

Existence and uniqueness theorem for a 3-dimensional polytope of \mathbb{R}^3 with prescribed directions and perimeters of the facets

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Abstract

We give a necessary and sufficient condition for the existence and uniqueness up to translations of a 3-dimensional polytope P in \mathbb{R}^3 having N facets with given unit outward normal vectors n_1, \dots, n_N and corresponding facet perimeters L_1, \dots, L_N .

In this paper, a polytope of \mathbb{R}^3 is the convex hull of finitely many points in \mathbb{R}^3 . The classical Minkowski problem for polytopes in \mathbb{R}^3 concerns the following question:

Given a collection n_1, \dots, n_N of N pairwise distinct unit vectors in \mathbb{R}^3 and F_1, \dots, F_N a collection of N positive real numbers, is there a polytope P in \mathbb{R}^3 having the n_i as its facet unit outward normals and the F_i as the corresponding facet areas ($1 \leq i \leq N$), and, if so, is P unique up to translations?

H. Minkowski proved that a polytope is uniquely determined, up to translations, by the directions and the areas of its facets (see [1, Theorem 9, p. 107]):

Theorem 1 (*H. Minkowski, 1897: [5] and [6, pp. 103-121]*)

A polytope in \mathbb{R}^3 is uniquely determined, up to translations, by the directions and the areas of its facets.

A well-known necessary condition for the existence of a polytope having facet unit outward normals n_1, \dots, n_N and corresponding facet areas F_1, \dots, F_N is that:

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$$\sum_{i=1}^N F_i n_i = 0.$$

An existence theorem of H. Minkowski ensures that this condition is both necessary and sufficient:

Theorem 2 (*H. Minkowski, 1897: [5] and [6, pp. 103-121]*)

Let $n_1, \dots, n_N \in \mathbb{R}^3$ be N pairwise distinct unit vectors linearly spanning \mathbb{R}^3 and let F_1, \dots, F_N be N positive real numbers. There exists a polytope P in \mathbb{R}^3 having N facets with unit outward normals n_1, \dots, n_N and corresponding facet areas F_1, \dots, F_N if, and only if, we have

$$\sum_{i=1}^N F_i n_i = 0.$$

Here, we have to mention that Theorem 2 is only the 3-dimensional version of the classical Minkowski existence and uniqueness theorem [7, p. 455], which is valid in \mathbb{R}^d for all $d \geq 2$. In our main proof, we shall make use of the 2-dimensional version, which is almost trivial.

This paper is devoted to the analogue of the classical Minkowski problem obtained by replacing areas by perimeters. For this analogue, the following uniqueness result is known (see [1, p. 108]):

Theorem 3 *A polytope in \mathbb{R}^3 is uniquely determined, up to translations, by the directions and the perimeters of its facets.*

Theorems 1 and 3 are similar uniqueness theorems which both follow from a same general result by A.D. Alexandrov (see [1, Theorem 8, p. 107]). Thus, we are led to the natural question of the existence of an analogue to Theorem 2 for the existence of a polytope with prescribed directions and perimeters of the facets. For convenience, we will restrict ourselves to 3-dimensional polytopes. Recall that the dimension of a convex body of \mathbb{R}^n is simply the dimension of its affine hull.

This problem has caught the attention of geometers and a paper by V. Alexandrov has recently highlighted its difficulty in giving a result that is interpreted by the author as an obstacle for finding an existence theorem for a polytope with prescribed directions and perimeters of the facets [2]. This paper was the major source of inspiration for the author.

Let n_1, \dots, n_N be a collection of N pairwise distinct unit vectors linearly spanning \mathbb{R}^3 and let L_1, \dots, L_N be a collection of N positive real numbers. A series of necessary conditions for the existence of a

3-dimensional polytope P in \mathbb{R}^3 having the n_i as its facet outward unit normals and the L_i as the corresponding facet perimeters is the following:

(i) For each $i \in \{1, \dots, N\}$, there exists a decomposition of L_i into a sum of N non-negative real numbers,

$$L_i = \sum_{j=1}^N l_{ij},$$

in such a way that:

(ii) for all $(i, j) \in (\{1, \dots, N\})^2$, $l_{ij} = 0$ if n_i and n_j are collinear;

(iii) for all $(i, j) \in (\{1, \dots, N\})^2$, $l_{ji} = l_{ij}$.

