Convergence of numerical schemes for a conservation equation with convection and degenerate diffusion

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Abstract

The approximation of problems with linear convection and degenerate nonlinear diffusion, which arise in the framework of the transport of energy in porous media with thermodynamic transitions, is done using a θ-scheme based on the centred gradient discretisation method. The convergence of the numerical scheme is proved, although the test functions which can be chosen are restricted by the weak regularity hypotheses on the convection field, owing to the application of a discrete Gronwall lemma and a general result for the time translate in the gradient discretisation setting. Some numerical examples, using both the Control Volume Finite Element method and the Vertex Approximate Gradient scheme, show the role of θ for stabilising the scheme.

keywords: linear convection, degenerate diffusion, gradient discretisation method, θ-scheme.

1 Introduction

The development of geothermal energy leads to increasing needs for simulating the displacement of the water in a porous medium, accounting for the liquid-vapour change of phase [4]. This is achieved by writing the system of the conservation equation of the mass of water and that of the conservation of energy, together with a system of equations and inequalities expressing the thermodynamic equilibrium between the two phases when they are simultaneously present [7]. Let us consider a simplification of this system, which may be considered as a reasonable approximation in some physical cases:

$$\partial_t(\rho_l(1-S)+\rho_v S) - \mathrm{div}\left(\frac{K}{\mu}(\rho_l(1-S)+\rho_v S)\nabla P\right) = w, \tag{1}$$

$$\partial_t(e_l(1-S)+e_v S) - \mathrm{div}\left(\frac{K}{\mu}(e_l(1-S)+e_v S)\nabla P + \Lambda\nabla T\right) = f, \tag{2}$$

$$(T < T_e \text{ and } S = 0) \text{ or } (T = T_e \text{ and } 0 \le S \le 1) \text{ or } (T > T_e \text{ and } S = 1). \tag{3}$$

In the preceding system, the indices l, v respectively stand for the liquid and vapour phases, S is the saturation of the vapour phase (hence $1-S$ is that of the liquid phase), P the pressure assumed to be common for the two fluids (we neglect the capillary pressure), and for $\alpha = l, v$, ρ_α and e_α are respectively the density and the internal energy per mass unit of the phase α, assumed to be given functions of T. In System (2)-(3), the mobilities of the phases l and v are assumed to be equal to $(1-S)/\mu$ and S/μ, assuming the same viscosity μ for the two phases, and K is the absolute permeability field. The thermal conductivity is denoted by Λ. The right hand sides w and f are respectively the source terms of water and energy. The thermodynamic equilibrium between

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the two fluid phases l and v is assumed to hold when the temperature is equal to the equilibrium temperature T_e, assumed to be a constant; otherwise, one of the two fluid phases is missing.
Now denoting by $\bar{u} = e_l(1-S) + e_v S$, we notice that, from (3), it is possible to express $T - T_e$ as a function ζ of $\bar{u}$. For example, if $e_l = C_l(T - T_e)$ and $e_v = L + C_v(T - T_e)$, where L is the latent heat and C_α the thermal capacity of phase α, then there holds,

$$\zeta(\bar{u}) = \begin{cases} \dfrac{\bar{u}}{C_l} & \bar{u} < 0, \\ 0 & 0 \le \bar{u} \le L, \\ \dfrac{\bar{u} - L}{C_v} & \bar{u} > L. \end{cases}$$

Therefore, denoting by $\vec{\boldsymbol{v}} = -\frac{K}{\mu}\nabla P$ and only focusing on the energy conservation (we assume that the water conservation equation (1) is in some way decoupled from this problem), we consider the following linear convection – degenerate diffusion problem, issued from (2)-(3):

$$\partial_t \bar{u}(\boldsymbol{x},t) \; + \; \mathrm{div}(\, \bar{u}(\boldsymbol{x},t)\vec{\boldsymbol{v}}(\boldsymbol{x},t) - \Lambda(\boldsymbol{x})\nabla\zeta(\bar{u}(\boldsymbol{x},t))\,) \; = \; f(\boldsymbol{x},t), \text{ for a.e. } (\boldsymbol{x},t) \in \Omega \times (0,T), \tag{4a}$$

with the initial condition:

$$\bar{u}(\boldsymbol{x},0) = u_{\mathrm{ini}}(\boldsymbol{x}), \text{ for a.e. } \boldsymbol{x} \in \Omega, \tag{4b}$$

together with the homogeneous Dirichlet boundary condition:

$$\zeta(\bar{u}(\boldsymbol{x},t)) = 0 \text{ on } \partial\Omega \times (0,T). \tag{4c}$$

In (4), we consider the following hypotheses.

- Ω is an open bounded connected polyhedral subset of $\mathbb{R}^d$, $d \in \mathbb{N}^\star$ and $T > 0$ is now the final time, (5a)
- $u_{\mathrm{ini}} \in L^2(\Omega)$, (5b)
- $\vec{\boldsymbol{v}} \in L^\infty(\Omega \times (0,T))$, (5c)
- Λ is a measurable function from Ω to the set of $d \times d$ symmetric matrices and there exist $\underline{\lambda}, \overline{\lambda} > 0$ such that, for a.e. $\boldsymbol{x} \in \Omega$, $\Lambda(\boldsymbol{x})$ has eigenvalues in $[\underline{\lambda}, \overline{\lambda}]$, (5d)
- $f \in L^2(\Omega \times (0,T))$, (5e)
- $\zeta \in C^0(\mathbb{R})$ is non–decreasing, Lipschitz continuous with constant $\overline{L}_\zeta$ and s.t. $\zeta(0) = 0$, and (5f)
  $|\zeta(s)| \ge \underline{L}_\zeta |s| - C_\zeta$ for all $s \in \mathbb{R}$ for some given values $\underline{L}_\zeta, C_\zeta \in (0, +\infty)$. (5g)

We remark that (5f) and (5g) imply $0 < \underline{L}_\zeta \le \overline{L}_\zeta$. Note that, inspired by the properties which can be expected from (1), Hypothesis (5c) only prescribes poor regularity properties for the velocity, that we only assume to be bounded, without regularity hypotheses on its derivatives. This weak regularity at least implies that Problem (4) must be considered in a weak sense (see (6) below). Hypotheses (5f)-(5g) on ζ are classically satisfied in the framework described in the beginning of this section. The assumptions on Λ are taking into account heterogeneous and anisotropic porous media.
A function $\bar{u}$ is said to be a weak solution of Problem (4) if the following holds:

$$\begin{aligned} &\bar{u} \in L^2(\Omega \times (0,T)),\ \zeta(\bar{u}) \in L^2(0,T; H_0^1(\Omega)),\ \forall \varphi \in C_c^\infty(\Omega \times [0,T)), \\ &-\int_0^T \int_\Omega \bar{u}(\boldsymbol{x},t)\partial_t\varphi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}t - \int_\Omega u_{\mathrm{ini}}(\boldsymbol{x})\varphi(\boldsymbol{x},0)\mathrm{d}\boldsymbol{x} \\ &+\int_0^T \int_\Omega (\Lambda(\boldsymbol{x})\nabla\zeta(\bar{u})(\boldsymbol{x},t) - \bar{u}(\boldsymbol{x},t)\vec{\boldsymbol{v}}(\boldsymbol{x},t)) \cdot \nabla\varphi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}t = \int_0^T \int_\Omega f(\boldsymbol{x},t)\varphi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}t, \end{aligned} \tag{6}$$

where we denote by $C_c^\infty(\Omega \times [0,T))$ the set of the restrictions of functions of $C_c^\infty(\Omega \times (-\infty, T))$ to $\Omega \times [0,T)$.

Let us first comment the question of the existence and uniqueness of a solution to (6). Since the case where ζ is constant on an interval is included in our study, (6) includes the case of linear scalar hyperbolic equations whose solutions are valued in such an interval (such cases occur in the numerical examples of Section 4). So in this framework, the lack of regularity of $\vec{\boldsymbol{v}}$ prevents from applying the existence and uniqueness results of the literature. In [8], existence is proved with very low regularity for $\vec{\boldsymbol{v}}$, but uniqueness only holds if $\vec{\boldsymbol{v}} \in W^{1,1}$ and $\mathrm{div}\vec{\boldsymbol{v}} \in L^\infty$, and in [2], the preceding results are extended to the case where $\vec{\boldsymbol{v}} \in BV$, and $\mathrm{div}\vec{\boldsymbol{v}} \in L^\infty$. Hence in this paper, we only provide an existence result (obtained here through the convergence of a numerical scheme); we are not able to prove a uniqueness result for a solution to (6). We follow in this paper the same mathematical steps for this proof as the ones which would be considered in the study of the convergence of u_ε, solution in a weak sense to the problem

$$\partial_t u_\varepsilon + \mathrm{div}\big(u_\varepsilon \vec{\boldsymbol{v}}(\boldsymbol{x},t) - \Lambda \nabla \zeta(u_\varepsilon)\big) = f_\varepsilon, \text{ in } \Omega \times (0,T),$$

together with the same initial and boundary conditions as those included in Problem (4), assuming that f_ε converges to f in $L^2(\Omega \times (0,T))$ as $\varepsilon \to 0$. In the case where the velocity $\vec{\boldsymbol{v}}$ is sufficiently regular (for example, $\vec{\boldsymbol{v}} \in H^1(\Omega)^d$, or even more regular as in [6]), one multiplies the equation by $\eta'(u_\varepsilon)$, in order to get estimates on $\eta(u_\varepsilon)$, for well-chosen convex functions η. Then it is possible, under suitable boundary conditions, to get a bound on the convection term. In the present situation, this choice for the test function does not lead to any estimate, and the only test function that we can take is $\zeta(u_\varepsilon)$, using the equation which provides a control on $\nabla\zeta(u_\varepsilon)$ in $L^2(\Omega \times (0,T))^d$. Let us detail this point. We introduce the function $Z(s) = \int_0^s \zeta(x)\mathrm{d}x$ (as in (14)), and we obtain, after integration on $\Omega \times (0,T)$, and an integrate by parts,

$$\int_\Omega Z(u_\varepsilon(\boldsymbol{x},T))\mathrm{d}\boldsymbol{x} - \int_\Omega Z(u_{\mathrm{ini}})\mathrm{d}\boldsymbol{x} + \int_0^T \int_\Omega \big(\Lambda\nabla\zeta(u_\varepsilon) - u_\varepsilon \vec{\boldsymbol{v}}(\boldsymbol{x},t)\big)\cdot\nabla\zeta(u_\varepsilon)\mathrm{d}\boldsymbol{x}\mathrm{d}t = \int_0^T \int_\Omega f_\varepsilon \zeta(u_\varepsilon)\mathrm{d}\boldsymbol{x}\mathrm{d}t. \quad (7)$$

Thanks to Hypothesis (5g), we show that $\int_\Omega Z(u_\varepsilon(\boldsymbol{x},T))\mathrm{d}\boldsymbol{x} \geq C(\|u_\varepsilon(\cdot,T)\|^2_{L^2(\Omega)} - 1)$ for some constant C. We then write, thanks to Young's inequality,

$$\left|\int_0^T \int_\Omega u_\varepsilon \vec{\boldsymbol{v}}(\boldsymbol{x},t)\cdot\nabla\zeta(u_\varepsilon)\mathrm{d}\boldsymbol{x}\mathrm{d}t\right| \leq \alpha \int_0^T \|u_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)}\mathrm{d}t + \beta\|\nabla\zeta(u_\varepsilon)\|^2_{L^2(\Omega\times(0,T))},$$

with β small enough for controlling the second term by $\int_0^T\int_\Omega \Lambda\nabla\zeta(u_\varepsilon)\cdot\nabla\zeta(u_\varepsilon)\mathrm{d}\boldsymbol{x}\mathrm{d}t$, and we write the Cauchy-Schwarz and the Poincaré inequality for handling the right hand side of (7). This leads to an equation under the form

$$\|u_\varepsilon(\cdot,T)\|^2_{L^2(\Omega)} \leq A\int_0^T \|u_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)}\mathrm{d}t + B,$$

allowing for the application of Gronwall's lemma. Gathering these results, we raise the following estimates:

(E1) an estimate on $\zeta(u_\varepsilon)$ in $L^2(0,T;H^1_0(\Omega))$,

(E2) an estimate on u_ε in $L^\infty(0,T;L^2(\Omega))$,

(E3) an estimate on $\partial_t u_\varepsilon$ in $L^2(0,T;H^{-1}(\Omega))$.

Once these estimates are proved, it only remains to prove that, under the extraction of a subsequence from a sequence of approximate solutions, u_ε converges to a weak solution of the problem. One remarkable idea in [1] is the use of the following result:

$$\|\zeta(\bar{u}_\varepsilon(\cdot,\cdot+\tau)) - \zeta(\bar{u}_\varepsilon)\|^2_{L^2(\Omega\times(0,T-\tau))} \leq \ \tau\ 2\overline{L}_\zeta \|\partial_t u_\varepsilon\|_{L^2(0,T;H^{-1}(\Omega))}\|\zeta(u_\varepsilon)\|_{L^2(0,T;H^1_0(\Omega))}$$

(we give a discrete equivalent of this result in Lemma B.1). This result, in addition to (E1) and (E3), allows to apply Kolmogorov's theorem, and therefore to extract a sequence $(\zeta(\bar{u}_{\varepsilon_m}))_{m\in\mathbb{N}}$ which converges to some function χ in $L^2(\Omega\times(0,T))$. Estimate (E2) allows to extract a subsequence from the preceding one such that there exists $\bar{u} \in L^\infty(0,T;L^2(\Omega))$ with $(u_{\varepsilon_m})_{m\in\mathbb{N}}$ weakly converges to $\bar{u}$. Then, thanks to the monotonicity of ζ, Minty's trick provides that $\chi = \zeta(\bar{u})$.

