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# Some new bounds for ratio functions of trigonometric and hyperbolic functions

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**Abstract.** This paper is devoted to obtain sharp bounds for the ratio functions  $\cos(x)/\cosh(x)$  and  $\sin(x)/\sinh(x)$ . The proofs are based on the use of a refinement of a special case of the Bernoulli inequality (which can be of independent interest) and infinite products.

**Keywords:** Trigonometric function; Bernoulli inequality; infinite product; exponential bounds.

**Mathematics Subject Classification(2010):** 26D07, 33B10, 33B20.

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## 1 Introduction

Various kinds of connections between hyperbolic and trigonometric functions have always received a lot of attention by the mathematicians. There is the case for ratio functions involving hyperbolic functions in the numerator and trigonometric functions in the denominator, or vice-versa. For instance, concerning the ratio function  $\cosh(x)/\cos(x)$ , see [3] and [2] and for the ratio function  $\sinh(x)/\sin(x)$ , see [2], [4] and [8], and the references therein. On the other side, let us notice that the graphs of ratio functions  $\cos(x)/\cosh(x)$  and  $\sin(x)/\sinh(x)$  are somewhat bell shaped. Due to the importance of bell shaped curves in probability theory and statistics, all their mathematical properties deserve to be studied. An important aspect is to explore tractable bounds for these functions, which can be done by using existing independent results on bounds for  $\cosh(x)$ ,  $\sinh(x)$ ,  $\cos(x)$  and  $\sin(x)$ . In this study, we provide new exponential and polynomial bounds to the ratio functions mentioned above, improving immediate bounds derived to those established in the literature. They are obtained by the use of a special type of Bernoulli inequality, improving the well-known Bernoulli inequality on the unit interval.

The organization of our paper is as follows. In Section 2, a refinement of the Bernoulli inequality in the special case is proposed. In Section 3, exponential and polynomial bounds for  $\cosh(x)/\cos(x)$  (and  $\cos(x)/\cosh(x)$ ) as well as exponential and polynomial bounds for  $\sinh(x)/\sin(x)$  (and  $\sin(x)/\sinh(x)$ ) are established.

## 2 A refinement of the Bernoulli inequality

First of all, let us present the so-called Bernoulli inequality in the case where  $u, v \in (0, 1)$ .

**Theorem 1.** [*A Bernoulli inequality*] For  $u, v \in (0, 1)$ , we have

$$1 - uv \geq (1 - v)^u.$$

We refer to [7, Theorem A] and [5], [6] for generalized version and other variants of Bernoulli inequality. Under the same context, we establish a more sharp lower bound, which can be of independent interest. It is presented in Proposition 1 below.

**Proposition 1.** For  $u, v \in (0, 1)$ , we have

$$1 - uv \geq f(u, v)(1 - v)^u,$$

where  $f(u, v)$  denotes the function defined by

$$f(u, v) = \frac{1 + uv}{(1 + v)^u},$$

and we have  $f(u, v) \geq 1$  (implying the standard Bernoulli inequality for  $u, v \in (0, 1)$ ).

*Proof of Proposition 1.* Using the logarithmic series expansion, for  $u, v \in (0, 1)$ , we have

$$\begin{aligned} \ln\left(\frac{1 + uv}{1 - uv}\right) &= \ln(1 + uv) - \ln(1 - uv) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}(uv)^n}{n} + \sum_{n=1}^{+\infty} \frac{(uv)^n}{n} \\ &= \sum_{n=1}^{+\infty} \frac{[(-1)^{n-1} + 1](uv)^n}{n} = 2 \sum_{k=0}^{+\infty} \frac{(uv)^{2k+1}}{2k+1}. \end{aligned}$$

For  $u, v \in (0, 1)$  and  $k \geq 0$ , we have  $u^{2k+1} \leq u$ , so  $(uv)^{2k+1} \leq uv^{2k+1}$ . Therefore

$$2 \sum_{k=0}^{+\infty} \frac{(uv)^{2k+1}}{2k+1} \leq u \left( 2 \sum_{k=0}^{+\infty} \frac{v^{2k+1}}{2k+1} \right) = u \ln\left(\frac{1 + v}{1 - v}\right).$$

Hence

$$\ln \left( \frac{1+uv}{1-uv} \right) \leq u \ln \left( \frac{1+v}{1-v} \right).$$

Then composing by the exponential function, we obtain

$$\frac{1+uv}{1-uv} \leq \left( \frac{1+v}{1-v} \right)^u,$$

which is equivalent to the desired inequality after rearrangement, i.e.

$$1-uv \geq f(u,v)(1-v)^u.$$

To prove that  $f(u,v) \geq 1$ , we need the general version of the Bernoulli inequality: for  $x > -1$  and  $r \in (0,1)$ , we have  $1+rx \geq (1+x)^r$  (see [7, Theorem A]). By applying it with  $r = u \in (0,1)$  and  $x = v \in (0,1)$ , we obtain the desired result.  $\square$

Therefore, Proposition 1 gives a more sharp lower bound for  $1-uv$  than the classical Bernoulli inequality.

