



HAL
open science

Mathematical analysis of informal arguments: A case-study in teacher-training context

Renaud Chorlay

► **To cite this version:**

Renaud Chorlay. Mathematical analysis of informal arguments: A case-study in teacher-training context. CERME 10, Feb 2017, Dublin, Ireland. hal-01938818

HAL Id: hal-01938818

<https://hal.science/hal-01938818>

Submitted on 28 Nov 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Mathematical analysis of informal arguments: A case-study in teacher-training context

Renaud Chorlay

LDAR, Uni. Paris Diderot, France & ESPE de l'Académie de Paris, France

renaud.chorlay@espe-paris.fr

*We will endeavour to show that some historical documents which are probably too difficult to be used in the classroom can nevertheless be fruitfully used in teacher-training, in order to provide teachers with tools for the analysis of informal or semi-formal justifications. We will analyze an extract from the *Nine Chapters*, so as to spell out tool for the analysis of three school documents bearing on the multiplication of decimal or general fractions.*

Keywords: History of mathematics, fractions, teacher-training, argumentation, generic examples.

Introduction

We will endeavour to show that some historical documents which are probably too difficult to be used in the classroom can nevertheless be fruitfully used in teacher-training, in order to provide teachers with tools for the analysis of informal or semi-formal justifications. We will analyze an extract from the *Nine Chapters*, so as to spell out tool for the analysis of three school documents bearing on the multiplication of decimal or general fractions.

A historical example: Multiplying fractions in the *Nine Chapters*

The *Nine Chapters on the Mathematical Art* (九章算術) is a Classic that was compiled during the Han Dynasty (206 BC – 220 AD). It contains an organized list of problems with general procedures to solve them, yet with no attempts at justification. In 263 AD, scholar Liu Hui wrote an extensive commentary in which endeavoured to justify all the procedures, and identify key subprocedures of general scope. We will comment on the passage in chapter one which deals with the multiplication of fractions, on the basis of the critical edition and French translation (Chemla & Shuchun, 2004)¹.

Problem 1.19. Now given a [rectangular] field, $\frac{4}{7}$ *bu* in breadth and $\frac{3}{5}$ *bu* in length. Tell: what is the area? Answer: $\frac{12}{35}$ *bu*.

(...)² Procedure for the multiplication of parts: multiply the denominators to make up the divisor; multiply the numerators to make up the dividend. Divide the dividend by the divisor.

(...) [Liu Hui]: In all cases when a dividend does not fill a divisor, then they are called denominator and numerator. If there are parts, expand the corresponding dividend by multiplication, then, when the divisor is filled, the division yields an integer. If, moreover, one multiplies something by the numerator, the denominator must therefore divide in return [*baochu*]. To divide in return is to “divide the dividend by the divisor”. Now, “the numerators are multiplied one by the other”, hence both

¹ An independent English translation is also available (Crossley, Lun & Shen, 1999). As far as this passage is concerned, the two translations differ significantly; our interpretation is based on Chemla’s translation and interpretation.

² I edited out two similar examples (with proper fractions), as well as Li Chunfeng’s commentary.

denominators have to divide in return. Whence the multiplication of the two denominators, and the division at one go [by their product].

The mere length and width of a rectangular field leave no room for a general explanation. Let someone ask: “20 horses are valued at 12 *jin* of gold. Now 20 horses are sold and the proceeds are shared by 35 persons. How much does each one get?” The answer is $12/35$ *jin*. To solve this by the procedure for dividing into parts, take 12 *jin* of gold as dividend, and 35 persons as divisor. Now, change [the problem] to: “5 horses are valued at 3 *jin* of gold; now 4 horses are sold, and [the proceeds] are shared by 7 persons. How much does each one get?” Answer: each one gets $12/35$ of *jin*. To solve this, you need to homogenize these quantities (*shu*) of people and of gold; it is then completely similar to the first problem, that of dividing into parts. If so, “multiply the numerators to make up the dividend” does homogenize this quantity of gold; “multiply the numerators to make up the dividend” does homogenize this quantity of people. Equalizing the denominators yields 20, but that plays no part: we just need the homogenized. Moreover, when 5 horses are valued at 3 *jin* of gold, these are the *liu* in whole numbers. If we express them with parts, 1 horse is valued at $3/5$ *jin* of gold. Let 7 people sell 4 horses, 1 person does sell $4/7$ of a horse. (...) As far as expression is concerned, that’s different; yet as far as quantities are concerned, the three procedures boil down to the same.