In the convergence part of this paper, we therefore derive discrete equivalents of (E1)-(E2)-(E3), from similar computations only resulting from the multiplication of the discrete scheme by $\zeta(u_\varepsilon)$. Note that, in any case, this choice is also the only one which could provide an estimate at the discrete level, even in the case of more regular $\vec{\boldsymbol{v}}$, since our aim is to approximate Problem (6) in some framework which includes, in addition to conforming finite elements (see [3], [10], [18] for application to degenerate parabolic problems), non-conforming methods such as mixed finite element methods, or discontinuous Galerkin methods, and other more recent methods [9]. In such a general framework, the continuous gradient operator is replaced by a discrete one, denoted by $\nabla_{\mathcal{D}}$. Then Stampacchia's result [19], which allows to write in the continuous case $\Lambda\nabla\zeta(u_\varepsilon)\cdot\nabla\eta(u_\varepsilon) = \Lambda\nabla\varphi(u_\varepsilon)\cdot\nabla\varphi(u_\varepsilon)$ with $(\varphi')^2 = \zeta'\eta'$, does not hold in the discrete framework (note that Stampacchia's result has a discrete counterpart if the scheme is based on the two-point flux approximation, but in this case, the meshes are restricted, and Λ should be isotropic, see the discussion in [13]).

We emphasize that the scheme which is considered below includes a parameter θ, such that, if $\theta = 0$, the convection term is explicit, if $\theta = 0.5$, the convection term is centred in time, and if $\theta = 1.$, the convection term is implicit. We consider in this paper the case $\theta \in \mathbb{R}$ since we show in the numerical examples that values $\theta > 1$ lead to a kind of stabilisation in the case where oscillations occur for $\theta \in [0, 1]$. In all cases, the degenerate diffusion term is taken implicit. We recall that, for a centred finite volume scheme for the convection, the θ-scheme is L^2- stable only if $\theta \geq 0.5$. We prove below that this limitation does not apply in the framework of this paper. Nevertheless, for any $\theta \in \mathbb{R}$, the degenerate diffusion is sufficient for leading to weakly convergent schemes and in some particular cases to a strong convergence property.

This paper is organised as follows. We first apply the gradient discretisation tools to the continuous Problem in Section 2, and derive some estimates, which are used in Section 3 for the convergence analysis. Finally in Section 4, numerical examples show the behaviour of the Control Volume Finite Element scheme (CVFE) and the Vertex Approximate Gradient (VAG) scheme [14] which present some interesting characteristics for coupled flows in porous media.

2 Approximation by space-time gradient discretisations

In the same manner as in [12], we perform the discretisation of Problem (6) with replacing the continuous operators by discrete ones, following the Gradient Discretisation Method [9]. Let $\mathcal{D}_T := (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}}, \mathcal{I}_{\mathcal{D}}, (t^{(n)})_{n=0,\dots,N})$ be a space-time discretisation in the sense of [9, Definition 4.1], such that $\mathcal{D} := (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ is a space gradient discretisation for Dirichlet boundary conditions in the sense of [9, Definition 2.1] (we then denote by $\delta t_{\mathcal{D}} = \max_{n=0,\dots,N-1} t^{(n+1)} - t^{(n)}$). Then, [9, Definition 2.1] specifies that $\|\cdot\|_{\mathcal{D}} := \|\nabla_{\mathcal{D}}\cdot\|_{L^2(\Omega)^d}$ is a norm on $X_{\mathcal{D},0}$, and the following quantities: $C_{\mathcal{D}} \in [0, +\infty)$, $S_{\mathcal{D}} : H_0^1(\Omega) \to [0, +\infty)$ and $W_{\mathcal{D}}$: $H_{\rm div}(\Omega) \to [0, +\infty)$, are defined by

$$C_{\mathcal{D}} = \max_{v \in X_{\mathcal{D},0}\setminus\{0\}} \frac{\|\Pi_{\mathcal{D}} v\|_{L^2(\Omega)}}{\|v\|_{\mathcal{D}}}, \tag{8}$$

$$\forall \varphi \in H_0^1(\Omega), \quad S_{\mathcal{D}}(\varphi) = \min_{v \in X_{\mathcal{D},0}} \left(\|\Pi_{\mathcal{D}} v - \varphi\|_{L^2(\Omega)} + \|\nabla_{\mathcal{D}} v - \nabla\varphi\|_{L^2(\Omega)^d} \right), \tag{9}$$

$$\forall \boldsymbol{\varphi} \in H_{\rm div}(\Omega),\ W_{\mathcal{D}}(\boldsymbol{\varphi}) = \max_{u \in X_{\mathcal{D},0}\setminus\{0\}} \frac{1}{\|u\|_{\mathcal{D}}} \left| \int_\Omega (\nabla_{\mathcal{D}} u(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(\boldsymbol{x}) + \Pi_{\mathcal{D}} u(\boldsymbol{x}) {\rm div}\boldsymbol{\varphi}(\boldsymbol{x}))\, {\rm d}\boldsymbol{x} \right|. \tag{10}$$

These quantities are respectively involved in [9, Definition 2.2] of coercivity, [9, Definition 2.4] of GD-consistency and in [9, Definition 2.5] of limit-conformity.

We assume that $\Pi_{\mathcal{D}}$ is a piecewise constant function reconstruction in the sense of [9, Definition 2.12]. Let

$\theta \in \mathbb{R}$ be given. The scheme consists in finding $u := (u^{(n)})_{n=0,\dots,N}$ such that:

$$\left\{\begin{array}{l} u^{(0)} = \mathcal{I}_{\mathcal{D}} u_{\rm ini} \in X_{\mathcal{D},0}, \\ u^{(n+1)} \in X_{\mathcal{D},0},\ \delta_{\mathcal{D}}^{(n+\frac{1}{2})} u = \Pi_{\mathcal{D}} \dfrac{u^{(n+1)} - u^{(n)}}{\delta t^{(n+\frac{1}{2})}}, \qquad u^{(n+\theta)} = \theta\ u^{(n+1)} + (1-\theta)\ u^{(n)}, \\ \displaystyle\int_\Omega \Big(\delta_{\mathcal{D}}^{(n+\frac{1}{2})} u(\boldsymbol{x}) \Pi_{\mathcal{D}} v(\boldsymbol{x}) - \Pi_{\mathcal{D}} u^{(n+\theta)}(\boldsymbol{x}) \vec{\boldsymbol{v}}(\boldsymbol{x},t) \cdot \nabla_{\mathcal{D}} v(\boldsymbol{x}) \\ \qquad\qquad + \Lambda(\boldsymbol{x}) \nabla_{\mathcal{D}} \zeta(u^{(n+1)})(\boldsymbol{x}) \cdot \nabla_{\mathcal{D}} v(\boldsymbol{x}) \Big) \,\mathrm{d}\boldsymbol{x} \\ \displaystyle = \frac{1}{\delta t^{(n+\frac{1}{2})}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f(\boldsymbol{x},t) \Pi_{\mathcal{D}} v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \mathrm{d}t, \quad \forall v \in X_{\mathcal{D},0},\ \forall n = 0,\dots,N-1. \end{array}\right. \tag{11}$$

Let us observe that all the degenerate diffusion terms are implicit, whereas a θ-scheme is used for the time discretisation of the convection term. This difference leads us to introduce the following notations for the definition of the discrete space-time dependent functions:

$$\begin{array}{l} \Pi_{\mathcal{D}}^{(\theta)} u(\boldsymbol{x},0) = \Pi_{\mathcal{D}} u^{(0)}(\boldsymbol{x}) \text{ and } \Pi_{\mathcal{D}}^{(1)} u(\boldsymbol{x},0) = \Pi_{\mathcal{D}} u^{(0)}(\boldsymbol{x}) \text{ for a.e. } \boldsymbol{x} \in \Omega, \\ \Pi_{\mathcal{D}}^{(\theta)} u(\boldsymbol{x},t) = \Pi_{\mathcal{D}} u^{(n+\theta)}(\boldsymbol{x}) \text{ and } \Pi_{\mathcal{D}}^{(1)} u(\boldsymbol{x},t) = \Pi_{\mathcal{D}} u^{(n+1)}(\boldsymbol{x}), \\ \Pi_{\mathcal{D}}^{(1)} \zeta(u)(\boldsymbol{x},t) = \Pi_{\mathcal{D}} \zeta(u^{(n+1)})(\boldsymbol{x}), \\ \nabla_{\mathcal{D}}^{(1)} \zeta(u)(\boldsymbol{x},t) = \nabla_{\mathcal{D}} \zeta(u^{(n+1)})(\boldsymbol{x}), \text{ for a.e. } \boldsymbol{x} \in \Omega,\ \forall t \in (t^{(n)}, t^{(n+1)}],\ \forall n = 0,\dots,N-1. \end{array} \tag{12}$$

We also denote

$$\delta_{\mathcal{D}} u(\boldsymbol{x},t) = \delta_{\mathcal{D}}^{(n+\frac{1}{2})} u(\boldsymbol{x}), \text{ for a.e. } (\boldsymbol{x},t) \in \Omega \times (t^{(n)}, t^{(n+1)}),\ \forall n = 0,\dots,N-1. \tag{13}$$

We can notice that, in the spirit of the estimates which are proved below, the space-time functions reconstructions for u are defined for all time $t \in [0,T]$, whereas $\delta_{\mathcal{D}} u$, $\Pi_{\mathcal{D}}^{(1)} \zeta(u)$ and $\nabla_{\mathcal{D}}^{(1)} \zeta(u)$ are only defined for a.e. $t \in (0,T)$. We finally introduce the function

$$Z(s) = \int_0^s \zeta(x) \mathrm{d}x,\ \forall s \in \mathbb{R}. \tag{14}$$

which is used several times in the convergence proofs. We then have

$$Z(s) = \int_0^s \zeta(x) \mathrm{d}x = \int_0^s (\zeta(x) - \zeta(0)) \mathrm{d}x \le \overline{L}_\zeta \int_0^s x \mathrm{d}x = \overline{L}_\zeta \frac{s^2}{2},\ \forall s \in \mathbb{R}, \tag{15}$$

and, from Hypotheses (5f) and (5g), and using Young's inequality,

$$Z(s) \ge \int_0^{|s|} (\underline{L}_\zeta x - C_\zeta) \mathrm{d}x = \frac{1}{2} \underline{L}_\zeta |s|^2 - C_\zeta |s| \ge \frac{1}{4} \underline{L}_\zeta s^2 - \frac{C_\zeta^2}{\underline{L}_\zeta},\ \forall s \in \mathbb{R}. \tag{16}$$

Lemma 2.1 (Discrete versions of (E1) and (E2) and existence of a discrete solution)**.** *Under Hypotheses* (5)*, let* $\mathcal{D}_T = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}}, \mathcal{I}_{\mathcal{D}}, (t^{(n)})_{n=0,\dots,N})$ *be a space-time gradient discretisation such that* $\Pi_{\mathcal{D}}$ *is a piecewise constant function reconstruction. Let* $\gamma \in (0,1)$ *be given and let* $\theta \in \mathbb{R}$ *be such that*

$$4 \|\vec{\boldsymbol{v}}\|_{L^\infty(\Omega\times(0,T))}^2 \theta^2 \delta t_{\mathcal{D}} \le \gamma \underline{L}_\zeta \underline{\lambda}. \tag{17}$$

Then there exists at least one solution to Scheme (11)*, and there exists* $C_1 > 0$*, only depending on* $\overline{L}_\zeta$*,* $\underline{L}_\zeta$*,* C_ζ*,* $C_P > C_{\mathcal{D}}$*,* $C_{\rm ini} > \|u_{\rm ini} - \Pi_{\mathcal{D}} u^{(0)}\|_{L^2(\Omega)}$*,* f*,* $\vec{\boldsymbol{v}}$*,* $\underline{\lambda}$*,* θ *and* γ *such that, for any solution* u *to this scheme,*

$$\|\Pi_{\mathcal{D}}^{(1)} \zeta(u)\|_{L^\infty(0,T;L^2(\Omega))} \le C_1,\ \|\Pi_{\mathcal{D}}^{(1)} u\|_{L^\infty(0,T;L^2(\Omega))} \le C_1,\ \text{and } \|\Pi_{\mathcal{D}}^{(\theta)} u\|_{L^\infty(0,T;L^2(\Omega))} \le C_1, \tag{18}$$

and

$$\|\nabla_{\mathcal{D}}^{(1)} \zeta(u)\|_{L^2(\Omega\times(0,T))} \le C_1. \tag{19}$$

Remark 2.2 (On condition (17))**.** *For any consistent sequence $((\mathcal{D}_T)_m)_{m\in\mathbb{N}}$ of space-time gradient discretisations, condition* (17) *is necessarily satisfied for m large enough for any $\theta \in \mathbb{R}$, since the consistency property implies that $\delta t_{\mathcal{D}_m}$ tends to* 0 *as $m \to \infty$. For a given space-time gradient discretisation, it is always possible to choose $\theta \in \mathbb{R}$ such that condition* (17) *holds (see Section 4 for an example of numerical maximum value for $|\theta|$ such that this condition holds).*

