only B squares.

But Quentin disagreed with this explanation:

Quentin: And because red has it also (One B square)

But Hadrien and Louis did not agree with this addition:

Hadrien: No, he has A squares and B squares.

Quentin: Yes!

Louis: I think that I should just say that there exists a red square.

The sequence above shows that these students do not need to disagree on the fact that it is a winning grid or not to begin to explain their own interpretation: “the same symbol as the red square” (Quentin), or “the same symbol for all black squares” (Loïc, Hadrien). Then Fabio, in the same way, explained that nothing must be added contrary to the sayings of Quentin:

Fabio: I do think that what Quentin added is not necessary.

Then Quentin proposed a grid to strengthen the differences of interpretations:



Figure 3: Quentin’s grid (As in black and Bs in red)

Leaving aside for a while the problem of the winning grid, the teacher asked whether the statement S1 was true or not, according to the grid of Quentin. Twelve students thought that the statement was true, ten false, while six students voted something else. Some conclusions were then raised :

Fabio: There are some, like me, that can think that all the black squares have got the same symbol is enough and some other, like Sébastien, who are thinking that there must be a red square that has got exactly the same symbol as any black square.

Mickaël: I'm asking: if you who think that only black squares have the same symbol is enough, this red square exists such that what?

This made Hadrien change his opinion, but Louis did not agree with this change:

Louis: There are two opposite opinions just because we're not thinking the same.

Teacher: Ok, you are not reading the same way... That is what Fabio said...

Louis: Exactly!

Juliette and Maxime then explained why they voted something else:

Maxime: That is exactly why, from the beginning, I voted Other.

Juliette: So do I.

These two interpretations are not directly linked to an EA/AE inversion but to the interpretation of the predicate. It would be interesting to investigate the role that the vernacular language (the linguistic subtleties that may eventually vary in French or English) may have played in leading students to interpret the two variables in the predicate as bound variables ("the same as red squares"), as shall be done in AE statements, where as it isn't the case in such an EA statement. In any case, the discussion among students led them to work on these interpretations and to get aware of the complexity of having a unique interpretation. The teacher then gave the conventions of interpretation of EA and AE logical statements. These conventions were then used successfully for the applications of the fourth period. One month later, the same students had to face double quantified statements in a situation aiming at the definition of limit. They experienced, once again, the need of conventions when they encountered another conflict of interpretation of these statements, so they checked the conventions to decide on their own (Lecorre, 2016b).

Later, another experimentation was realized with some other students using a "complete" rule ("...such that all the black squares have the same symbol as the red square") and this led students, with a grid of eight squares filled with symbols appearing twice, to the predicted conflict (EA/AE interpretation). A new didactic variable is then identified, the constitution of quantified sets (how they are filled relatively to the predicate), which raises an unexpected question: is the students' choice of convention between EA and AE convention for the interpretation more guided by the constitution of the sets of quantification than by the statement itself?

Conclusion

The Q^2 situation, with the scientific debate methodology, were used to make the interpretation of the double quantified statement the main object of students' discussion. The contradictions that the lack of conventions of interpretation was bound to imply then emerged in that discussion. This lack which was invisible to students suddenly came into light with these contradictions.

More precisely, the questionnaire led to identify some variables playing a role in the interpretation of students. There is a domain of the value of those variables which gives a good interpretation.

This Q^2 situation led students to be ready to receive the conventions of interpretation of double quantified statements. Indeed, they have experienced the need of shared conventions. The described situation Q^2 mainly aims the recognition that there are two kinds of double quantifications that should be differenced according to the order of the quantifiers in the statement and the convention of interpretation of the predicate but the validation using variables defined in function of the other variable remains a difficulty.

References

- Chellougui, F. (2009). L'utilisation des quantificateurs universel et existentiel en première année d'université, entre l'explicite et l'implicite. *Recherches en Didactique des Mathématiques*, 29(2), 123–154.
- Dubinsky, E., & Yiparaki, O. (2000). On student understanding of AE and EA Quantification. In E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), *Research in Collegiate Mathematics Education IV* (pp. 239–289). Providence, RI: American Mathematical Society.
- Lorenzen, P. (1967). *Métamathématiques*. Paris: Gauthier-Villars.
- Lecorre, T. (2016 a). *Des conditions de conception d'une ingénierie relative à la définition de la notion de limite. Elaboration d'un cadre basé sur un modèle de rationalité pour l'accès aux objets mathématiques complexes*. Doctoral thesis. Grenoble, France: University of Grenoble-Alpes,.
- Lecorre, T. (2016b). Rationality and concept of limit. In Nardi et al. (Eds.), *Proceedings of the first conference of the International Network for Didactic Research in University Mathematics (INDRUM 2016)* (pp. 63–72). Montpellier, France: University of Montpellier and INDRUM.
- Legrand, M. (2001). Scientific debate in mathematics courses. In D. Holton (Ed.), *The teaching and learning of mathematics at university level* (pp. 127–136). Dordrecht, the Netherlands: Kluwer.
- Piatek-Jimenez, K. (2010). Students' interpretations of mathematical statements involving quantification. *Mathematics Education Research Journal*, 22(3), 41–56.