In the sequel, we use this result to provide new bounds for ratio of trigonometric and hyperbolic functions.

### 3 Some new bounds

#### 3.1 Bounds for ratio of cosine and hyperbolic cosine functions

The result below presents an exponential upper bound for the ratio function  $\cosh(x)/\cos(x)$ .

**Proposition 2.** For  $\alpha \in (0, \pi/2)$  and  $x \in (0, \alpha)$ , we have

$$\frac{\cosh(x)}{\cos(x)} \leq e^{\beta x^2},$$

with  $\beta = [\ln(\cosh(\alpha))/\cos(\alpha)]/\alpha^2$ .

*Proof of Proposition 2.* First of all, let us recall some known results on infinite product series. We have the following expressions for  $\cosh(x)$  and  $\cos(x)$ : for all  $x \in \mathbb{R}$ , we have

$$\cosh(x) = \prod_{k=1}^{+\infty} \left( 1 + \frac{4x^2}{\pi^2(2k-1)^2} \right), \quad \cos(x) = \prod_{k=1}^{+\infty} \left( 1 - \frac{4x^2}{\pi^2(2k-1)^2} \right).$$

So we can express the ratio function  $\cosh(x)/\cos(x)$  as

$$\frac{\cosh(x)}{\cos(x)} = \prod_{k=1}^{+\infty} \left( \frac{1 + 4x^2/[\pi^2(2k-1)^2]}{1 - 4x^2/[\pi^2(2k-1)^2]} \right).$$

Secondly, note that Proposition 1 can be reformulated as follows: for  $u, v \in (0, 1)$ , we have

$$\frac{1+uv}{1-uv} \leq \left(\frac{1+v}{1-v}\right)^u. \quad (3.1)$$

It follows from these results that for  $\alpha \in (0, \pi/2)$  we have

$$\begin{aligned} \frac{\cosh(x)}{\cos(x)} &= \prod_{k=1}^{+\infty} \left( \frac{1 + (4\alpha^2/[\pi^2(2k-1)^2])(x^2/\alpha^2)}{1 - (4\alpha^2/[\pi^2(2k-1)^2])(x^2/\alpha^2)} \right) \\ &\leq \prod_{k=1}^{+\infty} \left( \frac{1 + 4\alpha^2/[\pi^2(2k-1)^2]}{1 - 4\alpha^2/[\pi^2(2k-1)^2]} \right)^{x^2/\alpha^2} = \left[ \prod_{k=1}^{+\infty} \left( \frac{1 + 4\alpha^2/[\pi^2(2k-1)^2]}{1 - 4\alpha^2/[\pi^2(2k-1)^2]} \right) \right]^{x^2/\alpha^2} \\ &= \left[ \frac{\cosh(\alpha)}{\cos(\alpha)} \right]^{x^2/\alpha^2} = e^{\beta x^2}. \end{aligned}$$

This ends the proof of Proposition 2.  $\square$

**Note:** Let us observe that we can write  $\beta$  as  $\beta = \beta_1 - \beta_2$  with  $\beta_1 = \ln(\cosh(\alpha))/\alpha^2$  and  $\beta_2 = \ln(\cos(\alpha))/\alpha^2$ . On the other hand, it follows from [1, Proposition 2 and the related note] that, for

$$e^{\beta_1 x^2} \leq \cosh(x) \leq e^{x^2/2}, \quad e^{\beta_2 x^2} \leq \cos(x) \leq e^{-x^2/2},$$

which implies that

$$\frac{\cosh(x)}{\cos(x)} \leq e^{\gamma x^2},$$

with  $\gamma = (1/2) - \beta_2$ . So Proposition 2 is not a trivial consequence of these results, and give a more sharp upper bound since  $\beta < \gamma$ .

**Note:** Proposition 2 can be reformulated in term of lower bound: For  $\alpha \in (0, \pi/2)$  and  $x \in (0, \alpha)$ , we have

$$\frac{\cos(x)}{\cosh(x)} \geq e^{-\beta x^2}.$$

An interest of the upper bound in Proposition 3 in comparison to the one in Proposition 2 is to not depend on the chosen  $\alpha$ .

Using similar arguments, we can provide a ratio polynomial upper bound for the ratio function  $\cosh(x)/\cos(x)$ .

