A vector is a line segment between two points? - Students' concept definitions of a vector during the transition from school to university

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The vector concept is an important concept students are confronted with in their first year at university. To be able to build on students' previous knowledge it is important to find out what they have learned about vectors from school. This study aims at exploring university freshmen's personal concept definitions of a vector. We therefore analyzed common German school textbooks to find out how vectors are introduced and what conception of a vector students might have developed at school. In addition, we administered a short pretest in which students were asked what a vector is and to explain vector addition and its properties. We ascertained that freshmen stated a lot of individual concept definitions. The majority of students stated geometric ones, which were mostly not fully adequate, i.e. improperly formalized to be embedded into the theory of abstract vector spaces. Furthermore, various misconceptions were identified.

Keywords: vector, concept definition, transition, textbook analysis.

Introduction

Many students face problems connected with the transition from school mathematics to university mathematics. To reduce students' difficulties during this transition, the Ministry of Innovation, Science and Research initiated the Studifinder project. Part of this project are the studiVEMINT learning materials. We have developed these learning materials (which are an e-learning based bridging course for mathematics) at the University of Paderborn since 2014 (Colberg, Mai, Wilms, & Biehler, 2017). The development was completed in summer 2016, although quality assurance still takes place. Students can use the course for several purposes: to fill gaps in their mathematical school knowledge, or to get used to elaborated forms of school content. In any case a focus lies on accurate language and notions and a mathematical discourse based on the definitions introduced, as it is expected at university.

An important concept students are confronted with at school and university (at a more abstract level) is the concept of vectors. In early 2016 we started the development of a chapter on vectors for the studiVEMINT learning materials by looking into school textbooks to explore how the concept could be introduced. While looking into many of these textbooks we observed the following:

1. The formal definition – that is either geometric as an infinite set of arrows with the same direction and length or symbolic as a triple of real numbers – is often not referred to again in the chapters following the introduction of the vector concept.
2. During the mathematical discourse related to vectors, several models (in the sense of Dörfler (2000)) of the concept are used without arguing about isomorphism.
3. The symbols labeled as vectors and used are not always conceptually coherent with the axiomatic definition of a vector space.

These observations led us to question which of the many representations of a vector that were labelled as vectors at school the students actually consider to be a vector. This question was examined by analyzing the common German school textbooks dealing with the vector concept and by administering a short test to university freshmen at the beginning of their university studies. The results are presented in this paper.

Theoretical background

Although every formalized mathematical concept has a precise definition, students need to give it a meaning by operating with the concept (maybe just mentally) in order to understand it. Tall and Vinner (1981) use the term *concept image* to describe all associations students may have acquired by operating with it. These include examples, counterexamples, visualizations as well as properties of the concept. In order to specify the concept with words it has a *concept definition*. This can either be the formal definition accepted by the mathematical community, or student's reconstruction of a definition of the concept from their *concept image* (more precisely from the parts of the concept image that were activated during this reconstruction process, which Tall and Vinner (1981) call *evoked concept image*). In the latter case Tall and Vinner (1981) call it *personal concept definition.*

The formal definitions of the vector concept the students might have learned at school are the following ones, which were discovered when analyzing German school textbooks:

1. A vector is an infinite set of arrows with equal direction and length (Bigalke & Köhler, 2012; Bossek & Heinrich, 2007; Brandt & Reinelt, 2009; Weber & Zillmer, 2014).
2. A vector is a triple of real numbers or a matrix with one row (Alpers et al., 2003; Artmann & Törner, 1984; Griesel, Andreas, & Suhr, 2012; Griesel & Postel, 1990).

However, students' personal concept definitions, which they reconstruct from their concept images, may differ depending on individual experiences with the concept.

In the following we present for each of the two formal definitions of the vector concept, how they are introduced in German school textbooks and what *possible personal concept definitions* university freshmen might have, assuming these introductions formed their concept image at school from which they reconstructed their personal concept definitions. Then we discuss how the formal definitions are referred to further in the books when operating with vectors and how this might again influence the students' concept definitions reconstructed from their evoked concept image.