Proof. Before showing the existence of at least one discrete solution to Scheme (11), let us first prove (18) and (19). From properties (15) and (16), and using $\int_a^b \zeta(s)\mathrm{d}s = Z(b) - Z(a) = \zeta(b)(b-a) - \int_a^b \zeta'(s)(s-a)\mathrm{d}s$, we get, since (5f) implies $\zeta'(s) \geq 0$ and using the piecewise constant reconstruction hypothesis, that

$$(\Pi_{\mathcal{D}} u^{(n+1)}(\boldsymbol{x}) - \Pi_{\mathcal{D}} u^{(n)}(\boldsymbol{x}))\ \Pi_{\mathcal{D}}\zeta(u^{(n+1)})(\boldsymbol{x}) \geq \Pi_{\mathcal{D}} Z(u^{(n+1)})(\boldsymbol{x}) - \Pi_{\mathcal{D}} Z(u^{(n)})(\boldsymbol{x}) \text{ for a.e. } \boldsymbol{x} \in \Omega.$$

We then let $v = \delta t^{(n+\frac{1}{2})}\zeta(u^{(n+1)})$ in (11), and we sum the obtained equation on $n = 0, \ldots, m-1$ for a given $m = 1, \ldots, N$. Accounting for the above inequality yields that there holds

$$\begin{aligned}\int_\Omega (Z(\Pi_{\mathcal{D}} u^{(m)}(\boldsymbol{x})) - Z(\Pi_{\mathcal{D}} u^{(0)}(\boldsymbol{x})))\mathrm{d}\boldsymbol{x} + \underline{\lambda}\int_0^{t^{(m)}}\int_\Omega |\nabla_{\mathcal{D}}^{(1)}\zeta(u)(\boldsymbol{x},t)|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t \\ \leq \int_0^{t^{(m)}}\int_\Omega \left(\Pi_{\mathcal{D}}^{(\theta)} u(\boldsymbol{x},t)\vec{\boldsymbol{v}}(\boldsymbol{x},t)\cdot\nabla_{\mathcal{D}}^{(1)}\zeta(u)(\boldsymbol{x},t) + f(\boldsymbol{x},t)\Pi_{\mathcal{D}}^{(1)}\zeta(u)(\boldsymbol{x},t)\right)\mathrm{d}\boldsymbol{x}\mathrm{d}t,\ \forall m = 0,\ldots,N, \quad (20)\end{aligned}$$

since for $m = 0$ the above inequality reduces to $0 \leq 0$. Thanks to Young's inequality and applying the definition of the coercivity constant (8), we get that, for any $n = 0, \ldots, m-1$, for every $\eta_1 > 0$ and a.e. $(\boldsymbol{x}, t) \in \Omega \times (t^{(n)}, t^{(n+1)})$,

$$\begin{aligned} f(\boldsymbol{x},t)\ \Pi_{\mathcal{D}}\zeta(u^{(n+1)})(\boldsymbol{x}) &\leq \frac{1}{2\eta_1}|f(\boldsymbol{x},t)|^2 + \frac{\eta_1}{2}|\Pi_{\mathcal{D}}\zeta(u^{(n+1)})(\boldsymbol{x})|^2, \\ &\leq \frac{1}{2\eta_1}|f(\boldsymbol{x},t)|^2 + \frac{C_{\mathcal{D}}^2\eta_1}{2}|\nabla_{\mathcal{D}}\zeta(u^{(n+1)})(\boldsymbol{x})|^2.\end{aligned}$$

Summing two Young's inequalities applied to both terms $\theta\ \Pi_{\mathcal{D}} u^{(n+1)}(\boldsymbol{x})\vec{\boldsymbol{v}}(\boldsymbol{x},t)\cdot\nabla_{\mathcal{D}}\zeta(u^{(n+1)})(\boldsymbol{x})$ and $(1-\theta)\ \Pi_{\mathcal{D}} u^{(n)}(\boldsymbol{x})\vec{\boldsymbol{v}}(\boldsymbol{x},t)\cdot\nabla_{\mathcal{D}}\zeta(u^{(n+1)})(\boldsymbol{x})$, we get that for every $\eta_2 > 0$ and a.e. $(\boldsymbol{x}, t) \in \Omega \times (t^{(n)}, t^{(n+1)})$,

$$\begin{aligned}&|\Pi_{\mathcal{D}} u^{(n+\theta)}(\boldsymbol{x})\vec{\boldsymbol{v}}(\boldsymbol{x},t)\cdot\nabla_{\mathcal{D}}\zeta(u^{(n+1)})(\boldsymbol{x})| \\ &\qquad \leq \|\vec{\boldsymbol{v}}\|_{L^\infty(\Omega\times(0,T))}\left(\frac{1}{2\eta_2}\left(\theta^2\ (\Pi_{\mathcal{D}} u^{(n+1)}(\boldsymbol{x}))^2 + (1-\theta)^2\ (\Pi_{\mathcal{D}} u^{(n)}(\boldsymbol{x}))^2\right) + \eta_2|\nabla_{\mathcal{D}}\zeta(u^{(n+1)})(\boldsymbol{x})|^2\right).\end{aligned}$$

Thus from (20), using the two above inequalities with $\eta_1 = \underline{\lambda}/(2C_{\mathcal{D}}^2)$ and $\eta_2 = \underline{\lambda}/(2\|\vec{\boldsymbol{v}}\|_{L^\infty(\Omega\times(0,T))})$, we get that, for all $m = 0, \ldots, N$ (with the convention that an empty sum is equal to 0),

$$\begin{aligned}&\|\Pi_{\mathcal{D}} Z(u^{(m)})\|_{L^1(\Omega)} + \frac{1}{4}\underline{\lambda}\|\nabla_{\mathcal{D}}^{(1)}\zeta(u)\|_{L^2(\Omega\times(0,t^{(m)}))}^2 \\ &\qquad \leq \frac{\|\vec{\boldsymbol{v}}\|_{L^\infty(\Omega\times(0,T))}^2}{\underline{\lambda}}\sum_{n=0}^{m-1}\delta t^{(n+\frac{1}{2})}\left(\theta^2\ \|\Pi_{\mathcal{D}} u^{(n+1)}\|_{L^2(\Omega)}^2 + (1-\theta)^2\ \|\Pi_{\mathcal{D}} u^{(n)}\|_{L^2(\Omega)}^2\right) \\ &\qquad\qquad + \frac{C_{\mathcal{D}}^2}{\underline{\lambda}}\|f\|_{L^2(\Omega\times(0,t^{(m)}))}^2 + \|\Pi_{\mathcal{D}} Z(u^{(0)})\|_{L^1(\Omega)}. \quad (21)\end{aligned}$$

which in turn yields, thanks to (15) and (16) and to the non-negativity of the second term of the left hand side,

$$\left(\frac{\underline{L}_\zeta}{4} - \frac{\|\vec{\boldsymbol{v}}\|^2_{L^\infty(\Omega\times(0,T))}}{\underline{\lambda}}\delta t^{(m-\frac{1}{2})}\theta^2\right)\|\Pi_{\mathcal{D}}u^{(m)}\|^2_{L^2(\Omega)}$$
$$\leq \frac{\|\vec{\boldsymbol{v}}\|^2_{L^\infty(\Omega\times(0,T))}}{\underline{\lambda}}\sum_{n=1}^{m-1}\left(\theta^2\ \delta t^{(n-\frac{1}{2})} + (1-\theta)^2\delta t^{(n+\frac{1}{2})}\right)\ \|\Pi_{\mathcal{D}}u^{(n)}\|^2_{L^2(\Omega)}$$
$$+ \frac{C^2_{\mathcal{D}}}{\underline{\lambda}}\|f\|^2_{L^2(\Omega\times(0,T))} + \left(\frac{\|\vec{\boldsymbol{v}}\|^2_{L^\infty(\Omega\times(0,T))}}{\underline{\lambda}}(1-\theta)^2\delta t^{(\frac{1}{2})} + \frac{\overline{L}_\zeta}{2}\right)\|\Pi_{\mathcal{D}}u^{(0)}\|^2_{L^2(\Omega)} + \frac{C^2_\zeta}{\underline{L}_\zeta},\ \forall m = 1,\dots,N. \quad (22)$$

Let us notice that (17) has been designed in order that the coefficient of $\|\Pi_{\mathcal{D}}u^{(m)}\|^2_{L^2(\Omega)}$ at the left hand side remains strictly positive. Indeed, (17) is equivalent to $\frac{\underline{L}_\zeta}{4} - \theta^2\ \frac{\|\vec{\boldsymbol{v}}\|^2_{L^\infty(\Omega\times(0,T))}}{\underline{\lambda}}\delta t_{\mathcal{D}} > \frac{\underline{L}_\zeta(1-\gamma)}{4}$. Thus, using

$$\|\Pi_{\mathcal{D}}u^{(0)}\|_{L^2(\Omega)} \leq \|u_{\rm ini}\|_{L^2(\Omega)} + C_{\rm ini}, \quad (23)$$

we obtain, denoting $a_m = \|\Pi_{\mathcal{D}}u^{(m)}\|^2_{L^2(\Omega)}$ for all $m = 1,\dots,N$, that

$$\forall m \in \{1,\dots,N\},\ a_m \leq \sum_{n=1}^{m-1} b_n a_n + B,$$

with the convention that an empty sum is equal to zero, and denoting by

$$B = \frac{4}{\underline{L}_\zeta(1-\gamma)}\left(\frac{C^2_{\mathcal{D}}}{\underline{\lambda}}\|f\|^2_{L^2(\Omega\times(0,T))} + \left(\frac{\overline{L}_\zeta}{2} + (1-\theta)^2 T\frac{\|\vec{\boldsymbol{v}}\|^2_{L^\infty(\Omega\times(0,T))}}{\underline{\lambda}}\right)(\|u_{\rm ini}\|_{L^2(\Omega)} + C_{\rm ini})^2 + \frac{C^2_\zeta}{\underline{L}_\zeta}\right),$$

and

$$b_n = A(\theta^2\delta t^{(n-\frac{1}{2})} + (1-\theta)^2\delta t^{(n+\frac{1}{2})}) \text{ with } A = \frac{4\|\vec{\boldsymbol{v}}\|^2_{L^\infty(\Omega\times(0,T))}}{(1-\gamma)\underline{L}_\zeta\underline{\lambda}}.$$

Using that $\sum_{n=1}^m \delta t^{(n-\frac{1}{2})} = t^{(m)}$, we get that $\sum_{n=1}^{m-1} b_n \leq AT(\theta^2 + (1-\theta)^2)$. Therefore, applying the discrete Gronwall lemma A.1, we prove that

$$\|\Pi_{\mathcal{D}}u^{(m)}\|^2_{L^2(\Omega)} \leq B\exp\left(AT(\theta^2 + (1-\theta)^2)\right),\ \forall m = 1,\dots,N.$$

Together with (23), this shows (18). Reporting this estimate in (21), we deduce (19).

Let us now turn to the proof of existence of a solution to Scheme (11). Let us introduce, for any $\mu \in [0,1]$, the function $\zeta_\mu(s) = (1-\mu)\underline{L}_\zeta s + \mu\zeta(s)$. Then all the hypotheses of the lemma are satisfied with the same values $\overline{L}_\zeta$, $\underline{L}_\zeta$ and C_ζ, which implies that the bounds in (18)-(19) hold with the same constant for all $\mu \in [0,1]$. For $\mu = 0$, Scheme (11) leads to a square linear system; repeating the above computations with $Z(s) = \frac{1}{2}\underline{L}_\zeta s^2$, $u^{(0)} = 0$, $f = 0$ and $C_\zeta = 0$ implies that the solution is equal to 0 if the right hand side vanishes, which implies the existence and uniqueness of the solution of this linear system. By continuity of the discrete equations with respect to μ, we apply the constancy of the topological degree by an homotopy which allow to conclude the existence of at least one solution for $\mu = 1$ (see [11] for the topological degree theory).

□

In view of the study of the time translates, for fulfilling the hypotheses of Kolmogorov's compactness theorem, let us prove an estimate on the dual norm of the discrete time derivative, defined as follows. We define the following semi-norm on $L^2(\Omega)$:

$$\forall w \in L^2(\Omega),\ |w|_{\star,\mathcal{D}} = \sup\left\{\int_\Omega w(\boldsymbol{x})\Pi_{\mathcal{D}}v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}; v \in X_{\mathcal{D},0}, \|\nabla_{\mathcal{D}}v\|_{L^2(\Omega)^d} = 1\right\}, \quad (24)$$

and we introduce the following semi-norm on $L^2(0,T;L^2(\Omega))$:

$$\forall w \in L^2(0,T;L^2(\Omega)),\ |w|_{L^2(0,T;\star,\mathcal{D})} = \left(\int_0^T |w(t)|^2_{\star,\mathcal{D}}\mathrm{d}t\right)^{1/2}. \tag{25}$$

