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To be or not to be an inflection point

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This paper describes students' grasp of inflection points. The participants were asked to define inflection points, to judge the validity of related statements, and to find inflection points by investigating (1) an algebraic representation of a function, and (2) the graph of the derivative. We found that participants provided their own "filtering conditions" to determine or deny the existence of an inflection point. In order to analyze participants' conceptions of inflection points, we used the lenses of Fischbein's theoretical framework.

Keywords: Inflection point, formal knowledge, algorithmic knowledge, intuitive knowledge.

Introduction

Functions receive considerable attention in secondary school, commonly in algebra and calculus lessons. Inflection point is one of the function-related notions addressed in high school and in further mathematics studies. In preliminary studies, we found some indications of common erroneous conceptions of the notion (e.g. Ovodenko & Tsamir, 2005; Tsamir & Ovodenko, 2004; 2013). These findings encouraged us to expand our research regarding the grasp of the notion of inflection point, and regarding possible sources of related common errors, while addressing a larger and diverse population who was given richer types of tasks (elaborated upon in the methodology section). In this paper, we present part of the findings from the extended study (Ovodenko, 2016). The research tools were designed and the findings analyzed with reference to a number of theoretical frameworks, including Fischbein's theory of algorithmic, formal, and intuitive components of mathematical knowledge (Fischbein, 1987, 1993a) and his theory of figural concepts (Fischbein, 1993b). Specifically, our research questions are: In the students' opinion, (1) When is a point an inflection point? (2) When is a point a non-inflection point?

What does research tell us about students' conceptions of inflection points?

Literature gives some indications of students' difficulties when using the notion of inflection point. Some researchers (e.g. Carlson, 1998) have reported that students tend to use fragments of phrases taken from earlier-learned theorems, such as "if the second derivative equals zero [then] it is an inflection point" when solving problems in the context of dynamic real-world situations.

Other researchers have reported that early experiences with the tangent to a circle contribute to the creation of the intuitive grasp of the tangent as a line that touches the graph only at one point and does not cross it (e.g. Vinner, 1982). This intuition was evoked when students were asked to identify and draw a tangent line to a curve's points that included non-differentiable and differentiable inflection points (e.g. Biza & Zachariades, 2010).

In a previous study, we examined students' conceptions of inflection points, in which we came across a novel tendency to regard a "peak point" as an inflection point (e.g. Tsamir & Ovodenko, 2004). We found tendencies to regard $f'(x) = 0$ as a necessity for the existence of an inflection point (Ovodenko & Tsamir, 2005), as well as tendencies to view $f'(x) \neq 0$ as a necessary condition and

$f''(x) = 0$ as a sufficient condition for an inflection point (Tsamir & Ovodenko, 2013). Consequently, we designed a large study to examine students' conceptions of the inflection point when solving a rich variety of problems. Here we report on part of the findings (Ovodenko, 2016).

Methodology

The research population included 223 participants from different educational levels of mathematics: high school students studying mathematics at the intermediate level, high school students studying mathematics at the advanced level, university students and university graduates (the latter majoring in mathematics-rich subjects, such as mathematics, computer science, and electronic engineering). All participants had studied the notion of inflection point during their calculus lessons.

According to the Israeli mathematics curriculum for secondary schools, an inflection point is defined as a point on a curve at which the curve changes from being concave up to concave down or vice versa, usually relating to functions that are at least twice differentiable in a small neighborhood of the point. It is important to note that the first encounter with inflection points occurs before the term is defined. It happens when students start investigating functions: they solve $f'(x) = 0$ to find possible x -s of extreme points and accidentally encounter cases where $f'(x) = 0$, but there is no extreme point because the function is monotonic in the interval that includes this point. In such cases $-f'(x) = 0$ but the point is non-extreme – students are first guided to label these points as inflection points for purposes of communication and to distinguish them from extreme points. Afterwards, useful theorems are introduced, e.g., a necessary condition for x_0 to be an inflection point is $f''(x_0) = 0$; a sufficient condition may be: (1) $f''(x_0^+)$, and $f''(x_0^-)$ have opposite signs in the neighborhood of x_0 ; or, (2) $f'''(x_0)$ exists and $f'''(x_0) \neq 0$.

In order to widen the scope of the gathered data, two types of tasks were designed: Produce-a-Solution (production) tasks (i.e., solve mathematical problems), and Evaluate-a-Solution (identification) tasks (i.e., examine the correctness of given solutions). Tasks in each of the two types were presented in verbal, graphic, and algebraic representations (see Figure 1).