Analysis of books using the geometric definition of a vector

The geometric definition as an infinite set of arrows with same length and direction is often motivated by translations (Bigalke & Köhler, 2012; Weber & Zillmer, 2014), and sometimes even defined by these (Brandt & Reinelt, 2009). The translations are then represented by arrows with the same length and direction. Afterwards, students are told that all of these arrows describe the same translation and can therefore be identified as the same object (e.g., see (Weber & Zillmer, 2014)). This path would lead to the adequate concept definition of a *vector as an infinite set of arrows with equal length and direction (D1)*. However, this identification step is rather difficult as is denoted in the literature, and may result in the incomplete conception that a *vector is considered as a single arrow (D2)* (Malle, 2005). The motivation of the formal geometric definition as a set of arrows with equal length and direction may also lead students to think that a vector is a translation. While a definition of a vector

as a translation mapping operation on the whole plane is consistent with its formal concept, literature shows that translations are often understood as the motion of an object (Yanik, 2011). So the students might think of a *vector as a translation of an object or translation of a point (D3)*. Yanik (2011) also found out that the connection between a vector and a translation is often not understood, and that many teacher students thought that a vector only gives the direction of a translation. This might lead to the following misconception: *vector as direction indicator (D4)*.

Besides using translations, some books also motivate arrows in space as a quantity characterized by length and direction in physical contexts like speed or force (Bossek & Heinrich, 2007; Weber & Zillmer, 2014). The recognition that two of these arrows can be considered as the same, since only the magnitude and direction matter (e.g., for the resulting movement of an object) leads to the adequate concept definition of *a free vector, which is a quantity characterized by length and direction and represented by a free movable arrow (D5)* (Watson, Spyrou, & Tall, 2003). But since forces are normally considered as dependent also on the point of origin (Watson et al., 2003), this approach can again lead students to the consideration that *a vector is a single arrow (D2)*.

The geometric definition requires not only a lot of effort in its introduction, it is also difficult to handle afterwards. In literature, this is denoted as a lack of operability of the definition (Bills & Tall, 1998). For example the geometric definition is difficult to handle when defining vector operations because for all operations the independence from the chosen representative of the vector has to be justified. In some books, this problem is discussed (Weber & Zillmer, 2014), others ignore it, and vectors are simply identified with arrows when defining vector operations geometrically (Bossek & Heinrich, 2007). This can again lead to the conception *of a vector as a single arrow (D2)*. Another option to deal with these difficulties is highlighted in Bigalke and Köhler (2012): the addition of vectors is defined via the addition of the components in the symbolic representation as an *n*-tuple (directly after its introduction) and from then on the geometric addition only serves as a visualization. This does not result in a misconception but becomes problematic when trying to embed the geometric vectors with operations defined between triples into the formal theory of vector spaces because in the formal theory the operations have to be defined on the set, whose elements will be the vectors if the axioms are satisfied. After the introduction of the vector operations and their properties, the definition as a set of arrows (or as a translation) is not referred to again (Bigalke & Köhler, 2012; Brandt & Reinelt, 2009; Weber & Zillmer, 2014). Instead, in the following chapters on analytical geometry, single arrows and their corresponding number triples are used to describe geometric objects. This can lead to a loose connection between the formal definition and students' concept image from which they might deduce their own personal concept definition (Vinner, 2002). The resulting personal concept definitions in this case would be: *vector as a single arrow (D1) or vector as a number triple (D6).*

In summary, if the vector concept was introduced geometrically as an infinite sets of arrows with equal length and direction, the following concept definitions can be expected: *vector as an infinite set of arrows with the same length and direction (D1), vector as a single arrow (D2), vector as a translation of an object or translation of a point (D3), vector as direction indicator (D4), vector as a quantity characterized by length and direction (D5), or vector as a triple of numbers (D6).* The personal concept definitions *D1* and *D6* correspond directly to possible formal definitions of the vector concept, *D5* is also an adequate conception, in which the equivalence of arrows with equal length and direction is realized by independence from the space, *D2* and *D3* are incomplete concept

definitions (*D2* does not take into account that vectors are equivalence classes, *D3* does not take into account that a translation is a mapping on the whole plane) and *D4* is a misconception.

Analysis of books using the symbolic definition of a vector as *n*-tuples

The symbolic definition is often motivated geometrically by translations or arrows (Alpers et al., 2003; Griesel et al., 2012) or as coordinates of the points in the space (Griesel & Postel, 1990). Sometimes the symbolic definition is introduced earlier in connection with the theory of systems of linear equations (Artmann & Törner, 1984). The symbolic definition of a vector has the advantage of allowing a flexible interpretation as a point or an arrow. This can avoid the discussion about the equivalence of arrows (e.g., see Alpers et al. (2003)). However, besides the already mentioned incomplete conception of a vector as a single arrow, this flexibility can lead to another inadequate conception: *vector as a point (D7)*. The identification of vectors and points becomes problematic in higher mathematics, e.g., in the theory of affine spaces, in which they are considered different objects (Henn & Filler, 2015).

