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Meta-representational competence with linear algebra in quantum mechanics

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In this report we analyze one student's meta-representational competence as he engages in solving a quantum mechanics problem involving the linear algebra concepts of basis, eigenvectors, and eigenvalues. We provide detail on student A25, who serves as a paradigmatic example of a student's power and flexibility in thinking in and using different notation systems. This case study, which lays the groundwork for future analysis, provides evidence that meta-representational competence (MRC) is beneficial to a student's ability to make sense of and use concepts from linear algebra while solving quantum mechanics problems.

Keywords: Linear algebra, meta-representational competence, physics.

Introduction

The National Research Council's (2012) report, which charges the United States to improve its undergraduate Science, Technology, Engineering, and Mathematics (STEM) education, specifically recommends "interdisciplinary studies of cross-cutting concepts and cognitive processes" (p. 3) in undergraduate STEM courses. It further states that "gaps remain in the understanding of student learning in upper division courses" (p. 199), and that interdisciplinary studies "could help to increase the coherence of students' learning experience across disciplines ... and could facilitate an understanding of how to promote the transfer of knowledge from one setting to another" (p. 202). Our work contributes towards this need by investigating student understanding of linear algebra in quantum mechanics. Two research questions that guide us in this paper are: what are the various ways in which students reason about and symbolize concepts related to eigentheory in quantum physics, and in what ways might meta-representational competence impact how they make sense of linear algebra concepts in quantum mechanics?

In this paper, we focus on one student's reflection on symbolizing choices he makes while solving a quantum mechanics problem that involves linear algebra. In particular, we analyze his reasons for how and why he chooses a specific symbol system – either Dirac notation or matrix notation – for solving an expectation value problem. We align our analysis with the frameworks of meta-representational competence (diSessa, Hammer, Sherin, & Kolpakowski, 1991) and of structural features of algebraic quantum notations (Gire & Price, 2015). This case study, which lays the groundwork for future analysis, explores in what ways MRC might aid a student's ability to make sense of and use concepts from linear algebra while solving quantum mechanics problems.

Background and theoretical framework

In this section, we give an overview of research conducted on student understanding of symbols and representations in mathematics and physics, as well as our theoretical orientation. We conclude with a brief introduction to eigentheory in Quantum Mechanics and Dirac notation.

Student understanding of symbols and representations

The recognition of the importance of students' understanding of symbols used in mathematics and physics has grown over the past few decades. Arcavi (1994, 2005) coined this as “symbol sense,” which includes aspects such as being “friendly” with symbols, engineering symbolic expressions, choosing which aspects of a mathematical situation to symbolize, using symbolic manipulations flexibly, and sensing the different roles symbols can play in various contexts. Other research along this vein include: an explication of how different perspectives, such as cognitivist, situationist, and social-psychological, provide vastly different ways to understand how students make sense of and use inscriptions and symbols (Kaput, 1998); a study of how students mathematize their language from a Vygotskian perspective (Van Oers, 2002); and an exploration of how notational systems can serve as a mediational tool which triggers and sustains mathematical activity (Meira, 2002).

Research into students' competence with symbols and representations is not limited to primary and secondary school studies. For example, Hillel (2000) described three modes of description (abstract, algebraic, and geometric) of the basic objects and operations in linear algebra and pointed out that “the ability to understand how vectors and transformation in one mode are differently represented, either within the same mode, or across modes is essential in coping with linear algebra” (p. 199). Thomas and Stewart (2011) found that students struggle to coordinate the two mathematical processes captured in $A\mathbf{x} = \lambda\mathbf{x}$, where A is an $n \times n$ matrix, \mathbf{x} is a vector in \mathbb{R}^n , and λ is a scalar, to make sense of equality as “yielding the same result.” This interpretation of the “equals” symbol is often novel and nontrivial for students (Harel, 2000). Harel also posits that the interpretation of “solution” in this setting, the set of all vectors \mathbf{x} that make the equation true, entails a new level of complexity than does solving equations such as $c\mathbf{x} = d$, with each taking values from the reals. Thomas and Stewart (2011) conjecture that this complexity may prevent students from progressing symbolically from $A\mathbf{x} = \lambda\mathbf{x}$ to $(A - \lambda I)\mathbf{x} = \mathbf{0}$, which is particularly useful when solving for the eigenvalues and eigenvectors of a matrix A .