Evaluate -a- Solution	Verbal representations <ul style="list-style-type: none"> Evaluate judgement and proof 	Verbal representations <ul style="list-style-type: none"> Define Prove 	Produce - a- Solution
	Graphic representations <ul style="list-style-type: none"> Evaluate marked points on the graph 	Graphic representations <ul style="list-style-type: none"> Draw example Mark inflection points Connection $f - f'$ 	
	Algebraic representations <ul style="list-style-type: none"> Evaluate the result and solution 	Algebraic representations <ul style="list-style-type: none"> Find inflection point 	

Figure 1: Structure of research questionnaire

This systematic structure of tasks provided insight to the participants' ideas and reasoning. It was developed and empirically validated (during preliminary pilot studies) as a tool that allows students to explain related formal, algorithmic and intuitive components of their mathematical knowledge (Fischbein, 1987, 1993a). The contribution of these structured tools may go beyond the broad exploration of students' conception of inflection points; such structured tools could be useful to

reveal students' conceptions of additional mathematical notions. That is, the research tools offered in this study may serve as a model when designing research tools aimed at investigating students' conceptions of other mathematical notions.

In the following, we present four tasks (Produce-a-Solution) from the questionnaire.

Task 1: Define: What is an inflection point?

Task 2: True or false? Prove:

Statement 1: $f: \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable function.

If $f'(x_0) = 0$, then $P(x_0, f(x_0))$ is an inflection point.

Statement 2: $f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous, (at least twice) differentiable function.

If $f''(x_0) = 0$, then $P(x_0, f(x_0))$ is an inflection point.

Statement 3: $f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous, (at least twice) differentiable function.

If $f'(x_0) = 0$ and $f''(x_0) = 0$, then $P(x_0, f(x_0))$ is an inflection point.

Statement 4: $f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous, (at least twice) differentiable function.

If $f''(x_0) = 0$ and the function is monotonically increasing (decreasing) in the neighborhood of x_0 , then $P(x_0, f(x_0))$ is an inflection point.

Task 3: Find (if possible) the inflection points of the functions:

$$(1) f(x) = x^4 + 2x^3 - 1; \quad (2) f(x) = x^4 + 32x; \quad (3) f(x) = |x^3 - 1|;$$

$$(4) f(x) = \begin{cases} x^3, & x \leq 0 \\ -x^3, & x > 0 \end{cases} \quad (5) f(x) = \begin{cases} x^2, & x \leq 0 \\ -x^2, & x > 0 \end{cases}$$

Task 4: Find (if possible) the inflection points of $f(x)$, $g(x)$, $t(x)$, based on the graphs of $f'(x)$, $g'(x)$, $t'(x)$ – see Figure 2.

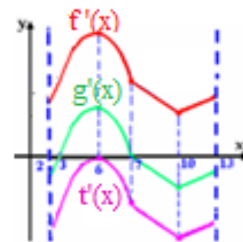


Figure 2: Graphs in Task 4

Task 1 and Task 2 are in a verbal representation, addressing participants' formal knowledge, Task 3: in an algebraic representation, addressing what we expected to be algorithmic knowledge; and Task 4 in a graphic representation, addressing participants' figural conceptions. Intuitive knowledge may be expressed in all four tasks.

Before giving our high school participants the tasks, we asked their teachers whether these tasks would be familiar to them. We learnt that Task 3 was expected to be most familiar – students usually investigated algebraic expressions of functions. They seldom analyzed graphs of the derivative, as required in Task 4; and were rarely asked to determine and prove the validity of a statement, as required in Task 2. Students' modest experience with such tasks and the impossibility of applying routine algorithms in the related solutions led us to assume that students' knowledge might be challenged.

Based on the analysis of their solutions, 20 participants were invited to individual, semi-structured, follow-up interviews, where they were asked by the researchers to elaborate on their written solutions. Interviewees were chosen according to their solutions in the questionnaires, focusing on interesting, correct, and incorrect ideas, while aiming to understand their reasoning. Interviewees were asked, among other things, to explain their solutions and to analyze solutions proposed by other participants. The interviews took 30-45 minutes and were audiotaped and transcribed.