Figure 1: (Chemla & Shuchun, 2004, 169-171). Free translation: R. Chorlay

Even if the procedure is quite easy to understand – and familiar to the reader – Liu Hui’s commentary is probably hard to decipher. Even without a clear understanding of the details, one can probably see at least two things: the goal of Liu is to justify the general procedure of the classic; he gives two different justifications, one which does not rely on a semantic context nor on specific numbers, and one which does.

Let us first focus on the first justification. Liu first distinguishes between two cases: a basic case in which the fractions represent whole numbers; in this case, the procedure for multiplication is already known. Of course, the goal is here to justify the new procedure – in the case where the divisions have non-zero remainders – on the basis of the basic procedure. The word “procedure” is important, since what is to be justified is the validity of an algorithm. More precisely, two sequences of calculations are to be compared:

Sequence 1: divide a by b ; divide c by d ; *multiply* the two quotients.

Sequence 2: multiply a by c ; multiply b by d ; divide the first product by the second product.

If the two sequences relied on operations defined previously and independently, the goal would have been to prove the universal equivalence of these two algorithms (that is, same entries yield the same output). However, if one is not to beg the question, the first sequence actually contains the undefined operation *multiply*, since the final multiplication may involve a multiplication of fractions³. So the goal here is rather to justify that sequence 2 (which involves only well defined operations on integers)

³ When the remainder in the Euclidean division is non-zero, the quotient, the remainder and the dividend were read as a mixed number. For instance, the division of 7 by 3 has quotient 2 and remainder 1, so its result is 2 and $1/3$.

should be universally equivalent to sequence 1, thereby implicitly defining “multiply” in terms of “multiply.”

To perform this, Liu uses terms which are specific to the commentary: whereas the Classic uses “divide” and “multiply”, Liu uses “expand by multiplication”, and “divide in return” (*baochu*); these terms point to the role of these operations. However, it is important to know that throughout the *Nine Chapters*, “*baochu*” denotes a division whose role is to compensate for a multiplication by the same number, so that these two steps of the procedure do not affect the final output of the sequence of operations. On this basis, one can see that Liu’s justification involves a third sequence of operations:

Sequence 3: 1. multiply a by b , so that the first factor becomes ab / b , which is equal to a
 2. multiply c by d , so that the second factor becomes cd / d , which is equal to c
 3. multiply the two factors, which yields ac
 4. divide by bd , which is the product of the two denominators

This sequence is not to be *actually* performed, it is mentioned for justificatory purposes only. Since step 4 is a *baochu*-division, it compensates steps 1 and 2. The line of argumentation can thus be reconstructed as: sequence 1 (which is yet undefined) should be universally equivalent to sequence 3, and sequence 3 boils down to sequence 2.

This line of argumentation may seem far-fetched. However, it is quite close to the following contemporary proof of the following theorem: the only map f from $\mathbb{Q} \times \mathbb{Q}^*$ which is \mathbb{Z} -bilinear and which is a prolongation of integer multiplication (i.e. if m and n are integers, then $f(m, n) = m \times n$), is $f\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{a \times c}{b \times d}$ (where a, b, c, d denote integers, with $bd \neq 0$). The proof goes as follows:

$$\begin{aligned} (b \times d) \times f\left(\frac{a}{b}, \frac{c}{d}\right) &= f\left(b \times \frac{a}{b}, d \times \frac{c}{d}\right) \text{ by } \mathbb{Z}\text{-bilinearity} \\ &= f(a, c) \text{ by a Lemma}^4 \text{ which relies only on addition in } \mathbb{Q} \\ &= (a \times c) \quad (\text{prolongation requirement}) \end{aligned}$$

Diving both sides by (bd) completes the proof. One could argue that, although Liu Hui states the prolongation requirement quite explicitly, never does he say anything resembling the bilinearity requirement. Even if this requirement is not explicit, it should be stressed that the bilinearity of the operation being defined (i.e. *multiply*) can be rephrased as follows: multiplying either of the two factors by an integer n should multiply the product by n ; which is exactly the property which justifies the role of the *baochu* division.

The second part of Liu Hui’s commentary clearly follows another line of argumentation. The first part does not rely on a semantic context of interpretation, nor does it use specific numbers. The second part, however, depends on a semantic context which Liu Hui introduces out of the blue after dismissing the context provided by the Classic (rectangular areas), and uses the numbers from Problem 19.

In the second part, several in-context problems are mentioned, and their relationships discussed.

⁴ $b \times \frac{a}{b} = \frac{a}{b} + \dots + \frac{a}{b}$ (with b terms) $= \frac{ba}{b} = a$ (assuming $b \in \mathbb{N}^*$).