Research into students' understanding of quantum mechanics also investigates student use of symbols, such as how students make sense of and use a novel notation, called Dirac notation (explained in the subsequent section). Most closely related with this current study, Gire and Price (2015) looked at structural features of three different notation systems used in quantum mechanics (Dirac, matrix, and wave function) and how students' reasoning interacts with these features. The features identified by the authors are: (a) individuation, or “the degree to which important features are represented as separate and elemental” (p. 5); (b) externalization, or “the degree to which elements and features are externalized with markings included in the representation” (p. 7); (c) compactness; and (d) symbolic support for computation. Using problem-solving interviews with students as insight into these features, Gire and Price found that students readily used Dirac notation, and that the structural features vary across the different notations and among contexts.

Relatedly, diSessa et al. (1991) importantly discovered that students have a great deal of knowledge about what good representations are and are able to critique and refine them, which the authors defined as Meta-Representational Competence (MRC). diSessa and Sherin (2000) explained that MRC includes inventing and designing new representations, judging and comparing the quality of representations, understanding the general and specific functions of representations, and quickly learning to use and understand new representations. Furthermore, diSessa (2002, 2004) offered a

variety of critical resources students possess as part of their MRC for judging the strength of representations, such as compactness, parsimony, and conventionality. Two particular resources encompassed by MRC that we focus on in our data are “critique and compare the adequacy of representations and judge their suitability for various tasks,” and “understand the purposes of representations generally and in particular contexts and understand how representations do the work they do for us (diSessa, 2004, p. 94).

In this study, we align ourselves with the theory that representations are a sense-making tool, in that “the construction of representations on paper during problem solving mediates and organizes one's understanding of mathematical concepts” (Meira, 2002, p. 101). We couple this with a framing of MRC, specific to two particular notational systems, to investigate a student's reflection on his own notational preferences in quantum mechanics and what that may reveal about his understanding of change of basis and eigentheory in that context.

Brief introduction to eigentheory and Dirac notation in quantum mechanics

In quantum mechanics, certain physical systems are modeled and made sense of using eigentheory. To a physical system we associate a Hilbert space (such as \mathbb{C}^2), to every possible state of the physical system we associate a vector in the Hilbert space, and to every possible observable (i.e., measurable physical quantity) we associate a Hermitian operator (usually given in its matrix form). The only possible result of a measurement is an eigenvalue of the operator, and after the measurement the system will be found in the corresponding eigenstate.

Dirac notation, also known as bra-ket or just ket notation, is a commonly used notational system in quantum mechanics. A vector representing a possible state is symbolized with a ket, such as $|\psi\rangle$. Mathematically, kets behave like column vectors, such as $|\psi\rangle \doteq \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $a_1, a_2 \in \mathbb{C}$, and are usually normalized. The complex conjugate transpose of a ket is called a *bra*, which behaves mathematically like a row vector, such as $\langle\psi| \doteq [a_1^* \ a_2^*]$. In addition, the eigenvalue equations for observables are central to many calculations. For example, the eigenvalue equations for S_x (the operator measuring the x -component of intrinsic angular momentum) of a spin- $1/2$ particle are $S_x|\pm\rangle_x = \pm\frac{\hbar}{2}|\pm\rangle_x$, where $|+\rangle_x$ and $|-\rangle_x$ form an orthonormal eigenbasis of S_x , and $\pm\frac{\hbar}{2}$ are the two possible measurement results of the observable. When symbolized in terms of this eigenbasis, the matrix representation of S_x is $\begin{bmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{bmatrix}$. One can also measure spin along other directions, such as z ; similarly, the eigenvalue equations are $S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$ (it is common for no subscript to be used for the z -direction). Thus, “within its own basis,” the matrix representation of S_z would be identical to the aforementioned diagonal one for S_x . It is often beneficial to change between bases; for example, $|+\rangle_x = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$ and $|-\rangle_x = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle$, so S_x in the “ z -basis” is $\begin{bmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{bmatrix}$. Finally, inner products are involved in computing the expectation value of observable A for state ψ , $\langle\psi|A|\psi\rangle$. These calculations require the bra and ket expansion to be in the same eigenbasis as the matrix representation of A . As such, expectation value problems present a rich setting for investigating students' symbolizing of eigentheory and change of basis in a physics context.