The way the vector concept is introduced in Artmann and Törner (1984) can also lead to another adequate concept definition. Artmann and Törner (1984) restrict their visualizations of vectors on points and arrows starting at the origin. If students identify the number triples with these arrows starting at the origin, they might consider *a vector as an arrow starting at the origin (D8).* These arrows starting at the origin can serve as elements of a vector space (with suitable operations defined between them).Unlike the geometric definition of a vector as a set of arrows, the symbolic definition is operable when defining vector operations and justifying their properties like the commutative law. However, some books do not mention these properties explicitly (Alpers et al., 2003; Artmann & Törner, 1984; Griesel et al., 2012). One reason, which is also noted in literature, might be their self-evidence (Harel, 2000). However, the symbolic definition can also be difficult to handle in the case of the definition of geometric concepts related to vectors such as the norm of a vector. Purely algebraic definitions of these concepts seem unnatural without further explanation (e.g., see Alpers et al. (2003)). Geometric definitions of these concepts on the other hand (e.g., see Griesel et al. (2012)), have the danger that the vector defined as a triple is again identified with just a single arrow, which is an at least incomplete vector conception.

After the introduction of vector operations, the concept of a vector is mainly used in geometrical settings (describing lines and planes in the space). This might cause students to not identify vectors with the originally defined ‘triple’ but with its geometrical representations such as *points (D4)* or *single arrows (D2)* (students might reconstruct their personal concept definitions of vectors from these representations and not from the formal symbolic definitions).

In summary, the symbolic approach can lead to two further personal concept definitions besides the intended definition of *a vector as a triple (D6)*, which have not been mentioned yet: *vector as a point (D7)* or *vector as an arrow starting at the origin (D8)*. The identification of symbolic vectors with arrows starting at the origin is not problematic because the latter ones can truly serve as objects, which the vector operations can be defined upon. The identification of vectors with points, however, can cause conflicts later in the theory of affine spaces, where these two objects have to be distinguished.

Methodology of the empirical study

Research question

On entering university, what personal concept definition of the vector concept do students have?

Data Collection

In September 2016 a short test was administered to 103 university freshmen in a mathematics bridging course at the University of Paderborn. These students were either freshmen majoring in mathematics or in mathematics for teachers at grammar schools. The pretest consisted of three open questions:

1. What is a vector?
2. Explain how you add two vectors $\vec{a}$ and $\vec{b}$.
3. Explain, why for all vectors $\vec{a}$ and $\vec{b}$ the following is valid: $\vec{a}+\vec{b}=\vec{b}+\vec{a}$.

The first question was asked to identify what the students' personal concept definition of a vector in the sense of Tall and Vinner (1981) is. We did not ask for a definition because we did not want the students to try to recall the formal definition they had learned at school, but rather to specify the concept in their own words. We also did not use the term "definition" because we suspected that many students might not be familiar with the term and therefore might get confused.

The other two questions were asked to further analyze if the students used the defined objects to explain vector operations and their properties. This is important for a proper embedding of the old vector concept into the abstract notion of a vector space, which is a set with operations defined on its elements. However, this problem will be investigated later.

Data Analysis

The answers to the first question "What is a vector?" were categorized by using possible personal concept definitions deduced from the analysis of the textbooks (see theoretical background, categories *D1,…, D8)*. Furthermore, four additional categories have been added. The first one, a *vector as an element of a vector space* was added before the analysis because, although this generalization is not taught at school, it may happen that some students had heard about it (e.g., in mathematical clubs at school). The other categories depict inadequate personal concept definitions that often showed up during the analysis: a *vector as a line segment, a vector as a line* and a category containing other inadequate concept definitions not yet mentioned.

The whole typology of 12 categories is shown in Table 1. The first five categories can be considered adequate, which means that objects described in the definition can serve as concrete examples of vectors in a vector space (if suitable operations are defined on them) or if vectors are already considered as elements of vector spaces. Categories 6 and 7 contain incomplete concept definitions, categories 8 to l2 contain inadequate concept image definitions, which can be considered as misconceptions.