Pb. 1: 20 horses are valued at 12 *jin* of gold [for all 20 horses]. Now 20 horses are sold and the proceeds are shared by 35 persons. How much does each one get?

Pb. 2: 5 horses are valued at 3 *jin* of gold [for all 5]. Now 4 horses are sold and [the proceeds] are shared by 7 persons. How much does each one get?

Problems 1 and 2 can be connected by a series of problems. Starting from Problem 2, one can consider:

Pb. 2': 5x4 horses are valued at 3x4 *jin* of gold. Now 4 horses are sold and [the proceeds] are shared by 7 persons. How much does each one get?

Although Pb2 and 2' differ from a semantic viewpoint, they are equivalent in the following sense: the numerical answers to Pb 2 and 2' are equal, since the second parts are the same, and the answer depends only on the ratio between horses and *jin* of gold. The argument would hold for any factor, it so happens that "4" is more relevant than others for what follows⁵. This is quite explicit in the text, since throughout the *Nine Chapters* the technical term *lii* denotes either a ratio, or numbers which are to be considered up to multiplication by a common factor. The relationship between Pb2 and Pb 2' is the same as that between Pb2' and Pb2'':

Pb. 2'': 5x4 horses are valued at 3x4 *jin* of gold. Now 4x5 horses are sold and [the proceeds] are shared by 7x5 persons. How much does each one get?

Now, Pb 2'' is the same as Pb 1, so their numerical answers are the same: 12/35.

But there is another way to solve Pb 2. The numerical answer to Pb 2 is the same as that of

Pb 2''': 1 horse is valued at 3/5 *jin* of gold. Now 4/7 horses are sold and [the proceeds] are shared by 1 person. How much does this person one get?

We want to define multiplication of fractions so that the following property of integer multiplication remains valid: total value = unit value times quantity. As a consequence, the product $\frac{3}{5} \times \frac{4}{7}$ should be the numerical answer to Pb 2''', hence to Pb 2 just as well, hence equal to 12/35.

A more formal summary would go as follows (we use \otimes to denote to multiplication of fractions, to be defined here in terms of \times):

$$\frac{3}{5} \otimes \frac{4}{7} = \frac{3 \times 4}{5 \times 4} \otimes \frac{4 \times 5}{7 \times 5} = \frac{3 \times 4}{\cancel{5 \times 4}} \otimes \frac{\cancel{4 \times 5}}{7 \times 5} = \frac{3 \times 4}{7 \times 5}$$

As for the extra-steps mentioned in the first argument, none of these intermediary operations are to be performed when actually multiplying fractions; they are here for justificatory purposes only. This formal summary does not do justice to the nature of the argumentation, since it is the context only which justifies the equalities: dependence on ratios only (first equality); then reduction to a problem involving only two integer data instead of four, and which can be solved by ordinary division. Just as well, the fact that any given fraction can be freely replaced by any equivalent fraction can be justified

⁵We will not comment on the use of technical terms such as "equalizing" and "homogenizing", which Liu Hui introduced in the context of the addition of fractions: $\frac{a}{b} + \frac{c}{d} = \frac{(ad)+(bc)}{(bd)}$ ← this is what you get by "homogenizing".

by the context; for instance, the unit price depends only on the ratio between the number of horses and their total value.

Tools for the analysis of arguments in today's textbooks

Reflecting on the *Nine Chapters* from a teacher-training perspective

The obscurity of the Chinese text probably makes it impossible to use in the classroom. In this paper, we would like to show how the mathematical analysis of this text that we carried out in the first part can help raise the awareness of prospective teachers as to several general features of argumentation in rather informal contexts (this is our Hypothesis #1); and even supply them with tools which are useful in everyday teaching, when they are to decide how to (somehow) justify some mathematical definition or property (this is our Hypothesis #2)⁶.