**Proposition 3.** For  $x \in (0, \pi/2)$ , we have

$$\frac{\cosh(x)}{\cos(x)} \leq \left( \frac{\pi^2 + 4x^2}{\pi^2 - 4x^2} \right)^{\pi^2/8}.$$

*Proof of Proposition 3.* Using the infinite product expression of  $\cosh(x)/\cos(x)$  and (3.1), with the equality  $\sum_{k=1}^{+\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$ , we obtain

$$\begin{aligned} \frac{\cosh(x)}{\cos(x)} &= \prod_{k=1}^{+\infty} \left( \frac{1 + (4x^2/\pi^2)[1/(2k-1)^2]}{1 - (4x^2/\pi^2)[1/(2k-1)^2]} \right) \\ &\leq \prod_{k=1}^{+\infty} \left( \frac{1 + 4x^2/\pi^2}{1 - 4x^2/\pi^2} \right)^{1/(2k-1)^2} = \left( \frac{1 + 4x^2/\pi^2}{1 - 4x^2/\pi^2} \right)^{\sum_{k=1}^{+\infty} 1/(2k-1)^2} \\ &= \left( \frac{1 + 4x^2/\pi^2}{1 - 4x^2/\pi^2} \right)^{\pi^2/8} = \left( \frac{\pi^2 + 4x^2}{\pi^2 - 4x^2} \right)^{\pi^2/8}. \end{aligned}$$

Proposition 3 is proved. □

We provide a graphical illustration of Proposition 3 in Figure 1.

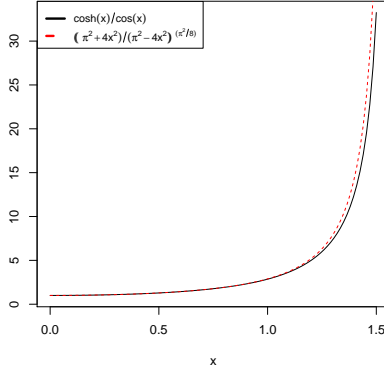


Figure 1: Graphs of the functions in Proposition 3 for  $x \in (0, 1.5)$ .

**Note:** It is proved in [9, Theorems 2 and 5] that for  $x \in (0, \pi/2)$ ,  $\epsilon \geq 1$  and  $\theta \geq \pi^2/16$ , we have

$$\left( \frac{\pi^2 - 4x^2}{\pi^2 + 4x^2} \right)^\epsilon \leq \cos(x), \quad \cosh(x) \leq \left( \frac{\pi^2 + 4x^2}{\pi^2 - 4x^2} \right)^\theta.$$

This implies that

$$\frac{\cosh(x)}{\cos(x)} \leq \left( \frac{\pi^2 + 4x^2}{\pi^2 - 4x^2} \right)^{\theta + \epsilon}$$

and since  $\pi^2 + 4x^2/(\pi^2 - 4x^2) \geq 1$  and  $\theta + \epsilon \geq 1 + \pi^2/16 \geq \pi^2/8$ , our upper bound is more sharp.

**Note:** Proposition 3 can be reformulated in term of lower bound: For  $x \in (0, \pi/2)$ , we have

$$\frac{\cos(x)}{\cosh(x)} \geq \left( \frac{\pi^2 - 4x^2}{\pi^2 + 4x^2} \right)^{\pi^2/8}.$$

Similar results to Proposition 2 and 3 can be proved for the ratio function  $\sinh(x)/\sin(x)$ . This is developed in the subsection below.

### 3.2 Bounds for ratio of sine and hyperbolic sine functions

**Proposition 4.** For  $\alpha \in (0, \pi)$  and  $x \in (0, \alpha)$ , we have

$$\frac{\sinh(x)}{\sin(x)} \leq e^{\zeta x^2},$$

with  $\zeta = [\ln(\sinh(\alpha)/\sin(\alpha))]/\alpha^2$ .

*Proof of Proposition 4.* We use the following expressions for  $\sinh(x)/x$  and  $\sin(x)/x$ : for all  $x \in \mathbb{R}$ , we have

$$\frac{\sinh(x)}{x} = \prod_{k=1}^{+\infty} \left( 1 + \frac{x^2}{\pi^2 k^2} \right), \quad \frac{\sin(x)}{x} = \prod_{k=1}^{+\infty} \left( 1 - \frac{x^2}{\pi^2 k^2} \right).$$

Hence

$$\frac{\sinh(x)}{\sin(x)} = \prod_{k=1}^{+\infty} \left( \frac{1 + x^2/(\pi^2 k^2)}{1 - x^2/(\pi^2 k^2)} \right).$$