Lemma 2.3 (Discrete version of (E3))**.**
Under Hypotheses (5)*, let* $\mathcal{D}_T = (X_{\mathcal{D},0}, \Pi_\mathcal{D}, \nabla_\mathcal{D}, \mathcal{I}_\mathcal{D}, (t^{(n)})_{n=0,\dots,N})$ *be a space-time gradient discretisation such that* $\Pi_\mathcal{D}$ *is a piecewise constant function reconstruction. Let* $\gamma \in (0,1)$ *be given and let* $\theta \in \mathbb{R}$ *be such that condition* (17) *holds. Then there exists* $C_2 > 0$*, only depending on* $\overline{L}_\zeta$*,* $\underline{L}_\zeta$*,* C_ζ*,* $C_P > C_\mathcal{D}$*,* $C_{\rm ini} > \|u_{\rm ini} - \Pi_\mathcal{D} u^{(0)}\|_{L^2(\Omega)}$*,* f*,* $\vec{\boldsymbol{v}}$*,* $\underline{\lambda}$*,* $\overline{\lambda}$*,* θ *and* γ *such that, for any solution* $u := (u^{(n)})_{n=0,\dots,N}$ *to Scheme* (11)*,*

$$|\delta_\mathcal{D} u|_{L^2(0,T;\star,\mathcal{D})} \le C_2. \tag{26}$$

Proof. From (11), we can write, for any $v \in X_{\mathcal{D},0}$ and any $n = 0,\dots,N-1$,

$$\begin{aligned}\int_\Omega \delta_\mathcal{D}^{(n+\frac12)} u(\boldsymbol{x})\Pi_\mathcal{D} v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} &= \int_\Omega \Pi_\mathcal{D} u^{(n+\theta)}(\boldsymbol{x})\vec{\boldsymbol{v}}(\boldsymbol{x},t)\cdot\nabla_\mathcal{D} v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\\ &\quad + \frac{1}{\delta t^{(n+\frac12)}}\int_{t^{(n)}}^{t^{(n+1)}}\int_\Omega f(\boldsymbol{x},t)\,\Pi_\mathcal{D} v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\mathrm{d}t - \int_\Omega \Lambda(\boldsymbol{x})\nabla_\mathcal{D}\zeta(u^{(n+1)})(\boldsymbol{x})\nabla_\mathcal{D} v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Applying the Cauchy-Schwarz inequality and applying the definition of the coercivity constant (8), we obtain

$$\begin{aligned}|\delta_\mathcal{D}^{(n+\frac12)} u|_{\star,\mathcal{D}} &\le \|\vec{\boldsymbol{v}}\|_{L^\infty(\Omega\times(0,T))}\|\Pi_\mathcal{D} u^{(n+\theta)}\|_{L^2(\Omega)}\\ &\quad + C_\mathcal{D}\frac{1}{\delta^{(n+\frac12)}}\int_{t^{(n)}}^{t^{(n+1)}}\|f(\cdot,t)\|_{L^2(\Omega)}\mathrm{d}t + \overline{\lambda}\|\nabla_\mathcal{D}\zeta(u^{(n+1)})\|_{L^2(\Omega)}.\end{aligned}$$

Then on one hand we use that for all $x,y,z \in \mathbb{R}$, $(x+y+z)^2 \le 3\left(x^2+y^2+z^2\right)$. On the other hand, the definition $|\delta_\mathcal{D} u|^2_{L^2(0,T;\star,\mathcal{D})} = \sum_{n=0}^{N-1}\delta^{(n+\frac12)}|\delta_\mathcal{D}^{(n+\frac12)}u|^2_{\star,\mathcal{D}}$ leads to

$$|\delta_\mathcal{D} u|^2_{L^2(0,T;\star,\mathcal{D})} \le 3\Big(\|\vec{\boldsymbol{v}}\|^2_{L^\infty(\Omega\times(0,T))}\|\Pi^{(\theta)}_\mathcal{D} u\|^2_{L^2(\Omega\times(0,T))} + C^2_\mathcal{D}\|f\|^2_{L^2(\Omega\times(0,T))} + \overline{\lambda}^2\|\nabla^{(1)}_\mathcal{D}\zeta(u)\|^2_{L^2(\Omega\times(0,T))}\Big).$$

Thus (18) and (19) of Lemma 2.1 imply (26), where the dependence of C_2 with respect to the data of the problem are resulting from the one arising in Lemma 2.1. □

The next lemma concerns the study of the time translates of $\Pi^{(1)}_\mathcal{D}\zeta(u)$. Note that the estimate (19) provides an estimate on the space translate of the same function, and the combination of these two estimates allows the application of Kolmogorov's theorem for deriving a strong convergence property.

Lemma 2.4 (Estimate on the time translates)**.**
Under Hypotheses (5)*, let* $\mathcal{D}_T = (X_{\mathcal{D},0}, \Pi_\mathcal{D}, \nabla_\mathcal{D}, \mathcal{I}_\mathcal{D}, (t^{(n)})_{n=0,\dots,N})$ *be a space-time gradient discretisation such that* $\Pi_\mathcal{D}$ *is a piecewise constant function reconstruction. Let* $\gamma \in (0,1)$ *be given and let* $\theta \in \mathbb{R}$ *be such that condition* (17) *holds. Then there exists* $C_3 > 0$*, only depending on* $\overline{L}_\zeta$*,* $\underline{L}_\zeta$*,* C_ζ*,* $C_P > C_\mathcal{D}$*,* $C_{\rm ini} > \|u_{\rm ini} - \Pi_\mathcal{D} u^{(0)}\|_{L^2(\Omega)}$*,* f*,* $\vec{\boldsymbol{v}}$*,* $\underline{\lambda}$*,* $\overline{\lambda}$*,* θ *and* γ *such that, for any solution* $u := (u^{(n)})_{n=0,\dots,N}$ *to Scheme* (11)*,*

$$\|\Pi^{(1)}_\mathcal{D}\zeta(u)(\cdot,\cdot+\tau) - \Pi^{(1)}_\mathcal{D}\zeta(u)(\cdot,\cdot)\|^2_{L^2(\Omega\times(0,T-\tau))} \le C_3\sqrt{\tau(\tau+\delta t_\mathcal{D})}, \forall \tau \in (0,T). \tag{27}$$

Proof. Let $\tau \in (0,T)$. Similarly using that $\overline{L}_\zeta$ is a Lipschitz constant of ζ and ζ is non-decreasing, and using the fact that $\Pi_\mathcal{D}$ is piecewise constant, the following inequality holds:

$$\int_{\Omega\times(0,T-\tau)}\Big(\Pi^{(1)}_\mathcal{D}\zeta(u)(\boldsymbol{x},t+\tau) - \Pi^{(1)}_\mathcal{D}\zeta(u)(\boldsymbol{x},t)\Big)^2\mathrm{d}\boldsymbol{x}\mathrm{d}t \le \overline{L}_\zeta\int_0^{T-\tau} A(t)\mathrm{d}t, \tag{28}$$

where, for almost every $t \in (0, T-\tau)$,

$$A(t) = \int_\Omega \Big(\Pi_{\mathcal{D}}^{(1)}\zeta(u)(\boldsymbol{x}, t+\tau) - \Pi_{\mathcal{D}}^{(1)}\zeta(u)(\boldsymbol{x}, t)\Big)\Big(\Pi_{\mathcal{D}}^{(1)}u(\boldsymbol{x}, t+\tau) - \Pi_{\mathcal{D}}^{(1)}u(\boldsymbol{x}, t)\Big)\mathrm{d}\boldsymbol{x}.$$

We apply lemma B.1 and we get

$$\int_0^{T-\tau} A(t)dt \leq 2\sqrt{\tau(\tau+\delta t)}\ |\delta_{\mathcal{D}}u|_{L^2(0,T;\star,\mathcal{D})}\ \|\nabla_{\mathcal{D}}^{(1)}\zeta(u)\|_{L^2(0,T;L^2(\Omega)^d)}. \tag{29}$$

Using (26), (19) in (29), we get the result. □

3 Convergence analysis

Let us begin with the weak convergence of $\Pi_{\mathcal{D}}^{(\theta)}u(t)$ and $\Pi_{\mathcal{D}}^{(1)}u(t)$, for all $t \in [0,T]$ to an element of $C_w([0,T]; L^2(\Omega))$, denoting the set of functions from $[0,T]$ to $L^2(\Omega)$, continuous for the weak topology of $L^2(\Omega)$).

Lemma 3.1 (Time pointwise weak convergence of $\Pi_{\mathcal{D}}^{(\theta)}u(t)$ and $\Pi_{\mathcal{D}}^{(1)}u(t)$)**.**
Let Hypotheses (5) *be fulfilled. Let* $((\mathcal{D}_T)_m)_{m\in\mathbb{N}}$ *be a consistent sequence of space-time gradient discretisations, such that the associated sequence of approximate gradient approximations is limit–conforming (it is then coercive thanks to [9, Lemma 2.6]), and such that, for all* $m \in \mathbb{N}$, $\Pi_{\mathcal{D}_m}$ *is a piecewise constant function reconstruction. Let* $\gamma \in (0,1)$ *be given and let* $\theta \in \mathbb{R}$ *be such that condition* (17) *holds for all* $m \in \mathbb{N}$*. For any* $m \in \mathbb{N}$*, let* u_m *be a solution to Scheme* (11)*.*
Then there exists $\bar{u} \in L^\infty(0,T;L^2(\Omega)) \cap C_w([0,T];L^2(\Omega))$ *such that, up to a subsequence, for all* $t \in [0,T]$, $\Pi_{\mathcal{D}_m}^{(\theta)}u_m(t)$ *and* $\Pi_{\mathcal{D}_m}^{(1)}u_m(t)$ *weakly converges in* $L^2(\Omega)$ *to* $\bar{u}(t)$ *as* $m \to \infty$*.*

Proof. Applying Lemma 2.1, we get that there exists $\bar{u}^{(\theta)} \in L^\infty(0,T;L^2(\Omega))$ (resp. $\bar{u}^{(1)} \in L^\infty(0,T;L^2(\Omega))$) such that $\Pi_{\mathcal{D}_m}^{(\theta)}u_m$ (resp. $\Pi_{\mathcal{D}_m}^{(1)}u_m$) weakly converges, up again to the extraction of a subsequence, to $\bar{u}^{(\theta)}$ (resp. $\bar{u}^{(1)}$) in $L^2(\Omega\times(0,T))$.

Let $\varphi \in C_c^\infty([0,T))$ and $w \in C_c^\infty(\Omega)$, and let $w_m \in X_{\mathcal{D}_m,0}$ be such that

$$w_m = \operatorname*{argmin}_{z\in X_{\mathcal{D}_m,0}} S_{\mathcal{D}_m}(z). \tag{30}$$

Since the definitions of $|\cdot|_{\star,\mathcal{D}}$ and of $\delta_{\mathcal{D}}^{(n+\frac12)}u$ imply

$$|\int_\Omega (\Pi_{\mathcal{D}_m}u_m^{(n+1)} - \Pi_{\mathcal{D}_m}u_m^{(n+\theta)})\Pi_{\mathcal{D}_m}w_m\mathrm{d}\boldsymbol{x}| \leq \delta t_{\mathcal{D}_m}|1-\theta|\ |\delta_{\mathcal{D}_m}^{(n+\frac12)}u|_{\star,\mathcal{D}_m}\|w_m\|_{\mathcal{D}_m},$$

we obtain, multiplying the above inequality by $\delta t^{(n+\frac12)}\varphi(t^{(n)})w_m$, summing the resulting equation on $n = 0,\dots,N_m-1$ and using the Cauchy-Schwarz inequality, that

$$|\sum_{n=0}^{N_m-1}\int_\Omega (\Pi_{\mathcal{D}_m}u_m^{(n+1)} - \Pi_{\mathcal{D}_m}u_m^{(n+\theta)})\delta t^{(n+\frac12)}\varphi(t^{(n)})\Pi_{\mathcal{D}_m}w_m\mathrm{d}\boldsymbol{x}|$$
$$\leq \delta t_{\mathcal{D}_m}|1-\theta|\ |\delta_{\mathcal{D}_m}u|_{L^2(0,T;\star,\mathcal{D}_m)}\|w_m\|_{\mathcal{D}_m}\sqrt{T}\|\varphi\|_{L^\infty([0,T])}.$$

Denoting by $\psi_m(\boldsymbol{x},t) = \varphi(t^{(n)})\Pi_{\mathcal{D}_m}w_m(\boldsymbol{x})$ for a.e. $t \in (t^{(n)}, t^{(n+1)})$ and $\boldsymbol{x}\in\Omega$, we get

$$|\int_0^T\int_\Omega (\Pi_{\mathcal{D}_m}^{(1)}u_m - \Pi_{\mathcal{D}_m}^{(\theta)}u_m)\psi_m\mathrm{d}\boldsymbol{x}\mathrm{d}t| \leq \delta t_{\mathcal{D}_m}|1-\theta|\ |\delta_{\mathcal{D}_m}u|_{L^2(0,T;\star,\mathcal{D}_m)}\|w_m\|_{\mathcal{D}_m}\sqrt{T}\|\varphi\|_{L^\infty([0,T])}.$$

Using Lemma 2.3, we get that the right hand side of the above inequality tends to 0 as $m \to \infty$. Passing to the limit in the above inequality, and using weak/strong convergence in the left hand side, we obtain that

$$\int_0^T \int_\Omega (\bar{u}^{(1)}(\boldsymbol{x},t) - \bar{u}^{(\theta)}(\boldsymbol{x},t))\varphi(t)w(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\mathrm{d}t = 0.$$

Since the set $\mathcal{T} = \{\sum_{i=1}^q \varphi_i(t)w_i(\boldsymbol{x}) \,:\, q \in \mathbb{N}, \varphi_i \in C_c^\infty[0,T), w_i \in C_c^\infty(\Omega)\}$ is dense in $C_c^\infty(\Omega \times [0,T))$, we conclude that $\bar{u}^{(1)} = \bar{u}^{(\theta)}$. We now denote by $\bar{u} \in L^\infty(0,T;L^2(\Omega))$ the common limit of $\Pi^{(1)}_{\mathcal{D}_m} u_m$ and $\Pi^{(\theta)}_{\mathcal{D}_m} u_m$.

The fact that $\bar{u} \in C_w([0,T];L^2(\Omega))$ and that, up to a subsequence, for all $t \in [0,T]$, $\Pi^{(\theta)}_{\mathcal{D}_m} u_m(t)$ and $\Pi^{(1)}_{\mathcal{D}_m} u_m(t)$ weakly converges in $L^2(\Omega)$ to $\bar{u}(t)$ as $m \to \infty$ is proved by [9, theorem 4.19], since its hypotheses hold thanks to Lemmas 2.1 and 2.3. □

We can now state the concluding convergence theorem.