The analysis of the *Nine Chapters* drew our attention to several facts, some of which bear specifically on fraction multiplication, and some of which are of a more general scope:

- There are several ways to justify the rule for the multiplication of fractions.
- The use of letters is not the only way to express a general line of argumentation, as Liu Hui's first argument illustrates.
- The rule for fraction multiplication is not only an equality between *formulae*; it can also be seen as the equivalence of two different algorithms (one with two divisions followed by a multiplication, and one with two multiplications followed by a division). This equivalence of algorithm can be established using general arguments within the algorithmic framework (i.e. arguments *about* algorithms), as opposed to the rewriting of formulae within an algebraic framework.
- Argumentation is not limited to properties or theorems: the choice of a definition (here: for the multiplication fractions) can also be justified. With fraction multiplication we are dealing with a case domain extension: some notion was already defined for a given class of objects; the to-be-defined notion has to apply to a class of objects that encompasses the first class, and to coincide with the former notion on the first class. In our case, a very weak form of justification would be: this definition of fraction multiplication boils down to ordinary multiplication (of integers) should the fractions denote integers (this could be called the *control case*). The *Nine Chapters* suggests a stronger kind of justification: the extended definition of "multiply" should preserve some structural properties of the former definition, properties that we value. For instance: bilinearity; or the validity of the formula "total value = unit value times quantity". On the other hand, we are ready to give up on some properties, such as: the product of two numbers is greater than or equal to either of them. To some extent, when justify the choice of a definition, these required properties play the part that hypotheses play in the proof of a theorem.
- In some cases, from a mathematical viewpoint, these requirements completely characterize the new notion. In these cases, whether or not this uniqueness property should be proved in

⁶ The decisions as to *how* to justify a given definition or property is closely related to the decision as to whether or not it *should* be, and to the knowledge of ways to justify it (*can* it be defined, or not, in a given mathematical context?). We will not have time to discuss the interplay of these three aspects in this short paper.

the classroom calls for another decision. Even if the teacher thinks the uniqueness aspect cannot / need not be made explicit, the *Nine Chapters* suggest another lead. Making explicit the properties that we want to extend also has a *heuristic* value, suggesting the path to a definition of the extended notion.

- Liu Hui's second line of argumentation also suggests many fruitful leads, which we can only mention in passing for lack of space:
 - A rather unusual form of argument: proving that two formulae or two algorithms are universally equal / equivalent by exhibiting a class of in-context problems for which the answer is unique – for any given set of initial data –, and for which – for reasons that can be spelled out – both formulae or algorithms work out the solution. Of course, this form of argumentation – that can be called *semantic immersion* – raises questions as to the context-dependence of the argument. For instance, multiplication plays a part in the context of commercial transactions and in the context of area calculation, and nothing proves that the fact that the relevant operations coincide in both contexts when the data are integers will still hold with all rational data.
 - Liu Hui's second argument is not only in-context (a horse-deal), it is also an arguments which clearly claims full generality while dwelling on a single specific case, that of $\frac{3}{5}$ times $\frac{4}{7}$. This raises the well-known but deep questions pertaining to the notion of *generic* example (Balachef, 1987, 157). Let us mention two such questions: (1) from a mathematical viewpoint, which conditions guarantee the minimal level of non-specificity that an example should enjoy to be potentially seen as generic (for instance: dealing with $\frac{1}{5}$, $\frac{7}{5}$, or $\frac{15}{5}$ instead of $\frac{3}{5}$ would probably make this case less generic)? (2) The *genericity* of a case is not a mathematical property but is a multilayered property of the relationship between it and the students; it involves pragmatic aspects (students should be able to *adapt* the reasoning to other cases), and epistemological aspects (students should *regard* this case as a mere representative of all cases, thus providing some form a general argument); what are the general conditions for a didactical genesis of this relationship?

Field-work

We shall end with some applied work. The following three documents are extracts from contemporary French textbooks or standard teacher-training documents, for students in the second year of middle-school (Figures 2 and 3) or in the last year of primary school (Figure 4). In cases 2 and 3 they are meant for students who are somewhat familiar with fractions and their addition, but with no knowledge of multiplication. As to Figure 4, students are familiar with integer multiplication, and with multiplication and division of decimals by powers of 10. These extracts are what most French textbooks call “introductory activities”, in which students are to experience a new notion from a hands-on, guided but non-dogmatic approach. We suggest these documents be analyzed in TWG12 in order to test Hypotheses #1 and #2.

ACTIVITÉ 3 Multiplication de deux fractions

1. L'unité de longueur est la longueur du côté du carré ABCD et l'unité d'aire est l'aire de ce carré.

a. S'aider du quadrillage pour déterminer l'aire du rectangle AMNP à l'aide d'une fraction.

b. Exprimer les dimensions de ce rectangle AMNP à l'aide de fractions.
Quelle opération faut-il effectuer sur ces fractions pour obtenir l'aire du rectangle AMNP ?

c. Recopier et compléter : $\frac{4}{5} \times \frac{2}{3} = \dots$

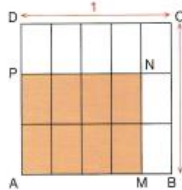



Figure 2: Two ways to work out a rectangular area (Malaval, 2014, 69)

1 Fraction de fraction

a) Thomas a mangé $\frac{1}{2}$ des $\frac{3}{5}$ d'une tarte rectangulaire. Quelle fraction de la tarte Thomas a-t-il mangé ?

b) Donner l'écriture décimale de $\frac{1}{2}$, de $\frac{3}{5}$ et du résultat de la question a).

c) Quelle opération sur les deux fractions $\frac{1}{2}$ et $\frac{3}{5}$ permet de trouver le résultat ?