Methods

Participants for this study were third year undergraduate physics majors at a large, public, research-intensive university in the Pacific Northwestern United States. They were drawn on a volunteer basis from a class of 35 students in a Spin and Quantum Measurements course; this course met for 7 class-hours per week for three weeks and involved many student-centered activities and discussions. The data for this report come from individual, semi-structured interviews (Bernard, 1988) conducted with 8 students at the end of the course. The goals of the interview questions were to learn how students reasoned about linear algebra concepts (e.g., normalization, basis, and especially eigentheory), how they reasoned with these concepts as they discussed quantum mechanics concepts and solved quantum mechanics problems, and how they symbolized their work.

To begin our analysis, we viewed the video and observed how students navigated the interview problems, while we kept in mind the overarching research questions regarding students' reasoning about and symbolizing eigentheory in quantum physics. We noticed some students were particularly fluent in how they talked about and worked with both matrix and Dirac notations. This compelled us to investigate the literature about student use of symbols and notations, the most relevant of which were discussed above. Our analysis draws most heavily on the work of diSessa and colleagues regarding MRC, and that of Gire and Price (2015) regarding structural features of algebraic quantum notations. In particular, we coded for instances of students mentioning structural features of the mathematics or students making explicit meta-commentary on the representations they chose to use. This allowed us to integrate our analysis of students' MRC with Gire and Price's types of structural features in a way novel to the physics and mathematics education fields.

In this report, we focus on one student: A25, a double major in physics and nuclear engineering who had completed two 10-week courses in linear algebra. The reason we chose to focus on participant A25 was his demonstrated ability to articulate his thinking. During the interview, he exhibited flexibility in reasoning about the concepts we were probing, and through his explanation a great deal of MRC seemed visible and analyzable.

Results

In the beginning of the interview, student A25 volunteered that he sometimes explicitly chooses between doing calculations in matrix notation or in Dirac notation:

I: So how do you feel like, using eigenvectors and eigenvalues, in spins has been similar to and different from how you've experienced those in other classes?

A25: Uh, well, it's very similar because you're doing a lot of the same math ...the difference especially in physics, you're looking at kets. In, in at first I was kind of jarring, like to- to try to do the math in kets. But now, it's kinda- it's kinda easier, there's problems, there certain problems...where there's two ways to do them, they're kind of parallel, you can do it and you can expand the- the state in- in like as a- and expand them as kets in a different basis, or you can write that state as a- as a, as a vector, in that basis, and you can either do the matrix math for the like expectation values for example, you can do the matrix math or you can do the ket math, and sometimes it's, I'm finding that I, rather expand something in the ket.

From the transcript we see that A25 was aware multiple legitimate ways exist to solve the problem, seemingly understanding the various mathematical nuances and implications of his notational choices. His brief explanation highlights sentiments that are consistent with Arcavi's characteristics of symbol sense, such as being "friendly" with symbols and using them flexibly. Also, A25's self-reflection on his symbol usage adds a metacognitive aspect to the symbol sense characterization.