Category	Prototypical Examples
1. Set of equivalent arrows	"An infinite set of arrows in the space, which are equal in their length and their direction."
2. Quantity with direction and magnitude	"A vector gives information about the length and the direction of an element in an at least two-dimensional space."
3. *N*-tuple	"A matrix with n rows and just one column."
4. Arrow starting at the origin	"A vector describes a journey to a point of $\mathbb{R}^3$."
5. Element of a vector space	"An element of a vector space."
6. Translation of a point or an object	"A vector is a translation of a point in the coordinate system to another point."
7. Arrow or line segment with direction	1. "Visually: an arrow." 2. "A vector is a line segment with direction. "
8. Direction	"A vector points from a point into a direction."
9. Connection between points or line segment	1. "A connection between two points in the 3-dimensional space." 2. "A line segment in the 3-dimensional space"
10. Point	"A vector describes a point in the coordinate system."
11. Line	"A vector describes a line in $\mathbb{R}^3$."
12. Other wrong definition	"A multidimensional object, which is located in a multidimensional space."

Figure 1: Answer categories to the question "What is a vector?"

Two of the authors separately coded the data from the questionnaire. The interrater-reliability coefficient, Cohen's Kappa, was κ=0.803, which is good. Afterwards, they discussed the answers they had coded differently and agreed on a categorization.

Results of the study

The students' personal concept definitions of the vector concept that were identified form the students' answers to the question "What is a vector?" are shown in Figure 2.

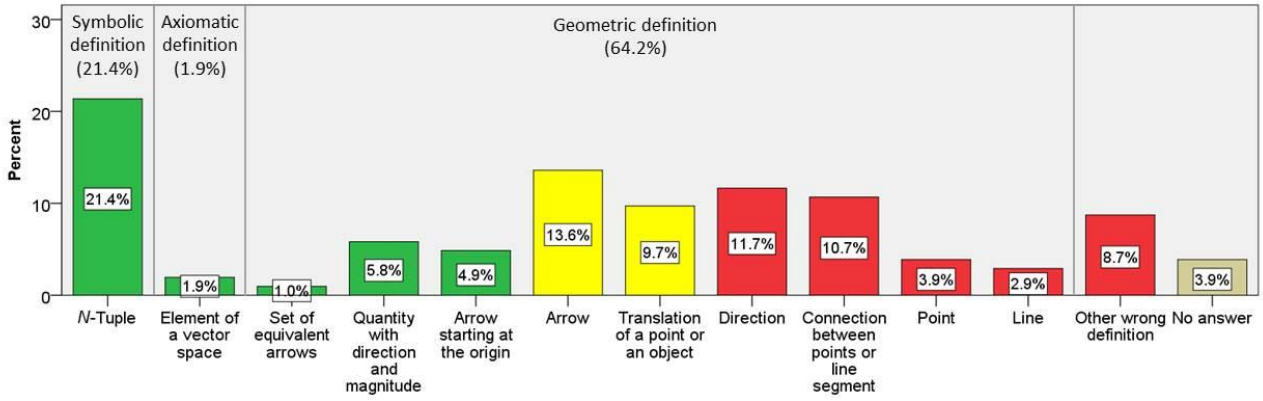

Figure 2: Students' answers to the question "What is a vector?" (N=103)

The bars of adequate personal concept definitions (which correspond roughly to possible formal concept definitions of models of the vector concept) are marked green, not fully adequate concept (i.e. they cannot be properly formalized or embedded into the abstract theory of vector space)

definitions are marked yellow, inadequate concept definitions are marked red. As can be seen in figure 2, the students had a variety of individual concept definitions of the vector concept when entering university. Most of them had a geometrical basis. However, in most cases these geometric concept definitions were either incomplete (the yellow bars, in which either the nature of a vector being an equivalence class was not mentioned or in which a vector was considered as a translation of points or objects and not as translations of the whole space) or inadequate (the red bars). Nevertheless, even the inadequate conceptions of a vector like “a direction”, “a connection between two points or a line segment” or “a point” have some properties of the adequate conceptions of vectors (e.g. if a vector is considered as a line segment, it has the property “finite length”, which is as basic property of the arrows, which represent a vector geometrically.