Results

In this section, we answer the research questions: In the students' opinion (1) When is a point an inflection point? (2) When is a point a non-inflection point? We discuss each of the ideas as presented in relevant tasks. More specifically, we present each of the conceptions found as expressed in the four tasks, in descending frequency of the phenomenon.

When is the point an inflection point?

Our data indicates six sets of conditions that in the participants' opinions guarantee the presence of an inflection point.

Passage point from convex to concave or vice versa \Rightarrow Inflection point – This conception was mainly expressed (55%) in Task 1 (Define): “Inflection point is a point where the graph shifts from concave to convex (or vice versa)”. However, no reference was made to characteristics of the function (domain, continuity or differentiability).

$f''(x_0) = 0$ and a passage from convex to concave \Rightarrow Inflection point – This conception prevented participants from finding the non-derivative inflection point of the functions, based on the graphs of derivatives in Task 4 (24%). In addition, it was also expressed in solutions to the algebraic representation of function $f(x) = |x^3 - 1|$ (16%) in Task 3.

$f''(x_0) = 0 \Rightarrow$ Inflection point – This conception was evident in the participants' solutions to three tasks. In Task 3, $f''(x) = 0$ considerations were erroneously used as sufficient for inflection points of polynomial functions and even of piecewise functions. For example, participants correctly found A(0, -1) and B(-1, -2) as inflection points of $f(x) = x^4 + 2x^3 - 1$ by examining only $f''(x) = 0$ (30%).

Similarly, the point (0, 0) was erroneously claimed to be an inflection point of $f(x) = \begin{cases} x^3, & x \leq 0 \\ -x^3, & x > 0 \end{cases}$ by examining only $f''(x) = 0$ (15%). About 30% incorrectly claimed that Statement 2 is correct (Task 2). Some of them added: “That is the definition: P is an inflection point if and only if $f''(x_P) = 0$ ”. Others provided algorithmic considerations, mentioning solutions to investigate-the-function and stopping after solving $f''(x_0) = 0$, e.g.: “We find inflection points when looking for extreme points. If $f''(x_0) = 0$, then the point is not minima or maxima, and that is why it is an inflection point”. In their reactions to Statement 3 (Task 2) participants (10%) incorrectly answered that the statement is valid and explained that “ $f''(x_0) = 0$ is a sufficient condition for the existence of an inflection point”.

$f'(x_0) = 0$ and $f''(x_0) = 0 \Rightarrow$ Inflection point – This conception was mainly expressed in Task 2; about half of the participants incorrectly claimed that Statement 3 was valid. Most of them provided the methods that they used to solve investigate-the-function tasks. They wrote: “[We] always find inflection points when looking for extreme points, thus starting this search with $f'(x) = 0$ ”, then: “An inflection point is a point where $f''(x) = 0$ ”. In addition, in Task 4 (Investigate the graphs of derivative), about 20% incorrectly found an inflection point only at $x = 6$, exhibiting an erroneous assumption: “We see that $t'(6) = 0$, that is, the slope of the tangent of $y = t(x)$ at $x = 6$ is zero, so, $t''(6) = 0$, therefore at $x = 6$ there is an inflection point.”

$f''(x_0) = 0$ and monotonicity in the neighborhood of $x_0 \Rightarrow$ Inflection point – Participants (42%) incorrectly claimed that Statement 4 was valid (Task 2). They explained: “Those are sufficient

conditions for an inflection point”, or gave a supporting example, like $f(x) = x^3 + 5$. In their interviews, several of the latter explained their solution in terms of: “If the second derivative is zero and the function continues to increase when increasing and to decrease when decreasing, there is a change of convexity-concavity”. The combination of these two conditions is likely to determine an inflection point. However, this answer can be refuted by a counter example, like $f(x) = x^4 + 32x$ (this function appeared in Task 3, but it was not used to refute this statement), $f''(0) = 0$ and the function is increasing monotonically in the neighborhood of $x = 0$ but $(0, 0)$ is not an inflection point. Evidence about this set of insufficient conditions that seem to be “allegedly certain” for ensuring an inflection point, to the best of our knowledge, was found for the first time in this research.