Indeed, if such a polytope P exists, then denoting by f_1, \dots, f_N the N facets with respective unit outward normals n_1, \dots, n_N , the required relationships hold if we put:

$$l_{ij} := \begin{cases} 0 & \text{if } i = j \text{ or if } f_i \text{ and } f_j \text{ have no common edge} \\ \text{the length of the common edge} & \text{otherwise.} \end{cases}$$

Moreover, we must also have:

$$\text{(iv) For every } i \in \{1, \dots, N\}, \quad \sum_{j \in \{k \mid l_{ik} \neq 0\}} l_{ij} \left[\frac{n_i \times n_j}{\sin(n_i, n_j)} \right] = 0,$$

where \times denotes the cross product (here, we of course assume that \mathbb{R}^3 is oriented by its canonical basis), and (n_i, n_j) the length of the shortest arc of great circle joining n_i to n_j on the unit sphere \mathbb{S}^2 of \mathbb{R}^3 (remind that n_i and n_j are non collinear by condition (ii) since $l_{ij} \neq 0$). Indeed, each facet f_i of P is a convex polygon the boundary of which is a closed polygonal line. Here, it is worth noting that, for all $j \in \{k \mid l_{ik} \neq 0\}$,

$$\overrightarrow{u_{i,j}} = (n_i \times n_j) / \sin(n_i, n_j)$$

is a unit vector that is such that the vector $\overrightarrow{v_{i,j}} := l_{ij} \overrightarrow{u_{i,j}}$ is of the form $\overrightarrow{MM'}$, where M and M' are two consecutive vertices of the oriented boundary of the face f_i (see the figure).

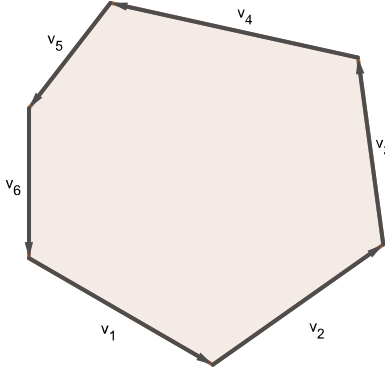


Figure. Illustration of the fact that

$$\sum_{j \in \{k | l_{ik} \neq 0\}} l_{ij} \vec{u}_{ij} = \sum_{l=1}^m v_l = 0$$

Our last necessary condition will follow from the Steinitz's theorem (e.g. see [4, Chapter 4, p. 103]), which characterizes in purely graph-theoretic terms those graphs that can be represented as the 1-skeleton of some 3-dimensional polytope:

Theorem (Steinitz's theorem). *A graph can be represented as the 1-skeleton of some 3-dimensional polytope if, and only if, it is simple, planar, and 3-connected.*

For the convenience of the reader, we summarize some basic definitions and facts on graphs and 1-skeletons:

- The *1-skeleton* of a polytope P is the graph whose vertices and edges are just the vertices and edges of P with the same incidence relation.
- A graph is said to be *polyhedral* if it can be represented as the 1-skeleton of some 3-dimensional polytope.
- A graph is said to be *simple* if it contains neither multiple edges nor loops.
- A graph is said to be *planar* if it can be drawn in the plane so that no two of the edges intersect at a point other than a vertex.
- A graph is called *3-connected* if it is connected, has at least 4 vertices, and remains connected whenever fewer than 3 vertices are removed.
- The dual of a polyhedral graph is also a polyhedral graph. More precisely, every polyhedral graph G has a well-defined dual graph G^* (independent of the plane embedding), corresponding to the 1-skeleton of the dual polytope.

From Steinitz's theorem, the following last condition is also necessary in our case:

(v) The datum of the matrix $(l_{ij})_{1 \leq i, j \leq N}$ determines as follows a simple 3-connected planar graph G drawn on the unit sphere \mathbb{S}^2 (so that no two of the edges intersect at a point other than a vertex): the vertices of G are the unit vectors n_1, \dots, n_N , and any pair of non-collinear vertices $\{n_i, n_j\}$ of G is connected by an edge that is given by the shortest arc of great circle joining the two vertices on \mathbb{S}^2 if, and only if, $l_{ij} \neq 0$.