Because A25 volunteered expectation value problems as a situation in which he could use either notation, the interviewer had him work on such a problem right away, even though it was prepared to be at the end of the interview: "Consider the state $|\psi\rangle = -\frac{4}{5}|+\rangle_x + i\frac{3}{5}|-\rangle_x$ in a spin-1/2 system. Calculate the expectation value for the measurement of S_x ." A25 immediately worked on the problem using Dirac notation, saying, "basically to find the expectation value... it's like denoted that way [writes $\langle A \rangle$] but really what you're doing is you're taking the, the bra of the state, and then you're putting the operator [writes $= \langle \psi | A | \psi \rangle$] in the middle of the inner product." He continued to explain his work as he proceeded, with statements such as "you know that S_x is just going to um, like apply it's eigenvalues to these, so, so like the eigenvalue corresponding to plus x is going to be $+\hbar/2$ and the, the eigenvalue corresponding to $-x$ is going to be $-\hbar/2$, so you end up with this equation that looks like this [points to the second half of line 2 in Figure 1a]. Note that his work in Figure 1a, which led him to the correct answer of $7\hbar/50$, involved the state's expansion and use of eigenvector equations for S_x in ket notation. He did not need to physically write the expansion of $|\psi\rangle$ in the x basis kets, nor did he write out the eigenvector equations; however, his verbal description of his process relied on his understanding of both basis and the eigenvector relationships at play. Furthermore, this notation was novel to the students during this course; as such, A25 was clearly quick to use and understand this representation (a quality of MRC, diSessa & Sherin, 2000).

After discussing his work and solution, the interviewer asked: "Before you were telling about bra-ket versus matrix notation, you brought up an expectation value as an example of like, either or both, so can you, now that you had this problem, kinda revisit that?" A25 immediately solved the problem completely within matrix notation. He began by saying "if we're strictly in the plus and minus x basis" and wrote the column vector $\begin{bmatrix} -\frac{4}{5} \\ i\frac{3}{5} \end{bmatrix}$ associated with the given ket $|\psi\rangle$. He then said, "and then the bra would be, um, minus 4 over 5 and then minus i 3 over 5," writing out the row vector $\begin{bmatrix} -\frac{4}{5} & -i\frac{3}{5} \end{bmatrix}$ as he spoke (see Line 1 in Figure 1b). He then said, "and so what you do is take this [copies the column vector]...and then you have the operator in the middle [writes an empty 2x2 matrix], and then you have the bra here [copies the row vector], and the operator in this case is S_x and we're in the x basis so it's just $\hbar/2$ and $-\hbar/2$, 0, 0" [fills in the 2x2 matrix values] (see line 2 in Figure 1b). Impressively, he was able to fluidly move from his original ket notation to matrix notation, flawlessly making translations from the bras, kets, and operators in ket notation to row vectors, column vectors, and matrices in the matrix notation, further evidence of his strong MRC. Next, he explained his process for computing the matrix times the column vector before he did the computation, noting that "you're gonna get a vector." Again in line 3 he explained "then I do it again, so, um, this time you're gonna get a number out," meaning he anticipated that a row vector times a column vector would be a number. This shows two aspects of A25's strong understanding: first, a fluency in the calculations and computations within matrix notation similar to his ease in working in ket notation, including the ability to anticipate results before actually carrying out a computation (as in anticipating the result of

a matrix times a vector); and second, an ability to compare the two notations as well as an understanding that the two notations represent two ways to conceptualize the quantum physical calculation of expectation value. We see this as flexibly using symbolic manipulations (Arcavi, 1995) and an anticipation of results.

$\langle A \rangle = \langle \psi A \psi \rangle_{S_x}$ $= \left(\langle + \frac{-4}{5} + \langle - \frac{-3i}{5} \right) \left(\frac{-\hbar 4}{10} + \right) + \left(\frac{-\hbar i 3}{10} + \right)$ $\frac{\hbar 16}{50} - \frac{9\hbar}{50} = \left \frac{7\hbar}{50} \right $ <p style="text-align: center;">(a)</p>	$\begin{pmatrix} -4/5 \\ i3/5 \end{pmatrix} \begin{pmatrix} -4/5 & -i3/5 \end{pmatrix}$ $\begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix} \begin{pmatrix} -4/5 \\ i3/5 \end{pmatrix}$ $\begin{pmatrix} -4/5 & -i3/5 \end{pmatrix} \begin{pmatrix} -4\hbar/10 \\ -i3\hbar/10 \end{pmatrix} = \frac{16\hbar}{50} - \frac{9\hbar}{50} = \frac{7\hbar}{50}$ <p style="text-align: center;">(b)</p>
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Figure 1: A25's expectation value problem, in ket notation (a) and matrix notation (b)