Theorem 3.2 (Convergence of Scheme (11))**.**
Let Hypotheses (5) *be fulfilled. Let* $((\mathcal{D}_T)_m)_{m\in\mathbb{N}}$ *be a consistent sequence of space-time gradient discretisations, such that the associated sequence of approximate gradient approximations is limit–conforming (it is then coercive) and compact ([9, Definition 2.8]), and such that, for all* $m \in \mathbb{N}$, $\Pi_{\mathcal{D}_m}$ *is a piecewise constant function reconstruction. Let* $\gamma \in (0,1)$ *be given and let* $\theta \in \mathbb{R}$ *be such that condition* (17) *holds for all* $m \in \mathbb{N}$. *For any* $m \in \mathbb{N}$, *let* u_m *be a solution to Scheme* (11).
Then there exists $\bar{u} \in L^\infty(0,T;L^2(\Omega)) \cap C_w([0,T];L^2(\Omega))$ *such that* $\bar{u}$ *is a solution of Problem* (6) *and, up to a subsequence,*

1. *for all* $t \in [0,T]$, $\Pi^{(\theta)}_{\mathcal{D}_m} u_m(t)$ *and* $\Pi^{(1)}_{\mathcal{D}_m} u_m(t)$ *weakly converges in* $L^2(\Omega)$ *to* $\bar{u}(t)$ *as* $m \to \infty$,
2. $\Pi^{(1)}_{\mathcal{D}_m} \zeta(u_m)$ *converges in* $L^2(\Omega \times (0,T))$ *to* $\zeta(\bar{u})$ *as* $m \to \infty$,
3. $\nabla^{(1)}_{\mathcal{D}_m} \zeta(u_m)$ *weakly converges in* $L^2(\Omega \times (0,T))^d$ *to* $\nabla\zeta(\bar{u})$ *as* $m \to \infty$.

Proof. We first apply Lemma 3.1, and we consider the corresponding extracted subsequence. The compactness hypothesis of $(\mathcal{D}_m)_{m\in\mathbb{N}}$ allows to enter into the framework of Kolmogorov's theorem. Indeed, prolonging $\Pi^{(1)}_{\mathcal{D}} \zeta(u)$ by 0 outside $\Omega \times (0,T)$, from [9, Lemma 2.21], we get that the space translates of $\Pi^{(1)}_{\mathcal{D}} \zeta(u)$ uniformly tend to 0. For the time translates, in addition to Lemma B.1, we show that the terms $\int_{\Omega\times(-\tau,0)} \left(\Pi^{(1)}_{\mathcal{D}} \zeta(u)(\boldsymbol{x},t+\tau) - \Pi^{(1)}_{\mathcal{D}} \zeta(u)(\boldsymbol{x},t)\right)^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t$ and $\int_{\Omega\times(T-\tau,T)} \left(\Pi^{(1)}_{\mathcal{D}} \zeta(u)(\boldsymbol{x},t+\tau) - \Pi^{(1)}_{\mathcal{D}} \zeta(u)(\boldsymbol{x},t)\right)^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t$ are of order less than τ thanks to Estimate (18). Therefore, there exists $\chi \in L^2(\Omega\times(0,T))$ such that $\Pi^{(1)}_{\mathcal{D}_m} \zeta(u_m)$ converges, up to the extraction of a subsequence, to χ in $L^2(\Omega \times (0,T))$. Thanks to the limit-conformity of the sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$, we get that $\chi \in L^2(0,T;H^1_0(\Omega))$.
This allows to apply to Minty's trick [9, Lemma D.10], for concluding that $\chi(\boldsymbol{x},t) = \zeta(\bar{u}(\boldsymbol{x},t))$ for a.e. $(\boldsymbol{x},t) \in \Omega \times (0,T)$. It now remains to prove that $\bar{u}$ is the weak solution of Problem (6).
Let $\varphi \in C_c^\infty([0,T))$ and $w \in C_c^\infty(\Omega)$. Let $m \in \mathbb{N}$, and let $v_m \in X_{\mathcal{D}_m,0}$ be such that

$$v_m = \operatorname*{argmin}_{z\in X_{\mathcal{D}_m,0}} \left(\|\Pi_{\mathcal{D}_m} z - w\|_{L^2(\Omega)} + \|\nabla_{\mathcal{D}_m} z - \nabla w\|_{L^2(\Omega)^d}\right).$$

Thanks to the consistency hypothesis, we have that $\Pi_{\mathcal{D}_m} v_m$ (resp. $\nabla_{\mathcal{D}_m} v_m$) converges in L^2 to w (resp. ∇w). We take as test function v in (11) the function $\delta t_m^{(n+\frac{1}{2})} \varphi(t^{(n)})\, v_m$, and we sum the resulting equation on $n = 0, \dots, N-1$. We get, denoting $\mathcal{D} = \mathcal{D}_m$ and dropping some indices m for the simplicity of the notation,

$$T_1^{(m)} + T_2^{(m)} + T_3^{(m)} = T_4^{(m)}, \tag{31}$$

with

$$T_1^{(m)} = \sum_{n=0}^{N-1} \delta t^{(n+\frac{1}{2})} \varphi(t^{(n)}) \int_\Omega \delta_{\mathcal{D}}^{(n+\frac{1}{2})} u(\boldsymbol{x}) \Pi_{\mathcal{D}} v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x},$$

$$T_2^{(m)} = \sum_{n=0}^{N-1} \delta t^{(n+\frac{1}{2})} \varphi(t^{(n)}) \int_\Omega \nabla_{\mathcal{D}} \zeta(u^{(n+1)})(\boldsymbol{x}) \cdot \nabla_{\mathcal{D}} v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x},$$

$$T_3^{(m)} = \sum_{n=0}^{N-1} \delta t^{(n+\frac{1}{2})} \varphi(t^{(n)}) \Pi_{\mathcal{D}} u^{(n+\theta)}(\boldsymbol{x}) \vec{\boldsymbol{v}}(\boldsymbol{x},t) \cdot \nabla_{\mathcal{D}} v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x},$$

and

$$T_4^{(m)} = -\sum_{n=0}^{N-1} \varphi(t^{(n)}) \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f(\boldsymbol{x},t) \Pi_{\mathcal{D}} v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \mathrm{d}t.$$

Writing

$$T_1^{(m)} = -\int_0^T \varphi'(t) \int_\Omega \Pi_{\mathcal{D}}^{(1)} u(\boldsymbol{x},t) \Pi_{\mathcal{D}} v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \mathrm{d}t - \varphi(0) \int_\Omega \Pi_{\mathcal{D}} u^{(0)}(\boldsymbol{x}) \Pi_{\mathcal{D}} v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x},$$

we get that

$$\lim_{m\to\infty} T_1^{(m)} = -\int_0^T \varphi'(t) \int_\Omega \bar{u}(\boldsymbol{x},t) w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \mathrm{d}t - \varphi(0) \int_\Omega u_{\mathrm{ini}}(\boldsymbol{x}) w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}.$$

We also immediately get that

$$\lim_{m\to\infty} T_2^{(m)} = \int_0^T \varphi(t) \int_\Omega \nabla \zeta(\bar{u})(\boldsymbol{x},t) \cdot \nabla w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \mathrm{d}t,$$

and

$$\lim_{m\to\infty} T_4^{(m)} = \int_0^T \varphi(t) \int_\Omega f(\boldsymbol{x},t) w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \mathrm{d}t.$$

Applying Lemma 3.1 stating the weak convergence of $\Pi_{\mathcal{D}}^{(\theta)} u$ to $\bar{u}$, we also obtain that

$$\lim_{m\to\infty} T_3^{(m)} = \int_0^T \varphi(t) \int_\Omega \bar{u}(\boldsymbol{x},t) \vec{\boldsymbol{v}}(\boldsymbol{x},t) \cdot \nabla w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \mathrm{d}t$$

Since the set $\mathcal{T} = \{\sum_{i=1}^q \varphi_i(t) w_i(\boldsymbol{x}) \,:\, q \in \mathbb{N}, \varphi_i \in C_c^\infty[0,T), w_i \in C_c^\infty(\Omega)\}$ is dense in $C_c^\infty(\Omega \times [0,T))$, we conclude the proof that $\bar{u} \in L^\infty(0,T;L^2(\Omega))$ is a solution of Problem (6), which concludes the proof of the theorem.

□

3.1 A strong convergence result in a particular case

This section concerns a particular case, which holds in the numerical results of Section 4. In the case of the function ζ defined by (32), we define the "mushy" zone $\mathcal{M}(t)$ for any $t \in [0,T]$ by $\mathcal{M}(t) = \{\boldsymbol{x} \in \Omega,\ 0 < \bar{u}(\boldsymbol{x},t) < 1\}$. It is known [5, 17] that, for regular velocity fields and if there is no source term, the solution $\bar{u}$ of (4a) is such that the measure of the mushy zone $|\mathcal{M}(t)|$ is decreasing with t, and therefore, if $\mathcal{M}(0) = 0$, we have $\mathcal{M}(t) = 0$ for all $t \in [0,T]$. Letting $Q = \Omega \times (0,T)$, $w = \bar{u}$ and $w_n = \Pi_{\mathcal{D}_n}^{(\theta)} u_n$, the following lemma shows that the weak convergence result of Theorem 3.2 becomes a strong convergence result in this case.

Lemma 3.3. *Let $N \in \mathbb{N}^\star$ be given, let Q be a non empty bounded open subset of $\mathbb{R}^N$. Let $w \in L^\infty(Q)$ be such that $w(\boldsymbol{x}) \notin (0,1)$ for a.e. $\boldsymbol{x} \in Q$. Let ζ be defined by*

$$\forall s \in \mathbb{R},\ \zeta(s) = \begin{cases} s & \text{if } s < 0, \\ 0 & \text{if } 0 \le s \le 1, \\ s-1 & \text{if } 1 < s. \end{cases} \tag{32}$$

Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of functions of $L^2(Q)$ such that, as $m \to \infty$:

1. *$(w_n)_{n\in\mathbb{N}}$ weakly converges to w in $L^2(Q)$,*
2. *$(\zeta(w_n))_{n\in\mathbb{N}}$ converges to $\zeta(w)$ in $L^2(Q)$.*

Then $(w_n)_{n\in\mathbb{N}}$ converges to w in $L^2(Q)$.

Proof. We first $|s| = 2\max(s,0) - s$ for writing, for a.e. $\boldsymbol{x} \in Q$,

$$\begin{aligned} \max(w(\boldsymbol{x}) - \frac{1}{2}, 0)|w_n(\boldsymbol{x}) - w(\boldsymbol{x})| &= \max(w(\boldsymbol{x}) - \frac{1}{2}, 0)\big(2\max(w_n(\boldsymbol{x}) - w(\boldsymbol{x}), 0) - (w_n(\boldsymbol{x}) - w(\boldsymbol{x}))\big) \\ &= \max(w(\boldsymbol{x}) - \frac{1}{2}, 0)\big(2\max(\zeta(w_n(\boldsymbol{x})) - \zeta(w(\boldsymbol{x})), 0) - (w_n(\boldsymbol{x}) - w(\boldsymbol{x}))\big). \end{aligned}$$

Then, using the strong convergence of $\zeta(w_n)$ and the weak convergence of w_n in $L^2(Q)$, we get

$$\lim_{n\to\infty} \int_Q \max(w(\boldsymbol{x}) - \frac{1}{2}, 0)|w_n(\boldsymbol{x}) - w(\boldsymbol{x})| \mathrm{d}\boldsymbol{x} = 0.$$

Similarly, we use $|s| = s - 2\min(s,0)$ for writing, for a.e. $\boldsymbol{x} \in Q$,

$$\begin{aligned} \max(\frac{1}{2} - w(\boldsymbol{x}), 0)|w_n(\boldsymbol{x}) - w(\boldsymbol{x})| &= \max(\frac{1}{2} - w(\boldsymbol{x}), 0)\big(w_n(\boldsymbol{x}) - w(\boldsymbol{x}) - 2\min(w_n(\boldsymbol{x}) - w(\boldsymbol{x}), 0)\big) \\ &= \max(\frac{1}{2} - w(\boldsymbol{x}), 0)\big(w_n(\boldsymbol{x}) - w(\boldsymbol{x}) - 2\min(\zeta(w_n(\boldsymbol{x})) - \zeta(w(\boldsymbol{x})), 0)\big), \end{aligned}$$

and we therefore obtain

$$\lim_{n\to\infty} \int_Q \max(\frac{1}{2} - w(\boldsymbol{x}), 0)|w_n(\boldsymbol{x}) - w(\boldsymbol{x})| \mathrm{d}\boldsymbol{x} = 0. \tag{33}$$

Adding the two limits such proved, we obtain

$$\lim_{n\to\infty} \int_Q |w(\boldsymbol{x}) - \frac{1}{2}||w_n(\boldsymbol{x}) - w(\boldsymbol{x})| \mathrm{d}\boldsymbol{x} = 0.$$

We have, for any $x, y \in \mathbb{R}$, $|x - y| \le 1 + |\zeta(x) - \zeta(y)|$. For a.e. $\boldsymbol{x} \in Q$, since $|w(\boldsymbol{x}) - \frac{1}{2}| \ge \frac{1}{2}$, we get that

$$(w_n(\boldsymbol{x}) - w(\boldsymbol{x}))^2 \le |w_n(\boldsymbol{x}) - w(\boldsymbol{x})|(2|w(\boldsymbol{x}) - \frac{1}{2}| + (w_n(\boldsymbol{x}) - w(\boldsymbol{x}))(\zeta(w_n(\boldsymbol{x})) - \zeta(w(\boldsymbol{x})).$$

Hence we obtain, using (33) for the first term and weak/strong convergence for the second one,

$$\begin{aligned} &\lim_{n\to\infty} \int_Q (w_n(\boldsymbol{x}) - w(\boldsymbol{x}))^2 \mathrm{d}\boldsymbol{x} \\ &\qquad \le \lim_{n\to\infty} \left(2\int_Q |w(\boldsymbol{x}) - \frac{1}{2}||w_n(\boldsymbol{x}) - w(\boldsymbol{x})| \mathrm{d}\boldsymbol{x} + \int_Q (w_n(\boldsymbol{x}) - w(\boldsymbol{x}))(\zeta(w_n(\boldsymbol{x})) - \zeta(w(\boldsymbol{x})) \mathrm{d}\boldsymbol{x}\right) = 0, \end{aligned}$$

which concludes the proof. □

4 Numerical examples

We compare in this section two schemes which enter into the framework of this paper. Let us briefly recall each of them.