Theorem 4 *Let $n_1, \dots, n_N \in \mathbb{R}^3$ be N distinct unit vectors linearly spanning \mathbb{R}^3 and let L_1, \dots, L_N be N positive real numbers. There exists a 3-dimensional polytope P in \mathbb{R}^3 having facet unit outward normals n_1, \dots, n_N and corresponding facet perimeters L_1, \dots, L_N if, and only if, the set of conditions $\{\mathbf{(i)}, \mathbf{(ii)}, \mathbf{(iii)}, \mathbf{(iv)}, \mathbf{(v)}\}$ holds.*

Proof. We have already seen that it is necessary that the set of conditions $\{\mathbf{(i)}, \mathbf{(ii)}, \mathbf{(iii)}, \mathbf{(iv)}, \mathbf{(v)}\}$ holds for such a polytope P to exist in \mathbb{R}^3 . Conversely assume that the set of conditions $\{\mathbf{(i)}, \mathbf{(ii)}, \mathbf{(iii)}, \mathbf{(iv)}, \mathbf{(v)}\}$ holds. By the Steinitz's theorem, the simple 3-connected planar graph G that is constructed on \mathbb{S}^2 in accordance with condition (v) and its geometric dual graph G^* (which is also simple, planar, and 3-connected) are polyhedral: they can be represented in \mathbb{R}^3 as the 1-skeletons of two dual 3-dimensional polytopes, say Q and Q^* , respectively. Moreover, by the following refinement by Barnette and Grünbaum, we can preassign the shape of a face of one of these two polytopes [3]:

Theorem (Barnette and Grünbaum). *If one face of a 3-dimensional Q is an n -gon, then there exists a polytope Q' combinatorially equivalent to Q , of which the corresponding face is any prescribed n -gon.*

Recall that two polytopes P and P' are said to be *combinatorially equivalent* if there is a bijection between their faces that preserves the inclusion relation. It is well-known that the combinatorial structure of a 3-dimensional polytope is completely determined by its 1-skeleton [4, p. 105]. Moreover, as noticed by Barnette and Grünbaum [3, p. 305]: "By an obvious application of duality, it follows from the theorem that the shape of one vertex-figure may be prescribed". Our desired polytope P will be combinatorially equivalent to Q^* .

In the sequel of the proof, the assembly of all the facets of a given 3-dimensional polytope that share a same vertex will be called a *corner* of the polytope. The *spherical representation* of such a corner, or of a 3-dimensional polytope, is defined as follows:

- A facet f corresponding to a unit normal n is represented on \mathbb{S}^2 by n ;
- An edge is represented on \mathbb{S}^2 by the arc of great circle joining the two points corresponding to the two adjacent facets of the edge;
- A vertex is represented on \mathbb{S}^2 by the spherical polygon that is bounded by the spherical arcs corresponding to the edges that are adjacent at the vertex.

Our aim is to prove the existence of a 3-dimensional polytope P that satisfies the set of conditions $\{(\mathbf{i}), (\mathbf{ii}), (\mathbf{iii}), (\mathbf{iv}), (\mathbf{v})\}$, and thus, the spherical representation of which is given by G .

Since condition (\mathbf{iv}) is satisfied, the Minkowski existence and uniqueness theorem for convex polygons ensures that, for any $i \in \{1, \dots, N\}$, there exists in n_i^\perp (the linear plane with unit normal n_i , endowed with the induced orientation) a positively oriented convex polygon f_i whose edges e_{ij} are directed by the unit vectors

$$\overrightarrow{u_{i,j}} = (n_i \times n_j) / \sin(n_i, n_j), \quad (j \in \{k \mid l_{ik} \neq 0\}),$$

and have corresponding lengths l_{ij} ; and moreover, that this polygon f_i , of which the perimeter is L_i by condition (\mathbf{i}) , is unique up to translations in n_i^\perp . Note that in the above expression of $\overrightarrow{u_{i,j}}$, the vectors n_i and n_j are non-collinear by condition (\mathbf{ii}) since $l_{ij} \neq 0$.