2 Multiplication de fractions Exercices 8 à 18 p. 55

a) Calculer sans passer par l'écriture décimale et donner le résultat sous forme d'une fraction :

$A = \frac{1}{5} \times \frac{3}{4}$; $B = \frac{7}{10} \times \frac{3}{10}$; $C = 2 \times \frac{3}{5}$

b) Contrôler le résultat en revenant à l'écriture décimale des fractions.

c) Énoncer la règle qui semble permettre de multiplier deux fractions.

Figure 3: A justification in terms of combination of operators, and a conjecture relying on the control-case of decimal fractions (Mante, 2006, 50)

Multiplication	Technique opératoire
Multiplication de nombres décimaux	
<p>Calcul de $12,7 \times 35,2$</p> <pre style="font-family: monospace; font-size: small;"> 1 2 , 7 x 3 5 , 2 ----- 2 5 4 6 3 5 0 3 8 1 0 0 ----- 4 4 7 , 0 4 </pre>	<p>Tout se déroule comme dans l'exemple précédent. Je fais l'opération sans tenir compte de la virgule et je la positionne au résultat.</p> <p>En fait, je vais multiplier 12,7 et 35,2 par 10</p> <p>Le résultat doit donc être divisé par $10 \times 10 = 100$</p>

Figure 4: Generalizing multiplication from integers to decimals using bilinearity⁷

Conclusion

The goal of this short paper was to contribute to the ongoing reflection on the use of original historical sources in the classroom or in teacher-training; a reflection which bears both on the goals of this use, and on the relevant ways to use such documents.

Within this general field of research, the main features of this specific contribution are: we dealt with teacher-training only, since we do not think this excerpt from the *Nine Chapters* can be used with

⁷ Document retrieved from a teacher-training slideshow: <http://docslide.fr/documents/techniques-operatoires-cycles-2-et-3-multiplication-jean-luc-despretz-cpc-landivisiau-avril-2010-lacquisition-des-mecanismes-en-mathematiques-est-toujours.html> (slide 22). Accessed September 14, 2016.

students who learn to multiply fractions; we endeavoured to show that even for such documents, they can be used in teacher-training for other purposes than simply teaching some history of mathematics, or enriching teacher's image of mathematics. As teachers, it is or will be part of their everyday work to analyze/assess teaching documents and make decisions on the basis of this analysis/assessment. Among these documents, some will be of an argumentative nature: sometimes in the form of pretty formal and academic proofs; more often than not, these documents will explicitly display or rather implicitly point to other forms of argumentation. We think the conceptual analysis – an analysis which has a mathematical component and an epistemological component, and which is to be distinguished from both the historical analysis and the didactical analysis – of some historical documents can give teacher-trainers opportunities to make explicit some tools which can be of constant use in the analysis and assessment of argumentative documents.

In this paper, we dealt with pretty informal arguments and elementary mathematics. However, we do not think this general scheme is specific to such contexts, since this work is a continuation of our CERME8 paper (Chorlay, 2013), in which a similar approach was used in a formal and rather advanced mathematical context (proof of the relationship between the variations of a function and the sign of its derivative).

References

- Balacheff, N. (1987). Processus de preuve et situations de validation. *Educational Studies in Mathematics*, 18, 147–176.
- Chemla, K., Shuchun, G. (2004). *Les neuf chapitres : Le classique mathématique de la Chine ancienne et ses commentaires*. Paris: Dunod.
- Chorlay, R. (2013). The making of a proof-chain: Epistemological and didactical perspectives. In B. Ubuz, Ç. Haser, & M.-A. Mariotti (Eds.), *Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education* (pp. 106–115). Ankara, Turkey: Middle East Technical University and ERME.
- Crossley, N., Lun, A., Shen, K. (1999). *The nine chapters on the mathematical art: Companion and commentary*. Oxford: O.U.P.
- Malaval, J. (Ed.) (2014). *Transmath 5ème*. Paris: Nathan.
- Mante, M. (Ed.) (2006). *Triangle 5ème*. Paris: Hatier.