The Control Volume Finite Element (CVFE) scheme

This scheme, also called the mass-lumped $\mathbb{P}^1$ finite element method, is detailed in [9, Section 8.4]. We consider a conforming triangular mesh of Ω, and we define a dual mesh by joining the centre of gravity of the triangles with the middle of the edges. Denote by $\mathcal{V}$ the set of the vertices of the mesh, and for $\boldsymbol{v} \in \mathcal{V}$ define $K_{\boldsymbol{v}}$ as the dual cell around the vertex $\boldsymbol{v}$.

1. We then define $X_{\mathcal{D},0}$ as the set of all families $u = (u_{\boldsymbol{v}})_{\boldsymbol{v}\in\mathcal{V}}$ such that $u_{\boldsymbol{v}} = 0$ for all vertices $\boldsymbol{v}$ located on the boundary of the domain.
2. For every $u \in X_{\mathcal{D},0}$, $\boldsymbol{v} \in \mathcal{V}$ and for almost-every $x \in K_{\boldsymbol{v}}$, $\Pi_{\mathcal{D}} u(x) = u_{\boldsymbol{v}}$ (piecewise constant reconstruction in all $K_{\boldsymbol{v}}$);
3. For every $u \in X_{\mathcal{D},0}$, define $\nabla_{\mathcal{D}} u$ as the gradient of the conforming piecewise affine function reconstructed in the triangles from the values at the vertices of the triangles.

The Vertex Approximate Gradient (VAG) scheme

The properties of the VAG scheme, introduced in [14], are detailed in [9, Section 8.5]. In the 2D case, a polygonal mesh $\mathcal{M}$ is given, such that each element $K \in \mathcal{M}$ is strictly star-shaped with respect to some point $\boldsymbol{x}_K$. We denote by $\mathcal{V}$ the set of all the vertices of the mesh, and by $\mathcal{E}$ the set of all the edges of the elements of the mesh, assumed to be linear segments. For any $\sigma \in \mathcal{E}$, we denote by $\boldsymbol{x}_\sigma$ its middle point. We consider two meshes of Ω. The first one is a triangular mesh, where the vertices of the triangles are the points $\boldsymbol{x}_K, \boldsymbol{x}_\sigma, \boldsymbol{v}$, for all $K \in \mathcal{M}$, for all σ of K and $\boldsymbol{v}$ common vertex of K and σ. The second one is a dual mesh, associated to all points $(\boldsymbol{x}_K)_{K\in\mathcal{M}}$ and $(\boldsymbol{v})_{\boldsymbol{v}\in\mathcal{V}}$.

1. We then define $X_{\mathcal{D},0}$ as the set of all families $u = ((u_K)_{K\in\mathcal{M}}, (u_{\boldsymbol{v}})_{\boldsymbol{v}\in\mathcal{V}})$ such that $u_{\boldsymbol{v}} = 0$ for all $\boldsymbol{v} \in \mathcal{V} \cap \partial\Omega$.
2. The mapping $\Pi_{\mathcal{D}}$ is defined by piecewise constant functions having the values u_K in the dual control volume associated to $\boldsymbol{x}_K$ for all $K \in \mathcal{M}$ and $u_{\boldsymbol{v}}$ in the dual control volume associated to $\boldsymbol{v} \in \mathcal{V}$.
3. Considering the value $u_\sigma = \frac{1}{2}(u_{\boldsymbol{v}} + u_{\boldsymbol{v}'})$ at the middle $\boldsymbol{x}_\sigma$ of an edge $\sigma = [\boldsymbol{v}, \boldsymbol{v}']$, the mapping $\nabla_{\mathcal{D}}$ is defined as the gradient of the $\mathbb{P}^1$ affine reconstruction with the values $u_K, u_\sigma, u_{\boldsymbol{v}}$ at the vertices $\boldsymbol{x}_K, \boldsymbol{x}_\sigma, \boldsymbol{v}$ of any triangle of the triangular mesh.

This scheme has two advantages. Firstly, adjusting the dual mesh with respect to the heterogeneous properties of the domain, it allows accurate computations of coupled conservation equations in porous media [14]. Secondly, it leads to cheap computations with the elimination of all values $(u_K)_{K\in\mathcal{M}}$ with respect to the values $(u_{\boldsymbol{v}})_{\boldsymbol{v}\in\mathcal{V}}$.

Data for the numerical tests

In the following numerical examples, we consider the function ζ presented in the introduction of this paper, letting $C_v = C_l = 1$, $T_f = 0$ and $L = 1$, which leads to

$$\forall s \in \mathbb{R},\ \zeta(s) = \begin{cases} s & \text{if } s < 0, \\ 0 & \text{if } 0 \le s \le 1, \\ s - 1 & \text{if } 1 < s. \end{cases} \tag{34}$$

In this case, the data involved in (5f) and (5g) can be chosen as $\overline{L}_\zeta = 1$, $\underline{L}_\zeta = 1$ and $C_\zeta = 1$.

We let $d = 2$, $\Omega = (0,1) \times (0,1)$, $\vec{\boldsymbol{v}} = (1,1)$ (hence $\|\vec{\boldsymbol{v}}\|_\infty = \sqrt{2}$), $\Lambda = \mathrm{Id}$ (we then choose $\underline{\lambda} = \overline{\lambda} = 1$ in (5d)). Letting γ tend to 1, and considering the case $\delta t_{\mathcal{D}} = 0.001$ (this value is selected in most of the numerical tests

below) the maximum value of θ^2 such that Condition (17) holds is therefore equal to $125\sqrt{2}$), which enables $|\theta| \leq 10$.

We finally let $f = 0$ and $u_{\rm ini}(\boldsymbol{x}) = \alpha_{\rm ini}$ if $\boldsymbol{x} = (x_1, x_2) \in (0.1, 0.4) \times (0.1, 0.4)$ and $u_{\rm ini}(\boldsymbol{x}) = 0$ elsewhere, with the three cases $\alpha_{\rm ini} = 1.5$, $\alpha_{\rm ini} = 1$ and $\alpha_{\rm ini} = .5$. These three choices are aimed to provide the behaviour of the numerical schemes in three cases: in the first and second (which is a limit case) cases a strong convergence property holds for $\Pi_{\mathcal{D}} u$ (see Section (3.1)), and in the third one is a case where oscillations are expected. The computations are done with a final time $T = 0.5$, with a constant time step equal to 0.001 and a family of triangular meshes extracted from the benchmark [16]. The coarsest grid, respectively the finest one, has a space step equal to 0.25, respectively 0.016.

4.1 Case $\alpha_{\rm ini} = 1.5$

The mushy zone is such that $\mathcal{M}(0) = 0$. So the solution is given at any time $t \in (0, T]$ by $\bar{u}(\cdot, t) > 1$ a.e. inside a moving domain $\widetilde{\Omega}(t)$, with $\widetilde{\Omega}(0) = (0.1, 0.4) \times (0.1, 0.4)$, and $\bar{u}(\cdot, t) = 0$, a.e. inside $\Omega \setminus \widetilde{\Omega}(t)$. Due to the choice of the velocity $\vec{\boldsymbol{v}}$, this domain $\widetilde{\Omega}(t)$ moves along the principal diagonal.

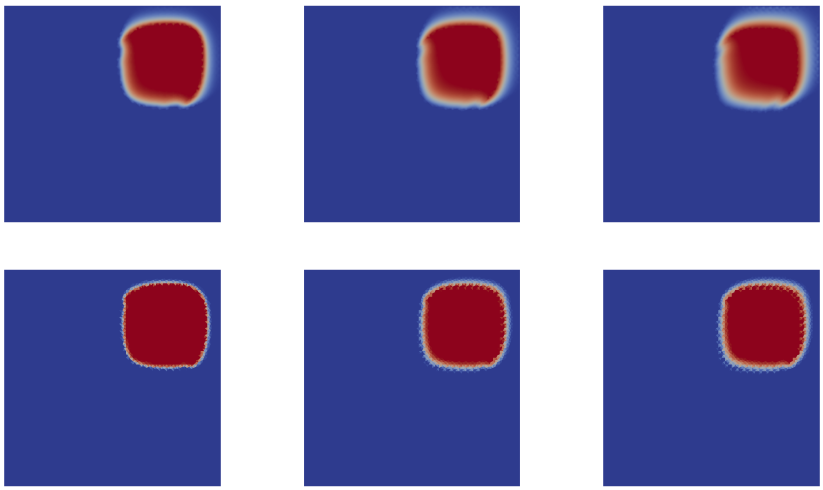

Figure 1: Case $\alpha_{\rm ini} = 1.5$, solutions at $T = 0.5$. Top: CVFE scheme (from left to right: $\theta = 0$, $\theta = 0.5$, $\theta = 1$). Bottom: same for the VAG scheme (cell center values). The colour blue is associated to the value 0, whereas the colour red corresponds to 1. Results obtained on the finest grid.

We then observe in Figures 1 and 2 that the numerical solutions obtained with the CVFE and VAG schemes respect the expected physical features of the problem. There is a good agreement between the two schemes, for the different values of θ on both figures.

4.2 Case $\alpha_{\rm ini} = 1$

This case has the advantage to be such that $\bar{u}(\boldsymbol{x}, t) \in \{0, 1\}$ so $\bar{u}(\boldsymbol{x}, t) \notin (0, 1)$, for any $t \in [0, 0.5]$ and a.e. $\boldsymbol{x} \in \Omega$, and is therefore such that $\zeta(\bar{u}(\boldsymbol{x}, t)) = 0$ for a.e. $(\boldsymbol{x}, t) \in \Omega \times (0, T)$. Therefore the solution $\bar{u}$ to Problem (6) is also the solution of the following pure convection problem:

$$\partial_t \bar{u} + \mathrm{div}(\bar{u}\vec{\boldsymbol{v}}) = 0, \text{ in } \Omega \times (0, T), \tag{35}$$

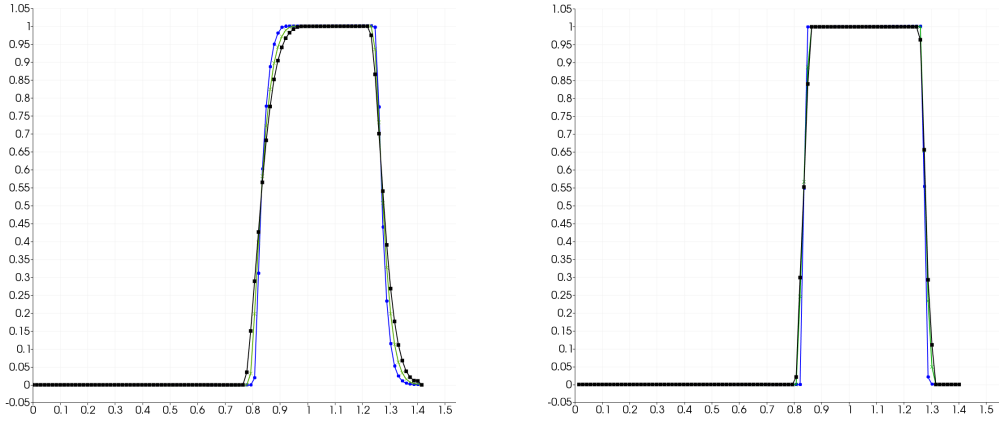

Figure 2: Case $\alpha_{\rm ini} = 1.5$, solutions at $T = 0.5$. Comparison of profiles along the first diagonal. Circle blue for $\theta = 0$, cross green for $\theta = 0.5$ and square black for $\theta = 1$. Left: CVFE scheme. Right: same for the VAG scheme (cell center values). Results obtained on the finest grid.

with the same initial and boundary conditions. It is given by $\bar{u}(\boldsymbol{x}, 0.5) = 1$ for a.e. $\boldsymbol{x} \in (0.6, 0.9) \times (0.6, 0.9)$, and 0 elsewhere. Although it is the limit case, the convergence of $\Pi_{\mathcal{D}_m} u_m$ in $L^2(\Omega \times (0,T))$ to $\bar{u}$ is again an immediate consequence of Theorem 3.2 and Lemma 3.3.