It follows from (3.1) that for  $\alpha \in (0, \pi)$  we have

$$\begin{aligned} \frac{\sinh(x)}{\sin(x)} &= \prod_{k=1}^{+\infty} \left( \frac{1 + (\alpha^2/(\pi^2 k^2))(x^2/\alpha^2)}{1 - (\alpha^2/(\pi^2 k^2))(x^2/\alpha^2)} \right) \\ &\leq \prod_{k=1}^{+\infty} \left( \frac{1 + \alpha^2/(\pi^2 k^2)}{1 - \alpha^2/(\pi^2 k^2)} \right)^{x^2/\alpha^2} = \left[ \prod_{k=1}^{+\infty} \left( \frac{1 + \alpha^2/(\pi^2 k^2)}{1 - \alpha^2/(\pi^2 k^2)} \right) \right]^{x^2/\alpha^2} \\ &= \left[ \frac{\sinh(\alpha)}{\sin(\alpha)} \right]^{x^2/\alpha^2} = e^{\zeta x^2}. \end{aligned}$$

This ends the proof of Proposition 4. □

**Note:** Proposition 4 can be reformulated in term of lower bound: For  $\alpha \in (0, \pi/2)$  and  $x \in (0, \alpha)$ , we have

$$\frac{\sin(x)}{\sinh(x)} \geq e^{-\zeta x^2}.$$

Using similar arguments, we can get a ratio polynomial upper bound for the ratio function  $\sinh(x)/\sin(x)$ .

**Proposition 5.** For  $x \in (0, \pi/2)$ , we have

$$\frac{\sinh(x)}{\sin(x)} \leq \left( \frac{\pi^2 + x^2}{\pi^2 - x^2} \right)^{\pi^2/6}.$$

*Proof of Proposition 3.* Using the infinite product expression of  $\cosh(x)/\cos(x)$  and (3.1), with the equality  $\sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ , we obtain

$$\begin{aligned} \frac{\sinh(x)}{\sin(x)} &= \prod_{k=1}^{+\infty} \left( \frac{1 + (x^2/\pi^2)(1/k^2)}{1 - (x^2/\pi^2)(1/k^2)} \right) \\ &\leq \prod_{k=1}^{+\infty} \left( \frac{1 + x^2/\pi^2}{1 - x^2/\pi^2} \right)^{1/k^2} = \left( \frac{1 + x^2/\pi^2}{1 - x^2/\pi^2} \right)^{\sum_{k=1}^{+\infty} 1/k^2} \\ &= \left( \frac{1 + x^2/\pi^2}{1 - x^2/\pi^2} \right)^{\pi^2/6} = \left( \frac{\pi^2 + x^2}{\pi^2 - x^2} \right)^{\pi^2/6}. \end{aligned}$$

Proposition 5 is proved. □

We provide a graphical illustration of Proposition 5 in Figure 2.

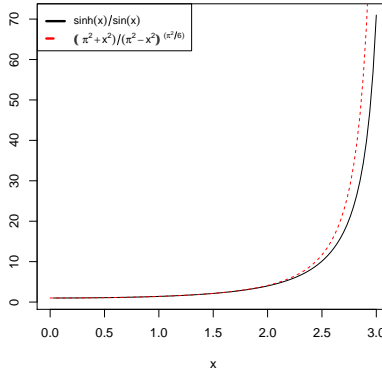


Figure 2: Graphs of the functions in Proposition 5 for  $x \in (0, 3)$ .

**Note:** It is proved in [9, Theorems 1 and 4] that for  $x \in (0, \pi)$ ,  $\xi \geq 1$  and  $\phi \geq \pi^2/12$ , we have

$$\left( \frac{\pi^2 - x^2}{\pi^2 + x^2} \right)^\xi \leq \frac{\sin(x)}{x}, \quad \frac{\sinh(x)}{x} \leq \left( \frac{\pi^2 + x^2}{\pi^2 - x^2} \right)^\phi.$$

Hence

$$\frac{\sinh(x)}{\sin(x)} \leq \left( \frac{\pi^2 + x^2}{\pi^2 - x^2} \right)^{\phi + \xi}$$



and since  $\pi^2 + x^2/(\pi^2 - x^2) \geq 1$  and  $\phi + \xi \geq 1 + \pi^2/12 \geq \pi^2/6$ , our upper bound is more sharp.

**Note:** Proposition 5 can be reformulated in term of lower bound: For  $x \in (0, \pi/2)$ , we have

$$\frac{\sin(x)}{\sinh(x)} \geq \left( \frac{\pi^2 - x^2}{\pi^2 + x^2} \right)^{\pi^2/6}.$$

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