Thus, for any $i \in \{1, \dots, N\}$, the desired oriented convex polygon f_i , with unit normal n_i , is well-defined and unique up to translations in \mathbb{R}^3 .

Now, given any positively oriented n -gon \mathcal{P}_n on \mathbb{S}^2 that is the oriented boundary of the closure of a connected component of the complementary of the graph G , which is drawn on \mathbb{S}^2 according to condition (\mathbf{v}) , ($n \geq 3$), we deduce from Girard's theorem that

$$\sum_{k=1}^n \alpha_k = (n-2)\pi + \text{Area}(\mathcal{P}_n),$$

where $\alpha_1, \dots, \alpha_n$ denote the interior angles of \mathcal{P}_n . Because of condition (\mathbf{iv}) , \mathcal{P}_n has no reflex angle so that:

$$\sum_{k=1}^n \beta_k = 2\pi - \text{Area}(\mathcal{P}_n) < 2\pi,$$

where β_1, \dots, β_n denote the exterior angles of \mathcal{P}_n , (that is, $\beta_k := \pi - \alpha_k$ for all $k \in \{1, \dots, n\}$). Now, for any $k \in \{1, \dots, n\}$, $\beta_k > 0$ can be regarded as the interior angle of one of the desired convex polygon f_{i_k} (corresponding to a unit vector n_{i_k} that is a vertex of \mathcal{P}_n) at the vertex of f_{i_k} that corresponds to \mathcal{P}_n . The above inequality, which says that

the sum $\sum_{i=1}^n \beta_k$ of these angles, is less than 2π can be regarded as a *nonnegative curvature condition* that is satisfied from our conditions. Therefore, taking into account condition **(iii)**, the convex polygons f_{i_k} that correspond to the unit vectors n_{i_k} that are the vertices of \mathcal{P}_n can be assembled (by gluing together their sides that correspond to a same edge of \mathcal{P}_n) to form a corner (of a 3-dimensional polytope), the spherical representation of which corresponds to \mathcal{P}_n in \mathbb{S}^2 . Here, the "convexity" at the corner is of course due to the nonnegative curvature condition.

Thus *all the corners of the desired polytope P are well-defined and unique up to translations in \mathbb{R}^3 .*

Starting from any of these corners, we can construct by induction the desired polytope P , which is combinatorially equivalent to Q^* and satisfies the set of conditions **{(i), (ii), (iii), (iv), (v)}**, by assembling at each step, an adjacent corner to the part of P that has already been constructed. Here, by "an adjacent corner" we mean "a corner that is not included in the part of P already constructed but sharing two facets with it". At each step, the spherical representation of the part of P that is constructed is controlled by condition **(v)**, and the construction can continue until completion since we made sure that all the pieces had the required shape and dimensions. Here, it is worth noting that, as soon as the position of the first corner is fixed (with, for example, its vertex placed at the origin O), the position of any other vertex S of P is deduced from that of O by a succession of translations from a vertex of P to another: consider any succession of adjacent regions of $\mathbb{S}^2 \setminus G$ from that corresponding to O to that corresponding to S (two regions of $\mathbb{S}^2 \setminus G$ are said to be adjacent if their boundaries share an edge of G) and note that each crossing on \mathbb{S}^2 from one region \mathcal{R}_i of $\mathbb{S}^2 \setminus G$ to an adjacent one \mathcal{R}_j corresponds on P to the translation from a vertex to another by a translation by a vector \vec{v}_{ij} whose direction is determined by the arc of great circle γ_{ij} separating the two regions on \mathbb{S}^2 (\vec{v}_{ij} is orthogonal to it and oriented in the sense of the crossing) and whose norm $\|\vec{v}_{ij}\|$ is the length l_{ij} corresponding to γ_{ij} . Of course, thanks to condition **(iv)**, the final position of the vertex (i.e. that of S) does not depend on the succession of adjacent regions that has been considered. ■

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