$f'(x_0) = 0$ and monotonicity in the neighborhood of $x_0 \Rightarrow$ Inflection point – Participants (16%) wrote, for instance: “A point where $f'(x) = 0$ and the graph keeps increasing (or decreasing) before the point and after it is an inflection point” (Task 1). In reaction to Statement 1 (Task 2), participants correctly answered that the statement is false, but their explanation was: “ $f'(x_0) = 0$ is necessary but not a sufficient condition for an inflection point. If in addition to $f'(x_0) = 0$ the function increases (or decreases) before and after the point, only then is the point an inflection point” (10%). It should be noted that a combination of these conditions define a particular type of notion – a horizontal inflection point. This grasp of inflection point probably ignores non-horizontal inflection points (Task 3), like the inflection point of the function $f(x) = x^4 + 2x^3 - 1$ (10%).

When is the point a non-inflection point?

We found three conditions that deny the existence of an inflection point:

No differentiability \Rightarrow No inflection point – In reactions to Task 3, investigate the function $f(x) = |x^3 - 1|$, most participants (63%) found an inflection point only at $x = 0$, providing algorithmic considerations of solutions to investigate-the-function, such as: “ $f'(0) = 0, f''(0) = 0$, before $x = 0, f''(x)$ is positive, so $f(x)$ is convex and after $x = 0, f''(x)$ is negative, so $f(x)$ is concave, thus $x = 0$ is an inflection point”. Some of them added a correct graph to their investigation and, with relation to $x = 1$, wrote that “although at this point the function changes from concave to convex, it is a non-inflection point, because the function is not differentiable at $x = 1$ ”. In their interviews, several participants explained: “The function is non differentiable at $x = 1$ and therefore there is no inflection point”, or: “No differentiability, no inflection point.” Following this, a quarter of the participants defined “inflection point” (Task 1) as requiring differentiability. For example: “A point where $f''(x) = 0$ and the function turns from concave to convex or vice versa”, or: “The slope of the tangent of the function at this point is zero, and the function is either increasing on both sides of the point, or decreasing on both sides of the point.”

No second derivative \Rightarrow No inflection point – This conception, that is consistent with the condition $f''(x_0) = 0$, is used as a filter to reactions to Task 3, Investigate the function $f(x) = \begin{cases} x^2, & x \leq 0 \\ -x^2, & x > 0 \end{cases}$, included explanations (23%) such as: “When there is no second derivative – there is no inflection point. All inflection points must satisfy the condition $f''(x_0) = 0$. Otherwise there is no inflection point”. It was also found in reactions to Task 4 (15%), Investigate the graphs of $f'(x)$, that only the point $x = 6$ that satisfies the condition $f''(x_0) = 0$ was identified as an inflection point. The other point ($x = 10$), where the second derivative is not defined, was ignored.

$f''(x_0) = 0$ and $f'''(x_0) = 0 \Rightarrow$ **No inflection point** – expressed in the investigation of the function $f(x) = x^4 + 32x$ (7%). Note that this function really has no inflection points. Yet, this correct judgment was based on a wrong consideration. A counter-example is $f(x) = x^5$ has an inflection point, yet both $f''(0) = 0$ and $f'''(0) = 0$.

Commonly studies report of students' conception of mathematical notions by reporting on criteria that lead to regard the notion as defined. Here we show a new angle of criteria that regards the notion as undefined.

Discussion

We discuss the findings by using Fischbein's (1993a, 1993b) theoretical framework for analyzing students' errors and for examining possible related sources.

What are possible sources of students' mathematical errors?

Fischbein studied broad aspects of students' mathematical reasoning, claiming that an analysis of students' performance has to take into account three basic aspects: algorithmic, formal and intuitive (Fischbein, 1987, 1993a). The algorithmic aspect includes knowledge of (a) "how" to solve a problem, and (b) "why" a certain sequence of steps is correct. The formal aspect includes knowledge of axioms, definitions, theorems, proofs and knowledge of how the mathematical realm works. The intuitive aspect of mathematical knowledge is an immediate and self-evident, though not necessary correct, knowledge, accepted with certainty. Fischbein's three components of mathematical knowledge and their interrelations play a vital role in students' mathematical performances. However, "sometimes, the intuitive background manipulates and hinders the formal interpretation or the use of algorithmic procedures", causing inconsistencies in students' solutions (Fischbein, 1993a, p. 14). Fischbein further addressed the impact of drawings (e.g., in geometry) on learners' mathematical reasoning by explaining that the figural structure may dominate one's reasoning instead of being controlled by the corresponding formal constraints (Fischbein, 1993b).

What are the possible sources for students' errors with the concept of inflection points?