The interviewer then asked A25 to reflect on any preference between the two notations:

A25: Uh...To be honest, I don't really, I don't really know why I prefer this [Figure 1a], I think it's just because, um, I like this notation. This specific notation [Figure 1a line 1] like this to me is like a cleaner way of writing that [Figure 1b line 2] because that- I mean this and that [touches Figure 1a line 1 and Figure 1b line 2 simultaneously] I feel like are your starting points, so you, you start here with this nice, like, looking thing [traces one finger under $\langle \psi | A | \psi \rangle$], or you start here with this big array of numbers [puts two open hands around Figure 1b line 2], and I prefer this [Figure 1a line 1], even though you have to expand this into basically the same amount of information [Figure 1a line 2]. And also, the nice thing about, about this [Figure 1a line 1], is it—actually this is really why it's better—is because you can, you can say ok S_x works- acts directly on these kets, you can just get rid of the matrix altogether...

We see his use of “nice looking thing” and “big array of numbers” in comparison to one another are an example of compactness. He also compares Figure 1a line 1 and Figure 1b line 3 regarding the “amount” of information they convey, which involves reflection on the physical and mathematical content expressed in the compared notations. Finally, acting directly on the expansion in terms of the eigenstates of the operator allow him to forego the matrix calculation entirely, which speaks to A25's view of compactness, parsimony, and symbolic support for ket notation for this problem.

When asked about his notation preferences if the basis expansion of a given state vector and the operator “didn't match,” A25 recalled a problem from his last homework that was “actually easier...to do the matrix multiplication,” stating “you don't want to have to change these kets into different bases all over the place 'cause they're already all written in the same basis and you know what the operator is in that basis so you might as well just, do the matrix multiplication.” Here we see how strong A25's understanding is of the important linear algebra concepts of bases and change of bases, and how they relate to the matrix multiplication within expectation value quantum mechanics problems.

Furthermore, we see another aspect of his MRC, namely his understanding that different notations have different strengths and weaknesses, and his ability to leverage these strengths and weaknesses depending upon the particular quantum mechanics situation. This speaks to his awareness of symbolic support as well as using symbols flexibly. Finally, when asked if the concepts of basis or eigenvectors/eigenvalues come up more in one notation than the other, A25 stated, “certainly...every time you write down a ket you're, you're very conscious of what basis you're in. In this one [points to Figure 1b] it's just kinda implied...all this [is] in the same basis, so you're just, you're just writing out numbers, an arrays of numbers, but here [in Figure 1a] you're thinking ok, this is the S_x operator, this is the x plus ket, this is the x minus bra...so I think that you're definitely more aware of what basis you're in when you're using this, because you have to be.” This explanation is consistent with externalization (Gire & Price, 2015), in that the ket notation allows features of the problem, namely basis, to be externalized in a way that matrix notation did not provide for A25. This again attests to his understanding that notations have different strengths and weaknesses, an element of MRC that seems particularly important within quantum mechanics.

Conclusion

In this report we analyzed one student’s MRC and his understanding of change of basis and eigentheory as he solved an expectation value problem in quantum mechanics. This case study lays the groundwork for future analysis by being a paradigmatic example of a student’s power and flexibility for thinking in and using different notation systems. In addition, it provides evidence that MRC seemed to positively impact this student’s ability to make sense of and use concepts from linear algebra while solving quantum mechanics problems. In addition to analyzing the other students from our data set, future research includes investigating how classroom interactions may have influenced students regarding their notational choices, what aspects of MRC seem most important to success in using linear algebra when solving quantum mechanics problems, and what that implies regarding students’ understanding of the mathematics and physics content involved.

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