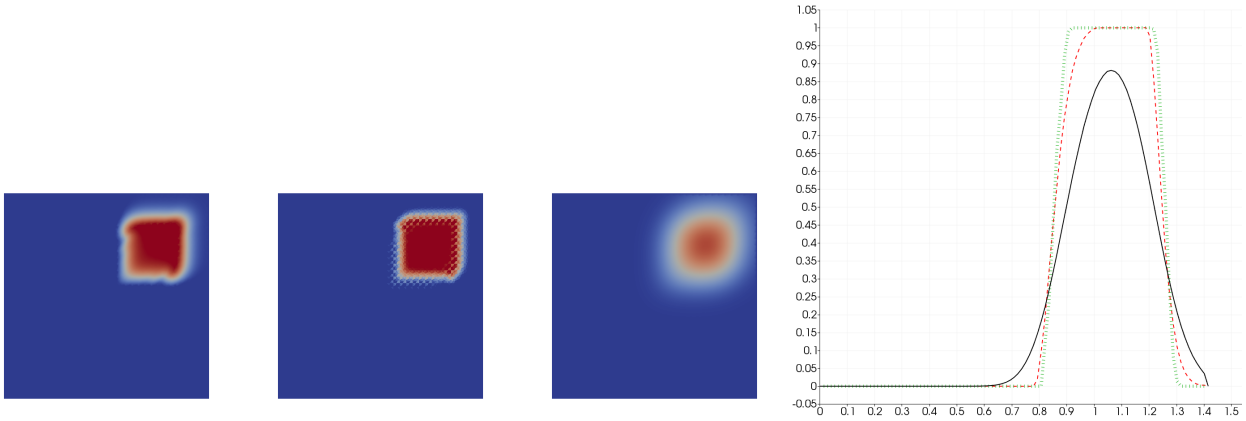

Figure 3: Case $\alpha_{\rm ini} = 1$, solutions at $T = 0.5$. From left to right: CVFE, VAG (cell centre values), upstream schemes (the colour blue is associated to the value 0, whereas the colour red corresponds to 1) and comparison of profiles along the first diagonal (dash red for CVE, dot green for VAG, black line for upstream). Results obtained on the finest grid.

We compare in Figure 3 the numerical solution obtained at $T = 0.5$ with $\theta = 1$ by the CVFE and the VAG schemes, compared to an upstream implicit weighting scheme. The accurateness of centred schemes compared to upstream scheme is confirmed by the L^1 and L^2 errors respectively shown for these two schemes in Tables 1. We also again observe that the bounds are not strictly respected by the centred scheme, but that the error committed on the bounds tends to 0 rapidly. As expected in the case of a discontinuous analytical solution, the numerical orders of convergence are nevertheless closer to 1/2 than to 1.

h	δt	L^2-error	ratio	L^1-error	ratio	$u_{\min}$	$u_{\max}$
0.250	0.01024	0.202E+00	-	0.769E-01	-	-0.020	1.032
0.125	0.00512	0.136E+00	0.572	0.378E-01	1.023	-0.021	1.024
0.062	0.00256	0.992E-01	0.454	0.239E-01	0.662	-0.012	1.019
0.031	0.00128	0.748E-01	0.408	0.149E-01	0.685	-0.007	1.009
0.016	0.00064	0.574E-01	0.383	0.922E-02	0.689	-0.004	1.005
0.250	0.01024	0.182E+00	-	0.572E-01	-	-0.040	1.045
0.125	0.00512	0.126E+00	0.535	0.314E-01	0.864	-0.018	1.019
0.062	0.00256	0.942E-01	0.415	0.171E-01	0.876	-0.013	1.016
0.031	0.00124	0.708E-01	0.412	0.102E-01	0.751	-0.007	1.006
0.016	0.00064	0.550E-01	0.362	0.619E-02	0.717	-0.003	1.004
0.250	0.01024	0.170E+00	-	0.625E-01	-	0.000	0.995
0.125	0.00512	0.151E+00	0.170	0.488E-01	0.358	0.000	1.000
0.062	0.00256	0.124E+00	0.293	0.401E-01	0.281	0.000	1.000
0.031	0.00128	0.102E+00	0.275	0.311E-01	0.369	0.000	1.000
0.016	0.00064	0.854E-01	0.258	0.229E-01	0.442	0.000	1.000

Table 1: Case $\alpha_{\rm ini} = 1$ with $\theta = 1$. $L^2(\Omega \times (0,T))$ and $L^1(\Omega \times (0,T))$ discrete errors. Results for CVFE method (top array), VAG scheme (middle array) and upstream scheme (bottom array).

4.3 Case $\alpha_{\rm ini} = 0.5$

We now turn to the case where $\alpha_{\rm ini} = 0.5$. In this case, $\bar{u}(\boldsymbol{x},t) \in \{0, 0.5\}$, for any $t \in [0, 0.5]$ and a.e. $\boldsymbol{x} \in \Omega$, and is again such that $\zeta(\bar{u}(\boldsymbol{x},t)) = 0$ for a.e. $(\boldsymbol{x},t) \in \Omega \times (0,T)$ (the property $\bar{u}(\boldsymbol{x},t) \notin (0,1)$ is no longer true, and we do no longer expect strong convergence properties for $\Pi_{\mathcal{D}} u$). Therefore the solution $\bar{u}$ to Problem (6) is also the solution of (35) with the same initial and boundary conditions as the problem studied in this paper, and it is given by $\bar{u}(\boldsymbol{x}, 0.5) = 0.5$ for a.e. $\boldsymbol{x} \in (0.6, 0.9) \times (0.6, 0.9)$, and is equal to 0 elsewhere. As mentioned in the introduction of this paper, although the numerical results show oscillations, the weak convergence for $\Pi_{\mathcal{D}} u$ holds for any value $\theta \in \mathbb{R}$. We compare in Figure 4 the numerical solution obtained at $T = 0.5$ with the CVFE and VAG schemes on the finest mesh. We observe that the height of the oscillations is decreasing with respect to θ, and that high values for θ stabilise the oscillations, but introduce some numerical diffusion. The solution obtained with the VAG scheme and $\theta = 1.5$ appears to be quite good.

4.4 The θ-scheme: another way to introduce numerical diffusion

In numerical codes such as the one used in [4], the upstream weighting scheme is used for the energy equation, leading to the addition of numerical diffusion in order to stabilise the convection terms. Let us show how the parameter θ can play a similar role (as we noticed in the numerical Section 4.3), by using an analogy with continuous equations.
We assume here that $\vec{\boldsymbol{v}}$ is a constant vector and we consider formally the transport equation, in which $\bar{u}(t)$ is replaced by $\bar{u}(t + \theta\delta t)$ in the convection term:

$$\partial_t \bar{u}(\boldsymbol{x},t) + \mathrm{div}(\bar{u}(\boldsymbol{x}, t+\theta\delta t)\vec{\boldsymbol{v}}) = \partial_t \bar{u}(\boldsymbol{x},t) + \vec{\boldsymbol{v}} \cdot \nabla \bar{u}(\boldsymbol{x}, t+\theta\delta t) = 0, \text{ in } \Omega \times (0,T). \tag{36}$$

We use the following approximation

$$\bar{u}(\boldsymbol{x}, t+\theta\delta t) \simeq \bar{u}(\boldsymbol{x},t) + \theta\delta t \partial_t \bar{u}(\boldsymbol{x},t) = \bar{u}(\boldsymbol{x},t) - \theta\delta t \vec{\boldsymbol{v}} \cdot \nabla \bar{u}(\boldsymbol{x}, t+\theta\delta t).$$

We then have, reporting this value in (36) and setting $\varepsilon = \theta\delta t |\vec{\boldsymbol{v}}|$ (which is homogeneous with a length),

$$\partial_t \bar{u}(\boldsymbol{x},t) + \vec{\boldsymbol{v}} \cdot \nabla \bar{u}(\boldsymbol{x},t) - \varepsilon \ \mathrm{div}(D \nabla \bar{u}(\boldsymbol{x}, t+\theta\delta t)) = 0,$$

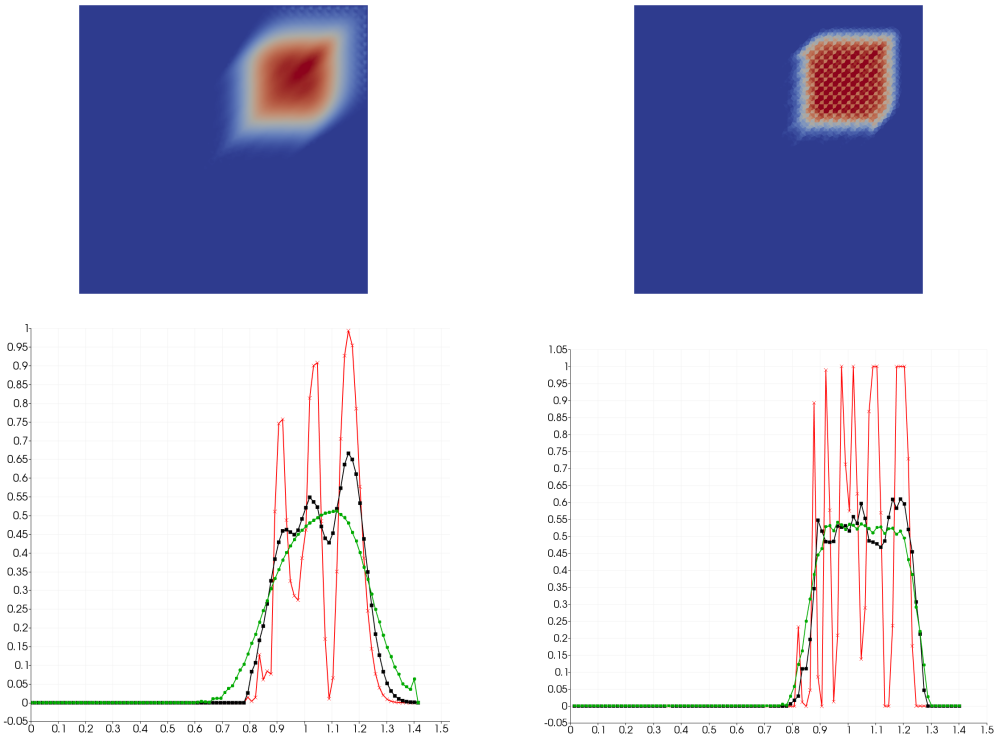

Figure 4: Case $\alpha_{\text{ini}} = 0.5$, solutions at $T = 0.5$. Results obtained on the finest grid. Left: CVFE scheme. Right: same for the VAG scheme (cell center values). Top: colour blue is associated to the value 0, whereas colour red corresponds to 0.5, CVFE for $\theta = 4$ and VAG for $\theta = 1.5$. Bottom: comparison of profiles along the diagonal, cross red for $\theta = 0$, square black for $\theta = 1$ and circle green for $\theta = 4$ (CVFE) or $\theta = 1.5$ (VAG).

with

$$D = \frac{1}{|\vec{v}|}\vec{v} \otimes \vec{v}.$$

This equation shows a transport equation, with an anisotropic diffusion term which applies in the direction of the velocity, as does an upstream weighting scheme.

5 Conclusion

The mathematical study of the approximation of a linear convection – degenerate diffusion problem by a centered θ-scheme for the convection term shows the following features:

1. The strong convergence of the scheme is observed in the case where the continuous solution is such that the measure of the mushy region is equal to 0.
2. A weak convergence always holds for any value of θ, due to the degenerate diffusion term, which is strongly different with the properties of a centred finite volume scheme without (degenerate) diffusion.
3. The centred scheme happens to be much more precise than upstream schemes in some situations.
4. The parameter θ can be numerically used for stabilising the centred scheme as an artificial viscosity.

These results show that a track which remains to be studied is the use of such a θ-centred scheme in practical applications (such as the ones handled in [4]), with the goal to tune the diffusion in such a way that the precision is sufficiently respected.

A Discrete Gronwall's lemma

Although there exist numerous papers providing discrete formulations of Gronwall's inequality [15], let us provide the statement of the precise inequality that we use in this paper, as well as its very short proof.

Lemma A.1 (Discrete Gronwall's lemma)**.** *Let $N \in \mathbb{N}^\star$ be given, let B, $(a_n)_{n=1,\dots,N}$ and $(b_n)_{n=1,\dots,N}$ be non-negative reals such that*

$$\forall m \in \{1,\dots,N\},\ a_m \le \sum_{n=1}^{m-1} b_n a_n + B,$$

with the convention that an empty sum is equal to zero. Then there holds

$$\forall m \in \{1,\dots,N\}, \quad a_m \le B \exp\left(\sum_{n=1}^{m-1} b_n\right).$$

Proof. We define $v_m = \sum_{n=1}^{m-1} b_n a_n$, and we prove by induction that $v_m \le B\exp\left(\sum_{n=1}^{m-1} b_n\right) - B$. Since we have

$$v_{m+1} - v_m = b_m a_m \le b_m(v_m + B) \le b_m B \exp\left(\sum_{n=1}^{m-1} b_n\right),$$

by applying the hypothesis of the lemma (first inequality) and the induction hypothesis (second inequality), we get, using again the induction hypothesis,

$$v_{m+1} \le B(1+b_m)\exp\left(\sum_{n=1}^{m-1} b_n\right) - B \le B\exp(b_m)\exp\left(\sum_{n=1}^{m-1} b_n\right) - B,$$

hence proving the induction hypothesis for $m+1$. □

B Gradient discretisations and time translates

The next two results are used in the proof of estimate of time translates

Lemma B.1. *Let $(X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}}, \mathcal{I}_{\mathcal{D}}, (t^{(n)})_{n=0,\dots,N})$ be a space-time gradient discretisation. Let $\delta t_{\mathcal{D}} = \max_{n=0,\dots,N-1} \delta t^{(n+\frac12)}$. Let $(u^{(n)})_{n=0,\dots,N}$ and $(v^{(n)})_{n=0,\dots,N}$ be two sequences of elements of $X_{\mathcal{D},0}$. Then there holds*

$$\int_0^{T-\tau}\int_\Omega \Pi_{\mathcal{D}}^{(1)}(v(\boldsymbol{x},t+\tau)-v(\boldsymbol{x},t))\ \Pi_{\mathcal{D}}^{(1)}(u(\boldsymbol{x},t+\tau)-u(\boldsymbol{x},t))\mathrm{d}\boldsymbol{x}\mathrm{d}t \\ \le 2\sqrt{\tau(\tau+\delta t_{\mathcal{D}})}\ |\delta_{\mathcal{D}} v|_{L^2(0,T;\star,\mathcal{D})}\ \|\nabla_{\mathcal{D}}^{(1)} u\|_{L^2(0,T;L^2(\Omega)^d)}. \quad (37)$$