Conclusion and outlook on possible further research

Our study shows that the students have a variety of concept definitions of what a vector is when entering university. Thus we should keep in mind that freshmen do not come to university with a shared idea on what a vector is. The majority of students stated geometric definitions which were mostly inadequate definitions in the sense that they cannot be properly formalized or embedded into the common definition of a vector in mathematics. This indicates that it is difficult for students to fully grasp the concept of a vector. However, many students seemed to be familiar with the symbolic definition of a vector as an *n*-tuple and that it can be interpreted in manifold representations. This property of the *n*-tuple approach seems very appealing. Dealing consistently with equivalence classes including the independence from the chosen representative can be circumvented with this approach. Hence, we chose this approach for the studiVEMINT course. We utilized the connection between the symbolic and geometric representations as often as possible. However, we avoided using the geometric representations while introducing the mathematical discourse on vectors that we wanted to be consistent with the provided definition (Sfard, 2000), similar to what is required from the students in their upcoming lectures about linear algebra.

For further research we will look into the students’ answers to questions 2 and 3 more thoroughly with a more elaborated theoretical framework. We also intend to do a follow-up study to investigate the influence that the linear algebra course achieved on students at the end of the currently ongoing winter term 2016/17. Including a semiotic point of view and an analysis of textbooks from school as well as the introduction of vectors within the linear algebra course will improve the theoretical framework and provide further insights.

References

Alpers, K., Dietzsch, V., Jahnke, T., Janßen, M., Siekmann, A., Simanowsky, U., & Wuttke, H. (2003). *Mathematik, Analytische Geometrie/Lineare Algebra*. Berlin, Germany: Cornelsen.

Artmann, B., & Törner, G. (1984). *Lineare Algebra und Geometrie. Grund-und Leistungskurs*. Göttingen, Germany: Vandenhoeck & Ruprecht.

Bigalke, A., & Köhler, H. (2012). *Mathematik Gymnasiale Oberstufe, Nordrhein Westphalen, Leistungskurs*. Berlin, Germany: Cornelsen.

Bills, L., & Tall, D. (1998). *Operable definitions in advanced mathematics: The case of the least upper bound.* Paper presented at the Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education, Stellenbosch, South Africa.

Bossek, H., & Heinrich, R. (2007). *Lehrbuch Analytische Geometrie, gymnasiale Oberstufe*: Duden Paetec Verlag.

Brandt, D., & Reinelt, G. (2009). *Lambacher Schweizer*: Klett-Verlag.

Colberg, C., Mai, T., Wilms, D., & Biehler, R. (2017). Studifinder: Developing e-learning materials for the transition from secondary school to university. In Göller, R., Biehler, R., Hochmuth, R., Rück, H.-G. (2017): *Didactics of Mathematics in Higher Education as a Scientific Discipline – Conference Proceedings* (pp. 462-465). Kassel, Germany: Universitätsbibliothek Kassel.

Dörfler, W. (2000). Means for meaning. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design* (pp. 99-132). Mahwah, NJ: Lawrence Erlbaum Associates.

Griesel, H., Andreas, G., & Suhr, F. (2012). *Elemente der Mathematik Qualifikationsphase, Grund- und Leitungskurs*. Hannover, Germany: Schroedel.

Griesel, H., & Postel, H. (1990). *Mathematik heute*. Hannover, Germany: Schroedel.

Harel, G. (2000). Three Principles of Learning and Teaching Mathematics. In J.-L. Dorier (Eds.), *On the Teaching of Linear Algebra* (pp. 177-189). Dordrecht: Springer Netherlands.

Henn, H.-W., & Filler, A. (2015). Analytische Geometrie *Didaktik der Analytischen Geometrie und Linearen Algebra* (pp. 149-238). Heidelberg, Germany: Springer.

Malle, G. (2005). Schwierigkeiten mit Vektoren. *Mathematik Lehren, 133*, 16-19.

Sfard, A. (2000). Symbolizing mathematical reality into being–or how mathematical discourse and mathematical objects create each other. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design* (pp. 37-98). Mahwah, NJ: Lawrence Erlbaum Associates.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.

Vinner, S. (2002). The role of definitions in the teaching and learning of mathematics *Advanced mathematical thinking* (pp. 65-81). Dordrecht, The Netherlands: Springer.

Watson, A., Spyrou, P., & Tall, D. (2003). The relationship between physical embodiment and mathematical symbolism: The concept of vector. *The Mediterranean Journal of Mathematics Education, 1*(2), 73-97.

Weber, K., & Zillmer, W. (2014). *Mathematik Leistungskurs-Lehrbuch. Analysis, Analytische Geometrie und lineare Algebra, Stochastik*. Berlin, Germany: Paetec.

Yanik, H. B. (2011). Prospective middle school mathematics teachers' preconceptions of geometric translations. *Educational Studies in Mathematics, 78*(2), 231-260.