We found tendencies to determine or deny the existence of an inflection point under certain conditions. It is important to note that during the study we did not ask directly: Under which conditions, does or does not one get an inflection point? Participants provided "filtering conditions" by their own initiative. So, if one of the following sets is true: (1) convex-concave; (2) $f''(x_0) = 0$ and convex-concave; (3) $f''(x_0) = 0$; (4) $f'(x_0) = 0$ and $f''(x_0) = 0$; (5) $f''(x_0) = 0$ and monotonicity in the neighborhood of x_0 ; (6) $f'(x_0) = 0$ and monotonicity in the neighborhood of x_0 ; on the other hand, if there is (7) No differentiability; (8) No second derivative; (9) $f''(x_0) = 0$ and $f'''(x_0) = 0$ – then there is no inflection point. During the interviews, students reinforced these views.

An initial evaluation of the reasons underlying erroneous conceptions suggested two main causes: algorithmic experience with investigations of functions, and the impact of the drawing. Students tended to explain that: "This is how I find an inflection point when I investigate a function", or, [in relation to Task 4] "According to the graphs, each function has one inflection point at $x = 7$ where the graphs shift from concave down to concave up". Thus, it seems that the answers may intuitively evolve from the participants' mathematical, algorithmic experiences (Fischbein, 1993a) and from their figural concept of inflection point (Fischbein, 1993b).

Four of the six sets of conditions that participants presented for “being an inflection point” do not necessarily lead to inflection points (sets 1, 3-5); the other two sets determine inflection points only for a limited family of functions (sets 2, 6). For example, in set 1, the participants provided intuitive definitions, without reference to the type of the functions (e.g., continuous or differentiable). In set 4, participants exhibited slope-zero figural concepts (Fischbein, 1993b) in their reactions to Task 4, when they incorrectly found an inflection point “where slope of the tangent is zero...”, or, “where the first derivative and second derivative cross the x -axis”. In set 5, the necessary condition $f''(x) = 0$ was presented as a critical step in the algorithmic offering (Fischbein, 1993a), but in combination with the condition of monotonicity, that at first sight seems “sufficient” for an inflection point, surprisingly this does not necessarily lead to an inflection point (as presented in the results section). In set 6, the unnecessary condition $f'(x) = 0$ was possibly used as a result of the “primacy effect”. That is, it might be the case because these are usually the first inflection points addressed in calculus lessons (Fischbein, 1987); but it was presented with the condition of monotonicity, and thus defined a particular type of notion – a horizontal inflection point.

The three sets of “denying conditions” can shed some additional light on students’ conception of inflection point from two perspectives: (1) types of functions that are usually investigated, and, (2) logical constraints of their knowledge. Requesting the necessity of differentiability (set 7) can be related to functions that are usually investigated or presented graphically in textbooks – most of these are differentiable at the inflection point. Thus, an intuitive image of a “smooth inflection point” was created. Here, as in many other cases, students recognize the concept “by experience and usage in appropriate contexts” (Tall & Vinner, 1981, p. 151; their emphasis). The necessity of twice differentiability (set 8) might be rooted in intuitive ideas that interfere with students’ formal knowledge (Fischbein, 1987; 1993a). That is, from the theorem, “If $f(x)$ is twice differentiable in some neighborhood of x_0 , and if x_0 is an inflection point, then $f''(x_0) = 0$ ” students erroneously conclude that “if no second derivative then there is no inflection point”. Here, this answer can be refuted by a counter example, like $f(x) = x^{5/3}$. In set 9, the inadequate declaration: ‘ $f''(x_0) = 0$ and $f'''(x_0) = 0 \Rightarrow$ No inflection point’, might be rooted in intuitive ideas that interfere with students’ algorithmic knowledge (Fischbein, 1987). That is, from the theorem “ $f''(x_0) = 0$ and $f'''(x_0) \neq 0 \Rightarrow$ Inflection point” students erroneously create the rule “if ... then..., if not ... then not...”.

This study considerably enriches the existing body of knowledge regarding high school students’, university students’, and university graduates’ conceptions of inflection points. Only a small number of studies have dealt with students’ conceptions of inflection points directly (e.g. Rivel, 2004; Tsamir & Ovodenko, 2004) and indirectly (e.g. Biza & Zachariades, 2010; Vinner, 1982). Reported studies usually addressed a limited population and dealt only with specific conceptions. The current research offers a broad collection of related correct and incorrect conceptions found among participants with suitable mathematical backgrounds (as specified in the methodology section), with reference to the type and the representation of the given tasks.

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