Proof. Let us define

$$A(t) = \int_\Omega \Pi_{\mathcal{D}}^{(1)}(u(\boldsymbol{x},t+\tau)-u(\boldsymbol{x},t))\Pi_{\mathcal{D}}^{(1)}(v(\boldsymbol{x},t+\tau)-v(\boldsymbol{x},t))\mathrm{d}\boldsymbol{x}.$$

Let $t \in (0, T-\tau)$. Denoting $n_0(t)$, $n_1(t) = 0,\dots,N-1$ such that $t^{(n_0(t))} \le t < t^{(n_0(t)+1)}$ and $t^{(n_1(t))} \le t+\tau < t^{(n_1(t)+1)}$, we may write

$$A(t) = \int_\Omega \Big(\Pi_{\mathcal{D}} u^{(n_1(t)+1)}(\boldsymbol{x}) - \Pi_{\mathcal{D}} u^{(n_0(t)+1)}(\boldsymbol{x})\Big)\Big(\sum_{n=n_0(t)+1}^{n_1(t)} \delta t^{(n+\frac12)} \delta_{\mathcal{D}}^{(n+\frac12)} v(\boldsymbol{x})\Big)\mathrm{d}\boldsymbol{x},$$

which also reads

$$A(t) = \int_\Omega \Big(\Pi_{\mathcal{D}} u^{(n_1(t)+1)}(\boldsymbol{x}) - \Pi_{\mathcal{D}} u^{(n_0(t)+1)}(\boldsymbol{x})\Big)\Big(\sum_{n=1}^{N-1} \chi_n(t,t+\tau)\delta t^{(n+\frac{1}{2})}\delta_{\mathcal{D}}^{(n+\frac{1}{2})} v(\boldsymbol{x})\Big)\mathrm{d}\boldsymbol{x},$$

with $\chi_n(t,t+\tau) = 1$ if $t^{(n)} \in (t,t+\tau]$ and $\chi_n(t,t+\tau) = 0$ if $t^{(n)} \notin (t,t+\tau]$.
This leads to

$$\begin{aligned} A(t) \leq \quad & \Big(\|\nabla_{\mathcal{D}} u^{(n_1(t)+1)}\|_{L^2(\Omega)^d} + \|\nabla_{\mathcal{D}} u^{(n_0(t)+1)}\|_{L^2(\Omega)^d}\Big) \\ & \times \Big(\sum_{n=1}^{N-1} \chi_n(t,t+\tau)\delta t^{(n+\frac{1}{2})} |\delta_{\mathcal{D}}^{(n+\frac{1}{2})} v|_{\star,\mathcal{D}}\Big). \end{aligned}$$

Using the inequality $ab \leq \frac{1}{2}(\alpha a^2 + \frac{1}{\alpha} b^2)$ for some $\alpha > 0$ chosen later, this yields:

$$A(t) \leq \frac{\alpha}{2}(A_0(t) + A_1(t)) + \frac{1}{\alpha} A_2(t). \tag{38}$$

with

$$A_0(t) = \sum_{n=1}^{N-1} \chi_n(t,t+\tau)\delta t^{(n+\frac{1}{2})} \|\nabla_{\mathcal{D}} u^{(n_0(t)+1)}\|^2_{L^2(\Omega)^d},$$

$$A_1(t) = \sum_{n=1}^{N-1} \chi_n(t,t+\tau)\delta t^{(n+\frac{1}{2})} \|\nabla_{\mathcal{D}} u^{(n_1(t)+1)}\|^2_{L^2(\Omega)^d},$$

and

$$A_2(t) = \sum_{n=1}^{N-1} \chi_n(t,t+\tau)\delta t^{(n+\frac{1}{2})} |\delta_{\mathcal{D}}^{(n+\frac{1}{2})} v|^2_{\star,\mathcal{D}}.$$

Applying (42) in lemma B.2 yields

$$\int_0^{T-\tau} A_0(t)\mathrm{d}t \leq (\tau + \delta t_{\mathcal{D}})\|\nabla_{\mathcal{D}}^{(1)} u\|^2_{L^2(0,T;L^2(\Omega)^d)} \text{ and } \int_0^{T-\tau} A_1(t)\mathrm{d}t \leq (\tau + \delta t_{\mathcal{D}})\|\nabla_{\mathcal{D}}^{(1)} u\|^2_{L^2(0,T;L^2(\Omega)^d)}, \tag{39}$$

and applying (41) in lemma B.2 gives.

$$\int_0^{T-\tau} A_2(t)\mathrm{d}t \leq \tau \int_0^T |\delta_{\mathcal{D}} v(t)|^2_{\star,\mathcal{D}}\mathrm{d}t = \tau |\delta_{\mathcal{D}} v|^2_{L^2(0,T;\star,\mathcal{D})}, \tag{40}$$

hence leading to (37), letting $\alpha = \sqrt{\tau/(\tau+\delta t_{\mathcal{D}})}|\delta_{\mathcal{D}} v|_{L^2(0,T;\star,\mathcal{D})}/\|\nabla_{\mathcal{D}}^{(1)} u\|_{L^2(0,T;L^2(\Omega)^d)}$. □

The following lemma is used in the course of the proof of the preceding lemma.

Lemma B.2. *Let $(t^{(n)})_{n\in\mathbb{Z}}$ be a stricly increasing sequence of real values such that $\delta t^{(n+\frac{1}{2})} := t^{(n+1)} - t^{(n)}$ is uniformly bounded by $\delta t > 0$, $\lim_{n\to-\infty} t^{(n)} = -\infty$ and $\lim_{n\to\infty} t^{(n)} = \infty$. For all $t \in \mathbb{R}$, we denote by $n(t)$ the element $n \in \mathbb{Z}$ such that $t \in [t^{(n)}, t^{(n+1)})$. Let $(a^{(n)})_{n\in\mathbb{Z}}$ be a family of non negative real values with a finite number of non zero values. Then*

$$\int_{\mathbb{R}} \sum_{n=n(t)+1}^{n(t+\tau)} (\delta t^{(n+\frac{1}{2})} a^{(n+1)})\mathrm{d}t = \tau \sum_{n\in\mathbb{Z}} (\delta t^{(n+\frac{1}{2})} a^{(n+1)}),\ \forall \tau \in (0,+\infty), \tag{41}$$

and

$$\int_{\mathbb{R}} \left(\sum_{n=n(t)+1}^{n(t+\tau)} \delta t^{(n+\frac{1}{2})}\right) a^{n(t+\zeta)+1}\mathrm{d}t \leq (\tau + \delta t) \sum_{n\in\mathbb{Z}} (\delta t^{(n+\frac{1}{2})} a^{(n+1)}),\ \forall \tau \in (0,+\infty),\ \forall \zeta \in \mathbb{R}. \tag{42}$$

Proof. Let us define the function $\chi(t,n,\tau)$ by $\chi(t,n,\tau)=1$ if $t<t^{(n)}$ and $t+\tau\geq t^{(n)}$, else $\chi(t,n,\tau)=0$. We have

$$\begin{aligned}\int_{\mathbb{R}}\sum_{n=n(t)+1}^{n(t+\tau)}(\delta t^{(n+\frac{1}{2})}a^{(n+1)})\mathrm{d}t &= \int_{\mathbb{R}}\sum_{n\in\mathbb{Z}}(\delta t^{(n+\frac{1}{2})}a^{(n+1)}\chi(t,n,\tau))\mathrm{d}t\\ &= \sum_{n\in\mathbb{Z}}\left(\delta t^{(n+\frac{1}{2})}a^{(n+1)}\int_{\mathbb{R}}\chi(t,n,\tau)\mathrm{d}t\right).\end{aligned}$$

Since $\int_{\mathbb{R}}\chi(t,n,\tau)\mathrm{d}t=\int_{t^{(n)}-\tau}^{t^{(n)}}\mathrm{d}t=\tau$, thus (41) is proven.
We now turn to the proof of (42). We define the function $\tilde{\chi}(n,t)$ by $\tilde{\chi}(n,t)=1$ if $n(t)=n$, else $\tilde{\chi}(n,t)=0$. We have

$$\int_{\mathbb{R}}\left(\sum_{n=n(t)+1}^{n(t+\tau)}\delta t^{(n+\frac{1}{2})}\right)a^{(n(t+\zeta)+1)}\mathrm{d}t=\int_{\mathbb{R}}\left(\sum_{n=n(t)+1}^{n(t+\tau)}\delta t^{(n+\frac{1}{2})}\right)\sum_{m\in\mathbb{Z}}a^{(m+1)}\tilde{\chi}(m,t+\zeta)\mathrm{d}t,$$

which yields

$$\int_{\mathbb{R}}\left(\sum_{n=n(t)+1}^{n(t+\tau)}\delta t^{(n+\frac{1}{2})}\right)a^{(n(t+\zeta)+1)}\mathrm{d}t=\sum_{m\in\mathbb{Z}}a^{(m+1)}\int_{t^m-\zeta}^{t^{m+1}-\zeta}\left(\sum_{n=n(t)+1}^{n(t+\tau)}\delta t^{(n+\frac{1}{2})}\right)\mathrm{d}t. \tag{43}$$

Since we have

$$\sum_{n=n(t)+1}^{n(t+\tau)}\delta t^{(n+\frac{1}{2})}=\sum_{n\in\mathbb{Z},\ t<t^{(n)}\leq t+\tau}(t^{(n+1)}-t^{(n)})\leq\tau+\delta t,$$

we can write from (43)

$$\begin{aligned}\int_{\mathbb{R}}\left(\sum_{n=n(t)+1}^{n(t+\tau)}\delta t^{(n+\frac{1}{2})}\right)a^{(n(t+\zeta)+1)}\mathrm{d}t &\leq(\tau+\delta t)\sum_{m\in\mathbb{Z}}a^{(m+1)}\int_{t^{(m)}-\zeta}^{t^{(m+1)}-\zeta}\mathrm{d}t\\ &=(\tau+\delta t)\sum_{m\in\mathbb{Z}}a^{(m+1)}\delta t^{(m+\frac{1}{2})},\end{aligned}$$

which is exactly (42).

□

References

[1] H. W. Alt and S. Luckhaus. Quasilinear elliptic-parabolic differential equations. *Math. Z.*, 183(3):311–341, 1983.

[2] L. Ambrosio. Transport equation and Cauchy problem for BV vector fields. *Invent. Math.*, 158(2):227–260, 2004.

[3] G. Amiez and P.-A. Gremaud. On a numerical approach to Stefan-like problems. *Numer. Math.*, 59(1):71–89, 1991.

[4] L. Beaude, K. Brenner, S. Lopez, R. Masson, and F. Smai. Non-isothermal compositional two-phase Darcy flow: formulation and outflow boundary condition. In *Finite volumes for complex applications VIII—hyperbolic, elliptic and parabolic problems*, volume 200 of *Springer Proc. Math. Stat.*, pages 317–325. Springer, Cham, 2017.

[5] M. Bertsch, P. de Mottoni, and L. A. Peletier. The Stefan problem with heating: appearance and disappearance of a mushy region. *Trans. Amer. Math. Soc.*, 293(2):677–691, 1986.

[6] J. Carrillo. Entropy solutions for nonlinear degenerate problems. *Arch. Ration. Mech. Anal.*, 147(4):269–361, 1999.

[7] D. De Vries. Thermal properties of soils. In W. R. van Wijk, editor, *Physics of Plant Environment (W. R. van Wijk ed.)*, pages 210–235. North-Holland Publishing Co., Amsterdam, 1964.

[8] R. J. DiPerna and P.-L. Lions. On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. of Math. (2)*, 130(2):321–366, 1989.

[9] J. Droniou, R. Eymard, T. Gallouët, C. Guichard, and R. Herbin. *The gradient discretisation method*, volume 82 of *Mathematics & Applications*. Springer International Publishing, 2018. `https://hal.archives-ouvertes.fr/hal-01382358`.

[10] C. Elliott. Error analysis of the enthalpy method for the Stefan problem. *IMA journal of numerical analysis*, 7(1):61–71, 1987.

[11] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[12] R. Eymard, P. Feron, T. Gallouët, R. Herbin, and C. Guichard. Gradient schemes for the Stefan problem. *International Journal on Finite Volumes*, page Volume 10 special, June 2013.

[13] R. Eymard, T. Gallouët, C. Guichard, R. Herbin, and R. Masson. TP or not TP, that is the question. *Comput. Geosci.*, 18(3-4):285–296, 2014.

[14] R. Eymard, C. Guichard, and R. Herbin. Small-stencil 3d schemes for diffusive flows in porous media. *M2AN Math. Model. Numer. Anal.*, 46:265–290, 2012.

[15] T. H. Gronwall. Note on the derivatives with respect to a parameter of the solutions of a system of differential equations. *Ann. of Math. (2)*, 20(4):292–296, 1919.

[16] R. Herbin and F. Hubert. Benchmark on discretization schemes for anisotropic diffusion problems on general grids. In *Finite volumes for complex applications V*, pages 659–692. ISTE, London, 2008.

[17] I. C. Kim and N. Požár. Viscosity solutions for the two-phase Stefan problem. *Comm. Partial Differential Equations*, 36(1):42–66, 2011.

[18] R. Nochetto and C. Verdi. Approximation of degenerate parabolic problems using numerical integration. *SIAM Journal on Numerical Analysis*, 25(4):784–814, 1988.

[19] G. Stampacchia. Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. *Ann. Inst. Fourier (Grenoble)*, 15(fasc. 1):189–258, 1965.